Obtención de operadores bases de n-orden luego de la conmutación del operador de n-orden diferencial general con la ecuación de Schrödinger

Clasificación, propiedades y recurrencia.

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Tabla de Contenido

Halmitoneano modificado de Schrodinger:

$$\hat{H} = a\partial_x^2 - i\hbar\partial_t \tag{1}$$

Con: $a = \frac{-\hbar^2}{2m}$ y $\hbar = 1$. Definimos un operador diferencial $\hat{L}$ de 1-orden:

1. $$\hat{L} = A(x,t)\partial_x + B(x,t)\partial_t + C(x,t) \tag{2}$$
2. Imponemos la conmutación y cumpliendo con un 'Conjunto Completo de Observables Compatibles' (CCOC):
$$\left[\hat{H}, \hat{L}\right]\Psi = 0 \tag{3}$$
3. Siendo: $\Psi = \{\Psi(x,t) \in \mathbb{C} | x \wedge t \in \mathbb{R}\}$, La eigenfunción asociada al hamiltoniano y a la solución del problema de valores propios:
$$\hat{H}\left|\Psi\right\rangle = \lambda\left|\Psi\right\rangle \tag{4}$$

Teorema: CCOC

Teorema de la compatibilidad

El teorema de la compatibilidad nos dice que dado dos observables, representados por sus operadores, en este caso $\hat{H}$ y $\hat{L}$, los observables asociados son compatibles si y solo si sus operadores correspondientes tienen una base propia comun. Estos operadores deben conmutar (ec. 3).

Tabla de Contenido

Sistema de ecuaciones L1

1. $$(2a\partial_x A)\partial_x^2\Psi = 0 \tag{5}$$
2. $$(2a\partial_x B)\partial_{x,t}^2\Psi = 0 \tag{6}$$
3. $$(a\partial_x^2 A - i\partial_t A + 2a\partial_x C)\partial_x\Psi = 0 \tag{7}$$
4. $$(a\partial_x^2 B - i\partial_t B)\partial_t\Psi = 0 \tag{8}$$
5. $$(a\partial_x^2 C - i\partial_t C)\Psi = 0 \tag{9}$$

Operadores base de L1

La clasificación por constantes luego de la conmutación nos da como resultado cuatro operadores independientes definidos como:

- $$\hat{D}_1 = (-2ai)t\partial_x + x \tag{10}$$
- $$\hat{D}_2 = \partial_x \tag{11}$$
- $$\hat{D}_3 = \partial_t \tag{12}$$
- $$\hat{D}_4 = \hat{I} \tag{13}$$

El operador diferencial de primer orden puede ser escrito (en forma base de operadores independientes) como:

$$\hat{L} = d_1\hat{D}_1 + d_2\hat{D}_2 + d_3\hat{D}_3 + d_4\hat{D}_4 \tag{14}$$

Las constantes $(d_i)_{i=1}^{4}$ son numeros complejos. En mayoría números imaginarios puros o números reales. Esto último para asegurar hermiticidad de los operadores por las derivadas parciales.

Todos los operadores base $\hat{D}_i$ cumplen la conmutacion con $\hat{H}$:

$$\left[\hat{H}, \hat{D}_i\right] = 0 \tag{15}$$

Sin embargo las conmutaciones internas, $\left[\hat{D}_i, \hat{D}_j\right]$:

$$\left[\hat{D}_2, \hat{D}_1\right] = \hat{D}_4 \wedge \left[\hat{D}_3, \hat{D}_1\right] = \hat{D}_2 \wedge \left[\hat{D}_3, \hat{D}_2\right] = 0 \tag{16}$$

$$\left[\hat{D}_i, \hat{D}_4\right] = 0 \tag{17}$$

Notemos el algebra de Lie $\mathfrak{g}$ de **G**, característica de nuestra elección de $\hat{H}$, tal que el conjunto $\left\{\hat{D}_i\right\}_{i=1}^{4}$ forma un grupo de generadores de simetria, el cual $e^{\alpha D_i} \in$ **G** para $\alpha \in \mathbb{R}$.

