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Introduction

This report is a two part study of:

I) Pricing a convertible bond contract in which, **at expiry** T the holder has the option to choose between receiving the principle F or alternatively receiving R underlying stocks with price S

II) An extension to the above contract where the holder is able to exercise the decision to convert the bond in stock at **any time before** the maturity of the contract. This is known as an American embedded option

through the use of advanced numerical methods such as Crank-Nicolson with PSOR.

1 European Type Option Convertible Bond

The PDE describing such a convertible bond contract is given by

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^{2\beta}\frac{\partial^2 V}{\partial S^2} + \kappa(\theta(t) - S)\frac{\partial V}{\partial S} - rV + Ce^{-\alpha t} = 0 \tag{1}$$

To use advanced numerical methods of (approximately) solving such PDEs we need a numerical scheme. This is a method of rewriting Equation 1 as a matrix equation as in Equation 2.

$$\begin{pmatrix} b_0 & c_0 & 0 & 0 & . & . & . & . & 0 \\ a_1 & b_1 & c_1 & 0 & . & . & . & . & . \\ 0 & a_2 & b_2 & c_2 & 0 & . & . & . & . \\ . & . & . & . & . & . & . & . & . \\ . & . & . & . & a_j & b_j & c_j & . & . \\ . & . & . & . & . & . & . & . & . \\ 0 & . & . & . & . & . & . & a_{jmax} & b_{jmax} \end{pmatrix} \begin{pmatrix} V_j^0 \\ V_j^1 \\ V_j^2 \\ . \\ V_j^i \\ . \\ V_{jmax}^i \end{pmatrix} = \begin{pmatrix} d_j^0 \\ d_j^1 \\ d_j^2 \\ . \\ d_j^i \\ . \\ d_{jmax}^i \end{pmatrix} \tag{2}$$

where j represents the steps in S and i the steps in t. The Crank-Nicolson method takes approximations of derivatives by Taylor expanding at the half time steps thus yielding

$$\frac{\partial V}{\partial t} \approx \frac{V_j^{i+1} - V_j^i}{\Delta t} \tag{3}$$

$$\frac{\partial V}{\partial S} \approx \frac{1}{4\Delta S}(V_{j+1}^i - V_{j-1}^i + V_{j+1}^{i+1} - V_{j-1}^{i+1}) \tag{4}$$

$$\frac{\partial^2 V}{\partial S^2} \approx \frac{1}{2\Delta S^2}(V_{j+1}^i - 2V_j^i + V_{j+1}^{i+1} - 2V_j^{i+1} + V_{j-1}^{i+1}) \tag{5}$$

$$V \approx \frac{1}{2}(V_j^i + V_j^{i+1}). \tag{6}$$

So substituting Equations 3 - 6 into Equation 1 gives the numerical scheme for the non-boundary regime $1 \le j < jmax$.

$$a_j = \frac{\sigma^2 S^{2\beta}}{4\Delta S^2} - \frac{\kappa(\theta - S)}{4\Delta S} \tag{7}$$

$$b_j = \frac{1}{\Delta t} - \frac{\sigma^2 S^{2\beta}}{2\Delta S^2} - \frac{r}{2} \tag{8}$$

$$c_j = \frac{\sigma^2 S^{2\beta}}{4\Delta S^2} + \frac{\kappa(\theta - S)}{4\Delta S} \tag{9}$$

$$d_j = -\frac{V_j^{i+1}}{\Delta t} - \frac{\sigma^2 S^{2\beta}}{4\Delta S^2}(V_{j+1}^{i+1} - 2V_j^{i+1} + V_{j-1}^{i+1}) - \frac{\kappa(\theta - S)}{4\Delta S}(V_{j+1}^{i+1} - V_{j-1}^{i+1}) + \frac{r}{2}V_j^{i+1} - Ce^{-\alpha t} \tag{10}$$

The boundary conditions are problem dependent so for this particular we have two boundaries at $S = 0$ and $\lim_{S \to +\infty}$. Consider the first boundary, when $S = 0$ i.e $j = 0$. Using Equations 3 and 6 and a modified Equation 4 which becomes

$$\frac{\partial V}{\partial S} \approx \frac{1}{\Delta S}(V_{j+1}^i - V_j^i). \tag{11}$$

The numerical scheme after substituing the approximated derivates is now given by

$$a_0 = 0 \tag{12}$$

$$b_0 = -\frac{1}{\Delta t} - \frac{\kappa\theta}{\Delta S} - \frac{r}{2} \tag{13}$$

$$c_0 = \frac{\kappa\theta}{\Delta S} \tag{14}$$

$$d_0 = (-\frac{1}{\Delta t} + \frac{r}{2})V_j^{i+1} - Ce^{-\alpha t} \tag{15}$$

For the $\lim_{S \to +\infty}$ we have the condition that

$$\frac{\partial V}{\partial t} + \kappa(X - S)\frac{\partial V}{\partial S} - rV + Ce^{-\alpha t} = 0 \tag{16}$$

with the ansatz

$$V(S, t) = SA(t) + B(t). \tag{17}$$

It can be shown (See Appendix 1) by partial differentiation and integrating using an integrating factor method that

$$A(t) = Re^{(\kappa + r)(t - T)} \tag{18}$$

and

$$B(t) = -XRe^{(\kappa + r)(t - T)} + \frac{C}{\alpha + r}e^{-\alpha t} - \frac{C}{\alpha + r}e^{-(\alpha + r)T + rt} + XRe^{r(t - T)}. \tag{19}$$

Finally we have the last part of the numerical scheme as

$$a_0 = 0 \tag{20}$$

$$b_0 = 1 \tag{21}$$

$$c_0 = 0 \tag{22}$$

$$d_0 = SA(t) + B(t). \tag{23}$$

Using this complete numerical scheme, the method is to solve backwards in time from $i = imax$ to $i = 0$ where at each time step the Equation 2 is solved using a method such as Successive Over Relaxation (SOR) for $j = 0 \to jmax$.

1.1 Investigating β and σ

For the rest of this section assume these values were used unless otherwise specified: $T = 2$, $F = 95$, $R = 2$, $r = 0.0229$, $\kappa = 0.125$, $\mu = 0.0113$, $X = 47.66$, $C = 1.09$, $\alpha = 0.02$, $\beta = 0.486$ and $\sigma = 3.03$. The value of the option $V(S, t)$ was investigated as a function of the initial underlying asset price S_0 for two cases:

1) ($\beta = 1$, $\sigma = 0.416$) with all other paramaters as previously defined

2) ($\beta = 0.486$, $\sigma = 3.03$) with all other paramaters as previously defined

The Crank-Nicolson method with the numerical scheme as calculated previously, combined with a SOR iterative method of solving the matrix equation, was implemented in code. This produced the plots seen in Figure 1.

