THE ESSENTIAL SKELETON OF A PRODUCT OF DEGENERATIONS

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ABSTRACT. We study the problem of how the dual complex of the special fiber of an snc degeneration $\mathscr{X}_R$ changes under products. We view the dual complex as a skeleton inside the Berkovich space associated to X_K. Using the Kato fan, we define a skeleton $\mathrm{Sk}(\mathscr{X}_R)$ when the model $\mathscr{X}_R$ is log-regular. We show that if $\mathscr{X}_R$ and $\mathscr{Y}_R$ are log-regular, and at least one is semistable, then $\mathrm{Sk}(\mathscr{X}_R \times_R \mathscr{Y}_R) \simeq \mathrm{Sk}(\mathscr{X}_R) \times \mathrm{Sk}(\mathscr{Y}_R)$.

The essential skeleton $\mathrm{Sk}(X_K)$, defined by Mustaţă and Nicaise, is a birational invariant of X_K and is independent of the choice of R-model. We extend their definition to pairs, and show that if both X_K and Y_K admit semistable models, $\mathrm{Sk}(X_K \times_K Y_K) \simeq \mathrm{Sk}(X_K) \times \mathrm{Sk}(Y_K)$.

As an application, we compute the homeomorphism type of the dual complex of some degenerations of hyper-Kähler varieties. We consider both the case of the Hilbert scheme of a semistable degeneration of K3 surfaces, and the generalized Kummer construction applied to a semistable degeneration of abelian surfaces. In both cases we find that the dual complex of the $2n$-dimensional degeneration is homeomorphic to either a point, n-simplex, or $\mathbb{CP}^n$, depending on the type of the degeneration.

1. INTRODUCTION

Let R be a discrete valuation ring with quotient field K and residue field k, and let X be a smooth proper variety over K. While there may be no way to extend X to a smooth proper variety over R, resolution of singularities guarantees that we can always produce an R model $\mathscr{X}$ where the special fiber $\mathscr{X}_k$ is a strict normal crossings (snc) divisor. Given such a model, we associate the dual complex $\mathcal{D}(\mathscr{X}_k)$, which is the intersection complex of the components of the special fiber.

The dual complex of the special fiber of a such degeneration reflects the geometry of the generic fiber. If the generic fiber is rationally connected, then the special fiber is contractible [dKX12]. For Calabi-Yau varieties, degenerations are classified by the action of monodromy on the cohomology. The principle is that the degenerations with maximally unipotent actions have the most rich combinatorial structure in the dual complex. In this case, the dual complex is always a $\mathbb{Q}$-homology sphere, and Kollár and Xu [KX16] show that it is a sphere if $n \leqslant 3$ or $n \leqslant 4$ and the special fiber is snc.

The goal of this work is to understand the dual complex of a model for the product of two smooth proper varieties over K. We consider this problem from two perspectives.

1.1. **Skeleta of Berkovich spaces.** The first is via the theory of Berkovich spaces. In this setting we assume that K is complete with respect to the valuation induced by R, which gives rise to a non-archimedean norm on K. In [Ber90], Berkovich develops a theory of analytic geometry over K. He associates a K-analytic space to X; each point corresponds to a real valuation on the residue field of a point of X, extending the discrete valuation on K. This space, denoted by X^{an}, is called the Berkovich space associated to X.

From any snc model $\mathscr{X}$ of X one can construct a subspace of X^{an}, called the Berkovich skeleton of $\mathscr{X}$ and denoted by $\mathrm{Sk}(\mathscr{X})$: it is homeomorphic to the dual intersection complex of the divisor $\mathscr{X}_k$ [MN15]. The Berkovich skeleta turn out to be relevant in the study of the topology of X^{an}. They shape the Berkovich space, as X^{an} is homeomorphic to the inverse limit $\varprojlim \mathrm{Sk}(\mathscr{X})$ where $\mathscr{X}$ runs through all snc

models of X. Also, the homotopy type of X^{an} is determined by any snc model $\mathscr{X}$: indeed, Berkovich and Thuillier prove that $\mathrm{Sk}(\mathscr{X})$ is a strong deformation retract of X^{an} [Ber90; Thu07].

1.2. **The dual complex of a dlt model.** The other approach to the study of the dual complexes comes from birational geometry. In this setting, we consider a pair $(\mathscr{X}, \Delta_{\mathscr{X}})$ over the germ of a curve. In the log general type case, running the minimal model program distinguishes a canonical model for the degeneration $(\mathscr{X}, \Delta_{\mathscr{X}})$ [Ale96; HMX16; KSB88], at the cost of possibly introducing worse singularities. If we are willing to tolerate some ambiguity in our choice of model, we can choose instead to produce a dlt minimal model. One advantage of dlt models is that they are expected to exist for all pairs admitting a log pluricanonical form. The singularities are mild enough that it is possible to define the dual complex as the intersection complex of divisors of coefficient 1, denoted $\mathcal{D}^{=1}(\mathscr{X}, \Delta_{\mathscr{X}})$. In [dKX12], de Fernex, Kollár, and Xu investigate how the dual complex is affected by the operations of the minimal model program. They show, under mild hypotheses, that every step of the MMP induces a homotopy equivalence between dual complexes. Moreover, $\mathcal{D}^{=1}(\mathscr{X}, \Delta_{\mathscr{X}})$ is a PL invariant under log crepant birational maps.

1.3. **The essential skeleton.** Recently there has been much interest in a synthesis of the two approaches. Kontsevich and Soibelmann [KS06] define a version of the skeleton of a K3 surface which detects the locus of simple poles along the special fiber of the distinguished canonical form. Mustaţă and Nicaise [MN15] extend their definition to any variety with non-negative Kodaira dimension. The key technical tool is the definition, for a rational pluricanonical form, of a weight function on the Berkovich space. The essential skeleton $\mathrm{Sk}(X)$ is the union over all regular pluricanonical forms of the minimality loci of the associated weight function.

Thus the essential skeleton has the advantage of being intrinsic to the variety X, with no dependence on a choice of model. As the weight function is closely related to the log discrepancy from birational geometry, it is natural to expect that the essential skeleton in some way encodes some of the minimal model theory of X. Nicaise and Xu [NX16] show, when X is a smooth projective variety with K_X semiample, and $\mathscr{X}$ is a good dlt minimal model, that the dual complex of $\mathscr{X}_k$ can be identified with the essential skeleton of X.

1.4. **Skeleta for log-regular models.** To produce nice models of the product, we work in the context of the logarithmic geometry. To any log-regular scheme $\mathscr{X}^+$, in [Kat94] Kato attaches a combinatorial structure $F_{\mathscr{X}}$ called a *fan*: if we denote by $D_{\mathscr{X}}$ the locus where the log structure is non-trivial, then the fan $F_{\mathscr{X}}$ consists of the set of the generic points of intersections of irreducible components of $D_{\mathscr{X}}$, equipped with a sheaf of monoids. We define a logarithmic version of the Berkovich skeleton for a log-regular model $\mathscr{X}^+$ of X over R: it gives rise to a polyhedral complex in X^{an} whose faces correspond to the point of $F_{\mathscr{X}}$.

Given two log-regular log schemes $\mathscr{X}^+$ and $\mathscr{Y}^+$ over R, their product $\mathscr{Z}^+$ in the category of fine and saturated log schemes is naturally log-regular, hence $\mathscr{Z}^+$ has an associated skeleton, and it is a model of the product $\mathscr{X}_K \times_K \mathscr{Y}_K$ of the generic fibers. If one of the two underlying schemes $\mathscr{X}$ or $\mathscr{Y}$ is semistable, which means it has reduced special fiber, then we show that the skeleton of the product $\mathscr{Z}^+$ is the product of the skeleta, with the projection maps given by restricting the valuation to the corresponding function fields (Proposition 3.4.3).

1.5. **Skeleta for pairs.** Working in the logarithmic setting, we may also allow a non-trivial log structure over the generic fiber. Geometrically this corresponds to adding horizontal divisors to the special fiber and yields to the addition of some unbounded faces in the skeleton. In [GRW16] Gubler, Rabinoff, and Werner construct a skeleton for strictly semistable snc models with suitable horizontal divisors. Both the constructions recover the Berkovich skeleton when there is no horizontal component and the special fiber is snc.

Pairs arise frequently in the minimal model program. Taking advantage of a construction that admits horizontal components, the definition of essential skeleton extends to the case of a pair (X, Δ_X) over K, where Δ_X has $\mathbb{Q}$-coefficients in $[0, 1]$, the support of Δ_X is snc, and to pluricanonical forms of some positive index r with divisor of poles no worse than $r\Delta_X$. We extend to pairs the result of Mustață and Nicaise [MN15] on the birational invariance of the essential skeleton (Proposition 5.1.5), as well as Nicaise and Xu's result [NX16] that the essential skeleton is homeomorphic to the dual complex of a good minimal dlt model (Proposition 5.1.7). It follows from these results that we can define the notion of essential skeleton for a dlt pair.

1.6. **Main Result.** Our main result establishes the behavior of essential skeleta under products.

Theorem 1.6.1. *Assume that the residue field k is algebraically closed. Let (X, Δ_X) and (Y, Δ_Y) be pairs that induce log-regular structures. Suppose that both pair have non-negative Kodaira-Iitaka dimension and admit semistable log-regular models $\mathscr{X}^+$ and $\mathscr{Y}^+$ over S^+. Then, the PL homeomorphism of skeleta induces a PL homeomorphism of essential skeleta*

$$\mathrm{Sk}(Z, \Delta_Z) \xrightarrow{\sim} \mathrm{Sk}(X, \Delta_X) \times \mathrm{Sk}(Y, \Delta_Y)$$

where Z and Δ_Z are the respective products.

Semistability is a key assumption; without it the projection map might fail to be injective, see Example 3.4.4. As expected, we get a corresponding result for dual complexes of semistable good dlt minimal models (Theorem 5.3.5). Unfortunately semistability is not well behaved under birational transformations so it seems possible that a degeneration admits a semistable good dlt minimal model but no semistable log-regular model.

1.7. **Application to degenerations of hyper-Kähler varieties.** As an application of Theorem 1.6.1, we study certain degenerations of hyper-Kähler varieties. One way to produce hyper-Kähler varieties is by taking the Hilbert scheme of points on a K3 surface. Another is to extend the Kummer construction to higher dimensional abelian varieties. Aside from two other examples found by O'Grady in dimensions 6 [O'G03] and 10 [O'G99] there are no other known examples, up to deformation equivalence.

Just as for Calabi-Yau varieties, degenerations of hyper-Kähler varieties can be understood in terms of the monodromy operator on cohomology, with classification into types I, II, III. Type I is the case where the dual complex is just a single point, but types II and III have more interesting structure. Kollár, Laza, Saccà and Voisin [KLSV17] show that in the type II case the dual complex is contractible, and in the type III case it has the rational homology type of $\mathbb{CP}^n$. Gulbrandsen, Halle, and Hulek [GHH15] use GIT to construct a model of the degeneration of nth order Hilbert schemes arising from some type II degenerations of K3 surfaces, and show that the dual complex is an n-simplex. There are considerations from mirror symmetry that suggest that for a type III degeneration the dual complex should be homeomorphic to $\mathbb{CP}^n$ [Hwa08; KS06].

Theorem 1.7.1. *Assume that the residue field k is algebraically closed. Let S be a K3 surface over K. If S admits a semistable log-regular model or a semistable good dlt minimal model, then the essential skeleton of the Hilbert scheme of n points on S is isomorphic to the n-th symmetric product of the essential skeleton of S*

$$\mathrm{Sk}(\mathrm{Hilb}^n(S)) \xrightarrow{\sim} \mathrm{Sym}^n(\mathrm{Sk}(S)).$$

Computing these complexes gives a single point in the type I case, an n-simplex in the type II case, and $\mathbb{CP}^n$ in the type III case. The same types arise in the Kummer case.

Theorem 1.7.2. *Assume the residue field k is algebraically closed. Let A be an abelian surface over K. Suppose that A admits a semistable log-regular model or a semistable good dlt minimal model. If the*

essential skeleton of A is homeomorphic to a point, the circle S^1 or the torus $S^1 \times S^1$, then the essential skeleton of the n-th generalised Kummer variety $K_n(A)$ is isomorphic to a point, the standard n-simplex or $\mathbb{CP}^n$ respectively.

Analysis of the weight function gives a powerful yet accessible approach to controlling the skeleta of these varieties. In both cases we use Theorem 1.6.1 to establish that the skeleton of the hyper-Kähler variety is a finite quotient of the n-fold product of the skeleton of the original surface under the action of a symmetric group. In the case of Hilbert schemes we can get a complete description of the action using functoriality of the projection maps, but in the Kummer case we additionally need to understand the restriction of the multiplication map to the essential skeleton of an abelian surface ([Ber90; HN17; Tem16]).

To our knowledge these are the first examples of type III degenerations of hyper-Kähler varieties where the PL homeomorphism type of the dual complex is known.

1.8. **Structure of the paper.** In section 2 we recall the Kato fan, and develop necessary background for the rest of the paper. Section 3 contains the definition and properties of the skeleton of a log regular scheme. These first two sections require no hypothesis on the characteristic of K.

For the remainder of the paper, we assume that K has equicharacteristic 0. In section 4 we introduce the weight function and essential skeleton from [MN15], extending their definition to the case of a pair. In section 5 we give connections to birational geometry, along with proofs of the main theorems. We discuss an application to degenerations of hyper-Kähler varieties in section 6.

1.9. **Notation.**

(1.9.1) Let R be a complete discrete valuation ring with maximal ideal $\mathfrak{m}$, residue field $k = R/\mathfrak{m}$ and quotient field K. We assume that the valuation v_K is normalized, namely $v_K(\pi) = 1$ for any uniformizer π of R. We define by $|\cdot|_K = \exp(-v_K(\cdot))$ the absolute value on K corresponding to v_K; this turns K into a non-archimedean complete valued field.

(1.9.2) We write $S = \operatorname{Spec} R$ and we denote by s the closed point of S. Let $\mathscr{X}$ be an R-scheme of finite type. We will denote by $\mathscr{X}_k$ the special fiber of $\mathscr{X}$ and by $\mathscr{X}_K$ the generic fiber.

(1.9.3) Let X be a proper K-scheme. A model for X over R is a flat separated R-scheme $\mathscr{X}$ of finite type endowed with an isomorphism of K-schemes $\mathscr{X}_K \to X$. If X is smooth over K, we say that $\mathscr{X}$ is an snc model for X if it is proper, regular over R, and the special fiber $\mathscr{X}_k$ is a strict normal crossings divisor on $\mathscr{X}$. In equicharacteristic 0, such a model always exists, by Hironaka's resolution of singularities.

We say that a model $\mathscr{X}$ over R is semistable if the special fiber $\mathscr{X}_k$ is reduced.

(1.9.4) All log schemes in this paper are fine and saturated (*fs*) log schemes and defined with respect to the Zariski topology. We denote a log scheme by $\mathscr{X}^+ = (\mathscr{X}, \mathcal{M}_\mathscr{X})$, where $\mathcal{M}_\mathscr{X}$ is the structural sheaf of monoids. We denote by

$$\mathcal{C}_\mathscr{X} = \mathcal{M}_\mathscr{X}/\mathcal{O}_\mathscr{X}^\times$$

the characteristic sheaf of $\mathscr{X}^+$. The sheaf $\mathcal{C}_\mathscr{X}$ is a Zariski sheaf on $\mathscr{X}^+$, supported on $\mathscr{X}_k$; if $\mathscr{X}^+$ is log-regular, then $\mathcal{C}_\mathscr{X}$ is a constructible sheaf. For every point x of $\mathscr{X}_k$, we denote by $\mathcal{I}_{\mathscr{X},x}$ the ideal in $\mathcal{O}_{\mathscr{X},x}$ generated by

$$\mathcal{M}_{\mathscr{X},x} \setminus \mathcal{O}_{\mathscr{X},x}^\times.$$

If a log scheme $\mathscr{X}^+$ has divisorial log structure induced from a divisor D, we denote it by $\mathscr{X}^+ = (\mathscr{X}, D)$.

We denote by S^+ the scheme S endowed with the standard log structure (the divisorial log structure induced by s), namely $S^+ = (S, s)$. If an R-scheme $\mathscr{X}$ is given, we will always denote by $\mathscr{X}^+$ the log scheme over S^+ that we obtain by endowing $\mathscr{X}$ with the divisorial log structure associated with $\mathscr{X}_k$.

(1.9.5) A log scheme is log-regular at a point x if the following two conditions are satisfied: $\mathcal{O}_{\mathscr{X},x}/\mathcal{I}_{\mathscr{X},x}$ is a regular local ring, and $\dim \mathcal{O}_{\mathscr{X},x} = \dim \mathcal{O}_{\mathscr{X},x}/\mathcal{I}_{\mathscr{X},x} + \operatorname{rank} \mathcal{C}^{\mathrm{gp}}_{\mathscr{X},x}$. For example, a toric variety with its toric logarithmic structure is log-regular. More generally, working over perfect fields, toroidal embeddings can be identified as log-regular varieties.

If $\mathscr{X}^+$ is a log-regular log scheme over S^+, then the locus where the log structure is non-trivial is a divisor that we will denote by $D_{\mathscr{X}}$. Thus, the log structure on $\mathscr{X}^+$ is the divisorial log structure induced by $D_{\mathscr{X}}$, by [Kat94], Theorem 11.6.

(1.9.6) Let (X, Δ_X) be a pair where X is a proper K-scheme, Δ_X is an effective $\mathbb{Q}$-divisor such that $\Delta_X = \sum a_i \Delta_{X,i}$ with $0 \leqslant a_i \leqslant 1$, and $X^+ = (X, \lceil \Delta_X \rceil)$ is a log-regular log scheme over K. A log-regular log scheme $\mathscr{X}^+$ over S^+ is a model for $(X, \lceil \Delta_X \rceil)$ over S^+ if $\mathscr{X}$ is a model of X over R, the closure of any component of Δ_X in $\mathscr{X}$ has non-empty intersection with $\mathscr{X}_k$, and $D_{\mathscr{X}} = \overline{\lceil \Delta_X \rceil} + \mathscr{X}_{k,\mathrm{red}}$.

(1.9.7) We denote by $(\cdot)^{\mathrm{an}}$ the analytification functor from the category of K-schemes of finite type to Berkovich's category of K-analytic spaces. For every K-scheme of finite type X, as a set, X^{an} consists of the pairs $x = (\xi_x, |\cdot|_x)$ where ξ_x is a point of X and $|\cdot|_x$ is an absolute value on the residue field $\kappa(\xi_x)$ of X at ξ_x extending the absolute value $|\cdot|_K$ on K. We endow X^{an} with the Berkovich topology, i.e. the weakest one such that

  (i) the forgetful map $\phi : X^{\mathrm{an}} \to X$, defined as $(\xi_x, |\cdot|_x) \mapsto \xi_x$, is continuous,
  (ii) for any Zariski open subset U of X and any regular function f on U the map $|f| : \phi^{-1}(U) \to \mathbb{R}$ defined by $|f|(\xi_x, |\cdot|_x) = |f(\xi_x)|$ is continuous.

The set $\mathrm{Bir}(X)$ of birational points of X^{an} is defined as the inverse image under ϕ of the generic point of X. By definition, it is a birational invariant of X.

2. The Kato fan of a log-regular log scheme

2.1. Definition of Kato fans.

(2.1.1) According to [Kat94], Definition 9.1, a monoidal space $(T, \mathcal{M}_T)$ is a topological space T endowed with a sharp sheaf of monoids $\mathcal{M}_T$, where *sharp* means that $\mathcal{M}^{\times}_{T,t} = \{1\}$ for every $t \in T$. We often simply denote the monoidal space by T.

A morphism of monoidal spaces is a pair $(f, \varphi) : (T, \mathcal{M}_T) \to (T', \mathcal{M}_{T'})$ such that $f : T \to T'$ is a continuous function of topological spaces and $\varphi : f^{-1}(\mathcal{M}_T) \to \mathcal{M}_{T'}$ is a sheaf homomorphism such that $\varphi_t^{-1}(\{1\}) = \{1\}$ for every $t \in T$.

Example 2.1.2. If $\mathscr{X}^+$ is a log scheme then the Zariski topological space $\mathscr{X}$ is equipped with a sheaf of sharp monoids $\mathcal{C}_{\mathscr{X}}$, namely the characteristic sheaf of $\mathscr{X}^+$. Thus $(\mathscr{X}, \mathcal{C}_{\mathscr{X}})$ is a monoidal space. Moreover, morphisms of log schemes induce morphisms of characteristic sheaves, hence morphism of monoidal spaces. We therefore obtain a functor from the category of log schemes to the category of monoidal spaces.

Example 2.1.3. Given a monoid P, we may associate to it a monoidal space called the spectrum of P. As a set, $\operatorname{Spec} P$ is the set of all prime ideals of P. The topology is characterized by the basis open sets $D(f) = \{\mathfrak{p} \in \operatorname{Spec} P | f \notin \mathfrak{p}\}$ for any $f \in P$. The monoidal sheaf is defined by

$$\mathcal{M}_{\operatorname{Spec} P}(D(f)) = S^{-1}P/(S^{-1}P)^{\times}$$

where $S = \{f^n | n \geqslant 0\}$.

(2.1.4) A monoidal space isomorphic to the monoidal space $\operatorname{Spec} P$ for some monoid P is called an affine Kato fan. A monoidal space is called a Kato fan if it has an open covering consisting of affine Kato fans.

In particular, we call a Kato fan integral, saturated, of finite type or fs if it admits a cover by the spectra of monoids with the respective properties.

(2.1.5) A morphism of fs Kato fans $F' \to F$ is called a *subdivision* if it has finite fibers and the morphism

$$\mathrm{Hom}(\mathrm{Spec}\,\mathbb{N}, F') \to \mathrm{Hom}(\mathrm{Spec}\,\mathbb{N}, F)$$

is a bijection. By allowing subdivisions, a Kato fan might take the following shape.

Proposition 2.1.6. *([Kat94], Proposition 9.8) Let F be a fs Kato fan. Then there is a subdivision $F' \to F$ such that F' has an open cover $\{U_i'\}$ by Kato cones with $U_i' \simeq \mathrm{Spec}\,\mathbb{N}^{r_i}$.*

The strategy of the proof of Proposition 2.1.6 goes back to [KKMSD73] and relies on a sequence of particular subdivisions of the Kato fan, the so-called star and barycentric subdivisions ([ACMUW15], Example 4.10).

