TITLE

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1. Introduction

1.1. Notation.

(1.1.1) Let R be a complete discrete valuation ring with maximal ideal $\mathfrak{m}$, residue field $k = R/\mathfrak{m}$ and quotient field K. We assume that the valuation v_K is normalized, namely $v_K(\pi) = 1$ for any uniformizer π of R. We define by $|\cdot|_K = \exp(-v_K(\cdot))$ the absolute value on K corresponding to v_K; this turns K into a non-archimedean complete valued field.

(1.1.2) We write $S = \operatorname{Spec} R$ and we denote by s the closed point of S. Let $\mathscr{X}$ be an R-scheme of finite type. We will denote by $\mathscr{X}_k$ the special fiber of $\mathscr{X}$ and by $\mathscr{X}_K$ the generic fibre. Moreover, we will denote by $\widehat{\mathscr{X}}$ the $\mathfrak{m}$-adic completion of $\mathscr{X}$ and by $\widehat{\mathscr{X}_\eta}$ the generic fiber of $\widehat{\mathscr{X}}$ in the category of K-analytic spaces.

(1.1.3) Let X be a proper K-scheme. A model for X over R is a flat separated R-scheme $\mathscr{X}$ of finite type endowed with an isomorphism of K-schemes $\mathscr{X}_K \to X$. If X is smooth over K, we say that $\mathscr{X}$ is an snc model for X if it is regular over R, and the special fiber $\mathscr{X}_k$ is a strict normal crossings divisor on $\mathscr{X}$. In equicharacteristic 0, such a model always exists, by Hironaka's resolution of singularities.

(1.1.4) All log schemes in this paper are fine and saturated (*fs*) log schemes and defined with respect to the Zariski topology. We denote a log scheme by $\mathscr{X}^+ = (\mathscr{X}, \mathcal{M}_{\mathscr{X}})$, where $\mathcal{M}_{\mathscr{X}}$ is the structural sheaf of monoids. We denote by

$$\mathcal{C}_{\mathscr{X}} = \mathcal{M}_{\mathscr{X}} / \mathcal{O}_{\mathscr{X}}^{\times}$$

the characteristic sheaf of $\mathscr{X}^+$. The sheaf $\mathcal{C}_{\mathscr{X}}$ is a Zariski sheaf on $\mathscr{X}^+$, supported on $\mathscr{X}_k$; if $\mathscr{X}^+$ is log-regular, then $\mathcal{C}_{\mathscr{X}}$ is a constructible sheaf. For every point x of $\mathscr{X}_k$, we denote by $\mathcal{I}_{\mathscr{X},x}$ the ideal in $\mathcal{O}_{\mathscr{X},x}$ generated by

$$\mathcal{M}_{\mathscr{X},x} \setminus \mathcal{O}_{\mathscr{X},x}^{\times}.$$

We denote by S^+ the scheme S endowed with the standard log structure (the divisorial log structure induced by s). If an R-scheme $\mathscr{X}$ is given, we will always denote by $\mathscr{X}^+$ the log scheme over S^+ that we obtain by endowing $\mathscr{X}$ with the divisorial log structure associated with $\mathscr{X}_k$.

If $\mathscr{X}^+$ is a log-regular log scheme over S^+, then the locus where the log structure is non-trivial is a divisor that we will denote by $D_{\mathscr{X}}$. Thus, the log structure on $\mathscr{X}^+$ is the divisorial log structure induced by $D_{\mathscr{X}}$, by [Kat94], Theorem 11.6.

(1.1.5) Let (X, Δ_X) be a pair where X is a proper K-scheme, Δ_X is an effective $\mathbb{Q}$-divisor such that $\Delta_X = \sum a_i \Delta_{X,i}$ with $0 \leqslant a_i \leqslant 1$, and $X^+ = (X, \lceil \Delta_X \rceil)$ is a log-regular log scheme over K. A log-regular log scheme $\mathscr{X}^+$ over S^+ is a model for $(X, \lceil \Delta_X \rceil)$ over S^+ if $\mathscr{X}$ is a model of X over R, the closure of any component of Δ_X in $\mathscr{X}$ has non-empty intersection with $\mathscr{X}_k$, and $D_{\mathscr{X}} = \overline{\lceil \Delta_X \rceil} + \mathscr{X}_{k,\mathrm{red}}$.