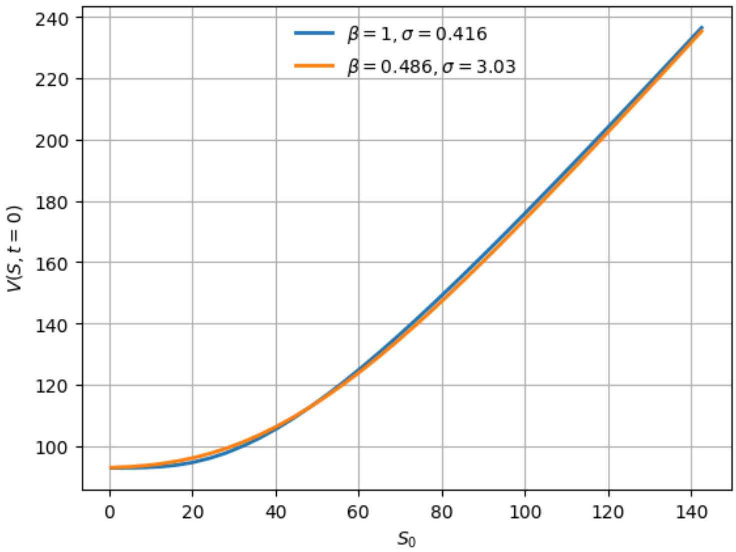


Figure 1: Value of the convertible bond $V(S, t = 0)$ against inital underlying asset price at time S_0 for two combinations of β and σ.

The two configurations were therefore found to have the same effect and produce plots for the price of the bond which were very close. This prompted further analysis on the linked relationship between β, σ and $V(S, t)$. A 3D graph of the value of the portfolio for a particular S_0, here chosen to be equal to X, and the two other parameters was plotted. Figure 2 illustrates such a relationship which is interesting both in shape and in what it can be modelled by. Going back a few steps, the risk-neutral process followed by the underlying stock price is given by

$$dS = \kappa(\theta(t) - S)dt + \sigma S^{\beta} dW \tag{24}$$

which is an Ornstein-Uhlenbeck (OU) process [1] with a drift term of function $\theta(t)$, together with a Constant Elasticity of Variance [2] model where the local variance is a powerlaw of elasticity. Using this model, σ is defined to be the actual Black-Scholes volatility or standard deviation of the underlying asset, while β is the elasticity parameter of the local volatility. Moreover, using this model the values of

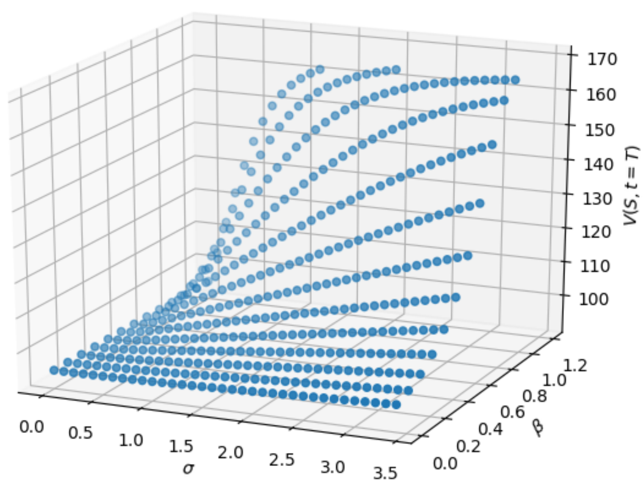


Figure 2: Value of the convertible bond $V(S = X, t = T)$ against parameters β and σ.

β which should be used are for ≤ 1. Above this, there are implications on the inaccessibility of the origin which for a stock price means it cannot go bankrupt which is not true. Thus, we shall stick for values of $\beta \leq 1$ here. In this regime, the model captures the so-called 'leverage effect' which practically means that stock price and volatily are inversely proportional [3]. The parameter β in Equation 24 controls the steepness of the implied volatility skew which is something seen in Figure 2. The parameter σ is now part of a scale parameter which fixes the 'at-the-money' (S close to X regime) volatility level. So, there are $\sigma - \beta$ planes on Figure 2 which have close values of the convertible bond for multiple combinations of (σ , β). This happens since having a steep implied variance skew but a lower actual variation, which is an average of a time period, and the other way round counteract each other.

1.2 Varying step sizes

As will be described later though, an increase in i_{max} is preferrable rather than j_{max} for convergence due to computational time requirements. Finally, for this section the parameters i_{max}, j_{max} and S_{max}

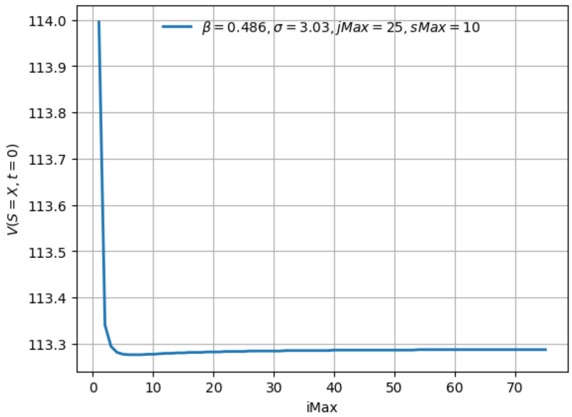
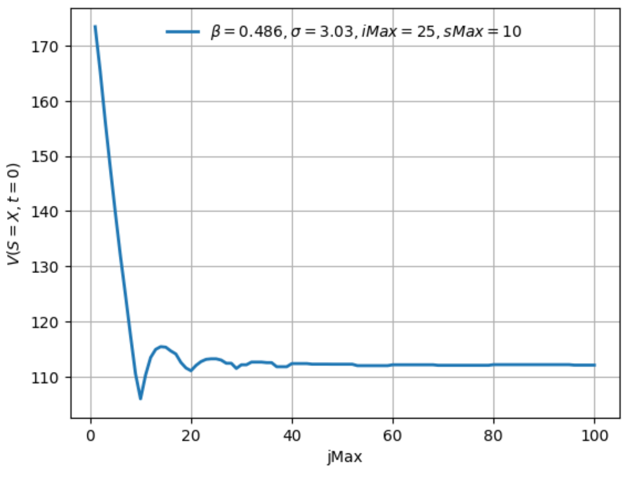


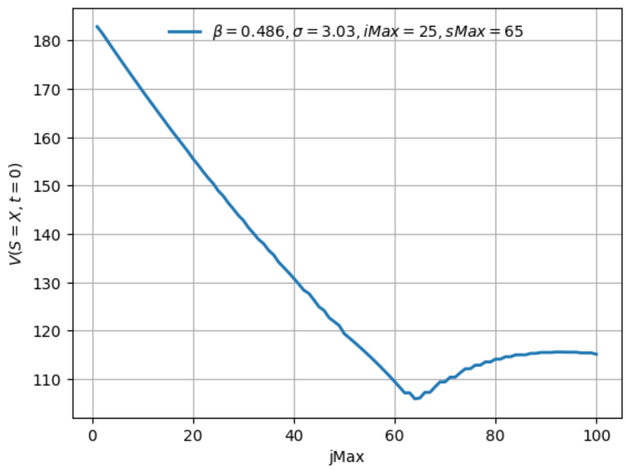
Figure 3: Trend of the convertible bond $V(S = X, t = T)$ as parameter i_{max} is varied.

were investigated to study how a variation in their value affected the result. The region selected was the at-the-money $S = X$ region of stock price to have comparable results across all three parameters. Starting with the variation in the time steps, Figure 3 illustrates such relationship. Here, it is clear that increasing i_{max} rapidly converges towards a single value of $V(X, T)$ and after $i_{max} = 25$ there is really no point in increasing this parameter too much.

