**The Simulated Structure of neutron stars**

**Abstract**

Neutron stars are simulated by using the principle of hydrostatic equilibrium with Newtonian and general relativistic gravitational models. The star is 'built' using numerical methods of the Runge Kuta $4^{th}$ order method with an Ideal Non-interacting Fermi Gas and Soft-core Interacting Equation of state. Radius and Mass Dependence on the Equation of state used is investigating as well as the density structure for comparison against observations. Using the Tolman-Oppenheimer-Volkov equation, with Bethe & Johnson's equation of state, the maximum mass of a neutron star was found to be …. With radius…. And the maximum radius was found to be …. With mass…. ( errors) .

**Intro**

'It doesn't matter how beautiful your theory is, it doesn't matter how smart you are. If it doesn't agree with experiment, its wrong'-Richard P.Feynman. Neutron Stars are perhaps the only loophole in todays physics to this argument. It would be experimentally difficult to look inside a neutron star and most are very difficult to detect unless accreting matter, emitting radiation ( in some cases a pulsar) or gravitational waves. After a star can no longer sustain nuclear fusion to counter the gravitational contraction on its own mass it collapses. If the stellar core is more massive than the Chandrasekar limit electron degeneracy pressure due will not stop the star from further collapse. The matter collapses in gravitational freefall until it hits the core, most often creating a supernova, if the core is less than approximately three solar masses a neutron star is formed, if above this limit a black hole is formed[c]. Currently equations of state theories are matched to the observed parameters of a neutron star.

In this report Neutron stars are simulated using numerical integration, the principle of hydrostatic equilibrium and two equations of state. The theorized matter distribution of compact stellar objects is stated before examining the method of 'building 'the star. We then aim to show which of the equation of states tested best suits observation and why. Additionally, special relativistic rotation is added as a fictitious force to give understanding of rotational affects.

**The Theoretical structure**

The structure of nuclear matter with increasing density is fundamental to equating observation to simulation. So, we form a theoretical spherically symmetrical density distribution based on current standard model physics.

On the surface/ outer crust for densities less than $4.3x10^{14}\ kgm^{-3}$ a coulomb lattice structure of neutron rich nuclei is formed, induced by inverse beta decay from the relativistic degenerate electrons. The Nuclear Binding energy peak representing the most stable nuclei peak for elements will be slightly skewed by this effect. Beyond $4.3x10^{14}\ kgm^{-3}$ the inverse beta decay process becomes more prominent, this causes the neutron drip, a process in which the neutron rich nuclei expel the neutrons into a neutron gas. In which case free electrons and free neutrons co-exist in an equilibrium of superfluid gas at this point. The neutron and electron gas with neutron rich nuclei can exist in the state up until the 'nuclear' density of $2.3x10^{17}\ kgm^{-3}$. Passed the nuclear density the nuclei being to merge, called nuclear saturation, causing complex inter-nucleon interactions. There will now be a superfluid of elections neutrons and protons, neutron degeneracy exerts an outward pressure due to the Pauli exclusion principle. At densities of order magnitude $1x10^{18}\ kgm^{-3}$ the pressure may cause the pion condensation, interactions will be very important between particles at these densities. Passed this density, quark matter could be formed[a][b][m].

As the we pass nuclear density interactions between the particles become increasingly important.

**Building a star.**

For a star not to collapse or explode each infinitesimally thick spherical shell of the neutron stars matter must be in an equilibrium of the pressure outward caused degeneracy pressure of the matter and the gravitational forces inward due to the mass inside that shell. This is the principle of hydrostatic equilibrium.

Using the Newtonian gravitational model, the pressure gradient and therefore condition for stellar equilibrium at radius $r$ due to gravity is

$$\frac{dP}{dr} = \frac{-Gm(r)\rho(r)}{r^2} \tag{1}$$

, where$\rho(r)$is the density of matter at $r$, $G$ is the universal gravitation constant of gravitation and $P$ is the pressure. $m(r)$ is the mass within $r$ given by

$$\frac{dm}{dr} = 4\pi r^2 \rho(r) \tag{2}$$

, due to the mass enclosed by a sphere of radius $r$.