2.2. Kato fans associated to log-regular log schemes.

Theorem 2.2.1. *([Kat94], Proposition 10.2) Let $\mathscr{X}^+$ be a log-regular log scheme. Then there is an initial strict morphism $(\mathscr{X}, \mathcal{C}_{\mathscr{X}}) \to F$ to a Kato fan in the category of monoidal spaces. Explicitly, there exist a Kato fan F and a morphism $\varrho : (\mathscr{X}, \mathcal{C}_{\mathscr{X}}) \to F$ such that $\varrho^{-1}(\mathcal{M}_F) \simeq \mathcal{C}_{\mathscr{X}}$ and any other morphism to a Kato fan factors through ϱ.*

The Kato fan F in Theorem 2.2.1 is called the Kato fan associated to $\mathscr{X}^+$; concretely, it is the topological subspace of $\mathscr{X}$ consisting of the points x such that the maximal ideal $\mathfrak{m}_x$ of $\mathcal{O}_{\mathscr{X},x}$ is equal to $\mathcal{I}_{\mathscr{X},x}$, and $\mathcal{M}_F$ is the inverse image of $\mathcal{C}_{\mathscr{X}}$ on F, henceforth we write $\mathcal{C}_F$ for $\mathcal{M}_F$.

Example 2.2.2. Assume that $\mathscr{X}$ is regular, of finite type over S and $\mathscr{X}_k$ is a divisor with strict normal crossings. Then $\mathscr{X}^+$ is log-regular and F is the set of generic points of intersections of irreducible components of $\mathscr{X}_k$. For each point x of F, the stalk of $\mathcal{C}_F$ is isomorphic to $(\mathbb{N}^r, +)$, with r the number of irreducible components of $\mathscr{X}_k$ that pass through x.

This example admits the following partial generalisation.

Lemma 2.2.3. *Let $\mathscr{X}^+$ be a log-regular log scheme. Then the fan F consists of the generic points of intersections of irreducible components of $D_{\mathscr{X}}$.*

Proof. First, we show that every such generic point is a point of F. Let $E_1, \ldots, E_r$ be irreducible components of $D_{\mathscr{X}}$ and let x be a generic point of the intersection $E_1 \cap \ldots \cap E_r$. We set $d = \dim \mathcal{O}_{\mathscr{X},x}$. Since $\mathscr{X}^+$ is log-regular, we know that $\mathcal{O}_{\mathscr{X},x}/\mathcal{I}_{\mathscr{X},x}$ is regular and that

(2.2.4) $$d = \dim \mathcal{O}_{\mathscr{X},x}/\mathcal{I}_{\mathscr{X},x} + \mathrm{rank}\,\mathcal{C}^{\mathrm{gp}}_{\mathscr{X},x}.$$

We denote by $V(\mathcal{I}_{\mathscr{X},x})$ the vanishing locus of the ideal $\mathcal{I}_{\mathscr{X},x}$ in $\mathscr{X}$. We want to prove that $\mathcal{I}_{\mathscr{X},x} = \mathfrak{m}_x$. We assume the contrary, hence that $\mathcal{I}_{\mathscr{X},x} \subsetneq \mathfrak{m}_x$. This assumption implies that there exists j such that $V(\mathcal{I}_{\mathscr{X},x}) \nsubseteq E_j$: indeed, if the vanishing locus is contained in each irreducible component E_i, i.e.

$$V(\mathcal{I}_{\mathscr{X},x}) \subseteq E_1 \cap \ldots \cap E_r \subseteq \overline{\{x\}},$$

then $\mathcal{I}_{\mathscr{X},x} \supseteq \mathfrak{m}_x$. From the assumption of log-regularity it follows that the vanishing locus $V(\mathcal{I}_{\mathscr{X},x})$ is a regular subscheme, and moreover that $\mathscr{X}^+$ is Cohen-Macaulay by [Kat94], Theorem 4.1. Thus, there exists a regular sequence $(f_1, \ldots, f_l)$ in $\mathcal{I}_{\mathscr{X},x}$, where l is the codimension of $V(\mathcal{I}_{\mathscr{X},x})$, i.e.

$$\dim \mathcal{O}_{\mathscr{X},x}/\mathcal{I}_{\mathscr{X},x} = d - l.$$

Moreover by the equality (2.2.4), $\mathrm{rank}\,\mathcal{C}^{\mathrm{gp}}_{\mathscr{X},x} = l$.

We claim that the residue classes of these elements f_i in $\mathcal{C}^{\mathrm{gp}}_{\mathscr{X},x}$ are linearly independent. Assume the contrary. Then, up to renumbering the f_i, there exist an integer e with $1 < e < l$, non-negative integers $a_1, \ldots, a_l$, not all zero, and a unit u in $\mathcal{O}_{\mathscr{X},x}$ such that

$$ f_1^{a_1} \cdot \ldots \cdot f_{e-1}^{a_{e-1}} = u \cdot f_e^{a_e} \cdot \ldots \cdot f_l^{a_l}. $$

This contradicts the fact that $(f_1, \ldots, f_l)$ is a regular sequence in $\mathcal{I}_{\mathscr{X},x}$. Thus, the classes $\overline{f_1}, \ldots, \overline{f_l}$ are independent in $\mathcal{C}^{\mathrm{gp}}_{\mathscr{X},x}$. As we also have the equality $\mathrm{rank}\, \mathcal{C}^{\mathrm{gp}}_{\mathscr{X},x} = l$, it follows that these classes generate $\mathcal{C}^{\mathrm{gp}}_{\mathscr{X},x} \otimes_{\mathbb{Z}} \mathbb{Q}$.

Let g_j be a non-zero element of the ideal $\mathcal{I}_{\mathscr{X},x}$ that vanishes along E_j: it necessarily exists as otherwise E_j is not a component of the divisor $D_{\mathscr{X}}$. Then g_j satisfies

$$ g_j^N = v \cdot f_1^{b_1} \cdot \ldots \cdot f_l^{b_l} $$

with $b_i \in \mathbb{Z}$, v is a unit in $\mathcal{O}_{\mathscr{X},x}$ and N is a positive integer. As g_j vanishes along the irreducible component E_j, at least one of the functions $f_1, \ldots, f_l$ has to vanish along E_j: assume that is f_1.

On the one hand, as f_1 is identically zero on E_j, the trace of E_j on $V(\mathcal{I}_{\mathscr{X},x})$ has at most codimension $l - 1$ in E_j at the point x. On the other hand, we assumed that $V(\mathcal{I}_{\mathscr{X},x})$ is not contained in E_j and it has codimension l in $\mathcal{O}_{\mathscr{X},x}$. Then, the trace of E_j on $V(\mathcal{I}_{\mathscr{X},x})$ has codimension l in E_j at x. This is a contradiction. We conclude that the ideal $\mathcal{I}_{\mathscr{X},x}$ is equal to the maximal ideal $\mathfrak{m}_x$, therefore x is a point of F.

It remains to prove the converse implication: every point x of the fan F is a generic point of an intersection of irreducible components of $D_{\mathscr{X}}$. Let x be a point of F: by construction of Kato fan F, the maximal ideal of $\mathcal{O}_{\mathscr{X},x}$ is equal to $\mathcal{I}_{\mathscr{X},x}$, thus it is generated by elements in $\mathcal{M}_{\mathscr{X},x}$. The zero locus of such an element is contained in $D_{\mathscr{X}}$ by definition of the logarithmic structure on $\mathscr{X}^+$. Therefore, the zero locus of a generator of $\mathfrak{m}_x$ in $\mathcal{M}_{\mathscr{X},x}$ is a union of irreducible components of the trace of $D_{\mathscr{X}}$ on $\mathrm{Spec}\, \mathcal{O}_{\mathscr{X},x}$ and x is a generic point of the intersection of all such irreducible components. $\qquad\square$

Remark 2.2.5. By convention, the generic point of the empty intersection of irreducible components is the generic point of $\mathscr{X}$. By definition, this point is also included in the Kato fan F. Thus, for example, the Kato fan associated to S^+ consists of two points: the generic point of S that corresponds to the empty intersection, and the closed point s corresponding to the unique irreducible component of the logarithmic divisorial structure.

Moreover, the example 2.2.2 also leads to the following characterization.

Proposition 2.2.6. *([GR04], Corollary 12.5.35) Let $\mathscr{X}^+$ be a log-regular log scheme over S^+ and F its associated Kato fan. The following are equivalent:*

  (1) for every $x \in F$, $M_{F,x} \simeq \mathbb{N}^{r(x)}$,
  (2) the underlying scheme $\mathscr{X}$ is regular.

If this is the case, then the special fiber $\mathscr{X}_k$ is a strict normal crossing divisor.

(2.2.7) The construction of the Kato fan of a log scheme defines a functor from the category of log-regular log schemes to the category of Kato fans. Indeed, given a morphism of log schemes $\mathscr{X}^+ \to \mathscr{Y}^+$, we consider the embedding of the associated Kato fan $F_{\mathscr{X}}$ in $\mathscr{X}^+$ and the canonical morphism $\mathscr{Y}^+ \to F_{\mathscr{Y}}$: the composition

$$ F_{\mathscr{X}} \hookrightarrow \mathscr{X}^+ \to \mathscr{Y}^+ \to F_{\mathscr{Y}} $$

functorially induces a map between associated Kato fans. Moreover, this association preserves strict morphisms ([Uli13], Lemma 4.9).

2.3. **Resolutions of log schemes via Kato fan subdivisions.**

Proposition 2.3.1. *([Kat94], Proposition 9.9) Let $\mathscr{X}^+$ be a log-regular log scheme and let F be its associated Kato fan. Let $F' \to F$ be a subdivision of fans. Then there exist a log scheme $\mathscr{X}'^+$, a morphism of log schemes $\mathscr{X}'^+ \to \mathscr{X}^+$ and a commutative diagram*

$$
\begin{array}{ccc}
(\mathscr{X}', \mathcal{C}_{\mathscr{X}'}) & \xrightarrow{\;p\;} & F' \\
\downarrow & & \downarrow \\
(\mathscr{X}, \mathcal{C}_{\mathscr{X}}) & \xrightarrow{\;\pi_{\mathscr{X}}\;} & F
\end{array}
$$

such that $p^{-1}(\mathcal{M}_{F'}) \simeq \mathcal{C}_{\mathscr{X}'}$; they define a final object in the category of such diagrams and the refinement $F' \to F$ is induced by the morphism of log-regular log schemes $\mathscr{X}'^+ \to \mathscr{X}^+$.

(2.3.2) It follows that given any subdivision $F' \to F$ of the Kato fan F associated with a log-regular log scheme $\mathscr{X}^+$, we can construct a log scheme over $\mathscr{X}^+$ with prescribed associated Kato fan F'. Combining this fact with Proposition 2.1.6 and Proposition 2.2.6 yields to the construction of resolutions of log schemes in the following sense: for any log-regular log scheme over S^+ we can find a birational modification by a regular log scheme with strict normal crossings special fiber. Moreover, the morphism of log schemes $\mathscr{X}'^+ \to \mathscr{X}^+$ is obtained by a log blow-up ([Niz06], Theorem 5.8).

2.4. **Fibred products and associated Kato fans.**

(2.4.1) Given morphisms of fs log schemes $f_1 : \mathscr{X}_1^+ \to \mathscr{Y}^+$ and $f_2 : \mathscr{X}_2^+ \to \mathscr{Y}^+$, their fibred product exists in the category of log schemes. It is obtained by endowing the usual fibred product of schemes

(2.4.2)
$$
\begin{array}{ccc}
\mathscr{X}_1 \times_{\mathscr{Y}} \mathscr{X}_2 & \xrightarrow{\;p_1\;} & \mathscr{X}_1 \\
\Big\downarrow{\scriptstyle p_2} & \searrow{\scriptstyle p_{\mathscr{Y}}} & \Big\downarrow{\scriptstyle f_1} \\
\mathscr{X}_2 & \xrightarrow{\;f_2\;} & \mathscr{Y}
\end{array}
$$

with the log structure associated to $p_1^{-1}\mathcal{M}_{\mathscr{X}_1} \oplus_{p_{\mathscr{Y}}^{-1}\mathcal{M}_{\mathscr{Y}}} p_2^{-1}\mathcal{M}_{\mathscr{X}_2}$. If $u_1 : P \to Q_1$ and $u_2 : P \to Q_2$ are charts for the morphisms f_1 and f_2 respectively, then the induced morphism $\mathscr{X}_1 \times_{\mathscr{Y}} \mathscr{X}_2 \to \operatorname{Spec} \mathbb{Z}[Q_1 \oplus_P Q_2]$ is a chart for $\mathscr{X}_1^+ \times_{\mathscr{Y}^+} \mathscr{X}_2^+$.

(2.4.3) In general, the fibred product is not fs, but the category of fs log schemes also admits fibred products. Keeping the same notations, the following is a chart of the fibred product in the category of fine and saturated log schemes

$$
\mathscr{X}_1^+ \times_{\mathscr{Y}^+}^{\mathrm{fs}} \mathscr{X}_2^+ = (\mathscr{X}_1^+ \times_{\mathscr{Y}^+} \mathscr{X}_2^+) \times_{\mathbb{Z}[Q_1 \oplus_P Q_2]} \mathbb{Z}[(Q_1 \oplus_P Q_2)^{\mathrm{sat}}]
$$

([Bul15], 3.6.16). We remark that the two fibred products above may not only have different log structures, but also the underlying schemes may differ. Nevertheless, this obviously does not occur when the monoid $Q_1 \oplus_P Q_2$ is saturated.

(2.4.4) Log-smoothness is preserved under fs base change and composition ([GR04], Proposition 12.3.24). In particular, if $f_1 : \mathscr{X}_1^+ \to \mathscr{Y}^+$ is log-smooth and $\mathscr{X}_2^+$ is log-regular, then $\mathscr{X}_1^+ \times_{\mathscr{Y}^+}^{\mathrm{fs}} \mathscr{X}_2^+$ is log-regular, by [Kat94], Theorem 8.2.

Consider log-smooth morphisms of fs log schemes $\mathscr{X}_1^+ \to \mathscr{Y}^+$ and $\mathscr{X}_2^+ \to \mathscr{Y}^+$. The sheaves of logarithmic differentials are related by the following isomorphism

(2.4.5)
$$
p_1^* \Omega^{\log}_{\mathscr{X}_1^+/\mathscr{Y}^+} \oplus p_2^* \Omega^{\log}_{\mathscr{X}_2^+/\mathscr{Y}^+} \simeq \Omega^{\log}_{\mathscr{X}_1^+ \times_{\mathscr{Y}^+}^{\mathrm{fs}} \mathscr{X}_2^+/\mathscr{Y}^+}
$$

by [GR04], Proposition 12.3.13. Furthermore, by assumption of log-smoothness over S^+ the logarithmic differential sheaves are locally free of finite rank ([Kat94], Proposition 3.10) and we can consider their determinants; they are called log canonical bundles and denoted by $\omega^{\log}$. The following isomorphism is a direct consequence of (2.4.5)

$$(2.4.6) \qquad p_1^* \omega^{\log}_{\mathscr{X}_1^+/\mathscr{Y}^+} \otimes p_2^* \omega^{\log}_{\mathscr{X}_2^+/\mathscr{Y}^+} \simeq \omega^{\log}_{\mathscr{X}_1^+ \times^{\mathrm{fs}}_{\mathscr{Y}^+} \mathscr{X}_2^+/\mathscr{Y}^+}.$$

(2.4.7) Similarly to the construction of fibred products of fs log schemes, the category of fs Kato fans admits fibred products: on affine Kato fans $F = \operatorname{Spec} P$ and $G = \operatorname{Spec} Q$ over $H = \operatorname{Spec} T$, $F \times_H G$ is the spectrum of the amalgamated sum $(P \oplus_T Q)^{\mathrm{sat}}$ in the category of fs monoids ([Uli16], Proposition 2.4) and on the underlying topological spaces, this coincides with the usual fibred product.

We seek to compare the Kato fan associated to the fibred product of log-regular log schemes with the fibred product of associated Kato fans.

Proposition 2.4.8. *([Sai04], Lemma 2.8) Given $\mathscr{T}^+$ a log-regular log scheme, let $\mathscr{X}^+$ and $\mathscr{Y}^+$ be log-smooth log schemes over $\mathscr{T}^+$. We denote by $\mathscr{Z}^+$ the fs fibred product $\mathscr{X}^+ \times^{\mathrm{fs}}_{\mathscr{T}^+} \mathscr{Y}^+$. Then, the natural morphisms $F_{\mathscr{Z}} \to F_{\mathscr{X}}$ and $F_{\mathscr{Z}} \to F_{\mathscr{Y}}$ induce a morphism of Kato fans*

$$(2.4.9) \qquad\qquad F_{\mathscr{Z}} \to F_{\mathscr{X}} \times_{F_{\mathscr{T}}} F_{\mathscr{Y}}$$

that is locally an isomorphism.

(2.4.10) For any pair of points (x, y) in $F_{\mathscr{X}} \times_{F_{\mathscr{T}}} F_{\mathscr{Y}}$, we denote by $n(x, y)$ the number of preimages of (x, y) in the Kato fan of $\mathscr{Z}^+$ under the local isomorphism (2.4.9).

Lemma 2.4.11. *If x' is in the closure of x, and y' in the closure of y, then $n(x', y') \geqslant n(x, y)$.*

Proof. Let z' be a preimage of the pair (x', y'). By Proposition 2.4.8, there exists an open neighbourhood $U_{z'}$ of z' such that the restriction of $F_{\mathscr{Z}} \to F_{\mathscr{X}} \times_{F_{\mathscr{T}}} F_{\mathscr{Y}}$ to $U_{z'}$ is an isomorphism onto its image. In particular, (x, y) lies in this image. Thus, there exists a unique preimage of (x, y) that is contained in $U_{z'}$. It follows that $n(x', y') \geqslant n(x, y)$. $\qquad\square$

2.5. **Semistability and Kato fans associated to the fibred products.**

(2.5.1) We recall that a log-regular log scheme $\mathscr{X}^+$ is said to be semistable if the special fiber is reduced. We will see that semistability is a sufficient condition to establish injectivitiy of the local isomorphism (2.4.9).

(2.5.2) We recall that, locally around a point x of $\mathscr{X}_k$, the morphism of characteristic monoids $\mathbb{N} \to \mathcal{C}_{\mathscr{X},x}$ is a saturated morphism if, for any morphism $u : \mathbb{N} \to P$ of fs monoids, the amalgamated sum $\mathcal{C}_{\mathscr{X},x} \oplus_{\mathbb{N}} P$ is still saturated.

Following the work by T. Tsuji in an unpublished 1997 preprint, Vidal in [Vid04] defines the saturation index of a morphism of fs monoids. In the case of log-regular log scheme over S^+ it can be easily computed at point in the special fiber: it is the least common multiple of the multiplicities of the prime components of $\mathscr{X}_k$. The following criterion holds.

Lemma 2.5.3. *([Vid04], Section 1.3) A morphism of fs monoids is saturated if and only if the saturation index is equal to 1.*

Proposition 2.5.4. *Assume that the residue field k is algebraically closed. Let $\mathscr{X}^+$ and $\mathscr{Y}^+$ be log-smooth log scheme over S^+. Let $\mathscr{Z}^+$ be their fs fibred product. If $\mathscr{X}^+$ is semistable, then for any pair of points (x, y) in $F_{\mathscr{X}} \times_{F_S} F_{\mathscr{Y}}$ whose closures intersect the special fibers $\mathscr{X}_k$ and $\mathscr{Y}_k$ respectively, the morphism $F_{\mathscr{Z}} \to F_{\mathscr{X}} \times_{F_S} F_{\mathscr{Y}}$, induced by the projections $\mathscr{Z}^+ \to \mathscr{X}^+$ and $\mathscr{Z}^+ \to \mathscr{Y}^+$, is a bijection above the pair (x, y), namely $n(x, y) = 1$.*

Proof. By hypothesis $\mathscr{X}^+$ is a semistable log-regular log scheme over S^+, hence the morphism $\mathbb{N} \to \mathcal{C}_{\mathscr{X},x}$ has saturation index 1 around any point $x \in F_\mathscr{X} \cap \mathscr{X}_k$, so by Lemma 2.5.3 it is a saturated morphism of characteristic monoids. The saturation condition implies that the fibred product in the category of log schemes coincides with the fibred product in the category of fs log schemes. In particular, the underlying scheme of the special fiber of $\mathscr{Z}^+$ coincides with the usual schematic fibred product, hence its points are characterized as follows:

$$z = (x, y, s, \mathfrak{p}) \text{ and } \mathcal{O}_{\mathscr{Z},z} = (\mathcal{O}_{\mathscr{X},x} \otimes_R \mathcal{O}_{\mathscr{Y},y})_\mathfrak{p}$$

where x and y are points of $\mathscr{X}^+$ and $\mathscr{Y}^+$ both mapped to s, while $\mathfrak{p}$ is a prime ideal of the tensor product of residue fields $\kappa(x) \otimes_k \kappa(y)$. We look for a characterization of points z in $\mathscr{Z}^+$ that lie in the Kato fan $F_\mathscr{Z}$.

If the point z lies in $F_\mathscr{Z}$, then the maximal ideal $\mathfrak{m}_z$ is equal to the ideal $\mathcal{I}_{\mathscr{Z},z}$ by definition. By the flatness of the models $\mathscr{X}^+$ and $\mathscr{Y}^+$ over S^+, the morphisms of local rings $\mathcal{O}_{\mathscr{X},x} \to \mathcal{O}_{\mathscr{Z},z}$ and $\mathcal{O}_{\mathscr{Y},y} \to \mathcal{O}_{\mathscr{Z},z}$ are injective. Hence, the equalities $\mathfrak{m}_x = \mathcal{I}_{\mathscr{X},x}$ and $\mathfrak{m}_y = \mathcal{I}_{\mathscr{Y},y}$ hold. Thus, the points z in $\mathscr{Z}^+$ that lie in the Kato fan $F_\mathscr{Z}$ are necessarily points such that the projections x and y to $\mathscr{X}^+$ and $\mathscr{Y}^+$ lie in the respective associated Kato fans. Therefore, we may assume $x \in F_\mathscr{X}$, $y \in F_\mathscr{Y}$, and it remains to characterize the prime ideals $\mathfrak{p}$ such that $z = (x, y, s, \mathfrak{p}) \in F_\mathscr{Z}$.

