

# TITLE

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## 1. INTRODUCTION

### 1.1. Notation.

**(1.1.1)** Let  $R$  be a complete discrete valuation ring with maximal ideal  $\mathfrak{m}$ , residue field  $k = R/\mathfrak{m}$  and quotient field  $K$ . We assume that the valuation  $v_K$  is normalized. We define by  $|\cdot|_K = \exp(-v_K(\cdot))$  the absolute value on  $K$  corresponding to  $v_K$ ; this turns  $K$  into a non-archimedean complete valued field.

**(1.1.2)** We write  $S = \operatorname{Spec} R$  and we denote by  $s$  the closed point of  $S$ . Let  $\mathcal{X}$  be an  $R$ -scheme of finite type. We will denote by  $\mathcal{X}_k$  the special fiber of  $\mathcal{X}$  and by  $\mathcal{X}_K$  the generic fiber. Moreover, we will denote by  $\widehat{\mathcal{X}}$  the  $\mathfrak{m}$ -adic completion of  $\mathcal{X}$  and by  $\widehat{\mathcal{X}}_\eta$  the generic fiber of  $\widehat{\mathcal{X}}$  in the category of  $K$ -analytic spaces.

**(1.1.3)** Let  $X$  be a proper  $K$ -scheme. A model for  $X$  over  $R$  is a flat separated  $R$ -scheme  $\mathcal{X}$  of finite type endowed with an isomorphism of  $K$ -schemes  $\mathcal{X}_K \rightarrow X$ . If  $X$  is smooth over  $K$ , we say that  $\mathcal{X}$  is an snc model for  $X$  if it is regular over  $R$ , and the special fiber  $\mathcal{X}_k$  is a strict normal crossings divisor on  $\mathcal{X}$ . Such a model always exists, by Hironaka's resolution of singularities.

**(1.1.4)** All log schemes in this paper are fine and saturated (*fs*) log schemes and defined with respect to the Zariski topology. We denote a log scheme by  $\mathcal{X}^+ = (\mathcal{X}, \mathcal{M}_{\mathcal{X}})$ , where  $\mathcal{M}_{\mathcal{X}}$  is the structural sheaf of monoids. We denote by

$$\mathcal{C}_{\mathcal{X}} = \mathcal{M}_{\mathcal{X}} / \mathcal{O}_{\mathcal{X}}^\times$$

the characteristic sheaf of  $\mathcal{X}^+$ . The sheaf  $\mathcal{C}_{\mathcal{X}}$  is a Zariski sheaf on  $\mathcal{X}^+$ , supported on  $\mathcal{X}_k$ ; if  $\mathcal{X}^+$  is log-regular, then  $\mathcal{C}_{\mathcal{X}}$  is a constructible sheaf. For every point  $x$  of  $\mathcal{X}_k$ , we denote by  $\mathcal{I}_{\mathcal{X},x}$  the ideal in  $\mathcal{O}_{\mathcal{X},x}$  generated by

$$\mathcal{M}_{\mathcal{X},x} \setminus \mathcal{O}_{\mathcal{X},x}^\times.$$

We denote by  $S^+$  the scheme  $S$  endowed with the standard log structure (the divisorial log structure induced by  $s$ ). If an  $R$ -scheme  $\mathcal{X}$  is given, we will always denote by  $\mathcal{X}^+$  the log scheme over  $S^+$  that we obtain by endowing  $\mathcal{X}$  with the divisorial log structure associated with  $\mathcal{X}_k$ .

If  $\mathcal{X}^+$  is a log-regular log scheme over  $S^+$ , then the locus where the log structure is non-trivial is a divisor that we will denote by  $D_{\mathcal{X}}$ . Thus, the log structure on  $\mathcal{X}^+$  is the divisorial log structure induced by  $D_{\mathcal{X}}$ , by [Kat94], Theorem 11.6.

**(1.1.5)** We say that a log-regular log scheme  $\mathcal{X}^+$  over  $S^+$  is semistable if the divisor  $D_{\mathcal{X}}$  is reduced. We say that a proper  $K$ -variety  $X$  has semistable reduction if it admits an  $R$ -model  $\mathcal{X}$  such that  $\mathcal{X}^+$  is log-smooth over  $S^+$ , with reduced special fiber; such a model is called a semistable model of  $X$ . This is a weaker notion than requiring the existence of an snc-model with reduced special fibre.

(1.1.6) We denote by  $(\cdot)^{\text{an}}$  the analytification functor from the category of  $K$ -schemes of finite type to Berkovich's category of  $K$ -analytic spaces. For every  $K$ -scheme of finite type  $X$ , as a set,  $X^{\text{an}}$  consists of the pairs  $x = (\xi_x, |\cdot|_x)$  where  $\xi_x$  is a point of  $X$  and  $|\cdot|_x$  is an absolute value on the residue field  $\kappa(\xi_x)$  of  $X$  at  $\xi_x$  extending the absolute value  $|\cdot|_K$  on  $K$ . We endow  $X^{\text{an}}$  with the Berkovich topology, i.e. the weakest one such that

- (i) the forgetful map  $\phi : X^{\text{an}} \rightarrow X$ , defined as  $(\xi_x, |\cdot|_x) \mapsto \xi_x$ , is continuous,
- (ii) for any Zariski open subset  $U$  of  $X$  and any regular function  $f$  on  $U$  the map  $|f| : \phi^{-1}(U) \rightarrow \mathbb{R}$  defined by  $|f|(\xi_x, |\cdot|_x) = |f(\xi_x)|$  is continuous.

## 2. THE KATO FAN OF A LOG-REGULAR LOG SCHEME

### 2.1. Definition of Kato fans.

(2.1.1) According to [Kat94], Definition 9.1, a monoidal space  $(T, \mathcal{M}_T)$  is a topological space  $T$  endowed with a sharp sheaf of monoids  $\mathcal{M}_T$ , where *sharp* means that  $\mathcal{M}_{T,t}^\times = \{1\}$  for every  $t \in T$ . We often simply denote the monoidal space by  $T$ .

A morphism of monoidal spaces is a pair  $(f, \varphi) : (T, \mathcal{M}_T) \rightarrow (T', \mathcal{M}_{T'})$  such that  $f : T \rightarrow T'$  is a continuous function of topological spaces and  $\varphi : f^{-1}(\mathcal{M}_{T'}) \rightarrow \mathcal{M}_T$  is a sheaf homomorphism such that  $\varphi_t^{-1}(\{1\}) = \{1\}$  for every  $t \in T$ .

**Example 2.1.2.** If  $\mathcal{X}^+$  is a log scheme then the Zariski topological space  $\mathcal{X}$  is equipped with a sheaf of sharp monoids  $\mathcal{C}_{\mathcal{X}}$ , the characteristic sheaf of  $\mathcal{X}^+$ . Thus  $(\mathcal{X}, \mathcal{C}_{\mathcal{X}})$  is a monoidal space. Moreover, morphisms of log schemes induce morphisms of characteristic sheaves, hence morphism of monoidal spaces. We therefore obtain a functor from the category of log schemes to the category of monoidal spaces.

**Example 2.1.3.** Given a monoid  $P$ , we may associate to it a monoidal space called the spectrum of  $P$ . As a set,  $\text{Spec } P$  is the set of all prime ideals of  $P$ . The topology is characterized by the basis open sets  $D(f) = \{\mathfrak{p} \in \text{Spec } P \mid f \notin \mathfrak{p}\}$  for any  $f \in P$ . The monoidal sheaf is defined by

$$\mathcal{M}_{\text{Spec } P}(D(f)) = S^{-1}P / (S^{-1}P)^\times$$

where  $S = \{f^n \mid n \geq 0\}$ .

(2.1.4) A monoidal space isomorphic to the monoidal space  $\text{Spec } P$  for some monoid  $P$  is called an affine Kato fan. A monoidal space is called a Kato fan if it has an open covering consisting of affine Kato fans. In particular, we call a Kato fan integral, saturated, of finite type or *fs* if it admits a cover by the spectra of monoids with the respective properties.

(2.1.5) A morphism of *fs* Kato fans  $F' \rightarrow F$  is called a *subdivision* if it has finite fibres and the morphism

$$\text{Hom}(\text{Spec } \mathbb{N}, F') \rightarrow \text{Hom}(\text{Spec } \mathbb{N}, F)$$

is a bijection. Allowing subdivisions a Kato fan might take the following shape.

**Proposition 2.1.6.** ([Kat94], Proposition 9.8) *Let  $F$  be a *fs* Kato fan. Then there is a subdivision  $F' \rightarrow F$  such that  $F'$  has an open cover  $\{U'_i\}$  by Kato cones with  $U'_i \simeq \text{Spec } \mathbb{N}^{r_i}$ .*

The strategy of the proof of Proposition 2.1.6 goes back to [KKMSD73] and relies on a sequence of particular subdivisions of the Kato fan, the so-called star and barycentric subdivisions ([ACMUW15], Example 4.10).

## 2.2. Kato fans associated to log-regular log schemes.

**Theorem 2.2.1.** ([Kat94], Proposition 10.2) *Let  $\mathcal{X}^+$  be a log-regular log scheme. Then there is an initial strict morphism  $(\mathcal{X}, \mathcal{C}_{\mathcal{X}}) \rightarrow F$  to a Kato fan in the category of monoidal spaces. Explicitly, there exist a Kato fan  $F$  and a morphism  $\pi : (\mathcal{X}, \mathcal{C}_{\mathcal{X}}) \rightarrow F$  such that  $\pi^{-1}(\mathcal{M}_F) \simeq \mathcal{C}_{\mathcal{X}}$  and any other morphism to a Kato fan factors through  $\pi$ .*

The Kato fan  $F$  in Theorem 2.2.1 is called the Kato fan associated to  $\mathcal{X}^+$ ; it is the topological subspace of  $\mathcal{X}$  consisting of the points  $x$  such that the maximal ideal  $\mathfrak{m}_x$  of  $\mathcal{O}_{\mathcal{X},x}$  is equal to  $\mathcal{I}_{\mathcal{X},x}$ , and  $\mathcal{M}_F$  is the inverse image of  $\mathcal{C}_{\mathcal{X}}$  on  $F$ , henceforth we write  $\mathcal{C}_F$  for  $\mathcal{M}_F$ .

**Example 2.2.2.** Assume that  $\mathcal{X}$  is regular, of finite type over  $S$  and  $\mathcal{X}_k$  is a divisor with strict normal crossings. Then  $\mathcal{X}^+$  is log-regular and  $F$  is the set of generic points of intersections of irreducible components of  $\mathcal{X}_k$ . For each point  $x$  of  $F$ , the stalk of  $\mathcal{C}_F$  is isomorphic to  $(\mathbb{N}^r, +)$ , with  $r$  the number of irreducible components of  $\mathcal{X}_k$  that pass through  $x$ .

This example admits the following partial generalisation.

**Lemma 2.2.3.** *Let  $\mathcal{X}^+$  be a log-regular log scheme. Then the fan  $F$  consists of the generic points of intersections of irreducible components of  $D_{\mathcal{X}}$ .*

*Proof.* First, we show that every such generic point is a point of  $F$ . Let  $E_1, \dots, E_r$  be irreducible components of  $D_{\mathcal{X}}$  and let  $x$  be a generic point of the intersection  $E_1 \cap \dots \cap E_r$ . We set  $d = \dim \mathcal{O}_{\mathcal{X},x}$ . Since  $\mathcal{X}^+$  is log-regular, we know that  $\mathcal{O}_{\mathcal{X},x}/\mathcal{I}_{\mathcal{X},x}$  is regular and that

$$(2.2.4) \quad d = \dim \mathcal{O}_{\mathcal{X},x}/\mathcal{I}_{\mathcal{X},x} + \text{rank } \mathcal{C}_{\mathcal{X},x}^{\text{gp}}.$$

We denote by  $V(\mathcal{I}_{\mathcal{X},x})$  the vanishing locus of the ideal  $\mathcal{I}_{\mathcal{X},x}$  in  $\mathcal{X}$ . We want to prove that  $\mathcal{I}_{\mathcal{X},x} = \mathfrak{m}_x$ . We assume the contrary, hence that  $\mathcal{I}_{\mathcal{X},x} \subsetneq \mathfrak{m}_x$ . This assumption implies that there exists  $j$  such that  $V(\mathcal{I}_{\mathcal{X},x}) \not\subseteq E_j$ : indeed, if the vanishing locus is contained in each irreducible component  $E_i$ , i.e.

$$V(\mathcal{I}_{\mathcal{X},x}) \subseteq E_1 \cap \dots \cap E_r \subseteq \overline{\{x\}},$$

then  $\mathcal{I}_{\mathcal{X},x} \supseteq \mathfrak{m}_x$ . From the assumption of log-regularity it follows that the vanishing locus  $V(\mathcal{I}_{\mathcal{X},x})$  is a regular subscheme, and moreover that  $\mathcal{X}^+$  is Cohen-Macaulay by [Kat94], Theorem 4.1. Thus, there exists a regular sequence  $(f_1, \dots, f_l)$  in  $\mathcal{I}_{\mathcal{X},x}$  where  $l$  is the codimension of  $V(\mathcal{I}_{\mathcal{X},x})$ , i.e.

$$\dim \mathcal{O}_{\mathcal{X},x}/\mathcal{I}_{\mathcal{X},x} = d - l.$$

Moreover by the equality (2.2.4),  $\text{rank } \mathcal{C}_{\mathcal{X},x}^{\text{gp}} = l$ .

