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TDII 2021 - Secondo compito

ESERCIZIO 1

$X = \{\text{numero di studenti che si laureano}\}$

$x_i = i \qquad i = 0, \dots, 5$

$p = \frac{3}{5} \qquad 1-p = \frac{2}{5}$

a) $P\{X=0\} = \binom{5}{0}\left(\frac{3}{5}\right)^0\left(\frac{2}{5}\right)^5 = \left(\frac{2}{5}\right)^5 = 0,01024$

b) $P\{X=1\} = \binom{5}{1}\left(\frac{3}{5}\right)^1\left(\frac{2}{5}\right)^4 = \frac{5\cdot\cancel{4!}}{\cancel{4!}}\left(\frac{3}{5}\right)\left(\frac{2}{5}\right)^4 = 0,0768$

c) $P\{X \geq 1\} = 1 - P\{X=0\} = 1 - 0,01024 = 0,98976$

ESERCIZIO 2

$X = \{\text{tentativo superato}\}$

$p = \frac{7}{10} \qquad 1-p = \frac{3}{10}$

$P\{X=4\} = \left(\frac{3}{10}\right)^3\left(\frac{7}{10}\right) = 0,0189$

ESERCIZIO 3

$f(x) = Cx^3 \qquad \left[0, \frac{1}{2}\right]$

a) $\int_{-\infty}^{+\infty} dx\, f(x) = \int_0^{\frac{1}{2}} dx\, Cx^3 = C\left.\frac{x^4}{4}\right|_0^{\frac{1}{2}} = \frac{1}{64}C$

$\frac{1}{64}C = 1 \longrightarrow C = 64$

b) $P\left\{\frac{1}{3} \leq X \leq 1\right\} = \int_{1/3}^{1/2} dx\, f(x) = \int_{1/3}^{1/2} 64x^3 = 64\left.\frac{x^4}{4}\right|_{1/3}^{1/2} =$

$= 1 - \frac{\cancel{64}^{\,16}}{\cancel{324}_{\,81}} = \frac{65}{81}$

c) $E[x^2] = \int_{-\infty}^{+\infty} dx\, f(x)\, x^2 = \int_0^{\frac{1}{2}} 64 x^5\, dx = 64 \frac{x^6}{6}\Big|_0^{1/2} =$

$= \cancel{64} \cdot \frac{1}{\cancel{64}} \cdot \frac{1}{6} = \frac{1}{6}$

$Var(X) = \int_{-\infty}^{+\infty} dx\, f(x)\, x^2 - \left(\int_{-\infty}^{+\infty} dx\, f(x)\, x\right)^2$

$E[x] = \int_{-\infty}^{+\infty} dx\, f(x) x = \int_0^{1/2} 64 x^4\, dx = 64 \frac{x^5}{5}\Big|_0^{1/2} =$

$= \cancel{64}^{2} \cdot \frac{1}{\cancel{32}_1} \cdot \frac{1}{5} = \frac{2}{5}$

$Var(x) = \frac{1}{6} - \frac{4}{25} = \frac{1}{150}$

ESERCIZIO 4

$f(x) = \begin{cases} \frac{1}{2} & x \in [0, 2] \\ 0 & x \notin [0, 2] \end{cases}$

a) $\int_0^2 e^{\frac{1}{2}}\, dx = e^{\frac{1}{2}} x\Big|_0^2 = 2\sqrt{e}$

b) $E[e^x] = \int_0^2 e^{\frac{1}{2}} x\, dx = e^{\frac{1}{2}} \int_0^2 x\, dx = \sqrt{e}\, \frac{x^2}{2}\Big|_0^2 = 2\sqrt{e}$

$E[e^{x^2}] = \int_0^2 dx\, f(x) x^2 = \int_0^2 e^x x^2\, dx = \sqrt{e}\, \frac{x^3}{3}\Big|_0^2 = \frac{8\sqrt{e}}{3}$

$Var(X) = \frac{8\sqrt{e}}{3} - (2\sqrt{e})^2 = \frac{8\sqrt{e}}{3} - 4e$

ESERCIZIO 5

$X = N(3, 9)$ $\qquad Y = aX + b \implies Y = N(a\mu + b; a^2 \sigma^2)$

$= N(a3 + b, a^2 9)$

$a = \frac{1}{6} = \frac{1}{3}$ $\qquad b = -\frac{\mu}{6} = -\frac{\cancel{3}}{\cancel{3}} = -1$