In Einstein's theory of gravitation, the corresponding condition for hydrostatic equilibrium is

$$\frac{dP}{dr} = \frac{-Gm(r)}{r^2} \frac{\left(1 + {P(r)}/{\rho(r)c}\right)\left(1 + 4\pi r^3\, {P(r)}/{m(r)c}\right)}{\left(1 - {2Gm(r)}/{rc^2}\right)} \qquad (3).$$

Equation (3) is called the Tolman–Oppenheimer–Volkoff equation (TOV). The TOV shows pressure dependence on the pressure gradient leading to a gravitational collapse at a smaller mass that equation (1). The It was predicted by Oppenheimer and Volkoff in 1939 using a cold Fermi gas (a very similar to our non-interacting model in the following section) a maximum neutron mass of 0.7 solar masses using the TOV[h]. The TOV has an upper limit of 3 solar mases.

These equations have been used with numerical integration and equations of state (EOS) to model neutron stars. We compare the solutions of equations (1) and (3) with corresponding EOS's with observation.

**The equation of state**

The equation of state gives a relation between the pressure exerted at a given mass density. In our models for simulations of neutron star two equations of state are used, The Bethe-Johnson (BJ) and an Ideal neutron gas EOS.

The Bethe Johnson EOS [d] is a Modified Reid soft core interaction model, using the N-N (N-N) potentials to produce a repulsive core via meson exchange. The BJ EOS uses Yukawa functions with parameters set to model experimental N-N scatting data [b]. The BJ equation on state is

$$P(n) = 363.44\, n^{2.54} \tag{4.a}$$

$$\rho(n) = 236\, n^{2.54} + n\, m_n \tag{4.b}$$

, where $n$ is the number density of neutrons, $P(n)$ is the pressure in $MeVc^{-2}fm^{-3}$ and $\rho(n)$ is the density in $MeVc^{-2}fm^{-3}$. A conversion factor to SI units was applied to equation (4) before application in the computational section. The BJ EOS is valid for the central density range of $(1.7\ \times 10^{17}\ \leq\ \rho\ \leq 3.2 \times 10^{19})kgm^{-3}$ [b].

The Ideal Neutron gas EOS (NI) is the simplest model that can be used [a], it is based on a non-interacting Fermi gas of neutrons. There are two parts to this equation of state: In the nonrelativistic regime the EOS is

$$P = \frac{\hbar}{5\, m_n^{8/3}} (3\pi^2)^{2/3} \rho^{5/3} \qquad (5.a).$$

The second part of the NI EOS in the extreme relativity approximation is given by

$$P = \frac{1}{3}\, \hbar c\, m_n^{-4/3} (3\pi^2)^{1/3} \rho^{4/3} \qquad (5.b).$$

The NI EOS is valid for $(0\ \leq\ \rho\ \leq \infty)kgm^{-3}$[b], with the advised upper bound to be $(\ \rho\ \leq 5 \times 10^{17})kgm^{-3}$ due to the softening of the EOS relations due to N-N interactions. There is a critical density of value $\rho_{\ c} = 5 \times 10^{17}$ [j] specified for computational processes that acts as a bound between equations (5.a) and (5.b). The value $\rho_{\ c}$ is derived as an approximate figure using

$$\varepsilon_F = \frac{\hbar^2}{2m_n} (3\pi^2 \frac{N}{V})^{2/3} \ll E_{mn} \qquad (6).$$

[k], where $\varepsilon_F$ is the fermi of the neutron in this case, $\frac{N}{V}$ is the density of neutrons per unit volume in 'k space' and $E_{mn}$is the rest mass of a neutron in joules. The maximum mass obtainable due to an ideal gas of degenerate neutrons is 5.8 solar masses.

We apply the BJ EOS central density religion to comply simultaneous with both limits of the EOS's in the simulation. Equations (4) and (5) will be used in conjunction with the differential equations in the previous section to simulate neutron stars. Both the BJ and NI equation of state assume a non-rotating neutron star.

$$E_{RK4} = \frac{|X_{2h} - X_h|}{(2^n - 1)}$$

**Computational method**

The code was written in Python 3.5.3 using numerical integration as the base method to build neutron star from a range of initial central densities. The primary numerical integration method used was the Runge Kuta 4th order method (RK4). A range of central densities were given and the central mass and radius were both initialized as zero [f]. Pressure is initialized and central density is passed through a multistep RK4. First solving equation (2) to find mass at the next step and then equation (1) or (3) to find the pressure at the next step dependent on what model of gravitation was required. The new density used to loop this method building up the step as a radius until the pressure is less than zero, then the program breaks the loop. If the program was to continue it would try to compute a density that that was complex. The code then takes the last two points, including the negative pressure point, and interpolates using a linear fit and Newton Raphson to find the exact radius at which the pressure goes to zero. A linear fit is then applied to the respective two points on the RK4 masses and uses a linear fit and interpolation to find the final mass on the neutron star at the final radius.

