```
    k1[1], k2[1] are the parameters for v
  */

  int i;
  double t_h;
  double yout[2],y_h[2]; //k1[2],k2[2],k3[2],k4[2],y_k[2];

  t_h=0;
  y_h[0]=y[0];   //phi
  y_h[1]=y[1];   //v
  ofstream fout("rk4.out");
  fout.setf(ios::scientific);
  fout.precision(20);
  for(i=1; i<=n; i++){
    rk4_step(t_h,y_h,yout,delta_t_roof);
    fout<<i*delta_t<<"\t\t"<<yout[0]<<"\t\t"<<yout[1]<<"\n";
    t_h+=delta_t_roof;
    y_h[0]=yout[0];
    y_h[1]=yout[1];
  }
  fout.close;
}

int main()
{
  pendelum testcase;
  testcase.initialise();
  testcase.rk4();
  return 0;
}  // end of main function
```

13.8 Exercises and projects

Project 13.1: studies of neutron stars

In the pendulum example we rewrote the equations as two differential equations in terms of so-called dimensionless variables. One should always do that. There are at least two good reasons for doing this.

– By rewriting the equations as dimensionless ones, the program will most likely be easier to read, with hopefully a better possibility of spotting eventual errors. In addtion, the various constants which are pulled out of the equations in the process of rendering the equations dimensionless, are reintroduced at the end of the calculation. If one of these constants is not correctly defined, it is easier to spot an eventual error.

– In many physics applications, variables which enter a differential equation, may differ by orders of magnitude. If we were to insist on not using dimensionless quantities, such differences can cause serious problems with respect to loss of numerical precision.

An example which demonstrates these features is the set of equations for gravitational equilibrium of a neutron star. We will not solve these equations numerically here, rather, we will limit ourselves to merely rewriting these equations in a dimensionless form.

The equations for a neutron star

The discovery of the neutron by Chadwick in 1932 prompted Landau to predict the existence of neutron stars. The birth of such stars in supernovae explosions was suggested by Baade and Zwicky 1934. First theoretical neutron star calculations were performed by Tolman, Oppenheimer and Volkoff in 1939 and Wheeler around 1960. Bell and Hewish were the first to discover a neutron star in 1967 as a *radio pulsar*. The discovery of the rapidly rotating Crab pulsar (rapidly rotating neutron star) in the remnant of the Crab supernova observed by the chinese in 1054 A.D. confirmed the link to supernovae. Radio pulsars are rapidly rotating with periods in the range 0.033 s $\leq P \leq 4.0$ s. They are believed to be powered by rotational energy loss and are rapidly spinning down with period derivatives of order $\dot{P} \sim 10^{-12} - 10^{-16}$. Their high magnetic field B leads to dipole magnetic braking radiation proportional to the magnetic field squared. One estimates magnetic fields of the order of $B \sim 10^{11} - 10^{13}$ G. The total number of pulsars discovered so far has just exceeded 1000 before the turn of the millenium and the number is increasing rapidly.

The physics of compact objects like neutron stars offers an intriguing interplay between nuclear processes and astrophysical observables, see Refs. [80–82] for further information and references on the physics of neutron stars. Neutron stars exhibit conditions far from those encountered on earth; typically, expected densities ρ of a neutron star interior are of the order of 10^3 or more times the density $\rho_d \approx 4 \cdot 10^{11}$ g/cm^3 at 'neutron drip', the density at which nuclei begin to dissolve and merge together. Thus, the determination of an equation of state (EoS) for dense matter is essential to calculations of neutron star properties. The EoS determines properties such as the mass range, the mass-radius relationship, the crust thickness and the cooling rate. The same EoS is also crucial in calculating the energy released in a supernova explosion.

Clearly, the relevant degrees of freedom will not be the same in the crust region of a neutron star, where the density is much smaller than the saturation density of nuclear matter, and in the center of the star, where density is so high that models based solely on interacting nucleons are questionable. Neutron star models including various so-called realistic equations of state result in the following general picture of the interior of a neutron star. The surface region, with typical densities $\rho < 10^6$ g/cm^3, is a region in which temperatures and magnetic fields may affect the equation of state. The outer crust for 10^6 g/cm^3 $< \rho < 4 \cdot 10^{11}$g/cm^3 is a solid region where a Coulomb lattice of heavy nuclei coexist in β-equilibrium with a relativistic degenerate electron gas. The inner crust for $4 \cdot 10^{11}$ g/cm^3 $< \rho < 2 \cdot 10^{14}$g/cm^3 consists of a lattice of neutron-rich nuclei together with a superfluid neutron gas and an electron gas. The neutron liquid for $2 \cdot 10^{14}$ g/cm^3 $< \rho < \cdot 10^{15}$g/cm^3 contains mainly superfluid neutrons with a smaller concentration of superconducting protons and normal electrons. At higher densities, typically $2-3$ times nuclear matter saturation density, interesting phase transitions from a phase with just nucleonic degrees of freedom to quark matter may take place. Furthermore, one may have a mixed phase of quark and nuclear matter, kaon or pion condensates, hyperonic matter, strong magnetic fields in young stars etc.

