Project 1
Quark DSE

ECT* Doctoral Training Program, May 2–18, 2022
Hadron Physics with Functional Methods

Introduction

The basic quantity that describes a particle in a quantum field theory (QFT) is its propagator. For example, the propagator of a free scalar particle has the form

$$\frac{1}{p^2+m^2}\,, \tag{1}$$

where m is the mass of the particle and p^2 is the squared momentum (which is Lorentz invariant). This doesn't tell us much except for the fact that it has a *pole* for some negative value of $p^2=-m^2$, which is the free particle pole. Very roughly speaking, this is how a QFT allows us to extract the mass of some particle or intermediate resonance: When you see a bump or a peak in an experimental cross section, then this corresponds to a pole in some propagator or some scattering amplitude, and the momentum or energy where the peak appears defines the mass of that particle. These poles do not necessarily sit on the real negative p^2 axis but they can also appear in the complex plane $p^2\in\mathbb{C}$; in fact, the resonance peaks we observe in experiments correspond to poles in the complex plane on higher Riemann sheets.

Quarks are spin-1/2 particles, so the most general possible form of the quark propagator $S(p)$ according to Lorentz invariance (using Euclidean conventions) is

$$S(p)^{-1}=A(p^2)\left(i\not{p}+M(p^2)\right) \quad\Leftrightarrow\quad S(p)=\frac{1}{A(p^2)}\,\frac{-i\not{p}+M(p^2)}{p^2+M(p^2)^2}=-i\not{p}\,\sigma_v(p^2)+\sigma_s(p^2)\,. \tag{2}$$

It depends on two dressing functions, $A(p^2)$ and the **quark mass function** $M(p^2)$, or equivalently the two dressing functions $\sigma_v(p^2)$ and $\sigma_s(p^2)$. The variable p^2 can take any value $p^2\in\mathbb{C}$, although in the following we are mainly interested in spacelike momenta, $p^2\in\mathbb{R}_+$. For a free spin-1/2 particle, Eq. (2) simplifies to $A(p^2)=1$ and $M(p^2)=m$, where m is the mass of the particle; in that case, the propagator becomes

$$S_0(p)=\frac{-i\not{p}+m}{p^2+m^2}\quad\Leftrightarrow\quad \sigma_v(p^2)=\frac{1}{p^2+m^2}\,,\quad \sigma_s(p^2)=\frac{m}{p^2+m^2}\,. \tag{3}$$

For interacting particles we should not expect to find a structure like in Eq. (3), i.e., $A(p^2)\neq 1$ and $M(p^2)\neq m$, although in practice they can look quite similar. What makes quarks rather special is that QCD is a gauge theory and quarks are confined inside hadrons, so their analytic structure may look drastically different; in principle, the functions $\sigma_v(p^2)$ and $\sigma_s(p^2)$ could have (many) poles or branch cuts in the complex plane of p^2.

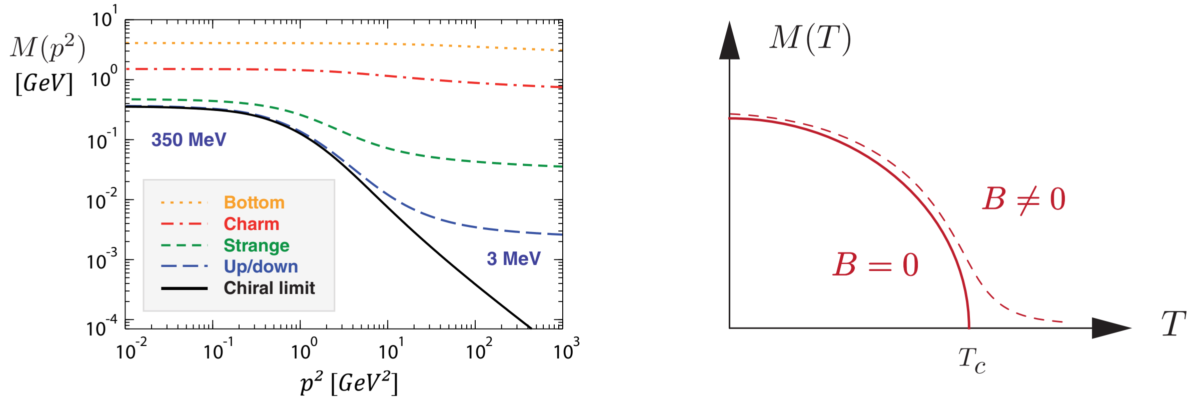

Figure 1: Quark mass function for different flavors [1]; magnetization as a function of temperature