When it came to varying j_{max} which is the number of steps in S per timestep, it was noticed that since the stepsize in S is calculated by dividing S_{max} by the number of steps then these had to go hand



(a) Stability and convergence can be obverved after $j_{max} = 40$



(b) The plot from 4a is stretched and since S_{max} is x6.5 as much, the first minimum is also stretched by that much.

Figure 4: Plots of the price of the convertible bond $V(X,T)$ against changing j_{max} for different values of S_{max}.

in hand when varying one of them. Figure 4 illustrates this very clearly. Keeping the range of j_{max} the same and increasing S_{max} shows the same plot but being stretched out in the x-axis. This happens since increasing the maximum cutoff S from which to start at each timestep but keeping the number of steps constant would mean larger jumps thus a less accurate result everytime. Instead, ensuring that the overally stepsize in S is constant or small enough is paramount in keeping the result accurate. Recall that the error in the Crank-Nicolson method is $\mathcal{O}(\Delta S^2, \Delta t^2)$.

Following this, the natural progression is to vary S_{max} for a given value of i_{max}. However, due to the results in Figure 4 we have to make sure we increase j_{max} as we go along. Figure 6 clearly shows

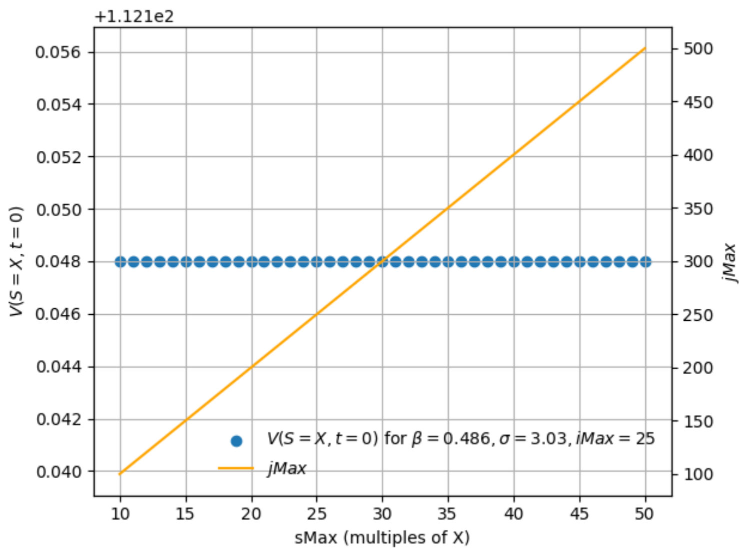


Figure 5: Trend of the convertible bond $V(S = X, t = 0)$ as parameter S_{max} is varied and j_{max} is varied to be kept at a comparable value.

that already at $S_{max} = 10X$ the value of the convertible bond converges. In fact, for higher S_{max} the result is identical to 7 significant figures since this is the residual error limit that was set on the SOR method, with a maximum cap of 10,000 iterations which all iterations of S_{max} shown stay under. The computational requirement of increasing S_{max} is linked to that of increasing j_{max} and since the return is an unreasonable amount of significant figures precision then it is not worth using an $S_{max} > 10X$.

The last issue left to investigate is the time requirements and processing complexity of varying these parameters. As can be infered from Figure 6 i_max follows a linear time increase while j_max is exponential. This is because of the fact that a single loop from time $t = T$ to $t = 0$ is done but a further loop of j_{max} length is done per time step. Since the error of Crank-Nicolson is given by $\mathcal{O}(\Delta S^2, \Delta t^2)$ both quantities are important and are dependent directly on i_{max} and j_{max} however j_{max} has the highest influence on the value converging to the analytic value so a compromise must be made.

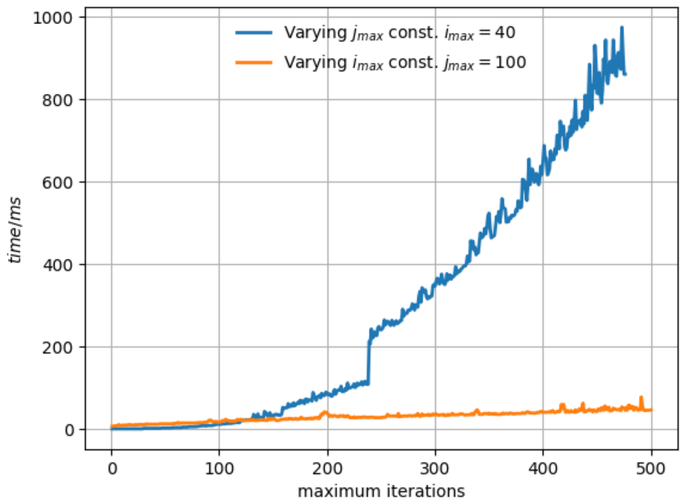


Figure 6: Variation of the time required to calculate the convertible bond price as j_{max} and i_{max} are varied.

The final value for $\sigma = 3.03$ and $\beta = 0.486$ was calculated to be $V(S = X, t = 0) = 112.163$ with $S_{max} = 10X$, $j_{max} = 400$, $i_{max} = 50$.

2 American Type Option Convertible Bond with Embedded Option

For this section the numerical method used was Crank-Nicolson Method with Penalty method and Thomas method for the matrix equation solver. The extension to the european option type convertible bond detailed in Section 1 is to change it to an American type option. By this we mean that the holder has the option to convert the bond in stock at any time before the maturity of the contract. To ensure this, the inequality

$$V \geq RS \tag{25}$$

must hold for all $t < T$. This means the $V_{american} > V_{european}$. A last addition is to embed a put option in this contract which means the holder has the option to sell the bond back to the issuer over some time period such that

$$V(S,t) \geq P_p \quad \text{for} \quad t \leq t_0 \tag{26}$$

must hold.

Figure 7 shows the results of adding these conditions in the code. The limit for large S is observed as expected to tend to RS and compared to Figure 1 the value of the option is higher. This is due to the fact that the effective increase in power given to the holder increases the price. Furthermore, adding the put option increases further the price since again this gives more power to the holder. This put option might be a sort of safety net in case the value of the stock decreases too much and as with most financial contracts, a decrease in risk must increase the price. The bond floor is thus observed to be raised when compared to the no-option case.

