The Look-and-Say Function

Enrico Borba

May 2015

Abstract

The Look-and-Say sequence was originally introduced and analyzed by John Conway. The sequence begins as follows:

$$1, 11, 21, 1211, 111221, 312211, 13112221, 1113213211...$$

To generate a member of the sequence from the previous member, read the digits of the previous member, counting the number of digits in groups of the same digit. For example:

1. 1 is read off as "one 1" or 11.
2. 11 is read off as "two 1s" or 21.
3. 21 is read off as "one 2, then one 1" or 1211.
4. 1211 is read off as "one 1, then one 2, then two 1s" or 111221.

This sequence has been extensively analyzed in terms of its properties, but no formal definition for a function which generates its terms has been developed. Until now. Let us venture to where no man has gone before...

1 Introduction

Before attempting to create a function which can generate the terms of the Look-and-Say sequence, a more rigorous algorithmic method of arriving at the terms should be described.

1.1 Differences

Some points of interest regarding the generation of a member of the sequence are the differences found between the digits of the previous member which was used to generate it. "Quantifying" these differences would consist of creating a number whose digits could be used to represent a member of the sequence. For example, embedding "comparisons" in between each of the digits of a member such as $L_0 = 111221$ may look something like this:

$$1_0 1_0 1_1 2_0 2_1 1$$

Here, a 0 represents an "equality" in the two digits it resides between, while a 1 designates a "difference". Consequently, the number generated using this technique, called Δ_0, would be

$$\begin{gathered} 00101 \\ \text{or} \\ 101 \end{gathered}$$

Now, it is important to preserve the leading 0s as they have significant value in regards to the current member being used to generate Δ_0. A simple way to preserve these leading zeros would be to "pad" the resulting Δ_0 with 1s as such:

$${}_1 1_0 1_0 1_1 2_0 2_1 1_1$$

$$\Delta_0 = 1001011$$

With Δ_0, the differences between the digits of L_0 can be identified and used to generate the next member.

1.2 Counting

In between each instance of a 1 in Δ_0 are digits of the same kind within L_0. Put simply, the instances of 1s in Δ_0 can be used to separate L_0 into its "groups" of digits.

$$\begin{gathered} {}_1 1_0 1_0 1_1 2_0 2_1 1_1 \\ {}_1 111_1 22_1 1_1 \end{gathered}$$

For example, here are L_0 and Δ_0 with their digits' indices labeled:

$$L_0 = \underset{5}{1}\underset{4}{1}\underset{3}{1}\underset{2}{2}\underset{1}{2}\underset{0}{1} \text{ and } \Delta_0 = \underset{6}{1}\underset{5}{0}\underset{4}{0}\underset{3}{1}\underset{2}{0}\underset{1}{1}\underset{0}{1}$$

The first instance of a 1 in Δ_0 (starting from the right) occurs at index 0.

$$\Delta_0 = 100101\underset{0}{1}$$

The second instance of a 1 occurs at index 1.

$$\Delta_0 = 10010\underset{1}{1}1$$

The difference in these indices is 1. This indicates that these two 1s surround a "group" of digits only one digit long. In this case, these first two instances of 1s "surround" a 1 in L_0 (placed in brackets for clarity).

$${}_1111_122[{}_11_1]$$

The next pair of 1s in Δ_0 occur in indices 3 and 1.

$$\Delta_0 = 100\underset{3}{1}0\underset{1}{1}1$$

This pair of 1s correspond to the second group in L_0. These two indices have a difference of 2 and therefore "surround" a "group" of digits in L_0 that is two digits long.

$${}_1111[{}_122_1]1_1$$

Carrying this forward to the final group of digits in L_0, the difference in indices is 3

$$\Delta_0 = \underset{6}{1}00\underset{3}{1}011$$

Since this is the third pair of 1s in Δ_0, it corresponds to the third "group" in L_0, which is composed of 1s.

$$[{}_1111_1]22_11_1$$

Appending each of these counts and digits in L_0 together, from right to left, generates the next member, L_1

$$\begin{aligned} &{}_1111_122[{}_11_1] && \text{one one} \\ &{}_1111[{}_122_1]1_1 && \text{two two} \\ &[{}_1111_1]22_11_1 && \text{three one} \\ &L_1 = 312211 && \end{aligned}$$

2 The Function

Creating a function, $L(x)$, that generates a term, L_{x+1}, given a previous term, x, would first involve creating the comparison function, $\Delta(x)$, which produces a corresponding Δ_x. Then, by iterating through each pairs of 1s within Δ_x and appending each digit of L_x and its "count" together, the next term, L_{x+1}, can be produced.

2.1 Writing $\Delta(x)$

As previously explained, $\Delta(x)$ consists of appending comparisons for each pair of digits within x and padding them with 1s on the ends. Given that $q(x, y)$ is the equality function[1], and $d(x, i)$ is the digit-at function[2], a comparison at digit i can be done as follows:

$$comp(x, i) = q(d(x, i), d(x, i + 1))$$

If digit i and digit $i+1$ in x are equal, this comparison results in a 1, otherwise, it evaluates to 0. This is the reverse of what was described earlier, as the differences in between the digits should result in 1s, while the equalities should result in 0s. Taking this into account:

$$comp(x, i) = 1 - q(d(x, i), d(x, i + 1))$$

Now the comparisons are consistent with what was described earlier. In order to append these comparisons, it's important to realize that any number in base 10 can be written as the sum of it's digits each multiplied by 10 raised to that digit's index. For example:

$$15012 = 1 \cdot 10^4 + 5 \cdot 10^3 + 0 \cdot 10^2 + 1 \cdot 10^1 + 2 \cdot 10^0$$

Applying this to the comparisons, and given that $n(x)$ is the number-of-digits function[3], the unpadded Δ_x can be calculated as follows:

$$\sum_{i=0}^{n(x)} comp(x, i) \cdot 10^i$$