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TdII 2021 - Terzo Compito

① X variabile casuale continua con pdf $f_X(x)$

$Y := |X| \qquad f_X \longrightarrow f_Y$

$$F_Y(a) = P(|X| \leq a) = P(-a \leq X \leq a) = \int_{-a}^{a} dx\, f_X(x) =$$
$$= \int_{-\infty}^{a} dx\, f(x) - \int_{-\infty}^{-a} dx\, f(x) = F_X(a) - F_X(-a)$$

$$f_Y(a) = \frac{d\, F_Y(a)}{da} = \frac{d\, F_X(a)}{da} - \frac{d\, F_X(-a)}{da} = f_X(a) + f_X(-a)$$

② X e Y due variabili casuali discrete

$P\{X=2, Y=3\} = \frac{1}{3} \qquad P\{X=3, Y=3\} = \frac{1}{4}$

$P\{X=3, Y=4\} = \frac{1}{4} \qquad P\{X=2, Y=1\} = \frac{1}{6}$

(a) $P_X(2) = \frac{1}{6} + \frac{1}{3} = \frac{1}{2} \qquad P_X(3) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$

$P_Y(1) = \frac{1}{6} \qquad P_Y(3) = \frac{1}{3} + \frac{1}{4} = \frac{7}{12} \qquad P_Y(4) = \frac{1}{4}$

(b) $E[X] = \cancel{2} \cdot \frac{1}{\cancel{2}} + 3 \cdot \frac{1}{2} = \frac{5}{2} \qquad E[Y] = 1 \cdot \frac{1}{6} + 3 \cdot \frac{7}{12} + \cancel{4} \cdot \frac{1}{\cancel{4}} = \frac{35}{12}$

(c) $E[XY] = 2 \cdot 1 \cdot \frac{1}{6} + 2 \cdot \cancel{3} \cdot \frac{1}{\cancel{3}} + 3 \cdot 3 \cdot \frac{1}{4} + 3 \cdot \cancel{4} \cdot \frac{1}{\cancel{4}} = \frac{91}{12}$

(d) $\mathrm{cov}(X, Y) = E[XY] - E[X] \cdot E[Y] = \frac{91}{12} - \frac{175}{24} = \frac{7}{24}$

(e) Le variabili X e Y non sono indipendenti poichè $P(i,j) \neq P(i) \cdot P(j)$

$P(2,1) = \frac{1}{6}$

$\neq$

$P(i) \cdot P(j) = \frac{1}{6} \cdot \frac{1}{2} = \frac{1}{8}$

(f) $P\{X \leq 3, Y \leq 3\} = \frac{1}{6} + \frac{1}{3} + \frac{1}{3} = \frac{5}{6}$

③ (a) $Cov(X,Y) = E[XY] - E[X]\cdot E[Y] = 0 \iff E[XY] = E[X]\cdot E[Y]$

Se X, Y sono indipendenti, cioè $p(i,j) = p(i)\cdot p(j)$

$$E[XY] = \sum_{i=1}^{N}\sum_{j=1}^{M} (x_i \cdot y_j)\, p(i,j) = \sum_{i=1}^{N}\sum_{j=1}^{M} (x_i \cdot y_j)\, p(i)\cdot p(j) =$$

$$= \sum_{i=1}^{N}\sum_{j=1}^{M} (x_i\, p(i))\cdot(y_j - p(j)) = \sum_{i=1}^{N} x_i \cdot p_x(i) \cdot \sum_{j=1}^{M} y_j\, p_y(j) =$$

$$= E[X]\cdot E[Y]$$

(b) Due variabili casuali dipendenti a covarianza nulla

X variabile aleatoria uniformemente distribuita su $[-1,1]$

$Y = X^2$

$Cov(X,Y) = Cov(X,X^2) = E[X^3] - E[X]\,E[X^2] = 0 - 0\,E[X^2] = 0$

④ X_1, X_2, X_3, X_4 variabili casuali indipendenti $E[X]=0$ $Var(X)=1$

$Y_1 := X_1 + X_2$ $Y_2 := X_2 + X_3$ $Y_3 := X_3 + X_4$

(a) $$\rho(X,Y) = \frac{cov(X,Y)}{\sqrt{var(X)}\sqrt{var(Y)}}$$

$E[Y_1] = E[X_1 + X_2] = 0$ $Var(Y_1) = Var(X_1 + X_2) = 2$

$E[Y_2] = E[X_2 + X_3] = 0$ $Var(Y_2) = Var(X_2 + X_3) = 2$

$Cov(Y_1, Y_2) = E[Y_1 Y_2]$ Y_1 e Y_2 <u>non</u> sono Indipendenti

$$\rho(\cancel{X_1}Y_1, Y_2) = \frac{E[Y_1 Y_2]}{\sqrt{2}\sqrt{2}} = \frac{E[Y_1 Y_2]}{2}$$

(b) Y_1 e Y_3 sono Indipendenti

$Cov(Y_1, Y_3) = 0$

$$\rho(Y_1, Y_3) = \frac{0}{2} = 0$$

⑤ $X_1, \dots, X_n$ X_i indipendenti e identicamente distribuite

$E[X] = \mu$ $\quad Var(X) = \sigma^2$ $\quad \sigma = 1$

Disuguaglianza di Chebyshev: poiché $E\left[\frac{1}{n} \sum_{i=1}^{n} X_i\right] = \mu$

$Var\left[\frac{1}{n} \sum_{i=1}^{n} X_i\right] = \frac{\sigma^2}{n} = \frac{1}{n}$

$$P\left\{\left|\frac{1}{n} \sum_{i=1}^{n} X_i - \mu\right| \geq 0,5\right\} \leq \frac{1}{0,25n} = \frac{4}{n}$$

$$\frac{4}{n} = 10\% \rightarrow \frac{4}{n} = \frac{1}{10} \longrightarrow n = 40$$

⑥

Teorema del limite centrale: $Z_n = \frac{\sum_{i=1}^{n} X_i - n\mu}{1\sqrt{n}} \sim N(0,1)$ per $n \rightarrow \infty$

$$P\left\{-0,5 \leq \left[\left(\frac{1}{n} \sum_{i=1}^{n} X_i\right) - \mu\right] \leq 0,5\right\} = P\left\{-0,5 \leq \frac{\sum_{i=1}^{n} X_i - n\mu}{n} \leq 0,5\right\}$$

$$= P\{-0,5\sqrt{n} \leq Z_n \leq 0,5\sqrt{n}\} = F_X\left(\frac{\sqrt{n}}{2}\right) - F_X\left(-\frac{\sqrt{n}}{2}\right)$$

$$= 2F_X\left(\frac{\sqrt{n}}{2}\right) - 1$$

$$2F_X\left(\frac{\sqrt{n}}{2}\right) = 90\%$$