Operador de segundo orden

Definimos el operador general de 2-orden:

$$\hat{L_2} = A(x,t)\partial_x^2 + B(x,t)\partial_t^2 + C(x,t)\partial_{x,t}^2 + D(x,t)\partial_x + E(x,t)\partial_t + F(x,t) \tag{18}$$

Conmutamos otra vez con $\hat{H} = a\partial_x^2 - i\partial_t$:

$$\left[\hat{H}, \hat{L_2}\right]\Psi = 0 \tag{19}$$

Esta vez la complejidad de resolución aumenta ya que resultan en nueve ecuaciones diferenciales parciales.

Sistema de ecuaciones de L2

1. $$(2a\partial_x A)\partial_x^3\Psi = 0 \tag{20}$$

2. $$(2a\partial_x B)\partial_{x,x,t}^3\Psi = 0 \tag{21}$$

3. $$(2a\partial_x C)\partial_{x,t,t}^3\Psi = 0 \tag{22}$$

4. $$(a\partial_x^2 A - i\partial_t A + 2a\partial_x D)\partial_x^2\Psi = 0 \tag{23}$$

5. $$(a\partial_x^2 B - i\partial_t B + 2a\partial_x E)\partial_{x,t}^2\Psi = 0 \tag{24}$$

6. $$(a\partial_x^2 C - i\partial_t C)\partial_t^2\Psi = 0 \tag{25}$$

7. $$(a\partial_x^2 D - i\partial_t D + 2a\partial_x F)\partial_x\Psi = 0 \tag{26}$$

8. $$(a\partial_x^2 E - i\partial_t E)\partial_t\Psi = 0 \tag{27}$$

9. $$(a\partial_x^2 F - i\partial_t F)\Psi = 0 \tag{28}$$

Operadores de L2

- $$\hat{D}_{2,1} = \partial_t^2 \qquad (29)$$

- $$\hat{D}_{2,2} = -ait^2\partial_t^2 + tx\partial_x + \frac{i}{2a}(\frac{x^2}{2} - \frac{i}{2at}) \qquad (30)$$

- $$\hat{D}_{2,3} = -2ait\partial_x^2 + x\partial_x - \frac{1}{4a^2} \qquad (31)$$

- $$\hat{D}_{2,4} = -2ait\partial_x + x \qquad (32)$$

- $$\hat{D}_{2,5} = \partial_x \qquad (33)$$

- $$\hat{D}_{2,6} = \hat{I} \quad (34)$$

- $$\hat{D}_{2,7} = -2ait\partial^2_{x,t} + x\partial_t \quad (35)$$

- $$\hat{D}_{2,8} = \partial_t \quad (36)$$

- $$\hat{D}_{2,9} = -2ai\partial^2_{x,t} \quad (37)$$

- $$\hat{D}_{2,10} = -2ai\partial^2_{x} \quad (38)$$

De la misma forma que el caso de $\hat{L}$, todos los operadores de $\hat{L}_2$ conmutan con $\hat{H}$.

$$[\hat{H}, \hat{D}_{2,i}] = 0$$

(39)

Operador de tercer orden

Definimos el operador general de 3-orden:

$$\hat{L}_3 = A(x,t)\partial_x^3 + B(x,t)\partial_t^3 + C(x,t)\partial_{x,x,t}^3 + D(x,t)\partial_{x,t,t}^3 + \\ E(x,t)\partial_x^2 + F(x,t)\partial_t^2 + G(x,t)\partial_{x,t}^2 + H(x,t)\partial_x + I(x,t)\partial_t + J(x,t) \tag{40}$$

Conmutamos otra vez con $\hat{H} = a\partial_x^2 - i\partial_t$:

$$\left[\hat{H}, \hat{L}_3\right]\Psi = 0 \tag{41}$$

La complejidad de la resolución de ec. 41 aumento de manera que ahora resultan en 14 ecuaciones diferenciales parciales, con 3 sistemas de ecuaciones de la misma forma.

Sistema de ecuaciones de L3

Simplificando las ecuaciones en 3 sistemas de ecuaciones diferenciales parciales.