We claim that the residue classes of these elements  $f_i$  in  $\mathcal{C}_{\mathcal{X},x}^{\text{gp}}$  are linearly independent. Assume the contrary. Then, up to renumbering the  $f_i$ , there exist an integer  $e$  with  $1 < e < l$ , non-negative integers  $a_1, \dots, a_l$ , not all zero, and a unit  $u$  in  $\mathcal{O}_{\mathcal{X},x}$  such that

$$f_1^{a_1} \cdot \dots \cdot f_{e-1}^{a_{e-1}} = u \cdot f_e^{a_e} \cdot \dots \cdot f_l^{a_l}.$$

This contradicts the fact that  $(f_1, \dots, f_l)$  is a regular sequence in  $\mathcal{I}_{\mathcal{X},x}$ . Thus, the classes  $\overline{f_1}, \dots, \overline{f_l}$  are independent in  $\mathcal{C}_{\mathcal{X},x}^{\text{gp}}$ . As we also have the equality  $\text{rank } \mathcal{C}_{\mathcal{X},x}^{\text{gp}} = l$ , it follows that these classes generate  $\mathcal{C}_{\mathcal{X},x}^{\text{gp}} \otimes_{\mathbb{Z}} \mathbb{Q}$ .

Let  $g_j$  be a non-zero element of the ideal  $\mathcal{I}_{\mathcal{X},x}$  that vanishes along  $E_j$ : it necessarily exists as otherwise  $E_j$  is not a component of the divisor  $D_{\mathcal{X}}$ . Then  $g_j$  satisfies

$$g_j^N = v \cdot f_1^{b_1} \cdot \dots \cdot f_l^{b_l}$$

with  $b_i \in \mathbb{Z}$ ,  $v$  a unit in  $\mathcal{O}_{\mathcal{X},x}$  and  $N$  a positive integer. As  $g_j$  vanishes along the irreducible component  $E_j$ , at least one of the functions  $f_1, \dots, f_l$  has to vanish along  $E_j$ : assume that is  $f_1$ .

On the one hand, as  $f_1$  is identically zero on  $E_j$ , the trace of  $E_j$  on  $V(\mathcal{I}_{\mathcal{X},x})$  has at most codimension  $l - 1$  in  $E_j$  at the point  $x$ . On the other hand, we assumed that  $V(\mathcal{I}_{\mathcal{X},x})$  is not contained in  $E_j$  and it has codimension  $l$  in  $\mathcal{O}_{\mathcal{X},x}$ . Then, the trace of  $E_j$  on  $V(\mathcal{I}_{\mathcal{X},x})$  has codimension  $l$  in  $E_j$  at  $x$ . This is a contradiction. We conclude that the ideal  $\mathcal{I}_{\mathcal{X},x}$  is equal to the maximal ideal  $\mathfrak{m}_x$ , therefore  $x$  is a point of  $F$ .

It remains to prove the converse implication: every point  $x$  of the fan  $F$  must be a generic point of an intersection of irreducible components of  $D_{\mathcal{X}}$ . Let  $x$  be a point of  $F$ : by construction of Kato fan  $F$ , the maximal ideal of  $\mathcal{O}_{\mathcal{X},x}$  is equal to  $\mathcal{I}_{\mathcal{X},x}$ , thus it is generated by elements in  $\mathcal{M}_{\mathcal{X},x}$ . The zero locus of such an element is contained in  $D_{\mathcal{X}}$  by definition of the logarithmic structure on  $\mathcal{X}^+$ . Therefore, the zero locus is a union of irreducible components of the trace of  $D_{\mathcal{X}}$  on  $\text{Spec } \mathcal{O}_{\mathcal{X},x}$  and  $x$  is a generic point of the intersection of all such irreducible components.  $\square$

Moreover, the example 2.2.2 also leads to the following characterization.

**Proposition 2.2.5.** (*[GR04], Corollary 12.5.35*) *Let  $\mathcal{X}^+$  be a log-regular log scheme over  $S^+$  and  $F$  its associated Kato fan. The following are equivalent:*

- (1) *for every  $x \in F$ ,  $M_{F,x} \simeq \mathbb{N}^{r(x)}$ ,*
- (2) *the underlying scheme  $\mathcal{X}$  is regular.*

*If this is the case, then the special fibre  $\mathcal{X}_k$  is a strict normal crossing divisor.*

**(2.2.6)** The construction of the Kato fan of a log scheme defines a functor from the category of log-regular log schemes to the category of Kato fans. Indeed, given a morphism of log schemes  $\mathcal{X}^+ \rightarrow \mathcal{Y}^+$ , we consider the embedding of the associated Kato fan  $F_{\mathcal{X}}$  in  $\mathcal{X}^+$  and the canonical morphism  $\mathcal{Y}^+ \rightarrow F_{\mathcal{Y}}$ : the composition

$$F_{\mathcal{X}} \hookrightarrow \mathcal{X}^+ \rightarrow \mathcal{Y}^+ \rightarrow F_{\mathcal{Y}}$$

functorially induces a map between associated Kato fans. Moreover, this association preserves strict morphisms ([Uli13], Lemma 4.9).

### 2.3. Resolutions of log schemes via Kato fan subdivisions.

**Proposition 2.3.1.** (*[Kat94], Proposition 9.9*) *Let  $\mathcal{X}^+$  be a log-regular log scheme and let  $F$  be its associated Kato fan. Let  $F' \rightarrow F$  be a subdivision of fans. Then there exist a log scheme  $\mathcal{X}'^+$ , a morphism of log schemes  $\mathcal{X}'^+ \rightarrow \mathcal{X}^+$  and a commutative diagram*

$$\begin{array}{ccc}
(\mathcal{X}', \mathcal{C}_{\mathcal{X}'} ) & \xrightarrow{p} & F' \\
\downarrow & & \downarrow \\
(\mathcal{X}, \mathcal{C}_{\mathcal{X}} ) & \xrightarrow{\pi_{\mathcal{X}}} & F
\end{array}$$

such that  $p^{-1}(\mathcal{M}_{F'}) \simeq \mathcal{C}_{\mathcal{X}'}$ , they define a final object in the category of such diagrams and the refinement  $F' \rightarrow F$  is induced by the morphism of log-regular log schemes  $\mathcal{X}'^+ \rightarrow \mathcal{X}^+$ .

**(2.3.2)** It follows that given any subdivision  $F' \rightarrow F$  of the Kato fan  $F$  associated with a log regular log scheme  $\mathcal{X}^+$ , we can construct a log scheme over  $\mathcal{X}^+$  with prescribed associated Kato fan  $F'$ . Combining this fact with Proposition 2.1.6 and Proposition 2.2.5 yields to the construction of resolutions of log schemes in the following sense: for any log-regular log scheme over  $S^+$  we can find a birational modification by a regular log scheme with strict normal special fibre. Moreover, the morphism of log schemes  $\mathcal{X}'^+ \rightarrow \mathcal{X}^+$  is obtained by a log blow-up ([Niz06], Theorem 5.8).

#### 2.4. Fibred products and associated Kato fans.

**(2.4.1)** Given morphisms of fs log schemes  $f_1 : \mathcal{X}_1^+ \rightarrow \mathcal{Y}^+$  and  $f_2 : \mathcal{X}_2^+ \rightarrow \mathcal{Y}^+$ , their fibred product exists in the category of log schemes. It is obtained by endowing the usual fibred product of schemes

$$\begin{array}{ccccc}
\mathcal{X}_1 \times_{\mathcal{Y}} \mathcal{X}_2 & \xrightarrow{p_1} & \mathcal{X}_1 & & \\
\downarrow p_2 & \searrow p_{\mathcal{Y}} & \downarrow f_1 & & \\
\mathcal{X}_2 & \xrightarrow{f_2} & \mathcal{Y} & & 
\end{array}
\tag{2.4.2}$$

with the log structure associated to  $p_1^{-1}\mathcal{M}_{\mathcal{X}_1} \oplus_{p_{\mathcal{Y}}^{-1}\mathcal{M}_{\mathcal{Y}}} p_2^{-1}\mathcal{M}_{\mathcal{X}_2}$ . If  $u_1 : P \rightarrow Q_1$  and  $u_2 : P \rightarrow Q_2$  are charts for the morphisms  $f_1$  and  $f_2$  respectively, then the induced morphism  $\mathcal{X}_1 \times_{\mathcal{Y}} \mathcal{X}_2 \rightarrow \text{Spec } \mathbb{Z}[Q_1 \oplus_P Q_2]$  is a chart for  $\mathcal{X}_1^+ \times_{\mathcal{Y}^+} \mathcal{X}_2^+$ .

**(2.4.3)** In general, the fibred product is not fs, but the category of fs log schemes also admits fibred products. Keeping the same notations, the following is a chart of the fibred product in the category of fine and saturated log schemes

$$\mathcal{X}_1^+ \times_{\mathcal{Y}^+}^{\text{fs}} \mathcal{X}_2^+ = (\mathcal{X}_1^+ \times_{\mathcal{Y}^+} \mathcal{X}_2^+) \times_{\mathbb{Z}[Q_1 \oplus_P Q_2]} \mathbb{Z}[(Q_1 \oplus_P Q_2)^{\text{sat}}]$$

([Bul15], 3.6.16). We remark that the two fibre products above may not only have different log structures, but also the underlying schemes may differ.

**(2.4.4)** Log smoothness is preserved under fs base change and composition ([GR04], Proposition 12.3.24). In particular, if  $f_1 : \mathcal{X}_1^+ \rightarrow \mathcal{Y}^+$  is log-smooth and  $\mathcal{X}_2^+$  is log-regular, then  $\mathcal{X}_1^+ \times_{\mathcal{Y}^+}^{\text{fs}} \mathcal{X}_2^+$  is log-regular, by [Kat94], Theorem 8.2.

Consider log-smooth morphisms of fs log schemes  $\mathcal{X}_1^+ \rightarrow \mathcal{Y}^+$  and  $\mathcal{X}_2^+ \rightarrow \mathcal{Y}^+$ . The sheaves of logarithmic differentials are related by the following isomorphism

$$p_1^* \Omega_{\mathcal{X}_1^+/\mathcal{Y}^+}^{\log} \oplus p_2^* \Omega_{\mathcal{X}_2^+/\mathcal{Y}^+}^{\log} \simeq \Omega_{\mathcal{X}_1^+ \times_{\mathcal{Y}^+}^{\text{fs}} \mathcal{X}_2^+/\mathcal{Y}^+}^{\log} \tag{2.4.5}$$

by [GR04], Proposition 12.3.13. Furthermore, by assumption of log-smoothness over  $S^+$  the logarithmic differential sheaves are locally free of finite rank ([Kat94], Proposition 3.10) and we can consider their determinants; they are called log

canonical bundles and denoted by  $\omega^{\log}$ . The following isomorphism is a direct consequence of (2.4.5)

$$(2.4.6) \quad p_1^* \omega_{\mathcal{X}_1^+/\mathcal{Y}^+}^{\log} \otimes p_2^* \omega_{\mathcal{X}_2^+/\mathcal{Y}^+}^{\log} \simeq \omega_{\mathcal{X}_1^+ \times_{\mathcal{Y}^+}^{\text{fs}} \mathcal{X}_2^+/\mathcal{Y}^+}^{\log}.$$

(2.4.7) Similarly to the construction of fibred products of *fs* log schemes, the category of *fs* Kato fans admits fibred products: on affine Kato fans  $F = \text{Spec } P$  and  $G = \text{Spec } Q$  over  $H = \text{Spec } T$ ,  $F \times_H G$  is the spectrum of the amalgamated sum  $(P \oplus_T Q)^{\text{sat}}$  in the category of *fs* monoids ([Uli16], Proposition 2.4) and on the underlying topological spaces, this coincides with the usual fibred product.

We seek to compare the Kato fan associated to the fibred product of log-regular log schemes with the fibred product of associated Kato fans.

**Proposition 2.4.8.** ([Sai04], Lemma 2.8) *Given  $\mathcal{T}^+$  a log-regular log scheme, let  $\mathcal{X}^+$  and  $\mathcal{Y}^+$  be log-smooth log schemes over  $\mathcal{T}^+$ . We denote by  $\mathcal{Z}^+$  the *fs* fibred product  $\mathcal{X}^+ \times_{\mathcal{T}^+}^{\text{fs}} \mathcal{Y}^+$ . Then, the natural morphisms  $F_{\mathcal{X}} \rightarrow F_{\mathcal{X}}$  and  $F_{\mathcal{X}} \rightarrow F_{\mathcal{Y}}$  induce a morphism of Kato fans*

$$F_{\mathcal{Z}} \rightarrow F_{\mathcal{X}} \times_{F_{\mathcal{T}}} F_{\mathcal{Y}}$$

*that is locally an isomorphism.*

## 2.5. Semistability and Kato fans associated to the fibred products.

(2.5.1) We investigate a sufficient condition to turn the local isomorphism of Proposition 2.4.8 into an isomorphism: it concerns the notion of semistability. We recall that a log-regular log scheme  $\mathcal{X}^+$  is said to be semistable if the divisor  $D_{\mathcal{X}}$ , where the log structure is non-trivial, is reduced.

(2.5.2) In order to see the relevance of the assumption of semistability, we need some results on saturated morphism of log schemes. We recall that, locally around a point  $x$  of the divisor  $D_{\mathcal{X}}$ , the morphism of characteristic monoids  $\mathbb{N} \rightarrow \mathcal{C}_{\mathcal{X},x}$  is a saturated morphism of monoids if, for any morphism  $u : \mathbb{N} \rightarrow P$  of *fs* monoids, the amalgamated sum  $\mathcal{C}_{\mathcal{X},x} \oplus_{\mathbb{N}} P$  is still saturated.

Following the work by T. Tsuji in an unpublished 1997 preprint, Vidal in [Vid04] defines the saturation index of a morphism of *fs* monoids. In the case of log-regular log scheme over  $S^+$  it can be easily computed: it is the least common multiple of the multiplicities of the prime components of the divisor  $D_{\mathcal{X}}$ . The following criterion holds.