Equilibrium equations

If the star is in thermal equilibrium, the gravitational force on every element of volume will be balanced by a force due to the spacial variation of the pressure P. The pressure is defined by the equation of state (EoS), recall e.g., the ideal gas $P = Nk_BT$. The gravitational force which acts on an element of volume at a distance r is given by

$$F_{Grav} = -\frac{Gm}{r^2}\rho/c^2, \tag{13.90}$$

where G is the gravitational constant, $\rho(r)$ is the mass density and $m(r)$ is the total mass inside a radius r. The latter is given by

$$m(r) = \frac{4\pi}{c^2}\int_0^r \rho(r')r'^2 dr' \tag{13.91}$$

which gives rise to a differential equation for mass and density

$$\frac{dm}{dr} = 4\pi r^2 \rho(r)/c^2. \tag{13.92}$$

When the star is in equilibrium we have

$$\frac{dP}{dr} = -\frac{Gm(r)}{r^2}\rho(r)/c^2. \tag{13.93}$$

The last equations give us two coupled first-order differential equations which determine the structure of a neutron star when the EoS is known.

The initial conditions are dictated by the mass being zero at the center of the star, i.e., when $r = 0$, we have $m(r = 0) = 0$. The other condition is that the pressure vanishes at the surface of the star. This means that at the point where we have $P = 0$ in the solution of the differential equations, we get the total radius R of the star and the total mass $m(r = R)$. The mass-energy density when $r = 0$ is called the central density ρ_s. Since both the final mass M and total radius R will depend on ρ_s, a variation of this quantity will allow us to study stars with different masses and radii.

Dimensionless equations

When we now attempt the numerical solution, we need however to rescale the equations so that we deal with dimensionless quantities only. To understand why, consider the value of the gravitational constant G and the possible final mass $m(r = R) = M_R$. The latter is normally of the order of some solar masses $M_\odot$, with $M_\odot = 1.989 \times 10^{30}$ Kg. If we wish to translate the latter into units of MeV/c^2, we will have that $M_R \sim 10^{60}$ MeV/c^2. The gravitational constant is in units of $G = 6.67 \times 10^{-45} \times \hbar c\ (MeV/c^2)^{-2}$. It is then easy to see that including the relevant values for these quantities in our equations will most likely yield large numerical roundoff errors when we add a huge number $\frac{dP}{dr}$ to a smaller number P in order to obtain the new pressure. We list here the units of the various quantities and in case of physical constants, also their values. A bracketed symbol like $[P]$ stands for the unit of the quantity inside the brackets.

We introduce therefore dimensionless quantities for the radius $\hat{r} = r/R_0$, mass-energy density $\hat{\rho} = \rho/\rho_s$, pressure $\hat{P} = P/\rho_s$ and mass $\hat{m} = m/M_0$.

The constants M_0 and R_0 can be determined from the requirements that the equations for $\frac{dm}{dr}$ and $\frac{dP}{dr}$ should be dimensionless. This gives

$$\frac{dM_0\hat{m}}{dR_0\hat{r}} = 4\pi R_0^2 \hat{r}^2 \rho_s \hat{\rho}, \tag{13.94}$$

yielding

$$\frac{d\hat{m}}{d\hat{r}} = 4\pi R_0^3 \hat{r}^2 \rho_s \hat{\rho}/M_0. \tag{13.95}$$

If these equations should be dimensionless we must demand that

$$4\pi R_0^3 \rho_s/M_0 = 1. \tag{13.96}$$

Quantity	Units
$[P]$	MeVfm^{-3}
$[\rho]$	MeVfm^{-3}
$[n]$	fm^{-3}
$[m]$	MeVc^{-2}
$M_\odot$	1.989×10^{30} Kg= 1.1157467×10^{60} MeVc^{-2}
1 Kg	= $10^{30}/1.78266270D0$ MeVc^{-2}
$[r]$	m
G	$\hbar c 6.67259 \times 10^{-45}$ $\mathrm{MeV}^{-2}\mathrm{c}^{-4}$
$\hbar c$	197.327 MeVfm

