# TITLE

ENRICA MAZZON

## 1. Introduction

### 1.1. Notation.

**(1.1.1)** Let $R$ be a complete discrete valuation ring with maximal ideal $\mathfrak{m}$, residue field $k = R/\mathfrak{m}$ and quotient field $K$. We assume that the valuation $v_K$ is normalized. We define by $|\cdot|_K = \exp(-v_K(\cdot))$ the absolute value on $K$ corresponding to $v_K$; this turns $K$ into a non-archimedean complete valued field.

**(1.1.2)** We write $S = \operatorname{Spec} R$ and we denote by $s$ the closed point of $S$. Let $\mathscr{X}$ be an $R$-scheme of finite type. We will denote by $\mathscr{X}_k$ the special fiber of $\mathscr{X}$ and by $\mathscr{X}_K$ the generic fiber. Moreover, we will denote by $\widehat{\mathscr{X}}$ the $\mathfrak{m}$-adic completion of $\mathscr{X}$ and by $\widehat{\mathscr{X}_\eta}$ the generic fiber of $\widehat{\mathscr{X}}$ in the category of $K$-analytic spaces.

**(1.1.3)** Let $X$ be a proper $K$-scheme. A model for $X$ over $R$ is a flat separated $R$-scheme $\mathscr{X}$ of finite type endowed with an isomorphism of $K$-schemes $\mathscr{X}_K \to X$. If $X$ is smooth over $K$, we say that $\mathscr{X}$ is an snc model for $X$ if it is regular over $R$, and the special fiber $\mathscr{X}_k$ is a strict normal crossings divisor on $\mathscr{X}$. Such a model always exists, by Hironaka's resolution of singularities.

**(1.1.4)** All log schemes in this paper are fine and saturated (*fs*) log schemes and defined with respect to the Zariski topology. We denote a log scheme by $\mathscr{X}^+ = (\mathscr{X}, \mathcal{M}_{\mathscr{X}})$, where $\mathcal{M}_{\mathscr{X}}$ is the structural sheaf of monoids. We denote by

$$\mathcal{C}_{\mathscr{X}} = \mathcal{M}_{\mathscr{X}}/\mathcal{O}_{\mathscr{X}}^{\times}$$

the characteristic sheaf of $\mathscr{X}^+$. The sheaf $\mathcal{C}_{\mathscr{X}}$ is a Zariski sheaf on $\mathscr{X}^+$, supported on $\mathscr{X}_k$; if $\mathscr{X}^+$ is log-regular, then $\mathcal{C}_{\mathscr{X}}$ is a constructible sheaf. For every point $x$ of $\mathscr{X}_k$, we denote by $\mathcal{I}_{\mathscr{X},x}$ the ideal in $\mathcal{O}_{\mathscr{X},x}$ generated by

$$\mathcal{M}_{\mathscr{X},x} \setminus \mathcal{O}_{\mathscr{X},x}^{\times}.$$

We denote by $S^+$ the scheme $S$ endowed with the standard log structure (the divisorial log structure induced by $s$). If an $R$-scheme $\mathscr{X}$ is given, we will always denote by $\mathscr{X}^+$ the log scheme over $S^+$ that we obtain by endowing $\mathscr{X}$ with the divisorial log structure associated with $\mathscr{X}_k$.

If $\mathscr{X}^+$ is a log-regular log scheme over $S^+$, then the locus where the log structure is non-trivial is a divisor that we will denote by $D_{\mathscr{X}}$. Thus, the log structure on $\mathscr{X}^+$ is the divisorial log structure induced by $D_{\mathscr{X}}$, by [Kat94], Theorem 11.6.

**(1.1.5)** Let $(X, \Delta)$ be a pair where $X$ is a proper $K$-scheme, $\Delta$ is a divisor and $X^+ = (X, \Delta)$ is a log-regular log scheme over $K$. A log-regular log scheme over $S^+$ is a model for $(X, \Delta)$ over $S^+$ if $\mathscr{X}$ is a model of $X$ over $R$, the closure $\overline{\Delta}$ of $\Delta$ in $\mathscr{X}$ has non-empty intersection with $\mathscr{X}_k$, and $D_{\mathscr{X}} = \overline{\Delta} + \mathscr{X}_k$.

**(1.1.6)** We say that a log-regular log scheme $\mathscr{X}^+$ over $S^+$ is semistable if the divisor $D_{\mathscr{X}}$ is reduced. We say that a proper $K$-variety X has semistable reduction if it admits an $R$-model $\mathscr{X}$ such that $\mathscr{X}^+$ is log-smooth over $S^+$, with reduced special fiber; such a model is called a semistable model of $X$. This is a weaker notion than requiring the existence of an snc-model with reduced special fibre.

**(1.1.7)** We denote by $(\cdot)^{\mathrm{an}}$ the analytification functor from the category of $K$-schemes of finite type to Berkovich's category of $K$-analytic spaces. For every $K$-scheme of finite type $X$, as a set, $X^{\mathrm{an}}$ consists of the pairs $x = (\xi_x, |\cdot|_x)$ where $\xi_x$ is a point of $X$ and $|\cdot|_x$ is an absolute value on the residue field $\kappa(\xi_x)$ of $X$ at $\xi_x$ extending the absolute value $|\cdot|_K$ on $K$. We endow $X^{\mathrm{an}}$ with the Berkovich topology, i.e. the weakest one such that

(i) the forgetful map $\phi : X^{\mathrm{an}} \to X$, defined as $(\xi_x, |\cdot|_x) \mapsto \xi_x$, is continuous,
(ii) for any Zariski open subset $U$ of $X$ and any regular function $f$ on $U$ the map $|f| : \phi^{-1}(U) \to \mathbb{R}$ defined by $|f|(\xi_x, |\cdot|_x) = |f(\xi_x)|$ is continuous.

## 2. The Kato fan of a log-regular log scheme

### 2.1. Definition of Kato fans.

**(2.1.1)** According to [Kat94], Definition 9.1, a monoidal space $(T, \mathcal{M}_T)$ is a topological space $T$ endowed with a sharp sheaf of monoids $\mathcal{M}_T$, where *sharp* means that $\mathcal{M}_{T,t}^{\times} = \{1\}$ for every $t \in T$. We often simply denote the monoidal space by $T$.

A morphism of monoidal spaces is a pair $(f, \varphi) : (T, \mathcal{M}_T) \to (T', \mathcal{M}_{T'})$ such that $f : T \to T'$ is a continuous function of topological spaces and $\varphi : f^{-1}(\mathcal{M}_T) \to \mathcal{M}_{T'}$ is a sheaf homomorphism such that $\varphi_t^{-1}(\{1\}) = \{1\}$ for every $t \in T$.

**Example 2.1.2.** If $\mathscr{X}^+$ is a log scheme then the Zariski topological space $\mathscr{X}$ is equipped with a sheaf of sharp monoids $\mathcal{C}_{\mathscr{X}}$, the characteristic sheaf of $\mathscr{X}^+$. Thus $(\mathscr{X}, \mathcal{C}_{\mathscr{X}})$ is a monoidal space. Moreover, morphisms of log schemes induce morphisms of characteristic sheaves, hence morphism of monoidal spaces. We therefore obtain a functor from the category of log schemes to the category of monoidal spaces.

**Example 2.1.3.** Given a monoid $P$, we may associate to it a monoidal space called the spectrum of $P$. As a set, $\operatorname{Spec} P$ is the set of all prime ideals of $P$. The topology is characterized by the basis open sets $D(f) = \{\mathfrak{p} \in \operatorname{Spec} P | f \notin \mathfrak{p}\}$ for any $f \in P$. The monoidal sheaf is defined by

$$\mathcal{M}_{\operatorname{Spec} P}(D(f)) = S^{-1}P/(S^{-1}P)^{\times}$$

where $S = \{f^n | n \geqslant 0\}$.

**(2.1.4)** A monoidal space isomorphic to the monoidal space $\operatorname{Spec} P$ for some monoid $P$ is called an affine Kato fan. A monoidal space is called a Kato fan if it has an open covering consisting of affine Kato fans. In particular, we call a Kato fan integral, saturated, of finite type or $fs$ if it admits a cover by the spectra of monoids with the respective properties.

**(2.1.5)** A morphism of fs Kato fans $F' \to F$ is called a *subdivision* if it has finite fibres and the morphism

$$\operatorname{Hom}(\operatorname{Spec} \mathbb{N}, F') \to \operatorname{Hom}(\operatorname{Spec} \mathbb{N}, F)$$

is a bijection. Allowing subdivisions a Kato fan might take the following shape.

**Proposition 2.1.6.** *([Kat94], Proposition 9.8) Let $F$ be a $fs$ Kato fan. Then there is a subdivision $F' \to F$ such that $F'$ has an open cover $\{U_i'\}$ by Kato cones with $U_i' \simeq \operatorname{Spec} \mathbb{N}^{r_i}$.*

The strategy of the proof of Proposition 2.1.6 goes back to [KKMSD73] and relies on a sequence of particular subdivisions of the Kato fan, the so-called star and barycentric subdivisions ([ACMUW15], Example 4.10).

### 2.2. **Kato fans associated to log-regular log schemes.**

**Theorem 2.2.1.** *([Kat94], Proposition 10.2) Let $\mathscr{X}^+$ be a log-regular log scheme. Then there is an initial strict morphism $(\mathscr{X}, \mathcal{C}_{\mathscr{X}}) \to F$ to a Kato fan in the category of monoidal spaces. Explicitly, there exist a Kato fan $F$ and a morphism $\pi : (\mathscr{X}, \mathcal{C}_{\mathscr{X}}) \to F$ such that $\pi^{-1}(\mathcal{M}_F) \simeq \mathcal{C}_{\mathscr{X}}$ and any other morphism to a Kato fan factors through $\pi$.*

The Kato fan $F$ in Theorem 2.2.1 is called the Kato fan associated to $\mathscr{X}^+$; it is the topological subspace of $\mathscr{X}$ consisting of the points $x$ such that the maximal ideal $\mathfrak{m}_x$ of $\mathcal{O}_{\mathscr{X},x}$ is equal to $\mathcal{I}_{\mathscr{X},x}$, and $\mathcal{M}_F$ is the inverse image of $\mathcal{C}_{\mathscr{X}}$ on $F$, henceforth we write $\mathcal{C}_F$ for $\mathcal{M}_F$.

**Example 2.2.2.** Assume that $\mathscr{X}$ is regular, of finite type over $S$ and $\mathscr{X}_k$ is a divisor with strict normal crossings. Then $\mathscr{X}^+$ is log-regular and $F$ is the set of generic points of intersections of irreducible components of $\mathscr{X}_k$. For each point $x$ of $F$, the stalk of $\mathcal{C}_F$ is isomorphic to $(\mathbb{N}^r, +)$, with $r$ the number of irreducible components of $\mathscr{X}_k$ that pass through $x$.

This example admits the following partial generalisation.

**Lemma 2.2.3.** *Let $\mathscr{X}^+$ be a log-regular log scheme. Then the fan $F$ consists of the generic points of intersections of irreducible components of $D_{\mathscr{X}}$.*

*Proof.* First, we show that every such generic point is a point of $F$. Let $E_1, \dots, E_r$ be irreducible components of $D_{\mathscr{X}}$ and let $x$ be a generic point of the intersection $E_1 \cap \ldots \cap E_r$. We set $d = \dim \mathcal{O}_{\mathscr{X},x}$. Since $\mathscr{X}^+$ is log-regular, we know that $\mathcal{O}_{\mathscr{X},x}/\mathcal{I}_{\mathscr{X},x}$ is regular and that

$$d = \dim \mathcal{O}_{\mathscr{X},x}/\mathcal{I}_{\mathscr{X},x} + \operatorname{rank} \mathcal{C}^{\mathrm{gp}}_{\mathscr{X},x}. \tag{2.2.4}$$

We denote by $V(\mathcal{I}_{\mathscr{X},x})$ the vanishing locus of the ideal $\mathcal{I}_{\mathscr{X},x}$ in $\mathscr{X}$. We want to prove that $\mathcal{I}_{\mathscr{X},x} = \mathfrak{m}_x$. We assume the contrary, hence that $\mathcal{I}_{\mathscr{X},x} \subsetneq \mathfrak{m}_x$. This assumption implies that there exists $j$ such that $V(\mathcal{I}_{\mathscr{X},x}) \nsubseteq E_j$: indeed, if the vanishing locus is contained in each irreducible component $E_i$, i.e.

$$V(\mathcal{I}_{\mathscr{X},x}) \subseteq E_1 \cap \ldots \cap E_r \subseteq \overline{\{x\}},$$

then $\mathcal{I}_{\mathscr{X},x} \supseteq \mathfrak{m}_x$. From the assumption of log-regularity it follows that the vanishing locus $V(\mathcal{I}_{\mathscr{X},x})$ is a regular subscheme, and moreover that $\mathscr{X}^+$ is Cohen-Macaulay by [Kat94], Theorem 4.1. Thus, there exists a regular sequence $(f_1, \dots, f_l)$ in $\mathcal{I}_{\mathscr{X},x}$ where $l$ is the codimension of $V(\mathcal{I}_{\mathscr{X},x})$, i.e.

$$\dim \mathcal{O}_{\mathscr{X},x}/\mathcal{I}_{\mathscr{X},x} = d - l.$$

Moreover by the equality (2.2.4), $\operatorname{rank} \mathcal{C}^{\mathrm{gp}}_{\mathscr{X},x} = l$.

We claim that the residue classes of these elements $f_i$ in $\mathcal{C}^{\mathrm{gp}}_{\mathscr{X},x}$ are linearly independent. Assume the contrary. Then, up to renumbering the $f_i$, there exist an integer $e$ with $1 < e < l$, non-negative integers $a_1, \dots, a_l$, not all zero, and a unit $u$ in $\mathcal{O}_{\mathscr{X},x}$ such that

$$f_1^{a_1} \cdot \dots \cdot f_{e-1}^{a_{e-1}} = u \cdot f_e^{a_e} \cdot \dots \cdot f_l^{a_l}.$$

This contradicts the fact that $(f_1, \dots, f_l)$ is a regular sequence in $\mathcal{I}_{\mathscr{X},x}$. Thus, the classes $\overline{f_1}, \dots, \overline{f_l}$ are independent in $\mathcal{C}^{\mathrm{gp}}_{\mathscr{X},x}$. As we also have the equality $\operatorname{rank} \mathcal{C}^{\mathrm{gp}}_{\mathscr{X},x} = l$, it follows that these classes generate $\mathcal{C}^{\mathrm{gp}}_{\mathscr{X},x} \otimes_{\mathbb{Z}} \mathbb{Q}$.