**Lemma 2.5.3.** ([Vid04], Section 1.3) *A morphism of *fs* monoids is saturated if and only if the saturation index is equal to 1.*

**Proposition 2.5.4.** *Assume that the residue field  $k$  is algebraically closed. Let  $\mathcal{X}^+$  and  $\mathcal{Y}^+$  be log-smooth log scheme over  $S^+$ . Let  $\mathcal{Z}^+$  be their *fs* fibred product. If  $\mathcal{X}^+$  is semistable, then the morphism*

$$F_{\mathcal{Z}} \rightarrow F_{\mathcal{X}} \times F_{\mathcal{Y}},$$

*induced by the projections  $\mathcal{Z}^+ \rightarrow \mathcal{X}^+$  and  $\mathcal{Z}^+ \rightarrow \mathcal{Y}^+$ , is an isomorphism of Kato fans.*

*Proof.* By hypothesis  $\mathcal{X}^+$  is a semistable log-regular log scheme over  $S^+$ , hence the saturation index of  $\mathcal{X}^+ \rightarrow S^+$  is 1. Thus, by Lemma 2.5.3 the morphism of log schemes  $\mathcal{X}^+ \rightarrow S^+$  induces a saturated morphism of characteristic monoids at

every point of  $\mathcal{X}^+$ . The saturation condition implies that the fibred product in the category of log schemes coincides with the fibred product in the category of  $fs$  log schemes. In particular, the underlying scheme of  $\mathcal{Z}^+$  coincides with the usual schematic fibred product, hence its points are characterized as follows:

$$z = (x, y, s, \mathfrak{p}) \text{ and } \mathcal{O}_{\mathcal{Z},z} = (\mathcal{O}_{\mathcal{X},x} \otimes_R \mathcal{O}_{\mathcal{Y},y})_{\mathfrak{p}}$$

where  $x$  and  $y$  are points of  $\mathcal{X}^+$  and  $\mathcal{Y}^+$  both mapped to the same point  $s$  of  $S$ , while  $\mathfrak{p}$  is a prime ideal of the tensor product of residue fields  $\kappa(x) \otimes_{\kappa(s)} \kappa(y)$ . We look for a characterization of points  $z$  in  $\mathcal{Z}^+$  that lie in the Kato fan  $F_{\mathcal{Z}}$ .

If the point  $z$  lies in  $F_{\mathcal{Z}}$ , then the maximal ideal  $\mathfrak{m}_z$  is equal to the ideal  $\mathcal{I}_{\mathcal{Z},z}$  by definition. By the flatness of the models  $\mathcal{X}^+$  and  $\mathcal{Y}^+$  over  $S^+$ , the morphisms of local rings  $\mathcal{O}_{\mathcal{X},x} \rightarrow \mathcal{O}_{\mathcal{Z},z}$  and  $\mathcal{O}_{\mathcal{Y},y} \rightarrow \mathcal{O}_{\mathcal{Z},z}$  are injective. Hence, the equalities  $\mathfrak{m}_x = \mathcal{I}_{\mathcal{X},x}$  and  $\mathfrak{m}_y = \mathcal{I}_{\mathcal{Y},y}$  hold. Thus, the points  $z$  in  $\mathcal{Z}^+$  that lie in the Kato fan  $F_{\mathcal{Z}}$  are necessarily points such that the projections  $x$  and  $y$  to  $\mathcal{X}^+$  and  $\mathcal{Y}^+$  lie in their associated Kato fans. Therefore, we may assume  $x \in F_{\mathcal{X}}$ ,  $y \in F_{\mathcal{Y}}$ , and it remains to characterize the prime ideals  $\mathfrak{p}$  such that  $z = (x, y, s, \mathfrak{p}) \in F_{\mathcal{Z}}$ .

By log-regularity of  $\mathcal{Z}^+$ , the point  $z$  lies in the associated Kato fan if and only if  $\dim \mathcal{O}_{\mathcal{Z},z} = \text{rank} \mathcal{C}_{\mathcal{Z},z}^{\text{gp}}$ . At the level of characteristic sheaves it holds that

$$\text{rank} \mathcal{C}_{\mathcal{Z},z}^{\text{gp}} = \text{rank} \mathcal{C}_{\mathcal{X},x}^{\text{gp}} + \text{rank} \mathcal{C}_{\mathcal{Y},y}^{\text{gp}} - 1.$$

Since  $x$  and  $y$  are both assumed to be points in the associated Kato fans, the equality between dimension of local rings and rank of the groupifications of characteristic sheaves lead to the equivalence

$$\begin{aligned} z \in F_{\mathcal{Z}} &\Leftrightarrow \dim \mathcal{O}_{\mathcal{Z},z} = \text{rank} \mathcal{C}_{\mathcal{Z},z}^{\text{gp}} \\ &= \text{rank} \mathcal{C}_{\mathcal{X},x}^{\text{gp}} + \text{rank} \mathcal{C}_{\mathcal{Y},y}^{\text{gp}} - 1 \\ &= \dim \mathcal{O}_{\mathcal{X},x} + \dim \mathcal{O}_{\mathcal{Y},y} - 1. \end{aligned}$$

By log-regularity of  $\mathcal{Z}^+$ , it holds that  $\dim \mathcal{O}_{\mathcal{Z},z} \geq \text{rank} \mathcal{C}_{\mathcal{Z},z}^{\text{gp}}$ , thus the inequality

$$\dim \mathcal{O}_{\mathcal{Z},z} \geq \dim \mathcal{O}_{\mathcal{X},x} + \dim \mathcal{O}_{\mathcal{Y},y} - 1$$

is always true and equality holds only for minimal prime ideals  $\mathfrak{p}$  of  $\kappa(x) \otimes_{\kappa(s)} \kappa(y)$ . Therefore, in order to conclude that there exists a unique point  $z$  whose projections are the points  $x$  and  $y$  and that lies in the Kato fan  $F_{\mathcal{Z}}$ , we need to prove the following property: the tensor product  $\kappa(x) \otimes_{\kappa(s)} \kappa(y)$  has a unique minimal prime ideal.

If  $s$  is the closed point of  $S$ , then its residue field is the algebraically closed field  $k$ . It follows that the tensor product  $\kappa(x) \otimes_k \kappa(y)$  is a domain, hence it has a unique minimal prime ideal, namely 0.

Otherwise, we denote by  $\mathcal{V}$  the closure of  $x$  in  $\mathcal{X}^+$ : it is still a log-smooth scheme over  $S^+$  with reduced special fibre  $\mathcal{V}_k$ . We denote by  $L$  the separable closure of  $\kappa(s)$  in  $\kappa(x)$  and by  $\mathcal{O}_L$  its valuation ring. Let  $\mathcal{V}_{\mathcal{O}_L}$  be the base change of  $\mathcal{V}$  to  $\text{Spec}(\mathcal{O}_L)$ : the generic fibre  $\mathcal{V}_L$  is normal, as normality is preserved under separable field extension, and the special fibre is still reduced. By [Liu02], Lemma 4.1.18, it follows that  $\mathcal{V}_{\mathcal{O}_L}$  is normal. Since  $\mathcal{V}_{\mathcal{O}_L}$  is normal and proper with reduced special fibre,  $\mathcal{V}_L$  is connected. We deduce that  $\kappa(s)$  is separably algebraically closed in  $\kappa(x)$ . Finally by , Proposition 4.3.2, the tensor product  $\kappa(x) \otimes_{\kappa(s)} \kappa(y)$  has a unique minimal prime ideal.  $\square$

### 3. THE SKELETON OF A LOG-REGULAR LOG SCHEME

#### 3.1. Construction of the skeleton of a log-regular log scheme.

**(3.1.1)** Let  $\mathcal{X}^+$  be a log-regular log scheme over  $S^+$ . Let  $x$  be a point of the associated Kato fan  $F$ . Denote by  $F(x)$  the set of points  $y$  of  $F$  such that  $x$  lies in the closure of  $\{y\}$ , and by  $\mathcal{C}_{F(x)}$  the restriction of  $\mathcal{C}_F$  to  $F(x)$ . Denote by  $\text{Spec } \mathcal{C}_{\mathcal{X},x}$  the spectrum of the monoid  $\mathcal{C}_{\mathcal{X},x} = \mathcal{C}_{F(x)}$ . Then there exists a canonical isomorphism of monoid spaces

$$(F(x), \mathcal{C}_{F(x)}) \rightarrow \text{Spec } \mathcal{C}_{\mathcal{X},x} : y \mapsto \{s \in \mathcal{C}_{\mathcal{X},x} \mid s(y) = 0\}$$

where the expression  $s(y) = 0$  means that  $s'(y) = 0$  for any representative  $s'$  of  $s$  in  $\mathcal{M}_{\mathcal{X},x}$ . In particular, we obtain a bijective correspondence between the faces of the monoid  $\mathcal{C}_{\mathcal{X},x}$  and the points of  $F(x)$ , and for every point  $y$  of  $F(x)$ , a surjective cospecialization morphism of monoids

$$\tau_{x,y} : \mathcal{C}_{\mathcal{X},x} \rightarrow \mathcal{C}_{\mathcal{X},y}$$

which induces an isomorphism of monoids

$$S^{-1}\mathcal{C}_{\mathcal{X},x}/(S^{-1}\mathcal{C}_{\mathcal{X},x})^\times \cong \mathcal{C}_{\mathcal{X},x}/S \xrightarrow{\sim} \mathcal{C}_{\mathcal{X},y}$$

where  $S$  denotes the monoid of elements  $s$  in  $\mathcal{C}_{\mathcal{X},x}$  such that  $s(y) \neq 0$ .

**(3.1.2)** For each point  $x$  in  $F$ , we denote by  $\sigma_x$  the set of morphisms of monoids

$$\alpha : \mathcal{C}_{\mathcal{X},x} \rightarrow (\mathbb{R}_{\geq 0}, +)$$

such that  $\alpha(\pi) = 1$  for every uniformizer  $\pi$  in  $R$ . We endow  $\sigma_x$  with the topology of pointwise convergence, where  $\mathbb{R}_{\geq 0}$  carries the usual Euclidean topology. Note that  $\sigma_x$  is a polygon in the real affine space

$$\{\alpha : \mathcal{C}_{\mathcal{X},x}^{\text{gp}} \rightarrow (\mathbb{R}, +) \mid \alpha(\pi) = 1 \text{ for every uniformizer } \pi \text{ in } R\}.$$

If  $y$  is a point of  $F(x)$ , then the surjective cospecialization morphism  $\tau_{x,y}$  induces a topological embedding  $\sigma_y \rightarrow \sigma_x$  that identifies  $\sigma_y$  with a face of  $\sigma_x$ .

**(3.1.3)** We denote by  $T$  the disjoint union of the topological spaces  $\sigma_x$  with  $x$  in  $F$ . On the topological space  $T$ , we consider the equivalence relation  $\sim$  generated by couples of the form  $(\alpha, \alpha \circ \tau_{x,y})$  where  $x$  and  $y$  are points in  $F$  such that  $x$  lies in the closure of  $\{y\}$  and  $\alpha$  is a point of  $\sigma_y$ .

The skeleton of  $\mathcal{X}^+$  is defined as the quotient of the topological space  $T$  by the equivalence relation  $\sim$ . We denote this skeleton by  $\text{Sk}(\mathcal{X}^+)$ . It is clear that  $\text{Sk}(\mathcal{X}^+)$  has the structure of a polyhedral complex with closed cells  $\{\sigma_x, x \in F\}$  and that the faces of a cell  $\sigma_x$  are precisely the cells  $\sigma_y$  with  $y$  in  $F(x)$ .

#### 3.2. Embedding the skeleton in the non-archimedean generic fiber.

**(3.2.1)** Let  $\mathcal{X}^+$  be a log-regular log scheme over  $S^+$ . Let  $x$  be a point of the associated Kato fan  $F$ . As the log structure on  $\mathcal{X}^+$  is of finite type, the characteristic monoid  $\mathcal{C}_{\mathcal{X},x}$  is of finite type too, and thus  $\mathcal{C}_{\mathcal{X},x}^{\text{gp}}$  is a free abelian group of finite rank. Hence there exists a section

$$\zeta : \mathcal{M}_{\mathcal{X},x}^{\text{gp}} / \mathcal{M}_{\mathcal{X},x}^\times \rightarrow \mathcal{M}_{\mathcal{X},x}^{\text{gp}}.$$

The section  $\zeta$  restricts to  $\mathcal{C}_{\mathcal{X},x} \rightarrow \mathcal{M}_{\mathcal{X},x}$ ; indeed, if  $x \in \mathcal{M}_{\mathcal{X},x}$  then  $\zeta(\bar{x}) - x \in \mathcal{M}_{\mathcal{X},x}^\times$ . Therefore we may choose a section

$$(3.2.2) \quad \mathcal{C}_{\mathcal{X},x} \rightarrow \mathcal{M}_{\mathcal{X},x}$$



of the projection homomorphism

$$\mathcal{M}_{\mathcal{X},x} \rightarrow \mathcal{C}_{\mathcal{X},x}$$

and use this section to view  $\mathcal{C}_{\mathcal{X},x}$  as a submonoid of  $\mathcal{M}_{\mathcal{X},x}$ . Note that  $\mathcal{C}_{\mathcal{X},x} \setminus \{0\}$  generates the ideal  $\mathcal{I}_{\mathcal{X},x}$  of  $\mathcal{O}_{\mathcal{X},x}$ .

We propose a generalisation of [MN15], Lemma 2.4.4.

**Lemma 3.2.3.** *Let  $A$  be a Noetherian ring, let  $I$  be an ideal of  $A$  and let  $(y_1, \dots, y_m)$  be a system of generators for  $I$ . We denote by  $\hat{A}$  the  $I$ -adic completion of  $A$ . Let  $B$  be a subring of  $A$  such that the elements  $y_1, \dots, y_m$  belong to  $B$  and generate the ideal  $B \cap I$  in  $B$ . Then, in the ring  $\hat{A}$ , every element  $f$  of  $B$  can be written as*

$$(3.2.4) \quad f = \sum_{\beta \in \mathbb{Z}_{\geq 0}^m} c_\beta y^\beta$$

where the coefficients  $c_\beta$  belong to  $((A \setminus I) \cap B) \cup \{0\}$ .