- $$(2a\partial_x E - i\partial_t A)\partial_x^3\Psi = 0 \tag{42}$$
  $$(a\partial_x^2 E - i\partial_t E + 2a\partial_x H)\partial_x^2\Psi = 0 \tag{43}$$
  $$(a\partial_x^2 H - i\partial_t H + 2a\partial_x J)\partial_x\Psi = 0 \tag{44}$$
  $$(2a\partial_x^2 J - i\partial_t J)\Psi = 0 \tag{45}$$

- $$(2a\partial_x G - i\partial_t C)\partial_{x,x,t}^3\Psi = 0 \tag{46}$$
  $$(a\partial_x^2 G - i\partial_t G + 2a\partial_x I)\partial_{x,t}^2\Psi = 0 \tag{47}$$
  $$(a\partial_x^2 I - i\partial_t I)\partial_t\Psi = 0 \tag{48}$$

- $$(2a\partial_x F - i\partial_t D)\partial_{x,t,t}^3\Psi = 0 \tag{49}$$
  $$(a\partial_x^2 - i\partial_t F)\partial_t^2\Psi = 0 \tag{50}$$

- $$\hat{D}_{3,1} = ai\frac{t^3}{3}\partial_x^3 + \frac{t^2x}{2}\partial_x^2 + (\frac{-itx^2}{4a} + \frac{t^2}{2})\partial_x + \frac{i}{4a^2}(\frac{ix^3}{6} - atx) \quad (51)$$

- $$\hat{D}_{3,2} = ait^2\partial_x^3 + tx\partial_x^2 + (\frac{-ix^2}{4a} + t)\partial_x - \frac{ix}{4a} \quad (52)$$

- $$\hat{D}_{3,3} = 2ait\partial_x^3 + x\partial_x^2 - \frac{i}{2a}\partial_x \quad (53)$$

- $$\hat{D}_{3,4} = \frac{t^2}{2}\partial_x^2 - \frac{itx}{2a}\partial_x + \frac{i}{4a^2}(\frac{ix^2}{2} - at) \quad (54)$$

- $$\hat{D}_{3,5} = t\partial_x^2 - \frac{ix}{2a}\partial_x - \frac{i}{4a} \quad (55)$$

- $$\hat{D}_{3,6} = \partial_x^2 \qquad (56)$$

- $$\hat{D}_{3,7} = 2ait\partial_x^3 \qquad (57)$$

- $$\hat{D}_{3,8} = \frac{1}{2ai}\partial_x \qquad (58)$$

- $$\hat{D}_{3,9} = \hat{I} \qquad (59)$$

- $$\hat{D}_{3,10} = ai^2\partial^3_{x,x,t} + tx\partial^2_{x,t} + (\frac{-ix^2}{4a} + \frac{t}{2})\partial_t \qquad (60)$$

- $$\hat{D}_{3,11} = 2ait\partial^3_{x,x,t} + x\partial^2_{x,t} + \frac{1}{2}\partial_t \tag{61}$$

- $$\hat{D}_{3,12} = t\partial^2_{x,t} - \frac{ix}{2a}\partial_t \tag{62}$$

- $$\hat{D}_{3,13} = i\partial^2_{x,t} \tag{63}$$

- $$\hat{D}_{3,14} = 2a\partial^3_{x,x,t} \tag{64}$$

- $$\hat{D}_{3,15} = i\partial_t \tag{65}$$

- $$\hat{D}_{3,16} = i\partial_t^3 \tag{66}$$

- $$\hat{D}_{3,17} = 2ait\partial_{x,t,t}^3 - x\partial_t^2 \tag{67}$$

- $$\hat{D}_{3,18} = i\partial_t^2 \tag{68}$$

- $$\hat{D}_{3,19} = 2ai\partial_{x,t,t}^3 \tag{69}$$

- $$\hat{D}_{3,20} = -2ait\partial_x + x \tag{70}$$

De la misma forma que el caso de $\hat{L}$ y $\hat{L}_3$, todos los operadores de $\hat{L}_2$ conmutan con $\hat{H}$.