By log-regularity of $\mathscr{Z}^+$, the point z lies in the associated Kato fan if and only if $\dim \mathcal{O}_{\mathscr{Z},z} = \operatorname{rank} \mathcal{C}_{\mathscr{Z},z}^{\mathrm{gp}}$. At the level of characteristic sheaves it holds that

$$\operatorname{rank}\mathcal{C}_{\mathscr{Z},z}^{\mathrm{gp}} = \operatorname{rank}\mathcal{C}_{\mathscr{X},x}^{\mathrm{gp}} + \operatorname{rank}\mathcal{C}_{\mathscr{Y},y}^{\mathrm{gp}} - 1.$$

Since x and y are both assumed to be points in the associated Kato fans, the equality between dimension of local rings and rank of the groupifications of characteristic sheaves lead to the equivalence

$$z \in F_\mathscr{Z} \Leftrightarrow \dim \mathcal{O}_{\mathscr{Z},z} = \operatorname{rank}\mathcal{C}_{\mathscr{Z},z}^{\mathrm{gp}} = \operatorname{rank}\mathcal{C}_{\mathscr{X},x}^{\mathrm{gp}} + \operatorname{rank}\mathcal{C}_{\mathscr{Y},y}^{\mathrm{gp}} - 1$$
$$= \dim \mathcal{O}_{\mathscr{X},x} + \dim \mathcal{O}_{\mathscr{Y},y} - 1.$$

By log-regularity of $\mathscr{Z}^+$, it holds that $\dim \mathcal{O}_{\mathscr{Z},z} \geqslant \operatorname{rank}\mathcal{C}_{\mathscr{Z},z}^{\mathrm{gp}}$, thus the inequality

$$\dim \mathcal{O}_{\mathscr{Z},z} \geqslant \dim \mathcal{O}_{\mathscr{X},x} + \dim \mathcal{O}_{\mathscr{Y},y} - 1$$

is always true and equality holds only for minimal prime ideals $\mathfrak{p}$ of $\kappa(x) \otimes_k \kappa(y)$. Therefore, in order to determine the number $n(x,y)$ of preimages of (x,y) in $F_\mathscr{Z}$, we need to study the number of minimal prime ideals of the tensor product $\kappa(x) \otimes_k \kappa(y)$. By assumption, the residue field k is algebraically closed field. It follows that the tensor product $\kappa(x) \otimes_k \kappa(y)$ is a domanin, hence it has a unique minimal prime ideal, namely 0. We obtain that $n(x,y) = 1$ for any pair of points of $F_\mathscr{X} \times_{F_S} F_\mathscr{Y}$ that lie in the special fibers.

Let (x,y) be a pair of points in $F_\mathscr{X} \times_{F_S} F_\mathscr{Y}$ whose closures intersect the special fibers, namely there exist $x' \in F_\mathscr{X} \cap \mathscr{X}_k$ and $y' \in F_\mathscr{Y} \cap \mathscr{Y}_k$ such that x' is in the closure of x and y' in the closure of y. Then, by the previous part of the proof and by Lemma 2.4.11, we have $n(x,y) \leqslant n(x',y') = 1$. $\qquad\square$

3. The skeleton of a log-regular log scheme

3.1. Construction of the skeleton of a log-regular log scheme.

(3.1.1) Let $\mathscr{X}^+$ be a log-regular log scheme over S^+. Let x be a point of the associated Kato fan F. Denote by $F(x)$ the set of points y of F such that x lies in the closure of $\{y\}$, and by $\mathcal{C}_{F(x)}$ the restriction of $\mathcal{C}_F$ to $F(x)$. Denote by $\operatorname{Spec}\mathcal{C}_{\mathscr{X},x}$ the spectrum of the monoid $\mathcal{C}_{\mathscr{X},x} = \mathcal{C}_{F,x}$. Then there exists a canonical isomorphism of monoid spaces

$$(F(x), \mathcal{C}_{F(x)}) \to \operatorname{Spec}\mathcal{C}_{\mathscr{X},x} : y \mapsto \{s \in \mathcal{C}_{\mathscr{X},x} \mid s(y) = 0\}$$

where the expression $s(y) = 0$ means that $s'(y) = 0$ for any representative s' of s in $\mathcal{M}_{\mathscr{X},x}$. In particular, we obtain a bijective correspondence between the faces of the monoid $\mathcal{C}_{\mathscr{X},x}$ and the points of $F(x)$, and for every point y of $F(x)$, a surjective cospecialization morphism of monoids

$$\tau_{x,y} : \mathcal{C}_{\mathscr{X},x} \to \mathcal{C}_{\mathscr{X},y}$$

which induces an isomorphism of monoids

$$S^{-1}\mathcal{C}_{\mathscr{X},x}/(S^{-1}\mathcal{C}_{\mathscr{X},x})^{\times} \cong \mathcal{C}_{\mathscr{X},x}/S \xrightarrow{\sim} \mathcal{C}_{\mathscr{X},y}$$

where S denotes the monoid of elements s in $\mathcal{C}_{\mathscr{X},x}$ such that $s(y) \neq 0$.

(3.1.2) For each point x in F, we denote by σ_x the set of morphisms of monoids

$$\alpha : \mathcal{C}_{\mathscr{X},x} \to (\mathbb{R}_{\geqslant 0}, +)$$

such that $\alpha(\pi) = 1$ for every uniformizer π in R. We endow σ_x with the topology of pointwise covergence, where $\mathbb{R}_{\geqslant 0}$ carries the usual Euclidean topology. Note that σ_x is a polyhedron, possibly unbounded, in the real affine space

$$\{\alpha : \mathcal{C}^{\mathrm{gp}}_{\mathscr{X},x} \to (\mathbb{R}, +) \,|\, \alpha(\pi) = 1 \text{ for every uniformizer } \pi \text{ in } R\}.$$

If y is a point of $F(x)$, then the surjective cospecialization morphism $\tau_{x,y}$ induces a topological embedding $\sigma_y \to \sigma_x$ that identifies σ_y with a face of σ_x.

(3.1.3) We denote by T the disjoint union of the topological spaces σ_x with x in F. On the topological space T, we consider the equivalence relation $\sim$ generated by couples of the form $(\alpha, \alpha \circ \tau_{x,y})$ where x and y are points in F such that x lies in the closure of $\{y\}$ and α is a point of σ_y.

The skeleton of $\mathscr{X}^+$ is defined as the quotient of the topological space T by the equivalence relation $\sim$. We denote this skeleton by $\mathrm{Sk}(\mathscr{X}^+)$. It is clear that $\mathrm{Sk}(\mathscr{X}^+)$ has the structure of a polyhedral complex with cells $\{\sigma_x, x \in F\}$, so it comes equipped with a piecewise linear (PL) structure, and that the faces of a cell σ_x are precisely the cells σ_y with y in $F(x)$.

(3.1.4) We note that σ_x is empty for any point x that does not lie in the special fiber $\mathscr{X}_k$: indeed, outside the special fiber any uniformizer is an invertible element, so it is trivial in $\mathcal{C}_{\mathscr{X},x}$ and is mapped to 0 by any morphism of monoids. Therefore, the construction of the skeleton associated to $\mathscr{X}^+$ only concerns the points in the Kato fans F that lie in the special fiber. In particular, given a generic point $x \in \mathscr{X}_k$ of an intersection of components of $D_{\mathscr{X}}$, where at least one component is not in the special fiber, the corresponding face σ_x is unbounded.

In other words, the skeleton associated to a log-regular scheme $\mathscr{X}^+$, where $D_{\mathscr{X}}$ allows horizontal components, generalizes Berkovich's skeleta by admitting unbounded faces in the direction of the horizontal components as well as by allowing singularities in the special fiber. It also generalizes the construction performed by Gubler, Rabinoff and Werner in [GRW16] of a skeleton associated to a strictly semistable snc pair.

3.2. Embedding the skeleton in the non-archimedean generic fiber.

(3.2.1) Let $\mathscr{X}^+$ be a log-regular log scheme over S^+. Let x be a point of the associated Kato fan F. As the log structure on $\mathscr{X}^+$ is of finite type, the characteristic monoid $\mathcal{C}_{\mathscr{X},x}$ is of finite type too, and thus $\mathcal{C}^{\mathrm{gp}}_{\mathscr{X},x}$ is a free abelian group of finite rank. Hence there exists a section

$$\zeta : \mathcal{M}^{\mathrm{gp}}_{\mathscr{X},x}/\mathcal{M}^{\times}_{\mathscr{X},x} \to \mathcal{M}^{\mathrm{gp}}_{\mathscr{X},x}.$$

The section ζ restricts to $\mathcal{C}_{\mathscr{X},x} \to \mathcal{M}_{\mathscr{X},x}$; indeed, if $x \in \mathcal{M}_{\mathscr{X},x}$ then $\zeta(\overline{x}) - x \in \mathcal{M}^{\times}_{\mathscr{X},x}$. Therefore we may choose a section

(3.2.2) $$\mathcal{C}_{\mathscr{X},x} \to \mathcal{M}_{\mathscr{X},x}$$

of the projection homomorphism

$$\mathcal{M}_{\mathscr{X},x} \to \mathcal{C}_{\mathscr{X},x}$$

and use this section to view $\mathcal{C}_{\mathscr{X},x}$ as a submonoid of $\mathcal{M}_{\mathscr{X},x}$. Note that $\mathcal{C}_{\mathscr{X},x} \setminus \{0\}$ generates the ideal $\mathcal{I}_{\mathscr{X},x}$ of $\mathcal{O}_{\mathscr{X},x}$.

We propose a generalisation of [MN15], Lemma 2.4.4.

Lemma 3.2.3. *Let A be a Noetherian ring, let I be an ideal of A and let $(y_1, \ldots, y_m)$ be a system of generators for I. We denote by $\hat{A}$ the I-adic completion of A. Let B be a subring of A such that the elements $y_1, \ldots, y_m$ belong to B and generate the ideal $B \cap I$ in B. Then, in the ring $\hat{A}$, every element f of B can be written as*

$$(3.2.4) \qquad\qquad f = \sum_{\beta \in \mathbb{Z}_{\geq 0}^m} c_\beta y^\beta$$

where the coefficients c_β belong to $((A \setminus I) \cap B) \cup \{0\}$.

Proof. Let f be an element of B, we construct an expansion for f of the form (3.2.4) by induction. If f belongs to the complement of I, the conclusion trivially holds. Otherwise, f belongs to I and we can write f as a linear combination of the elements $y_1, \ldots, y_m$ with coefficients in B:

$$f = \sum_{j=1}^m b_j y_j, \quad b_j \in B.$$

By induction hypothesis, we suppose that i is a positive integer and that we can write every f in B as a sum of an element f_i of the form (3.2.4) and a linear combination of degree i monomials in the elements $y_1, \ldots, y_m$ with coefficients in B. We apply this assumption to the coefficients b_j, hence

$$b_j = b_{j,i} + \sum_{\substack{\beta \in \mathbb{Z}_{\geq 0}^m \\ |\beta|=i}} b_{j,\beta} y^\beta, \quad b_{j,\beta} \in B.$$

Then we can write f as a sum of an element f_{i+1} of the form (3.2.4) and a linear combination of degree $i+1$ monomials in the elements $y_1, \ldots, y_m$ with coefficients in B

$$f = \underbrace{\sum_{j=1}^m b_{j,i} y_j}_{f_{i+1}} + \sum_{j=1}^m \Big(\sum_{\substack{\beta \in \mathbb{Z}_{\geq 0}^m \\ |\beta|=i}} b_{j,\beta} y^\beta \Big) y_j$$

such that f_i and f_{i+1} have the same coefficients in degree less or equal to i. Iterating this construction we finally find an expansion of f of the required form. $\qquad\square$

(3.2.5) Let f be an element of $\mathcal{O}_{\mathscr{X},x}$. Considering $A = B = \mathcal{O}_{\mathscr{X},x}$, $I = \mathfrak{m}_x$ and a system of generators for $\mathfrak{m}_x$ in $\mathcal{C}_{\mathscr{X},x} \setminus \{1\}$, by Lemma 3.2.3 we can write f as a formal power series

$$(3.2.6) \qquad\qquad f = \sum_{\gamma \in \mathcal{C}_{\mathscr{X},x}} c_\gamma \gamma$$

in $\widehat{\mathcal{O}}_{\mathscr{X},x}$, where each coefficient c_γ is either zero or a unit in $\mathcal{O}_{\mathscr{X},x}$. We call this formal series an *admissible expansion* of f. We set

$$(3.2.7) \qquad\qquad S = \{ \gamma \in \mathcal{C}_{\mathscr{X},x} \mid c_\gamma \neq 0 \}$$

and we denote by $\Gamma_x(f)$ the set of elements of S that lie on a compact face of the convex hull of $S + \mathcal{C}_{\mathscr{X},x}$ in $\mathcal{C}_{\mathscr{X},x}^{\mathrm{gp}} \otimes_{\mathbb{Z}} \mathbb{R}$. We call $\Gamma_x(f)$ the *initial support* of f at x, notation which is justified by the next proposition.

Proposition 3.2.8.

(1) *The element*

$$f_x = \sum_{\gamma \in \Gamma_x(f)} c_\gamma(x)\gamma \quad \in k(x)[\mathcal{C}_{\mathscr{X},x}]$$

depends on the choice of the section (3.2.2), *but not on the expansion* (3.2.6).

(2) *The subset* $\Gamma_x(f)$ *of* $\mathcal{C}_{\mathscr{X},x}$ *only depends on* f *and* x, *and not on the choice of the section* (3.2.2) *or the expansion* (3.2.6).

Proof. If we denote by I the ideal of $k(x)[\mathcal{C}_{\mathscr{X},x}]$ generated by $\mathcal{C}_{\mathscr{X},x} \setminus \{1\}$, then it follows from [Kat94] that there exists an isomorphism of $k(x)$-algebras

$$(3.2.9) \qquad \mathrm{gr}_I \, k(x)[\mathcal{C}_{\mathscr{X},x}] \to \mathrm{gr}_{\mathfrak{m}_x} \mathcal{O}_{\mathscr{X},x}.$$

Using this result and following the argument of [MN15] Proposition 2.4.6, we show now that f_x does not depend on the expansion of f. Let

$$f = \sum_{\gamma \in \mathcal{C}_{\mathscr{X},x}} c'_\gamma \gamma$$

be another admissible expansion of f with associated set $\Gamma_x(f)'$ and element f'_x. Then

$$0 = \sum_{\gamma \in \mathcal{C}_{\mathscr{X},x}} (c_\gamma - c'_\gamma)\gamma = \sum_{\gamma \in \mathcal{C}_{\mathscr{X},x}} d_\gamma \gamma$$

where the right hand side is an admissible expansion obtained by choosing admissible expansions for the elements $c_\gamma - c'_\gamma$ that do not lie in $\mathcal{O}^\times_{\mathscr{X},x} \cup \{0\}$. In particular $d_\gamma(x) = c_\gamma(x) - c'_\gamma(x)$ for any γ in $\Gamma_x(f) \cup \Gamma_x(f)'$. The isomorphism of graded algebras in (3.2.9) implies that the elements d_γ must all vanish, hence $\Gamma_x(f) = \Gamma_x(f)'$ and $f_x = f'_x$.

Point (2) follows from the fact that the coefficients c_γ of f_x are independent of the chosen section up to multiplication by a unit in $\mathcal{O}_{\mathscr{X},x}$, so that the support $\Gamma_x(f)$ of f_x only depends on f and x. $\qquad\square$

Proposition 3.2.10. *Let* x *be a point of* F *and let*

$$\alpha : \mathcal{C}_{\mathscr{X},x} \to (\mathbb{R}_{\geqslant 0}, +)$$

be an element of σ_x. *Then there exists a unique minimal real valuation*

$$v : \mathcal{O}_{\mathscr{X},x} \setminus \{0\} \to \mathbb{R}_{\geqslant 0}$$

such that $v(m) = \alpha(\overline{m})$ *for each element* m *of* $\mathcal{M}_{\mathscr{X},x}$.

Proof. We will prove that the map

$$(3.2.11) \qquad v : \mathcal{O}_{\mathscr{X},x} \setminus \{0\} \to \mathbb{R} : f \mapsto \min\{\alpha(\gamma) \,|\, \gamma \in \Gamma_x(f)\}$$

satisfies the requirements in the statement. We fix a section

$$\mathcal{C}_{\mathscr{X},x} \to \mathcal{M}_{\mathscr{X},x}.$$

It is straightforward to check that $(f \cdot g)_x = f_x \cdot g_x$ for all f and g in $\mathcal{O}_{\mathscr{X},x}$. This implies that v is a valuation. It is obvious that $v(m) = \alpha(\overline{m})$ for all m in $\mathcal{M}_{\mathscr{X},x}$, since we can write m as the product of an element of $\mathcal{C}_{\mathscr{X},x}$ and a unit in $\mathcal{O}_{\mathscr{X},x}$.

Now we prove minimality. Consider any real valuation

$$w : \mathcal{O}_{\mathscr{X},x} \to \mathbb{R}$$

such that $w(m) = \alpha(\overline{m})$ for each element m of $\mathcal{M}_{\mathscr{X},x}$, and let f be an element of $\mathcal{O}_{\mathscr{X},x}$. We must show that $w(f) \geqslant v(f)$.

We set

$$C_\alpha = \mathcal{C}_{\mathscr{X},x} \setminus \alpha^{-1}(0).$$

We denote by I the ideal in $\mathcal{O}_{\mathscr{X},x}$ generated by C_α and by A the I-adic completion of $\mathcal{O}_{\mathscr{X},x}$. By Lemma 3.2.3, we see that we can write f in A as

$$(3.2.12) \qquad \sum_{\beta \in C_\alpha \cup \{1\}} d_\beta \beta$$

where d_β is either zero or contained in the complement of I in $\mathcal{O}_{\mathscr{X},x}$.

Since $\alpha(\beta) > 0$ for every $\beta \in C_\alpha$, we can find an integer $N > 0$ such that $w(g) > w(f)$ for every element g in I^N. So we have

$$w(f) \geqslant \min\{\alpha(\beta) \,|\, d_\beta \neq 0\}$$

recalling that $w(\beta) = \alpha(\beta)$ for all β in $\mathcal{C}_{\mathscr{X},x}$.

We consider the coefficients in the expansion $(3.2.12)$ of f. Applying Lemma 3.2.3 as in paragraph $(3.2.5)$, we can write admissible expansions of these coefficients in $\widehat{\mathcal{O}}_{\mathscr{X},x}$ as

$$d_\beta = \sum_{\gamma \in \mathcal{C}_{\mathscr{X},x}} c_{\gamma,\beta}\gamma, \quad c_{\gamma,\beta} \in \mathcal{O}_{\mathscr{X},x}^\times \cup \{0\},$$

with $\alpha(\gamma) = 0$ in the expansions of d_β that belong to $\mathfrak{m}_x \setminus I$.

Therefore we obtain an admissible expansion of f

$$f = \sum_{\substack{\beta \in C_\alpha \cup \{1\} \\ \gamma \in \mathcal{C}_{\mathscr{X},x}}} c_{\gamma,\beta} \, \gamma\beta$$

and we have $v(f) = \min\{\alpha(\gamma\beta) \,|\, c_{\gamma,\beta} \neq 0\} \geqslant \min\{\alpha(\beta) \,|\, d_\beta \neq 0\} \geqslant w(f)$. $\qquad\square$

Remark 3.2.13. In the definition $(3.2.11)$ of the valuation v, we compute the minimum over the terms in the initial support of f: these elements are a finite number and they only depends on x and f by Proposition 3.2.8. Therefore, this minimum provides a well-defined function on $\mathcal{O}_{\mathscr{X},x} \setminus \{0\}$. Nevertheless, it is equivalent to consider the minimum over all the terms of an admissible expansion of f, i.e. for any admissible expansion $f = \sum_{\gamma \in \mathcal{C}_{\mathscr{X},x}} c_\gamma \gamma$

$$\min\{\alpha(\gamma) \,|\, \gamma \in \Gamma_x(f)\} = \min\{\alpha(\gamma) \,|\, \gamma \in S\},$$

where $S = \{\gamma \in \mathcal{C}_{\mathscr{X},x} \,|\, c_\gamma \neq 0\}$ as in $(3.2.7)$. Indeed, any element that belongs to S can be written as a sum of an element of the initial support of f and an element of $\mathcal{C}_{\mathscr{X},x}$. Since the morphism α is additive and takes positive real values, then the minimum is necessarily attained by the elements in the initial support.

(3.2.14) We will denote the valuation v from Proposition 3.2.10 by $v_{x,\alpha}$. Since $v_{x,\alpha}$ induces a real valuation on the function field of $\mathscr{X}_K$ that extends the discrete valuation v_K on K, it defines a point of the K-analytic space $\mathscr{X}_K^{\mathrm{an}}$, which we will denote by the same symbol $v_{x,\alpha}$. We now show that the characterization of $v_{x,\alpha}$ in Proposition 3.2.10 implies that

$$v_{y,\alpha'} = v_{x,\alpha'\circ\tau_{x,y}}$$

for every y in $F(x)$ and every α' in σ_y.