Light quarks are also special for another reason, because most of their mass is dynamically generated in QCD through their interactions with gluons. The light u and d quarks have current-quark masses of about 3–5 MeV, which arise from the Higgs mechanism and are an external input to QCD. However, the mass of a proton is 940 MeV, which means that about 99% of the mass of the proton (and therefore nuclei and atoms) must be somehow produced in QCD. This can be understood through **spontaneous chiral symmetry breaking**, which is a non-perturbative effect that emerges through the dynamics of quarks and gluons, i.e., their interactions cause the quarks to gain mass. This is visible in the mass function $M(p^2)$: At large momenta, $M(p^2)$ becomes the current-quark mass m, whereas at small momenta it is much larger by several hundred MeV and thereby defines a 'constituent-quark mass' (Fig. 1), which is the relevant mass scale for the proton and other hadrons. An analogy for this is the spontaneous magnetization of a magnet below a critical temperature. If an external magnetic field B is switched on, the magnetization persists also for large temperatures. For the quark propagator in QCD, the role of T is played by the momentum and the analogue of B is the current-quark mass in the Lagrangian.

The fact that this effect is non-perturbative means that we cannot produce it at any order in **perturbation theory**, which is one of the main tools to make a QFT useful in practice. When we sum up Feynman diagrams, where each comes with a power in the coupling, then if the coupling is small enough one can stop the series after a few terms. For QCD this only works at large momenta where the strong coupling α_{QCD} is small; this is what allows us to describe high-energy scattering processes. However, for small momenta α_{QCD} becomes large and perturbation theory no longer works. A simple analogue is the geometric series

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \qquad \text{for} \quad |x| < 1\,, \tag{4}$$

which only converges for $|x| < 1$. But suppose someone *gave* you the equation

$$f(x) = 1 + x f(x) \quad \Leftrightarrow \quad f(x)^{-1} = 1 - x\,, \tag{5}$$

which has the solution $f(x) = 1/(1-x)$ for any x. If you insert the l.h.s. into the r.h.s., then

$$f(x) = 1 + x f(x) = 1 + x + x^2 f(x) = 1 + x + x^2 + x^3 f(x) = \dots \tag{6}$$

is exact at every step, whereas in the geometric series we drop the last term which otherwise pulls the result back even if x becomes large, and this is what makes the perturbative expansion fail.

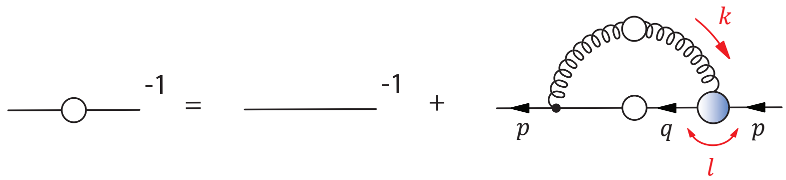

Figure 2: Momentum routing in the quark DSE.

Quark DSE

The analogues of Eq. (5) in QCD are the Dyson-Schwinger equations (DSEs), which are the exact, non-perturbative quantum equations of motion of a QFT. The **quark DSE** is shown in Fig. 2 and determines the quark propagator $S(p)$:

$$S(p)^{-1} = Z_2 \, (i\not{p} + m_0) + \Sigma(p) \, , \tag{7}$$

where $\Sigma(p)$ is the quark self-energy that depends on the gluon propagator, the quark-gluon vertex and again the quark propagator. m_0 is the bare current quark mass that enters in the QCD Lagrangian, and Z_2 is the quark renormalization constant. The equation has the structural form of Eq. (5), $f(x)^{-1} = 1 - x$ (the minus is absorbed in the self-energy). Thus, if the QCD coupling contained in $\Sigma(p)$ becomes small, we can expand $S(p)$ into a series like in Eq. (6) by reinserting the equation at every instance where the quark propagator appears inside the loop, and stop after a few terms — this is the perturbative series for the quark propagator shown in Fig. 3. However, if the coupling becomes large we have no choice but to solve the equation directly.