Finally, the arrows are pointing to two decision points at which the price of the contract becomes more than P_p thereafter and becomes more than RS_0 thereafter, respectively. These are important points since the holder would only ever buy the contract for values of S_0 between those two points otherwise they would just buy the contract to sell it again or would buy the underlying equity.

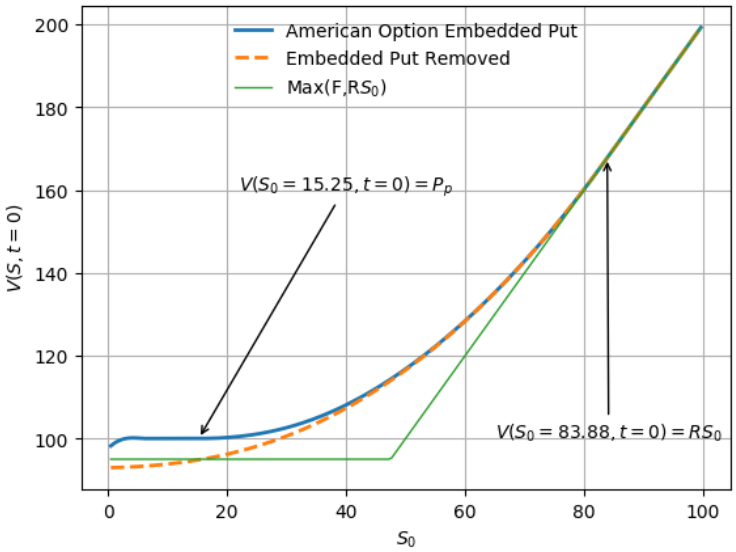


Figure 7: Value of the american type option convertible bond $V(S, t = 0)$ against inital underlying asset price S_0 with and without the embedded put option.

The sensitivity to the mean reversion rate [4] κ was studied. Refering back to Equation 24, this is the rate at which the stock will revert back to the long term mean price described by $\theta(t)$. As can be seen from Figure 8 an increase in κ decreases the value of the bond in the at-the-money region of the underlying stock. This is expected since less fluctuations in the stock price movements make it less attractive to buy this contract due to the probabilities of the stock price changing drastically in the future being lower thus the convertibility of the bond being unused.

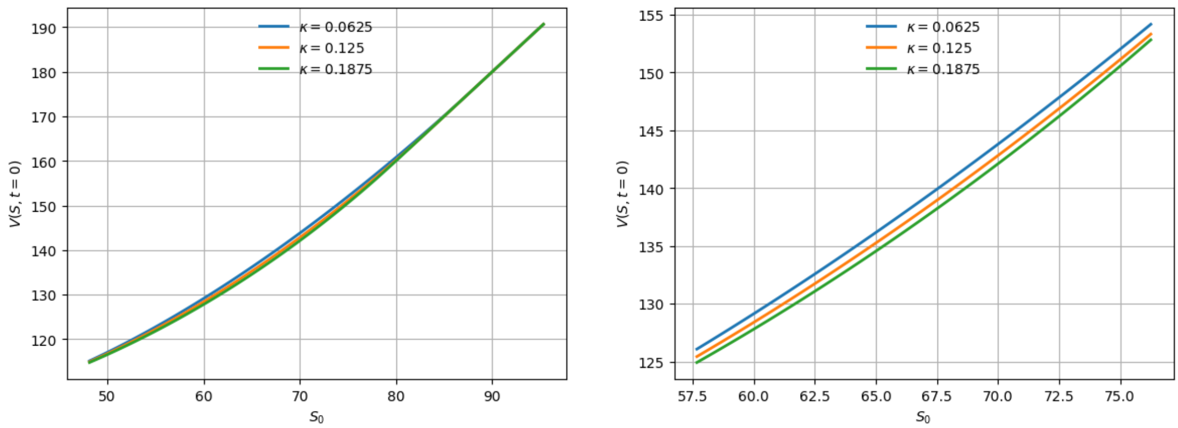


Figure 8: Value of the american type option convertible bond $V(S, t = 0)$ with embedded put option for different values of parameter κ. Right side plot is a zoomed version of the left side plot.

Lastly, it was requested to obtain the most accurate value possible in one second of processing. Referring to accuracy first, the important thing to eliminate is any source of error. There are errors on the boundary (finite-element) and errors in discontinuities of the domain. Since there is a discontinuity at time t_0 one must change the timestep used before and now split the domain effectively in two. The timestep was chosen such that

$$\Delta t = \frac{T - t_0}{i_{max} f} \qquad \text{for} \qquad t_0 < t \leq T \tag{27}$$

$$\Delta t = \frac{t_0}{i_{max}(1 - f)} \qquad \text{for} \qquad 0 < t \leq t_0 \tag{28}$$

where $f = \frac{T - t_0}{T}$. A better timestepping system would have been Rannacher smoothing [5] however this was beyond the scope of the analysis and as such literature was used to check that the expected convergence ratios were in the right ranges. The results, comparing the method PSOR to the penalty method are detailed in Table 1.

	Penalty				PSOR			
N	$V(X,0)$	Iters	Diff.Ratio	Time(ms)	$V(X,0)$	Iters	Diff.Ratio	Time(ms)
100	114.5230426	126		11	114.5230426	1635		18
200	114.4983047	225		34	114.4983048	2211		38
400	114.4814288	427	1.47	97	114.4814289	3208	1.47	123
800	114.4729397	827	1.99	379	114.4729397	4806	1.99	440
1600	114.4702297	1627	3.13	1455	114.4702297	8005	3.13	1807
3200	114.4688601	3227	1.98	6685	114.4688602	12804	1.98	6098
6400	114.4681719	8838.6	1.99	25486	114.4681722	19249	1.99	23284

Table 1: Table comparing convergence results and efficiencies of PSOR and Penalty methods with a Crank-Nicolson numerical scheme. Here, $j_{max} = i_{max} = N$ and $S_{max} = NX/20$

The results show the convergence rate is nowhere near square with timestep as it should be for a crank nicolson scheme but rather linear. This is may be due to the increased complexity the convertibility in the whole life of the bond brings with it. Because of the smoothing method taken combined with other errors such as interpolation errors the rate is less than optimal however it is within expected ranges [6]. Finally, using the information from Table 1 the most accurate value in the given time was found to be: $V(X, t_0 = 0) = 114.472554$ in 965ms with $j_{max} = i_{max} = 1300$ and $s_{max} = 65X$.

References

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[6] L. X. Li, *Pricing Convertible Bonds using Partial Differential Equations.* University of Toronto, 2005.