Firstly we note that $\mathcal{O}_{\mathscr{X},y}$ is the localization of $\mathcal{O}_{\mathscr{X},x}$ with respect to the elements of $m \in \mathcal{M}_{\mathscr{X},x}$ in the kernel of $\tau_{x,y}$. Indeed, by construction of $\tau_{x,y}$, the kernel is given by

$$\ker(\tau_{x,y}) = \{s \in \mathcal{C}_{\mathscr{X},x} | s(y) \neq 0\};$$

to obtain $\mathcal{O}_{\mathscr{X},y}$ from $\mathcal{O}_{\mathscr{X},x}$, we localize by

$$S = \{a \in \mathcal{O}_{\mathscr{X},x} | a(y) \neq 0\};$$

therefore we can identify the set of elements of $\mathcal{M}_{\mathscr{X},x}$ whose reduction is in $\ker(\tau_{x,y})$ with the set S, recalling that, for points in the Kato fan, $\mathcal{C}_{\mathscr{X},x} \setminus \{1\}$ generates the maximal ideal of $\mathcal{O}_{\mathscr{X},x}$. Therefore we are dealing with these two morphisms:

$$\mathcal{O}_{\mathscr{X},x} \hookrightarrow S^{-1}\mathcal{O}_{\mathscr{X},x} = \mathcal{O}_{\mathscr{X},y},$$

$$\mathcal{C}_{\mathscr{X},x} \twoheadrightarrow \mathcal{C}_{\mathscr{X},x}/S = \mathcal{C}_{\mathscr{X},y}.$$

Let f be an element of $\mathcal{O}_{\mathscr{X},x}$. Under the notations of Lemma 3.2.3, we apply the lemma to $A = \mathcal{O}_{\mathscr{X},y}$ and $B = \mathcal{O}_{\mathscr{X},x}$, choosing a system of generators of $\mathfrak{m}_y$ in $\mathcal{C}_{\mathscr{X},x}$: we can find an admissible expansion of f of the form

$$f = \sum_{\delta \in \mathcal{C}_{\mathscr{X},y}} d_\delta \delta \quad \text{with } d_\delta \in (\mathcal{O}_{\mathscr{X},x} \cap \mathcal{O}_{\mathscr{X},y}^\times) \cup \{0\}.$$

Admissible expansions of coefficients d_δ induce an admissible expansion for f by

$$f = \sum_{\delta \in \mathcal{C}_{\mathscr{X},y}} \Big(\sum_{\gamma \in S} c_{\gamma\delta}\gamma\Big)\delta \quad \text{with } c_{\gamma\delta} \in \mathcal{O}_{\mathscr{X},x}^\times \cup \{0\},$$

where γ runs through the set S since $d_\delta \in \mathcal{O}_{\mathscr{X},y}^\times$. Thus we have

$$\begin{aligned}
v_{y,\alpha'}(f) &= \min\{\alpha'(\delta) \,|\, \delta \in \Gamma_y(f)\} \\
&= \min\{\alpha' \circ \tau_{x,y}(\gamma\delta) \,|\, \delta \in \Gamma_y(f), \gamma \in S\} \\
&= \min\{\alpha' \circ \tau_{x,y}(\gamma\delta) \,|\, \gamma\delta \in \Gamma_x(f)\} \\
&= v_{x,\alpha' \circ \tau_{x,y}}(f).
\end{aligned}$$

Hence, we obtain a well-defined map

$$\iota : \mathrm{Sk}(\mathscr{X}^+) \to \mathscr{X}_K^{\mathrm{an}}$$

by sending α to $v_{x,\alpha}$ for every point x of F and every $\alpha \in \sigma_x$.

Proposition 3.2.15. *The map*

$$\iota : \mathrm{Sk}(\mathscr{X}^+) \to \mathscr{X}_K^{\mathrm{an}}$$

is a topological embedding.

Proof. First, we show that ι is injective. Let x be a point of F and α an element of σ_x. Let y be the point of $F(x)$ corresponding to the face $\mathcal{C}_{\mathscr{X},x} \setminus \alpha^{-1}(0)$ of $\mathcal{C}_{\mathscr{X},x}$. Then α factors through an element

$$\alpha' : \mathcal{C}_{\mathscr{X},y} \to \mathbb{R}_{\geqslant 0}$$

of σ_y. Note that $\alpha = \alpha'$ in $\mathrm{Sk}(\mathscr{X}^+)$ because $\alpha = \alpha' \circ \tau_{x,y}$. Moreover, since $(\alpha')^{-1}(0) = \{1\}$, the center of the valuation $v_{y,\alpha'}$ is the point y, so that $\mathrm{red}_{\mathscr{X}}(v_{y,\alpha'}) = y$. Thus we can recover y from $v_{y,\alpha'}$. Then we can also reconstruct α' by looking at the values of $v_{y,\alpha'}$ at the elements of $\mathcal{M}_{\mathscr{X},y}$. We conclude that ι is injective.

Now, we show that ι is a homeomorphism onto its image. For any valuation v in $\mathrm{Sk}(\mathscr{X}^+)$ and any small open neighbourhood U of $\iota(v)$ in $\mathscr{X}_K^{\mathrm{an}}$, there exists a closed subset C in $\mathrm{Sk}(\mathscr{X}^+)$ such that $U \cap \iota(\mathrm{Sk}(\mathscr{X}^+)) \subseteq \iota(C)$ and, up to subdivisions, we can assume that the C is a closed cell of $\mathrm{Sk}(\mathscr{X}^+)$. Therefore, it suffices to prove that the restriction of ι to any closed cell σ_x of $\mathrm{Sk}(\mathscr{X}^+)$ is an homeomorphism. The restriction $\iota_{|\sigma_x}$ is an injective map from a compact set to the Hausdorff space $\mathscr{X}_K^{\mathrm{an}}$, so we reduce to show that $\iota_{|\sigma_x}$ is continuous, to conclude that $\iota_{|\sigma_x}$ is a homeomorphism. By definition of the Berkovich topology, it is enough to prove that the map

$$\sigma_x \to \mathbb{R} : \alpha \mapsto v_{x,\alpha}(f)$$

is continuous for every f in $\mathcal{O}_{\mathscr{X},x}$. This is obvious from the formula (3.2.11). $\square$

(3.2.16) From now on, we will view $\mathrm{Sk}(\mathscr{X}^+)$ as a topological subspace of $\mathscr{X}_K^{\mathrm{an}}$ by means of the embedding ι in Proposition 3.2.15. If $\mathscr{X}$ is regular over R and $\mathscr{X}_k$ is a divisor with strict normal crossings, the skeleton $\mathrm{Sk}(\mathscr{X}^+)$ was described in [MN15], Section 3.1.

3.3. **Contracting the generic fiber to the skeleton.**

(3.3.1) We denote by $D_{\mathscr{X},\mathrm{hor}}$ the component of $D_{\mathscr{X}}$ not contained in the special fiber $\mathscr{X}_k$. The inclusion $\iota : \mathrm{Sk}(\mathscr{X}^+) \to \mathscr{X}_K^{\mathrm{an}}$ is actually an inclusion in $(\mathscr{X}_K \setminus D_{\mathscr{X},\mathrm{hor}})^{\mathrm{an}}$ and it admits a continuous retraction

$$\rho_{\mathscr{X}} : (\mathscr{X}_K \setminus D_{\mathscr{X},\mathrm{hor}})^{\mathrm{an}} \to \mathrm{Sk}(\mathscr{X}^+)$$

constructed as follows. Let x be a point of $(\mathscr{X}_K \setminus D_{\mathscr{X},\mathrm{hor}})^{\mathrm{an}}$ and consider the reduction map

$$\mathrm{red}_{\mathscr{X}} : (\mathscr{X}_K \setminus D_{\mathscr{X},\mathrm{hor}})^{\mathrm{an}} \to \mathscr{X}_k.$$

Let $E_1, \ldots, E_r$ be the irreducible components of $D_{\mathscr{X}}$ passing through the point $\mathrm{red}_{\mathscr{X}}(x)$. We denote by ξ the generic point of the connected component of $E_1 \cap \ldots \cap E_r$ that contains $\mathrm{red}_{\mathscr{X}}(x)$. By Lemma 2.2.3, ξ is a point in the associated Kato fan F. We set α to be the morphism of monoids

$$\alpha : \mathcal{C}_{\mathscr{X},\xi} \to \mathbb{R}_{\geqslant 0}$$

such that $\alpha(\overline{m}) = v_x(m)$ for any element m of $\mathcal{M}_{\mathscr{X},\xi}$. In particular $\alpha(\pi) = v_x(\pi) = 1$ as we assumed the normalization of all valuations in the Berkovich space. Then $\rho_{\mathscr{X}}(x)$ is the point of $\mathrm{Sk}(\mathscr{X}^+)$ corresponding to the couple (ξ, α). By construction $\rho_{\mathscr{X}}$ is continuous and right inverse to the inclusion ι.

(3.3.2) Given a dominant morphism $f : \mathscr{X}^+ \to \mathscr{Y}^+$ of integral flat separated log-regular log schemes over S, it induces a map between the set of birational points $\mathrm{Bir}(\mathscr{X}_K) \to \mathrm{Bir}(\mathscr{Y}_K)$. As $\mathrm{Bir}(\mathscr{X}_K) \subseteq (\mathscr{X}_K \setminus D_{\mathscr{X},\mathrm{hor}})^{\mathrm{an}}$, we can employ the retraction ρ to define a map of skeleta as follows

$$
\begin{array}{ccc}
\mathrm{Bir}(\mathscr{X}_K) & \xrightarrow{\widehat{f}} & \mathrm{Bir}(\mathscr{Y}_K) \\
{\scriptstyle \rho_{\mathscr{X}}} \big\updownarrow {\scriptstyle \iota_{\mathscr{Y}}} & & \big\downarrow {\scriptstyle \rho_{\mathscr{Y}}} \\
\mathrm{Sk}(\mathscr{X}^+) & \dashrightarrow & \mathrm{Sk}(\mathscr{Y}^+).
\end{array}
$$

This association makes the skeleton construction $\mathrm{Sk}(\mathscr{X}^+)$ functorial in $\mathscr{X}^+$ with respect to dominant morphisms.

3.4. **Skeleton of a fs fibred product.**

(3.4.1) Let $\mathscr{X}^+$ and $\mathscr{Y}^+$ be log-smooth log schemes over S^+, let $\mathscr{Z}^+$ be their fs fibred product. Let

$$\mathrm{Sk}(\mathscr{Z}^+) \to \mathrm{Sk}(\mathscr{X}^+) \times \mathrm{Sk}(\mathscr{Y}^+)$$

be the continuous map of skeleta functorially associated to the projections $\mathrm{pr}_{\mathscr{X}} : \mathscr{Z}^+ \to \mathscr{X}^+$ and $\mathrm{pr}_{\mathscr{Y}} : \mathscr{Z}^+ \to \mathscr{Y}^+$. We denote this map by $\left(\mathrm{pr}_{\mathrm{Sk}(\mathscr{X})}, \mathrm{pr}_{\mathrm{Sk}(\mathscr{Y})}\right)$ and we recall that it is constructed considering the diagram

(3.4.2)
$$
\begin{array}{ccc}
\mathrm{Bir}(\mathscr{Z}_K) & \xrightarrow{(\widehat{\mathrm{pr}_{\mathscr{X}}}, \widehat{\mathrm{pr}_{\mathscr{Y}}})} & \mathrm{Bir}(\mathscr{X}_K) \times \mathrm{Bir}(\mathscr{Y}_K) \\
{\scriptstyle \rho_{\mathscr{Z}}} \big\updownarrow {\scriptstyle \iota_{\mathscr{Z}}} & & \big\downarrow {\scriptstyle (\rho_{\mathscr{X}}, \rho_{\mathscr{X}})} \\
\mathrm{Sk}(\mathscr{Z}^+) & \xrightarrow{(\mathrm{pr}_{\mathrm{Sk}(\mathscr{X})}, \mathrm{pr}_{\mathrm{Sk}(\mathscr{Y})})} & \mathrm{Sk}(\mathscr{X}^+) \times \mathrm{Sk}(\mathscr{Y}^+).
\end{array}
$$

Proposition 3.4.3. *Assume that the residue field k is algebraically closed. If $\mathscr{X}^+$ is semistable, then the map $\left(\mathrm{pr}_{\mathrm{Sk}(\mathscr{X})}, \mathrm{pr}_{\mathrm{Sk}(\mathscr{Y})}\right)$ is a PL homeomorphism.*

Proof. The surjectivity of the map $\big(\mathrm{pr}_{\mathrm{Sk}(\mathscr{X})}, \mathrm{pr}_{\mathrm{Sk}(\mathscr{Y})}\big)$ follows from the commutativity of the diagram (3.4.2) and the surjectivity of $(\rho_{\mathscr{X}}, \rho_{\mathscr{X}}) \circ (\widehat{\mathrm{pr}_{\mathscr{X}}}, \widehat{\mathrm{pr}_{\mathscr{Y}}})$. To prove the injectivity of $\big(\mathrm{pr}_{\mathrm{Sk}(\mathscr{X})}, \mathrm{pr}_{\mathrm{Sk}(\mathscr{Y})}\big)$, we provide an explicit description of the map $\mathrm{pr}_{\mathrm{Sk}(\mathscr{X})}$.

We recall that the projection $\widehat{\mathrm{pr}_{\mathscr{X}}}$ is such that a valuation v on the function field $K(\mathscr{Z}_K)$ maps to the composition $v \circ i$ where $i : K(\mathscr{X}_K) \hookrightarrow K(\mathscr{Z}_K)$.

Let $v_{z,\varepsilon}$ be the valuation in $\mathrm{Sk}(\mathscr{Z}^+)$ corresponding to a couple (z, ε) with $z \in F_{\mathscr{Z}} \cap \mathscr{Z}_k$ and $\varepsilon \in \sigma_z$. We consider the morphism of associated Kato fans

$$F_{\mathscr{Z}} \to F_{\mathscr{X}} \times_{F_S} F_{\mathscr{Y}}$$

as established in Proposition 2.4.8. We denote respectively by $\mathrm{pr}_{F_{\mathscr{X}}}$ and $\mathrm{pr}_{F_{\mathscr{Y}}}$ the projection to the first and second factor. Then $\mathrm{pr}_{F_{\mathscr{X}}}(z)$ is a point in the associated Kato fan $F_{\mathscr{X}}$, that we denote by x. We consider the morphism of monoids

$$i_x : \mathcal{C}_{\mathscr{X},x} \to \mathcal{C}_{\mathscr{Z},z}$$

and the composition

$$\mathrm{pr}_{\mathscr{X}}(\varepsilon): \quad \mathcal{C}_{\mathscr{X},x} \xrightarrow{i_x} \mathcal{C}_{\mathscr{Z},z} = (\mathcal{C}_{\mathscr{X},x} \oplus_{\mathbb{N}} \mathcal{C}_{\mathscr{Y},y})^{\mathrm{sat}} \xrightarrow{\varepsilon} \mathbb{R}_{\geqslant 0}$$
$$a \longmapsto [a, 1] \longmapsto \varepsilon([a, 1]).$$

It trivially satisfies $\varepsilon \circ i_x(\pi) = 1$. In order to conclude that it correctly defines a point in the skeleton $\mathrm{Sk}(\mathscr{X}^+)$, we need to check the compatibility with respect to the equivalence relation $\sim$. Indeed, suppose that $\varepsilon = \varepsilon' \circ \tau_{z,z'}$ for some $z' \in \overline{\{z\}}$. We denote by x' the projection of z' under the local isomorphism of associated Kato fans. The diagram

$$
\begin{array}{ccc}
\mathcal{C}_{\mathscr{X},x} & \xrightarrow{i_x} & \mathcal{C}_{\mathscr{Z},z} \\
{\scriptstyle\tau_{x,x'}}\big\downarrow & & \big\downarrow{\scriptstyle\tau_{z,z'}} \quad \xrightarrow{\varepsilon} \mathbb{R}_{\geqslant 0} \\
\mathcal{C}_{\mathscr{X},x'} & \xrightarrow{i_{x'}} & \mathcal{C}_{\mathscr{Z},z'} \quad {\scriptstyle\varepsilon'}
\end{array}
$$

is commutative as made up by a commutative square and a commutative triangle of arrows. Therefore, by commutativity

$$\mathrm{pr}_{\mathscr{X}}(\varepsilon) = \mathrm{pr}_{\mathscr{X}}(\varepsilon') \circ \tau_{x,x'}$$

and this implies that $\mathrm{pr}_{\mathscr{X}}(\varepsilon)$ defines a well-defined point $v_{x,\mathrm{pr}_{\mathscr{X}}(\varepsilon)}$ of $\mathrm{Sk}(\mathscr{X}^+)$.

We claim that $v_{x,\mathrm{pr}_{\mathscr{X}}(\varepsilon)}$ is indeed the image of $v_{z,\varepsilon}$ under the map $\mathrm{pr}_{\mathrm{Sk}(\mathscr{X})}$, hence that the equality in the following inner diagram holds

$$
\begin{array}{ccc}
\widehat{\mathscr{Z}_\eta} & \xrightarrow{\widehat{\mathrm{pr}_{\mathscr{X}}}} & \widehat{\mathscr{X}_\eta} \\
{\scriptstyle\rho_{\mathscr{X}}}\big\updownarrow\,{\scriptstyle\iota_{\mathscr{X}}} & \begin{array}{c} v_{z,\varepsilon} \longmapsto v_{z,\varepsilon} \circ i \\ \uparrow \qquad\qquad \downarrow \\ v_{z,\varepsilon} \blacktriangleright \rho_{\mathscr{X}}(v_{z,\varepsilon} \circ i) = v_{x,\mathrm{pr}_{\mathscr{X}}(\varepsilon)} \end{array} & \big\downarrow{\scriptstyle\rho_{\mathscr{X}}} \\
\mathrm{Sk}(\mathscr{Z}^+) & \xrightarrow{\mathrm{pr}_{\mathrm{Sk}(\mathscr{X})}} & \mathrm{Sk}(\mathscr{X}^+).
\end{array}
$$

We denote $\rho_{\mathscr{X}}(v_{z,\varepsilon} \circ i)$ by (x, α) as a point of $\mathrm{Sk}(\mathscr{X}^+)$. By definition of the retraction $\rho_{\mathscr{X}}$, the morphism α is characterized by the fact that $\alpha(\overline{m}) = (v_{z,\varepsilon} \circ i)(m)$ for any m in $\mathcal{M}_{\mathscr{X},x}$ and then we have

$$\alpha(\overline{m}) = (v_{z,\varepsilon} \circ i)(m) = v_{z,\varepsilon}(m) = \varepsilon(\overline{m}).$$

On the other hand, for any m in $\mathcal{M}_{\mathscr{X},x}$

$$v_{x,\mathrm{pr}_{\mathscr{X}}(\varepsilon)}(m) = \mathrm{pr}_{\mathscr{X}}(\varepsilon)(\overline{m}) = \varepsilon(\overline{m})$$

hence we obtain that α coincide with the morphism $\mathrm{pr}_{\mathscr{X}}(\varepsilon)$. It means that their associated points $\rho_{\mathscr{X}}(v_{z,\varepsilon} \circ i)$ and $v_{x,\mathrm{pr}_{\mathscr{X}}(\varepsilon)}$ coincide in $\mathrm{Sk}(\mathscr{X}^+)$.

Given a pair of points in $\mathrm{Sk}(\mathscr{X}^+) \times \mathrm{Sk}(\mathscr{Y}^+)$, we know by surjectivity of $\big(\mathrm{pr}_{\mathrm{Sk}(\mathscr{X})}, \mathrm{pr}_{\mathrm{Sk}(\mathscr{Y})}\big)$ that they are of the form

$$\big(v_{x,\mathrm{pr}_{\mathscr{X}}(\varepsilon)}, v_{y,\mathrm{pr}_{\mathscr{Y}}(\varepsilon)}\big).$$

The assumptions of semistability of $\mathscr{X}^+$ and algebraic closedness of k guarantee that there is a unique z in $F_{\mathscr{Z}}$ in the fiber of x and y, by Proposition 2.5.4 and Remark 3.1.4. Moreover, we can uniquely reconstruct ε by looking at the values of $v_{x,\mathrm{pr}_{\mathscr{X}}(\varepsilon)}$ at the elements of $\mathcal{M}_{\mathscr{X},x}$ and respectively of $v_{y,\mathrm{pr}_{\mathscr{Y}}(\varepsilon)}$ at the elements of $\mathcal{M}_{\mathscr{Y},y}$. We conclude that $\big(\mathrm{pr}_{\mathrm{Sk}(\mathscr{X})}, \mathrm{pr}_{\mathrm{Sk}(\mathscr{Y})}\big)$ is injective. $\square$

The assumption of semistability is crucial in the result of Proposition 3.4.3. To see this, it is helpful to consider an example.

Example 3.4.4. Let q be the equation a generic quartic curve in $\mathbb{P}^2_{\mathbb{C}((t))}$. Then $\mathscr{X} : tq + x^2 y^2 = 0$ gives the equation of a family of genus 3 curves, degenerating to two double lines. The dual complex $\mathcal{D}(\mathscr{X}_{\mathbb{C}})$ of the special fiber $\mathscr{X}_{\mathbb{C}}$ is a line segment and $\mathscr{X}$ has 4 singularities of type A_1 in each component of the special fiber, corresponding to the base points of the family. In this case taking a semistable model of $\mathscr{X}_{\mathbb{C}((t))}$ requires an order two base change, which induces coverings branched at each of these singular points (see [HM98], p.133 for details). Let $\mathscr{Y}$ be such a semistable reduction. Thus the special fiber of $\mathscr{Y}$ consists of two elliptic curves, call them E_1 and E_2, which intersect in two points, p_A and p_B, which are the preimages of the point $(0 : 0 : 1)$. The dual complex $\mathcal{D}(\mathscr{Y}_{\mathbb{C}})$ of the special fiber $\mathscr{Y}_{\mathbb{C}}$ is isomorphic to S^1.

We will compare the dual complex of $(\mathscr{X} \times_R \mathscr{X})_{\mathbb{C}}$ with that of $(\mathscr{Y} \times_R \mathscr{Y})_{\mathbb{C}}$. The models $\mathscr{X}$ and $\mathscr{Y}$ are not log-regular at every point, but from our prospective it is enough that they are log-regular at the generic point of each stratum. For the product with a semistable model, the dual complex is the product of the dual complexes, and $\mathcal{D}((\mathscr{Y} \times_R \mathscr{Y})_{\mathbb{C}})$ is therefore a real 2-torus $S^1 \times S^1$.

