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Introduction

This report is a two part study of:

I) Pricing a convertible bond contract in which, **at expiry** T the holder has the option to choose between receiving the principle F or alternatively receiving R underlying stocks with price S

II) An extension to the above contract where the holder is able to exercise the decision to convert the bond in stock at **any time before** the maturity of the contract. This is known as an American embedded option

through the use of advanced numerical methods such as Crank-Nicolson with PSOR.

1 European Option Convertible Bond

The PDE describing such a convertible bond contract is given by

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^{2\beta}\frac{\partial^2 V}{\partial S^2} + \kappa(\theta(t) - S)\frac{\partial V}{\partial S} - rV + Ce^{-\alpha t} = 0 \tag{1}$$

To use advanced numerical methods of (approximately) solving such PDEs we need a numerical scheme. This is a method of rewriting Equation **??** as a matrix equation as in Equation **??**.

$$\begin{pmatrix} b_0 & c_0 & 0 & 0 & . & . & . & . & 0 \\ a_1 & b_1 & c_1 & 0 & . & . & . & . & . \\ 0 & a_2 & b_2 & c_2 & 0 & . & . & . & . \\ . & . & . & . & . & . & . & . & . \\ . & . & . & . & a_j & b_j & c_j & . & . \\ . & . & . & . & . & . & . & . & . \\ 0 & . & . & . & . & . & . & a_{jmax} & b_{jmax} \end{pmatrix} \begin{pmatrix} V_j^0 \\ V_j^1 \\ V_j^2 \\ . \\ V_j^i \\ . \\ V_{jmax}^i \end{pmatrix} = \begin{pmatrix} d_j^0 \\ d_j^1 \\ d_j^2 \\ . \\ d_j^i \\ . \\ d_{jmax}^i \end{pmatrix} \tag{2}$$

where j represents the steps in S and i the steps in t. The Crank-Nicolson method takes approximations of derivatives by Taylor expanding at the half time steps thus yielding

$$\frac{\partial V}{\partial t} \approx \frac{V_j^{i+1} - V_j^i}{\Delta t} \tag{3}$$

$$\frac{\partial V}{\partial S} \approx \frac{1}{4\Delta S}(V_{j+1}^i - V_{j-1}^i + V_{j+1}^{i+1} - V_{j-1}^{i+1}) \tag{4}$$

$$\frac{\partial^2 V}{\partial S^2} \approx \frac{1}{2\Delta S^2}(V_{j+1}^i - 2V_j^i + V_{j+1}^{i+1} - 2V_j^{i+1} + V_{j-1}^{i+1}) \tag{5}$$

$$V \approx \frac{1}{2}(V_j^i + V_j^{i+1}). \tag{6}$$

So substituting Equations **??** - **??** into Equation **??** gives the numerical scheme for the non-boundary regime $1 \le j < jmax$.

$$a_j = \frac{\sigma^2 S^{2\beta}}{4\Delta S^2} - \frac{\kappa(\theta - S)}{4\Delta S} \tag{7}$$

$$b_j = \frac{1}{\Delta t} - \frac{\sigma^2 S^{2\beta}}{2\Delta S^2} - \frac{r}{2} \tag{8}$$

$$c_j = \frac{\sigma^2 S^{2\beta}}{4\Delta S^2} + \frac{\kappa(\theta - S)}{4\Delta S} \tag{9}$$

$$d_j = -\frac{V_j^{i+1}}{\Delta t} - \frac{\sigma^2 S^{2\beta}}{4\Delta S^2}(V_{j+1}^{i+1} - 2V_j^{i+1} + V_{j-1}^{i+1}) - \frac{\kappa(\theta - S)}{4\Delta S}(V_{j+1}^{i+1} - V_{j-1}^{i+1}) + \frac{r}{2}V_j^{i+1} - Ce^{-\alpha t} \tag{10}$$

The boundary conditions are problem dependent so for this particular we have two boundaries at $S = 0$ and $\lim_{S\to+\infty}$. Consider the first boundary, when $S = 0$ i.e $j = 0$. Using Equations **??** and **??** and a modified Equation **??** which becomes

$$\frac{\partial V}{\partial S} \approx \frac{1}{\Delta S}(V_{j+1}^i - V_j^i). \tag{11}$$

The numerical scheme after substituing the approximated derivates is now given by

$$a_0 = 0 \tag{12}$$

$$b_0 = -\frac{1}{\Delta t} - \frac{\kappa\theta}{\Delta S} - \frac{r}{2} \tag{13}$$

$$c_0 = \frac{\kappa\theta}{\Delta S} \tag{14}$$

$$d_0 = (-\frac{1}{\Delta t} + \frac{r}{2})V_j^{i+1} - Ce^{-\alpha t} \tag{15}$$

For the $\lim_{S\to+\infty}$ we have the condition that

$$\frac{\partial V}{\partial t} + \kappa(X - S)\frac{\partial V}{\partial S} - rV + Ce^{-\alpha t} = 0 \tag{16}$$

with the ansatz

$$V(S, t) = SA(t) + B(t). \tag{17}$$

It can be shown (See Appendix 1) by partial differentiation and integrating using an integrating factor method that

$$A(t) = Re^{(\kappa+r)(t-T)} \tag{18}$$

and

$$B(t) = -XRe^{(\kappa+r)(t-T)} + \frac{C}{\alpha + r}e^{-\alpha t} - \frac{C}{\alpha + r}e^{-(\alpha+r)T+rt} + XRe^{r(t-T)}. \tag{19}$$

Finally we have the last part of the numerical scheme as

$$a_0 = 0 \tag{20}$$

$$b_0 = 1 \tag{21}$$

$$c_0 = 0 \tag{22}$$

$$d_0 = SA(t) + B(t). \tag{23}$$

Using this complete numerical scheme, the method is to solve backwards in time from $i = imax$ to $i = 0$ where at each time step the Equation **??** is solved using a method such as Successive Over Relaxation (SOR) for $j = 0 \to jmax$.

1.1 Investigating β and σ

For the rest of this section assume these values were used unless otherwise specified: $T = 2$, $F = 95$, $R = 2$, $r = 0.0229$, $\kappa = 0.125$, $\mu = 0.0113$, $X = 47.66$, $C = 1.09$, $\alpha = 0.02$, $\beta = 0.486$ and $\sigma = 3.03$. The value of the option $V(S, t)$ was investigated as a function of the initial underlying asset price S_0 for two cases:

1) ($\beta = 1$, $\sigma = 0.416$) with all other paramaters as previously defined

2) ($\beta = 0.486$, $\sigma = 3.03$) with all other paramaters as previously defined

The Crank-Nicolson method with the numerical scheme as calculated previously, combined with a SOR iterative method of solving the matrix equation, was implemented in code. This produced the plots seen in Figure **??**.