(1.1.6) We say that a log-regular log scheme $\mathscr{X}^+$ over S^+ is semistable if the divisor $D_{\mathscr{X}}$ is reduced. We say that a proper K-variety X has semistable reduction if it admits an R-model $\mathscr{X}$ such that $\mathscr{X}^+$ is log-smooth over S^+, with reduced special fiber; such a model is called a semistable model of X. This is a weaker notion than requiring the existence of an snc-model with reduced special fibre.

(1.1.7) We denote by $(\cdot)^{\mathrm{an}}$ the analytification functor from the category of K-schemes of finite type to Berkovich's category of K-analytic spaces. For every K-scheme of finite type X, as a set, X^{an} consists of the pairs $x = (\xi_x, |\cdot|_x)$ where ξ_x is a point of X and $|\cdot|_x$ is an absolute value on the residue field $\kappa(\xi_x)$ of X at ξ_x extending the absolute value $|\cdot|_K$ on K. We endow X^{an} with the Berkovich topology, i.e. the weakest one such that

(i) the forgetful map $\phi : X^{\mathrm{an}} \to X$, defined as $(\xi_x, |\cdot|_x) \mapsto \xi_x$, is continuous,
(ii) for any Zariski open subset U of X and any regular function f on U the map $|f| : \phi^{-1}(U) \to \mathbb{R}$ defined by $|f|(\xi_x, |\cdot|_x) = |f(\xi_x)|$ is continuous.

2. The Kato fan of a log-regular log scheme

2.1. **Definition of Kato fans.**

(2.1.1) According to [Kat94], Definition 9.1, a monoidal space $(T, \mathcal{M}_T)$ is a topological space T endowed with a sharp sheaf of monoids $\mathcal{M}_T$, where *sharp* means that $\mathcal{M}_{T,t}^{\times} = \{1\}$ for every $t \in T$. We often simply denote the monoidal space by T.

A morphism of monoidal spaces is a pair $(f, \varphi) : (T, \mathcal{M}_T) \to (T', \mathcal{M}_{T'})$ such that $f : T \to T'$ is a continuous function of topological spaces and $\varphi : f^{-1}(\mathcal{M}_T) \to \mathcal{M}_{T'}$ is a sheaf homomorphism such that $\varphi_t^{-1}(\{1\}) = \{1\}$ for every $t \in T$.

Example 2.1.2. If $\mathscr{X}^+$ is a log scheme then the Zariski topological space $\mathscr{X}$ is equipped with a sheaf of sharp monoids $\mathcal{C}_{\mathscr{X}}$, the characteristic sheaf of $\mathscr{X}^+$. Thus $(\mathscr{X}, \mathcal{C}_{\mathscr{X}})$ is a monoidal space. Moreover, morphisms of log schemes induce morphisms of characteristic sheaves, hence morphism of monoidal spaces. We therefore obtain a functor from the category of log schemes to the category of monoidal spaces.

Example 2.1.3. Given a monoid P, we may associate to it a monoidal space called the spectrum of P. As a set, $\operatorname{Spec} P$ is the set of all prime ideals of P. The topology is characterized by the basis open sets $D(f) = \{\mathfrak{p} \in \operatorname{Spec} P | f \notin \mathfrak{p}\}$ for any $f \in P$. The monoidal sheaf is defined by

$$\mathcal{M}_{\operatorname{Spec} P}(D(f)) = S^{-1}P/(S^{-1}P)^{\times}$$

where $S = \{f^n | n \geqslant 0\}$.

(2.1.4) A monoidal space isomorphic to the monoidal space $\operatorname{Spec} P$ for some monoid P is called an affine Kato fan. A monoidal space is called a Kato fan if it has an open covering consisting of affine Kato fans. In particular, we call a Kato fan integral, saturated, of finite type or fs if it admits a cover by the spectra of monoids with the respective properties.

(2.1.5) A morphism of fs Kato fans $F' \to F$ is called a *subdivision* if it has finite fibres and the morphism

$$\mathrm{Hom}(\mathrm{Spec}\,\mathbb{N}, F') \to \mathrm{Hom}(\mathrm{Spec}\,\mathbb{N}, F)$$

is a bijection. Allowing subdivisions a Kato fan might take the following shape.