The self-energy has the explicit form

$$\Sigma(p) = -\frac{4g^2}{3} \, Z_\Gamma \int\limits_q i\gamma^\mu \, S(q) \, D^{\mu\nu}(k) \, \Gamma^\nu(l,k) \tag{8}$$

and contains the following ingredients:

- g is the strong coupling ($\alpha_{\text{QCD}} = g^2/(4\pi)$) and the prefactor $4/3$ comes from the color trace.
- The dressed quark propagator $S(q)$ appears again inside the loop, so the DSE is an integral equation which determines $S(p)$.
- The dressed gluon propagator $D^{\mu\nu}(k)$ depends on the gluon momentum $k = q - p$. We work in Landau gauge, where it is given by

$$D^{\mu\nu}(k) = \frac{Z(k^2)}{k^2} \, T_k^{\mu\nu}, \qquad T_k^{\mu\nu} = \delta^{\mu\nu} - \frac{k^\mu k^\nu}{k^2} \, . \tag{9}$$

  $Z(k^2)$ is the gluon dressing function and $T_k^{\mu\nu}$ is a transverse projector.
- The tree-level quark-gluon vertex $g \, Z_\Gamma \, i\gamma^\mu$ comes with a renormalization constant Z_Γ.
- The dressed quark-gluon vertex $g \, \Gamma^\mu(l,k)$ depends on the average quark momentum $l = (q+p)/2$ and the gluon momentum k. In principle it consists of 12 Lorentz-Dirac tensors, but we restrict ourselves to the **rainbow-ladder truncation** which is defined by the ansatz

$$\Gamma^\mu(l,k) = f(k^2) \, i\gamma^\mu \, , \tag{10}$$

  where the dressing function $f(k^2)$ depends on the gluon momentum only.

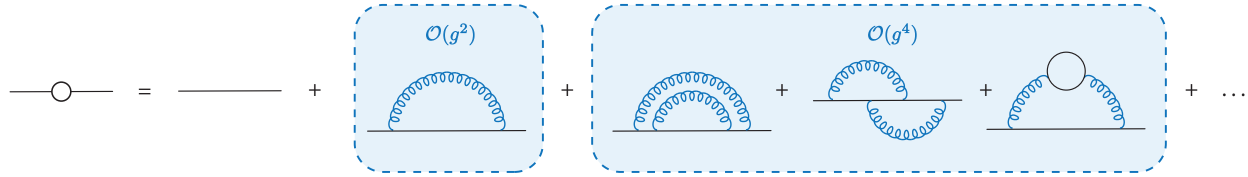

Figure 3: Perturbative expansion of the quark propagator

- The integral measure $\int_q$ is given by

$$\int_q = \int \frac{d^4 q}{(2\pi)^4} = \frac{1}{(2\pi)^4}\,\frac{1}{2}\int_0^{L^2} dq^2\,q^2 \int_{-1}^{1} dz\,\sqrt{1-z^2}\int_{-1}^{1} dy \int_0^{2\pi} d\phi\,, \tag{11}$$

where L is the cutoff in the system (a typical value is $L = 10^3$ GeV) and we use hyperspherical variables:

$$p^\mu = \sqrt{p^2}\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \qquad q^\mu = \sqrt{q^2}\begin{bmatrix} \sqrt{1-z^2}\,\sqrt{1-y^2}\,\sin\phi \\ \sqrt{1-z^2}\,\sqrt{1-y^2}\,\cos\phi \\ \sqrt{1-z^2}\,y \\ z \end{bmatrix}. \tag{12}$$

Because in the end we will break down the quark DSE into two Lorentz-invariant equations for $A(p^2)$ and $M(p^2)$ and the only Lorentz invariants in the system are p^2, q^2 and $p\cdot q = \sqrt{p^2}\sqrt{q^2}\,z$, the two integrations over the variables y and ϕ become trivial so that only the two integrations over q^2 and z remain. Therefore, we can equivalently work with the vector

$$q^\mu = \sqrt{q^2}\begin{bmatrix} 0 \\ 0 \\ \sqrt{1-z^2} \\ z \end{bmatrix}. \tag{13}$$

