1 a) $rk(A)=?$ $\begin{pmatrix} 1 & -1 & 0 & 3 \\ 0 & 3 & 1 & -5 \\ 2 & 7 & 3 & -9 \end{pmatrix}$

C_{123} $9-2-7=0$

C_{134} $\begin{vmatrix} 1 & 0 & 3 \\ 0 & 1 & -5 \\ 2 & 3 & -9 \end{vmatrix} = -9-6+15=0$

C_{124} $\begin{vmatrix} 1 & -1 & 3 \\ 0 & 3 & -5 \\ 2 & 7 & -9 \end{vmatrix} = -27+10-18+35=0$

C_{234} $9+27-21-15=0$ $\quad rk(A)\neq 3$

$\begin{vmatrix} 1 & -1 \\ 0 & 3 \end{vmatrix} = 3 \neq 0 \Rightarrow rk(A)=2$

b) $x \perp \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \Leftrightarrow x \cdot v = 0 \Leftrightarrow x_1 \cdot 1 + x_2 \cdot 0 + x_3 \cdot 0 + x_4 \cdot 0 = 0 \Leftrightarrow x_1 = 0$

1ª colonna non ci importa

$\begin{pmatrix} -1 & 0 & 3 \\ 3 & 1 & -5 \\ 7 & 3 & -9 \end{pmatrix}$ $\quad rk(\ ) \underset{=2}{\neq 3} \Rightarrow \infty$ soluz.

$rk\left(\ \middle| \begin{smallmatrix} 0 \\ 0 \\ 0 \end{smallmatrix} \right) \underset{=2}{\neq 3}$

$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ è soluz ed è $\perp$ a v!

senno?

R_2+3R_1, R_3+7R_1 $\begin{pmatrix} -1 & 0 & 3 \\ 0 & 1 & 4 \\ 0 & 3 & 12 \end{pmatrix} \to \begin{pmatrix} -1 & 0 & 3 \\ 0 & 1 & 4 \\ 0 & 3 & 12 \end{pmatrix} \to \begin{pmatrix} -1 & 0 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{cases} -x_2+3x_4=0 \\ x_3+4x_4=0 \end{cases} \begin{cases} x_2=3x_4 \\ x_3=-4x_4 \end{cases}$ $\begin{pmatrix} 0 \\ 3x_4 \\ -4x_4 \\ x_4 \end{pmatrix}$

ad es. $x_4=1 \to \begin{pmatrix} 0 \\ 3 \\ -4 \\ 1 \end{pmatrix}$

2 $rk(A)=?$ per quali λ?

$\begin{pmatrix} 2-\lambda & -1 & 0 & 3 \\ 1-\lambda & 3 & 1 & -5\lambda \\ 2 & 5+2\lambda & 3\lambda & -9 \end{pmatrix}$

$\det(C_{234})$ $9+27\lambda-15-6\lambda-15\lambda^2 = -15\lambda^2+21\lambda-6$

$\det=0 \Leftrightarrow -5\lambda^2+7\lambda-2=0 \Leftrightarrow \lambda_{1,2} = \frac{-7\pm\sqrt{49-40}}{-10} = \frac{-7\pm 3}{-10} = \begin{cases} 1 \\ 2/5 \end{cases}$

se $\lambda \neq 1$ e $\neq 2/5 \Rightarrow rk=3$

se $\lambda=1$? $\begin{pmatrix} 1 & -1 & 0 & 3 \\ 0 & 3 & 1 & -5 \\ 2 & 7 & 3 & -9 \end{pmatrix}$

$C_{123}=9-2-7=0$

$C_{134}=-9-6+15=0$

$C_{234}=9+27-21-15=0$

$\Rightarrow rk\neq 3$

se $\lambda=2/5$?

$$\begin{pmatrix} 2-\lambda & -1 & 0 & 3 \\ 1-\lambda & 3 & 1 & -5\lambda \\ 2 & 5+2\lambda & 3\lambda & -9 \end{pmatrix}$$

$\lambda = \frac{3}{5}$

$$\begin{pmatrix} 8/5 & -1 & 0 & 3 \\ 3/5 & 3 & 1 & -2 \\ 2 & 21/5 & 6/5 & -9 \end{pmatrix}$$

$9 + 9 \cdot \frac{6}{5} - \frac{63}{5} - \frac{6}{5} = \frac{45+54-63-6}{5} \neq 0 \Rightarrow rk = 3$

$\Rightarrow \forall \lambda \neq 1$

(b) $\lambda \neq 1$ $rk(A) = rk(A|b) \Rightarrow$ ha soluz

$\lambda = 1$?