The above method was implemented for a family of neutron stars constrained by the BJ EOS density limitations for comparative analysis with the NI EOS.

The BJ EOS central pressure was initialized by specifying the central density and using a Newton Raphson with equation (4). The NI EOS depends on two sub EOS's as mentioned in previous sections. Therefore, when the program iterates through the numerical integration, decreasing in density until the critical density is reached, the EOS then changes from equation (5.b) to (5.a) as lower density regions are reached.

Plots of final mass vs final radius, central density vs final mass and central density vs final radius were all plotted. In addition, four density/matter distribution graphs were plotted to show the internal structure on the star produced by their respective EOS and gravitational model.

Furthermore, Runge Kuta 5$^{th}$ order (RK5) and Euler numerical integration methods were implemented. The RK5 is technically more accurate than Rk4 but with more numerical rounding error due to extra valuations. The Euler method is technically less accurate RK4, these two methods provide an indicative bound for our simulation result.

**(Subsection)** – Errors

To calculate the errors on the mass and radius of the simulated neutron stars, two main elements of error were considered. The error in interpolation method is

$$E_{inter} = \pm(x_i - x_{i-1})^{n+1} \tag{7}$$

[g],where $x_i$ , $x_{i-1}$are the repespective points being interpolatedand $n$ is the order of polynomial, in our case this is linear and therefore $n = 1$. The error on the rk4 is calculated by simultaneously building the star with an rk4 that has a step of twice that of the step size used for the result. The equation for the RK4 error is

$$E_{RK4} = \frac{|X_{2h} - X_h|}{(2^n - 1)} \tag{8}$$

[h], where $X_h$ is the final value of numerical integration such as the mass or radius with single step of $h$ and $X_{2h}$ is the rk4 double stepped final value. Both the radius and the mass had both errors for equations (6) and (7) added in quadrature for their respective values.

**(Subsection)** - Fictitious forces.

The NI and BJ EOS's assume a non-rotating neutron star. To give an indication of to what effect rotation would have on the structure of neutron stars, the pressure gradients of equations (1) and (3) were adapted to include the special relativistic centrifugal fictitious force on a shell of mass at a given radius [i]. Therefore, our adaptation is an approximation the effect of rotation.

## Results And Discussion

<u>Family plots for Newtonian</u>

<u>Family Plots for TOV</u>

<u>Density Distribution NI EOS</u>

<u>Density Distribution BJ EOS</u>

Our results shows that the maximum mass of a neutron star for the BJ EOS to be …. With error and radius with error. And the maximum mass of the NI EOS to be … with…..

The TOV shows a more accurate representation when compared to the Newtonian of observed properties of neutron stars due to the high densities with the stars.

The BJ EOS Fits well (or not) with current observations of neutron stars( give range of observed values .with mass and radius).

We need to include the step size used(10m), but only that step size at the error we calculate is a difference in step size. We need to run the same step size through the rk4 to check the maximum mass obtained by the optimum model that fits observation (hopefully the BJ EQ of state). We need to briefly compare the density distributions to the theoretical structure section. We need to comment on the non-unique solutions for mass and radius of both EOS's

The main reason for the BJ EOS hopefully being a better model than the NI is the fact it considers N-N interactions.

Can you also please run the rk5 and Euler for the optimum model (hopefully BJ TOV) maximum mass for a comparison in numerical integration.

The effects of rotation using a first order special relativistic approximation and a frequency of **641** ***Hz*** corresponding too one of the fasted observed neutron star rotations [n]. The BJ EOS and TOV were used for this exercise, increased the size the modeled neutron star by ….. using a central density of….. This shows an increase although very small in comparison with the size of the star. More accurate approximations will involve treating the structure of the neutron star as a viscous fluid with a oblate spheroid shape and use fictitious forces derived from general relativity, from our exercise we would expect an increase in the radius size for a specific central density.

**Summary**

How does the result fit observation? What is the closet result and why?

The TOV gravitational model is seen to fit observed values and limitations as opposed to the Newtonian model

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Need to use ref m, b and a in the structure section,