Let $g_j$ be a non-zero element of the ideal $\mathcal{I}_{\mathscr{X},x}$ that vanishes along $E_j$: it necessarily exists as otherwise $E_j$ is not a component of the divisor $D_{\mathscr{X}}$. Then $g_j$ satisfies

$$g_j^N = v \cdot f_1^{b_1} \cdot \dots \cdot f_l^{b_l}$$

with $b_i \in \mathbb{Z}$, $v$ a unit in $\mathcal{O}_{\mathscr{X},x}$ and $N$ a positive integer. As $g_j$ vanishes along the irreducible component $E_j$, at least one of the functions $f_1, \dots, f_l$ has to vanish along $E_j$: assume that is $f_1$.

On the one hand, as $f_1$ is identically zero on $E_j$, the trace of $E_j$ on $V(\mathcal{I}_{\mathscr{X},x})$ has at most codimension $l-1$ in $E_j$ at the point $x$. On the other hand, we assumed that $V(\mathcal{I}_{\mathscr{X},x})$ is not contained in $E_j$ and it has codimension $l$ in $\mathcal{O}_{\mathscr{X},x}$. Then, the trace of $E_j$ on $V(\mathcal{I}_{\mathscr{X},x})$ has codimension $l$ in $E_j$ at $x$. This is a contradiction. We conclude that the ideal $\mathcal{I}_{\mathscr{X},x}$ is equal to the maximal ideal $\mathfrak{m}_x$, therefore $x$ is a point of $F$.

It remains to prove the converse implication: every point $x$ of the fan $F$ must be a generic point of an intersection of irreducible components of $D_{\mathscr{X}}$. Let $x$ be a point of $F$: by construction of Kato fan $F$, the maximal ideal of $\mathcal{O}_{\mathscr{X},x}$ is equal to $\mathcal{I}_{\mathscr{X},x}$, thus it is generated by elements in $\mathcal{M}_{\mathscr{X},x}$. The zero locus of such an element is contained in $D_{\mathscr{X}}$ by definition of the logarithmic structure on $\mathscr{X}^+$. Therefore, the zero locus is a union of irreducible components of the trace of $D_{\mathscr{X}}$ on $\operatorname{Spec} \mathcal{O}_{\mathscr{X},x}$ and $x$ is a generic point of the intersection of all such irreducible components. □

Moreover, the example 2.2.2 also leads to the following characterization.

**Proposition 2.2.5.** *([GR04], Corollary 12.5.35) Let $\mathscr{X}^+$ be a log-regular log scheme over $S^+$ and $F$ its associated Kato fan. The following are equivalent:*

*(1) for every $x \in F$, $M_{F,x} \simeq \mathbb{N}^{r(x)}$,*
*(2) the underlying scheme $\mathscr{X}$ is regular.*

*If this is the case, then the special fibre $\mathscr{X}_k$ is a strict normal crossing divisor.*

**(2.2.6)** The construction of the Kato fan of a log scheme defines a functor from the category of log-regular log schemes to the category of Kato fans. Indeed, given a morphism of log schemes $\mathscr{X}^+ \to \mathscr{Y}^+$, we consider the embedding of the associated Kato fan $F_{\mathscr{X}}$ in $\mathscr{X}^+$ and the canonical morphism $\mathscr{Y}^+ \to F_{\mathscr{Y}}$: the composition

$$F_{\mathscr{X}} \hookrightarrow \mathscr{X}^+ \to \mathscr{Y}^+ \to F_{\mathscr{Y}}$$

functorially induces a map between associated Kato fans. Moreover, this association preserves strict morphisms ([Uli13], Lemma 4.9).

2.3. **Resolutions of log schemes via Kato fan subdivisions.**

**Proposition 2.3.1.** *([Kat94], Proposition 9.9) Let $\mathscr{X}^+$ be a log-regular log scheme and let $F$ be its associated Kato fan. Let $F' \to F$ be a subdivision of fans. Then there exist a log scheme $\mathscr{X}'^+$, a morphism of log schemes $\mathscr{X}'^+ \to \mathscr{X}^+$ and a commutative diagram*

$$\begin{tikzcd} (\mathscr{X}', \mathcal{C}_{\mathscr{X}'}) \arrow[r, "p"] \arrow[d] & F' \arrow[d] \\ (\mathscr{X}, \mathcal{C}_{\mathscr{X}}) \arrow[r, "\pi_{\mathscr{X}}"] & F \end{tikzcd}$$

*such that $p^{-1}(\mathcal{M}_{F'}) \simeq \mathcal{C}_{\mathscr{X}'}$, they define a final object in the category of such diagrams and the refinement $F' \to F$ is induced by the morphism of log-regular log schemes $\mathscr{X}'^+ \to \mathscr{X}^+$.*

**(2.3.2)** It follows that given any subdivision $F' \to F$ of the Kato fan F associated with a log regular log scheme $\mathscr{X}^+$, we can construct a log scheme over $\mathscr{X}^+$ with prescribed associated Kato fan $F'$. Combining this fact with Proposition 2.1.6 and Proposition 2.2.5 yields to the construction of resolutions of log schemes in the following sense: for any log-regular log scheme over $S^+$ we can find a birational modification by a regular log scheme with strict normal special fibre. Moreover, the morphism of log schemes $\mathscr{X}'^+ \to \mathscr{X}^+$ is obtained by a log blow-up ([Niz06], Theorem 5.8).

2.4. **Fibred products and associated Kato fans.**

**(2.4.1)** Given morphisms of *fs* log schemes $f_1 : \mathscr{X}_1^+ \to \mathscr{Y}^+$ and $f_2 : \mathscr{X}_2^+ \to \mathscr{Y}^+$, their fibred product exists in the category of log schemes. It is obtained by endowing the usual fibred product of schemes

$$\begin{tikzcd} \mathscr{X}_1 \times_{\mathscr{Y}} \mathscr{X}_2 \arrow[r, "p_1"] \arrow[d, "p_2"] \arrow[dr, "p_{\mathscr{Y}}"] & \mathscr{X}_1 \arrow[d, "f_1"] \\ \mathscr{X}_2 \arrow[r, "f_2"] & \mathscr{Y} \end{tikzcd} \tag{2.4.2}$$

with the log structure associated to $p_1^{-1}\mathcal{M}_{\mathscr{X}_1} \oplus_{p_{\mathscr{Y}}^{-1}\mathcal{M}_{\mathscr{Y}}} p_2^{-1}\mathcal{M}_{\mathscr{X}_2}$. If $u_1 : P \to Q_1$ and $u_2 : P \to Q_2$ are charts for the morphisms $f_1$ and $f_2$ respectively, then the induced morphism $\mathscr{X}_1 \times_{\mathscr{Y}} \mathscr{X}_2 \to \operatorname{Spec} \mathbb{Z}[Q_1 \oplus_P Q_2]$ is a chart for $\mathscr{X}_1^+ \times_{\mathscr{Y}^+} \mathscr{X}_2^+$.

**(2.4.3)** In general, the fibred product is not $fs$, but the category of $fs$ log schemes also admits fibred products. Keeping the same notations, the following is a chart of the fibred product in the category of fine and saturated log schemes

$$\mathscr{X}_1^+ \times_{\mathscr{Y}^+}^{\mathrm{fs}} \mathscr{X}_2^+ = (\mathscr{X}_1^+ \times_{\mathscr{Y}^+} \mathscr{X}_2^+) \times_{\mathbb{Z}[Q_1 \oplus_P Q_2]} \mathbb{Z}[(Q_1 \oplus_P Q_2)^{\mathrm{sat}}]$$

([Bul15], 3.6.16). We remark that the two fibre products above may not only have different log structures, but also the underlying schemes may differ.

**(2.4.4)** Log smoothness is preserved under fs base change and composition ([GR04], Proposition 12.3.24). In particular, if $f_1 : \mathscr{X}_1^+ \to \mathscr{Y}^+$ is log-smooth and $\mathscr{X}_2^+$ is log-regular, then $\mathscr{X}_1^+ \times_{\mathscr{Y}^+}^{\mathrm{fs}} \mathscr{X}_2^+$ is log-regular, by [Kat94], Theorem 8.2.

Consider log-smooth morphisms of fs log schemes $\mathscr{X}_1^+ \to \mathscr{Y}^+$ and $\mathscr{X}_2^+ \to \mathscr{Y}^+$. The sheaves of logarithmic differentials are related by the following isomorphism

$$p_1^*\Omega^{\log}_{\mathscr{X}_1^+/\mathscr{Y}^+} \oplus p_2^*\Omega^{\log}_{\mathscr{X}_2^+/\mathscr{Y}^+} \simeq \Omega^{\log}_{\mathscr{X}_1^+ \times^{\mathrm{fs}}_{\mathscr{Y}^+} \mathscr{X}_2^+/\mathscr{Y}^+} \tag{2.4.5}$$

by [GR04], Proposition 12.3.13. Furthermore, by assumption of log-smoothness over $S^+$ the logarithmic differential sheaves are locally free of finite rank ([Kat94], Proposition 3.10) and we can consider their determinants; they are called log canonical bundles and denoted by $\omega^{\log}$. The following isomorphism is a direct consequence of (2.4.5)

$$p_1^*\omega^{\log}_{\mathscr{X}_1^+/\mathscr{Y}^+} \otimes p_2^*\omega^{\log}_{\mathscr{X}_2^+/\mathscr{Y}^+} \simeq \omega^{\log}_{\mathscr{X}_1^+ \times^{\mathrm{fs}}_{\mathscr{Y}^+} \mathscr{X}_2^+/\mathscr{Y}^+}. \tag{2.4.6}$$

**(2.4.7)** Similarly to the construction of fibred products of $fs$ log schemes, the category of $fs$ Kato fans admits fibred products: on affine Kato fans $F = \operatorname{Spec} P$ and $G = \operatorname{Spec} Q$ over $H = \operatorname{Spec} T$, $F \times_H G$ is the spectrum of the amalgamated sum $(P \oplus_T Q)^{\mathrm{sat}}$ in the category of $fs$ monoids ([Uli16], Proposition 2.4) and on the underlying topological spaces, this coincides with the usual fibred product.

We seek to compare the Kato fan associated to the fibred product of log-regular log schemes with the fibred product of associated Kato fans.

**Proposition 2.4.8.** *([Sai04], Lemma 2.8) Given $\mathscr{T}^+$ a log-regular log scheme, let $\mathscr{X}^+$ and $\mathscr{Y}^+$ be log-smooth log schemes over $\mathscr{T}^+$. We denote by $\mathscr{Z}^+$ the $fs$ fibred product $\mathscr{X}^+ \times^{fs}_{\mathscr{T}^+} \mathscr{Y}^+$. Then, the natural morphisms $F_{\mathscr{Z}} \to F_{\mathscr{X}}$ and $F_{\mathscr{Z}} \to F_{\mathscr{Y}}$ induce a morphism of Kato fans*

$$F_{\mathscr{Z}} \to F_{\mathscr{X}} \times_{F_{\mathscr{T}}} F_{\mathscr{Y}}$$

*that is locally an isomorphism.*

## 2.5. Semistability and Kato fans associated to the fibred products.

**(2.5.1)** We investigate a sufficient condition to turn the local isomorphism of Proposition 2.4.8 into an isomorphism: it concerns the notion of semistability. We recall that a log-regular log scheme $\mathscr{X}^+$ is said to be semistable if the divisor $D_{\mathscr{X}}$, where the log structure is non-trivial, is reduced.

**(2.5.2)** In order to see the relevance of the assumption of semistability, we need some results on saturated morphism of log schemes. We recall that, locally around a point $x$ of the divisor $D_{\mathscr{X}}$, the morphism of characteristic monoids $\mathbb{N} \to \mathcal{C}_{\mathscr{X},x}$ is a saturated morphism of monoids if, for any morphism $u : \mathbb{N} \to P$ of $fs$ monoids, the amalgamated sum $\mathcal{C}_{\mathscr{X},x} \oplus_{\mathbb{N}} P$ is still saturated.

Following the work by T. Tsuji in an unpublished 1997 preprint, Vidal in [Vid04] defines the saturation index of a morphism of *fs* monoids. In the case of log-regular log scheme over $S^+$ it can be easily computed: it is the least common multiple of the multiplicities of the prime components of the divisor $D_{\mathscr{X}}$. The following criterion holds.

**Lemma 2.5.3.** *([Vid04], Section 1.3) A morphism of fs monoids is saturated if and only if the saturation index is equal to 1.*

**Proposition 2.5.4.** *Assume that the residue field $k$ is algebraically closed. Let $\mathscr{X}^+$ and $\mathscr{Y}^+$ be log-smooth log scheme over $S^+$. Let $\mathscr{Z}^+$ be their $fs$ fibred product. If $\mathscr{X}^+$ is semistable, then the morphism*

$$F_{\mathscr{Z}} \to F_{\mathscr{X}} \times F_{\mathscr{Y}},$$

*induced by the projections $\mathscr{Z}^+ \to \mathscr{X}^+$ and $\mathscr{Z}^+ \to \mathscr{Y}^+$, is an isomorphism of Kato fans.*

*Proof.* By hypothesis $\mathscr{X}^+$ is a semistable log-regular log scheme over $S^+$, hence the saturation index of $\mathscr{X}^+ \to S^+$ is 1. Thus, by Lemma 2.5.3 the morphism of log schemes $\mathscr{X}^+ \to S^+$ induces a saturated morphism of characteristic monoids at every point of $\mathscr{X}^+$. The saturation condition implies that the fibred product in the category of log schemes coincides with the fibred product in the category of $fs$ log schemes. In particular, the underlying scheme of $\mathscr{Z}^+$ coincides with the usual schematic fibred product, hence its points are characterized as follows:

$$z = (x, y, s, \mathfrak{p}) \text{ and } \mathcal{O}_{\mathscr{Z},z} = (\mathcal{O}_{\mathscr{X},x} \otimes_R \mathcal{O}_{\mathscr{Y},y})_{\mathfrak{p}}$$

where $x$ and $y$ are points of $\mathscr{X}^+$ and $\mathscr{Y}^+$ both mapped to the same point $s$ of $S$, while $\mathfrak{p}$ is a prime ideal of the tensor product of residue fields $\kappa(x) \otimes_{\kappa(s)} \kappa(y)$. We look for a characterization of points $z$ in $\mathscr{Z}^+$ that lie in the Kato fan $F_{\mathscr{Z}}$.