$$\left(\begin{array}{cccc|c} 1 & -1 & 0 & 3 & 1 \\ 0 & 3 & 1 & -5 & 1 \\ 2 & 7 & 3 & -9 & 6 \end{array}\right)$$

$C_{125} = \det\begin{pmatrix} 1 & -1 & 1 \\ 0 & 3 & 1 \\ 2 & 7 & 6 \end{pmatrix} = 18 - 2 - 6 - 7 = 3 \neq 0 \Rightarrow rk(A|b) = 3 \neq \overset{\overset{2}{\shortparallel}}{rk}(A)$

$\Rightarrow \nexists$ soluz. per $\lambda = 1$

$\boxed{3}$ (a) $A^3 = I_m$ $\quad \underset{\overset{\shortparallel}{\det(I_m)=1}}{\det(A^3)} = (\det A)^3 \Rightarrow \det A = 1 \Rightarrow$ inv

(b) $A^2 = 0$ $\quad \det A^2 = (\det A)^2 = 0 \Rightarrow \det(A) = 0 \Rightarrow$ no inv

(c) $A^2 + A = I_m \Rightarrow A(A + I_m) = I_m \Rightarrow \det(A) \cdot \det(A + I_m) = \det(I_m) = 1$

$\Rightarrow \det(A) \neq 0 \Rightarrow$ invertibile

$\boxed{4}$ $v_1 = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$ $\quad v_2 = \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}$

$v_3 = ?$

per $b = 1$ $\quad \begin{pmatrix} 2 \\ -5 \\ 1 \end{pmatrix} \leftarrow \begin{pmatrix} 2c \\ -5c \\ c \end{pmatrix}$

(a) v_3 t.c. $v_3 \perp v_1$ e $v_2 \perp v_3$ e $v_3 \neq 0$

$v_3 = (a, b, c)$

$$\begin{cases} (a,b,c)\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} = 0 \\ (a,b,c)\begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} = 0 \end{cases} \quad \begin{cases} 2a + b + c = 0 \\ a + 0 - 2c = 0 \end{cases} \quad \begin{cases} 4c + b + c = 0 \\ a = 2c \end{cases} \quad \begin{cases} b = -5c \\ a = 2c \end{cases}$$

(b) $v_3 = v_1 + v_2 = \begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix}$

combinaz. lin

(c) come (a)

$$\begin{vmatrix} i & j & k \\ 2 & 1 & 1 \\ 1 & 0 & -2 \end{vmatrix} = -2i + j - k + 4j = (-2, 5, -1)$$

$$\begin{vmatrix} 0 & 1 & -1 \\ 1 & 0 & 2 \\ 2 & 3 & 1 \end{vmatrix} = 0+4-3-0-1=0$$

tutte le diagonali meno tutte le antidiagonali

$$\begin{vmatrix} 0 & 1 & 2 \\ 1 & 0 & -1 \\ 2 & 3 & 4 \end{vmatrix} = 0+6-2-0-0-4=0$$

$v \perp w \iff v \cdot w = 0$

$$\underbrace{(x_1\ x_2\ x_3\ x_4)}_{1\times 4} \underbrace{\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}}_{4\times 1} = x_1 = 0$$

$$\begin{pmatrix} 1 & -1 & 2 \\ 0 & 2 & -1 \\ 3 & 1 & 4 \end{pmatrix} \begin{pmatrix} x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$rk(A) = rk(A|b) \Rightarrow \exists$ soluz.

$\neq \Rightarrow \nexists$

n° soluz $= \infty^{n^{\circ} \text{colonne} - rk}$

$$\begin{pmatrix} 0 & 1 & -1 & 2 \\ 1 & 0 & 2 & -1 \\ 2 & 3 & 1 & 4 \end{pmatrix} \xrightarrow[R_2 \leftrightarrow R_1]{} \begin{pmatrix} 1 & 0 & 2 & -1 \\ 0 & 1 & -1 & 2 \\ 2 & 3 & 1 & 4 \end{pmatrix}$$

$$\xrightarrow[R_3-2R_1]{} \begin{pmatrix} 1 & 0 & 2 & -1 \\ 0 & 1 & -1 & 2 \\ 0 & 3 & -3 & 6 \end{pmatrix} \xrightarrow[R_3-3R_2]{} \begin{pmatrix} 1 & 0 & 2 & -1 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{cases} x_1 = 0 \\ 2x_3 - x_4 = 0 \\ x_2 - x_3 + 2x_4 = 0 \end{cases} \quad \begin{cases} x_1 = 0 \\ x_4 = 2x_3 \\ x_2 - x_3 + 4x_3 = 0 \end{cases} \quad \begin{cases} x_1 = 0 \\ x_4 = 2x_3 \\ x_2 = -3x_3 \end{cases}$$

$$\begin{pmatrix} 0 \\ -3x_3 \\ x_3 \\ 2x_3 \end{pmatrix} \xrightarrow{x_3 = 1} \begin{pmatrix} 0 \\ -3 \\ 1 \\ 2 \end{pmatrix}$$