On the other hand, the dual complex of the product $(\mathscr{X} \times_R \mathscr{X})_{\mathbb{C}}$ is given by a quotient of $S^1 \times S^1$ by the action of $\mathbb{Z}/2\mathbb{Z}$. $\mathcal{D}((\mathscr{Y} \times_R \mathscr{Y})_{\mathbb{C}})$ has the structure of a cell complex, whose cells correspond to ordered pairs of strata in $\mathcal{D}(\mathscr{Y}_{\mathbb{C}})$, so

zero-dimensional strata:	$(E_1, E_2), (E_1, E_1), (E_2, E_1), (E_2, E_2)$
one-dimensional strata:	$(E_1, p_A), (E_1, p_B), (E_2, p_A), (E_2, p_B)$
	$(p_A, E_1), (p_B, E_1), (p_A, E_2), (p_B, E_2)$
two-dimensional strata:	$(p_A, p_A), (p_A, p_B), (p_B, p_A), (p_B, p_B).$

The action of $\mathbb{Z}/2\mathbb{Z}$ fixes E_1 and E_2, while switching p_A and p_B. Therefore it fixes exactly the zero dimensional strata while acting freely on the other points. The quotient, the complex $\mathcal{D}((\mathscr{X} \times_R \mathscr{X})_{\mathbb{C}})$, is piecewise-linearly homeomorphic to the sphere S^2. In particular, it is not isomorphic to the product of two line segments.

4. THE WEIGHT FUNCTION

From now on we assume $\mathrm{char}(K) = \mathrm{char}(k) = 0$.

4.1. **Weight function associated to a logarithmic pluricanonical form.**

(4.1.1) Let X be a connected, smooth and proper K-variety of dimension n. We introduce the following notation: for any log-regular model $\mathscr{X}^+$ of X, for any point $x = (\xi_x, |\cdot|_x) \in X^{\mathrm{an}}$ and for any divisor D on $\mathscr{X}^+$ whose support does not contain ξ_x, we set

$$v_x(D) = -\ln|f(x)|$$

where f is any element of $K(X)^\times$ such that $D = \mathrm{div}(f)$ locally at $\mathrm{red}_{\mathscr{X}}(x)$.

(4.1.2) Let (X, Δ_X) be a pair where Δ_X is an effective $\mathbb{Q}$-divisor such that $\Delta_X = \sum a_i \Delta_{X,i}$ has $0 \leqslant a_i \leqslant 1$, and $X^+ = (X, \lceil \Delta_X \rceil)$ is a log-regular log scheme over K. Let ω be a regular m-pluricanonical form on X^+ with poles of order at most ma_i along $\Delta_{X,i}$, for some m such that $ma_i \in \mathbb{N}$ for any i. We call such forms Δ_X-logarithmic m-pluricanonical forms.

Moreover, ω is a regular section of the logarithmic m-pluricanonical bundle of X^+ and for each log-regular model $\mathscr{X}^+$ of X^+, the form ω defines a divisor $\mathrm{div}_{\mathscr{X}^+}(\omega)$ on $\mathscr{X}^+$. Note that the multiplicity in $\mathrm{div}_{\mathscr{X}^+}(\omega)$ of the closure $\overline{\Delta_{X,i}}$ in $\mathscr{X}$ of $\Delta_{X,i}$ is at least $m(1 - a_i)$.

(4.1.3) Given a Δ_X-logarithmic m-pluricanonical form ω, we can consider it as a rational section of $\omega_{X/K}^{\otimes m}$. Hence, we can associate to ω the weight function wt_ω as in [MN15]. If $\mathscr{X}^+$ is a log-regular model of X^+, then

$$\mathrm{wt}_\omega(x) = v_x(\mathrm{div}_{\mathscr{X}^+}(\omega)) + m$$

for every point x of $\mathrm{Sk}(\mathscr{X}^+)$. This gives us an interpretation of the weight function in terms of logarithmic differentials, that we will use in the following.

(4.1.4) We recall from [MN15], Section 4.5 that the Kontsevich-Soibelman skeleton $\mathrm{Sk}(X, \Delta_X, \omega)$ is the closure in $\mathrm{Bir}(X)$ of the set $\mathrm{Div}(X)$ of divisorial points of X^{an} where the weight function wt_ω reaches its minimal weight, namely

$$\mathrm{wt}_\omega(X, \Delta_X) = \inf\{\mathrm{wt}_\omega(x) \,|\, x \in \mathrm{Div}(X)\} \in \mathbb{R} \cup \{-\infty\}.$$

A priori the weight function associated to a rational pluricanonical form may have minimal weight $-\infty$, hence the corresponding Kontsevich-Soibelman skeleton would be empty. We prove that this does not occur for Δ_X-logarithmic pluricanonical forms.

Proposition 4.1.5. *Given a Δ_X-logarithmic m-pluricanonical form ω, for any log-regular model $\mathscr{X}^+$ of X^+ the inclusion $\mathrm{Sk}(X, \Delta_X, \omega) \subseteq \mathrm{Sk}(\mathscr{X}^+)$ holds.*

Proof. Let $\mathscr{X}^+$ be a log-regular model of X^+ and let y be a divisorial point of X^{an}. It suffices to prove that

$$\mathrm{wt}_\omega(y) \geqslant \mathrm{wt}_\omega(\rho_{\mathscr{X}}(y))$$

and that the equality holds if and only if y is in $\mathrm{Sk}(\mathscr{X}^+)$. We consider the blow-up of $\mathscr{X}^+$ at the closure of $\mathrm{red}_{\mathscr{X}}(y)$. By [KM08], Lemma 2.45[1] iterating this procedure a finite number of times, we obtain a log-regular model $\mathscr{Y}^+$ such that $y \in \mathrm{Sk}(\mathscr{Y}^+)$.

Let $h : \mathscr{X}^+ \to \mathscr{W}^+$ be a morphism of this sequence. If $\overline{\{\mathrm{red}_{\mathscr{W}}(y)\}}$ is a stratum of $D_{\mathscr{W}}$, then the morphism h induces a subdivision of the skeleton $\mathrm{Sk}(\mathscr{W}^+)$, so $\rho_{\mathscr{X}}(y) = \rho_{\mathscr{W}}(y)$.

Otherwise, $\overline{\{\mathrm{red}_{\mathscr{W}}(y)\}}$ is strictly contained in a stratum V of $\mathcal{D}_{\mathscr{W}}$. Let E be the exceptional divisor of h, $r = r_h + r_v$ be the codimension of V in $\mathscr{W}$, where r_h and r_v are the number of irreducible components of $\lceil \Delta_X \rceil$ and respectively of the special fiber $\mathscr{W}_k$, containing V. Let $r + j$ be the codimension of $\overline{\{\mathrm{red}_{\mathscr{W}}(y)\}}$,

---

[1]The proof in [KM08] considers the case of a variety over a field, but it generalizes to varieties of finite type over a discrete valuation ring.

where $j \geqslant 1$. We denote the projections onto S^+ by $s_{\mathscr{W}} : \mathscr{W}^+ \to S^+$ and $s_{\mathscr{Z}} : \mathscr{Z}^+ \to S^+$ and by π a uniformizer in R. Then we have that

$$
\begin{aligned}
h^*(\omega_{\mathscr{W}^+/S^+}^{\log}) &= h^*\big(\omega_{\mathscr{W}/R} \otimes \mathcal{O}_{\mathscr{W}}(\overline{\lceil \Delta_X \rceil}_{,\mathrm{red}} + \mathscr{W}_{k,\mathrm{red}} - s_{\mathscr{W}}^*(\pi))\big) \\
&= \omega_{\mathscr{Z}/R} \otimes \mathcal{O}_{\mathscr{Z}}((1 - r - j)E + \overline{\lceil \Delta_X \rceil}_{,\mathrm{red}} + r_h E + \mathscr{Z}_{k,\mathrm{red}} + (r_v - 1)E - s_{\mathscr{Z}}^*(\pi)) \\
&= \omega_{\mathscr{Z}^+/S^+}^{\log} \otimes \mathcal{O}_{\mathscr{Z}}(-jE).
\end{aligned}
$$

It follows that $\mathrm{div}_{\mathscr{Z}^+}(\omega) = h^*(\mathrm{div}_{\mathscr{W}^+}(\omega)) + mjE$, so

$$
\begin{aligned}
v_{\rho_{\mathscr{Z}}(y)}(\mathrm{div}_{\mathscr{Z}^+}(\omega)) &= v_{\rho_{\mathscr{Z}}(y)}\big(h^*(\mathrm{div}_{\mathscr{W}^+}(\omega)) + mjE\big) \\
&\geqslant v_{\rho_{\mathscr{W}}(y)}(\mathrm{div}_{\mathscr{W}^+}(\omega)) + mjv_{\rho_{\mathscr{W}}(y)}(E) \\
&> v_{\rho_{\mathscr{W}}(y)}(\mathrm{div}_{\mathscr{W}^+}(\omega))
\end{aligned}
$$

where for the first inequality we apply [MN15], Proposition 3.1.6, while the second strict inequality holds as $j > 0$ and $v_{\rho_{\mathscr{W}}(y)}(E) > 0$ since the center of the valuation $\rho_{\mathscr{W}}(y)$ is contained in E. Therefore, for any such morphism h, the weight is strictly increasing, namely $\mathrm{wt}_\omega(\rho_{\mathscr{Z}}(y)) > \mathrm{wt}_\omega(\rho_{\mathscr{W}}(y))$. This concludes the proof. $\qquad\square$

Only the components of Δ_X with coefficient $a_i = 1$ determine strata that are contained in the Kontsevich-Soibelman skeleta. The introduction of Δ_X-log pluricanonical forms allows us to construct non-empty Kontsevich-Soibelman skeleta even for varieties with Kodaira dimension $-\infty$, as in the following examples:

Example 4.1.6. Let X be the projective line $\mathbb{P}_K^1$ with affine coordinates x and y, and $\Delta_X = (0 : 1) + (1 : 0)$. Then $a_i = 1$ for any i and there exist Δ_X-logarithmic canonical forms. For example, we consider

$$
\omega = \frac{dx}{x} = -\frac{dy}{y}.
$$

Let $\mathscr{X} = \mathbb{P}_R^1$ and $D_{\mathscr{X}} = (0 : 1) + (1 : 0) + \mathbb{P}_k^1$. The log scheme $\mathscr{X}^+ = (\mathscr{X}, D_{\mathscr{X}})$ is a log-regular model of $X^+ = (X, \lceil \Delta_X \rceil)$ and the associated skeleton $\mathrm{Sk}(\mathscr{X}^+)$ looks like this:
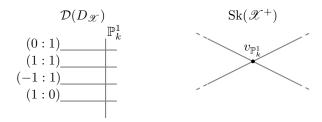

Since $\mathrm{div}_{\mathscr{X}^+}(\omega) = 0$, the weight associated to ω is minimal at any point of the skeleton $\mathrm{Sk}(\mathscr{X}^+)$. Thus $\mathrm{Sk}(X, \Delta_X, \omega) = \mathrm{Sk}(\mathscr{X}^+) \simeq \mathbb{R}$.

Example 4.1.7. Let $X = \mathbb{P}_K^1$ and $\Delta_X = \frac{1}{2}(0 : 1) + \frac{1}{2}(1 : 0) + \frac{1}{2}(1 : 1) + \frac{1}{2}(-1 : 1)$. So $a_i = \frac{1}{2}$ for any i and there exist Δ_X-logarithmic 2-pluricanonical forms. We set

$$
\omega = \frac{1}{x+1} \cdot \frac{1}{x-1} \cdot \frac{1}{x}(dx)^2 = \frac{1}{y+1} \cdot \frac{1}{1-y} \cdot \frac{1}{y}(dy)^2.
$$

We consider $\mathscr{X} = \mathbb{P}_R^1$ and $D_{\mathscr{X}} = (0 : 1) + (1 : 0) + (1 : 1) + (-1 : 1) + \mathbb{P}_k^1$, then $\mathscr{X}^+ = (\mathscr{X}, D_{\mathscr{X}})$ is a log-regular model of $X^+ = (X, \lceil \Delta_X \rceil)$ and $\mathrm{Sk}(\mathscr{X}^+)$ is

$$\mathcal{D}(D_{\mathscr{X}}) \qquad\qquad \mathrm{Sk}(\mathscr{X}^+)$$

Since $\mathrm{div}_{\mathscr{X}^+}(\omega) = (0:1) + (1:0) + (1:1) + (-1:1)$, the weight associated to ω is minimal at the divisorial point $v_{\mathbb{P}^1_k}$ corresponding to $\mathbb{P}^1_k$ and is strictly increasing with slope 1 along the unbounded edges, when we move away from the point $v_{\mathbb{P}^1_k}$. Therefore, $\mathrm{Sk}(X, \Delta_X, \omega) = \{v_{\mathbb{P}^1_k}\}$.

4.2. Weight function and Kontsevich-Soibelman skeleton for products.

(4.2.1) Let $\mathscr{X}^+$ and $\mathscr{Y}^+$ be log-smooth models over S^+ of $X^+ = (X, \lceil\Delta_X\rceil)$ and $Y^+ = (Y, \lceil\Delta_Y\rceil)$ respectively. Then, the fs fibred product $\mathscr{Z}^+ = \mathscr{X}^+ \times_{S^+}^{\mathrm{fs}} \mathscr{Y}^+$ is a log-regular model of $Z^+ := X^+ \times_K^{\mathrm{fs}} Y^+$. Therefore, given ω_{X^+} and ω_{Y^+} Δ_X-logarithmic and Δ_Y-logarithmic m-pluricanonical forms on (X, Δ_X) and (Y, Δ_Y) respectively, the form

$$\varpi = \mathrm{pr}^*_{X^+}\,\omega_{X^+} \otimes \mathrm{pr}^*_{Y^+}\,\omega_{Y^+}$$

is a Δ_Z-logarithmic m-pluricanonical form on (Z, Δ_Z), where $\Delta_Z = X \times_K \Delta_Y + \Delta_X \times_K Y$. Viewing these forms as rational sections of logarithmic m-pluricanonical bundles, we see that $\mathrm{div}_{\mathscr{Z}^+}(\varpi) = \mathrm{div}_{\mathscr{Z}^+}(\mathrm{pr}^*_{X^+}\,\omega_{X^+} \otimes \mathrm{pr}^*_{Y^+}\,\omega_{Y^+})$ according to (2.4.6).

(4.2.2) Let z be a point of $F_{\mathscr{Z}} \cap \mathscr{Z}_k$; as before, we denote by x and y the images of z under the local isomorphism $F_{\mathscr{Z}} \to F_{\mathscr{X}} \times_{F_S} F_{\mathscr{Y}}$. Any morphism $\varepsilon \in \sigma_z$ defines a point $v_{z,\varepsilon}$ in $\mathrm{Sk}(\mathscr{Z}^+)$. For the sake of convenience, we simply denote the valuations by the corresponding morphism and we denote $\alpha = \mathrm{pr}_{\mathscr{X}}(\varepsilon)$ and $\beta = \mathrm{pr}_{\mathscr{Y}}(\varepsilon)$. We aim to relate the valuation $v_\varepsilon\big(\mathrm{div}_{\mathscr{Z}^+}(\varpi)\big)$ to the values

$$v_\alpha\big(\mathrm{div}_{\mathscr{X}^+}(\omega_{X^+})\big),\ v_\beta\big(\mathrm{div}_{\mathscr{Y}^+}(\omega_{Y^+})\big).$$

(4.2.3) Let $f_x \in \mathcal{O}_{\mathscr{X},x}$ be a local equation of $\mathrm{div}_{\mathscr{X}^+}(\omega_{X^+})$ around x. In order to evaluate $v_{x,\alpha}$ on f_x, we consider an admissible expansion of f_x as in (3.2.6)

$$f_x = \sum_{\gamma \in \mathcal{C}_{\mathscr{X},x}} c_\gamma \gamma.$$

Furthermore, this expansion induces also an expansion of $\mathrm{pr}^*_{\mathscr{X}}(f_x)$ by

$$\mathrm{pr}^*_{\mathscr{X}}(f_x) = \sum_{\gamma \in \mathcal{C}_{\mathscr{X},x}} \mathrm{pr}^*_{\mathscr{X}}(c_\gamma) \gamma$$

as formal power series in $\widehat{\mathcal{O}}_{\mathscr{Z},z}$, since the morphism of characteristic sheaves $\mathcal{C}_{\mathscr{X},x} \hookrightarrow \mathcal{C}_{\mathscr{Z},z}$ is injective. Following the same procedure for a local equation $f_y \in \mathcal{O}_{\mathscr{Y},y}$ of $\mathrm{div}_{\mathscr{Y}^+}(\omega_{Y^+})$ around y, we get an expansion of f_y that extends to $\mathrm{pr}^*_{\mathscr{Y}}(f_y)$:

$$f_y = \sum_{\delta \in \mathcal{C}_{\mathscr{Y},y}} d_\delta \delta.$$

(4.2.4) A local equation of ϖ around z is determined by $\mathrm{pr}^*_{\mathscr{X}}(f_x)\,\mathrm{pr}^*_{\mathscr{Y}}(f_y)$. Thus

$$v_\varepsilon(\mathrm{div}_{\mathscr{Z}^+}(\varpi)) = v_\varepsilon\big(\mathrm{pr}^*_{\mathscr{X}}(f_x)\,\mathrm{pr}^*_{\mathscr{Y}}(f_y)\big)$$

and by multiplicativity of the valuation v_ε

$$v_\varepsilon\big(\mathrm{pr}^*_{\mathscr{X}}(f_x)\,\mathrm{pr}^*_{\mathscr{Y}}(f_y)\big) = v_\varepsilon\big(\mathrm{pr}^*_{\mathscr{X}}(f_x)\big) + v_\varepsilon\big(\mathrm{pr}^*_{\mathscr{Y}}(f_y)\big).$$

Recalling Remark 3.2.13, the valuation can be computed as follows

$$v_\varepsilon\big(\mathrm{pr}_{\mathscr{X}}^*(f_x)\big) = \min\{\varepsilon(\gamma)\,|\,c_\gamma \neq 0\};$$

as the elements γ belong to $\mathcal{C}_{\mathscr{X},x}$ and α is defined to be $\mathrm{pr}_{\mathscr{X}}(\varepsilon)$

$$\min\{\varepsilon(\gamma)\,|\,c_\gamma \neq 0\} = \min\{\alpha(\gamma)\,|\,c_\gamma \neq 0\} = v_{x,\alpha}(f_x).$$

Hence, we conclude that

(4.2.5)
$$\begin{aligned}
v_\varepsilon\big(\mathrm{div}_{\mathscr{Z}+}(\varpi)\big) &= v_\varepsilon\big(\mathrm{pr}_{\mathscr{X}}^*(f_x)\big) + v_\varepsilon\big(\mathrm{pr}_{\mathscr{Y}}^*(f_y)\big) \\
&= v_\alpha(f_x) + v_\beta(f_y) \\
&= v_\alpha\big(\mathrm{div}_{\mathscr{X}+}(\omega_{X+})\big) + v_\beta\big(\mathrm{div}_{\mathscr{Y}+}(\omega_{Y+})\big).
\end{aligned}$$

(4.2.6) This result turns out to be advantageous to compute the weight function wt_ϖ on divisorial points of $\mathrm{Sk}(\mathscr{Z}^+)$:

(4.2.7)
$$\begin{aligned}
\mathrm{wt}_\varpi(\varepsilon) &= v_\varepsilon\big(\mathrm{div}_{\mathscr{Z}+}(\varpi)\big) + m \\
&= v_\alpha\big(\mathrm{div}_{\mathscr{X}+}(\omega_{X+})\big) + v_\beta\big(\mathrm{div}_{\mathscr{Y}+}(\omega_{Y+})\big) + m \\
&= \mathrm{wt}_{\omega_{X+}}(\alpha) + \mathrm{wt}_{\omega_{Y+}}(\beta) - m.
\end{aligned}$$

(4.2.8) Under the notations of the previous paragraphs, our computations lead to the following result.

Theorem 4.2.9. *Suppose that the residue field k is algebraically closed and that $\mathscr{X}^+$ is semistable. Then, the PL homeomorphism of skeleta*

$$\mathrm{Sk}(\mathscr{Z}^+) \xrightarrow{\sim} \mathrm{Sk}(\mathscr{X}^+) \times \mathrm{Sk}(\mathscr{Y}^+)$$

given in Proposition 3.4.3 restricts to a PL homeomorphism of Kontsevich-Soibelman skeleta

$$\mathrm{Sk}(Z, \Delta_Z, \varpi) \xrightarrow{\sim} \mathrm{Sk}(X, \Delta_X, \omega_{X+}) \times \mathrm{Sk}(Y, \Delta_Y, \omega_{Y+}).$$

Proof. This follows immediately from the equality (4.2.7) that shows that a point in $\mathrm{Sk}(Z, \Delta_Z)$ has minimal value $\mathrm{wt}_\varpi(Z, \Delta_Z)$ if and only if its projections have minimal value $\mathrm{wt}_{\omega_{X+}}(X, \Delta_X)$ and $\mathrm{wt}_{\omega_{Y+}}(Y, \Delta_Y)$. $\qquad\square$

5. The essential skeleton of a product

We will need a few notions from birational geometry, see [KM08]. Let (X, Δ_X) be a pair such that X is normal, Δ_X is effective, and $K_X + \Delta_X$ is $\mathbb{Q}$-Cartier. Then we say that (X, Δ_X) is *log canonical* if for every log resolution $f\colon Z \to X$, in the formula

$$K_Z + \Delta_Z = f^*(K_X + \Delta_X) + \sum a_D D,$$

where Δ_Z is the strict transform of Δ_X plus the reduced exceptional divisor, all the a_D are non-negative. The sum ranges over all components of Δ_Z. In fact the quantity a_D, called the *log discrepancy* of D with respect to (X, Δ_X), depends only on the valuation corresponding to D, and this condition needs only be tested on a single log resolution.

We will be most interested in the case where (X, Δ_X) satisfies the stronger condition of being *divisorially log terminal*, or dlt. A closed subset $Y \subset X$ is called a *log canonical center* if for some (respectively any) log resolution, Y is the image of a divisor D with $a_D = 0$. The pair (X, Δ_X) is said to be dlt if for every log canonical center Y, there is a neighborhood of the generic point of Y where (X, Δ_X) is snc.

We say that a pair (X, Δ) is *Kawamata log terminal*, or klt, if it is dlt and the coefficients of Δ are all strictly less than 1.