$$[\hat{H}, \hat{D}_{3,i}] = 0 \tag{71}$$

Tabla de Contenido

De la misma forma operando sucesivamente a $\hat{L}$ (tomando en cuenta que $\hat{D}_4 = \hat{I}$),
$\hat{L}\hat{L}' = (d_1\hat{D}_1 + d_2\hat{D}_2 + d_3\hat{D}_3 + d_4\hat{D}_4) * (d_1'\hat{D}_1 + d_2'\hat{D}_2 + d_3'\hat{D}_3 + d_4'\hat{D}_4)$ nos da como resultado:

$$\begin{aligned}\hat{L}^2 = 2(&d_1d_1'\hat{D}_1^2 + d_2d_2'\hat{D}_2^2 + d_3d_3'\hat{D}_3^2 + 2d_1d_4'\hat{D}_1 + \\ &2d_2d_4'\hat{D}_2 + 2d_3d_4'\hat{D}_3 + 2d_4d_4'\hat{D}_4 + d_2d_3'(\hat{D}_2\hat{D}_3 + \hat{D}_3\hat{D}_2) \\ &+ d_1d_2'(\hat{D}_1\hat{D}_2 + \hat{D}_2\hat{D}_1) + d_1d_3'(\hat{D}_1\hat{D}_3 + \hat{D}_3\hat{D}_1))\end{aligned} \tag{72}$$

Entonces $\hat{L}^2 = \hat{L}_2$ si y solo si las constantes cumplen:

- $$d_{2,l} = 2d_id_j' \tag{73}$$
- $$d_id_j' = d_{i,j} \tag{74}$$
- $$d_{i,j} = d_{j,i} \tag{75}$$

Con $k \in \{1 \sim 10\}$ y $i \wedge j \in \{1,2,3,4\}$

Operadores en forma base de L3

Operando sucesivamente a $\hat{L}$ (tomando en cuenta que $\hat{D}_4 = \hat{I}$) $\hat{L}\hat{L}'\hat{L}'' = (d_1\hat{D}_1 + d_2\hat{D}_2 + d_3\hat{D}_3 + d_4\hat{D}_4) * (d_1'\hat{D}_1 + d_2'\hat{D}_2 + d_3'\hat{D}_3 + d_4'\hat{D}_4) * (d_1''\hat{D}_1 + d_2''\hat{D}_2 + d_3''\hat{D}_3 + d_4''\hat{D}_4)$ nos da como resultado:

$$
\begin{gathered}
\hat{L}^3 = 3(d_1 d_1' d_1'' \hat{D}_1^3 + d_2 d_2' d_2'' \hat{D}_2^3 + d_3 d_3' d_3'' \hat{D}_3^3 + d_1 d_1' d_2'' (\hat{D}_1^2 \hat{D}_2 + \hat{D}_1 \hat{D}_2 \hat{D}_1 + \hat{D}_2 \hat{D}_1^2 \\
+ d_1 d_2' d_2'' (\hat{D}_1 \hat{D}_2^2 + \hat{D}_2 \hat{D}_1 \hat{D}_2 + \hat{D}_2^2 \hat{D}_1) + d_1 d_1' d_3'' (\hat{D}_1^2 \hat{D}_3 + \hat{D}_1 \hat{D}_3 \hat{D}_1 + \hat{D}_3 \hat{D}_1^2) \\
+ d_1 d_3' d_3'' (\hat{D}_3^2 \hat{D}_1 + \hat{D}_3 \hat{D}_1 \hat{D}_3 + \hat{D}_1 \hat{D}_3^2) + d_2 d_2' d_3'' (\hat{D}_2^2 \hat{D}_3 + \hat{D}_2 \hat{D}_3 \hat{D}_2 + \hat{D}_3 \hat{D}_2^2) \\
+ d_2 d_3' d_3'' (\hat{D}_3^2 \hat{D}_2 + \hat{D}_3 \hat{D}_2 \hat{D}_3 + \hat{D}_2 \hat{D}_3^2) \\
+ d_1 d_2' d_3'' (\hat{D}_1 \hat{D}_2 \hat{D}_3 + \hat{D}_3 \hat{D}_1 \hat{D}_2 + \hat{D}_2 \hat{D}_3 \hat{D}_1 + \hat{D}_2 \hat{D}_1 \hat{D}_3)) + \\
2(d_1 d_1' d_4'' \hat{D}_1^2 + d_2 d_2' d_4'' \hat{D}_2^2 + d_3 d_3' d_4'' \hat{D}_3^2 + 2 d_1 d_4' \hat{D}_1 + \\
d_2 d_4' d_4'' \hat{D}_2 + d_3 d_4' d_4'' \hat{D}_3 + d_4 d_4' d_4'' \hat{D}_4 + \\
d_2 d_3' d_4'' (\hat{D}_2 \hat{D}_3 + \hat{D}_3 \hat{D}_2) + d_1 d_2' d_4'' (\hat{D}_1 \hat{D}_2 + \hat{D}_2 \hat{D}_1) + d_1 d_3' d_4'' (\hat{D}_1 \hat{D}_3 + \hat{D}_3 \hat{D}_1))
\end{gathered}
\tag{76}
$$