If the point $z$ lies in $F_{\mathscr{Z}}$, then the maximal ideal $\mathfrak{m}_z$ is equal to the ideal $\mathcal{I}_{\mathscr{Z},z}$ by definition. By the flatness of the models $\mathscr{X}^+$ and $\mathscr{Y}^+$ over $S^+$, the morphisms of local rings $\mathcal{O}_{\mathscr{X},x} \to \mathcal{O}_{\mathscr{Z},z}$ and $\mathcal{O}_{\mathscr{Y},y} \to \mathcal{O}_{\mathscr{Z},z}$ are injective. Hence, the equalities $\mathfrak{m}_x = \mathcal{I}_{\mathscr{X},x}$ and $\mathfrak{m}_y = \mathcal{I}_{\mathscr{Y},y}$ hold. Thus, the points $z$ in $\mathscr{Z}^+$ that lie in the Kato fan $F_{\mathscr{Z}}$ are necessarily points such that the projections $x$ and $y$ to $\mathscr{X}^+$ and $\mathscr{Y}^+$ lie in their associated Kato fans. Therefore, we may assume $x \in F_{\mathscr{X}}$, $y \in F_{\mathscr{Y}}$, and it remains to characterize the prime ideals $\mathfrak{p}$ such that $z = (x, y, s, \mathfrak{p}) \in F_{\mathscr{Z}}$.

By log-regularity of $\mathscr{Z}^+$, the point $z$ lies in the associated Kato fan if and only if $\dim \mathcal{O}_{\mathscr{Z},z} = \mathrm{rank}\mathcal{C}^{\mathrm{gp}}_{\mathscr{Z},z}$. At the level of characteristic sheaves it holds that

$$\mathrm{rank}\mathcal{C}^{\mathrm{gp}}_{\mathscr{Z},z} = \mathrm{rank}\mathcal{C}^{\mathrm{gp}}_{\mathscr{X},x} + \mathrm{rank}\mathcal{C}^{\mathrm{gp}}_{\mathscr{Y},y} - 1.$$

Since $x$ and $y$ are both assumed to be points in the associated Kato fans, the equality between dimension of local rings and rank of the groupifications of characteristic sheaves lead to the equivalence

$$\begin{aligned} z \in F_{\mathscr{Z}} \Leftrightarrow \dim \mathcal{O}_{\mathscr{Z},z} &= \mathrm{rank}\mathcal{C}^{\mathrm{gp}}_{\mathscr{Z},z} \\ &= \mathrm{rank}\mathcal{C}^{\mathrm{gp}}_{\mathscr{X},x} + \mathrm{rank}\mathcal{C}^{\mathrm{gp}}_{\mathscr{Y},y} - 1 \\ &= \dim \mathcal{O}_{\mathscr{X},x} + \dim \mathcal{O}_{\mathscr{Y},y} - 1. \end{aligned}$$

By log-regularity of $\mathscr{Z}^+$, it holds that $\dim \mathcal{O}_{\mathscr{Z},z} \geqslant \mathrm{rank}\mathcal{C}^{\mathrm{gp}}_{\mathscr{Z},z}$, thus the inequality

$$\dim \mathcal{O}_{\mathscr{Z},z} \geqslant \dim \mathcal{O}_{\mathscr{X},x} + \dim \mathcal{O}_{\mathscr{Y},y} - 1$$

is always true and equality holds only for minimal prime ideals $\mathfrak{p}$ of $\kappa(x) \otimes_{\kappa(s)} \kappa(y)$. Therefore, in order to conclude that there exists a unique point $z$ whose projections are the points $x$ and $y$ and that lies in the Kato fan $F_{\mathscr{Z}}$, we need to prove the following property: the tensor product $\kappa(x) \otimes_{\kappa(s)} \kappa(y)$ has a unique minimal prime ideal.

If $s$ is the closed point of $S$, then its residue field is the algebraically closed field $k$. It follows that the tensor product $\kappa(x) \otimes_k \kappa(y)$ is a domanin, hence it has a unique minimal prime ideal, namely 0.

Otherwise, we denote by $\mathscr{V}$ the closure of $x$ in $\mathscr{X}^+$: it is still a log-smooth scheme over $S^+$ with reduced special fibre $\mathscr{V}_k$. We denote by $L$ the separable closure of $\kappa(s)$ in $\kappa(x)$ and by $O_L$ its valuation ring. Let $\mathscr{V}_{O_L}$ be the base change of $\mathscr{V}$ to $\mathrm{Spec}\,(O_L)$: the generic fibre $\mathscr{V}_L$ is normal, as normality is preserved under separable field extension, and the special fibre is still reduced. By [Liu02], Lemma

4.1.18, it follows that $\mathscr{V}_{O_L}$ is normal. Since $\mathscr{V}_{O_L}$ is normal and proper with reduced special fibre, $\mathscr{V}_L$ is connected. We deduce that $\kappa(s)$ is separably algebraically closed in $\kappa(x)$. Finally by , Proposition 4.3.2, the tensor product $\kappa(x) \otimes_{\kappa(s)} \kappa(y)$ has a unique minimal prime ideal. $\square$

## 3. The skeleton of a log-regular log scheme

### 3.1. Construction of the skeleton of a log-regular log scheme.

**(3.1.1)** Let $\mathscr{X}^+$ be a log-regular log scheme over $S^+$. Let $x$ be a point of the associated Kato fan $F$. Denote by $F(x)$ the set of points $y$ of $F$ such that $x$ lies in the closure of $\{y\}$, and by $\mathcal{C}_{F(x)}$ the restriction of $\mathcal{C}_F$ to $F(x)$. Denote by $\operatorname{Spec}\mathcal{C}_{\mathscr{X},x}$ the spectrum of the monoid $\mathcal{C}_{\mathscr{X},x} = \mathcal{C}_{F,x}$. Then there exists a canonical isomorphism of monoid spaces

$$(F(x), \mathcal{C}_{F(x)}) \to \operatorname{Spec}\mathcal{C}_{\mathscr{X},x} : y \mapsto \{s \in \mathcal{C}_{\mathscr{X},x} \,|\, s(y) = 0\}$$

where the expression $s(y) = 0$ means that $s'(y) = 0$ for any representative $s'$ of $s$ in $\mathcal{M}_{\mathscr{X},x}$. In particular, we obtain a bijective correspondence between the faces of the monoid $\mathcal{C}_{\mathscr{X},x}$ and the points of $F(x)$, and for every point $y$ of $F(x)$, a surjective cospecialization morphism of monoids

$$\tau_{x,y} : \mathcal{C}_{\mathscr{X},x} \to \mathcal{C}_{\mathscr{X},y}$$

which induces an isomorphism of monoids

$$S^{-1}\mathcal{C}_{\mathscr{X},x}/(S^{-1}\mathcal{C}_{\mathscr{X},x})^{\times} \cong \mathcal{C}_{\mathscr{X},x}/S \xrightarrow{\sim} \mathcal{C}_{\mathscr{X},y}$$

where $S$ denotes the monoid of elements $s$ in $\mathcal{C}_{\mathscr{X},x}$ such that $s(y) \neq 0$.

**(3.1.2)** For each point $x$ in $F$, we denote by $\sigma_x$ the set of morphisms of monoids

$$\alpha : \mathcal{C}_{\mathscr{X},x} \to (\mathbb{R}_{\geq 0}, +)$$

such that $\alpha(\pi) = 1$ for every uniformizer $\pi$ in $R$. We endow $\sigma_x$ with the topology of pointwise covergence, where $\mathbb{R}_{\geq 0}$ carries the usual Euclidean topology. Note that $\sigma_x$ is a polygon in the real affine space

$$\{\alpha : \mathcal{C}^{\mathrm{gp}}_{\mathscr{X},x} \to (\mathbb{R}, +) \,|\, \alpha(\pi) = 1 \text{ for every uniformizer } \pi \text{ in } R\}.$$

If $y$ is a point of $F(x)$, then the surjective cospecialization morphism $\tau_{x,y}$ induces a topological embedding $\sigma_y \to \sigma_x$ that identifies $\sigma_y$ with a face of $\sigma_x$.

**(3.1.3)** We denote by $T$ the disjoint union of the topological spaces $\sigma_x$ with $x$ in $F$. On the topological space $T$, we consider the equivalence relation $\sim$ generated by couples of the form $(\alpha, \alpha \circ \tau_{x,y})$ where $x$ and $y$ are points in $F$ such that $x$ lies in the closure of $\{y\}$ and $\alpha$ is a point of $\sigma_y$.

The skeleton of $\mathscr{X}^+$ is defined as the quotient of the topological space $T$ by the equivalence relation $\sim$. We denote this skeleton by $\operatorname{Sk}(\mathscr{X}^+)$. It is clear that $\operatorname{Sk}(\mathscr{X}^+)$ has the structure of a polyhedral complex with closed cells $\{\sigma_x,\, x \in F\}$ and that the faces of a cell $\sigma_x$ are precisely the cells $\sigma_y$ with $y$ in $F(x)$.

## 3.2. **Embedding the skeleton in the non-archimedean generic fiber.**

**(3.2.1)** Let $\mathscr{X}^+$ be a log-regular log scheme over $S^+$. Let $x$ be a point of the associated Kato fan $F$. As the log structure on $\mathscr{X}^+$ is of finite type, the characteristic monoid $\mathcal{C}_{\mathscr{X},x}$ is of finite type too, and thus $\mathcal{C}^{\mathrm{gp}}_{\mathscr{X},x}$ is a free abelian group of finite rank. Hence there exists a section

$$\zeta : \mathcal{M}^{\mathrm{gp}}_{\mathscr{X},x}/\mathcal{M}^{\times}_{\mathscr{X},x} \to \mathcal{M}^{\mathrm{gp}}_{\mathscr{X},x}.$$

The section $\zeta$ restricts to $\mathcal{C}_{\mathscr{X},x} \to \mathcal{M}_{\mathscr{X},x}$; indeed, if $x \in \mathcal{M}_{\mathscr{X},x}$ then $\zeta(\overline{x}) - x \in \mathcal{M}^{\times}_{\mathscr{X},x}$. Therefore we may choose a section

$$\mathcal{C}_{\mathscr{X},x} \to \mathcal{M}_{\mathscr{X},x} \tag{3.2.2}$$

of the projection homomorphism

$$\mathcal{M}_{\mathscr{X},x} \to \mathcal{C}_{\mathscr{X},x}$$

and use this section to view $\mathcal{C}_{\mathscr{X},x}$ as a submonoid of $\mathcal{M}_{\mathscr{X},x}$. Note that $\mathcal{C}_{\mathscr{X},x} \setminus \{0\}$ generates the ideal $\mathcal{I}_{\mathscr{X},x}$ of $\mathcal{O}_{\mathscr{X},x}$.

We propose a generalisation of [MN15], Lemma 2.4.4.

**Lemma 3.2.3.** *Let $A$ be a Noetherian ring, let $I$ be an ideal of $A$ and let $(y_1, \dots, y_m)$ be a system of generators for $I$. We denote by $\hat{A}$ the $I$-adic completion of $A$. Let $B$ be a subring of $A$ such that the elements $y_1, \dots, y_m$ belong to $B$ and generate the ideal $B \cap I$ in $B$. Then, in the ring $\hat{A}$, every element $f$ of $B$ can be written as*

$$f = \sum_{\beta \in \mathbb{Z}^m_{\geqslant 0}} c_\beta y^\beta \tag{3.2.4}$$

*where the coefficients $c_\beta$ belong to $((A \setminus I) \cap B) \cup \{0\}$.*

*Proof.* Let $f$ be an element of $B$, we construct an expansion for $f$ of the form (3.2.4) by induction. If $f$ belongs to the complement of $I$, the conclusion trivially holds. Otherwise, $f$ belongs to $I$ and we can write $f$ as a linear combination of the elements $y_1, \dots, y_m$ with coefficients in $B$:

$$f = \sum_{j=1}^{m} b_j y_j, \quad b_j \in B.$$

By induction hypothesis, we suppose that $i$ is a positive integer and that we can write every $f$ in $B$ as a sum of an element $f_i$ of the form (3.2.4) and a linear combination of degree $i$ monomials in the elements $y_1, \dots, y_m$ with coefficients in $B$. We apply this assumption to the coefficients $b_j$, hence

$$b_j = b_{j,i} + \sum_{\substack{\beta \in \mathbb{Z}^m_{\geqslant 0} \\ |\beta| = i}} b_{j,\beta} y^\beta, \quad b_{j,\beta} \in B.$$

Then we can write $f$ as a sum of an element $f_{i+1}$ of the form (3.2.4) and a linear combination of degree $i+1$ monomials in the elements $y_1, \dots, y_m$ with coefficients in $B$

$$f = \underbrace{\sum_{j=1}^{m} b_{j,i} y_j}_{f_{i+1}} + \sum_{j=1}^{m} \Big( \sum_{\substack{\beta \in \mathbb{Z}^m_{\geqslant 0} \\ |\beta| = i}} b_{j,\beta} y^\beta \Big) y_j$$

such that $f_i$ and $f_{i+1}$ have the same coefficients in degree $i < 0$. Iterating this construction we finally find an expansion of $f$ of the required form. $\square$

**(3.2.5)** Let $f$ be an element of $\mathcal{O}_{\mathscr{X},x}$. Considering $A = B = \mathcal{O}_{\mathscr{X},x}$, $I = \mathfrak{m}_x$ and a system of generators for $\mathfrak{m}_x$ in $\mathcal{C}_{\mathscr{X},x} \setminus \{1\}$, by Lemma 3.2.3 we can write $f$ as a formal power series

$$f = \sum_{\gamma \in \mathcal{C}_{\mathscr{X},x}} c_\gamma \gamma \tag{3.2.6}$$

in $\widehat{\mathcal{O}}_{\mathscr{X},x}$, where each coefficient $c_\gamma$ is either zero or a unit in $\mathcal{O}_{\mathscr{X},x}$. We call this formal series an *admissible expansion* of $f$. We set

$$S = \{\gamma \in \mathcal{C}_{\mathscr{X},x} \,|\, c_\gamma \neq 0\} \tag{3.2.7}$$

and we denote by $\Gamma$ the set of elements of $S$ that lie on a compact face of the convex hull of $S + \mathcal{C}_{\mathscr{X},x}$ in $\mathcal{C}^{\mathrm{gp}}_{\mathscr{X},x} \otimes_{\mathbb{Z}} \mathbb{R}$.