5.1. **Essential skeleton of a pair.**

(5.1.1) Let (X, Δ_X) be a pair such that $X^+ = (X, \lceil \Delta_X \rceil)$ is a log-regular log scheme. Let ω be a non-zero regular Δ_X-logarithmic m-pluricanonical form on (X, Δ_X). Let $\mathscr{X}^+$ be a log-smooth model of X^+ over S^+. Then, the associated divisor $\operatorname{div}_{\mathscr{X}^+}(\omega)$ is effective. There exist minimal positive integers d and n such that the divisor $\operatorname{div}_{\mathscr{X}^+}(\omega^{\otimes d} \pi^{-n})$ is effective and the multiplicity of some component of the special fiber is zero: we denote this divisor by $D_{\min}(\mathscr{X}, \omega)$. It follows from the properties of the weight function (see [MN15], Proposition 4.4.5) that for any $x \in \operatorname{Sk}(\mathscr{X}^+)$

$$v_x(D_{\min}(\mathscr{X}, \omega)) + dm = \operatorname{wt}_{\omega^{\otimes d} \pi^{-n}}(x) = d \cdot \operatorname{wt}_\omega + v_x(\pi^{-n}) = d \cdot \operatorname{wt}_\omega(x) - n.$$

Lemma 5.1.2. *Let $v_{x,\alpha}$ be a divisorial point in $\operatorname{Sk}(\mathscr{X}^+)$, then $v_{x,\alpha} \in \operatorname{Sk}(X, \Delta_X, \omega)$ if and only if $D_{\min}(\mathscr{X}, \omega)$ does not contain x.*

Proof. We denote $v_{x,\alpha}$ simply by α. By the above series of equalities, the weight function wt_ω reaches its minimum at α if and only if $v_\alpha(D_{\min}(\mathscr{X}, \omega))$ is minimal, hence in particular equal to zero.

Let $h : \mathscr{Y}^+ \to \mathscr{X}^+$ be a sequence of blow-up morphisms of strata of $D_{\mathscr{X}}$ such that α corresponds to an irreducible component E of $D_{\mathscr{Y}}$. As in the proof of Proposition 4.1.5

$$h^*(D_{\min}(\mathscr{X}, \omega)) = h^*(\operatorname{div}_{\mathscr{X}^+}(\omega^{\otimes d} \pi^{-n})) = \operatorname{div}_{\mathscr{Y}^+}(\omega^{\otimes d} \pi^{-n}).$$

Therefore we have that $v_\alpha(D_{\min}(\mathscr{X}, \omega)) > 0$ if and only if $E \subseteq \operatorname{div}_{\mathscr{Y}^+}(\omega^{\otimes d} \pi^{-n})$, and this holds if and only if $x \in D_{\min}(\mathscr{X}, \omega)$. $\square$

(5.1.3) We define the essential skeleton $\operatorname{Sk}(X, \Delta_X)$ of (X, Δ_X) as the union of all Kontsevich-Soibelman skeleta $\operatorname{Sk}(X, \Delta_X, \omega)$, where ω ranges over all regular Δ_X-logarithmic pluricanonical forms. In the case of an empty boundary, this recovers the notions introduced in [MN15].

The reason to define the essential skeleton this way is that it behaves nicely under birational morphisms. Let $f : X' \to X$ be a log resolution. Then there is a $\mathbb{Q}$-divisor Γ with snc support, and no coefficient exceeding 1, such that $K_{X'} + \Gamma' = f^*(K_X + \Delta_X)$. Take $\Delta_{X'}$ to be the positive part of Γ' and write

$$K_{X'} + \Delta_{X'} = f^*(K_X + \Delta_X) + N.$$

for any m, pullback along with multiplication by the divisor of discrepancies N induces an isomorphism of vector spaces

$$(5.1.4) \qquad H^0(X, mK_X + m\Delta_X) \cong H^0(X', mK_{X'} + m\Delta_{X'}).$$

Let ω and ω' be corresponding forms via this isomorphism.

Proposition 5.1.5. *Under the identification of the birational points of X with those of X', $\operatorname{Sk}(X, \Delta_X, \omega)$ is identified with $\operatorname{Sk}(X', \Delta_{X'}, \omega')$.*

Proof. The Kontsevich-Soibelman skeleton $\operatorname{Sk}(X, \Delta_X, \omega)$ is contained in the skeleton associated to any log-regular model by Proposition 4.1.5, so we choose log-regular models $\mathscr{X}^+$ and $\mathscr{X}'^+$ so that f extends to a log resolution $f_R : \mathscr{X}' \to \mathscr{X}$. We denote by $\Delta_{\mathscr{X}} = \overline{\Delta_X} + \mathscr{X}_{k,\mathrm{red}}$ and $\Delta_{\mathscr{X}'} = \overline{\Delta_{X'}} + \mathscr{X}'_{k,\mathrm{red}}$. It suffices to check the proposition for divisorial valuations. By Lemma 5.1.2 a divisorial valuation v of $\operatorname{Sk}(\mathscr{X}^+)$ is in $\operatorname{Sk}(X, \Delta_X, \omega)$ if and only if it is a log canonical center of $(\mathscr{X}, \Delta_{\mathscr{X}})$ and the divisor $D_{\min}(\mathscr{X}, \omega)$ does not contain the center of v.

Suppose v is a divisorial point in $\operatorname{Sk}(X, \Delta_X, \omega)$. Without loss of generality, we can assume that the divisor $\operatorname{div}_{\mathscr{X}^+}(\omega)$ in $\mathscr{X}^+$ does not contain the center of v. As $(\mathscr{X}, \Delta_{\mathscr{X}})$ is a dlt pair, we have that

$$K_{\mathscr{X}'} + \Delta_{\mathscr{X}'} = f_R^*(K_{\mathscr{X}} + \Delta_{\mathscr{X}}) + M$$

where M is effective, thus $\operatorname{div}_{\mathscr{X}'+}(\omega') = f_R^*(\operatorname{div}_{\mathscr{X}+}(\omega)) + mM$. As v is a log canonical center of $(\mathscr{X}, \Delta_{\mathscr{X}})$, M does not vanish along v, so neither does $\operatorname{div}_{\mathscr{X}'+}(\omega')$. Likewise v is a log canonical center of $(\mathscr{X}', \Delta_{\mathscr{X}'})$. It follows that $v \in \operatorname{Sk}(X', \Delta_{X'}, \omega')$.

Conversely, if v is a divisorial point in $\operatorname{Sk}(X', \Delta_{X'}, \omega')$, it is a log canonical center of $(\mathscr{X}', \Delta_{\mathscr{X}'})$ and the divisor $D_{\min}(\mathscr{X}', \omega')$ does not contain the center of v. As a result, v is also a log canonical center of $(\mathscr{X}, \Delta_{\mathscr{X}})$, and the divisor $D_{\min}(\mathscr{X}, \omega)$ does not contain the center of v since its pullback does not. $\square$

We define the Kontsevich-Soibelman skeleton $\operatorname{Sk}(X, \Delta_X, \omega)$ of a dlt pair (X, Δ_X) as the Kontsevich-Soibelman skeleton $\operatorname{Sk}(X', \Delta_{X'}, \omega')$ where $(X', \Delta_{X'})$ is any log-resolution of (X, Δ_X), and ω' is the forms corresponding to ω under the isomorphism (5.1.4): Proposition 5.1.5 guarantees that this is well-defined. It follows that we can define the essential skeleton $\operatorname{Sk}(X, \Delta_X)$ of a dlt pair (X, Δ_X) as the essential skeleton of any log-resolution of (X, Δ_X).

Moreover, notice that our construction works more generally for log canonical pairs, hence the notions of Kontsevich-Soibelman skeleta and essential skeleton generalize to such pairs.

(5.1.6) Suppose that (X, Δ_X) is a dlt pair over K, such that $K_X + \Delta_X$ is semiample. Suppose thar $\mathscr{X}$ is a good dlt minimal model of (X, Δ_X) over R and let $\Delta_{\mathscr{X}} = \overline{\Delta_X} + \mathscr{X}_{k,\mathrm{red}}$. We denote by $\mathcal{D}_0^{=1}(\mathscr{X}, \Delta_{\mathscr{X}})$ the dual complex of the strata of the coefficient 1 part of $\Delta_{\mathscr{X}}$ that lie in the special fier.

We consider a log resolution $f : X' \to X$ that extends to a log resolution of $(\mathscr{X}, \Delta_{\mathscr{X}})$. We write $K_{X'} + \Gamma' = f^*(K_X + \Delta_X)$. Let $\Delta_{X'}$ be the positive part of Γ'. We may embed the open dual complex $\mathcal{D}_0^{=1}(\mathscr{X}, \Delta_{\mathscr{X}})$ into the birational points of X.

Proposition 5.1.7. *This embedding identifies $\mathcal{D}_0^{=1}(\mathscr{X}, \Delta_{\mathscr{X}})$ with $\operatorname{Sk}(X', \Delta_{X'})$.*

Proof. Choose a regular R-model $\mathscr{X}'$ for X' which is a log resolution of $(\mathscr{X}, \Delta_{\mathscr{X}})$ extending f and let $\Delta_{\mathscr{X}'} = \overline{\Delta_{X'}} + \mathscr{X}'_{k,\mathrm{red}}$, namely we have the log resolutions

$$(X', \Delta_{X'}) \to (X, \Delta_X) \qquad \text{where } (X, \Delta_X) \text{ dlt and } \Delta_{X'} \text{ is the positive part of } \Gamma'$$

$$(\mathscr{X}', \Delta_{\mathscr{X}'}) \to (\mathscr{X}, \Delta_{\mathscr{X}}) \qquad \text{where } \Delta_{\mathscr{X}} = \overline{\Delta_X} + \mathscr{X}_{k,\mathrm{red}} \text{ and } \Delta_{\mathscr{X}'} = \overline{\Delta_{X'}} + \mathscr{X}'_{k,\mathrm{red}}.$$

As in the previous proof, it suffices to check the proposition for divisorial valuations. Let v be a divisorial valuation, and suppose $v \in \mathcal{D}_0^{=1}(\mathscr{X}, \Delta_{\mathscr{X}})$. Then v is a log canonical center for $(\mathscr{X}, \Delta_{\mathscr{X}})$, so v is also a log canonical center of $(\mathscr{X}', \Delta_{\mathscr{X}'})$. For a sufficiently divisible index, we may find a Δ_X-logarithmic pluricanonical form on (X, Δ_X) whose associated divisor in $\mathscr{X}$ has vanishing locus C such that C is a divisor not containing the center of v. After pullback, we get a $\Delta_{X'}$-logarithmic pluricanonical form ω' whose associated divisor in $\mathscr{X}'$ is supported on the strict transform of C and the exceptional divisors of positive log discrepancy. But none of these contain v. Thus, $v \in \operatorname{Sk}(X', \Delta_{X'}, \omega')$.

Conversely, if v is a divisorial point in $\operatorname{Sk}(X', \Delta_{X'})$, then v is a log canonical center of $(\mathscr{X}', \Delta_{\mathscr{X}'})$, so v is a log canonical center of $(\mathscr{X}, \Delta_{\mathscr{X}})$, hence an element of the open dual complex $\mathcal{D}_0^{=1}(\mathscr{X}, \Delta_{\mathscr{X}})$. $\square$

Proposition 5.1.7 compares the essential skeleton of (X, Δ_X) to the skeleton of a good minimal dlt model of (X, Δ_X). Thus, the result can be restated as follows: if (X, Δ_X) is a dlt pair with $K_X + \Delta_X$ semiample and $(\mathscr{X}, \Delta_{\mathscr{X}})$ is a good dlt minimal model of (X, Δ_X) over R, then $\mathcal{D}_0^{=1}(\mathscr{X}, \Delta_{\mathscr{X}}) = \operatorname{Sk}(X, \Delta_X)$. This generalizes [NX16], Theorem 3.3.3 to dlt pairs.

5.2. **Essential skeleta and products of log-regular models.**

(5.2.1) We say that a pair (X, Δ_X) has non-negative Kodaira-Iitaka dimension if some multiple of the line bundle $K_X + \Delta_X$ has a regular section.

Theorem 5.2.2. *Assume that the residue field k is algebraically closed. Let (X, Δ_X) and (Y, Δ_Y) be pairs such that $X^+ = (X, \lceil \Delta_X \rceil)$ and $Y^+ = (Y, \lceil \Delta_Y \rceil)$ are log-regular log scheme over K. Suppose that both pairs have non-negative Kodaira-Iitaka dimension and both admit semistable log-regular models $\mathscr{X}^+$ and $\mathscr{Y}^+$ over S^+. Then, the PL homeomorphism of skeleta*

$$\mathrm{Sk}(\mathscr{Z}^+) \xrightarrow{\sim} \mathrm{Sk}(\mathscr{X}^+) \times \mathrm{Sk}(\mathscr{Y}^+)$$

of Proposition 3.4.3 induces a PL homeomorphism of essential skeleta

$$\mathrm{Sk}(Z, \Delta_Z) \xrightarrow{\sim} \mathrm{Sk}(X, \Delta_X) \times \mathrm{Sk}(Y, \Delta_Y)$$

where $\mathscr{Z}^+$, Z and Δ_Z are the respective products.

Proof. It follows immediately from Theorem 4.2.9 that we have the inclusion $\mathrm{Sk}(X, \Delta_X) \times \mathrm{Sk}(Y, \Delta_Y) \subseteq \mathrm{Sk}(Z, \Delta_Z)$. Thus, we reduce to prove the following statement. Let $v_{z,\varepsilon}$ be a divisorial point in $\mathrm{Sk}(\mathscr{Z}^+)$ and $(v_{x,\alpha}, v_{y,\beta})$ be the corresponding pair in $\mathrm{Sk}(\mathscr{X}^+) \times \mathrm{Sk}(\mathscr{Y}^+)$ under the isomorphism of Proposition 3.4.3; if $v_{z,\varepsilon}$ lies in the essential skeleton $\mathrm{Sk}(Z, \Delta_Z)$, then $v_{x,\alpha}$ lies in $\mathrm{Sk}(X, \Delta_X)$.

Assume that $v_{z,\varepsilon}$ lies in the essential skeleton $\mathrm{Sk}(Z, \Delta_Z)$. Then there exists a non-zero regular Δ_Z-logarithmic m-pluricanonical form ω on Z^+, such that $v_{z,\varepsilon} \in \mathrm{Sk}(Z, \Delta_Z, \omega)$. By Lemma 5.1.2, $D_{\min}(\mathscr{Z}, \omega)$ does not contain z.

Let E be an irreducible component of $\mathscr{Y}_k$ containing y and denote by ξ_E the generic point of E. Then the point in the Kato fan of $\mathscr{Z}^+$ corresponding to (x, ξ_E) is not contained in $D_{\min}(\mathscr{Z}, \omega)$, as otherwise z would be contained in it.

As k is algebraically closed, we can choose a k-rational point p in E such that p is contained in no other components of $D_{\mathscr{Y}}$ and $D_{\min}(\mathscr{Z}, \omega)$ does not contained the locus $\overline{\{x\}} \times_R \{p\}$. By Hensel's Lemma and the assumption of semistability, p can be lifted to an R-rational point of $\mathscr{Y}$. The pull-back of $\mathscr{Z}^+$ along this R-rational point is an embedding $i : \mathscr{X}^+ \to \mathscr{Z}^+$, so we have the diagram

$$
\begin{array}{ccccc}
\mathscr{X}^+ & \xrightarrow{\ i\ } & \mathscr{Z}^+ & \xrightarrow{\ \mathrm{pr}_{\mathscr{X}}\ } & \mathscr{X}^+ \\
\downarrow & & \downarrow{\scriptstyle \mathrm{pr}_{\mathscr{Y}}} & & \downarrow \\
S & \longrightarrow & \mathscr{Y}^+ & \longrightarrow & S.
\end{array}
$$

Since S has trivial normal bundle in $\mathscr{Y}$, we have that

$$\omega^{\log}_{X^+/K} = i^*\big(\omega^{\log}_{Z^+/K}\big),$$

so $i^*(\omega)$ is a non-zero regular logarithmic m-pluricanonical form on X^+ and in particular is a regular Δ_X-logarithmic m-pluricanonical form. Moreover, $D_{\min}(\mathscr{X}, i^*(\omega)) = i^*(D_{\min}(\mathscr{Z}, \omega))$. Finally, x is not contained in $D_{\min}(\mathscr{X}, i^*(\omega))$, as otherwise $i(x) = \overline{\{x\}} \times_R \{p\}$ would be contained in $D_{\min}(\mathscr{Z}, \omega)$. By Lemma 5.1.2, x is a point of $\mathrm{Sk}(X, \Delta_X, i^*(\omega))$ and this concludes the proof. $\qquad \square$

Remark 5.2.3. Consider the case where the line bundles $K_X + \Delta_X$ and $K_Y + \Delta_Y$ are semi-ample, i.e. some multiple of them is base point free. It follows from the arguments of [NX16], Theorem 3.3.3 that the essential skeleton of (Z, Δ_Z) is a finite union of Kontsevich-Soibelman skeleta where the union runs through a generating set of global sections of a sufficiently large multiple of $K_Z + \Delta_Z$. We can construct such a set from generating sets of global sections of multiples of $K_X + \Delta_X$ and $K_Y + \Delta_Y$ respectively, via tensor product. Then, in this case, the result of Theorem 5.2.2 follows directly from Theorem 4.2.9.

5.3. **Essential skeleta and products of dlt models.**

(5.3.1) We need a combinatorial lemma to understand the formally local behavior of products of semistable dlt models.

Lemma 5.3.2. *Let M be the monoid generated by $r_1 \ldots r_{n_1}$, $s_1 \ldots s_{n_2}$ with the single relation $\sum_{i=1}^{n_1} r_i = \sum_{j=1}^{n_2} s_j$. Then any small $\mathbb{Q}$-factorialization of the affine toric variety $W = \mathrm{Spec}\,(k[M])$ associated to M is a log resolution.*

Proof. We calculate the fan of W. Let N be the dual lattice of M. The fan associated to W is the cone of elements of $N \otimes \mathbb{R}$ which are non-negative on M. We consider these as linear functions l on the vector space spanned by the r_i and s_i, subject to the restriction that $l(\sum_{i=1}^{n_1} r_i) = l(\sum_{j=1}^{n_2} s_j)$. Let x_{ij} be the function which is 1 on r_i and s_j and 0 on all others. Then the fan of W is given by the single cone C_W spanned by the x_{ij}.

Any $\mathbb{Q}$-factorialization $\widetilde{W}$ corresponds to a simplicial subdivision of the cone C_W (see [Ful93], p.65). We now check that every choice of $\widetilde{W}$ is non-singular.

A maximal cone of $\widetilde{W}$ is spanned by $n = n_1 + n_2 - 1$ independent rays of C_W. Each ray of C_W corresponds to a choice of x_{ij}, and we can index these by edges of the complete bipartite graph B on the r_i and s_j. These x_{ij} are independent if and only if the corresponding edges form a spanning tree. Let $w_1 \ldots w_n$ span a maximal cone of $\widetilde{W}$. On this affine chart, $\widetilde{W}$ is smooth if and only if the w_i generate N as a lattice. We have shown already that the x_{ij} generate N. But every x_{ij} is either one of the w_i, or it completes a cycle in B, so that it is a $\mathbb{Z}$-linear combination of the w_i. $\qquad\square$

Proposition 5.3.3. *Let $(\mathscr{X}, \Delta_{\mathscr{X}})$ and $(\mathscr{Y}, \Delta_{\mathscr{Y}})$ be semistable good projective dlt minimal pairs over the germ of a pointed curve C. The product $(\mathscr{Z}, \Delta_{\mathscr{Z}})$ is a log canonical pair, $K_{\mathscr{Z}} + \Delta_{\mathscr{Z}}$ is semiample, and the log canonical centers of $(\mathscr{Z}, \Delta_{\mathscr{Z}})$ are strata of the coefficient 1 part of $\Delta_{\mathscr{Z}}$.*

Proof. The product $\mathscr{Z}$ is normal as $\mathscr{X}$ and $\mathscr{Y}$ are semistable. The divisor $K_{\mathscr{Z}} + \Delta_{\mathscr{Z}}$ is semiample by pullback of semiample divisors.

Let $\widetilde{\mathscr{X}}$ and $\widetilde{\mathscr{Y}}$ be log resolutions of $\mathscr{X}$ and $\mathscr{Y}$. Then, we have

$$K_{\widetilde{\mathscr{X}}} + \Delta_{\widetilde{\mathscr{X}}} = f_{\mathscr{X}}^*(K_{\mathscr{X}} + \Delta_{\mathscr{X}}) + \sum a_i E_{\mathscr{X},i}$$

$$K_{\widetilde{\mathscr{Y}}} + \Delta_{\widetilde{\mathscr{Y}}} = f_{\mathscr{Y}}^*(K_{\mathscr{Y}} + \Delta_{\mathscr{Y}}) + \sum b_j E_{\mathscr{Y},j}$$

where the coefficients a_i and b_j are non-negative. Let $\widetilde{\mathscr{Z}}$ be a toroidal log resolution of the fs product $\widetilde{\mathscr{X}} \times^{\mathrm{fs}} \widetilde{\mathscr{Y}}$. In particular, $\widetilde{\mathscr{Z}}$ is a log resolution of $\mathscr{Z}$ and we can write

$$K_{\widetilde{\mathscr{Z}}} + \Delta_{\widetilde{\mathscr{Z}}} = f_{\mathscr{Z}}^*(K_{\mathscr{Z}} + \Delta_{\mathscr{Z}}) + \sum c_h E_{\mathscr{Z},h}$$

where $\Delta_{\widetilde{\mathscr{Z}}}$ is effective. Over the generic fiber, $(\mathscr{Z}, \Delta_{\mathscr{Z}})$ is dlt, so we need only compute discrepancies over the special fiber, namely study the positivity of the coefficients c_h.

Let Γ be a divisor of $\widetilde{\mathscr{Z}}$ over the special fiber, denote by v_Γ the corresponding divisorial valuation in $\mathrm{Sk}(\widetilde{\mathscr{Z}}^+)$, by $\Gamma_{\mathscr{X}}$, $\Gamma_{\mathscr{Y}}$ and $\Gamma_{\mathscr{Z}}$ its images in $\mathscr{X}$, $\mathscr{Y}$ and $\mathscr{Z}$. The projections of v_Γ in $\mathrm{Sk}(\widetilde{\mathscr{X}}^+)$ and $\mathrm{Sk}(\widetilde{\mathscr{Y}}^+)$ are divisorial valuations. Up to subdivisions of the skeleta, we can assume without loss of generality that the projections correspond to divisors $\Gamma_{\widetilde{\mathscr{X}}}$ and $\Gamma_{\widetilde{\mathscr{Y}}}$.

