The Chandrasekhar limit

One curious feature of white-dwarf stars is that their radius decreases as their mass increases [see Eq. (696)]. It follows, from Eq. (689), that the mean energy of the degenerate electrons inside the star increases strongly as the stellar mass increases: in fact, $K \propto M^{4/3}$. Hence, if M becomes sufficiently large the electrons become *relativistic*, and the above analysis needs to be modified. Strictly speaking, the non-relativistic analysis described in the previous section is only valid in the low mass limit $M \ll M_\odot$. Let us, for the sake of simplicity, consider the ultra-relativistic limit in which $p \gg m\,c$.

The total electron energy (including the rest mass energy) can be written

$$K = \frac{3\,V}{\Lambda^3}\int_0^{p_F}(p^2\,c^2+m^2\,c^4)^{1/2}\,p^2\,dp, \tag{697}$$

by analogy with Eq. (688). Thus,

$$K \simeq \frac{3\,V\,c}{\Lambda^3}\int_0^{p_F}\left(p^3+\frac{m^2\,c^2}{2}\,p+\cdots\right)dp, \tag{698}$$

giving

$$K \simeq \frac{3}{4}\,\frac{V\,c}{\Lambda^3}\left[p_F^{\,4}+m^2\,c^2\,p_F^{\,2}+\cdots\right]. \tag{699}$$

It follows, from the above, that the total energy of an ultra-relativistic white-dwarf star can be written in the form

$$E \simeq \frac{A-B}{R}+C\,R, \tag{700}$$

where

$$A = \frac{3}{8}\left(\frac{9\pi}{8}\right)^{1/3}\hbar\,c\left(\frac{M}{m_p}\right)^{4/3}, \tag{701}$$

$$B = \tag{702}$$

$$\frac{3}{5}\,G\,M^2,$$

$$C = \frac{3}{4}\,\frac{1}{(9\pi)^{1/3}}\,\frac{m^2\,c^3}{\hbar}\left(\frac{M}{m_p}\right)^{2/3}. \tag{703}$$

As before, the equilibrium radius $R_\ast$ is that which minimizes the total energy E. However, in the ultra-relativistic case, a non-zero value of $R_\ast$ only exists for $A-B>0$. When $A-B<0$ the energy decreases monotonically with decreasing stellar radius: in other words, the degeneracy pressure of the electrons is incapable of halting the collapse of the star under gravity. The criterion which must be satisfied for a relativistic white-dwarf star to be maintained against gravity is that

$$\frac{A}{B} > 1. \tag{704}$$

This criterion can be re-written

$$M < M_C, \tag{705}$$

where

$$M_C = \frac{15}{64}\,(5\pi)^{1/2}\,\frac{(\hbar\,c/G)^{1/2}}{m_p^{\,2}} = 1.72\,M_\odot \tag{706}$$

is known as the *Chandrasekhar limit*, after A. Chandrasekhar who first derived it in 1931. A more realistic calculation, which does not assume constant density, yields

$$M_C = 1.4\,M_\odot. \tag{707}$$

Thus, if the stellar mass exceeds the Chandrasekhar limit then the star in question cannot become a white-dwarf when its nuclear fuel is exhausted, but, instead, must continue to collapse. What is the ultimate fate of such a star?