Appendix 2

Portfolio Pricing Program Listing

```cpp
#include <iostream>
#include <fstream>
#include <cmath>
#include <vector>
#include <algorithm>
#include <chrono>

using namespace std;

/*
 * ON INPUT:
 * a, b and c -- are the diagonals of the matrix
 * rhs        -- is the right hand side
 * x          -- is the initial guess
 * iterMax    -- is maximum iterations
 * tol        -- is the tolerance level
 * omega      -- is the relaxation parameter
 * sor        -- not used
 * ON OUTPUT:
 * a, b, c, rhs        -- unchanged
 * x                   -- solution to Ax=b
 * iterMax, tol, omega -- unchanged
 * sor                 -- number of iterations to converge
 */
void sorSolve(const std::vector<double> &a, const std::vector<double> &b, const std::vector<double> &c, const std::vector<double> &rhs,
              std::vector<double> &x, int iterMax, double tol, double omega, int &sorCount)
{
  // assumes vectors a,b,c,d,rhs and x are same size (doesn't check)
  int n = a.size() - 1;
  // sor loop
  for (sorCount = 0; sorCount < iterMax; sorCount++)
  {
    double error = 0.;
    // implement sor in here
    {
      double y = (rhs[0] - c[0] * x[1]) / b[0];
      x[0] = x[0] + omega * (y - x[0]);
    }
    for (int j = 1; j < n; j++)
    {
      double y = (rhs[j] - a[j] * x[j - 1] - c[j] * x[j + 1]) / b[j];
      x[j] = x[j] + omega * (y - x[j]);
    }
    {
      double y = (rhs[n] - a[n] * x[n - 1]) / b[n];
      x[n] = x[n] + omega * (y - x[n]);
    }
    // calculate residual norm ||r|| as sum of absolute values
    error += std::fabs(rhs[0] - b[0] * x[0] - c[0] * x[1]);
    for (int j = 1; j < n; j++)
      error += std::fabs(rhs[j] - a[j] * x[j - 1] - b[j] * x[j] - c[j] * x[j + 1]);
```

```
    error += std::fabs(rhs[n] - a[n] * x[n - 1] - b[n] * x[n]);
    // make an exit condition when solution found
    if (error < tol)
      break;
  }
}
std::vector<double> thomasSolve(const std::vector<double> &a, const std::vector<double> &b_, const std::vector<double> &c, std::vector<double> &d)
{
  int n = a.size();
  std::vector<double> b(n), temp(n);
  // initial first value of b
  b[0] = b_[0];
  for (int j = 1; j < n; j++)
  {
    b[j] = b_[j] - c[j - 1] * a[j] / b[j - 1];
    d[j] = d[j] - d[j - 1] * a[j] / b[j - 1];
  }
  // calculate solution
  temp[n - 1] = d[n - 1] / b[n - 1];
  for (int j = n - 2; j >= 0; j--)
    temp[j] = (d[j] - c[j] * temp[j + 1]) / b[j];
  return temp;
}
/* Template code for the Crank Nicolson Finite Difference
 */
double crank_nicolson(double S0, double X, double F, double T, double r, double sigma,
                      double R, double kappa, double mu, double C, double alpha, double beta, int iMax, int jMax, int S_max, double tol, double omega, int iterMax, int &sorCount)
{
  // declare and initialise local variables (ds,dt)
  double dS = S_max / jMax;
  double dt = T / iMax;
  // create storage for the stock price and option price (old and new)
  vector<double> S(jMax + 1), vOld(jMax + 1), vNew(jMax + 1);
  // setup and initialise the stock price
  for (int j = 0; j <= jMax; j++)
  {
    S[j] = j * dS;
  }
  // setup and initialise the final conditions on the option price
  for (int j = 0; j <= jMax; j++)
  {
    vOld[j] = max(F, R * S[j]);
    vNew[j] = max(F, R * S[j]);
  }
  // start looping through time levels
  for (int i = iMax - 1; i >= 0; i--)
  {
    // declare vectors for matrix equations
    vector<double> a(jMax + 1), b(jMax + 1), c(jMax + 1), d(jMax + 1);
    // set up matrix equations a[j]=
    double theta = (1 + mu) * X * exp(mu * i * dt);
    a[0] = 0;
    b[0] = (-1 / dt) - (r / 2) - (kappa * theta / dS);
```

```
    c[0] = (kappa * theta / dS);
    d[0] = (-C * exp(-alpha * i * dt)) + (vOld[0] * (-(1 / dt) + (r / 2)));
    for (int j = 1; j <= jMax - 1; j++)
    {
      //
      a[j] = (pow(sigma, 2) * pow(j * dS, 2 * beta) / (4 * pow(dS, 2))) - (kappa * (theta - j * dS) / (4 * dS));
      b[j] = (-1 / dt) - ((pow(sigma, 2.) * pow(j * dS, 2. * beta)) / (2. * pow(dS, 2))) - (r / 2.);
      c[j] = ((pow(sigma, 2.) * pow(j * dS, 2. * beta)) / (4. * pow(dS, 2.))) + ((kappa * (theta - j * dS)) / (4. * dS));
      d[j] = (-vOld[j] / dt) - ((pow(sigma, 2.) * pow(j * dS, 2. * beta) / (4. * pow(dS, 2.))) * (vOld[j + 1] - 2. * vOld[j] + vOld[j - 1])) - (((kappa * (theta - j * dS)) / (4. * dS)) * (vOld[j + 1] - vOld[j - 1])) + ((r / 2.) * vOld[j]) - (C * exp(-alpha * dt * i));
    }
    double A = R * exp((kappa + r) * (i * dt - T));
    double B = -X * A + C * exp(-alpha * i * dt) / (alpha + r) + X * R * exp(r * (i * dt - T)) - C * exp(-(alpha + r) * T + r * i * dt) / (alpha + r);
    a[jMax] = 0;
    b[jMax] = 1;
    c[jMax] = 0;
    d[jMax] = jMax * dS * A + B;
    // solve matrix equations with SOR
    sorSolve(a, b, c, d, vNew, iterMax, tol, omega, sorCount);
    //vNew = thomasSolve(a, b, c, d);

    if (sorCount == iterMax)
      return -1;

    // set old=new
    vOld = vNew;
  }
  // finish looping through time levels

  // output the estimated option price
  double optionValue;

  int jStar = S0 / dS;
  double sum = 0.;
  sum += (S0 - S[jStar]) / (dS)*vNew[jStar + 1];
  sum += (S[jStar + 1] - S0) / (dS)*vNew[jStar];
  optionValue = sum;

  return optionValue;
}

int main()
{
  //