③ $\begin{pmatrix} \lambda & 1 & -1 & 2 \\ 1 & \lambda & 2 & -1 \\ 2 & 3 & 2\lambda+1 & 4 \end{pmatrix}$

ⓐ $rk = 3$?

C_1, C_2, C_4 $\begin{vmatrix} \lambda & 1 & 2 \\ 1 & \lambda & -1 \\ 2 & 3 & 4 \end{vmatrix} = 4\lambda^2 + 6 - 2 - 4\lambda + 3\lambda - 4 = 4\lambda^2 - \lambda$

$= \lambda(4\lambda - 1)$

$\lambda = 0$, $\lambda = \frac{1}{4}$

se $\lambda \neq 0$ e $\neq \frac{1}{4} \Rightarrow rk(A) = 3$

se $\lambda = 0$ $\begin{pmatrix} 0 & 1 & -1 & 2 \\ 1 & 0 & 2 & -1 \\ 2 & 3 & 1 & 4 \end{pmatrix}$ $rk = 2$ (es 1)

se $\lambda = \frac{1}{4}$ $\begin{pmatrix} \frac{1}{4} & 1 & -1 & 2 \\ 1 & \frac{1}{4} & 2 & -1 \\ 2 & 3 & \underbrace{\frac{2}{4}+1}_{\frac{3}{2}} & 4 \end{pmatrix}$

$C_1\, C_3\, C_4$ $\begin{vmatrix} \frac{1}{4} & -1 & 2 \\ 1 & 2 & -1 \\ 2 & \frac{3}{2} & 4 \end{vmatrix} = 2 + 2 + 3 - 6 + \frac{3}{8} + 4$

$= 5 + \frac{3}{8} \neq 0 \Rightarrow rk = 3$

se $\lambda \neq 0 \rightarrow rk(A) = 3$

se $\lambda = 0 \rightarrow rk(A) = 2$

(b) $\left(\begin{array}{cccc|c} \lambda & 1 & -1 & 2 & 0 \\ 1 & \lambda & 2 & -1 & 0 \\ 2 & 3 & 2\lambda+1 & 4 & \lambda \end{array}\right)$

λ t.c. non ammette soluzioni?

$rk(A) \neq rk(A|b)$

A 3×4 $\quad$ $A|b$ 3×5

se $rk(A) = \underset{\text{valore MAX}}{3} \Rightarrow rk(A|b) = 3$ $\quad$ $(rk(A|b) \geq rk(A))$

se $\lambda \neq 0 \Rightarrow rk(A) = rk(A|b) \Rightarrow \exists$ soluz.

se $\lambda = 0$? $\left(\begin{array}{cccc|c} 0 & 1 & -1 & 2 & 0 \\ 1 & 0 & 2 & -1 & 0 \\ 2 & 3 & 1 & 4 & 0 \end{array}\right)$

omogenea

$rk(A) = rk(A|b) = 2$

$\Rightarrow rk(A) = rk(A|b) \quad \forall \lambda \in \mathbb{R} \Rightarrow \exists$ soluz. $\forall \lambda \in \mathbb{R}$

③ (a) A inv $\Rightarrow \det A > 0$

FALSO, $\det\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = -1 \neq 0$

(b) A, B inv $\Rightarrow AB$ inv

$\det(AB) = \det(A)\det(B)$

$\left[\det(A^{-1}) = \frac{1}{\det(A)}\right]$

$\det(A) \neq 0$ e $\det(B) \neq 0 \Rightarrow \underbrace{\det(A)\det(B)}_{\det(AB)} \neq 0 \Rightarrow AB$ inv

(c) $A^{13} = B$ e B inv $\Rightarrow A$ inv

B inv $\Rightarrow \det(B) \neq 0 \Rightarrow \det(A^{13}) \neq 0 \Rightarrow \det(A)^{13} \neq 0 \Rightarrow \det A \neq 0$