Putting things together, we arrive at

$$\Sigma(p) = \frac{16\pi}{3}\,Z_2^2 \int_q \frac{\alpha(k^2)}{k^2}\,T_k^{\mu\nu}\gamma^\mu S(q)\gamma^\nu\,, \qquad \alpha(k^2) = \frac{g^2}{4\pi}\,\frac{Z_\Gamma}{Z_2^2}\,Z(k^2)\,f(k^2)\,, \tag{14}$$

where the **effective interaction** $\alpha(k^2)$ absorbs the dressing functions of the gluon propagator and quark-gluon vertex. Since we do not know these quantities (we would need to solve their own DSEs for that), let's employ the following ansatz for $\alpha(k^2)$ to solve the quark DSE:

$$\alpha(k^2) = \pi\eta^7 x^2\,e^{-\eta^2 x} + \frac{2\pi\gamma_m\left(1-e^{-k^2/\Lambda_t^2}\right)}{\ln\left[e^2 - 1 + \left(1 + k^2/\Lambda_{\text{QCD}}^2\right)^2\right]}\,, \qquad x = \frac{k^2}{\Lambda^2}\,, \tag{15}$$

which is the Maris-Tandy model [2,3]. The second term with the parameters $\Lambda_t = 1$ GeV, $\Lambda_{\text{QCD}} = 0.234$ GeV and $\gamma_m = 12/25$ is only relevant for large momenta, where it ensures the correct perturbative behavior but is otherwise not essential. By contrast, the first term with the parameters $\Lambda = 0.72$ GeV and $1.6 \lesssim \eta \lesssim 2$ dominates the small-momentum behavior and is important for the dynamical generation of a quark mass (in practice, you can use $\eta = 1.8$).

To work out the explicit form of the quark DSE, let's abbreviate

$$g(k^2) = Z_2^2\,\frac{16\pi}{3}\,\frac{\alpha(k^2)}{k^2} \qquad \Rightarrow \qquad \Sigma(p) = \int\limits_q g(k^2)\,T_k^{\mu\nu}\gamma^\mu S(q)\gamma^\nu\,. \tag{16}$$

Like Eq. (2) for the propagator, the most general Lorentz-covariant form of the self-energy is

$$\Sigma(p) = i\not{p}\,\Sigma_A(p^2) + \Sigma_M(p^2)\,, \tag{17}$$

from where $\Sigma_A(p^2)$ and $\Sigma_M(p^2)$ follow by taking Dirac traces:

$$\frac{1}{4}\,\mathrm{Tr}\,\Sigma(p) = \Sigma_M(p^2)\,, \qquad \frac{1}{4p^2}\,\mathrm{Tr}\left\{-i\not{p}\,\Sigma(p)\right\} = \Sigma_A(p^2)\,. \tag{18}$$

Applying this to Eq. (16) yields the expressions

$$\Sigma_A(p^2) = \int\limits_q \sigma_v(q^2)\,g(k^2)\,F(p^2,q^2,z), \qquad \Sigma_M(p^2) = 3\int\limits_q \sigma_s(q^2)\,g(k^2) \tag{19}$$

which depend on the quark dressings $\sigma_v(q^2)$, $\sigma_s(q^2)$ that can be reconstructed from $A(q^2)$ and $M(q^2)$. The squared gluon momentum is $k^2 = p^2 + q^2 - 2\,p\cdot q = p^2+q^2-2pqz$ (we now abbreviate $p=\sqrt{p^2}$ and $q=\sqrt{q^2}$), and the dimensionless quantity F is given by

$$\begin{aligned} p^2\,F(p^2,q^2,z) &= -\tfrac{1}{4}\,\mathrm{Tr}\left\{\not{p}\,\gamma^\mu\,\not{q}\,\gamma^\nu\right\}T_k^{\mu\nu} = p\cdot q + \frac{2}{k^2}\,(p\cdot k)(q\cdot k) \\ &= 3\,p\cdot q - \frac{2}{k^2}\left(p^2\,q^2 - (p\cdot q)^2\right) = 3pqz - \frac{2p^2q^2}{k^2}\,(1-z^2) \\ &= -k^2 + \frac{p^2+q^2}{2} + \frac{(p^2-q^2)^2}{2k^2} = p^2 + 3\,p\cdot k + 2\,\frac{(p\cdot k)^2}{k^2}\,. \end{aligned} \tag{20}$$

Either of these forms are equally good to calculate the self-energy integrals, e.g.