*Proof.* Let  $f$  be an element of  $B$ , we construct an expansion for  $f$  of the form (3.2.4) by induction. If  $f$  belongs to the complement of  $I$ , the conclusion trivially holds. Otherwise,  $f$  belongs to  $I$  and we can write  $f$  as a linear combination of the elements  $y_1, \dots, y_m$  with coefficients in  $B$ :

$$f = \sum_{j=1}^m b_j y_j, \quad b_j \in B.$$

By induction hypothesis, we suppose that  $i$  is a positive integer and that we can write every  $f$  in  $B$  as a sum of an element  $f_i$  of the form (3.2.4) and a linear combination of degree  $i$  monomials in the elements  $y_1, \dots, y_m$  with coefficients in  $B$ . We apply this assumption to the coefficients  $b_j$ , hence

$$b_j = b_{j,i} + \sum_{\substack{\beta \in \mathbb{Z}_{\geq 0}^m \\ |\beta|=i}} b_{j,\beta} y^\beta, \quad b_{j,\beta} \in B.$$

Then we can write  $f$  as a sum of an element  $f_{i+1}$  of the form (3.2.4) and a linear combination of degree  $i+1$  monomials in the elements  $y_1, \dots, y_m$  with coefficients in  $B$

$$f = \underbrace{\sum_{j=1}^m b_{j,i} y_j}_{f_{i+1}} + \sum_{j=1}^m \left( \sum_{\substack{\beta \in \mathbb{Z}_{\geq 0}^m \\ |\beta|=i}} b_{j,\beta} y^\beta \right) y_j$$

such that  $f_i$  and  $f_{i+1}$  have the same coefficients in degree  $i < 0$ . Iterating this construction we finally find an expansion of  $f$  of the required form.  $\square$

**(3.2.5)** Let  $f$  be an element of  $\mathcal{O}_{\mathcal{X},x}$ . Considering  $A = B = \mathcal{O}_{\mathcal{X},x}$ ,  $I = \mathfrak{m}_x$  and a system of generators for  $\mathfrak{m}_x$  in  $\mathcal{C}_{\mathcal{X},x} \setminus \{1\}$ , by Lemma 3.2.3 we can write  $f$  as a formal power series

$$(3.2.6) \quad f = \sum_{\gamma \in \mathcal{C}_{\mathcal{X},x}} c_\gamma \gamma$$

in  $\widehat{\mathcal{O}}_{\mathcal{X},x}$ , where each coefficient  $c_\gamma$  is either zero or a unit in  $\mathcal{O}_{\mathcal{X},x}$ . We call this formal series an *admissible expansion* of  $f$ . We set

$$(3.2.7) \quad S = \{\gamma \in \mathcal{C}_{\mathcal{X},x} \mid c_\gamma \neq 0\}$$

and we denote by  $\Gamma$  the set of elements of  $S$  that lie on a compact face of the convex hull of  $S + \mathcal{C}_{\mathcal{X},x}$  in  $\mathcal{C}_{\mathcal{X},x}^{\text{gp}} \otimes_{\mathbb{Z}} \mathbb{R}$ .

**Proposition 3.2.8.**

(1) *The element*

$$f_x = \sum_{\gamma \in \Gamma} c_\gamma(x) \gamma \in k(x)[\mathcal{C}_{\mathcal{X},x}]$$

*depends on the choice of the section (3.2.2), but not on the expansion (3.2.6).*

(2) *The subset  $\Gamma$  of  $\mathcal{C}_{\mathcal{X},x}$  only depends on  $f$  and  $x$ , and not on the choice of the section (3.2.2) or the expansion (3.2.6).*

*Proof.* If we denote by  $I$  the ideal of  $k(x)[\mathcal{C}_{\mathcal{X},x}]$  generated by  $\mathcal{C}_{\mathcal{X},x} \setminus \{1\}$ , then it follows from [Kat94] that there exists an isomorphism of  $k(x)$ -algebras

$$(3.2.9) \quad \text{gr}_I k(x)[\mathcal{C}_{\mathcal{X},x}] \rightarrow \text{gr}_{\mathfrak{m}_x} \mathcal{O}_{\mathcal{X},x}.$$

Using this result and following the argument of [MN15] Proposition 2.4.6, we show that  $f_x$  does not depend on the expansion of  $f$ . Let

$$f = \sum_{\gamma \in \mathcal{C}_{\mathcal{X},x}} c'_\gamma \gamma$$

be another admissible expansion of  $f$  with associated set  $\Gamma'$  and element  $f'_x$ . Then

$$0 = \sum_{\gamma \in \mathcal{C}_{\mathcal{X},x}} (c_\gamma - c'_\gamma) \gamma = \sum_{\gamma \in \mathcal{C}_{\mathcal{X},x}} d_\gamma \gamma$$

where the right hand side is an admissible expansion obtained by choosing admissible expansions for the elements  $c_\gamma - c'_\gamma$  that do not lie in  $\mathcal{O}_{\mathcal{X},x}^\times \cup \{0\}$ . In particular  $d_\gamma(x) = c_\gamma(x) - c'_\gamma(x)$  for any  $\gamma$  in  $\Gamma_x \cup \Gamma'_x$ . The isomorphism of graded algebras in (3.2.9) implies that the elements  $d_\gamma$  must all vanish, hence  $\Gamma_x = \Gamma'_x$  and  $f_x = f'_x$ .

Point (2) follows from the fact that the coefficients  $c_\gamma$  of  $f_x$  are independent of the chosen section up to multiplication by a unit in  $\mathcal{O}_{\mathcal{X},x}$ , so that the support  $\Gamma$  of  $f_x$  only depends on  $f$  and  $x$ .  $\square$

(3.2.10) We will denote the subset  $\Gamma$  of  $\mathcal{C}_{\mathcal{X},x}$  by  $\Gamma_x(f)$  and call it the *initial support* of  $f$  at  $x$ .

**Proposition 3.2.11.** *Let  $x$  be a point of  $F$  and let*

$$\alpha : \mathcal{C}_{\mathcal{X},x} \rightarrow (\mathbb{R}_{\geq 0}, +)$$

*be an element of  $\sigma_x$ . Then there exists a unique minimal real valuation*

$$v : \mathcal{O}_{\mathcal{X},x} \setminus \{0\} \rightarrow \mathbb{R}_{\geq 0}$$

*such that  $v(m) = \alpha(\overline{m})$  for each element  $m$  of  $\mathcal{M}_{\mathcal{X},x}$ .*

*Proof.* We will prove that the map

$$(3.2.12) \quad v : \mathcal{O}_{\mathcal{X},x} \setminus \{0\} \rightarrow \mathbb{R} : f \mapsto \min\{\alpha(\gamma) \mid \gamma \in \Gamma_x(f)\}$$

satisfies the requirements in the statement. We fix a section

$$\mathcal{C}_{\mathcal{X},x} \rightarrow \mathcal{M}_{\mathcal{X},x}.$$

It is straightforward to check that  $(f \cdot g)_x = f_x \cdot g_x$  for all  $f$  and  $g$  in  $\mathcal{O}_{\mathcal{X},x}$ . This implies that  $v$  is a valuation. It is obvious that  $v(m) = \alpha(\overline{m})$  for all  $m$  in  $\mathcal{M}_{\mathcal{X},x}$ , since we can write  $m$  as the product of an element of  $\mathcal{C}_{\mathcal{X},x}$  and a unit in  $\mathcal{O}_{\mathcal{X},x}$ .

Now we prove minimality. Consider any real valuation

$$w : \mathcal{O}_{\mathcal{X},x} \rightarrow \mathbb{R}$$

such that  $w(f) = \alpha(\overline{m})$  for each element  $m$  of  $\mathcal{M}_{\mathcal{X},x}$ , and let  $f$  be an element of  $\mathcal{O}_{\mathcal{X},x}$ . We must show that  $w(f) \geq v(f)$ .

We set

$$C_\alpha = \mathcal{C}_{\mathcal{X},x} \setminus \alpha^{-1}(0).$$

We denote by  $I$  the ideal in  $\mathcal{O}_{\mathcal{X},x}$  generated by  $C_\alpha$  and by  $A$  the  $I$ -adic completion of  $\mathcal{O}_{\mathcal{X},x}$ . By Lemma 3.2.3, we see that we can write  $f$  in  $A$  as

$$(3.2.13) \quad \sum_{\beta \in C_\alpha \cup \{1\}} d_\beta \beta$$

where  $d_\beta$  is either zero or contained in the complement of  $I$  in  $\mathcal{O}_{\mathcal{X},x}$ .

Since  $\alpha(\beta) > 0$  for every  $\beta \in C_\alpha$ , we can find an integer  $N > 0$  such that  $w(g) > w(f)$  for every element  $g$  in  $I^N$ . Since  $w(\beta) = \alpha(\beta)$  for all  $\beta$  in  $\mathcal{C}_{\mathcal{X},x}$ , we can write

$$w(f) \geq \min\{\alpha(\beta) \mid d_\beta \neq 0\}.$$

We consider the coefficients in the expansion (3.2.13) of  $f$ . Applying Lemma 3.2.3 as in paragraph (3.2.5), we can write admissible expansions of these coefficients in  $\widehat{\mathcal{O}}_{\mathcal{X},x}$  as

$$d_\beta = \sum_{\gamma \in \mathcal{C}_{\mathcal{X},x}} c_{\gamma,\beta} \gamma, \quad c_{\gamma,\beta} \in \mathcal{O}_{\mathcal{X},x}^\times \cup \{0\},$$

with  $\alpha(\gamma) = 0$  in the expansions of  $d_\beta$  that belong to  $\mathfrak{m}_x \setminus I$ .

Therefore we obtain an admissible expansion of  $f$

$$f = \sum_{\substack{\beta \in C_\alpha \cup \{1\} \\ \gamma \in \mathcal{C}_{\mathcal{X},x}}} c_{\gamma,\beta} \gamma \beta$$

and we have

$$\begin{aligned} v(f) &= \min\{\alpha(\gamma\beta) \mid c_{\gamma,\beta} \neq 0\} \\ &= \min\{\alpha(\beta) \mid d_\beta \neq 0\} \\ &\geq w(f). \end{aligned}$$

□

**Remark 3.2.14.** In the definition (3.2.12) of the valuation  $v$ , we compute the minimum over the terms in the initial support of  $f$ : these elements are a finite number and they only depends on  $x$  and  $f$  by Proposition 3.2.8. Therefore, this minimum provides a well-defined function on  $\mathcal{O}_{\mathcal{X},x} \setminus \{0\}$ . Nevertheless, it is

equivalent to consider the minimum over all the terms of an admissible expansion of  $f$ , i.e. for any admissible expansion  $f = \sum_{\gamma \in \mathcal{C}_{\mathcal{X},x}} c_\gamma \gamma$

$$\min\{\alpha(\gamma) \mid \gamma \in \Gamma_x(f)\} = \min\{\alpha(\gamma) \mid \gamma \in S\},$$

where  $S = \{\gamma \in \mathcal{C}_{\mathcal{X},x} \mid c_\gamma \neq 0\}$  as in (3.2.7). Indeed, any element that belongs to  $S$  can be written as a sum of an element of the initial support of  $f$  and an element of  $\mathcal{C}_{\mathcal{X},x}$ . Since the morphism  $\alpha$  is additive and takes positive real values, then the minimum is necessarily attained by the elements in the initial support.

**(3.2.15)** We will denote the valuation  $v$  from Proposition 3.2.11 by  $v_{x,\alpha}$ . Since  $v_{x,\alpha}$  induces a real valuation on the function field of  $\mathcal{X}_K$  that extends the discrete valuation  $v_K$  on  $K$ , it defines a point of the  $K$ -analytic space  $\widehat{\mathcal{X}}_\eta$ , which we will denote by the same symbol  $v_{x,\alpha}$ . We now show that the characterization of  $v_{x,\alpha}$  in Proposition 3.2.11 implies that

$$v_{y,\alpha'} = v_{x,\alpha' \circ \tau_{x,y}}$$

for every  $y$  in  $F(x)$  and every  $\alpha'$  in  $\sigma_y$ .