**Proposition 3.2.8.**

*(1) The element*

$$f_x = \sum_{\gamma \in \Gamma} c_\gamma(x)\gamma \quad \in k(x)[\mathcal{C}_{\mathscr{X},x}]$$

*depends on the choice of the section (3.2.2), but not on the expansion (3.2.6).*

*(2) The subset $\Gamma$ of $\mathcal{C}_{\mathscr{X},x}$ only depends on $f$ and $x$, and not on the choice of the section (3.2.2) or the expansion (3.2.6).*

*Proof.* If we denote by $I$ the ideal of $k(x)[\mathcal{C}_{\mathscr{X},x}]$ generated by $\mathcal{C}_{\mathscr{X},x} \setminus \{1\}$, then it follows from [Kat94] that there exists an isomorphism of $k(x)$-algebras

$$\mathrm{gr}_I\, k(x)[\mathcal{C}_{\mathscr{X},x}] \to \mathrm{gr}_{\mathfrak{m}_x} \mathcal{O}_{\mathscr{X},x}. \tag{3.2.9}$$

Using this result and following the argument of [MN15] Proposition 2.4.6, we show that $f_x$ does not depend on the expansion of $f$. Let

$$f = \sum_{\gamma \in \mathcal{C}_{\mathscr{X},x}} c'_\gamma \gamma$$

be another admissible expansion of $f$ with associated set $\Gamma'$ and element $f'_x$. Then

$$0 = \sum_{\gamma \in \mathcal{C}_{\mathscr{X},x}} (c_\gamma - c'_\gamma)\gamma = \sum_{\gamma \in \mathcal{C}_{\mathscr{X},x}} d_\gamma \gamma$$

where the right hand side is an admissible expansion obtained by choosing admissible expansions for the elements $c_\gamma - c'_\gamma$ that do not lie in $\mathcal{O}^\times_{\mathscr{X},x} \cup \{0\}$. In particular $d_\gamma(x) = c_\gamma(x) - c'_\gamma(x)$ for any $\gamma$ in $\Gamma_x \cup \Gamma'_x$. The isomorphism of graded algebras in (3.2.9) implies that the elements $d_\gamma$ must all vanish, hence $\Gamma_x = \Gamma'_x$ and $f_x = f'_x$.

Point (2) follows from the fact that the coefficients $c_\gamma$ of $f_x$ are independent of the chosen section up to multiplication by a unit in $\mathcal{O}_{\mathscr{X},x}$, so that the support $\Gamma$ of $f_x$ only depends on $f$ and $x$. $\square$

**(3.2.10)** We will denote the subset $\Gamma$ of $\mathcal{C}_{\mathscr{X},x}$ by $\Gamma_x(f)$ and call it the *initial support* of $f$ at $x$.

**Proposition 3.2.11.** *Let $x$ be a point of $F$ and let*

$$\alpha : \mathcal{C}_{\mathscr{X},x} \to (\mathbb{R}_{\geq 0}, +)$$

*be an element of $\sigma_x$. Then there exists a unique minimal real valuation*

$$v : \mathcal{O}_{\mathscr{X},x} \setminus \{0\} \to \mathbb{R}_{\geq 0}$$

*such that $v(m) = \alpha(\overline{m})$ for each element $m$ of $\mathcal{M}_{\mathscr{X},x}$.*

*Proof.* We will prove that the map

$$v : \mathcal{O}_{\mathscr{X},x} \setminus \{0\} \to \mathbb{R} : f \mapsto \min\{\alpha(\gamma) \,|\, \gamma \in \Gamma_x(f)\} \tag{3.2.12}$$

satisfies the requirements in the statement. We fix a section

$$\mathcal{C}_{\mathscr{X},x} \to \mathcal{M}_{\mathscr{X},x}.$$

It is straightforward to check that $(f \cdot g)_x = f_x \cdot g_x$ for all $f$ and $g$ in $\mathcal{O}_{\mathscr{X},x}$. This implies that $v$ is a valuation. It is obvious that $v(m) = \alpha(\overline{m})$ for all $m$ in $\mathcal{M}_{\mathscr{X},x}$, since we can write $m$ as the product of an element of $\mathcal{C}_{\mathscr{X},x}$ and a unit in $\mathcal{O}_{\mathscr{X},x}$.

Now we prove minimality. Consider any real valuation

$$w : \mathcal{O}_{\mathscr{X},x} \to \mathbb{R}$$

such that $w(f) = \alpha(\overline{m})$ for each element $m$ of $\mathcal{M}_{\mathscr{X},x}$, and let $f$ be an element of $\mathcal{O}_{\mathscr{X},x}$. We must show that $w(f) \geq v(f)$.

We set

$$C_\alpha = \mathcal{C}_{\mathscr{X},x} \setminus \alpha^{-1}(0).$$

We denote by $I$ the ideal in $\mathcal{O}_{\mathscr{X},x}$ generated by $C_\alpha$ and by $A$ the $I$-adic completion of $\mathcal{O}_{\mathscr{X},x}$. By Lemma 3.2.3, we see that we can write $f$ in $A$ as

$$\sum_{\beta \in C_\alpha \cup \{1\}} d_\beta \beta \tag{3.2.13}$$

where $d_\beta$ is either zero or contained in the complement of $I$ in $\mathcal{O}_{\mathscr{X},x}$.

Since $\alpha(\beta) > 0$ for every $\beta \in C_\alpha$, we can find an integer $N > 0$ such that $w(g) > w(f)$ for every element $g$ in $I^N$. Since $w(\beta) = \alpha(\beta)$ for all $\beta$ in $\mathcal{C}_{\mathscr{X},x}$, we can write

$$w(f) \geq \min\{\alpha(\beta) \,|\, d_\beta \neq 0\}.$$

We consider the coefficients in the expansion (3.2.13) of $f$. Applying Lemma 3.2.3 as in paragraph (3.2.5), we can write admissible expansions of these coefficients in $\widehat{\mathcal{O}}_{\mathscr{X},x}$ as

$$d_\beta = \sum_{\gamma \in \mathcal{C}_{\mathscr{X},x}} c_{\gamma,\beta}\gamma, \quad c_{\gamma,\beta} \in \mathcal{O}^\times_{\mathscr{X},x} \cup \{0\},$$

with $\alpha(\gamma) = 0$ in the expansions of $d_\beta$ that belong to $\mathfrak{m}_x \setminus I$.

Therefore we obtain an admissible expansion of $f$

$$f = \sum_{\substack{\beta \in C_\alpha \cup \{1\} \\ \gamma \in \mathcal{C}_{\mathscr{X},x}}} c_{\gamma,\beta}\, \gamma\beta$$

and we have

$$\begin{aligned} v(f) &= \min\{\alpha(\gamma\beta) \,|\, c_{\gamma,\beta} \neq 0\} \\ &= \min\{\alpha(\beta) \,|\, d_\beta \neq 0\} \\ &\geq w(f). \end{aligned}$$

□

**Remark 3.2.14.** In the definition (3.2.12) of the valuation $v$, we compute the minimum over the terms in the initial support of $f$: these elements are a finite number and they only depends on $x$ and $f$ by Proposition 3.2.8. Therefore, this minimum provides a well-defined function on $\mathcal{O}_{\mathscr{X},x} \setminus \{0\}$. Nevertheless, it is equivalent to consider the minimum over all the terms of an admissible expansion of $f$, i.e. for any admissible expansion $f = \sum_{\gamma \in \mathcal{C}_{\mathscr{X},x}} c_\gamma \gamma$

$$\min\{\alpha(\gamma) \,|\, \gamma \in \Gamma_x(f)\} = \min\{\alpha(\gamma) \,|\, \gamma \in S\},$$

where $S = \{\gamma \in \mathcal{C}_{\mathscr{X},x} \,|\, c_\gamma \neq 0\}$ as in (3.2.7). Indeed, any element that belongs to $S$ can be written as a sum of an element of the initial support of $f$ and an element of $\mathcal{C}_{\mathscr{X},x}$. Since the morphism $\alpha$ is additive and takes positive real values, then the minimum is necessarily attained by the elements in the initial support.

**(3.2.15)** We will denote the valuation $v$ from Proposition 3.2.11 by $v_{x,\alpha}$. Since $v_{x,\alpha}$ induces a real valuation on the function field of $\mathscr{X}_K$ that extends the discrete valuation $v_K$ on $K$, it defines a point of the $K$-analytic space $\widehat{\mathscr{X}_\eta}$, which we will denote by the same symbol $v_{x,\alpha}$. We now show that the characterization of $v_{x,\alpha}$ in Proposition 3.2.11 implies that

$$v_{y,\alpha'} = v_{x,\alpha' \circ \tau_{x,y}}$$

for every $y$ in $F(x)$ and every $\alpha'$ in $\sigma_y$.

Firstly we note that $\mathcal{O}_{\mathscr{X},y}$ is the localization of $\mathcal{O}_{\mathscr{X},x}$ with respect to the elements of $m \in \mathcal{M}_{\mathscr{X},x}$ in the kernel of $\tau_{x,y}$. Indeed, by construction of $\tau_{x,y}$, the kernel is given by

$$\ker(\tau_{x,y}) = \{s \in \mathcal{C}_{\mathscr{X},x} | s(y) \neq 0\};$$

to obtain $\mathcal{O}_{\mathscr{X},y}$ from $\mathcal{O}_{\mathscr{X},x}$, we localize by

$$S = \{a \in \mathcal{O}_{\mathscr{X},x} | a(y) \neq 0\};$$

therefore we can identify the set of elements in $\mathcal{M}_{\mathscr{X},x}$ in $\ker(\tau_{x,y})$ with the set $S$, recalling that for points in the Kato fan $\mathcal{C}_{\mathscr{X},x} \setminus \{1\}$ generates the maximal ideal of $\mathcal{O}_{\mathscr{X},x}$. Therefore we are dealing with these two morphisms:

$$\mathcal{O}_{\mathscr{X},x} \hookrightarrow S^{-1}\mathcal{O}_{\mathscr{X},x} = \mathcal{O}_{\mathscr{X},y},$$

$$\mathcal{C}_{\mathscr{X},x} \twoheadrightarrow \mathcal{C}_{\mathscr{X},x}/S = \mathcal{C}_{\mathscr{X},y}.$$

Let $f$ be an element of $\mathcal{O}_{\mathscr{X},x}$. Under the notations of Lemma 3.2.3, we apply the lemma to $A = \mathcal{O}_{\mathscr{X},y}$ and $B = \mathcal{O}_{\mathscr{X},x}$, choosing a system of generators of $\mathfrak{m}_y$ in $\mathcal{C}_{\mathscr{X},x}$: we can find an admissible expansion of $f$ of the form

$$f = \sum_{\delta \in \mathcal{C}_{\mathscr{X},y}} d_\delta \delta \quad \text{ with } d_\delta \in (\mathcal{O}_{\mathscr{X},x} \cap \mathcal{O}^\times_{\mathscr{X},y}) \cup \{0\}.$$

Admissible expansions of coefficients $d_\delta$ induce an admissible expansion for $f$ by

$$f = \sum_{\delta \in \mathcal{C}_{\mathscr{X},y}} \big(\sum_{\gamma \in S} c_{\gamma\delta}\gamma\big)\delta \quad \text{ with } c_{\gamma\delta} \in \mathcal{O}^\times_{\mathscr{X},x} \cup \{0\},$$

where $\gamma$ runs through the set $S$ since $d_\delta \in \mathcal{O}^\times_{\mathscr{X},y}$. Thus we have

$$\begin{aligned} v_{y,\alpha'}(f) &= \min\{\alpha'(\delta)\,|\,\delta \in \Gamma_y(f)\} \\ &= \min\{\alpha' \circ \tau_{x,y}(\gamma\delta)\,|\,\delta \in \Gamma_y(f), \gamma \in S\} \\ &= \min\{\alpha' \circ \tau_{x,y}(\gamma\delta)\,|\,\gamma\delta \in \Gamma_x(f)\} \\ &= v_{x,\alpha'\circ\tau_{x,y}}(f). \end{aligned}$$

Hence, we obtain a well-defined map

$$\iota : \mathrm{Sk}(\mathscr{X}^+) \to \widehat{\mathscr{X}_\eta}$$

by sending $\alpha$ to $v_{x,\alpha}$ for every point $x$ of $F$ and every $\alpha \in \sigma_x$.

**Proposition 3.2.16.** *The map*

$$\iota : \mathrm{Sk}(\mathscr{X}^+) \to \widehat{\mathscr{X}_\eta}$$

*is a topological embedding.*

*Proof.* First, we show that $\iota$ is injective. Let $x$ be a point of $F$ and $\alpha$ an element of $\sigma_x$. Let $y$ be the point of $F(x)$ corresponding to the face $\mathcal{C}_{\mathscr{X},x} \setminus \alpha^{-1}(0)$ of $\mathcal{C}_{\mathscr{X},x}$. Then $\alpha$ factors through an element

$$\alpha' : \mathcal{C}_{\mathscr{X},y} \to \mathbb{R}_{\geq 0}$$

of $\sigma_y$. Note that $\alpha = \alpha'$ in $\mathrm{Sk}(\mathscr{X}^+)$ because $\alpha = \alpha' \circ \tau_{x,y}$. Moreover, since $(\alpha')^{-1}(0) = \{1\}$, the center of the valuation $v_{y,\alpha'}$ is the point $y$, so that $\mathrm{red}_{\mathscr{X}}(v_{y,\alpha'}) = y$. Thus we can recover $y$ from $v_{y,\alpha'}$. Then we can also reconstruct $\alpha'$ by looking at the values of $v_{y,\alpha'}$ at the elements of $\mathcal{M}_{\mathscr{X},y}$. We conclude that $\iota$ is injective.

Now, we show that $\iota$ is a homeomorphism onto its image. Since $\mathrm{Sk}(\mathscr{X}^+)$ is compact and $\widehat{\mathscr{X}_\eta}$ is Hausdorff, it suffices to show that $\iota$ is continuous. The family $\{\sigma_x,\, x \in F\}$ is a cover of $\mathrm{Sk}(\mathscr{X}^+)$ by closed subsets, so that we only have to prove that the restriction of $\iota$ to $\sigma_x$ is continuous, for every $x \in F$. By definition of the Berkovich topology, it is enough to prove that the map

$$\sigma_x \to \mathbb{R} : \alpha \mapsto v_{x,\alpha}(f)$$

is continuous for every $f$ in $\mathcal{O}_{\mathscr{X},x}$. This is obvious from the formula (3.2.12). □

**(3.2.17)** From now on, we will view $\mathrm{Sk}(\mathscr{X}^+)$ as a topological subspace of $\mathscr{X}_K^{\mathrm{an}}$ by means of the embedding $\iota$ in Proposition 3.2.16. If $\mathscr{X}$ is regular over $R$ and $\mathscr{X}_k$ is a divisor with strict normal crossings, the skeleton $\mathrm{Sk}(\mathscr{X}^+)$ was described in [MN15], Section 3.1.