Proposition 2.1.6. *([Kat94], Proposition 9.8) Let F be a fs Kato fan. Then there is a subdivision $F' \to F$ such that F' has an open cover $\{U_i'\}$ by Kato cones with $U_i' \simeq \mathrm{Spec}\,\mathbb{N}^{r_i}$.*

The strategy of the proof of Proposition 2.1.6 goes back to [KKMSD73] and relies on a sequence of particular subdivisions of the Kato fan, the so-called star and barycentric subdivisions ([ACMUW15], Example 4.10).

2.2. **Kato fans associated to log-regular log schemes.**

Theorem 2.2.1. *([Kat94], Proposition 10.2) Let $\mathscr{X}^+$ be a log-regular log scheme. Then there is an initial strict morphism $(\mathscr{X}, \mathcal{C}_{\mathscr{X}}) \to F$ to a Kato fan in the category of monoidal spaces. Explicitly, there exist a Kato fan F and a morphism $\pi : (\mathscr{X}, \mathcal{C}_{\mathscr{X}}) \to F$ such that $\pi^{-1}(\mathcal{M}_F) \simeq \mathcal{C}_{\mathscr{X}}$ and any other morphism to a Kato fan factors through π.*

The Kato fan F in Theorem 2.2.1 is called the Kato fan associated to $\mathscr{X}^+$; it is the topological subspace of $\mathscr{X}$ consisting of the points x such that the maximal ideal $\mathfrak{m}_x$ of $\mathcal{O}_{\mathscr{X},x}$ is equal to $\mathcal{I}_{\mathscr{X},x}$, and $\mathcal{M}_F$ is the inverse image of $\mathcal{C}_{\mathscr{X}}$ on F, henceforth we write $\mathcal{C}_F$ for $\mathcal{M}_F$.

Example 2.2.2. Assume that $\mathscr{X}$ is regular, of finite type over S and $\mathscr{X}_k$ is a divisor with strict normal crossings. Then $\mathscr{X}^+$ is log-regular and F is the set of generic points of intersections of irreducible components of $\mathscr{X}_k$. For each point x of F, the stalk of $\mathcal{C}_F$ is isomorphic to $(\mathbb{N}^r, +)$, with r the number of irreducible components of $\mathscr{X}_k$ that pass through x.

This example admits the following partial generalisation.

Lemma 2.2.3. *Let $\mathscr{X}^+$ be a log-regular log scheme. Then the fan F consists of the generic points of intersections of irreducible components of $D_{\mathscr{X}}$.*

Proof. First, we show that every such generic point is a point of F. Let $E_1, \dots, E_r$ be irreducible components of $D_{\mathscr{X}}$ and let x be a generic point of the intersection $E_1 \cap \ldots \cap E_r$. We set $d = \dim \mathcal{O}_{\mathscr{X},x}$. Since $\mathscr{X}^+$ is log-regular, we know that $\mathcal{O}_{\mathscr{X},x}/\mathcal{I}_{\mathscr{X},x}$ is regular and that

$$d = \dim \mathcal{O}_{\mathscr{X},x}/\mathcal{I}_{\mathscr{X},x} + \mathrm{rank}\,\mathcal{C}^{\mathrm{gp}}_{\mathscr{X},x}. \tag{2.2.4}$$

We denote by $V(\mathcal{I}_{\mathscr{X},x})$ the vanishing locus of the ideal $\mathcal{I}_{\mathscr{X},x}$ in $\mathscr{X}$. We want to prove that $\mathcal{I}_{\mathscr{X},x} = \mathfrak{m}_x$. We assume the contrary, hence that $\mathcal{I}_{\mathscr{X},x} \subsetneq \mathfrak{m}_x$. This assumption implies that there exists j such that $V(\mathcal{I}_{\mathscr{X},x}) \nsubseteq E_j$: indeed, if the vanishing locus is contained in each irreducible component E_i, i.e.