Firstly we note that  $\mathcal{O}_{\mathcal{X},y}$  is the localization of  $\mathcal{O}_{\mathcal{X},x}$  with respect to the elements of  $m \in \mathcal{M}_{\mathcal{X},x}$  in the kernel of  $\tau_{x,y}$ . Indeed, by construction of  $\tau_{x,y}$ , the kernel is given by

$$\ker(\tau_{x,y}) = \{s \in \mathcal{C}_{\mathcal{X},x} \mid s(y) \neq 0\};$$

to obtain  $\mathcal{O}_{\mathcal{X},y}$  from  $\mathcal{O}_{\mathcal{X},x}$ , we localize by

$$S = \{a \in \mathcal{O}_{\mathcal{X},x} \mid a(y) \neq 0\};$$

therefore we can identify the set of elements in  $\mathcal{M}_{\mathcal{X},x}$  in  $\ker(\tau_{x,y})$  with the set  $S$ , recalling that for points in the Kato fan  $\mathcal{C}_{\mathcal{X},x} \setminus \{1\}$  generates the maximal ideal of  $\mathcal{O}_{\mathcal{X},x}$ . Therefore we are dealing with these two morphisms:

$$\mathcal{O}_{\mathcal{X},x} \hookrightarrow S^{-1}\mathcal{O}_{\mathcal{X},x} = \mathcal{O}_{\mathcal{X},y},$$

$$\mathcal{C}_{\mathcal{X},x} \twoheadrightarrow \mathcal{C}_{\mathcal{X},x}/S = \mathcal{C}_{\mathcal{X},y}.$$

Let  $f$  be an element of  $\mathcal{O}_{\mathcal{X},x}$ . Under the notations of Lemma 3.2.3, we apply the lemma to  $A = \mathcal{O}_{\mathcal{X},y}$  and  $B = \mathcal{O}_{\mathcal{X},x}$ , choosing a system of generators of  $\mathfrak{m}_y$  in  $\mathcal{C}_{\mathcal{X},x}$ : we can find an admissible expansion of  $f$  of the form

$$f = \sum_{\delta \in \mathcal{C}_{\mathcal{X},y}} d_\delta \delta \quad \text{with } d_\delta \in (\mathcal{O}_{\mathcal{X},x} \cap \mathcal{O}_{\mathcal{X},y}^\times) \cup \{0\}.$$

Admissible expansions of coefficients  $d_\delta$  induce an admissible expansion for  $f$  by

$$f = \sum_{\delta \in \mathcal{C}_{\mathcal{X},y}} \left( \sum_{\gamma \in S} c_{\gamma\delta} \gamma \right) \delta \quad \text{with } c_{\gamma\delta} \in \mathcal{O}_{\mathcal{X},x}^\times \cup \{0\},$$

where  $\gamma$  runs through the set  $S$  since  $d_\delta \in \mathcal{O}_{\mathcal{X},y}^\times$ . Thus we have

$$\begin{aligned} v_{y,\alpha'}(f) &= \min\{\alpha'(\delta) \mid \delta \in \Gamma_y(f)\} \\ &= \min\{\alpha' \circ \tau_{x,y}(\gamma\delta) \mid \delta \in \Gamma_y(f), \gamma \in S\} \\ &= \min\{\alpha' \circ \tau_{x,y}(\gamma\delta) \mid \gamma\delta \in \Gamma_x(f)\} \\ &= v_{x,\alpha' \circ \tau_{x,y}}(f). \end{aligned}$$

Hence, we obtain a well-defined map

$$\iota : \text{Sk}(\mathcal{X}^+) \rightarrow \widehat{\mathcal{X}}_\eta$$

by sending  $\alpha$  to  $v_{x,\alpha}$  for every point  $x$  of  $F$  and every  $\alpha \in \sigma_x$ .

**Proposition 3.2.16.** *The map*

$$\iota : \text{Sk}(\mathcal{X}^+) \rightarrow \widehat{\mathcal{X}}_\eta$$

*is a topological embedding.*

*Proof.* First, we show that  $\iota$  is injective. Let  $x$  be a point of  $F$  and  $\alpha$  an element of  $\sigma_x$ . Let  $y$  be the point of  $F(x)$  corresponding to the face  $\mathcal{C}_{\mathcal{X},x} \setminus \alpha^{-1}(0)$  of  $\mathcal{C}_{\mathcal{X},x}$ . Then  $\alpha$  factors through an element

$$\alpha' : \mathcal{C}_{\mathcal{X},y} \rightarrow \mathbb{R}_{\geq 0}$$

of  $\sigma_y$ . Note that  $\alpha = \alpha'$  in  $\text{Sk}(\mathcal{X}^+)$  because  $\alpha = \alpha' \circ \tau_{x,y}$ . Moreover, since  $(\alpha')^{-1}(0) = \{1\}$ , the center of the valuation  $v_{y,\alpha'}$  is the point  $y$ , so that  $\text{red}_{\mathcal{X}}(v_{y,\alpha'}) = y$ . Thus we can recover  $y$  from  $v_{y,\alpha'}$ . Then we can also reconstruct  $\alpha'$  by looking at the values of  $v_{y,\alpha'}$  at the elements of  $\mathcal{M}_{\mathcal{X},y}$ . We conclude that  $\iota$  is injective.

Now, we show that  $\iota$  is a homeomorphism onto its image. Since  $\text{Sk}(\mathcal{X}^+)$  is compact and  $\widehat{\mathcal{X}}_\eta$  is Hausdorff, it suffices to show that  $\iota$  is continuous. The family  $\{\sigma_x, x \in F\}$  is a cover of  $\text{Sk}(\mathcal{X}^+)$  by closed subsets, so that we only have to prove that the restriction of  $\iota$  to  $\sigma_x$  is continuous, for every  $x \in F$ . By definition of the Berkovich topology, it is enough to prove that the map

$$\sigma_x \rightarrow \mathbb{R} : \alpha \mapsto v_{x,\alpha}(f)$$

is continuous for every  $f$  in  $\mathcal{O}_{\mathcal{X},x}$ . This is obvious from the formula (3.2.12).  $\square$

**(3.2.17)** From now on, we will view  $\text{Sk}(\mathcal{X}^+)$  as a topological subspace of  $\mathcal{X}_K^{\text{an}}$  by means of the embedding  $\iota$  in Proposition 3.2.16. If  $\mathcal{X}$  is regular over  $R$  and  $\mathcal{X}_k$  is a divisor with strict normal crossings, the skeleton  $\text{Sk}(\mathcal{X}^+)$  was described in [MN15], Section 3.1.

### 3.3. Contracting the generic fibre to the skeleton.

**(3.3.1)** The inclusion  $\iota : \text{Sk}(\mathcal{X}^+) \rightarrow \widehat{\mathcal{X}}_\eta$  admits a continuous retraction

$$\rho_{\mathcal{X}} : \widehat{\mathcal{X}}_\eta \rightarrow \text{Sk}(\mathcal{X}^+)$$

constructed as follows. Let  $x$  be a point of  $\widehat{\mathcal{X}}_\eta$  and consider the reduction map

$$\text{red}_{\mathcal{X}} : \widehat{\mathcal{X}}_\eta \rightarrow \mathcal{X}_k.$$

Let  $E_1, \dots, E_r$  be the irreducible components of  $D_{\mathcal{X}}$  passing through the point  $\text{red}_{\mathcal{X}}(x)$ . We denote by  $\xi$  the generic point of the connected component of  $E_1 \cap \dots \cap E_r$  that contains  $\text{red}_{\mathcal{X}}(x)$ . By Lemma 2.2.3,  $\xi$  is a point in the associated Kato fan  $F$ . We set  $\alpha$  to be the morphism of monoids

$$\alpha : \mathcal{C}_{\mathcal{X},\xi} \rightarrow \mathbb{R}_{\geq 0}$$

such that  $\alpha(\overline{m}) = v_x(m)$  for any element  $m$  of  $\mathcal{M}_{\mathcal{X},\xi}$ . In particular  $\alpha(\pi) = v_x(\pi) = 1$  as we assumed the normalization of all valuations in the Berkovich space. Then  $\rho_{\mathcal{X}}(x)$  is the point of  $\text{Sk}(\mathcal{X}^+)$  corresponding to the couple  $(\xi, \alpha)$ . By construction  $\rho_{\mathcal{X}}$  is continuous and right inverse to the inclusion  $\iota$ .

**(3.3.2)** Given a morphism  $f : \mathcal{X}^+ \rightarrow \mathcal{Y}^+$  of integral flat separated log-regular  $S$ -schemes, we can employ the retraction  $\rho$  to define a map of skeleta as follows

This association makes the skeleton construction  $\text{Sk}(\mathcal{X}^+)$  functorial in  $\mathcal{X}^+$ .

It trivially satisfies  $\varepsilon \circ i_x(\pi) = 1$ . In order to conclude that it correctly defines a point in the skeleton  $\mathrm{Sk}(\mathcal{X}^+)$ , we need to check the compatibility with respect to the equivalence relation  $\sim$ . Indeed, suppose that  $\varepsilon = \varepsilon' \circ \tau_{z, z'}$  for some  $z' \in \overline{\{z\}}$ . We denote by  $x'$  the projection of  $z'$  under the local isomorphism of associated Kato fans. The diagram

$$\begin{array}{ccccc}
\mathcal{C}_{\mathcal{X},x} & \xrightarrow{i_x} & \mathcal{C}_{\mathcal{X},z} & \xrightarrow{\varepsilon} & \mathbb{R}_{\geq 0} \\
\downarrow \tau_{x,x'} & & \downarrow \tau_{z,z'} & & \\
\mathcal{C}_{\mathcal{X},x'} & \xrightarrow{i_{x'}} & \mathcal{C}_{\mathcal{X},z'} & \xrightarrow{\varepsilon'} & 
\end{array}$$

is commutative as made up by a commutative square and a commutative triangle of arrows. Therefore, by commutativity

$$\mathrm{pr}_{\mathcal{X}}(\varepsilon) = \mathrm{pr}_{\mathcal{X}}(\varepsilon') \circ \tau_{x,x'}$$

and this implies that  $\mathrm{pr}_{\mathcal{X}}(\varepsilon)$  defines a well-defined point  $v_{x,\mathrm{pr}_{\mathcal{X}}(\varepsilon)}$  of  $\mathrm{Sk}(\mathcal{X}^+)$ .

We claim that  $v_{x,\mathrm{pr}_{\mathcal{X}}(\varepsilon)}$  is indeed the image of  $v_{z,\varepsilon}$  under the map  $\mathrm{pr}_{\mathrm{Sk}(\mathcal{X})}$ , hence that the equality in the following inner diagram holds

$$\begin{array}{ccc}
\widehat{\mathcal{X}}_{\eta} & \xrightarrow{\widehat{\mathrm{pr}}_{\mathcal{X}}} & \widehat{\mathcal{X}}_{\eta} \\
\downarrow \rho_{\mathcal{X}} & & \downarrow \rho_{\mathcal{X}} \\
\mathrm{Sk}(\mathcal{X}^+) & \xrightarrow{\mathrm{pr}_{\mathrm{Sk}(\mathcal{X})}} & \mathrm{Sk}(\mathcal{X}^+)
\end{array}
\quad
\begin{array}{ccc}
v_{z,\varepsilon} \vdash & \xrightarrow{\quad} & v_{z,\varepsilon} \circ i \\
\uparrow & & \downarrow \\
v_{z,\varepsilon} \vdash & \xrightarrow{\quad} & \rho_{\mathcal{X}}(v_{z,\varepsilon} \circ i) = v_{x,\mathrm{pr}_{\mathcal{X}}(\varepsilon)}
\end{array}$$

We denote  $\rho_{\mathcal{X}}(v_{z,\varepsilon} \circ i)$  by  $(x, \alpha)$  as a point of  $\mathrm{Sk}(\mathcal{X}^+)$ . By definition of the retraction  $\rho_{\mathcal{X}}$ , the morphism  $\alpha$  is characterized by the fact that  $\alpha(\overline{m}) = (v_{z,\varepsilon} \circ i)(m)$  for any  $m$  in  $\mathcal{M}_{\mathcal{X},x}$  and then we have

$$\alpha(\overline{m}) = (v_{z,\varepsilon} \circ i)(m) = v_{z,\varepsilon}(m) = \varepsilon(\overline{m}).$$

On the other hand, for any  $m$  in  $\mathcal{M}_{\mathcal{X},x}$

$$v_{x,\mathrm{pr}_{\mathcal{X}}(\varepsilon)}(m) = \mathrm{pr}_{\mathcal{X}}(\varepsilon)(\overline{m}) = \varepsilon(\overline{m})$$

hence we obtain that  $\alpha$  coincide with the morphism  $\mathrm{pr}_{\mathcal{X}}(\varepsilon)$ . It means that their associated points  $\rho_{\mathcal{X}}(v_{z,\varepsilon} \circ i)$  and  $v_{x,\mathrm{pr}_{\mathcal{X}}(\varepsilon)}$  coincide in  $\mathrm{Sk}(\mathcal{X}^+)$ .

Given a pair of points in  $\mathrm{Sk}(\mathcal{X}^+) \times \mathrm{Sk}(\mathcal{Y}^+)$ , we know by surjectivity of  $(\mathrm{pr}_{\mathrm{Sk}(\mathcal{X})}, \mathrm{pr}_{\mathrm{Sk}(\mathcal{Y})})$  that they are of the form

$$(v_{x,\mathrm{pr}_{\mathcal{X}}(\varepsilon)}, v_{y,\mathrm{pr}_{\mathcal{Y}}(\varepsilon)}).$$

The assumption of semistability of  $\mathcal{X}^+$  guarantees that there is a unique  $z$  in  $F_{\mathcal{X}}$  in the fibre of  $x$  and  $y$ , by Proposition 2.5.4. Moreover, we can uniquely reconstruct  $\varepsilon$  by looking at the values of  $v_{x,\mathrm{pr}_{\mathcal{X}}(\varepsilon)}$  at the elements of  $\mathcal{M}_{\mathcal{X},x}$  and respectively of  $v_{y,\mathrm{pr}_{\mathcal{Y}}(\varepsilon)}$  at the elements of  $\mathcal{M}_{\mathcal{Y},y}$ . We conclude that  $(\mathrm{pr}_{\mathrm{Sk}(\mathcal{X})}, \mathrm{pr}_{\mathrm{Sk}(\mathcal{Y})})$  is injective.  $\square$

#### 4. THE WEIGHT FUNCTION ON A LOG-REGULAR LOG SCHEME

##### 4.1. Weight function on log-regular skeleta.

(4.1.1) Let  $X$  be a connected, smooth and proper  $K$ -variety of dimension  $n$ . We introduce the following notation: for any log-smooth model  $\mathcal{X}^+$  of  $X$ , for any point  $x = (\xi_x, |\cdot|_x) \in \widehat{\mathcal{X}}_{\eta}$  and for any divisor  $D$  whose support does not contain  $\xi_x$ , we set

$$v_x(D) = -\ln |f(x)|$$

where  $f$  is any element of  $K(X)^\times$  such that, locally at  $\text{red}_{\mathcal{X}}(x)$ , we have  $D = \text{div}(f)$ .