Entonces $\hat{L}^3 = \hat{L}_3$ si y solo si las constantes cumplen:

- $$d_{3,l} = 3 d_i d'_j d''_k \tag{77}$$
- $$d_i d'_j d''_k = d_{i,j,k} \tag{78}$$
- $$d_{i,j,k} = d_{k,i,j} = d_{j,k,i} \tag{79}$$

Con $l \in \{1 \sim 20\}$ y $i, j \wedge k \in \{1, 2, 3, 4\}$

Operador de segundo orden:

$$\hat{L}_2 = d'_4\hat{L} + d_1d'_1\hat{D}_1^2 + d_2d'_2\hat{D}_2^2 + d_3d'_3\hat{D}_3^2 + d_2d'_3(\hat{D}_2\hat{D}_3) + d_1d'_2(\hat{D}_1\hat{D}_2) + d_1d'_3(\hat{D}_1\hat{D}_3) \quad (80)$$

Operador de tercer orden:

$$\begin{aligned}\hat{L}_3 = d''_4\hat{L}_2 + d_1d'_1d''_1\hat{D}_1^3 + d_2d'_2d''_2\hat{D}_2^3 + d_3d'_3d''_3\hat{D}_3^3 + d_1d'_1d''_2(\hat{D}_1^2\hat{D}_2 + \hat{D}_2\hat{D}_1^2)\\ + d_1d'_2d''_2(\hat{D}_1\hat{D}_2^2 + \hat{D}_2^2\hat{D}_1) + d_1d'_1d''_3(\hat{D}_1^2\hat{D}_3 + \hat{D}_3\hat{D}_1^2)\\ + d_1d'_3d''_3(\hat{D}_3^2\hat{D}_1 + \hat{D}_1\hat{D}_3^2) + d_2d'_2d''_3(\hat{D}_2^2\hat{D}_3 + \hat{D}_3\hat{D}_2^2)\\ + d_2d'_3d''_3(\hat{D}_3^2\hat{D}_2 + \hat{D}_2\hat{D}_3^2) + d_1d'_2d''_3(\hat{D}_1\hat{D}_2\hat{D}_3 + \hat{D}_2\hat{D}_3\hat{D}_1) \quad (81)\end{aligned}$$

La cantidad de operadores esta dada por: $a_n = n(n+1)(n+2)/6$. La forma general de recurrencia (Sin simplificacion M-N):

- Operador 1-orden ($i \in \{1,2,3,4\}$):

$$\hat{O}_i = \hat{D}_i \tag{82}$$

- Operador 2-orden ($i \wedge j \in \{1,2,3\}$):

$$\hat{O}_{i,j} = (\hat{D}_i\hat{D}_j + \hat{D}_j\hat{D}_i)/2 \tag{83}$$

- Operador 3-orden ($i, j \wedge k \in \{1,2,3\}$):

$$\hat{O}_{i,j,k} = (\hat{D}_i\hat{D}_j\hat{D}_k + \hat{D}_k\hat{D}_i\hat{D}_j + \hat{D}_j\hat{D}_k\hat{D}_i)/3 \tag{84}$$