$$V(\mathcal{I}_{\mathscr{X},x}) \subseteq E_1 \cap \ldots \cap E_r \subseteq \overline{\{x\}},$$

then $\mathcal{I}_{\mathscr{X},x} \supseteq \mathfrak{m}_x$. From the assumption of log-regularity it follows that the vanishing locus $V(\mathcal{I}_{\mathscr{X},x})$ is a regular subscheme, and moreover that $\mathscr{X}^+$ is Cohen-Macaulay by [Kat94], Theorem 4.1. Thus, there exists a regular sequence $(f_1, \dots, f_l)$ in $\mathcal{I}_{\mathscr{X},x}$ where l is the codimension of $V(\mathcal{I}_{\mathscr{X},x})$, i.e.

$$\dim \mathcal{O}_{\mathscr{X},x}/\mathcal{I}_{\mathscr{X},x} = d - l.$$

Moreover by the equality (2.2.4), $\operatorname{rank} \mathcal{C}^{\mathrm{gp}}_{\mathscr{X},x} = l$.

We claim that the residue classes of these elements f_i in $\mathcal{C}^{\mathrm{gp}}_{\mathscr{X},x}$ are linearly independent. Assume the contrary. Then, up to renumbering the f_i, there exist an integer e with $1 < e < l$, non-negative integers $a_1, \dots, a_l$, not all zero, and a unit u in $\mathcal{O}_{\mathscr{X},x}$ such that

$$f_1^{a_1} \cdot \dots \cdot f_{e-1}^{a_{e-1}} = u \cdot f_e^{a_e} \cdot \dots \cdot f_l^{a_l}.$$

This contradicts the fact that $(f_1, \dots, f_l)$ is a regular sequence in $\mathcal{I}_{\mathscr{X},x}$. Thus, the classes $\overline{f_1}, \dots, \overline{f_l}$ are independent in $\mathcal{C}^{\mathrm{gp}}_{\mathscr{X},x}$. As we also have the equality $\operatorname{rank} \mathcal{C}^{\mathrm{gp}}_{\mathscr{X},x} = l$, it follows that these classes generate $\mathcal{C}^{\mathrm{gp}}_{\mathscr{X},x} \otimes_{\mathbb{Z}} \mathbb{Q}$.

Let g_j be a non-zero element of the ideal $\mathcal{I}_{\mathscr{X},x}$ that vanishes along E_j: it necessarily exists as otherwise E_j is not a component of the divisor $D_{\mathscr{X}}$. Then g_j satisfies

$$g_j^N = v \cdot f_1^{b_1} \cdot \dots \cdot f_l^{b_l}$$

with $b_i \in \mathbb{Z}$, v a unit in $\mathcal{O}_{\mathscr{X},x}$ and N a positive integer. As g_j vanishes along the irreducible component E_j, at least one of the functions $f_1, \dots, f_l$ has to vanish along E_j: assume that is f_1.

On the one hand, as f_1 is identically zero on E_j, the trace of E_j on $V(\mathcal{I}_{\mathscr{X},x})$ has at most codimension $l-1$ in E_j at the point x. On the other hand, we assumed that $V(\mathcal{I}_{\mathscr{X},x})$ is not contained in E_j and it has codimension l in $\mathcal{O}_{\mathscr{X},x}$. Then, the trace of E_j on $V(\mathcal{I}_{\mathscr{X},x})$ has codimension l in E_j at x. This is a contradiction. We conclude that the ideal $\mathcal{I}_{\mathscr{X},x}$ is equal to the maximal ideal $\mathfrak{m}_x$, therefore x is a point of F.

It remains to prove the converse implication: every point x of the fan F must be a generic point of an intersection of irreducible components of $D_{\mathscr{X}}$. Let x be a point of F: by construction of Kato fan F, the maximal ideal of $\mathcal{O}_{\mathscr{X},x}$ is equal to $\mathcal{I}_{\mathscr{X},x}$, thus it is generated by elements in $\mathcal{M}_{\mathscr{X},x}$. The zero locus of such an element is contained in $D_{\mathscr{X}}$ by definition of the logarithmic structure on $\mathscr{X}^+$. Therefore, the zero locus is a union of irreducible components of the trace of $D_{\mathscr{X}}$ on $\operatorname{Spec} \mathcal{O}_{\mathscr{X},x}$ and x is a generic point of the intersection of all such irreducible components. $\square$

Remark 2.2.5. By convention, the generic point of the empty intersection of irreducible components is the generic point of $\mathscr{X}$. By definition, this point is also included in the Kato fan F. Thus, for example, the Kato fan associated to S^+ consists of two points: the generic point of S that corresponds to the empty intersection, and the closed point s corresponding to the unique irreducible component of the logarithmic divisorial structure.