**(4.1.2)** Let  $\omega$  be a canonical form on  $X$  and let  $x$  be a divisorial point in  $X^{\text{an}}$ . In order to compute the weight  $\text{wt}_\omega(x)$ , we can consider a snc model  $\mathcal{Y}$  of  $X$  such that  $x \in \text{Sk}(\mathcal{Y}^+)$ . Then according to [NX16], Section 3.2.2 we have

$$(4.1.3) \quad \text{wt}_\omega(x) = v_x(\text{div}_{\mathcal{Y}^+}(\omega)) + 1,$$

where we denote by  $\text{div}_{\mathcal{Y}^+}(\omega)$  the divisor on  $\mathcal{Y}$  associated to  $\omega$  viewed as a rational section of the line bundle  $\omega_{\mathcal{Y}^+/S^+}^{\log}$ . The following proposition shows that this formula can be applied, more generally, to log-smooth models.

**Proposition 4.1.4.** *Let  $\mathcal{X}^+$  be a log-smooth log scheme over  $S^+$  such that  $x$  lies in  $\text{Sk}(\mathcal{X}^+)$ . Then the weight of  $\omega$  at  $x$  is given by*

$$\text{wt}_\omega(x) = v_x(\text{div}_{\mathcal{X}^+}(\omega)) + 1.$$

*Proof.* As we noticed in Remark 2.3.2, we can obtain an snc model  $\mathcal{Y}$  adapted to  $x$  by means of a log blow-up  $h : \mathcal{Y}^+ \rightarrow \mathcal{X}^+$  of  $\mathcal{X}^+$  (Propositions 2.2.5 and 2.3.1). Moreover, the corresponding skeleton  $\text{Sk}(\mathcal{Y}^+)$  is given by a subdivision of  $\text{Sk}(\mathcal{X}^+)$  (Proposition 2.1.6).

Log blow-ups are log étale morphisms ([Sai04], Section 2.1) and for log étale morphisms the sheaf of log differentials is stable under pullback ([Kat94], Proposition 3.12), therefore

$$h^* \omega_{\mathcal{Y}^+/S^+}^{\log} \simeq \omega_{\mathcal{X}^+/S^+}^{\log}.$$

Then  $\text{div}_{\mathcal{Y}^+}(\omega) = h^* \text{div}_{\mathcal{X}^+}(\omega)$  and in particular for points  $x$  of the skeleton  $\text{Sk}(\mathcal{X}^+) = \text{Sk}(\mathcal{Y}^+)$ , it holds that

$$v_x(\text{div}_{\mathcal{Y}^+}(\omega)) = v_x(h^* \text{div}_{\mathcal{X}^+}(\omega)) = v_x(\text{div}_{\mathcal{X}^+}(\omega)).$$

We conclude that  $\text{wt}_\omega(x) = v_x(\text{div}_{\mathcal{Y}^+}(\omega)) + 1 = v_x(\text{div}_{\mathcal{X}^+}(\omega)) + 1$ .  $\square$

## 4.2. Weight function on skeleta associated to $fs$ fibred products.

**(4.2.1)** Let  $\mathcal{X}^+$  and  $\mathcal{Y}^+$  be log-smooth log schemes over  $S^+$ , let  $\mathcal{Z}^+$  be their  $fs$  fibred product. By the universal property of the fibred products, it results that  $\mathcal{Z}_K \simeq \mathcal{X}_K \times_K \mathcal{Y}_K$ . Therefore, given  $\omega_{\mathcal{X}_K}$  and  $\omega_{\mathcal{Y}_K}$  canonical forms on  $\mathcal{X}_K$  and  $\mathcal{Y}_K$  respectively, the form

$$\varpi = \text{pr}_{\mathcal{X}}^* \omega_{\mathcal{X}_K} \otimes \text{pr}_{\mathcal{Y}}^* \omega_{\mathcal{Y}_K}$$

is a canonical form on  $\mathcal{Z}_K$ . Viewing these forms as rational sections of log canonical bundles, we see that  $\text{div}_{\mathcal{Z}^+}(\varpi) = \text{div}_{\mathcal{Z}^+}(\text{pr}_{\mathcal{X}}^* \omega_{\mathcal{X}_K} \otimes \text{pr}_{\mathcal{Y}}^* \omega_{\mathcal{Y}_K})$  according to (2.4.6).

**(4.2.2)** Let  $z$  be a point of  $F_{\mathcal{Z}}$ ; as before, we denote by  $x$  and  $y$  the images of  $z$  under the local isomorphism  $F_{\mathcal{Z}} \rightarrow F_{\mathcal{X}} \times F_{\mathcal{Y}}$ . Any morphism  $\varepsilon \in \sigma_z$  defines a point  $v_{z,\varepsilon}$  in  $\text{Sk}(\mathcal{Z}^+)$ . For the sake of convenience, we simply denote the valuation by the corresponding morphism and we denote  $\alpha = \text{pr}_{\mathcal{X}}(\varepsilon)$  and  $\beta = \text{pr}_{\mathcal{Y}}(\varepsilon)$ . We aim to relate the valuation  $v_\varepsilon(\text{div}_{\mathcal{Z}^+}(\varpi))$  to the values

$$v_\alpha(\text{div}_{\mathcal{X}^+}(\omega_{\mathcal{X}_K})), v_\beta(\text{div}_{\mathcal{Y}^+}(\omega_{\mathcal{Y}_K})).$$



(4.2.3) Let  $f_x \in \mathcal{O}_{\mathcal{X},x}$  be a local equation of  $\text{div}_{\mathcal{X}^+}(\omega_{\mathcal{X}_K})$  around  $x$ . In order to evaluate  $v_{x,\alpha}$  on  $f_x$ , we consider an admissible expansion of  $f_x$  as in (3.2.6)

$$f_x = \sum_{\gamma \in \mathcal{C}_{\mathcal{X},x}} c_\gamma \gamma.$$

Furthermore, this expansion induces also an expansion of  $\text{pr}_{\mathcal{X}}^*(f_x)$  by

$$\text{pr}_{\mathcal{X}}^*(f_x) = \sum_{\gamma \in \mathcal{C}_{\mathcal{X},x}} \text{pr}_{\mathcal{X}}^*(c_\gamma) \gamma$$

as formal power series in  $\widehat{\mathcal{O}}_{\mathcal{X},z}$ , since the morphism of characteristic sheaves  $\mathcal{C}_{\mathcal{X},x} \hookrightarrow \mathcal{C}_{\mathcal{X},z}$  is injective. Following the same procedure for a local equation  $f_y \in \mathcal{O}_{\mathcal{Y},y}$  of  $\text{div}_{\mathcal{Y}^+}(\omega_{\mathcal{Y}_K})$  around  $y$ , we get an expansion of  $f_y$  that extends to  $\text{pr}_{\mathcal{Y}}^*(f_y)$ :

$$f_y = \sum_{\delta \in \mathcal{C}_{\mathcal{Y},y}} d_\delta \delta.$$

(4.2.4) A local equation of  $\varpi$  around  $z$  is determined by  $\text{pr}_{\mathcal{X}}^*(f_x) \text{pr}_{\mathcal{Y}}^*(f_y)$ . Thus

$$v_\varepsilon(\text{div}_{\mathcal{Z}^+}(\varpi)) = v_\varepsilon(\text{pr}_{\mathcal{X}}^*(f_x) \text{pr}_{\mathcal{Y}}^*(f_y))$$

and by multiplicativity of the valuation  $v_\varepsilon$

$$v_\varepsilon(\text{pr}_{\mathcal{X}}^*(f_x) \text{pr}_{\mathcal{Y}}^*(f_y)) = v_\varepsilon(\text{pr}_{\mathcal{X}}^*(f_x)) + v_\varepsilon(\text{pr}_{\mathcal{Y}}^*(f_y)).$$

Recalling Remark 3.2.14, the valuation can be computed as follows

$$v_\varepsilon(\text{pr}_{\mathcal{X}}^*(f_x)) = \min\{\varepsilon(\gamma) \mid c_\gamma \neq 0\};$$

as the elements  $\gamma$  belong to  $\mathcal{C}_{\mathcal{X},x}$  and  $\alpha$  is defined to be  $\text{pr}_{\mathcal{X}}(\varepsilon)$

$$\min\{\varepsilon(\gamma) \mid c_\gamma \neq 0\} = \min\{\alpha(\gamma) \mid c_\gamma \neq 0\} = v_{x,\alpha}(f_x).$$

Hence, we conclude that

$$\begin{aligned} v_\varepsilon(\text{div}_{\mathcal{Z}^+}(\varpi)) &= v_\varepsilon(\text{pr}_{\mathcal{X}}^*(f_x)) + v_\varepsilon(\text{pr}_{\mathcal{Y}}^*(f_y)) \\ &= v_\alpha(f_x) + v_\beta(f_y) \\ &= v_\alpha(\text{div}_{\mathcal{X}^+}(\omega_{\mathcal{X}_K})) + v_\beta(\text{div}_{\mathcal{Y}^+}(\omega_{\mathcal{Y}_K})). \end{aligned}$$

(4.2.5) This result turns out to be advantageous to compute the weight function  $\text{wt}_\varpi$  on divisorial points of  $\text{Sk}(\mathcal{Z}^+)$ :

$$\begin{aligned} \text{wt}_\varpi(\varepsilon) &= v_\varepsilon(\text{div}_{\mathcal{Z}^+}(\varpi)) + 1 \\ (4.2.6) \quad &= v_\alpha(\text{div}_{\mathcal{X}^+}(\omega_{\mathcal{X}_K})) + v_\beta(\text{div}_{\mathcal{Y}^+}(\omega_{\mathcal{Y}_K})) + 1 \\ &= \text{wt}_{\omega_{\mathcal{X}_K}}(\alpha) + \text{wt}_{\omega_{\mathcal{Y}_K}}(\beta) - 1. \end{aligned}$$

### 4.3. Kontsevich-Soibelman skeleta of $fs$ fibred products.

(4.3.1) In [MN15], Section 4.5 the Kontsevich-Soibelman skeleton  $\text{Sk}(\mathcal{Z}_K, \varpi)$  is defined as the closure of the set of divisorial points of  $\mathcal{Z}_K^{\text{an}}$  where the weight function  $\text{wt}_\varpi$  reaches its minimal value  $\text{wt}_\varpi(\mathcal{Z}_K)$ . Our computations lead to the following result.

**Theorem 4.3.2.** *Assume that the residue field  $k$  is algebraically closed. Let  $\mathcal{X}^+$  and  $\mathcal{Y}^+$  be log-smooth log schemes over  $S^+$ , let  $\mathcal{Z}^+$  be their  $fs$  fibred product. Suppose that  $\mathcal{X}^+$  is semistable. Then, given  $\omega_{\mathcal{X}_K}$  and  $\omega_{\mathcal{Y}_K}$  canonical forms on  $\mathcal{X}_K$  and  $\mathcal{Y}_K$  respectively, the homeomorphism of skeleta*

$$\text{Sk}(\mathcal{Z}^+) \xrightarrow{\sim} \text{Sk}(\mathcal{X}^+) \times \text{Sk}(\mathcal{Y}^+)$$

in Proposition 3.4.3 restricts to a homeomorphism of Kontsevich-Soibelman skeleta

$$\mathrm{Sk}(\mathcal{Z}_K, \varpi) \xrightarrow{\sim} \mathrm{Sk}(\mathcal{X}_K, \omega_{\mathcal{X}_K}) \times \mathrm{Sk}(\mathcal{Y}_K, \omega_{\mathcal{Y}_K}).$$

*Proof.* This follows immediately from the equality (4.2.6) that shows that a point in  $\mathrm{Sk}(\mathcal{Z}^+)$  has minimal value  $\mathrm{wt}_{\varpi}(\mathcal{Z}_K)$  if and only if its projections have minimal value  $\mathrm{wt}_{\omega_{\mathcal{X}_K}}(\mathcal{X}_K)$  and  $\mathrm{wt}_{\omega_{\mathcal{Y}_K}}(\mathcal{Y}_K)$ .  $\square$

#### 4.4. Essential skeleta of products of CY varieties.

(4.4.1) Let  $X$  be a smooth  $K$ -variety with trivial canonical sheaf, i.e. a Calabi-Yau variety. The essential skeleton of  $X$  is defined in [MN15], Definition 4.6.2 as

$$\mathrm{Sk}(X) := \bigcup_{\omega} \mathrm{Sk}(X, \omega) \subseteq X^{\mathrm{an}}$$

where  $\omega$  runs through the set of non-zero regular pluricanonical forms on  $X$ . Under the assumption of triviality of the canonical bundle, the essential skeleton of a Calabi-Yau variety is reduced to the Kontsevich-Soibelman skeleton of any non-zero canonical form.

(4.4.2) We characterize the essential skeleton of a fibred product of Calabi-Yau varieties with semistable reduction.