## 3.3. **Contracting the generic fibre to the skeleton.**

**(3.3.1)** The inclusion $\iota : \mathrm{Sk}(\mathscr{X}^+) \to \widehat{\mathscr{X}_\eta}$ admits a continuous retraction

$$\rho_{\mathscr{X}} : \widehat{\mathscr{X}_\eta} \to \mathrm{Sk}(\mathscr{X}^+)$$

constructed as follows. Let $x$ be a point of $\widehat{\mathscr{X}_\eta}$ and consider the reduction map

$$\mathrm{red}_{\mathscr{X}} : \widehat{\mathscr{X}_\eta} \to \mathscr{X}_k.$$

Let $E_1, \ldots, E_r$ be the irreducible components of $D_{\mathscr{X}}$ passing through the point $\mathrm{red}_{\mathscr{X}}(x)$. We denote by $\xi$ the generic point of the connected component of $E_1 \cap$

$\ldots \cap E_r$ that contains $\mathrm{red}_{\mathscr{X}}(x)$. By Lemma 2.2.3, $\xi$ is a point in the associated Kato fan $F$. We set $\alpha$ to be the morphism of monoids

$$\alpha : \mathcal{C}_{\mathscr{X},\xi} \to \mathbb{R}_{\geqslant 0}$$

such that $\alpha(\overline{m}) = v_x(m)$ for any element $m$ of $\mathcal{M}_{\mathscr{X},\xi}$. In particular $\alpha(\pi) = v_x(\pi) = 1$ as we assumed the normalization of all valuations in the Berkovich space. Then $\rho_{\mathscr{X}}(x)$ is the point of $\mathrm{Sk}(\mathscr{X}^+)$ corresponding to the couple $(\xi, \alpha)$. By construction $\rho_{\mathscr{X}}$ is continuous and right inverse to the inclusion $\iota$.

**(3.3.2)** Given a morphism $f : \mathscr{X}^+ \to \mathscr{Y}^+$ of integral flat separated log-regular $S$-schemes, we can employ the retraction $\rho$ to define a map of skeleta as follows

$$\begin{tikzcd}
\widehat{\mathscr{X}_\eta} \arrow[r, "\widehat{f}"] \arrow[d, shift right, "\rho_{\mathscr{X}}"'] & \widehat{\mathscr{Y}_\eta} \arrow[d, "\rho_{\mathscr{Y}}"] \\
\mathrm{Sk}(\mathscr{X}^+) \arrow[u, shift right, "\iota_{\mathscr{Y}}"'] \arrow[r, dotted] & \mathrm{Sk}(\mathscr{Y}^+).
\end{tikzcd}$$

This association makes the skeleton construction $\mathrm{Sk}(\mathscr{X}^+)$ functorial in $\mathscr{X}^+$.

## 3.4. **Skeleton of a $fs$ fibred product.**

**(3.4.1)** Let $\mathscr{X}^+$ and $\mathscr{Y}^+$ be log-smooth log schemes over $S^+$, let $\mathscr{Z}^+$ be their $fs$ fibred product. Let

$$\mathrm{Sk}(\mathscr{Z}^+) \to \mathrm{Sk}(\mathscr{X}^+) \times \mathrm{Sk}(\mathscr{Y}^+)$$

be the continuous map of skeleta functorially associated to the projections $\mathrm{pr}_{\mathscr{X}} : \mathscr{Z}^+ \to \mathscr{X}^+$ and $\mathrm{pr}_{\mathscr{Y}} : \mathscr{Z}^+ \to \mathscr{Y}^+$. We denote this map by $\left(\mathrm{pr}_{\mathrm{Sk}(\mathscr{X})}, \mathrm{pr}_{\mathrm{Sk}(\mathscr{Y})}\right)$ and we recall that it is constructed considering the diagram

$$\begin{tikzcd}
\widehat{\mathscr{Z}_\eta} \arrow[rr, "(\widehat{\mathrm{pr}_{\mathscr{X}}}, \widehat{\mathrm{pr}_{\mathscr{Y}}})"] \arrow[d, shift right, "\rho_{\mathscr{Z}}"'] & & \widehat{\mathscr{X}_\eta} \times \widehat{\mathscr{Y}_\eta} \arrow[d, "(\rho_{\mathscr{X}}, \rho_{\mathscr{X}})"] \\
\mathrm{Sk}(\mathscr{Z}^+) \arrow[u, shift right, "\iota_{\mathscr{Z}}"'] \arrow[rr, "(\mathrm{pr}_{\mathrm{Sk}(\mathscr{X})}, \mathrm{pr}_{\mathrm{Sk}(\mathscr{Y})})"] & & \mathrm{Sk}(\mathscr{X}^+) \times \mathrm{Sk}(\mathscr{Y}^+).
\end{tikzcd} \tag{3.4.2}$$

**Proposition 3.4.3.** *Assume that the residue field $k$ is algebraically closed. If $\mathscr{X}^+$ is semistable, then the map $\left(\mathrm{pr}_{\mathrm{Sk}(\mathscr{X})}, \mathrm{pr}_{\mathrm{Sk}(\mathscr{Y})}\right)$ is a homeomorphism.*

*Proof.* The surjectivity of the map $\left(\mathrm{pr}_{\mathrm{Sk}(\mathscr{X})}, \mathrm{pr}_{\mathrm{Sk}(\mathscr{Y})}\right)$ follows from the commutativity of the diagram (3.4.2) and the surjectivity of $(\rho_{\mathscr{X}}, \rho_{\mathscr{X}}) \circ (\widehat{\mathrm{pr}_{\mathscr{X}}}, \widehat{\mathrm{pr}_{\mathscr{Y}}})$. To prove the injectivity of $\left(\mathrm{pr}_{\mathrm{Sk}(\mathscr{X})}, \mathrm{pr}_{\mathrm{Sk}(\mathscr{Y})}\right)$, we provide an explicit description of the map $\mathrm{pr}_{\mathrm{Sk}(\mathscr{X})}$.

We recall that the projection $\widehat{\mathrm{pr}_{\mathscr{X}}}$ is such that a valuation $v$ on the function field $K(\mathscr{Z}_K)$ maps to the composition $v \circ i$ where $i : K(\mathscr{X}_K) \hookrightarrow K(\mathscr{Z}_K)$.

Let $v_{z,\varepsilon}$ be the valuation in $\mathrm{Sk}(\mathscr{Z}^+)$ corresponding to a couple $(z, \varepsilon)$ with $z \in F_{\mathscr{Z}}$ and $\varepsilon \in \sigma_z$. We consider the morphism of associated Kato fans

$$F_{\mathscr{Z}} \to F_{\mathscr{X}} \times F_{\mathscr{Y}}$$

as established in Proposition 2.4.8. We denote respectively by $\mathrm{pr}_{F_{\mathscr{X}}}$ and $\mathrm{pr}_{F_{\mathscr{Y}}}$ the projection to the first and second factor. Then $\mathrm{pr}_{F_{\mathscr{X}}}(z)$ is a point in the associated Kato fan $F_{\mathscr{X}}$, that we denote by $x$. We consider the morphism of monoids

$$i_x : \mathcal{C}_{\mathscr{X},x} \to \mathcal{C}_{\mathscr{Z},z}$$

and the composition

$$\begin{array}{rcl} \mathrm{pr}_{\mathscr{X}}(\varepsilon): & \mathcal{C}_{\mathscr{X},x} \xrightarrow{i_x} \mathcal{C}_{\mathscr{Z},z} = (\mathcal{C}_{\mathscr{X},x} \oplus_{\mathbb{N}} \mathcal{C}_{\mathscr{Y},y})^{\mathrm{sat}} \xrightarrow{\varepsilon} \mathbb{R}_{\geq 0} \\ & a \longmapsto [a,1] \longmapsto \varepsilon([a,1]). \end{array}$$

It trivially satisfies $\varepsilon \circ i_x(\pi) = 1$. In order to conclude that it correctly defines a point in the skeleton $\mathrm{Sk}(\mathscr{X}^+)$, we need to check the compatibility with respect to the equivalence relation $\sim$. Indeed, suppose that $\varepsilon = \varepsilon' \circ \tau_{z,z'}$ for some $z' \in \overline{\{z\}}$. We denote by $x'$ the projection of $z'$ under the local isomorphism of associated Kato fans. The diagram

$$\begin{tikzcd} \mathcal{C}_{\mathscr{X},x} \arrow[r, "i_x"] \arrow[d, "\tau_{x,x'}"] & \mathcal{C}_{\mathscr{Z},z} \arrow[d, "\tau_{z,z'}"] \arrow[dr, "\varepsilon"] & \\ \mathcal{C}_{\mathscr{X},x'} \arrow[r, "i_{x'}"] & \mathcal{C}_{\mathscr{Z},z'} \arrow[r, "\varepsilon'"'] & \mathbb{R}_{\geq 0} \end{tikzcd}$$

is commutative as made up by a commutative square and a commutative triangle of arrows. Therefore, by commutativity

$$\mathrm{pr}_{\mathscr{X}}(\varepsilon) = \mathrm{pr}_{\mathscr{X}}(\varepsilon') \circ \tau_{x,x'}$$

and this implies that $\mathrm{pr}_{\mathscr{X}}(\varepsilon)$ defines a well-defined point $v_{x,\mathrm{pr}_{\mathscr{X}}(\varepsilon)}$ of $\mathrm{Sk}(\mathscr{X}^+)$.

We claim that $v_{x,\mathrm{pr}_{\mathscr{X}}(\varepsilon)}$ is indeed the image of $v_{z,\varepsilon}$ under the map $\mathrm{pr}_{\mathrm{Sk}(\mathscr{X})}$, hence that the equality in the following inner diagram holds

$$\begin{tikzcd} \widehat{\mathscr{Z}_\eta} \arrow[rrr, "\widehat{\mathrm{pr}_{\mathscr{X}}}"] \arrow[ddd, "\rho_{\mathscr{X}}"'] & & & \widehat{\mathscr{X}_\eta} \arrow[ddd, "\rho_{\mathscr{X}}"] \\ & v_{z,\varepsilon} \arrow[r, mapsto] & v_{z,\varepsilon} \circ i \arrow[d, mapsto] & \\ & v_{z,\varepsilon} \arrow[u, mapsto] \arrow[r, mapsto] & \rho_{\mathscr{X}}(v_{z,\varepsilon} \circ i) = v_{x,\mathrm{pr}_{\mathscr{X}}(\varepsilon)} & \\ \mathrm{Sk}(\mathscr{Z}^+) \arrow[uuu, bend right, "\iota_{\mathscr{X}}"'] \arrow[rrr, "\mathrm{pr}_{\mathrm{Sk}(\mathscr{X})}"] & & & \mathrm{Sk}(\mathscr{X}^+). \end{tikzcd}$$

We denote $\rho_{\mathscr{X}}(v_{z,\varepsilon} \circ i)$ by $(x, \alpha)$ as a point of $\mathrm{Sk}(\mathscr{X}^+)$. By definition of the retraction $\rho_{\mathscr{X}}$, the morphism $\alpha$ is characterized by the fact that $\alpha(\overline{m}) = (v_{z,\varepsilon} \circ i)(m)$ for any $m$ in $\mathcal{M}_{\mathscr{X},x}$ and then we have

$$\alpha(\overline{m}) = (v_{z,\varepsilon} \circ i)(m) = v_{z,\varepsilon}(m) = \varepsilon(\overline{m}).$$

On the other hand, for any $m$ in $\mathcal{M}_{\mathscr{X},x}$

$$v_{x,\mathrm{pr}_{\mathscr{X}}(\varepsilon)}(m) = \mathrm{pr}_{\mathscr{X}}(\varepsilon)(\overline{m}) = \varepsilon(\overline{m})$$

hence we obtain that $\alpha$ coincide with the morphism $\mathrm{pr}_{\mathscr{X}}(\varepsilon)$. It means that their associated points $\rho_{\mathscr{X}}(v_{z,\varepsilon} \circ i)$ and $v_{x,\mathrm{pr}_{\mathscr{X}}(\varepsilon)}$ coincide in $\mathrm{Sk}(\mathscr{X}^+)$.

Given a pair of points in $\mathrm{Sk}(\mathscr{X}^+) \times \mathrm{Sk}(\mathscr{Y}^+)$, we know by surjectivity of $\big(\mathrm{pr}_{\mathrm{Sk}(\mathscr{X})}, \mathrm{pr}_{\mathrm{Sk}(\mathscr{Y})}\big)$ that they are of the form

$$\big(v_{x,\mathrm{pr}_{\mathscr{X}}(\varepsilon)}, v_{y,\mathrm{pr}_{\mathscr{Y}}(\varepsilon)}\big).$$

The assumption of semistability of $\mathscr{X}^+$ guarantees that there is a unique $z$ in $F_{\mathscr{Z}}$ in the fibre of $x$ and $y$, by Proposition 2.5.4. Moreover, we can uniquely reconstruct $\varepsilon$ by looking at the values of $v_{x,\mathrm{pr}_{\mathscr{X}}(\varepsilon)}$ at the elements of $\mathcal{M}_{\mathscr{X},x}$ and respectively of $v_{y,\mathrm{pr}_{\mathscr{Y}}(\varepsilon)}$ at the elements of $\mathcal{M}_{\mathscr{Y},y}$. We conclude that $\big(\mathrm{pr}_{\mathrm{Sk}(\mathscr{X})}, \mathrm{pr}_{\mathrm{Sk}(\mathscr{Y})}\big)$ is injective. $\square$

## 4. The weight function for pairs

Given a connected, smooth and proper $K$-variety $X$, in [MN15] Mustaţă and Nicaise define the *weight function* on $X^{\mathrm{an}}$ associated to a pluricanonical form $\omega$ on $X$. Following the arguments of Section 4 in [MN15], we will extend the construction of weight functions to pairs $(X, \Delta)$ with log-regular log scheme $X^+ = (X, \Delta)$ induced by $\Delta$, and to sections of the logarithmic pluricanonical line bundle $(\omega^{\log}_{X^+/K})^{\otimes m}$ on $X^+$, for any $m > 0$.

### 4.1. **Weight function associated to a logarithmic pluricanonical form.**

**(4.1.1)** Let $X$ be a connected, smooth and proper $K$-variety of dimension $n$. We introduce the following notation: for any log-regular model $\mathscr{X}^+$ of $X$, for any point $x = (\xi_x, |\cdot|_x) \in \widehat{\mathscr{X}_\eta}$ and for any divisor $D$ on $\mathscr{X}^+$ whose support does not contain $\xi_x$, we set

$$v_x(D) = -\ln |f(x)|$$

where $f$ is any element of $K(X)^\times$ such that $D = \mathrm{div}(f)$ locally at $\mathrm{red}_{\mathscr{X}}(x)$.

**(4.1.2)** Let $(X, \Delta)$ be a pair with log-regular log scheme $X^+ = (X, \Delta)$ over $K$. Let $\omega$ be a logarithmic $m$-pluricanonical form on $X^+$. For each log-regular model $\mathscr{X}^+$ of $X^+$, the form $\omega$ defines a divisor $\mathrm{div}_{\mathscr{X}^+}(\omega)$ on $\mathscr{X}^+$. For any point $x$ in $\mathrm{Sk}(\mathscr{X}^+)$ that is associated to an irreducible component of $\mathscr{X}_k$, the value

$$v_x(\mathrm{div}_{\mathscr{X}^+}(\omega)) + m \tag{4.1.3}$$

does not depend on the choice of the model $\mathscr{X}^+$; indeed, this follows from the same arguments of [MN15], Proposition 4.2.4. We denote this value by $\mathrm{wt}_\omega(x)$ and call it the weight of $x$ with respect to $\omega$. Since any snc model $\mathscr{X}$ of $X$ can be turned by resolution of singularities into a log-regular model of $X^+$, we can compute the weight with respect to $\omega$ of any divisorial point of $X^{\mathrm{an}}$. Thus, we obtain a function

$$\mathrm{wt}_\omega : \mathrm{Div}(X) \to \mathbb{Q}, \quad x \mapsto \mathrm{wt}_\omega(x)$$

on the set of divisorial points, called the weight function associated to $\omega$. We prove that the formula 4.1.3 expresses the weight of any divisorial point in $\mathrm{Sk}(\mathscr{X}^+)$.