  double T = 2., F = 95., R = 2., r = 0.0229, kappa = 0.125, altSigma = 0.416,
         mu = 0.0213, X = 47.66, C = 1.09, alpha = 0.02, beta = 0.486, sigma = 3.03, tol = 1.e-7, omega = 1., S_max = 10 * X;
  //
  /*
  double T = 3., F = 56., R = 1., r = 0.0038, kappa = 0.083333333, altSigma
```

```
 = 0.369,
      mu = 0.0073, X = 56.47, C = 0.106, alpha = 0.01, beta = 1., sigma = 3.73, S_max = 10 * X, tol = 1.e-7, omega = 1.;
*/
int iterMax = 10000, iMax = 100, jMax = 100;
//Create graph of varying S0 and beta and bond
int length = 300;
double S_range = 3 * X;
int sor;
/*
std::ofstream outFile7("./data/varying_s_beta.csv");

for (double beta = 0; beta < 1.3; beta += 0.1)
{
  for (int j = 1; j <= length - 1; j++)
  {
    outFile7 << beta << " , " << altSigma << " , " << j * S_range / length << " , " << crank_nicolson(j * S_range / length, X, F, T, r, altSigma, R, kappa, mu, C, alpha, beta, iMax, jMax, S_max, tol, omega, iterMax, sor) << "\n";
  }
}
outFile7.close();
*/
/*
std::ofstream outFile8("./data/varying_s_sigma.csv");
beta = 1;
for (double altSigma = 0; altSigma < 3.5; altSigma += 0.1)
{
  for (int j = 1; j <= length - 1; j++)
  {
    outFile8 << beta << " , " << altSigma << " , " << j * S_range / length << " , " << crank_nicolson(j * S_range / length, X, F, T, r, altSigma, R, kappa, mu, C, alpha, beta, iMax, jMax, S_max, tol, omega, iterMax, sor) << "\n";
  }
}
outFile8.close();
*/
/*
std::ofstream outFile9("./data/varying_s_sigma_beta.csv");
for (double altSigma = 0; altSigma < 3.5; altSigma += 0.1)
{
  for (double beta = 0; beta < 1.3; beta += 0.1)
  {
    double S0 = X;
    outFile9 << beta << " , " << altSigma << " , " << S0 << " , " << crank_nicolson(S0, X, F, T, r, altSigma, R, kappa, mu, C, alpha, beta, iMax, jMax, S_max, tol, omega, iterMax, sor) << "\n";
  }
}
outFile9.close();
*/
/*

std::ofstream outFile1("./data/varying_s_beta_1.csv");
std::ofstream outFile2("./data/varying_s_beta_0_4.csv");
for (int j = 1; j <= length - 1; j++)
```

```
  {
    vector<double> gamma(jMax + 1);
    outFile1 << j * S_range / length << " , " << crank_nicolson(j * S_range / length, X, F, T, r, altSigma, R, kappa, mu, C, alpha, 1, iMax, jMax, S_max, tol, omega, iterMax, sor, gamma) << "\n";
    outFile2 << j * S_range / length << " , " << crank_nicolson(j * S_range / length, X, F, T, r, sigma, R, kappa, mu, C, alpha, beta, iMax, jMax, S_max, tol, omega, iterMax, sor, gamma) << "\n";
  }
  outFile1.close();
  outFile2.close();
  */
  /*
  std::ofstream outFile3("./data/varying_imax.csv");
  jMax = 100;
  for (iMax = 1; iMax <= 500; iMax += 1)
  {
    double S = X;
    auto t1 = std::chrono::high_resolution_clock::now();
    double result = crank_nicolson(S, X, F, T, r, sigma, R, kappa, mu, C, alpha, beta, iMax, jMax, S_max, tol, omega, iterMax, sor);
    auto t2 = std::chrono::high_resolution_clock::now();
    auto time_taken =
        std::chrono::duration_cast<std::chrono::milliseconds>(t2 - t1)
            .count();
    outFile3 << S_max << "," << iMax << "," << jMax << "," << S << " , " << std::fixed << result << "," << time_taken << "\n";
  }
  outFile3.close();
  */
  /*
  for (int s_Mult = 10; s_Mult <= 10; s_Mult += 1)
  {
    double S = X;
    S_max = s_Mult * X;
    string title = "./data/smax_jmax/" + to_string(s_Mult) + "_varying_jmax.csv";
    std::ofstream outFile4(title);
    iMax = 40;
    for (jMax = 1; jMax <= 500; jMax += 1)
    {
      auto t1 = std::chrono::high_resolution_clock::now();
      double result = crank_nicolson(S, X, F, T, r, sigma, R, kappa, mu, C, alpha, beta, iMax, jMax, S_max, tol, omega, iterMax, sor);
      auto t2 = std::chrono::high_resolution_clock::now();
      auto time_taken =
          std::chrono::duration_cast<std::chrono::milliseconds>(t2 - t1)
              .count();
      outFile4 << S_max << "," << iMax << "," << jMax << "," << S << " , " << std::fixed << result << "," << time_taken << "\n";
    }
    outFile4.close();
  }
  */

  //std::ofstream outFile5("./data/varying_smax.csv");
  iMax = 25;
  tol = 1.e-7;
```

```
  for (int s_Mult = 10; s_Mult <= 50; s_Mult += 1)
  {
    jMax = s_Mult * 10;
    double S = X;
    S_max = s_Mult * X;
    int sorCount;
    auto t1 = std::chrono::high_resolution_clock::now();
    double result = crank_nicolson(S, X, F, T, r, sigma, R, kappa, mu, C, alpha, beta, iMax, jMax, S_max, tol, omega, iterMax, sorCount);
    auto t2 = std::chrono::high_resolution_clock::now();
    auto time_taken =
        std::chrono::duration_cast<std::chrono::milliseconds>(t2 - t1)
            .count();
    cout << S_max << "," << iMax << "," << jMax << "," << S << " , " << std::fixed << result << "," << time_taken << "\n";
  }
  //outFile5.close();

  /*
  S_max = 10 * X;
  std::ofstream outFile6("./data/analytic.csv");
  iMax = 25, jMax = 100;

  for (int j = 1; j <= length - 1; j++)
  {
    vector<double> gamma(jMax + 1);

    outFile6 << j * S_range / length << " , " << crank_nicolson(j * S_range / length, X, F, T, r, altSigma, R, 0, mu, C, alpha, 1, iMax, jMax, S_max, tol, omega, iterMax, sor, gamma) << "\n";
  }
  outFile6.close();
  */
  S_max = 10 * X;
  iMax = 50, jMax = 400;
  double S0 = X;
  std::cout << std::fixed << crank_nicolson(S0, X, F, T, r, sigma, R, kappa, mu, C, alpha, beta, iMax, jMax, S_max, tol, omega, iterMax, sor);
}
```