**Corollary 4.4.3.** *Assume that the residue field  $k$  is algebraically closed. Let  $X$  and  $Y$  be smooth Calabi-Yau varieties over  $K$  and denote by  $Z$  the fibred product  $X \times_K Y$ . Suppose that  $X$  has semistable reduction. Given  $\mathcal{X}^+$  a semistable log-smooth model of  $X$  and  $\mathcal{Y}^+$  a log-smooth model of  $Y$ , we denote by  $\mathcal{Z}^+$  their fs fibred product. Then the homeomorphism of skeleta*

$$\mathrm{Sk}(\mathcal{Z}^+) \xrightarrow{\sim} \mathrm{Sk}(\mathcal{X}^+) \times \mathrm{Sk}(\mathcal{Y}^+)$$

in Proposition 3.4.3 restricts to a homeomorphism of essential skeleta

$$\mathrm{Sk}(Z) \xrightarrow{\sim} \mathrm{Sk}(X) \times \mathrm{Sk}(Y).$$

*Proof.* This follows immediately from Theorem 4.3.2.  $\square$

### 5. THE ANALYTIFICATION OF THE $n$ -TH SYMMETRIC PRODUCT.

#### 5.1. Analytification of the quotient.

(5.1.1) Let  $X$  be a smooth  $K$ -variety and let  $S_n$  be  $n$ -th symmetric group. We describe an action of  $S_n$  on the  $n$ -fold fibred product  $X^n$ . Any point  $x$  of  $X^n$  is characterized as a tuple  $x = (x_1, \dots, x_n, s, \mathfrak{p})$  where  $s$  is the image of any of  $x_i$ 's and  $\mathfrak{p}$  is a prime ideal of the tensor algebra of residue fields  $\kappa(x_1) \otimes \dots \otimes \kappa(x_n)$ . Given a permutation  $\sigma$  of  $n$  elements, it induces an isomorphism of tensor algebras  $k(x_1) \otimes \dots \otimes k(x_n) \simeq k(x_{\sigma(1)}) \otimes \dots \otimes k(x_{\sigma(n)})$  still denoted by  $\sigma$ . Then the action of  $S_n$  on  $X^n$  is given by

$$\sigma \cdot x = (x_{\sigma(1)}, \dots, x_{\sigma(n)}, s, \sigma(\mathfrak{p})).$$

Let  $(X^n)^{\mathrm{an}}$  be the analytification of  $X^n$ . We recall that any point of  $(X^n)^{\mathrm{an}}$  is a pair  $(x, |\cdot|_x)$  with  $x \in X^n$  and  $|\cdot|_x$  an absolute value on the residue field  $\kappa(x)$  extending the absolute value on  $K$ . The action of  $S_n$  extends to  $(X^n)^{\mathrm{an}}$  by

$$\sigma \cdot (x, |\cdot|_x) = (x_{\sigma(1)}, \dots, x_{\sigma(n)}, s, \sigma(\mathfrak{p}), |\cdot|_x \circ \sigma^{-1}).$$

(5.1.2) By functoriality of the Berkovich analytification, the morphism of schemes  $f : X^n \rightarrow Y = X^n/S_n$  to the  $n$ -th symmetric product of  $X$  induces a surjective morphism of Berkovich spaces  $f^{\text{an}} : (X^n)^{\text{an}} \rightarrow Y^{\text{an}}$  and the image of a point  $(x, |\cdot|_x)$  is

$$f^{\text{an}}(x, |\cdot|_x) = ([x], |\cdot|_{[x]}),$$

with  $[x] = ([x_i], s, [\mathbf{p}]) \in X^n/S_n$  and  $|\cdot|_{[x]}$  an absolute value on  $k(x)^{S_n}$ , the field of  $S_n$ -invariant elements of  $k(x)$ . Two points  $(x, |\cdot|_x)$  and  $(x', |\cdot|_{x'})$  have the same image if and only if there exists  $\sigma \in S_n$  such that  $\sigma \cdot x = x'$  and  $|\cdot|_x \circ \sigma^{-1} = |\cdot|_{x'}$ .

This implies that  $f^{\text{an}}$  factors uniquely through the quotient map  $\pi : (X^n)^{\text{an}} \rightarrow (X^n)^{\text{an}}/S_n$ . So we may draw the diagram below

$$\begin{array}{ccc} (X^n)^{\text{an}} & \xrightarrow{f^{\text{an}}} & Y^{\text{an}} \\ & \searrow \pi & \uparrow \sim \\ & & (X^n)^{\text{an}}/S_n \xrightarrow{\sim} (X^{\text{an}})^n/S_n \end{array}$$

reminding that the analytification functor commutes with fibred products ([Ber93], Proposition 2.6.1), hence  $(X^n)^{\text{an}} \simeq (X^{\text{an}})^n$ .

## 5.2. Representation of divisorial points of the quotient.

(5.2.1) We keep the notation of the previous paragraph. Let  $y$  be a divisorial point of  $Y^{\text{an}}$  and consider a regular snc  $R$ -model  $\mathcal{Y}$  of  $Y$  adapted to  $y$ , i.e. such that  $y$  is the divisorial point associated to  $(\mathcal{Y}, E)$  for some irreducible component  $E$  of  $\mathcal{Y}_k$ . We denote by  $\mathcal{X}$  the normalization of  $\mathcal{Y}$  inside  $K(X^n)$ , where  $K(\mathcal{Y}) = K(Y) = K(X^n)^{S_n} \hookrightarrow K(X^n)$ .

(5.2.2) We check that  $\mathcal{X}$  is an  $R$ -model of  $X^n$ ; it is enough to show that the base change  $\mathcal{X}_K$  is isomorphic to  $X^n$ . We consider the following commutative diagram

$$\begin{array}{ccccc} & & \mathcal{X}_K & & \\ & & \swarrow & & \searrow \\ & X^n & \xrightarrow{\quad} & \mathcal{X} & \\ & \downarrow f & & \downarrow & \\ Y & \longrightarrow & \mathcal{Y} & & \\ \downarrow & & \downarrow & & \\ \text{Spec } K & \longrightarrow & S & & \end{array}$$

As the  $n$ -fold product  $X^n$  is a normal variety endowed with a morphism  $X^n \rightarrow \mathcal{Y}$ , by universal property of normalization, it factors uniquely through  $\mathcal{X}$  and the diagram is still commutative. Then by universal property of fibred product, there exists a morphism  $X^n \rightarrow \mathcal{X}_K$ . Therefore, it suffices to prove that

$$[K(X^n) : K(\mathcal{X}_K)] = 1.$$

Indeed, if this is the case, then  $X^n \rightarrow \mathcal{X}_K$  is a finite birational morphism between normal varieties, hence an isomorphism.

(5.2.3) The degree of the extension  $[K(X^n) : K(\mathcal{X}_K)]$  may be computed on an affine open, so we assume that  $\mathcal{Y}$  is an affine scheme with associated ring  $K[\mathcal{Y}]$ . Then we consider the diagram of inclusions

$$\begin{array}{ccccc}
& & K(\mathcal{X}) = K(\mathcal{X}_K) & & \\
& \nearrow & & \searrow & \\
\widehat{K[\mathcal{Y}]} = K[\mathcal{X}] & \xrightarrow{\quad\quad\quad} & K(X^n) & & \\
\uparrow & & \uparrow \text{finite deg} & & \\
K[\mathcal{Y}] & \xrightarrow{\quad\quad\quad} & K(\mathcal{Y}) = K(Y) = K(X^n)^{S_n} & & 
\end{array}$$

As  $K(X^n)$  is finite field extension of  $K(\mathcal{Y})$  and  $K[\mathcal{X}]$  the integral closure of  $K[\mathcal{Y}]$  in  $K(X^n)$ , then  $K(X^n)$  is the fraction field of  $K[\mathcal{X}]$ . Thus,  $K(\mathcal{X}) = \text{Frac}(K[\mathcal{X}]) = K(X^n)$  and in particular we conclude that  $[K(X^n) : K(\mathcal{X}_K)] = 1$ .

**Remark 5.2.4.** This procedure of normalization illustrates a way to start with a regular snc  $R$ -model  $\mathcal{Y}$  of  $Y$  adapted to a point  $y \in \text{Div}(Y)$  and construct an  $R$ -model  $\mathcal{X}$  of  $X^n$  that, by normality, is regular at generic points of the special fibre  $\mathcal{X}_k$ .

### 5.3. Weight function values along fibres of the quotient.

**(5.3.1)** As before, let  $y \in \text{Div}(Y)$  and let  $\mathcal{Y}$  be a regular snc  $R$ -model with divisorial representation  $(\mathcal{Y}, E)$  of  $y$ . Let  $\mathcal{X}$  be the normalization of  $\mathcal{Y}$  in  $K(X^n)$ : as we observed in Remark 5.2.4, it is an  $R$ -model of  $X^n$ , regular at generic points of the special fibre  $\mathcal{X}_k$ .

The preimage of  $E$  coincides with the pull-back of the Cartier divisor  $E$  on  $\mathcal{X}$ , hence  $f^{-1}(E)$  still defines a codimension one subset on  $\mathcal{X}$ . We denote by  $F_i$  the irreducible components of  $f^{-1}(E)$  and we associate to  $F_i$ 's their corresponding divisorial valuations  $x_i = (\mathcal{X}, F_i)$ .

**(5.3.2)** Let  $\omega_X$  be a canonical form on  $X$  and let  $\text{pr}_j : X^n \rightarrow X$  be the  $j$ -th canonical projection. We consider

$$\omega = \bigwedge_{1 \leq j \leq n} \text{pr}_j^* \omega_X.$$

It is a canonical form on  $X^n$  and moreover it is invariant under the action of  $S_n$ . Thus,  $\omega$  induces a canonical form on the  $n$ -th symmetric product  $Y$ .

We compare the values at  $y$  and  $x_i$  of weight functions attached to  $\omega$ :

$$\text{wt}_\omega(x_i) = v_{x_i}(\text{div}_{\mathcal{X}^+}(\omega)) + 1$$

$$\text{wt}_\omega(y) = v_y(\text{div}_{\mathcal{Y}^+}(\omega)) + 1.$$

We recall that for log étale morphisms the sheaves of logarithmic differentials are isomorphic ([Kat94], Proposition 3.12). Furthermore, it suffices to check that, locally around the generic point of  $F_i$ , the morphism  $\mathcal{X}^+ \rightarrow \mathcal{Y}^+$  is a log étale morphism of divisorial log structures, to conclude that the weight function values coincide. To this purpose, we will apply Kato's criterion for log étaleness ([Kat89], Theorem 3.5) to log schemes with respect to the étale topology.

**(5.3.3)** We denote by  $\xi_{F_i}$  the generic point of  $F_i$  and by  $\xi_E$  the generic point of  $E$ . The divisorial log structures on  $\mathcal{X}^+$  and  $\mathcal{Y}^+$  have charts  $\mathbb{N}$  at  $\xi_{F_i}$  and  $\xi_E$ . In the étale topology, the normalization morphism  $\mathcal{X}^+ \rightarrow \mathcal{Y}^+$  admits a chart induced by  $t : \mathbb{N} \rightarrow \mathbb{N}$  where  $1 \mapsto m$  for some positive integer  $m$ :

$$\begin{array}{ccc}
\mathrm{Spec} \mathcal{O}_{\mathcal{X}, \xi_{F_i}} & \longrightarrow & \mathrm{Spec} \mathbb{Z}[\mathbb{N}] \\
\downarrow & & \downarrow \\
\mathrm{Spec} \mathcal{O}_{\mathcal{Y}, \xi_E} & \longrightarrow & \mathrm{Spec} \mathbb{Z}[\mathbb{N}]
\end{array}$$

Firstly, by the universal property of the fibre product, we have a morphism

$$\mathrm{Spec} \mathcal{O}_{\mathcal{X}, \xi_{F_i}} \rightarrow \mathrm{Spec} \mathcal{O}_{\mathcal{Y}, \xi_E} \times_{\mathrm{Spec} \mathbb{Z}[\mathbb{N}]} \mathrm{Spec} \mathbb{Z}[\mathbb{N}]$$

and it corresponds to

$$\mathcal{O}_{\mathcal{Y}, \xi_E} \otimes_{\mathbb{Z}[\mathbb{N}]} \mathbb{Z}[\mathbb{N}] \rightarrow \mathcal{O}_{\mathcal{X}, \xi_{F_i}}.$$

This is a morphism of finite type with finite fibres between regular rings and by [Liu02], Lemma 4.3.20 and [Now97] it is flat and unramified, hence étale. One of the two conditions in Kato's criterion for log étaleness is then fulfilled. Secondly, the chart  $t : \mathbb{N} \mapsto \mathbb{N}$  induces a group homomorphism  $t^{\mathrm{gp}} : \mathbb{Z} \mapsto \mathbb{Z}$ ; in particular, it is injective and it has finite cokernel. Then  $t$  satisfies the second condition of Kato's criterion for log étaleness. Therefore we conclude that  $\mathrm{wt}_{\omega}(y) = \mathrm{wt}_{\omega}(x_i)$ .

**Remark 5.3.4.** Given a divisorial point  $y$ , this construction provides a divisorial point  $x \in (X^n)^{\mathrm{an}}$  such that  $f^{\mathrm{an}}(x) = y$  and with the property that they have the same weight function value with respect to an  $S_n$ -invariant canonical form  $\omega$ .

#### 5.4. Kontsevich-Soibelman skeleta of the quotient.

**(5.4.1)** Let  $\omega_X$  be a canonical form on  $X$  and  $\omega = \bigwedge_{1 \leq j \leq n} \mathrm{pr}_j^* \omega_X$  the induced  $S_n$ -invariant canonical form on  $X^n$  that passes to the quotient  $Y$ . We claim that the minimal values of the weight functions  $\mathrm{wt}_{\omega}$  on  $X^n$  and  $Y$  coincide, i.e.  $\mathrm{wt}_{\omega}(X^n) = \mathrm{wt}_{\omega}(Y)$  according to the notation of [MN15]. The key arguments to prove the claim are:

- (1) the Kontsevich-Soibelman skeleton  $\mathrm{Sk}(X^n, \omega)$  of the  $n$ -fold fibred product is invariant under the  $S_n$ -action, as a consequence of Theorem 4.3.2;
- (2) given a divisorial point  $y$ , we may construct a divisorial point  $x \in (X^n)^{\mathrm{an}}$  such that  $f^{\mathrm{an}}(x) = y$  and  $\mathrm{wt}_{\omega}(y) = \mathrm{wt}_{\omega}(x)$ , as showed in Remark 5.3.4.

**(5.4.2)** By the argument in (2), the following inclusion is true

$$\{\mathrm{wt}_{\omega}(y) \mid y \in \mathrm{Div}(Y)\} \subseteq \{\mathrm{wt}_{\omega}(x) \mid x \in \mathrm{Div}(X^n)\},$$

so  $\mathrm{wt}_{\omega}(X^n) = \inf\{\mathrm{wt}_{\omega}(x) \mid x \in \mathrm{Div}(X^n)\} \leq \inf\{\mathrm{wt}_{\omega}(y) \mid y \in \mathrm{Div}(Y)\} = \mathrm{wt}_{\omega}(Y)$ .