Moreover, the example 2.2.2 also leads to the following characterization.

Proposition 2.2.6. *([GR04], Corollary 12.5.35) Let $\mathscr{X}^+$ be a log-regular log scheme over S^+ and F its associated Kato fan. The following are equivalent:*

(1) for every $x \in F$, $M_{F,x} \simeq \mathbb{N}^{r(x)}$,
(2) the underlying scheme $\mathscr{X}$ is regular.

If this is the case, then the special fibre $\mathscr{X}_k$ is a strict normal crossing divisor.

(2.2.7) The construction of the Kato fan of a log scheme defines a functor from the category of log-regular log schemes to the category of Kato fans. Indeed, given a morphism of log schemes $\mathscr{X}^+ \to \mathscr{Y}^+$, we consider the embedding of the associated Kato fan $F_{\mathscr{X}}$ in $\mathscr{X}^+$ and the canonical morphism $\mathscr{Y}^+ \to F_{\mathscr{Y}}$: the composition

$$F_{\mathscr{X}} \hookrightarrow \mathscr{X}^+ \to \mathscr{Y}^+ \to F_{\mathscr{Y}}$$

functorially induces a map between associated Kato fans. Moreover, this association preserves strict morphisms ([Uli13], Lemma 4.9).

2.3. Resolutions of log schemes via Kato fan subdivisions.

Proposition 2.3.1. *([Kat94], Proposition 9.9) Let $\mathscr{X}^+$ be a log-regular log scheme and let F be its associated Kato fan. Let $F' \to F$ be a subdivision of fans. Then there exist a log scheme $\mathscr{X}'^+$, a morphism of log schemes $\mathscr{X}'^+ \to \mathscr{X}^+$ and a commutative diagram*

$$\begin{tikzcd} (\mathscr{X}', \mathcal{C}_{\mathscr{X}'}) \arrow[r, "p"] \arrow[d] & F' \arrow[d] \\ (\mathscr{X}, \mathcal{C}_{\mathscr{X}}) \arrow[r, "\pi_{\mathscr{X}}"] & F \end{tikzcd}$$

such that $p^{-1}(\mathcal{M}_{F'}) \simeq \mathcal{C}_{\mathscr{X}'}$, they define a final object in the category of such diagrams and the refinement $F' \to F$ is induced by the morphism of log-regular log schemes $\mathscr{X}'^+ \to \mathscr{X}^+$.

(2.3.2) It follows that given any subdivision $F' \to F$ of the Kato fan F associated with a log regular log scheme $\mathscr{X}^+$, we can construct a log scheme over $\mathscr{X}^+$ with prescribed associated Kato fan F'. Combining this fact with Proposition 2.1.6 and Proposition 2.2.6 yields to the construction of resolutions of log schemes in the following sense: for any log-regular log scheme over S^+ we can find a birational modification by a regular log scheme with strict normal special fibre. Moreover, the morphism of log schemes $\mathscr{X}'^+ \to \mathscr{X}^+$ is obtained by a log blow-up ([Niz06], Theorem 5.8).

2.4. Fibred products and associated Kato fans.

(2.4.1) Given morphisms of *fs* log schemes $f_1 : \mathscr{X}_1^+ \to \mathscr{Y}^+$ and $f_2 : \mathscr{X}_2^+ \to \mathscr{Y}^+$, their fibred product exists in the category of log schemes. It is obtained by endowing the usual fibred product of schemes

$$\begin{tikzcd} \mathscr{X}_1 \times_{\mathscr{Y}} \mathscr{X}_2 \arrow[r, "p_1"] \arrow[d, "p_2"] \arrow[dr, "p_{\mathscr{Y}}"] & \mathscr{X}_1 \arrow[d, "f_1"] \\ \mathscr{X}_2 \arrow[r, "f_2"] & \mathscr{Y} \end{tikzcd} \tag{2.4.2}$$