$$F(p^2,q^2,z) = \frac{3qz}{p} - \frac{2q^2}{k^2}\,(1-z^2)\,. \tag{21}$$

If we plug Eq. (17) into the quark DSE and compare the coefficients of the Dirac matrices, we arrive at two coupled integral equations for the dressing functions $A(p^2)$ and $M(p^2)$:

$$A(p^2) = Z_2 + \Sigma_A(p^2)\,, \qquad M(p^2)A(p^2) = Z_2\,m_0 + \Sigma_M(p^2)\,. \tag{22}$$

The self-energy integrals are logarithmically UV-divergent when the cutoff L is sent to infinity, so we must employ **renormalization**. To this end, we demand that

$$A(\mu^2) \stackrel{!}{=} 1\,, \qquad M(\mu^2) \stackrel{!}{=} m\,, \tag{23}$$

where m is the renormalized current-quark mass at some arbitrary renormalization point $p^2=\mu^2$ (this is the quark mass that we should be compared to high-energy scattering experiments). The condition $A(\mu^2)=1$ is an arbitrary renormalization condition; if we choose any other value this will multiplicatively renormalize $A(p^2)$ but it cannot affect any observable. Eq. (22) then yields

$$Z_2 = 1 - \Sigma_A(\mu^2)\,, \qquad m_0 = \frac{m - \Sigma_M(\mu^2)}{1-\Sigma_A(\mu^2)}\,, \tag{24}$$

and plugging this back into Eq. (22) yields the final form of the DSEs:

$$\begin{aligned} A(p^2) &= 1 + \Sigma_A(p^2) - \Sigma_A(\mu^2)\,, \\ M(p^2)A(p^2) &= m + \Sigma_M(p^2) - \Sigma_M(\mu^2)\,. \end{aligned} \tag{25}$$

By the subtraction all divergences cancel and the dressing functions $A(p^2)$ and $M(p^2)$ are finite.

Eqs. (25) can be solved iteratively. To do so, start with some guess for $A(p^2)$ and $M(p^2)$ (e.g., set them to 1) and calculate $\Sigma_A(p^2)$ and $\Sigma_M(p^2)$ from Eq. (19). From there, determine $\Sigma_A(\mu^2)$ and $\Sigma_M(\mu^2)$ at the renormalization point $\mu^2 = 19$ GeV^2 using cubic splines. Determine the new functions $A(p^2)$, $M(p^2)$ according to Eq. (25) as well as the renormalization constant Z_2 from Eq. (24), which enters again in the self-energy in the next step. Repeat the procedure until it converges. The renormalized current-quark mass m is an input, and because we do not distinguish between up and down quarks a typical value for both is $m = 4$ MeV.

Tasks:

- Solve the quark DSE as described above and determine $A(p^2)$ and $M(p^2)$. For the integration in q, use a Gauss-Legendre quadrature ($\gtrsim 200$ grid points should be sufficient) and for the integration in z a Gauss-Chebyshev quadrature ($\gtrsim 30$ points).
- The mass function $M(p^2)$ is independent of the renormalization point: while $A(p^2, \mu^2)$ also depends on μ^2, $M(p^2)$ depends on p^2 only. To see this, read off $M(\mu'^2) = m'$ at some arbitary value of μ'^2 from the converged solution (e.g., $\mu'^2 = 1$ GeV^2), and solve the system again with the new values of μ' and m'. The resulting mass function $M(p^2)$ must be identical to the one you obtained previously.
- Solve the DSE for a range of different current-quark masses up to the bottom quark and plot $M(p^2 = 0)$. Note that the chiral limit is defined by $m_0 = 0$.
- Solve the quark DSE in the complex plane, i.e., for complex values $p^2 \in \mathbb{C}$. To do so, you only need to solve Eqs. (25) *once more* for $p^2 \in \mathbb{C}$. Use a rectangular grid with $\mathrm{Re}\, p^2, \mathrm{Im}\, p^2 \in [-1, 1]$ GeV^2. Plot $\sigma_v(p^2)$ and $\sigma_s(p^2)$ and find the locations of the poles nearest to the origin $p^2 = 0$.
- **(Extra)** Develop an algorithm that determines these pole locations automatically. E.g., you can use the fact that a pole corresponds to the intersection of the curves $\mathrm{Re}\,[1/\sigma_v(p^2)] = 0$ and $\mathrm{Im}\,[1/\sigma_v(p^2)] = 0$.

References

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