Correspondingly, we have for the pressure equation

$$\frac{d\rho_s \hat{P}}{dR_0 \hat{r}} = -GM_0 \frac{\hat{m}\rho_s\hat{\rho}}{R_0^2\hat{r}^2} \tag{13.97}$$

and since this equation should also be dimensionless, we will have

$$GM_0/R_0 = 1. \tag{13.98}$$

This means that the constants R_0 and M_0 which will render the equations dimensionless are given by

$$R_0 = \frac{1}{\sqrt{\rho_s G 4\pi}}, \tag{13.99}$$

and

$$M_0 = \frac{4\pi\rho_s}{(\sqrt{\rho_s G 4\pi})^3}. \tag{13.100}$$

However, since we would like to have the radius expressed in units of 10 km, we should multiply R_0 by 10^{-19}, since 1 fm = 10^{-15} m. Similarly, M_0 will come in units of $\mathrm{MeV}/\mathrm{c}^2$, and it is convenient therefore to divide it by the mass of the sun and express the total mass in terms of solar masses $M_\odot$.

The differential equations read then

$$\frac{d\hat{P}}{d\hat{r}} = -\frac{\hat{m}\hat{\rho}}{\hat{r}^2}, \qquad \frac{d\hat{m}}{d\hat{r}} = \hat{r}^2\hat{\rho}. \tag{13.101}$$

In the solution of our problem, we will assume that the mass-energy density is given by a simple parametrization from Bethe and Johnson [83]. This parametrization gives ρ as a function of the number density $n = N/V$, with N the total number of baryons in a volume V. It reads

$$\rho(n) = 236 \times n^{2.54} + nm_n, \tag{13.102}$$

where $m_n = 938.926\mathrm{MeV/c}^2$, the mass of the neutron (averaged). This means that since $[n] = \mathrm{fm}^{-3}$, we have that the dimension of ρ is $[\rho] = \mathrm{MeV/c^2fm}^{-3}$. Through the thermodynamic relation

$$P = -\frac{\partial E}{\partial V}, \tag{13.103}$$

where E is the energy in units of MeV/c^2 we have

$$P(n) = n\frac{\partial \rho(n)}{\partial n} - \rho(n) = 363.44 \times n^{2.54}. \tag{13.104}$$

We see that the dimension of pressure is the same as that of the mass-energy density, i.e., $[P]$ =MeV/c^{2}fm^{-3}.

Here comes an important point you should observe when solving the two coupled first-order differential equations. When you obtain the new pressure given by

$$P_{new} = \frac{dP}{dr} + P_{old}, \tag{13.105}$$

this comes as a function of r. However, having obtained the new pressure, you will need to use Eq. (13.104) in order to find the number density n. This will in turn allow you to find the new value of the mass-energy density $\rho(n)$ at the relevant value of r.

In solving the differential equations for neutron star equilibrium, you should proceed as follows

1. Make first a dimensional analysis in order to be sure that all equations are really dimensionless.

2. Define the constants R_0 and M_0 in units of 10 km and solar mass $M_\odot$. Find their values. Explain why it is convenient to insert these constants in the final results and not at each intermediate step.

3. Set up the algorithm for solving these equations and write a main program where the various variables are defined.

4. Write thereafter a small function which uses the expressions for pressure and mass-energy density from Eqs. (13.104) and (13.102).

5. Write then a function which sets up the derivatives

$$-\frac{\hat{m}\hat{\rho}}{\hat{r}^2}, \qquad \hat{r}^2\hat{\rho}. \tag{13.106}$$

6. Employ now the fourth order Runge-Kutta algorithm to obtain new values for the pressure and the mass. Play around with different values for the step size and compare the results for mass and radius.

7. Replace the fourth order Runge-Kutta method with the simple Euler method and compare the results.

8. Replace the non-relativistic expression for the derivative of the pressure with that from General Relativity (GR), the so-called Tolman-Oppenheimer-Volkov equation

$$\frac{d\hat{P}}{d\hat{r}} = -\frac{(\hat{P}+\hat{\rho})(\hat{r}^3\hat{P}+\hat{m})}{\hat{r}^2 - 2\hat{m}\hat{r}}, \tag{13.107}$$

and solve again the two differential equations.

9. Compare the non-relatistic and the GR results by plotting mass and radius as functions of the central density.