$\det(AB) = \det(A)\det(B)$

$\Downarrow$

A inv

---

④ (a) $v_1 = \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix}$ $\quad v_2 = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$

$v_3 = v_1 + v_2 = \begin{pmatrix} 4 \\ -2 \\ 3 \end{pmatrix}$

è combinaz. lineare di $v_1 + v_2$ $\quad a_1 v_1 + a_2 v_2$

(b) $w = v_1 \wedge v_2 = \begin{vmatrix} i & j & k \\ 3 & -1 & 1 \\ 1 & -1 & 2 \end{vmatrix} = -2i + j - 3k + k + i - 6j = -i - 5j - 2k = \begin{pmatrix} -1 \\ -5 \\ -2 \end{pmatrix}$

v_1, v_2

$w \perp v_1$ e $w \perp v_2$

$\|w\|_2 = \sqrt{w_1^2 + w_2^2 + w_3^2} = \sqrt{1 + 25 + 4} = \sqrt{30}$

$w' = \frac{w}{\|w\|_2}$ $\quad \|w'\| = 1$ vettore normalizzato $\quad v_3 = 10w' = 10\frac{w}{\|w\|_2} = \frac{10}{\sqrt{30}}\begin{pmatrix} -1 \\ -5 \\ -2 \end{pmatrix}$

(c) Non possono esserci 4 vet. lin indip in $\mathbb{R}^3$

10-02-2022

(2) $A=\begin{pmatrix}1 & -1 & 0\\ 2 & 0 & 1\\ 3 & -2 & 1\end{pmatrix}$

$\det A=\begin{vmatrix}1 & -1 & 0\\ 2 & 0 & 1\\ 3 & -2 & 1\end{vmatrix} = 0 - 3 + 0 - 0 + 2 + 2 = 1 \neq 0 \Rightarrow \text{inv}$

$$\left(\underbrace{\begin{matrix}1 & -1 & 0\\ 2 & 0 & 1\\ 3 & -2 & 1\end{matrix}}_{A}\;\middle|\;\underbrace{\begin{matrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{matrix}}_{I}\right) \xrightarrow[R_3-3R_1]{R_2-2R_1} \left(\begin{matrix}1 & -1 & 0\\ 0 & +2 & 1\\ 0 & 1 & 1\end{matrix}\;\middle|\;\begin{matrix}1 & 0 & 0\\ -2 & 1 & 0\\ -3 & 0 & 1\end{matrix}\right)$$

$$\xrightarrow[R_3-\frac{1}{2}R_2]{} \left(\begin{matrix}1 & -1 & 0\\ 0 & 2 & 1\\ 0 & 0 & \frac{1}{2}\end{matrix}\;\middle|\;\begin{matrix}1 & 0 & 0\\ -2 & 1 & 0\\ -2 & -\frac{1}{2} & 1\end{matrix}\right) \xrightarrow[\substack{R_2\to\frac{1}{2}R_2\\ R_3\to 2R_3}]{} \left(\begin{matrix}1 & -1 & 0\\ 0 & 1 & \frac{1}{2}\\ 0 & 0 & 1\end{matrix}\;\middle|\;\begin{matrix}1 & 0 & 0\\ -1 & \frac{1}{2} & 0\\ -4 & -1 & 2\end{matrix}\right)$$

$$\xrightarrow[R_2-\frac{1}{2}R_3]{} \left(\begin{matrix}1 & -1 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{matrix}\;\middle|\;\begin{matrix}1 & 0 & 0\\ 1 & 1 & -1\\ -4 & -1 & 2\end{matrix}\right) \xrightarrow[R_1+R_2]{}$$

$$\left(\underbrace{\begin{matrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{matrix}}_{I}\;\middle|\;\underbrace{\begin{matrix}2 & 1 & -1\\ 1 & 1 & -1\\ -4 & -1 & 2\end{matrix}}_{A^{-1}}\right)$$

$$AA^{-1}=\begin{pmatrix}1 & -1 & 0\\ 2 & 0 & 1\\ 3 & -2 & 1\end{pmatrix}\begin{pmatrix}2 & 1 & -1\\ 1 & 1 & -1\\ -4 & -1 & 2\end{pmatrix}=\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}$$

$A^{-1}A = (\quad \dots \quad)(\quad \dots \quad) = I$

(b) $\|x\|_2 = 2 \qquad Ax = \begin{pmatrix} \vdots \end{pmatrix}$

$\left(\begin{matrix} rk(A) = 3 \\ \| \\ rk(A|b) \end{matrix} \qquad \infty^{\overset{0}{\widetilde{3-3}}} = 1 \right)$