- Operador n-orden ($i_1, i_2, ...i_n \in \{1,2,3\}$):

$$\hat{O}_{i,i_2,..,i_n} = \frac{1}{n}(\hat{D}_{i_1}\hat{D}_{i_2}\hat{D}_{i_3}...\hat{D}_{i_n}+\hat{D}_{i_n}\hat{D}_{i_1}\hat{D}_{i_2}...\hat{D}_{i_{n-1}}+\hat{D}_{i_{n-1}}\hat{D}_{i_n}\hat{D}_{i_1}...\hat{D}_{i_{n-2}}+...) \tag{85}$$

Forma simplificada:

- Operador 1-orden ($i \in \{1,2,3,4\}$):

$$\hat{O}_i = \hat{D}_i \tag{86}$$

- Operador 2-orden ($i \wedge j \in \{1,2,3\}$):

$$\hat{O}_{i,j} = (\hat{D}_i \hat{D}_j) \tag{87}$$

- Operador 3-orden ($i, j \wedge k \in \{1,2,3\}$):

$$\hat{O}_{i,j} = (\hat{D}_i^2 \hat{D}_j + \hat{D}_j \hat{D}_i^2) \tag{88}$$

- Operador n-orden ($i_1, i_2, ...i_n \in \{1,2,3\}$) para $n > 3$:

$$\begin{aligned}\hat{O}_{i,i_2,..,i_{n-1}} = (&\hat{D}_{i_1}^{n-1} \hat{D}_{i_2} ... \hat{D}_{i_{n-1}} + \hat{D}_{i_{n-1}} \hat{D}_{i_1}^{n-2} ... \hat{D}_{i_{n-2}} \\ &+ \hat{D}_{i_{n-1}} \hat{D}_{i_n} \hat{D}_{i_1}^{n-3} ... \hat{D}_{i_{n-2}} + ...)\end{aligned} \tag{89}$$

Operador de cuarto orden general derivado

Operador general de 4-orden:

$$\hat{L}_4 = \sum_{i=1}^{4} d_i \hat{D}_i + \frac{1}{2}\sum_{i=1}^{3}\sum_{j=1}^{3} d_{i,j}(\hat{D}_i\hat{D}_j + \hat{D}_j\hat{D}_i)$$
$$+ \frac{1}{3}\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} d_{i,j,k}(\hat{D}_i\hat{D}_j\hat{D}_k + \hat{D}_k\hat{D}_i\hat{D}_j + \hat{D}_j\hat{D}_k\hat{D}_i)$$
$$+\frac{1}{4}\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3} d_{i,j,k,l}(\hat{D}_i\hat{D}_j\hat{D}_k\hat{D}_l+\hat{D}_l\hat{D}_i\hat{D}_j\hat{D}_k+\hat{D}_k\hat{D}_l\hat{D}_i\hat{D}_j+\hat{D}_j\hat{D}_k\hat{D}_l\hat{D}_i) \tag{90}$$

El cual cumple con la conmutacion $\left[\hat{L}^4, H\right] = 0$ y las relaciones de simetria de las constantes:

$$d_{i,j,k,l} = d_{l,i,j,k} = d_{k,l,i,j} = d_{j,k,l,i} \tag{91}$$

Simplificacion M-N del 4-orden:

$$\hat{L}_4 = \sum_{i=1}^{4} d_i \hat{D}_i + \sum_{i=1}^{3}\sum_{j=1}^{3} d^*_{i,j}(\hat{D}_i\hat{D}_j + \hat{D}_j\hat{D}_i) + \sum_{i=1}^{3}\sum_{k=1}^{3} d^*_{i,k}(\hat{D}_i^2\hat{D}_k + \hat{D}_k\hat{D}_i^2)$$
$$+ \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} d^*_{i,j,k}(\hat{D}_i^3\hat{D}_j + \hat{D}_k\hat{D}_i^2\hat{D}_j + \hat{D}_j\hat{D}_i^3) \quad (92)$$

Luego de la simplificacion correspondiente, podemos concluir que:

$$\hat{L}^4 = \hat{L}_4 \quad (93)$$

¡Gracias!