**Proposition 4.1.4.** *If $x$ is a divisorial point in $\mathrm{Sk}(\mathscr{X}^+)$, then*

$$\mathrm{wt}_\omega(x) = v_x(\mathrm{div}_{\mathscr{X}^+}(\omega)) + m.$$

*Proof.* If the point $x$ is associated to an irreducible component of the special fibre, then the equality holds by definition. As in [MN15], Proposition 2.4.11, any divisorial point can be reduced to such a representation by a finite sequence of blow-ups of strata of $D_{\mathscr{X}}$. Therefore, it suffices to consider one blow-up morphism $h : \mathscr{Y} \to \mathscr{X}$ of a stratum $Z$ of $D_{\mathscr{X}}$ and check that

$$v_x(\mathrm{div}_{\mathscr{X}^+}(\omega)) + m = v_x(\mathrm{div}_{\mathscr{Y}^+}(\omega)) + m,$$

where $\mathscr{Y}^+$ is the log-regular log scheme with $D_{\mathscr{Y}}$ equals to the sum of the closure of the strict trasform $\Delta_Y$ of $\Delta_X$ and the special fibre $\mathscr{Y}_k$.

We denote by $E$ the exceptional divisor of the blow-up $h$, by $r = r_h + r_v$ the codimension of $Z$ in $\mathscr{X}$, where $r_h$ and $r_v$ are the number of irreducible components of $\overline{\Delta_X}$ and respectively of the special fibre $\mathscr{X}_k$, containing $Z$. We denote the projections onto $S^+$ by $s_{\mathscr{X}} : \mathscr{X}^+ \to S^+$ and $s_{\mathscr{Y}} : \mathscr{Y}^+ \to S^+$ and by $\pi$ a uniformizer in $R$. Then we have that

$$\begin{aligned} h^*(\omega^{\log}_{\mathscr{X}^+/S^+}) &= h^*\big(\omega_{\mathscr{X}/R} \otimes \mathcal{O}_{\mathscr{X}}(\overline{\Delta_X}_{,\mathrm{red}} + \mathscr{X}_{k,\mathrm{red}} - s^*_{\mathscr{X}}(\pi))\big) \\ &= \omega_{\mathscr{Y}/R} \otimes \mathcal{O}_{\mathscr{Y}}((1-r)E) \otimes \mathcal{O}_{\mathscr{Y}}(\overline{\Delta_Y}_{,\mathrm{red}} + r_h E + \mathscr{Y}_{k,\mathrm{red}} + (r_v - 1)E - s^*_{\mathscr{Y}}(\pi)) \\ &= \omega_{\mathscr{Y}/R} \otimes \mathcal{O}_{\mathscr{Y}}(\overline{\Delta_Y}_{,\mathrm{red}} + \mathscr{Y}_{k,\mathrm{red}} - s^*_{\mathscr{Y}}(\pi)) = \omega^{\log}_{\mathscr{Y}^+/S^+}. \end{aligned}$$

This implies that

$$v_x(\mathrm{div}_{\mathscr{X}^+}(\omega)) + m = v_x\big(h^*(\mathrm{div}_{\mathscr{X}^+}(\omega))\big) + m = v_x(\mathrm{div}_{\mathscr{Y}^+}(\omega)) + m$$

and concludes the proof. □

**(4.1.5)** We define a function

$$\mathrm{wt}_{\mathscr{X}^+,\omega} : \mathrm{Sk}(\mathscr{X}^+) \to \mathbb{R}, \quad x \mapsto v_x(\mathrm{div}_{\mathscr{X}^+}(\omega)) + m$$

on the skeleton associated to a log-regular model $\mathscr{X}^+$ and we call it the weight function associated to $\omega$ and $\mathscr{X}^+$. By Proposition 4.1.4, if $x$ is a divisorial point of $\mathrm{Sk}(\mathscr{X}^+)$, then the weight at $x$ associated to $\omega$ and $\mathscr{X}^+$ is actually independent on the choice of the model and equal to $\mathrm{wt}_\omega(x)$. Following the arguments of Lemma 4.2.8 and Proposition 4.3.4 in [MN15], we can prove that there exists a unique function on the set of birational points of $X^{\mathrm{an}}$

$$\mathrm{wt}_\omega : \mathrm{Bir}(X) \to \mathbb{R}, \text{ such that } \mathrm{wt}_\omega(x) = \mathrm{wt}_{\mathscr{X}^+,\omega}(x)$$

for every birational point $x$ and every log-regular model $\mathscr{X}^+$ such that $x \in \mathrm{Sk}(\mathscr{X}^+)$. Moreover, after the arguments of Proposition 4.4.5 in [MN15], we obtain a function on the Berkovich space $X^{\mathrm{an}}$ by setting

$$\mathrm{wt}_\omega(x) = \sup_{\mathscr{X}^+}\{\mathrm{wt}_\omega(\rho_{\mathscr{X}}(x))\}$$

where $\mathscr{X}^+$ runs through the set of all log-regular model $\mathscr{X}^+$ of $X^+$. We have that

$$\mathrm{wt}_\omega(x) = v_x(\mathrm{div}_{\mathscr{X}^+}(\omega)) + m \tag{4.1.6}$$

for every birational point $x$ and every log-regular model $\mathscr{X}^+$ such that $x \in \mathrm{Sk}(\mathscr{X}^+)$. We call it the weight function associated to $\omega$.

## 4.2. **Weight function for log-regular models.**

**(4.2.1)** In case the divisor $\Delta$ is empty, the forms we considered in the previous paragraph to construct the weight functions are simply the sections of the $m$-pluricanonical line bundle $\omega^{\otimes m}_{X/K}$ on $X$, for some $m > 0$. Given a $m$-pluricanonical form $\omega$ on $X$, we check that the weight function $\mathrm{wt}_\omega$ defined as in (4.1.6) coincides with the definition of the weight function associated to $\omega$ according to [MN15]. This results in a generalized formula for the usual weight of $\omega$ at the birational points, in terms of their representations in skeleta associated to log-regular models of $X$.

**Proposition 4.2.2.** *Let $\mathscr{X}$ be a model of $X$ over $R$ such that $\mathscr{X}^+$ is log-regular over $S^+$. If $x$ is a point of $\mathrm{Sk}(\mathscr{X}^+)$, then the weight of $\omega$ at $x$ as in [MN15] is given by*

$$\mathrm{wt}_\omega(x) = v_x(\mathrm{div}_{\mathscr{X}^+}(\omega)) + m.$$

*Proof.* In order to compute the weight of $\omega$ at $x$ according to the definition in [MN15], we can consider any snc model $\mathscr{Y}$ of $X$ such that $x$ is a point of $\mathrm{Sk}(\mathscr{Y})$. Then, by [NX16], Section 3.2.2, the weight is given by

$$v_x(\mathrm{div}_{\mathscr{Y}^+}(\omega)) + m.$$

As we noticed in Remark 2.3.2, we can obtain an snc model $\mathscr{Y}$ adapted to $x$ by means of a log blow-up $h : \mathscr{Y}^+ \to \mathscr{X}^+$ of $\mathscr{X}^+$ (Propositions 2.2.5 and 2.3.1). Moreover, the corresponding skeleton $\mathrm{Sk}(\mathscr{Y}^+)$ is given by a subdivision of $\mathrm{Sk}(\mathscr{X}^+)$ (Proposition 2.1.6) and coincides with $\mathrm{Sk}(\mathscr{Y})$. Therefore it suffices to prove that $v_x(\mathrm{div}_{\mathscr{Y}^+}(\omega)) = v_x(\mathrm{div}_{\mathscr{X}^+}(\omega))$ for such a model $\mathscr{Y}^+$.

Log blow-ups are log-étale morphisms ([Sai04], Section 2.1) and for log-étale morphisms the sheaf of log differentials is stable under pullback ([Kat94], Proposition 3.12), therefore

$$h^*\omega^{\log}_{\mathscr{X}^+/S^+} \simeq \omega^{\log}_{\mathscr{Y}^+/S^+}.$$

Then $\mathrm{div}_{\mathscr{Y}^+}(\omega) = h^*\mathrm{div}_{\mathscr{X}^+}(\omega)$ and in particular for points $x$ of the skeleton $\mathrm{Sk}(\mathscr{X}^+) = \mathrm{Sk}(\mathscr{Y})$, it holds that

$$v_x(\mathrm{div}_{\mathscr{Y}^+}(\omega)) = v_x(h^*\mathrm{div}_{\mathscr{X}^+}(\omega)) = v_x(\mathrm{div}_{\mathscr{X}^+}(\omega)).$$

□

4.3. **Weight function on skeleta associated to *fs* fibred products.**

**(4.3.1)** Let $\mathscr{X}^+$ and $\mathscr{Y}^+$ be log-smooth models over $S^+$ of $X^+ = (X, \Delta_X)$ and $Y^+ = (Y, \Delta_Y)$ respectively. Then, the $fs$ fibred product $\mathscr{Z}^+ = \mathscr{X}^+ \times^{\mathrm{fs}}_{S^+} \mathscr{Y}^+$ is a log-regular model of $Z^+ := X^+ \times^{\mathrm{fs}}_K Y^+$. Therefore, given $\omega_{X^+}$ and $\omega_{Y^+}$ logarithmic $m$-pluricanonical forms on $X^+$ and $Y^+$ respectively, the form

$$\varpi = \mathrm{pr}^*_{X^+}\,\omega_{X^+} \otimes \mathrm{pr}^*_{Y^+}\,\omega_{Y^+}$$

is a logarithmic $m$-pluricanonical form on $Z^+$. Viewing these forms as rational sections of log $m$-pluricanonical bundles, we see that $\mathrm{div}_{\mathscr{Z}^+}(\varpi) = \mathrm{div}_{\mathscr{Z}^+}(\mathrm{pr}^*_{X^+}\,\omega_{X^+} \otimes \mathrm{pr}^*_{Y^+}\,\omega_{Y^+})$ according to (2.4.6).

**(4.3.2)** Let $z$ be a point of $F_{\mathscr{Z}}$; as before, we denote by $x$ and $y$ the images of $z$ under the local isomorphism $F_{\mathscr{Z}} \to F_{\mathscr{X}} \times F_{\mathscr{Y}}$. Any morphism $\varepsilon \in \sigma_z$ defines a point $v_{z,\varepsilon}$ in $\mathrm{Sk}(\mathscr{Z}^+)$. For the sake of convenience, we simply denote the valuations by the corresponding morphisms and we denote $\alpha = \mathrm{pr}_{\mathscr{X}}(\varepsilon)$ and $\beta = \mathrm{pr}_{\mathscr{Y}}(\varepsilon)$. We aim to relate the valuation $v_\varepsilon\big(\mathrm{div}_{\mathscr{Z}^+}(\varpi)\big)$ to the values

$$v_\alpha\big(\mathrm{div}_{\mathscr{X}^+}(\omega_{X^+})\big)\ ,\ v_\beta\big(\mathrm{div}_{\mathscr{Y}^+}(\omega_{Y^+})\big).$$

**(4.3.3)** Let $f_x \in \mathcal{O}_{\mathscr{X},x}$ be a local equation of $\mathrm{div}_{\mathscr{X}^+}(\omega_{X^+})$ around $x$. In order to evaluate $v_{x,\alpha}$ on $f_x$, we consider an admissible expansion of $f_x$ as in (3.2.6)

$$f_x = \sum_{\gamma \in \mathcal{C}_{\mathscr{X},x}} c_\gamma \gamma.$$

Furthermore, this expansion induces also an expansion of $\mathrm{pr}^*_{\mathscr{X}}(f_x)$ by

$$\mathrm{pr}^*_{\mathscr{X}}(f_x) = \sum_{\gamma \in \mathcal{C}_{\mathscr{X},x}} \mathrm{pr}^*_{\mathscr{X}}(c_\gamma)\gamma$$

as formal power series in $\widehat{\mathcal{O}}_{\mathscr{Z},z}$, since the morphism of characteristic sheaves $\mathcal{C}_{\mathscr{X},x} \hookrightarrow \mathcal{C}_{\mathscr{Z},z}$ is injective. Following the same procedure for a local equation $f_y \in \mathcal{O}_{\mathscr{Y},y}$ of $\mathrm{div}_{\mathscr{Y}^+}(\omega_{Y^+})$ around $y$, we get an expansion of $f_y$ that extends to $\mathrm{pr}^*_{\mathscr{Y}}(f_y)$:

$$f_y = \sum_{\delta \in \mathcal{C}_{\mathscr{Y},y}} d_\delta \delta.$$

**(4.3.4)** A local equation of $\varpi$ around $z$ is determined by $\mathrm{pr}^*_{\mathscr{X}}(f_x)\ \mathrm{pr}^*_{\mathscr{Y}}(f_y)$. Thus

$$v_\varepsilon(\mathrm{div}_{\mathscr{Z}^+}(\varpi)) = v_\varepsilon\big(\mathrm{pr}^*_{\mathscr{X}}(f_x)\ \mathrm{pr}^*_{\mathscr{Y}}(f_y)\big)$$

and by multiplicativity of the valuation $v_\varepsilon$

$$v_\varepsilon\big(\mathrm{pr}^*_{\mathscr{X}}(f_x)\ \mathrm{pr}^*_{\mathscr{Y}}(f_y)\big) = v_\varepsilon\big(\mathrm{pr}^*_{\mathscr{X}}(f_x)\big) + v_\varepsilon\big(\mathrm{pr}^*_{\mathscr{Y}}(f_y)\big).$$

Recalling Remark 3.2.14, the valuation can be computed as follows

$$v_\varepsilon\big(\mathrm{pr}^*_{\mathscr{X}}(f_x)\big) = \min\{\varepsilon(\gamma) \mid c_\gamma \neq 0\};$$

as the elements $\gamma$ belong to $\mathcal{C}_{\mathscr{X},x}$ and $\alpha$ is defined to be $\mathrm{pr}_{\mathscr{X}}(\varepsilon)$

$$\min\{\varepsilon(\gamma) \mid c_\gamma \neq 0\} = \min\{\alpha(\gamma) \mid c_\gamma \neq 0\} = v_{x,\alpha}(f_x).$$

Hence, we conclude that

$$\begin{aligned} v_\varepsilon\big(\mathrm{div}_{\mathscr{Z}^+}(\varpi)\big) &= v_\varepsilon\big(\mathrm{pr}^*_{\mathscr{X}}(f_x)\big) + v_\varepsilon\big(\mathrm{pr}^*_{\mathscr{Y}}(f_y)\big) \\ &= v_\alpha(f_x) + v_\beta(f_y) \\ &= v_\alpha\big(\mathrm{div}_{\mathscr{X}^+}(\omega_{X^+})\big) + v_\beta\big(\mathrm{div}_{\mathscr{Y}^+}(\omega_{Y^+})\big). \end{aligned}$$

**(4.3.5)** This result turns out to be advantageous to compute the weight function $\mathrm{wt}_\varpi$ on divisorial points of $\mathrm{Sk}(\mathscr{Z}^+)$:

$$\begin{aligned} \mathrm{wt}_\varpi(\varepsilon) &= v_\varepsilon\big(\mathrm{div}_{\mathscr{Z}^+}(\varpi)\big) + m \\ &= v_\alpha\big(\mathrm{div}_{\mathscr{X}^+}(\omega_{X^+})\big) + v_\beta\big(\mathrm{div}_{\mathscr{Y}^+}(\omega_{Y^+})\big) + m \\ &= \mathrm{wt}_{\omega_{X^+}}(\alpha) + \mathrm{wt}_{\omega_{Y^+}}(\beta) - m. \end{aligned} \tag{4.3.6}$$

## 4.4. **Kontsevich-Soibelman skeleta of *fs* fibred products.**

**(4.4.1)** Let $X^+ = (X, \Delta)$ be a pair such that $X$ is a connected, smooth and proper $K$-variety and $X^+$ a log-regular log scheme over $K$. Let $\omega$ be a non-zero rational logarithmic $m$-pluricanonical form on $X^+$ for some $m > 0$. Similarly to [MN15], Section 4.5, we define the Kontsevich-Soibelman skeleton $\mathrm{Sk}(X^+, \omega)$ as the closure of the set of divisorial points of $X^{\mathrm{an}}$ where the weight function $\mathrm{wt}_\omega$ reaches its minimal value, denoted by $\mathrm{wt}_\omega(X^+)$.