with the log structure associated to $p_1^{-1}\mathcal{M}_{\mathscr{X}_1} \oplus_{p_{\mathscr{Y}}^{-1}\mathcal{M}_{\mathscr{Y}}} p_2^{-1}\mathcal{M}_{\mathscr{X}_2}$. If $u_1 : P \to Q_1$ and $u_2 : P \to Q_2$ are charts for the morphisms f_1 and f_2 respectively, then the induced morphism $\mathscr{X}_1 \times_{\mathscr{Y}} \mathscr{X}_2 \to \operatorname{Spec} \mathbb{Z}[Q_1 \oplus_P Q_2]$ is a chart for $\mathscr{X}_1^+ \times_{\mathscr{Y}^+} \mathscr{X}_2^+$.

(2.4.3) In general, the fibred product is not fs, but the category of fs log schemes also admits fibred products. Keeping the same notations, the following is a chart of the fibred product in the category of fine and saturated log schemes

$$\mathscr{X}_1^+ \times^{\mathrm{fs}}_{\mathscr{Y}^+} \mathscr{X}_2^+ = (\mathscr{X}_1^+ \times_{\mathscr{Y}^+} \mathscr{X}_2^+) \times_{\mathbb{Z}[Q_1 \oplus_P Q_2]} \mathbb{Z}[(Q_1 \oplus_P Q_2)^{\mathrm{sat}}]$$

([Bul15], 3.6.16). We remark that the two fibre products above may not only have different log structures, but also the underlying schemes may differ. Nevertheless, this does not occur when the monoid $Q_1 \oplus_P Q_2$ is saturated.

(2.4.4) Log smoothness is preserved under fs base change and composition ([GR04], Proposition 12.3.24). In particular, if $f_1 : \mathscr{X}_1^+ \to \mathscr{Y}^+$ is log-smooth and $\mathscr{X}_2^+$ is log-regular, then $\mathscr{X}_1^+ \times^{\mathrm{fs}}_{\mathscr{Y}^+} \mathscr{X}_2^+$ is log-regular, by [Kat94], Theorem 8.2.

Consider log-smooth morphisms of fs log schemes $\mathscr{X}_1^+ \to \mathscr{Y}^+$ and $\mathscr{X}_2^+ \to \mathscr{Y}^+$. The sheaves of logarithmic differentials are related by the following isomorphism

$$p_1^*\Omega^{\log}_{\mathscr{X}_1^+/\mathscr{Y}^+} \oplus p_2^*\Omega^{\log}_{\mathscr{X}_2^+/\mathscr{Y}^+} \simeq \Omega^{\log}_{\mathscr{X}_1^+ \times^{\mathrm{fs}}_{\mathscr{Y}^+} \mathscr{X}_2^+/\mathscr{Y}^+} \tag{2.4.5}$$

by [GR04], Proposition 12.3.13. Furthermore, by assumption of log-smoothness over S^+ the logarithmic differential sheaves are locally free of finite rank ([Kat94], Proposition 3.10) and we can consider their determinants; they are called log canonical bundles and denoted by $\omega^{\log}$. The following isomorphism is a direct consequence of (2.4.5)

$$p_1^*\omega^{\log}_{\mathscr{X}_1^+/\mathscr{Y}^+} \otimes p_2^*\omega^{\log}_{\mathscr{X}_2^+/\mathscr{Y}^+} \simeq \omega^{\log}_{\mathscr{X}_1^+ \times^{\mathrm{fs}}_{\mathscr{Y}^+} \mathscr{X}_2^+/\mathscr{Y}^+}. \tag{2.4.6}$$

(2.4.7) Similarly to the construction of fibred products of fs log schemes, the category of fs Kato fans admits fibred products: on affine Kato fans $F = \operatorname{Spec} P$ and $G = \operatorname{Spec} Q$ over $H = \operatorname{Spec} T$, $F \times_H G$ is the spectrum of the amalgamated sum $(P \oplus_T Q)^{\mathrm{sat}}$ in the category of fs monoids ([Uli16], Proposition 2.4) and on the underlying topological spaces, this coincides with the usual fibred product.

We seek to compare the Kato fan associated to the fibred product of log-regular log schemes with the fibred product of associated Kato fans.