$Ax = b \rightarrow \underbrace{A^{-1}A}_{I} x = A^{-1}b \Rightarrow x = A^{-1}b$

$\underbrace{\begin{pmatrix} 2 & 1 & -1 \\ 1 & 1 & -1 \\ -4 & -1 & 2 \end{pmatrix}}_{A^{-1}} \underbrace{\begin{pmatrix} \vdots \end{pmatrix}}_{b} = \underbrace{\begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix}}_{x}$

$b \rightarrow$ vettore dei termini noti

$\|x\|_2 \quad \sqrt{2^2 + 1^2 + 3^2} = \sqrt{14}$

Essendo A inv, esiste unica soluz $\begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix}$

e $\|x\|_2 \neq 2 \Rightarrow \nexists$ soluz di norma $= 2$

③

ⓐ A inv t.c. $A^2 = A \overset{?}{\Rightarrow} \det A = 1$

$\det A \neq 0$

$A^2 = A \Rightarrow \underset{\det(A)^2}{\underset{\parallel}{\det(A^2)}} = \det(A) \Rightarrow \det(A)^2 = \det A$

$\Rightarrow \det(A)^2 - \det A = 0 \Rightarrow \det A(\det A - 1) = 0$

$\underset{\det A \neq 0}{\overset{\uparrow}{\Rightarrow}} \det A = 1$

ⓑ FALSO, $\overset{\text{in } \mathbb{R}^4}{\text{posso}}$ avere al più 4 vet. lin. indip.

ⓒ $\begin{pmatrix} 0 & & A \\ & 0 & \\ & & 0 \\ 0 & & & 0 \end{pmatrix} - \begin{pmatrix} \lambda & & 0 \\ & \lambda & \\ & & \lambda \\ 0 & & & \lambda \end{pmatrix} \begin{pmatrix} 0 & & A \\ & 0 & \\ 0 & & \end{pmatrix}$ (1,2)

$\det(A - \lambda I) = \det \begin{pmatrix} -\lambda & & A \\ & -\lambda & \\ & & -\lambda \\ 0 & & & -\lambda \end{pmatrix} = \lambda^4 = 0 \Leftrightarrow \lambda = 0$

$\Rightarrow$ tutti gli autov. sono nulli $\Rightarrow$ è nilpotente

$\boxed{4}$ $v_1 = \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$ $v_2 = \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix}$

ⓐ $\begin{pmatrix} -\pi/2 \\ 3\pi \\ \frac{7}{2}\pi \end{pmatrix} \in \langle v_1, v_2 \rangle$?

combinaz. lineare $a \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} + b \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} -\pi/2 \\ 3\pi \\ \frac{7\pi}{2} \end{pmatrix} \rightarrow \begin{cases} -a + 3b = -\frac{\pi}{2} \\ a + 2b = 3\pi \\ 2a - b = \frac{7}{2}\pi \end{cases}$

$$\begin{cases} a = 3b + \frac{\pi}{2} \Rightarrow a = 3\frac{\pi}{2} + \frac{\pi}{2} = 2\pi \\ 3b + \frac{\pi}{2} + 2b = 3\pi \rightarrow \cancel{5}b = \frac{\cancel{5}}{2}\pi \\ 4\pi - \frac{\pi}{2} = \frac{7}{2}\pi \rightarrow \frac{7}{2}\pi = \frac{7}{2}\pi \Rightarrow v_3 \in \langle v_1, v_2 \rangle \end{cases}$$

(b) $v \perp v_1$ e $v \perp v_2$

prod vet
$$\begin{pmatrix} i & j & k \\ -1 & 1 & 2 \\ 3 & 2 & -1 \end{pmatrix} = -i+6j-2k-3k-4i-j = $$
$$= -5i+5j-5k = \begin{pmatrix} -5 \\ 5 \\ -5 \end{pmatrix}$$

$\langle v, v_1, v_2 \rangle = \mathbb{R}^3$

$v \perp v_1$ e $v \perp v_2$, $v_1 \perp v_2$?

prod scalare (è $=0$ se vet ortog) $\Rightarrow$

(prod vet. $=0$ se vet paralleli ovvero uno multiplo dell'altro)

$$\rightarrow \begin{pmatrix} -1 & 1 & 2 \end{pmatrix} \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix} = -3+2-2 = -3 \neq 0$$

$\Rightarrow$ v_1 e v_2 non sono $\perp$ $\Rightarrow$ NO BASE ORTOGONALE