**(4.4.2)** Under the notations of the previous paragraph, our computations lead to the following result.

**Theorem 4.4.3.** *Let $\mathscr{X}^+$ and $\mathscr{Y}^+$ be log-smooth models over $S^+$ of $X^+$ and $Y^+$ respectively, and let $\mathscr{Z}^+$ be their $fs$ fibred product. Let $\omega_{X^+}$ and $\omega_{Y^+}$ be $m$-pluricanonical forms on $X^+$ and $Y^+$ respectively. Suppose that the residue field $k$ is algebraically closed and that $\mathscr{X}^+$ is semistable. Then, the homeomorphism of skeleta*

$$\mathrm{Sk}(\mathscr{Z}^+) \xrightarrow{\sim} \mathrm{Sk}(\mathscr{X}^+) \times \mathrm{Sk}(\mathscr{Y}^+)$$

*given in Proposition 3.4.3 restricts to a homeomorphism of Kontsevich-Soibelman skeleta*

$$\mathrm{Sk}(Z^+, \varpi) \xrightarrow{\sim} \mathrm{Sk}(X^+, \omega_{X^+}) \times \mathrm{Sk}(Y^+, \omega_{Y^+}).$$

*Proof.* This follows immediately from the equality (4.3.6) that shows that a point in $\mathrm{Sk}(Z^+)$ has minimal value $\mathrm{wt}_\varpi(Z^+)$ if and only if its projections have minimal value $\mathrm{wt}_{\omega_{X^+}}(X^+)$ and $\mathrm{wt}_{\omega_{Y^+}}(Y^+)$. ☐

## 5. The essential skeleton of a product

## 6. Applications

### 6.1. Title.

**(6.1.1)** Let $X$ be a connected, smooth and proper $K$-variety and let $G$ be a group acting on $X$. Let $X^{\mathrm{an}}$ be the analytification of $X$. We recall that any point of $X^{\mathrm{an}}$ is a pair $(x, |\cdot|_x)$ with $x \in X$ and $|\cdot|_x$ an absolute value on the residue field $\kappa(x)$ extending the absolute value on $K$. For any point $x$ of $X$, an element $g$ of the group $G$ induces an isomorphism between the residue fields $\kappa(x)$ and $\kappa(g.\,x)$, that we still denote by $g$. Then, the action of $G$ extends to $X^{\mathrm{an}}$ in the following way

$$g.\,(x, |\cdot|_x) = (g.\,x, |\cdot|_x \circ g^{-1}).$$

In particular the action preserves the sets of divisorial and birational points of $X$. Let $f : X \to Y = X/G$ be the quotient map of $K$-schemes, let $f^{\mathrm{an}} : X^{\mathrm{an}} \to Y^{\mathrm{an}}$ be the map of Berkovich spaces induced by functoriality and let $\tilde{f} : X^{\mathrm{an}} \to X^{\mathrm{an}}/G$ be the quotient map of topological spaces.

**Proposition 6.1.2.** *([Ber95], Corollary 5) Under the above notations, there is a canonical homeomorphism between $X^{\mathrm{an}}/G$ and $Y^{\mathrm{an}}$ such that $\tilde{f}$ and $f^{\mathrm{an}}$ are identified.*

**Lemma 6.1.3.** *Let $\omega$ be a $m$-pluricanonical form on $X$. If $\omega$ is $G$-invariant, then the weight function associated to $\omega$ on the set of birational points factors through the quotient by the action of $G$.*

*Proof.* Let $x$ be a birational point of $X$ and $g$ an element of $G$. There exist snc models $\mathscr{X}$ and $\mathscr{X}'$ over $R$ such that $x \in \mathrm{Sk}(\mathscr{X})$ and $g.\,x \in \mathrm{Sk}(\mathscr{X}')$. By replacing them by an snc model $\mathscr{Y}$ that dominates both $\mathscr{X}$ and $\mathscr{X}'$, we can assume that both points lies in $\mathrm{Sk}(\mathscr{Y})$. The weights of $\omega$ at $x$ and $g.\,x$ can be computed using the formula 4.1.6, so

$$\begin{aligned}\mathrm{wt}_\omega(g.\,x) &= v_{g.x}(\mathrm{div}_{\mathscr{Y}^+}(\omega)) + m = v_x((g^{-1})^*\mathrm{div}_{\mathscr{Y}^+}(\omega)) + m \\ &= v_x(\mathrm{div}_{\mathscr{Y}^+}(\omega)) + m = \mathrm{wt}_\omega(x)\end{aligned}$$

as $\omega$ is a $G$-invariant form. Thus we see that birational points in the same $G$-orbit have the same weight with respect to $\omega$. ☐

**Corollary 6.1.4.** *If $\omega$ is a $G$-invariant pluricanonical form on $X$, then the Kontsevich-Soibelman skeleton $\mathrm{Sk}(X, \omega)$ is stable under the action of $G$.*

*Proof.* This follows immediately from Lemma 6.1.3. ☐

### 6.2. Analytification of the quotient.

**(6.2.1)** Let $X$ be a smooth $K$-variety and let $S_n$ be $n$-th symmetric group. We describe an action of $S_n$ on the $n$-fold fibred product $X^n$. Any point $x$ of $X^n$ is characterized as a tuple $x = (x_1, \ldots, x_n, s, \mathfrak{p})$ where $s$ is the image of any of $x_i$'s and $\mathfrak{p}$ is a prime ideal of the tensor algebra of residue fields $\kappa(x_1) \otimes \ldots \otimes \kappa(x_n)$. Given a permutation $\sigma$ of $n$ elements, it induces an isomorphism of tensor algebras

$k(x_1) \otimes \ldots \otimes k(x_n) \simeq k(x_{\sigma(1)}) \otimes \ldots \otimes k(x_{\sigma(n)})$ still denoted by $\sigma$. Then the action of $S^n$ on $X^n$ is given by

$$\sigma.\, x = (x_{\sigma(1)}, \ldots, x_{\sigma(n)}, s, \sigma(\mathfrak{p})).$$

Let $(X^n)^{\mathrm{an}}$ be the analytification of $X^n$. We recall that any point of $(X^n)^{\mathrm{an}}$ is a pair $(x, |\cdot|_x)$ with $x \in X^n$ and $|\cdot|_x$ an absolute value on the residue field $\kappa(x)$ extending the absolute value on $K$. The action of $S_n$ extends to $(X^n)^{\mathrm{an}}$ by

$$\sigma.\, (x, |\cdot|_x) = (x_{\sigma(1)}, \ldots, x_{\sigma(n)}, s, \sigma(\mathfrak{p}), |\cdot|_x \circ \sigma^{-1}).$$

**(6.2.2)** By functoriality of the Berkovich analytification, the morphism of schemes $f : X^n \to Y = X^n/S_n$ to the $n$-th symmetric product of $X$ induces a surjective morphism of Berkovich spaces $f^{\mathrm{an}} : (X^n)^{\mathrm{an}} \to Y^{\mathrm{an}}$ and the image of a point $(x, |\cdot|_x)$ is

$$f^{\mathrm{an}}(x, |\cdot|_x) = ([x], |\cdot|_{[x]}),$$

with $[x] = ([(x_i)], s, [\mathfrak{p}]) \in X^n/S_n$ and $|\cdot|_{[x]}$ is an absolute value on $k(x)^{S_n}$, the field of $S_n$-invariant elements of $k(x)$. Two points $(x, |\cdot|_x)$ and $(x', |\cdot|_{x'})$ have the same image if and only if there exists $\sigma \in S_n$ such that $\sigma.\, x = x'$ and $|\cdot|_x \circ \sigma^{-1} = |\cdot|_{x'}$.

This implies that $f^{\mathrm{an}}$ factors uniquely through the quotient map $\pi : (X^n)^{\mathrm{an}} \to (X^n)^{\mathrm{an}}/S_n$. So we can draw the diagram below

$$\begin{tikzcd}
(X^n)^{\mathrm{an}} \arrow[r, two heads, "f^{\mathrm{an}}"] \arrow[dr, "\pi"'] & Y^{\mathrm{an}} & \\
& (X^n)^{\mathrm{an}}/S_n \arrow[u, "\sim"'] \arrow[r, "\sim"] & (X^{\mathrm{an}})^n/S_n
\end{tikzcd}$$

reminding that the analytification functor commutes with fibred products ([Ber93], Proposition 2.6.1), hence $(X^n)^{\mathrm{an}} \simeq (X^{\mathrm{an}})^n$.

### 6.3. **Representation of divisorial points of the quotient.**

**(6.3.1)** We keep the notation of the previous paragraph. Let $y$ be a divisorial point of $Y^{\mathrm{an}}$ and consider a regular snc $R$-model $\mathscr{Y}$ of $Y$ adapted to $y$, i.e. such that $y$ is the divisorial point associated to $(\mathscr{Y}, E)$ for some irreducible component $E$ of $\mathscr{Y}_k$. We denote by $\mathscr{X}$ the normalization of $\mathscr{Y}$ inside $K(X^n)$, where $K(\mathscr{Y}) = K(Y) = K(X^n)^{S_n} \hookrightarrow K(X^n)$.

**(6.3.2)** We check that $\mathscr{X}$ is an $R$-model of $X^n$; it is enough to show that the base change $\mathscr{X}_K$ is isomorphic to $X^n$. We consider the following commutative diagram

$$\begin{tikzcd}
\mathscr{X}_K \arrow[drr] \arrow[ddd] & & \\
& X^n \arrow[ul, dotted] \arrow[r, dotted] \arrow[d, "f"] & \mathscr{X} \arrow[d] \\
& Y \arrow[r] \arrow[d] & \mathscr{Y} \arrow[d] \\
& \operatorname{Spec} K \arrow[r] & S
\end{tikzcd}$$

As the $n$-fold product $X^n$ is a normal variety endowed with a morphism $X^n \to \mathscr{Y}$, by universal property of normalization, it factors uniquely through $\mathscr{X}$ and the diagram is still commutative. Then by universal property of fibred product, there exists a morphism $X^n \to \mathscr{X}_K$. Therefore, it suffices to prove that

$$[K(X^n) : K(\mathscr{X}_K)] = 1.$$

Indeed, if this is the case, then $X^n \to \mathscr{X}_K$ is a finite birational morphism between normal varietes, hence an isomorphism.

**(6.3.3)** The degree of the extension $[K(X^n) : K(\mathscr{X}_K)]$ may be computed on an affine open, so we assume that $\mathscr{Y}$ is an affine scheme with associated ring $K[\mathscr{Y}]$. Then we consider the diagram of inclusions

$$\begin{tikzcd}
 & K(\mathscr{X}) = K(\mathscr{X}_K) \arrow[dr] & \\
\widehat{K[\mathscr{Y}]} = K[\mathscr{X}] \arrow[ur] \arrow[rr] & & K(X^n) \\
K[\mathscr{Y}] \arrow[u] \arrow[rr] & & K(\mathscr{Y}) = K(Y) = K(X^n)^{S_n} \arrow[u, "\text{finite deg}"']
\end{tikzcd}$$

As $K(X^n)$ is finite field extension of $K(\mathscr{Y})$ and $K[\mathscr{X}]$ the integral closure of $K[\mathscr{Y}]$ in $K(X^n)$, then $K(X^n)$ is the fraction field of $K[\mathscr{X}]$. Thus, $K(\mathscr{X}) = \mathrm{Frac}(K[\mathscr{X}]) = K(X^n)$ and in particular we conclude that $[K(X^n) : K(\mathscr{X}_K)] = 1$.

**Remark 6.3.4.** This procedure of normalization illustrates a way to start with a regular snc $R$-model $\mathscr{Y}$ of $Y$ adapted to a point $y \in \mathrm{Div}(Y)$ and construct an $R$-model $\mathscr{X}$ of $X^n$ that, by normality, is regular at generic points of the special fibre $\mathscr{X}_k$.

### 6.4. **Weight function values along fibres of the quotient.**

**(6.4.1)** As before, let $y \in \mathrm{Div}(Y)$ and let $\mathscr{Y}$ be a regular snc $R$-model with divisorial representation $(\mathscr{Y}, E)$ of $y$. Let $\mathscr{X}$ be the normalization of $\mathscr{Y}$ in $K(X^n)$: as we observed in Remark 6.3.4, it is an $R$-model of $X^n$, regular at generic points of the special fibre $\mathscr{X}_k$.

The preimage of $E$ coincides with the pull-back of the Cartier divisor $E$ on $\mathscr{X}$, hence $f^{-1}(E)$ still defines a codimension one subset on $\mathscr{X}$. We denote by $F_i$ the irreducible components of $f^{-1}(E)$ and we associate to $F_i$'s their corresponding divisorial valuations $x_i = (\mathscr{X}, F_i)$.