Choose a $\Delta_{\mathscr{X}}$-logarithmic and a $\Delta_{\mathscr{Y}}$-logarithmic pluricanonical forms $\omega_{\mathscr{X}}$ on $\mathscr{X}$ and $\omega_{\mathscr{Y}}$ on $\mathscr{Y}$ respectively, such that the divisors $\mathrm{div}_{\mathscr{X}^+}(\omega_{\mathscr{X}})$ and $\mathrm{div}_{\mathscr{Y}^+}(\omega_{\mathscr{Y}})$ do not contain $\Gamma_{\mathscr{X}}$ and $\Gamma_{\mathscr{Y}}$ respectively, where $\mathscr{X}^+ = (\mathscr{X}, \lceil \Delta_{\mathscr{X}} \rceil)$ and $\mathscr{Y}^+ = (\mathscr{Y}, \lceil \Delta_{\mathscr{Y}} \rceil)$. Then, the divisor $\mathrm{div}_{\mathscr{Z}^+}(\omega_{\mathscr{Z}})$, associated to the wedge

product $\omega_{\mathscr{Z}}$ of the pullbacks $\omega_{\mathscr{X}}$ and $\omega_{\mathscr{Y}}$ to $\mathscr{Z}$, does not contain $\Gamma_{\mathscr{Z}}$. Denote by $\omega_{\widetilde{\mathscr{X}}}$, $\omega_{\widetilde{\mathscr{Y}}}$ and $\omega_{\widetilde{\mathscr{Z}}}$ the pullback of the respective forms to $\widetilde{\mathscr{X}}$, $\widetilde{\mathscr{Y}}$ and $\widetilde{\mathscr{Z}}$. Then, we have

$$\operatorname{div}_{\widetilde{\mathscr{X}}+}(\omega_{\widetilde{\mathscr{X}}}) = f_{\mathscr{X}}^*(\operatorname{div}_{\mathscr{X}+}(\omega_{\mathscr{X}})) + \sum a_i E_{\mathscr{X},i}$$

$$\operatorname{div}_{\widetilde{\mathscr{Y}}+}(\omega_{\widetilde{\mathscr{Y}}}) = f_{\mathscr{Y}}^*(\operatorname{div}_{\mathscr{Y}+}(\omega_{\mathscr{Y}})) + \sum b_j E_{\mathscr{Y},j}$$

$$\operatorname{div}_{\widetilde{\mathscr{Z}}+}(\omega_{\widetilde{\mathscr{Z}}}) = f_{\mathscr{Z}}^*(\operatorname{div}_{\mathscr{Z}+}(\omega_{\mathscr{Z}})) + \sum c_h E_{\mathscr{Z},h}.$$

As $\operatorname{div}_{\mathscr{X}+}(\omega_{\mathscr{X}})$, $\operatorname{div}_{\mathscr{Y}+}(\omega_{\mathscr{Y}})$ and $\operatorname{div}_{\mathscr{Z}+}(\omega_{\mathscr{Z}})$ do not contain $\Gamma_{\mathscr{X}}$, $\Gamma_{\mathscr{Y}}$ and $\Gamma_{\mathscr{Z}}$ respectively, we have

$$v_{\Gamma_{\widetilde{\mathscr{X}}}}(\operatorname{div}_{\widetilde{\mathscr{X}}+}(\omega_{\widetilde{\mathscr{X}}})) = \sum a_i v_{\Gamma_{\widetilde{\mathscr{X}}}}(E_{\mathscr{X},i}) \geqslant 0$$

$$v_{\Gamma_{\widetilde{\mathscr{Y}}}}(\operatorname{div}_{\widetilde{\mathscr{Y}}+}(\omega_{\widetilde{\mathscr{Y}}})) = \sum b_j v_{\Gamma_{\widetilde{\mathscr{Y}}}}(E_{\mathscr{Y},j}) \geqslant 0$$

$$v_{\Gamma}(\operatorname{div}_{\widetilde{\mathscr{Z}}+}(\omega_{\widetilde{\mathscr{Z}}})) = \sum c_h v_{\Gamma}(E_{\mathscr{Z},h}).$$

From the formula 4.2.5, $v_{\Gamma}(\operatorname{div}_{\widetilde{\mathscr{Z}}+}(\omega_{\widetilde{\mathscr{Z}}})) = v_{\Gamma_{\widetilde{\mathscr{Y}}}}(\operatorname{div}_{\widetilde{\mathscr{Y}}+}(\omega_{\widetilde{\mathscr{Y}}})) + v_{\Gamma_{\widetilde{\mathscr{X}}}}(\operatorname{div}_{\widetilde{\mathscr{X}}+}(\omega_{\widetilde{\mathscr{X}}}))$, hence we obtain that the log discrepancy of Γ with respect to the pair $(\mathscr{Z}, \Delta_{\mathscr{Z}})$ is non-negative. Moreover, it is zero if and only if the log discrepancies of $\Gamma_{\widetilde{\mathscr{X}}}$ and $\Gamma_{\widetilde{\mathscr{Y}}}$ are both zero, namely if and only if $\Gamma_{\mathscr{X}}$ and $\Gamma_{\mathscr{Y}}$ are log canonical centers of $(\mathscr{X}, \Delta_{\mathscr{X}})$ and $(\mathscr{Y}, \Delta_{\mathscr{Y}})$ respectively. Since for dlt pairs the log canonical centers are the strata of the coefficient 1 part of the boundary, it follows that any log canonical center of $(\mathscr{Z}, \Delta_{\mathscr{Z}})$ is a product of such strata, hence a stratum of the coefficient part 1 of $(\mathscr{Z}, \Delta_{\mathscr{Z}})$. $\square$

(5.3.4) Let (X, Δ_X) and (Y, Δ_Y) be dlt pairs over the germ of a punctured curve C. We denote by (Z, Δ_Z) their product, where $\Delta_Z = X \times_C \Delta_Y + \Delta_X \times_C Y$. Let $(\mathscr{X}, \Delta_{\mathscr{X}})$ and $(\mathscr{Y}, \Delta_{\mathscr{Y}})$ be semistable good projective dlt minimal models over the pointed curve.

Theorem 5.3.5. *The product (Z, Δ_Z) has a semistable good projective dlt minimal model $(\mathscr{Z}', \Delta_{\mathscr{Z}'})$ and $\mathcal{D}_0^{=1}(\Delta_{\mathscr{Z}'}) \simeq \mathcal{D}_0^{=1}(\Delta_{\mathscr{X}}) \times \mathcal{D}_0^{=1}(\Delta_{\mathscr{Y}})$.*

Proof. Let $(\mathscr{Z}, \Delta_{\mathscr{Z}})$ be the product of $(\mathscr{X}, \Delta_{\mathscr{X}})$ and $(\mathscr{Y}, \Delta_{\mathscr{Y}})$ as in Proposition 5.3.3. Thus $K_{\mathscr{Z}} + \Delta_{\mathscr{Z}}$ is semiample and log canonical, so any minimal dlt model over $(\mathscr{Z}, \Delta_{\mathscr{Z}})$ will also have semiample log canonical divisor.

Let $\vartheta \colon \mathscr{W} \to \mathscr{Z}$ be a log resolution given by iterated blow-ups at centers of codimension at least 2. Then there exists an effective divisor D supported on all of the exceptional divisors, such that $-D$ is ϑ-ample.

Claim: there exist $B_{\mathscr{X}}$ and $B_{\mathscr{Y}}$ effective divisors on $\mathscr{X}$ and $\mathscr{Y}$ whose respective supports contain no log canonical centers of $\Delta_{\mathscr{X}}$ and $\Delta_{\mathscr{Y}}$, and such that $B_{\mathscr{X}} - \epsilon \Delta_{\mathscr{X}}^{=1}$ and $B_{\mathscr{Y}} - \epsilon \Delta_{\mathscr{Y}}^{=1}$ are ample, for ϵ small and rational.

Choose ϵ small and rational. Then $\Gamma_{\mathscr{X}} = \Delta_{\mathscr{X}} + B_{\mathscr{X}} - \epsilon \Delta_{\mathscr{X}}^{=1}$ and $\Gamma_{\mathscr{Y}} = \Delta_{\mathscr{Y}} + B_{\mathscr{Y}} - \epsilon \Delta_{\mathscr{Y}}^{=1}$ are effective, and $(\mathscr{X}, \Gamma_{\mathscr{X}})$ and $(\mathscr{Y}, \Gamma_{\mathscr{Y}})$ are klt. Let $\Gamma'_{\mathscr{Z}}$ be the log pullback to $\mathscr{W}$ of the product $\Gamma_{\mathscr{Z}}$ of $\Gamma_{\mathscr{X}}$ and $\Gamma_{\mathscr{Y}}$, namely $K_{\mathscr{W}} + \Gamma'_{\mathscr{Z}} = \vartheta^*(K_{\mathscr{Z}} + \Gamma_{\mathscr{Z}})$, we denote by $\Gamma_{\mathscr{W}}$ the positive part of $\Gamma'_{\mathscr{Z}}$. Since $(\mathscr{Z}, \Gamma_{\mathscr{Z}})$ was klt, so is $(\mathscr{W}, \Gamma_{\mathscr{W}})$. For sufficiently small δ, $(\mathscr{W}, \Gamma_{\mathscr{W}} + \delta D)$ is still klt.

Let α_i be arbitrary small rational coefficients, one for each divisor $\Delta_{\mathscr{Z},i}^{=1}$ of $\Delta_{\mathscr{Z}}$ with coefficient 1 and in the special fiber. We will recover the requested dlt model by running an MMP with scaling on the pair $(\mathscr{W}, \Gamma_{\mathscr{W}} + \delta D - \sum \alpha_i \Delta_{\mathscr{Z},i}^{=1})$, scaling with respect to an ample divisor A equivalent to $-D$. By [BCHM10] this MMP terminates in a log terminal model $\phi \colon \mathscr{W} \dashrightarrow \mathscr{Z}'$. Moreover, as long as the α_i are small relative to δ, when the MMP terminates it must be the case that every exceptional divisor is contracted. As running MMP induces birational contractions, the morphism $\psi \colon \mathscr{Z}' \to \mathscr{Z}$ is small. Hence

$(\mathscr{Z}', \psi^* \Delta_{\mathscr{Z}})$ is log canonical, and every log canonical center dominates a stratum of the coefficient 1 part of $(\mathscr{Z}, \Delta_{\mathscr{Z}})$.

By construction of $\mathscr{Z}'$, the divisor $-\sum \alpha_i \Delta^{=1}_{\mathscr{Z},i} + \mu \phi_* A$ is ψ-ample, where μ is arbitrarily small. But for μ small enough, we can absorb $\mu \phi_* A$ into the term δD. Thus in fact $-\sum \alpha_i \Delta^{=1}_{\mathscr{Z},i}$ is ψ-ample. As a result we can represent

$$\mathscr{Z}' = \mathrm{Proj}_{\mathscr{Z}} \bigoplus_{m \geqslant 0} \mathcal{O}\big(-m(\sum \alpha_i \Delta^{=1}_{\mathscr{Z},i})\big).$$

At this point we may take an arbitrarily large Veronese subring and assume the α_i are all integers.

Now, we show that $(\mathscr{Z}', \psi^* \Delta_{\mathscr{Z}})$ is a dlt pair by looking at a formal toric model. Indeed, $(\mathscr{Z}, \Delta_{\mathscr{Z}})$ is formally locally toric at the log canonical centers, and the condition of being dlt is a formally local property. Moreover, we reduce to check this property at the log canonical centers of $(\mathscr{Z}', \psi^* \Delta_{\mathscr{Z}})$ that lie in the special fiber, as the generic fibers of $\mathscr{Z}'$ and $\mathscr{Z}$ are isomorphic and the latter is dlt.

Let z be the generic point of the image in $\mathscr{Z}$ of a log canonical center of $(\mathscr{Z}', \psi^* \Delta_{\mathscr{Z}})$, hence z is the generic point of a log canonical center of $(\mathscr{Z}, \Delta_{\mathscr{Z}})$ and by Proposition 5.3.3 it is a stratum of the coefficient 1 part of $\Delta_{\mathscr{Z}}$. Let x and y be the generic points of the corresponding strata of $(\mathscr{X}, \Delta_{\mathscr{X}})$ and $(\mathscr{Y}, \Delta_{\mathscr{Y}})$. Let E_x and E_y be the monoids of effective Cartier divisors supported on the strata near x and y respectively. Then the corresponding monoid for z is $\langle E_x \oplus E_y \rangle / (t_x = t_y)$, where t_x and t_y are the respective sums of local equations of strata in the special fibers. This monoid has the form $M \oplus \mathbb{N}^l$, where l is the number of horizontal divisors containing z, and M is a monoid of the type considered in Lemma 5.3.2. The toric variety $T_{M,k} = \mathrm{Spec}\, k[M \oplus \mathbb{N}^l]$ is a formal local model for $\mathscr{Z}$ near z, so it suffices to consider

$$T' = \mathrm{Proj}_{T_{M,k}} \bigoplus_{m \geqslant 0} \mathcal{O}\big(-m(\sum \alpha_i \Delta_{T,i})\big),$$

where the divisors $\Delta_{T,i}$ range over the torus invariant Weil divisors corresponding to the $\Delta^{=1}_{\mathscr{Z},i}$.

The divisors corresponding to the l generators of $\mathbb{N}^l$ are Cartier, so their contribution to T is trivial and we can reduce to the case $l = 0$. For sufficiently general choices for the α_i, the toric variety T' is simplicial, its fan being induced by the simplicial subdivision of the fan of $T_{M,k}$ such that the α_i induce a strictly convex piecewise linear function. Thus, T' is a small $\mathbb{Q}$-factorialization of $T_{M,k}$. By Lemma 5.3.2, for any such model, T' is a smooth toric variety, hence its invariant divisors are snc. Thus $(\mathscr{Z}', \psi^* \Delta_{\mathscr{Z}})$ is dlt.

Finally, we compute the dual complex of the coefficient 1 part of $(\mathscr{Z}', \psi^* \Delta_{\mathscr{Z}})$ by looking at formal toric models again. Locally at the generic points of the log canonical centers, $(\mathscr{X}, \Delta_{\mathscr{X}})$ and $(\mathscr{Y}, \Delta_{\mathscr{Y}})$ are snc, hence log-regular, and are also semistable. By Proposition 2.5.4 there is a bijective correspondence between pairs of strata in $\Delta^{=1}_{\mathscr{X}}$ and $\Delta^{=1}_{\mathscr{Y}}$, and strata of their product, namely between pairs of points in the Kato fans of $\mathscr{X}$ and $\mathscr{Y}$ locally around the log canonical centers and the points in the Kato fan of the product. Consider a log canonical center z corresponding to (x,y) with associated monoid $E_z = \langle E_x \oplus E_y \rangle / (t_x = t_y)$. As the points of the Kato fan of $\mathscr{Z}$ corresponding to strata containing z are in one to one correspondence with prime ideals of E_z, there is a one-to-one correspondence between these strata in $\mathscr{Z}$ and the strata in the formal toric model $T_{M,k}$. rephrase more clearly Hence, we reduce to study the strata of the coefficient 1 part of $(\mathscr{Z}', \psi^* \Delta_{\mathscr{Z}})$ in the formal model T'. But T' is obtained by a simplicial subdivision of T. Thus, we conclude that the dual complex $\mathcal{D}^{=1}_0(\psi^* \Delta_{\mathscr{Z}})$ is identified with the product of the dual complexes of the coefficient 1 part of $(\mathscr{X}, \Delta_{\mathscr{X}})$ and $(\mathscr{Y}, \Delta_{\mathscr{Y}})$. $\square$

Proof of Claim. Let $(\mathscr{X}, \Delta_{\mathscr{X}})$ be a good projective minimal dlt model of (X, Δ_X) and let $A_{\mathscr{X}}$ be an ample divisor on $\mathscr{X}$. Let $\mathcal{J}$ be the ideal sheaf of $\Delta^{=1}_{\mathscr{X}}$ and let

$$\chi : \mathscr{X}' = \mathrm{Bl}_{\mathcal{J}} \mathscr{X} \to \mathscr{X}$$

be the blow-up of $\mathscr{X}$ with respect to $\mathcal{J}$. Then the trasform $-((\Delta_{\overline{\mathscr{X}}}^{=1})' + E$ is a χ-ample Cartier divisor, where $(\Delta_{\overline{\mathscr{X}}}^{=1})'$ denotes the strict trasform of $\Delta_{\overline{\mathscr{X}}}^{=1}$ and E an effective divisor supported on the exceptional divisor of χ. For ε small positive rational $\chi^*(A_{\mathscr{X}}) - \varepsilon(\Delta_{\overline{\mathscr{X}}}^{=1} + E)$ is ample. We choose such an ε. For a sufficiently large integer n, we can find

$$G_{\mathscr{X}'} \sim n\chi^*(A_{\mathscr{X}}) - n\varepsilon(\Delta_{\overline{\mathscr{X}'}}^{=1} + E)$$

such that $G_{\mathscr{X}'}$ is effective and contains no log canonical centers. Then the push-forward $G_{\mathscr{X}} \sim nA_{\mathscr{X}} - n\varepsilon\Delta_{\overline{\mathscr{X}}}^{=1}$ of $G_{\mathscr{X}'}$ is effective and contains no log canonical centers of $(\mathscr{X}, \Delta_{\mathscr{X}})$. In particular $G_{\mathscr{X}}$ is Cartier at the log canonical centers of $(\mathscr{X}, \Delta_{\mathscr{X}})$.

We can run the same construction for the ideal sheaf of the divisor $G_{\mathscr{X}}$. Notice that the blow-up with respect to $G_{\mathscr{X}}$ is an isomorphism at the log canonical centers of $(\mathscr{X}, \Delta_{\mathscr{X}})$ as there $G_{\mathscr{X}}$ is Cartier. We obtain that, for sufficiently small positive rationals δ and then sufficiently large integers m, we can find $H_{\mathscr{X}} \sim mA_{\mathscr{X}} - m\delta G_{\mathscr{X}}$ such that it is effective and not containing any log canonical center of $(\mathscr{X}, \Delta_{\mathscr{X}})$. Then

$$\frac{1}{\delta}A_{\mathscr{X}} - \frac{1}{m\delta}H_{\mathscr{X}} \sim G_{\mathscr{X}} \sim nA_{\mathscr{X}} - n\varepsilon\Delta_{\overline{\mathscr{X}}}^{=1}$$

implies that

$$\left(\frac{1}{\delta} - n\right)A_{\mathscr{X}} \sim \frac{1}{m\delta}H_{\mathscr{X}} - n\varepsilon\Delta_{\overline{\mathscr{X}}}^{=1}.$$

For δ sufficiently small the term $\left(\frac{1}{\delta} - n\right)$ is positive, and then for m sufficiently large $B_{\mathscr{X}} = \frac{1}{m\delta}H_{\mathscr{X}}$ concludes the proof. $\qquad\square$

6. Applications

6.1. **Weight functions and skeleta for finite quotients.**

(6.1.1) Let X be a connected, smooth and proper K-variety and let G be a finite group acting on X. Let X^{an} be the analytification of X. We recall that any point of X^{an} is a pair $(x, |\cdot|_x)$ with $x \in X$ and $|\cdot|_x$ an absolute value on the residue field $\kappa(x)$ that extends the absolute value on K. For any point x of X, an element g of the group G induces an isomorphism between the residue fields $\kappa(x)$ and $\kappa(g.x)$, that we still denote by g. Then, the action of G extends to X^{an} in the following way

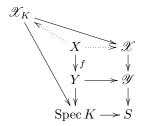$$g.(x, |\cdot|_x) = (g.x, |\cdot|_x \circ g^{-1}).$$

In particular the action preserves the sets of divisorial and birational points of X.

Let $f : X \to Y = X/G$ be the quotient map of K-schemes, let $f^{\mathrm{an}} : X^{\mathrm{an}} \to Y^{\mathrm{an}}$ be the map of Berkovich spaces induced by functoriality and let $\tilde{f} : X^{\mathrm{an}} \to X^{\mathrm{an}}/G$ be the quotient map of topological spaces.

Proposition 6.1.2. *([Ber95], Corollary 5) Under the above notations, there is a canonical homeomorphism between X^{an}/G and Y^{an} such that $\tilde{f}$ and f^{an} are identified.*

(6.1.3) Let y be a divisorial point of Y^{an} and consider a regular snc R-model $\mathscr{Y}$ of Y adapted to y, i.e. such that y is the divisorial point associated to $(\mathscr{Y}, E)$ for some irreducible component E of $\mathscr{Y}_k$. We denote by $\mathscr{X}$ the normalization of $\mathscr{Y}$ inside $K(X)$, where $K(\mathscr{Y}) = K(Y) = K(X)^G \hookrightarrow K(X)$.

We check that $\mathscr{X}$ is an R-model of X; it is enough to show that the base change $\mathscr{X}_K$ is isomorphic to X. We consider the following commutative diagram

As the X is a normal variety endowed with a morphism $X \to \mathscr{Y}$, by universal property of normalization, it factors uniquely through $\mathscr{X}$ and the diagram is still commutative. Then by universal property of fibred product, there exists a morphism $X \to \mathscr{X}_K$. Therefore, it suffices to prove that

$$[K(X) : K(\mathscr{X}_K)] = 1.$$

Indeed, if this is the case, then $X \to \mathscr{X}_K$ is a finite birational morphism between normal varietes, hence an isomorphism.

The degree of the extension $[K(X) : K(\mathscr{X}_K)]$ may be computed on an affine open, so we assume that $\mathscr{Y}$ is an affine scheme with associated ring $K[\mathscr{Y}]$. Then we consider the diagram of inclusions

As $K(X)$ is finite field extension of $K(\mathscr{Y})$ and $K[\mathscr{X}]$ the integral closure of $K[\mathscr{Y}]$ in $K(X)$, then $K(X)$ is the fraction field of $K[\mathscr{X}]$. Thus, $K(\mathscr{X}) = \mathrm{Frac}(K[\mathscr{X}]) = K(X)$ and in particular we conclude that $[K(X) : K(\mathscr{X}_K)] = 1$.