Graphing Program Listing

```
import scipy.stats as si
import numpy as np
import csv
import matplotlib.pyplot as plt

C = 1.09
alpha = 0.02
r = 0.0229
T = 2.
F = 95.
R = 2.
sigma = 0.416
K = 47.66
'''
T=3
C=0.106
alpha=0.01
```

```python
r=0.0038
R=1
F=56
sigma = 0.369
K=56.47
'''
#Calulate value of coupon
#Through integrating Cexp(-(alpha+r)t)dt from 0 to T
COUPON = C/(alpha+r) * (1- np.exp((-(alpha+r)*T)))

BOND = F*np.exp(-r*T)

variationData=[]
with open('data/analytic.csv', newline='\n') as csvfile:
    reader = csv.DictReader(csvfile,fieldnames=['S','V'],quoting=csv.QUOTE_NONNUMERIC)
    currentData={'x':[],'y':[]}
    for row in reader:
        currentData['x'].append(row['S'])
        currentData['y'].append(row['V'])
    variationData.append(currentData)

def euro_vanilla_call(S, K, T, r, sigma):

    #S: spot price
    #K: strike price
    #T: time to maturity
    #r: interest rate
    #sigma: volatility of underlying asset

    d1 = (np.log(S / K) + (r + 0.5 * sigma ** 2) * T) / (sigma * np.sqrt(T))
    d2 = (np.log(S / K) + (r - 0.5 * sigma ** 2) * T) / (sigma * np.sqrt(T))

    call = (S * si.norm.cdf(d1, 0.0, 1.0) - K * np.exp(-r * T) * si.norm.cdf(d2, 0.0, 1.0))

    return call

ANALYTIC_PRICE = []
STOCK_PRICE = []
BOND_FLOOR = []
CONV_BOND = []
for s in range(1,140):
    STOCK_PRICE.append(s)
    ANALYTIC_PRICE.append(R*euro_vanilla_call(s, K, T, r, sigma) + BOND + COUPON)

#plt.plot(S,V1, label = " beta = 1")
#plt.plot(STOCK_PRICE,BOND_FLOOR, label = "Bond")
plt.plot(STOCK_PRICE,ANALYTIC_PRICE, label = "Analytic")
plt.plot(variationData[0]['x'],variationData[0]['y'], label = "Crank")
plt.xlabel('Stock price')
plt.ylabel('Eurocall Option')
plt.legend()
plt.savefig('images/analytic.png',bbox_inches='tight', pad_inches=0.2)
```

```
import matplotlib.pyplot as plt
import numpy as np
import csv
import scipy.stats as si

X=47.66
R=2
F=95
T=2.0
C = 1.09
alpha = 0.02
r = 0.0229
T = 2.
sigma = 0.416
'''
def euro_vanilla_call(S, K, T, r, sigma):

    #S: spot price
    #K: strike price
    #T: time to maturity
    #r: interest rate
    #sigma: volatility of underlying asset

    d1 = (np.log(S / K) + (r + 0.5 * sigma ** 2) * T) / (sigma * np.sqrt(T)
    )
    d2 = (np.log(S / K) + (r - 0.5 * sigma ** 2) * T) / (sigma * np.sqrt(T)
    )

    call = (R*S * si.norm.cdf(d1, 0.0, 1.0) - K * np.exp(-r * T) * si.norm.
    cdf(d2, 0.0, 1.0))

    return call

variationData=[]
with open('data/varying_imax.csv', newline='\n') as csvfile:
    reader = csv.DictReader(csvfile,fieldnames=['sMax','iMax','jMax','S','V
    '],quoting=csv.QUOTE_NONNUMERIC)
    currentData={'x':[],'y':[],'jMax':0,'sMax':0}
    for row in reader:
        currentData['x'].append(row['iMax'])
        currentData['y'].append(row['V'])
        currentData['jMax']=row['jMax']
        currentData['sMax']=int(row['sMax']/X)
    variationData.append(currentData)

plt.figure()
plt.grid()
plt.plot(variationData[0]['x'],variationData[0]['y'],label=r'$\beta=0.486,\
    sigma=3.03,jMax=%i,sMax=%i$'%(variationData[0]['jMax'],variationData
    [0]['sMax']),linewidth=2)
plt.xlabel('iMax')
plt.ylabel(r'$V(S=X,t=0)$')
plt.legend(loc='upper center',fancybox=False, framealpha=0.0)
plt.savefig('images/european_varying_imax.png',bbox_inches='tight',
    pad_inches=0.2)

with open('data/varying_smax.csv', newline='\n') as csvfile:
    reader = csv.DictReader(csvfile,fieldnames=['sMax','iMax','jMax','S','V
```

```
'],quoting=csv.QUOTE_NONNUMERIC)
    currentData={'x':[],'y':[],'jMax':[],'iMax':0}
    for row in reader:
        currentData['x'].append(int(row['sMax']/X))
        currentData['y'].append(row['V'])
        currentData['jMax'].append(row['jMax'])
        currentData['iMax']=row['iMax']
    variationData.append(currentData)

fig, ax1 = plt.subplots()
ax1.set_xlabel(r'sMax (multiples of X)')
ax1.set_ylabel(r'$V(S=X,t=0)$')
ax1.grid()
ax1.scatter(np.asarray(variationData[1]['x'][:20]),variationData[1]['y'][:20],label=r'$V(S=X,t=0)$ for $\beta=0.486,\sigma=3.03,iMax=%i$'%(variationData[1]['iMax']))
ax2 = ax1.twinx()
ax2.set_ylabel(r'$jMax$')
fig.tight_layout()
ax2.plot(np.asarray(variationData[1]['x'][:20]),variationData[1]['jMax'][:20],label=r'$jMax$',color="orange")
lines, labels = ax1.get_legend_handles_labels()
lines2, labels2 = ax2.get_legend_handles_labels()
ax2.legend(lines + lines2, labels + labels2,loc='lower right',fancybox=False, framealpha=0.0)
plt.savefig('images/european_varying_smax_zoomed.png',bbox_inches='tight',pad_inches=0.2)

fig, ax1 = plt.subplots()
ax1.set_xlabel(r'sMax (multiples of X)')
ax1.set_ylabel(r'$V(S=X,t=0)$')
ax1.grid()
ax1.scatter(np.asarray(variationData[1]['x']),variationData[1]['y'],label=r'$V(S=X,t=0)$ for $\beta=0.486,\sigma=3.03,iMax=%i$'%(variationData[1]['iMax']))
ax2 = ax1.twinx()
ax2.set_ylabel(r'$jMax$')
fig.tight_layout()
ax2.plot(np.asarray(variationData[1]['x']),variationData[1]['jMax'],label=r'$jMax$',color="orange")
lines, labels = ax1.get_legend_handles_labels()
lines2, labels2 = ax2.get_legend_handles_labels()
ax2.legend(lines + lines2, labels + labels2,loc='lower right',fancybox=False, framealpha=0.0)
plt.savefig('images/european_varying_smax.png',bbox_inches='tight',pad_inches=0.2)

'''
'''
for smax in range(10,101):
    currentData={'x':[],'y':[],'iMax':0,'sMax':0}
    with open('data/smax_jmax/'+str(smax)+'_varying_jmax.csv', newline='\n') as csvfile:
        reader = csv.DictReader(csvfile,fieldnames=['sMax','iMax','jMax','S','V'],quoting=csv.QUOTE_NONNUMERIC)
        for row in reader:
            currentData['x'].append(row['jMax'])
            currentData['y'].append(row['V'])
```

```python
            currentData['iMax']=row['iMax']
            currentData['sMax']=int(row['sMax']/X)

    plt.figure()
    plt.plot(currentData['x'],currentData['y'],label=r'$\beta=0.486,\sigma=3.03,iMax=%i,sMax=%i$'%(currentData['iMax'],currentData['sMax']),linewidth=2)
    plt.xlabel('jMax')
    plt.ylabel(r'$V(S=X,t=0)$')
    plt.legend(loc='upper center',fancybox=False, framealpha=0.0)
    plt.grid()
    plt.savefig('images/smax_jmax/'+str(smax)+'_european_varying_jmax.png',bbox_inches='tight', pad_inches=0.2)
    plt.close()
'''

allData=[]
with open('data/varying_s_beta_1.csv', newline='\n') as csvfile:
    reader = csv.DictReader(csvfile,fieldnames=['S','V'],quoting=csv.QUOTE_NONNUMERIC)
    currentData={'S':[], 'V':[]}
    for row in reader:
        currentData['S'].append(row['S'])
        currentData['V'].append(row['V'])
    allData.append(currentData)

with open('data/varying_s_beta_0_4.csv', newline='\n') as csvfile:
    reader = csv.DictReader(csvfile,fieldnames=['S','V'],quoting=csv.QUOTE_NONNUMERIC)
    currentData={'S':[], 'V':[]}
    for row in reader:
        currentData['S'].append(row['S'])
        currentData['V'].append(row['V'])
    allData.append(currentData)

plt.figure()
plt.grid()
plt.plot(allData[0]['S'],allData[0]['V'],label=r'$\beta=1,\sigma=0.416$',linewidth=2)
plt.plot(allData[1]['S'],allData[1]['V'],label=r'$\beta=0.486,\sigma=3.03$',linewidth=2)
#plt.plot(allData[0]['S'],np.ones(len(allData[0]['S'])) * F,label=r'Principal Only',linewidth=2)
#equityOnly_1 = [ R*euro_vanilla_call(s, X, T, r, sigma) for s in allData[0]['S'] ]
#equityOnly_2 = [ R*euro_vanilla_call(s, X, T, r, 3.03) for s in allData[0]['S'] ]
#plt.plot(allData[0]['S'],equityOnly_1,label=r'Equity Only $\sigma=0.416$',linewidth=2)
#plt.plot(allData[0]['S'],equityOnly_2,label=r'Equity Only $\sigma=3.03$',linewidth=2)

plt.xlabel(r'$S_0$')
plt.ylabel(r'$V(S,t=0)$')
plt.legend(loc='upper center',fancybox=False, framealpha=0.0)
plt.savefig('images/european_varying_s.png',bbox_inches='tight', pad_inches=0.2)
'''
```

```
import matplotlib.pyplot as plt
from mpl_toolkits.mplot3d import Axes3D

currentData={'S':[], 'V':[], 'beta':[]}
with open('data/varying_s_beta.csv', newline='\n') as csvfile:
    reader = csv.DictReader(csvfile,fieldnames=['beta','altsigma','S','V'],
    quoting=csv.QUOTE_NONNUMERIC)
    currentData={'S':[], 'V':[], 'beta':[]}
    for row in reader:
        currentData['S'].append(row['S'])
        currentData['V'].append(row['V'])
        currentData['beta'].append(row['beta'])

fig = plt.figure()
ax = fig.add_subplot(111, projection='3d')
ax.scatter(currentData['S'],currentData['V'],currentData['beta'])
plt.savefig('images/european_varying_s_varying_beta.png',bbox_inches='tight', pad_inches=0.2)
'''
'''
currentData={'S':[], 'V':[], 'beta':[], 'sigma':[]}
with open('data/varying_s_sigma_beta.csv', newline='\n') as csvfile:
    reader = csv.DictReader(csvfile,fieldnames=['beta','sigma','S','V'],
    quoting=csv.QUOTE_NONNUMERIC)
    for row in reader:
        if(row['V']==-1):
            continue
        currentData['S'].append(row['S'])
        currentData['V'].append(row['V'])
        currentData['beta'].append(row['beta'])
        currentData['sigma'].append(row['sigma'])
'''
'''
fig = plt.figure()
ax = fig.add_subplot(111, projection='3d')
ax.scatter(currentData['sigma'],currentData['beta'],currentData['V'])
ax.set_xlabel(r'$\sigma$')
ax.set_ylabel(r'$\beta$')
ax.set_zlabel(r'$V(S,t=T)$')
plt.show()
plt.savefig('images/european_varying_s_varying_sigma_varying_beta.png',
    bbox_inches='tight', pad_inches=0.2)
'''
'''
fig = plt.figure()
ax2 = fig.add_subplot(111)
ax2.hist2d(currentData['sigma'],currentData['beta'],bins=100,weights=
    currentData['V'])
ax2.set_xlabel(r'$\sigma$')
ax2.set_ylabel(r'$\beta$')
#ax2.set_zlabel(r'$V(S,t=T)$')
plt.show()
plt.savefig('images/hist2d_european_varying_s_varying_sigma_varying_beta.
    png',bbox_inches='tight', pad_inches=0.2)
'''
'''
plt.figure()
plt.grid()
```

```
plt.xlabel(r'maximum iterations')
plt.ylabel(r'$time/ms$')

currentData={'x':[],'y':[],'iMax':0,'sMax':0,'time':[]}
with open('data/smax_jmax/10_varying_jmax.csv', newline='\n') as csvfile:
    reader = csv.DictReader(csvfile,fieldnames=['sMax','iMax','jMax','S','V','time'],quoting=csv.QUOTE_NONNUMERIC)
    for row in reader:
        currentData['x'].append(row['jMax'])
        currentData['y'].append(row['V'])
        currentData['iMax']=row['iMax']
        currentData['sMax']=int(row['sMax']/X)
        currentData['time'].append(row['time'])


plt.plot(currentData['x'],currentData['time'],label=r'Varying $j_{max}$ const. $i_{max}=40$',linewidth=2)



variationData=[]
currentData={'x':[],'y':[],'jMax':0,'sMax':0,'time':[]}

with open('data/varying_imax.csv', newline='\n') as csvfile:
    reader = csv.DictReader(csvfile,fieldnames=['sMax','iMax','jMax','S','V','time'],quoting=csv.QUOTE_NONNUMERIC)
    for row in reader:
        currentData['x'].append(row['iMax'])
        currentData['y'].append(row['V'])
        currentData['jMax']=row['jMax']
        currentData['sMax']=int(row['sMax']/X)
        currentData['time'].append(row['time'])

plt.plot(currentData['x'],currentData['time'],label=r'Varying $i_{max}$ const. $j_{max}=100$',linewidth=2)

plt.legend(loc='upper center',fancybox=False, framealpha=0.0)
plt.savefig('images/european_time.png',bbox_inches='tight', pad_inches=0.2)
'''
```