ESERCIZIO 6

$X = \{\text{tempo mancante al prossimo autobus}\}$

$P(x < 5) = \int_0^5 \frac{1}{15}\, dx = \frac{1}{15} x \Big|_0^5 = \frac{\cancel{5}^{1}}{\cancel{15}_{3}} = \frac{1}{3}$

$P(x > 10) = 1 - P(x < 10) = P(10 < x < 15) =$

$= \int_{10}^{15} \frac{1}{15}\, dx = \frac{1}{15} x \Big|_{10}^{15} = 1 - \frac{\cancel{10}^{2}}{\cancel{15}_{3}} = \frac{1}{3}$

ESERCIZIO 7

$f(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}$ $\qquad \lambda = 6{,}9$

a) $P(x < \frac{1}{4}) = \int_0^{\frac{1}{4}} \lambda e^{-\lambda x}\, dx = 1 - e^{-\lambda \frac{1}{4}} = 1 - e^{-\frac{69}{40}}$

b) $P(\frac{1}{4} < x < \frac{3}{4}) = \int_{\frac{1}{4}}^{\frac{3}{4}} \lambda e^{-\lambda x}\, dx = 1 - e^{-\frac{207}{40}} - \left(1 - e^{-\frac{69}{40}}\right) =$

$= \cancel{1} - e^{-\frac{207}{40}} \cancel{- 1} + e^{-\frac{69}{40}} = -\frac{1}{\sqrt[40]{e^{207}}} + \frac{1}{\sqrt[40]{e^{69}}}$

ESERCIZIO 8

$$Var(x) = E[X^2] - (E[X])^2 = \sum_{i=1}^{+\infty} i^2 p (1-p)^{i-1} - \frac{1}{p^2} \rightarrow E[X] = \frac{1}{p}$$

$$= p \sum_{i=1}^{+\infty} [i(i-1)+i](1-p)^{i-1} - \frac{1}{p^2} = p(1-p) \sum_{i=2}^{+\infty} i(i-1)(1-p)^{i-2} +$$

$$+ \underbrace{\sum_{i=1}^{+\infty} i (1-p)^{i-1} p}_{1/p} - \frac{1}{p^2} =$$

$\boxed{i(i-1)(1-p)^{i-2} \text{ è la derivata seconda rispetto a } p \text{ di } (1-p)^i}$

$$= p(1-p) \frac{d^2}{dp^2} \sum_{i=2}^{+\infty} (1-p)^i + \frac{1}{p} - \frac{1}{p^2}$$

$$= p(1-p) \frac{d^2(1/p)}{dp^2} + \frac{1}{p} - \frac{1}{p^2} = p(1-p)\frac{2}{p^3} + \frac{1}{p} - \frac{1}{p^2}$$

$$= \boxed{\frac{1-p}{p^2}}$$

ESERCIZIO 9

$$Var(x) = E[x^2] - (E[x])^2 = \sum_{i=0}^{+\infty} i^2 \frac{\lambda^i}{i!} e^{-\lambda} - \lambda^2 \rightarrow E[x] = \lambda$$

$$= \sum_{i=0}^{+\infty} i(i-1) \frac{\lambda^i}{i!} e^{-\lambda} + \underbrace{\sum_{i=0}^{+\infty} i \frac{\lambda^i}{i!} e^{-\lambda}}_{\lambda} - \lambda^2 =$$

$$= \sum_{i=2}^{+\infty} i(i-1) \frac{\lambda^i}{i!} e^{-\lambda} + \lambda - \lambda^2 = \lambda^2 e^{-\lambda} \sum_{i=2}^{+\infty} \frac{\lambda^{(i-2)}}{(i-2)!} + \lambda - \lambda^2 =$$

$$= \lambda^2 + \lambda - \lambda^2 = \boxed{\lambda}$$

ESERCIZIO 10

$M = \{\text{numero di maschi}\}$ $F = \{\text{numero di femmine}\}$

$N = \{\text{numero di figli}\}$

$- P\{N=0\} = \frac{1}{5} = P(M=0, F=0)$

M \ F	0	1	2
0	$\frac{1}{5}$	$\frac{1}{4}$	$\frac{3}{40}$
1	$\frac{1}{4}$	$\frac{3}{20}$	0
2	$\frac{3}{40}$	0	0

$P(M=1, F=0) = P(M=1 \mid N=1) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$

$P(F=2, M=0) = P(F=2 \mid N=2) = \left(\frac{1}{2}\right)^2 \cdot \frac{3}{10} = \frac{3}{40}$

$P(M=1, F=1) = P(F=1, M=1 \mid N=2) = \frac{3}{20}$

$P(M_i, F_j) = p(i, j)$ $i = 0 \dots 2$

$j = 0 \dots 2$