Conversely, let  $x \in \mathrm{Div}(X^n)$  such that  $\mathrm{wt}_{\omega}(x) = \mathrm{wt}_{\omega}(X^n)$ . Consider  $y := f^{\mathrm{an}}(x)$ ; it is a divisorial valuation since it is induced by restriction of  $v_x$  to  $K(Y) \hookrightarrow K(X^n) \xrightarrow{v_x} \mathbb{R}$  and its image is contained in the discrete image of  $v_x$  in  $\mathbb{R}$ , hence it is discrete too. Applying the construction of (2), we obtain  $x' \in \mathrm{Div}(X^n)$  such that  $f^{\mathrm{an}}(x') = y$  and  $\mathrm{wt}_{\omega}(x') = \mathrm{wt}_{\omega}(y)$ . This means that  $x$  and  $x'$  are in the same  $S_n$ -class; since the  $S_n$ -action preserves the Kontsevich-Soibelman skeleton  $\mathrm{Sk}(X^n, \omega)$  by (1) and  $x \in \mathrm{Sk}(X^n, \omega)$ , thus  $x' \in \mathrm{Sk}(X^n, \omega)$ . Therefore

$$\mathrm{wt}_{\omega}(y) = \mathrm{wt}_{\omega}(x') = \mathrm{wt}_{\omega}(X^n),$$

so  $\mathrm{wt}_{\omega}(Y) = \inf\{\mathrm{wt}_{\omega}(y) \mid y \in \mathrm{Div}(Y)\} \leq \mathrm{wt}_{\omega}(X^n)$ . Finally, we have equality of weights of  $X^n$  and  $Y$  with respect to  $\omega$ :

$$(5.4.3) \quad \mathrm{wt}_{\omega}(X^n) = \mathrm{wt}_{\omega}(Y).$$

**(5.4.4)** The equality of minimal weights leads to the main results of this paragraph.

**Proposition 5.4.5.** *Let  $X$  be a smooth  $K$ -variety and let  $Y = X^n/S_n$  be the  $n$ -th symmetric product of  $X$  with  $f : X^n \rightarrow Y$ . Let  $\omega_X$  be a canonical form on  $X$  and  $\omega = \bigwedge_{1 \leq j \leq n} \text{pr}_j^* \omega_X$  the induced canonical form on  $X^n$  and  $Y$ . Then the Kontsevich-Soibelman skeleton  $\text{Sk}(Y, \omega)$  is the image under  $f^{\text{an}}$  of the Kontsevich-Soibelman skeleton  $\text{Sk}(X^n, \omega)$ .*

*Proof.* We characterize divisorial points in the Kontsevich-Soibelman skeleton  $\text{Sk}(Y, \omega)$  in term of their preimages in  $X^{\text{an}}$  as follows: given  $y \in \text{Div}(Y)$ ,

$$y \in \text{Sk}(Y, \omega) \Leftrightarrow \text{for some/any } x \in (f^{\text{an}})^{-1}(y) \quad \text{wt}_\omega(x) = \text{wt}_\omega(X^n).$$

Indeed, if  $y \in \text{Sk}(Y, \omega)$ , we may construct  $x$  such that  $f^{\text{an}}(x) = y$  and  $\text{wt}_\omega(x) = \text{wt}_\omega(y)$ , i.e. a divisorial point  $x \in (f^{\text{an}})^{-1}(y)$  such that  $\text{wt}_\omega(x) = \text{wt}_\omega(y) = \text{wt}_\omega(Y) = \text{wt}_\omega(X^n)$  by the equality (5.4.3). By argument (1), this holds for any point in the preimage of  $y$ .

Conversely, suppose that for all  $x \in (f^{\text{an}})^{-1}(y)$  we have  $\text{wt}_\omega(x) = \text{wt}_\omega(X^n)$ . Then, this holds in particular for a divisorial point  $\tilde{x}$  constructed as in (2). Therefore  $\text{wt}_\omega(y) = \text{wt}_\omega(\tilde{x}) = \text{wt}_\omega(X^n) = \text{wt}_\omega(Y)$  again by equation (5.4.3); this means that  $y \in \text{Sk}(Y, \omega)$ .  $\square$

**Corollary 5.4.6.** *Assume that the residue field  $k$  is algebraically closed. Let  $X$  be a smooth  $K$ -variety and let  $\omega_X$  be a canonical form on  $X$ . If  $X$  has semistable reduction, then the Kontsevich-Soibelman skeleton of the  $n$ -th symmetric product of  $X$  is isomorphic to the  $n$ -th symmetric product of the Kontsevich-Soibelman skeleton of  $X$*

$$\text{Sk}(X^n/S_n, \omega) \xrightarrow{\sim} S^n(\text{Sk}(X, \omega_X)).$$

*Proof.* Iterating the result of Theorem 4.3.2, we have that the projection map defines an isomorphism of Kontsevich-Soibelman skeleta

$$\text{Sk}(X^n, \omega) \xrightarrow{\sim} \text{Sk}(X, \omega_X) \times \dots \times \text{Sk}(X, \omega_X).$$

Hence, by Proposition 5.4.5, the diagram

$$\begin{array}{ccc} (X^n)^{\text{an}} & \xrightarrow{f^{\text{an}}} & Y^{\text{an}} \simeq (X^{\text{an}})^n/S_n \\ \text{Sk}(X^n, \omega) & \xrightarrow{\quad \quad \quad} & \text{Sk}(Y, \omega) \\ \parallel & & \parallel \\ \text{Sk}(X, \omega_X) \times \dots \times \text{Sk}(X, \omega_X) & \xrightarrow{\quad \quad \quad} & S^n(\text{Sk}(X, \omega_X)) \end{array}$$

gives a concrete description of the Kontsevich-Soibelman skeleta of the quotient  $Y$  in terms of the Kontsevich-Soibelman skeleton of  $X$  as required.  $\square$

### 5.5. Essential skeleton of the $n$ -th symmetric product of a CY variety.

(5.5.1) These results on Kontsevich-Soibelman skeleta of the  $n$ -th symmetric products translate into properties of essential skeleta when we are dealing with Calabi-Yau varieties.

**Corollary 5.5.2.** *Let  $X$  be a smooth Calabi-Yau variety over  $K$ . Assume that  $X$  has semistable reduction and the residue field  $k$  is algebraically closed. Then the essential skeleton of the  $n$ -th symmetric product of  $X$  is isomorphic to the  $n$ -th symmetric product of the essential skeleton of  $X$*

$$\text{Sk}(X^n/S_n) \xrightarrow{\sim} S^n(\text{Sk}(X)).$$

*Proof.* This follows immediately from Corollary 5.4.6.  $\square$

## 6. THE ESSENTIAL SKELETON OF THE HILBERT SCHEME OF A K3 SURFACE

**(6.0.3)** Let  $S$  be a K3 surface over  $K$  (i.e.  $S$  is a complete non-singular variety of dimension two such that  $\Omega_{S/K}^2 \simeq \mathcal{O}_S$  and  $H^1(S, \mathcal{O}_S) = 0$ ). In particular  $S$  is a Calabi-Yau variety.

We consider  $\text{Hilb}^n(S)$  the Hilbert scheme of  $n$  points on  $S$ ; a concrete way to construct it is by first taking the  $n$ -th symmetric product of  $S$ , and by then resolving its singularities:

$$\begin{array}{ccc} & S^n & \\ & \downarrow f & \\ \text{Hilb}^n(S) & \xrightarrow{\rho} & S^n/S_n \end{array}$$

Indeed, the  $n$ -th symmetric product  $S^n/S_n$  has quotient singularities along the images via  $f$  of the loci

$$\Delta_{ij} = \{(x_1, \dots, x_n) \in S^n \mid x_i = x_j\},$$

which are precisely the fixed loci of the  $S_n$ -action on  $S^n$ . Then the morphism  $\rho : \text{Hilb}^n(S) \rightarrow S^n/S_n$  is a resolution of singularities and it can be seen explicitly as the map sending a zero-dimensional scheme  $Z \subseteq S$  to its associated zero-cycle  $\text{supp}(Z)$ . We refer to the morphism  $\rho$  as the Hilbert-Chow morphism. It follows that the Hilbert scheme of  $n$  points on  $S$  is birational to the  $n$ -th symmetric product of  $S$  [Fog68].

**(6.0.4)** We can finally illustrate the essential skeleton of the  $n$ -th Hilbert scheme of a K3 surface with semistable reduction.

**Theorem 6.0.5.** *Let  $S$  be a K3 surface over  $K$ . Assume that  $S$  has semistable reduction and the residue field  $k$  is algebraically closed. Then the essential skeleton of the Hilbert scheme of  $n$  points on  $S$  is isomorphic to the  $n$ -th symmetric product of the essential skeleton of  $S$*

$$\text{Sk}(\text{Hilb}^n(S)) \xrightarrow{\sim} S^n(\text{Sk}(S)).$$

*Proof.* In [MN15], Proposition 4.6.3, Mustata and Nicaise proved that the essential skeleton of a variety is a birational invariant. Therefore, the description of the essential skeleton of the  $n$ -th symmetric product of  $S$  in Corollary 5.5.2 entails a description of the essential skeleton of the Hilbert scheme of  $n$  points on  $S$ .  $\square$

## REFERENCES

- [ACMUW15] D. Abramovich, Q. Chen, S. Marcus, M. Ulirsch, and J. Wise. Skeletons and fans of logarithmic structures. In: *Nonarchimedean and Tropical Geometry*. Ed. by M. Baker and S. Payne. Springer International Publishing, 2015.
- [ACP15] D. Abramovich, L. Caporaso, and S. Payne. The tropicalization of the moduli space of curves. *Ann. Sci. Éc. Norm. Supér. (4)* 48.4 (2015), 765–809.
- [BR15] M. Baker and J. Rabinoff. The skeleton of the Jacobian, the Jacobian of the skeleton, and lifting meromorphic functions from tropical to algebraic curves. *Int. Math. Res. Not. IMRN* 16 (2015), 7436–7472.

- [Ber90] V. G. Berkovich. *Spectral theory and analytic geometry over non-Archimedean fields*. Vol. 33. American Mathematical Society, Providence, RI, 1990.
- [Ber93] V. G. Berkovich. Etale cohomology for non-Archimedean analytic spaces. eng. *Publications Mathématiques de l’IHS* 78 (1993), 5–161.
- [Bul15] E. Bultot. Motivic Integration and Logarithmic Geometry. *ArXiv e-prints* (May 2015). arXiv: 1505.05688 [math.AG].
- [dKX12] T. de Fernex, J. Kollár, and C. Xu. The dual complex of singularities. *ArXiv e-prints* (Dec. 2012). arXiv: 1212.1675 [math.AG].
- [Fog68] J. Fogarty. Algebraic Families on an Algebraic Surface. *American Journal of Mathematics* 90.2 (1968), 511–521.
- [GR04] O. Gabber and L. Ramero. Foundations for almost ring theory – Release 6.95. *ArXiv Mathematics e-prints* (Sept. 2004). eprint: math/0409584.
- [GHH16] M. G. Gulbrandsen, L. H. Halle, and K. Hulek. A GIT construction of degenerations of Hilbert schemes of points. *ArXiv e-prints* (Apr. 2016). arXiv: 1604.00215 [math.AG].
- [GHH15] M. G. Gulbrandsen, L. H. Halle, and K. Hulek. A relative Hilbert-Mumford criterion. *Manuscripta Math.* 148.3-4 (2015), 283–301.
- [Kat89] K. Kato. Logarithmic structures of Fontaine-Illusie. In: *Algebraic analysis, geometry, and number theory (Baltimore, MD, 1988)*. Johns Hopkins Univ. Press, Baltimore, MD, 1989, 191–224.
- [Kat94] K. Kato. Toric singularities. *Amer. J. Math.* 116.5 (1994), 1073–1099.
- [KKMSD73] G. Kempf, F. F. Knudsen, D. Mumford, and B. Saint-Donat. *Toroidal embeddings. I*. Springer-Verlag, Berlin-New York, 1973.
- [Li01] J. Li. Stable Morphisms to Singular Schemes and Relative Stable Morphisms. *J. Differential Geom.* 57.3 (Mar. 2001), 509–578.
- [Liu02] Q. Liu. *Algebraic geometry and arithmetic curves*. Vol. 6. Translated from the French by Reinie Ern , Oxford Science Publications. Oxford University Press, Oxford, 2002.
- [MN15] M. Mustata and J. Nicaise. Weight functions on non-Archimedean analytic spaces and the Kontsevich-Soibelman skeleton. *Algebr. Geom.* 2.3 (2015), 365–404.
- [NX16] J. Nicaise and C. Xu. The essential skeleton of a degeneration of algebraic varieties. *Amer. J. Math.* 138.6 (2016), 1645–1667.
- [Niz06] W. a. Niziol. Toric singularities: log-blow-ups and global resolutions. *J. Algebraic Geom.* 15.1 (2006), 1–29.
- [Now97] K. J. Nowak. Flat morphisms between regular varieties. *Univ. Iagel. Acta Math.* 35 (1997), 243–246.
- [Sai04] T. Saito. Log smooth extension of a family of curves and semi-stable reduction. *J. Algebraic Geom.* 13.2 (2004), 287–321.
- [Uli13] M. Ulirsch. Functorial tropicalization of logarithmic schemes: The case of constant coefficients. *ArXiv e-prints* (Oct. 2013). arXiv: 1310.6269 [math.AG].
- [Uli16] M. Ulirsch. Non-Archimedean geometry of Artin fans. *ArXiv e-prints* (Mar. 2016). arXiv: 1603.07589 [math.AG].
- [Vid04] I. Vidal. Monodromie locale et fonctions zeta des log sch mas. In: *Geometric aspects of Dwork theory. Vol. I, II*. 2004.