Remark 6.1.4. This procedure of normalization illustrates a way to start with a regular snc R-model $\mathscr{Y}$ of Y adapted to a point $y \in \mathrm{Div}(Y)$ and construct an R-model $\mathscr{X}$ of X that, by normality, is regular at generic points of the special fiber $\mathscr{X}_k$.

Lemma 6.1.5. *Let ω be a m-pluricanonical rational form on X. If ω is G-invariant, then the weight function associated to ω on the set of birational points is stable under the action of G.*

Proof. Let x be a birational point of X and g an element of G. There exist snc models $\mathscr{X}$ and $\mathscr{X}'$ over R such that $x \in \mathrm{Sk}(\mathscr{X})$ and $g.x \in \mathrm{Sk}(\mathscr{X}')$. By replacing them by an snc model $\mathscr{Y}$ that dominates both $\mathscr{X}$ and $\mathscr{X}'$, we can assume that both points lies in $\mathrm{Sk}(\mathscr{Y})$. The weights of ω at x and $g.x$ are such that

$$\mathrm{wt}_\omega(g.x) = v_{g.x}(\mathrm{div}_{\mathscr{Y}+}(\omega)) + m = v_x((g^{-1})^* \mathrm{div}_{\mathscr{Y}+}(\omega)) + m$$
$$= v_x(\mathrm{div}_{\mathscr{Y}+}(\omega)) + m = \mathrm{wt}_\omega(x)$$

as ω is a G-invariant form. Thus we see that birational points in the same G-orbit have the same weight with respect to ω. $\qquad\square$

Corollary 6.1.6. *Let ω be a G-invariant pluricanonical rational form on X. Then the Kontsevich-Soibelman skeleton $\mathrm{Sk}(X, \omega)$ is stable under the action of G.*

Proof. This follows immediately from Lemma 6.1.5. $\qquad\square$

(6.1.7) We denote respectively by R and by B the codimension 1 components of the ramification locus and of the branch locus of $f : X \to Y$. We set $X^+ = (X, R_{\mathrm{red}})$ and $Y^+ = (Y, B_{\mathrm{red}})$. Then f extends to a morphism of log schemes $f^+ : X^+ \to Y^+$.

Lemma 6.1.8. *The logarithmic canonical bundles $\omega_{X^+/K}^{\log}$ and $\omega_{Y^+/K}^{\log}$ are identified via the pullback $(f^+)^*$.*

Proof. From a generalization to higher dimension of [Har77], Proposition IV.2.3, we have that $\omega_{X/K} = f^*(\omega_{Y/K}) \otimes \mathcal{O}_X(R)$. It follows that

$$\omega_{X^+/K}^{\log} = \omega_{X/K} \otimes \mathcal{O}_X(R_{\mathrm{red}}) = f^*(\omega_{Y/K}) \otimes \mathcal{O}_Y(R + R_{\mathrm{red}}).$$

In order to study the divisor $R + R_{\mathrm{red}}$, we consider one irreducible component of B. Let D_B be an irreducible component of B, denote by e the ramification index of f at D_B, and by D_R the support of the preimage of D_B. As $\mathrm{mult}_{D_R}(R) = e - 1$, $\mathrm{mult}_{D_R}(f^*(D_B)) = e$ and $\mathrm{mult}_{D_R}(R_{\mathrm{red}}) = 1$, we conclude that $R + R_{\mathrm{red}} = f^*(B)$ and finally $\omega_{X^+/K}^{\log} = (f^+)^*(\omega_{Y^+/K}^{\log})$. $\qquad\square$

Proposition 6.1.9. *Let ω be a G-invariant R_{red}-logarithmic m-pluricanonical form on X^+ and let $\overline{\omega}$ be the corresponding B_{red}-logarithmic form on Y^+. Let y be a divisorial point of Y^{an}. Then, for any divisorial point $x \in (f^{\mathrm{an}})^{-1}(y)$, the weights of ω at x and of $\overline{\omega}$ at y coincide.*

Proof. Let $\mathscr{Y}$ be a regular snc R-model such that y has divisorial representation $(\mathscr{Y}, E)$. Let $\mathscr{X}$ be the normalization of $\mathscr{Y}$ in $K(X)$: as we observed in Remark 6.1.4, it is an R-model of X, regular at generic points of the special fiber $\mathscr{X}_k$. The preimage of E coincides with the pull-back of the Cartier divisor E on $\mathscr{X}$, hence $f^{-1}(E)$ still defines a codimension one subset on $\mathscr{X}$. We denote by F_i the irreducible components of $f^{-1}(E)$ and we associate to F_i's their corresponding divisorial valuations $x_i = (\mathscr{X}, F_i)$. By Lemma 6.1.5, it is enough to prove the result for one of the x_i's. We denote it by $x = (\mathscr{X}, F)$ and we compare the weights at y and x. As by Lemma 6.1.8 the sheaves of logarithmic differentials are stable under the pull-back along f, we have

$$\mathrm{div}_{\mathscr{Y}^+}(\overline{\omega}) = f^*(\mathrm{div}_{\mathscr{X}^+}(\omega))$$

and so $\mathrm{wt}_{\overline{\omega}}(y) = v_y(\mathrm{div}_{\mathscr{Y}^+}(\overline{\omega})) + m = v_x(\mathrm{div}_{\mathscr{X}^+}(\omega)) + m = \mathrm{wt}_{\omega}(x)$. $\qquad\square$

Proposition 6.1.10. *Let ω be a G-invariant R_{red}-logarithmic pluricanonical form on X and let $\overline{\omega}$ be the corresponding B_{red}-logarithmic form. Then the canonical homeomorphism between X^{an}/G and Y^{an} of Proposition 6.1.2 induces the homeomorphism*

$$\mathrm{Sk}(X, \omega)/G \simeq \mathrm{Sk}(X/G, \overline{\omega}).$$

Proof. This follows immediately from Corollary 6.1.6 and Proposition 6.1.9. $\qquad\square$

(6.1.11) Let X be a smooth K-variety and let ω_X be a pluricanonical form on X. Let $\mathrm{pr}_j : X^n \to X$ be the j-th canonical projection. Then

$$\omega = \bigwedge_{1 \leqslant j \leqslant n} \mathrm{pr}_j^* \omega_X$$

is a pluricanonical form on X^n and moreover it is invariant under the action of $\mathfrak{S}_n$. We denote by $\overline{\omega}$ the corresponding form on the quotient $X^n/\mathfrak{S}_n$ as in Lemma 6.1.8.

Proposition 6.1.12. *Assume that the residue field k is algebraically closed. If X admits a semistable log-regular model or a semistable good dlt minimal model, then the Kontsevich-Soibelman skeleton of the n-th symmetric product of X associated to $\overline{\omega}$ is PL homeomorphic to the n-th symmetric product of the Kontsevich-Soibelman skeleton of X associated to ω_X.*

Proof. Iterating the result of Theorem 5.2.2 and Theorem 5.3.5, we have that the projection map defines a PL homeomorphism of Kontsevich-Soibelman skeleta

$$\mathrm{Sk}(X^n, \omega) \xrightarrow{\sim} \mathrm{Sk}(X, \omega_X) \times \ldots \times \mathrm{Sk}(X, \omega_X).$$

Thus, applying Proposition 6.1.10 with the group $\mathfrak{S}_n$ acting on the product X^n, we obtain that

$$\mathrm{Sk}(X^n/\mathfrak{S}_n, \overline{\omega}) \simeq \mathrm{Sk}(X^n, \omega)/\mathfrak{S}_n \simeq \mathrm{Sk}(X, \omega_X)^n/\mathfrak{S}_n$$

Since the action on the Kontsevich-Soibelman skeleton $\mathrm{Sk}(X^n, \omega)$ is induced from the symmetric action on X^n, and the projections $\mathrm{pr}_j : X^n \to X$ functorially induce the projections $\overline{\mathrm{pr}_j} : \mathrm{Sk}(X, \omega_X)^n \to \mathrm{Sk}(X, \omega)$, the action of $\mathfrak{S}_n$ on $\mathrm{Sk}(X, \omega)^n$ is exactly by permutations of the components. Thus,

$$\mathrm{Sk}(X, \omega_X)^n/\mathfrak{S}_n \simeq \mathrm{Sym}^n(\mathrm{Sk}(X, \omega_X)).$$

$\square$

6.2. **The essential skeleton of Hilbert schemes of a K3 surface.**

(6.2.1) Let S be an irreducible regular surface. We consider $\mathrm{Hilb}^n(S)$ the Hilbert scheme of n points on S: by [Fog68] it is an irreducible regular variety of dimension $2n$. Moreover, the morphism

$$\rho_{HC} : \mathrm{Hilb}^n(S) \to S^n/\mathfrak{S}_n$$

that sends a zero-dimensional scheme $Z \subseteq S$ to its associated zero-cycle $\mathrm{supp}(Z)$ is a birational morphism, called the Hilbert-Chow morphism.

(6.2.2) Let S be a K3 surface over K, namely S is a complete non-singular variety of dimension two such that $\Omega_{S/K}^2 \simeq \mathcal{O}_S$ and $H^1(S, \mathcal{O}_S) = 0$. In particular S is a variety with trivial canonical line bundle.

Corollary 6.2.3. *Assume that the residue field k is algebraically closed. Suppose that S admits a semistable log-regular model or a semistable good dlt minimal model. Then the essential skeleton of the Hilbert scheme of n points on S is PL homeomorphic to the n-th symmetric product of the essential skeleton of S*

$$\mathrm{Sk}(\mathrm{Hilb}^n(S)) \xrightarrow{\sim} Sym^n(\mathrm{Sk}(S)).$$

Proof. This follows immediately from Corollary 6.1.12 and the birational invariance of the essential skeleton, [MN15], Proposition 4.6.3. $\square$

Proposition 6.2.4. *If the essential skeleton of S is PL homeomorphic to a point, a closed interval or the 2-dimensional sphere, then the essential skeleton of $\mathrm{Hilb}^n(S)$ is PL homeomorphic to a point, the standard n-simplex or $\mathbb{CP}^n$ respectively.*

Proof. Applying Corollary 6.2.3, we reduce to the computation of the symmetric product of a point, a closed interval or the sphere S^2. Then, the result follows from [Hat02], Section 4K. $\square$

6.3. **The essential skeleton of generalised Kummer varieties.**

(6.3.1) Let A be an abelian surface over K, namely a complete non-singular, connected group variety of dimension two. Since A is a group variety, the canonical line bundle is trivial and the group structure provides a multiplication morphism $m_{n+1} : A \times A \times \ldots \times A \to A$ that is invariant under the permutation action of $\mathfrak{S}_{n+1}$, hence it induces a morphism

$$\Sigma_{n+1} : \mathrm{Hilb}^{n+1}(A) \xrightarrow{\rho_{HC}} \mathrm{Sym}^{n+1}(A) \to A$$

by composition with the Hilbert-Chow morphism. Then $\mathrm{K}_n(A) = \Sigma_{n+1}^{-1}(1)$ is called the n-th generalised Kummer variety and is a hyper-Kähler manifold of dimension $2n$ ([Bea83]).

(6.3.2) In [HN17], Proposition 4.3.2, Halle and Nicaise, using Temkin's generalization of the weight function ([Tem16]), prove that the essential skeleton of an abelian variety A over K coincides with the construction of a skeleton of A done by Berkovich in [Ber90], Paragraph 6.5. It follows from this identification and [Ber90], Theorem 6.5.1 that the essential skeleton of A has a group structure, compatible with the group structure on A^{an} under the retraction ρ_A of A^{an} onto the essential skeleton, so the following diagram commutes

$$
\begin{array}{ccc}
(A^{\mathrm{an}})^{n+1} & \xrightarrow{\;m^{\mathrm{an}}_{n+1}\;} & A^{\mathrm{an}} \\
{\scriptstyle(\rho_A)^{n+1}}\big\downarrow & & \big\downarrow{\scriptstyle\rho_A} \\
\mathrm{Sk}(A)^n & \xrightarrow{\;\mu_{n+1}\;} & \mathrm{Sk}(A)
\end{array}
$$

where μ denotes the multiplication of $\mathrm{Sk}(A)$.

Proposition 6.3.3. *Assume that the residue field k is algebraically closed. Suppose that A admits a semistable log-regular model or a semistable good dlt minimal model. Then the essential skeleton of the n-th generalised Kummer variety is PL homeomorphic to the symmetric quotient of the kernel of the morphism μ, namely*

$$\mathrm{Sk}(K_n(A)) \simeq \mathrm{Sk}\!\left(m_{n+1}^{-1}(1)/\mathfrak{S}_{n+1}\right) \simeq \mu_{n+1}^{-1}(1)/\mathfrak{S}_{n+1}.$$

Proof. The first homeomorphism follows from the birational invariance of the essential skeleton ([MN15], Proposition 4.6.3). We denote by

$$L = m_{n+1}^{-1}(1) \quad \text{and} \quad \Lambda = \mu_{n+1}^{-1}(1).$$

For any choice of an $\mathfrak{S}_{n+1}$-invariant generating canonical form on A, it follows from Proposition 6.1.10 that $\mathrm{Sk}(L/\mathfrak{S}_{n+1}) \simeq \mathrm{Sk}(L)/\mathfrak{S}_{n+1}$. We reduce to study the quotients $\mathrm{Sk}(L)/\mathfrak{S}_{n+1}$ and $\Lambda/\mathfrak{S}_{n+1}$.

Let $\mathfrak{S}'_n$ and $\mathfrak{S}''_n$ be the subgroups of $\mathfrak{S}_{n+1}$ of the permutations that fix n and $n+1$ respectively. Then $\mathfrak{S}_{n+1}$ is generated by the two subgroups, so its action on $\mathrm{Sk}(L)$ and Λ is completely determined by the actions of these subgroups. We consider the following isomorphisms

$$f_n : L \xrightarrow{\sim} A^n \quad (z_1, \ldots, z_{n+1}) \mapsto (z_1, \ldots, z_{n-1}, z_{n+1})$$

$$f_{n+1} : L \xrightarrow{\sim} A^n \quad (z_1, \ldots, z_{n+1}) \mapsto (z_1, \ldots, z_{n-1}, z_n).$$

Then f_n is $\mathfrak{S}'_n$-equivariant, f_{n+1} is $\mathfrak{S}''_n$-equivariant and the morphism ψ

$$
\begin{array}{ccc}
 & L & \\
{\scriptstyle f_{n+1}}\nearrow & & \searrow{\scriptstyle f_n} \\
A^n & \xrightarrow[\;\psi\;]{} & A^n \\
(z_1, \ldots, z_{n-1}, z_n) & \longmapsto & (z_1, \ldots, \prod_{i=1}^{n} z_i^{-1}).
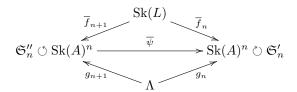\end{array}
$$

is equivariant with respect to the action of $\mathfrak{S}''_n$ on the source and of $\mathfrak{S}'_n$ on the target. Hence, we obtain a commutative diagram of equivariant isomorphisms. We denote by $\overline{f}_n$, $\overline{f}_{n+1}$ and $\overline{\psi}$ the isomorphisms induced on the essential skeleta. By Theorem 5.2.2 and Theorem 5.3.5 we can identify $\mathrm{Sk}(A^n)$ with $\mathrm{Sk}(A)^n$. Thus, we have the commutative diagram

$$
\begin{array}{ccc}
 & \mathrm{Sk}(L) & \\
{\scriptstyle \overline{f}_{n+1}}\nearrow & & \searrow{\scriptstyle \overline{f}_n} \\
\mathrm{Sk}(A)^n & \xrightarrow[\;\overline{\psi}\;]{} & \mathrm{Sk}(A)^n \\
(v_1, \ldots, v_{n-1}, v_n) & \longmapsto & (v_1, \ldots, \prod_{i=1}^{n} v_i^{-1}).
\end{array}
$$

Then the action of $\mathfrak{S}_{n+1}$ on $\mathrm{Sk}(L)$ is induced by the isomorphisms $\overline{f}_n$ and $\overline{f}_{n+1}$ from the actions of $\mathfrak{S}''_n$ and $\mathfrak{S}'_n$ on $\mathrm{Sk}(A)^n$ and these actions are compatible as $\overline{\psi}$ is equivariant.

In a similar way, Λ is isomorphic to n copies of $\mathrm{Sk}(A)$ and comes equipped with an action of $\mathfrak{S}_{n+1}$. So, we have equivariant projections g_n and g_{n+1} with respect to $\mathfrak{S}'_n$ and $\mathfrak{S}''_n$. The equivariant morphism that completes and makes the diagram commutative is $\overline{\psi}$. Finally, we have the equivariant commutative diagram

$$
\begin{array}{ccc}
& \mathrm{Sk}(L) & \\
{\scriptstyle \overline{f}_{n+1}} \swarrow & & \searrow {\scriptstyle \overline{f}_n} \\
\mathfrak{S}''_n \circlearrowleft \mathrm{Sk}(A)^n \xrightarrow{\ \ \overline{\psi}\ \ } & & \mathrm{Sk}(A)^n \circlearrowleft \mathfrak{S}'_n \\
{\scriptstyle g_{n+1}} \searrow & & \nearrow {\scriptstyle g_n} \\
& \Lambda &
\end{array}
$$

and we conclude that the quotients $\mathrm{Sk}(Z)/\mathfrak{S}_{n+1}$ and $\Lambda/\mathfrak{S}_{n+1}$ are homeomorphic. $\qquad\square$

Proposition 6.3.4. *If the essential skeleton of A is PL homeomorphic to a point, the circle S^1 or the torus $S^1 \times S^1$, then the essential skeleton of $K_n(A)$ is PL homeomorphic to a point, the standard n-simplex or $\mathbb{CP}^n$ respectively.*

Proof. The case of the point is trivial. For the circle S^1, it follows directly from [Mor67], Theorem. To prove the result for the torus $S^1 \times S^1$, we apply [Loo76], Theorem 3.4: the action of the symmetric group corresponds to the root system of A_n, the highest root is the sum of the simple roots, each with coefficient 1, and so the quotient is the complex projective space of dimension n. $\qquad\square$

6.4. **Remarks on hyper-Kähler varieties.**

(6.4.1) The cases we consider in Proposition 6.2.4 and Proposition 6.3.4 are motivated by the work of Kulikov, Persson and Pinkham. In [Kul77] and [PP81], they consider degenerations over the unit complex disk, of surfaces such that some power of the canonical bundle is trivial. They prove that, after base change and birational transformations, any such degeneration can be arranged to be semistable with trivial canonical bundle, namely a *Kulikov degeneration*. Then, they classify the possible special fibers of Kulikov degenerations according to the type of the degeneration.

We recall that the monodromy operator T on $\mathrm{H}^2(X_t, \mathbb{Q})$ of the fibers X_t of a Kulikov degeneration is unipotent, so we denote by ν the nilpotency index of $\log(T)$, namely the positive integer such that $\log(T)^\nu = 0$ and $\log(T)^{(\nu-1)} \neq 0$. The type of the Kulikov degeneration is defined as the nilpotency index ν and called type I, II or III accordingly to it.

It follows from [Kul77], Theorem II, that the dual complex of the special fiber of a Kulikov degeneration of a K3 surface is a point, a closed interval or the sphere S^2 according to the respective type. For a degeneration of abelian surfaces, the dual complex of the special fiber is homeomorphic to a point, the circle S^1 or the torus $S^1 \times S^1$ according to the three types (see an overview of these results in [FM83]). In all cases, the dimension of the dual complex is equal $\nu - 1$, hence determined by the type.

(6.4.2) Hilbert schemes of K3 surfaces and generalised Kummer varieties represent two families of examples of hyper-Kähler varieties. For a semistable degeneration of hyper-Kähler manifolds over the unit disk, it is possible to define the type as the nilpotency index of the monodromy operator on the second cohomology group. It naturally extends the definition for Kulikov degenerations.

In [KLSV17], Kollár, Laza, Saccà and Voisin study the essential skeleton of a degeneration of hyper-Kähler manifolds in terms of the type. More precisely, in Theorem 0.10, given a minimal dlt degeneration of $2n$-dimensional hyper-Kähler manifolds, firstly they prove that the dual complex of the special fiber has

dimension $(\nu - 1)n$, where ν denotes the type of the degeneration. Secondly, they prove that, in the type III case, the dual complex is a simply connected closed pseudo-manifold with the rational homology of $\mathbb{CP}^n$.

From this prospective, Proposition 6.2.4 and Proposition 6.3.4 confirm and strengthen their result for the specific cases of Hilbert schemes and generalized Kummer varieties. In particular, we turn the rational cohomological description of the essential skeleton (Theorem 0.10(ii)) into a topological characterization.

For Hilbert schemes associated to some type II degenerations of K3 surfaces, a complementary proof of our result is due to Gulbrandsen, Halle, Hulek and Zhang, see [GHH16] and [GHHZ17]. Their approach is based on the method of *expanded degenerations*, which first appeared in [Li01], and on the construction of suitable GIT quotients, in order to obtain an explicit minimal dlt degeneration for the associated family of Hilbert schemes.

(6.4.3) The structure of the essential skeleton of a degeneration of hyper-Kähler manifolds is relevant in the context of mirror symmetry and in view of the work of Kontsevich and Soibelman ([KS01], [KS06]). The SYZ fibration ([SYZ96]) is a conjectural geometric explanation for the phenomenon of mirror symmetry and, roughly speaking, asserts the existence of a special Lagrangian fibration, such that mirror pairs of manifolds with trivial canonical bundle should admit fiberwise dual special Lagrangian fibrations. Moreover, the expectation is that, for type III degenerations of $2n$-dimensional hyper-Kähler manifolds, the base of the SYZ fibration is $\mathbb{CP}^n$ (see for instance [Hwa08]).

The most relevant fact from our prospective is that Kontsevich and Soibelman predict that the base of the Lagrangian fibration of a type III degeneration is homeomorphic to the essential skeleton. So, the outcomes on the topology of the essential skeleton we obtain in Proposition 6.2.4 and Proposition 6.3.4 match the predictions of mirror symmetry about the occurrence of $\mathbb{CP}^n$ in the type III case.

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