**(6.4.2)** Let $\omega_X$ be a canonical form on $X$ and let $\mathrm{pr}_j : X^n \to X$ be the $j$-th canonical projection. We consider

$$\omega = \bigwedge_{1 \leqslant j \leqslant n} \mathrm{pr}_j^* \omega_X.$$

It is a canonical form on $X^n$ and moreover it is invariant under the action of $S_n$. Thus, $\omega$ induces a canonical form on the $n$-th symmetric product $Y$.

We compare the values at $y$ and $x_i$ of weight functions attached to $\omega$:

$$\mathrm{wt}_\omega(x_i) = v_{x_i}(\mathrm{div}_{\mathscr{X}^+}(\omega)) + 1$$

$$\mathrm{wt}_\omega(y) = v_y(\mathrm{div}_{\mathscr{Y}^+}(\omega)) + 1.$$

We recall that for log-étale morphisms the sheaves of logarithmic differentials are stable under pull-back ([Kat94], Proposition 3.12). Furthermore, it suffices to check that, locally around the generic point of $F_i$, the morphism $\mathscr{X}^+ \to \mathscr{Y}^+$ is a log-étale morphism of divisorial log structures, to conclude that the weights coincide. To this purpose, we will apply Kato's criterion for log étaleness ([Kat89], Theorem 3.5) to log schemes with respect to the étale topology.

**(6.4.3)** We denote by $\xi_{F_i}$ the generic point of $F_i$ and by $\xi_E$ the generic point of $E$. The divisorial log structures on $\mathscr{X}^+$ and $\mathscr{Y}^+$ have charts $\mathbb{N}$ at $\xi_{F_i}$ and $\xi_E$. In the étale topology, the normalization morphism $\mathscr{X}^+ \to \mathscr{Y}^+$ admits a chart induced by $t : \mathbb{N} \to \mathbb{N}$ where $1 \mapsto m$ for some positive integer $m$:

$$\begin{CD} \operatorname{Spec} \mathcal{O}_{\mathscr{X},\xi_{F_i}} @>>> \operatorname{Spec} \mathbb{Z}[\mathbb{N}] \\ @VVV @VVV \\ \operatorname{Spec} \mathcal{O}_{\mathscr{Y},\xi_E} @>>> \operatorname{Spec} \mathbb{Z}[\mathbb{N}] \end{CD}$$

Firstly, by the universal property of the fibre product, we have a morphism

$$\operatorname{Spec} \mathcal{O}_{\mathscr{X},\xi_{F_i}} \to \operatorname{Spec} \mathcal{O}_{\mathscr{Y},\xi_E} \times_{\operatorname{Spec} \mathbb{Z}[\mathbb{N}]} \operatorname{Spec} \mathbb{Z}[\mathbb{N}]$$

and it corresponds to

$$\mathcal{O}_{\mathscr{Y},\xi_E} \otimes_{\mathbb{Z}[\mathbb{N}]} \mathbb{Z}[\mathbb{N}] \to \mathcal{O}_{\mathscr{X},\xi_{F_i}}.$$

This is a morphism of finite type with finite fibres between regular rings and by [Liu02], Lemma 4.3.20 and [Now97] it is flat and unramified, hence étale. One of the two conditions in Kato's criterion for log étaleness is then fulfilled. Secondly, the chart $t : \mathbb{N} \mapsto \mathbb{N}$ induces a group homomorphism $t^{\mathrm{gp}} : \mathbb{Z} \mapsto \mathbb{Z}$; in particular, it is injective and it has finite cokernel. Then $t$ satisfies the second condition of Kato's criterion for log étaleness. Therefore we conclude that $\mathrm{wt}_\omega(y) = \mathrm{wt}_\omega(x_i)$.

**Remark 6.4.4.** Given a divisorial point $y$, this construction provides a divisorial point $x \in (X^n)^{\mathrm{an}}$ such that $f^{\mathrm{an}}(x) = y$ and with the property that they have the same weight with respect to an $S_n$-invariant canonical form $\omega$.

## 6.5. **Kontsevich-Soibelman skeleta of the quotient.**

**(6.5.1)** Let $\omega_X$ be a canonical form on $X$ and $\omega = \bigwedge_{1 \leqslant j \leqslant n} \mathrm{pr}_j^* \omega_X$ the induced $S_n$-invariant canonical form on $X^n$ that passes to the quotient $Y$. We claim that the minimal values of the weight functions $\mathrm{wt}_\omega$ on $X^n$ and $Y$ coincide, denoted by $\mathrm{wt}_\omega(X^n)$ and $\mathrm{wt}_\omega(Y)$ respectively. The key arguments to prove the claim are:

(1) the Kontsevich-Soibelman skeleton $\mathrm{Sk}(X^n, \omega)$ of the $n$-fold fibred product is invariant under the $S_n$-action, as a consequence of Theorem 4.4.3;
(2) given a divisorial point $y$, we may construct a divisorial point $x \in (X^n)^{\mathrm{an}}$ such that $f^{\mathrm{an}}(x) = y$ and $\mathrm{wt}_\omega(y) = \mathrm{wt}_\omega(x)$, as showed in Remark 6.4.4.

**(6.5.2)** By the argument in (2), the following inclusion is true

$$\{\mathrm{wt}_\omega(y) \,|\, y \in \mathrm{Div}(Y)\} \subseteq \{\mathrm{wt}_\omega(x) \,|\, x \in \mathrm{Div}(X^n)\},$$

so $\mathrm{wt}_\omega(X^n) = \inf\{\mathrm{wt}_\omega(x) \,|\, x \in \mathrm{Div}(X^n)\} \leqslant \inf\{\mathrm{wt}_\omega(y) \,|\, y \in \mathrm{Div}(Y)\} = \mathrm{wt}_\omega(Y)$.

Conversely, let $x \in \mathrm{Div}(X^n)$ such that $\mathrm{wt}_\omega(x) = \mathrm{wt}_\omega(X^n)$. Consider $y := f^{\mathrm{an}}(x)$; it is a divisorial valuation since it is induced by restriction of $v_x$ to $K(Y) \hookrightarrow K(X^n) \xrightarrow{v_x} \mathbb{R}$ and its image is contained in the discrete image of $v_x$ in $\mathbb{R}$, hence it is discrete too. Applying the construction of (2), we obtain $x' \in \mathrm{Div}(X^n)$ such that $f^{\mathrm{an}}(x') = y$ and $\mathrm{wt}_\omega(x') = \mathrm{wt}_\omega(y)$. This means that $x$ and $x'$ are in the same $S_n$-class; since the $S_n$-action preserves the Kontsevich-Soibelman skeleton $\mathrm{Sk}(X^n, \omega)$ by (1) and $x \in \mathrm{Sk}(X^n, \omega)$, thus $x' \in \mathrm{Sk}(X^n, \omega)$. Therefore

$$\mathrm{wt}_\omega(y) = \mathrm{wt}_\omega(x') = \mathrm{wt}_\omega(X^n),$$

so $\mathrm{wt}_\omega(Y) = \inf\{\mathrm{wt}_\omega(y) \,|\, y \in \mathrm{Div}(Y)\} \leqslant \mathrm{wt}_\omega(X^n)$. Finally, we have equality of weights of $X^n$ and $Y$ with respect to $\omega$:

$$\mathrm{wt}_\omega(X^n) = \mathrm{wt}_\omega(Y). \tag{6.5.3}$$

**(6.5.4)** The equality of minimal weights leads to the main results of this paragraph.

**Proposition 6.5.5.** *Let $X$ be a smooth $K$-variety and let $Y = X^n/S_n$ be the $n$-th symmetric product of $X$ with $f : X^n \to Y$. Let $\omega_X$ be a canonical form on $X$ and $\omega = \bigwedge_{1\leqslant j\leqslant n} \mathrm{pr}_j^*\omega_X$ the induced canonical form on $X^n$ and $Y$. Then the Kontsevich-Soibelman skeleton $\mathrm{Sk}(Y,\omega)$ is the image under $f^{\mathrm{an}}$ of the Kontsevich-Soibelman skeleton $\mathrm{Sk}(X^n,\omega)$.*

*Proof.* We characterize divisorial points in the Kontsevich-Soibelman skeleton $\mathrm{Sk}(Y,\omega)$ in term of their preimages in $X^{\mathrm{an}}$ as follows: given $y \in \mathrm{Div}(Y)$,

$$y \in \mathrm{Sk}(Y,\omega) \Leftrightarrow \text{ for some/any } x \in (f^{\mathrm{an}})^{-1}(y) \quad \mathrm{wt}_\omega(x) = \mathrm{wt}_\omega(X^n).$$

Indeed, if $y \in \mathrm{Sk}(Y,\omega)$, we may construct $x$ such that $f^{\mathrm{an}}(x) = y$ and $\mathrm{wt}_\omega(x) = \mathrm{wt}_\omega(y)$, i.e. a divisorial point $x \in (f^{\mathrm{an}})^{-1}(y)$ such that $\mathrm{wt}_\omega(x) = \mathrm{wt}_\omega(y) = \mathrm{wt}_\omega(Y) = \mathrm{wt}_\omega(X^n)$ by the equality (6.5.3). By argument (1), this holds for any point in the preimage of $y$.

Conversely, suppose that for all $x \in (f^{\mathrm{an}})^{-1}(y)$ we have $\mathrm{wt}_\omega(x) = \mathrm{wt}_\omega(X^n)$. Then, this holds in particular for a divisorial point $\tilde{x}$ constructed as in (2). Therefore $\mathrm{wt}_\omega(y) = \mathrm{wt}_\omega(\tilde{x}) = \mathrm{wt}_\omega(X^n) = \mathrm{wt}_\omega(Y)$ again by equation (6.5.3); this means that $y \in \mathrm{Sk}(Y,\omega)$. ☐

**Corollary 6.5.6.** *Assume that the residue field $k$ is algebraically closed. Let $X$ be a smooth $K$-variety and let $\omega_X$ be a canonical form on $X$. If $X$ has semistable reduction, then the Kontsevich-Soibelman skeleton of the $n$-th symmetric product of $X$ is isomorphic to the $n$-th symmetric product of the Kontsevich-Soibelman skeleton of $X$*

$$\mathrm{Sk}(X^n/S_n,\omega) \xrightarrow{\sim} S^n(\mathrm{Sk}(X,\omega_X)).$$

*Proof.* Iterating the result of Theorem 4.4.3, we have that the projection map defines an isomorphism of Kontsevich-Soibelman skeleta

$$\mathrm{Sk}(X^n,\omega) \xrightarrow{\sim} \mathrm{Sk}(X,\omega_X) \times \ldots \times \mathrm{Sk}(X,\omega_X).$$

Hence, by Proposition 6.5.5, the diagram

$$\begin{tikzcd}
(X^n)^{\mathrm{an}} \arrow[r, "f^{\mathrm{an}}"] & Y^{\mathrm{an}} \simeq (X^{\mathrm{an}})^n/S_n \\
\mathrm{Sk}(X^n,\omega) \arrow[r, mapsto] \arrow[d, equal] & \mathrm{Sk}(Y,\omega) \arrow[d, equal] \\
\mathrm{Sk}(X,\omega_X) \times \ldots \times \mathrm{Sk}(X,\omega_X) \arrow[r, mapsto] & S^n(\mathrm{Sk}(X^n,\omega))
\end{tikzcd}$$

gives a concrete description of the Kontsevich-Soibelman skeleta of the quotient $Y$ in terms of the Kontsevich-Soibelman skeleton of $X$ as required. ☐

## 6.6. Essential skeleton of the $n$-th symmetric product of a CY variety.

**(6.6.1)** These results on Kontsevich-Soibelman skeleta of the $n$-th symmetric products translate into properties of essential skeleta when we are dealing with Calabi-Yau varieties.

**Corollary 6.6.2.** *Let $X$ be a smooth Calabi-Yau variety over $K$. Assume that $X$ has semistable reduction and the residue field $k$ is algebraically closed. Then the essential skeleton of the $n$-th symmetric product of $X$ is isomorphic to the $n$-th symmetric product of the essential skeleton of $X$*

$$\mathrm{Sk}(X^n/S_n) \xrightarrow{\sim} S^n(\mathrm{Sk}(X)).$$

*Proof.* This follows immediately from Corollary 6.5.6. □

### 6.7. The essential skeleton of the Hilbert scheme of a K3 surface.

**(6.7.1)** Let $S$ be a K3 surface over $K$ (i.e. $S$ is a complete non-singular variety of dimension two such that $\Omega^2_{S/K} \simeq \mathcal{O}_S$ and $H^1(S, \mathcal{O}_S) = 0$). In particular $S$ is a Calabi-Yau variety.

We consider $\mathrm{Hilb}^n(S)$ the Hilbert scheme of $n$ points on $S$; a concrete way to construct it is by first taking the $n$-th symmetric product of $S$, and by then resolving its singularities:

$$\begin{tikzcd} & S^n \arrow[d, "f"] \\ \mathrm{Hilb}^n(S) \arrow[r, "\rho"] & S^n/S_n \end{tikzcd}$$

Indeed, the $n$-th symmetric product $S^n/S_n$ has quotient singularities along the images via $f$ of the loci

$$\Delta_{ij} = \{(x_1, \dots, x_n) \in S^n \mid x_i = x_j\},$$

which are precisely the fixed loci of the $S_n$-action on $S^n$. Then the morphism $\rho : \mathrm{Hilb}^n(S) \to S^n/S_n$ is a resolution of singularities and it can be seen explicitly as the map sending a zero-dimensional scheme $Z \subseteq S$ to its associated zero-cycle $\mathrm{supp}(Z)$. We refer to the morphism $\rho$ as the Hilbert-Chow morphism. It follows that the Hilbert scheme of $n$ points on $S$ is birational to the $n$-th symmetric product of $S$ [Fog68].

**(6.7.2)** We can finally illustrate the essential skeleton of the $n$-th Hilbert scheme of a K3 surface with semistable reduction.

**Theorem 6.7.3.** *Let $S$ be a K3 surface over $K$. Assume that $S$ has semistable reduction and the residue field $k$ is algebraically closed. Then the essential skeleton of the Hilbert scheme of $n$ points on $S$ is isomorphic to the $n$-th symmetric product of the essential skeleton of $S$*

$$\mathrm{Sk}(\mathrm{Hilb}^n(S)) \xrightarrow{\sim} S^n(\mathrm{Sk}(S)).$$

*Proof.* In [MN15], Proposition 4.6.3, Mustata and Nicaise proved that the essential skeleton of a variety is a birational invariant. Therefore, the description of the essential skeleton of the $n$-th symmetric product of $S$ in Corollary 6.6.2 entails a description of the essential skeleton of the Hilbert scheme of $n$ points on $S$. □