Proposition 2.4.8. *([Sai04], Lemma 2.8) Given $\mathscr{T}^+$ a log-regular log scheme, let $\mathscr{X}^+$ and $\mathscr{Y}^+$ be log-smooth log schemes over $\mathscr{T}^+$. We denote by $\mathscr{Z}^+$ the fs fibred product $\mathscr{X}^+ \times^{fs}_{\mathscr{T}^+} \mathscr{Y}^+$. Then, the natural morphisms $F_{\mathscr{Z}} \to F_{\mathscr{X}}$ and $F_{\mathscr{Z}} \to F_{\mathscr{Y}}$ induce a morphism of Kato fans*

$$F_{\mathscr{Z}} \to F_{\mathscr{X}} \times_{F_{\mathscr{T}}} F_{\mathscr{Y}} \tag{2.4.9}$$

that is locally an isomorphism.

(2.4.10) For any pair of points (x, y) in $F_{\mathscr{X}} \times_{F_{\mathscr{T}}} F_{\mathscr{Y}}$, we denote by $n(x, y)$ be the number of preimages of (x, y) in the Kato fan of $\mathscr{Z}^+$ under the local isomorphism (2.4.9).

Lemma 2.4.11. *If x' is in the closure of x, and y' in the closure of y, then $n(x', y') \geqslant n(x, y)$.*

Proof. Let z' be a preimage of the pair (x', y'). By Proposition 2.4.8, there exists an open neighbourhood $U_{z'}$ of z' such that the restriction of $F_{\mathscr{Z}} \to F_{\mathscr{X}} \times_{F_{\mathscr{T}}} F_{\mathscr{Y}}$ to $U_{z'}$ is an isomorphism onto its image. In particular, (x, y) lies in this image. Thus, there exists a unique preimage of (x, y) that is contained in $U_{z'}$. It follows that $n(x', y') \geqslant n(x, y)$. □

2.5. **Semistability and Kato fans associated to the fibred products.**

(2.5.1) change style We investigate a sufficient condition, for pairs of points whose closures intersect the special fibre, to turn the local isomorphism (2.4.9) into an isomorphism: it concerns the notion of semistability. We recall that a log-regular log scheme $\mathscr{X}^+$ is said to be semistable if the divisor $D_{\mathscr{X}}$, where the log structure is non-trivial, is reduced.

(2.5.2) In order to see the relevance of the assumption of semistability, we need some results on saturated morphism of log schemes. We recall that, locally around a point x of the divisor $D_{\mathscr{X}}$, the morphism of characteristic monoids $\mathbb{N} \to \mathcal{C}_{\mathscr{X},x}$ is a saturated morphism of monoids if, for any morphism $u : \mathbb{N} \to P$ of fs monoids, the amalgamated sum $\mathcal{C}_{\mathscr{X},x} \oplus_{\mathbb{N}} P$ is still saturated.

Following the work by T. Tsuji in an unpublished 1997 preprint, Vidal in [Vid04] defines the saturation index of a morphism of *fs* monoids. In the case of log-regular log scheme over S^+ it can be easily computed: it is the least common multiple of the multiplicities of the prime components of the divisor $D_{\mathscr{X}}$. The following criterion holds.

Lemma 2.5.3. *([Vid04], Section 1.3) A morphism of fs monoids is saturated if and only if the saturation index is equal to 1.*

Proposition 2.5.4. *Assume that the residue field k is algebraically closed. Let $\mathscr{X}^+$ and $\mathscr{Y}^+$ be log-smooth log scheme over S^+. Let $\mathscr{Z}^+$ be their fs fibred product. If $\mathscr{X}^+$ is semistable, then for any pair of points (x, y) in $F_{\mathscr{X}} \times_{F_S} F_{\mathscr{Y}}$ whose closures intersect the special fibres $\mathscr{X}_k$ and $\mathscr{Y}_k$ respectively, the morphism $F_{\mathscr{Z}} \to F_{\mathscr{X}} \times F_{\mathscr{Y}}$, induced by the projections $\mathscr{Z}^+ \to \mathscr{X}^+$ and $\mathscr{Z}^+ \to \mathscr{Y}^+$, is a bijection above the pair (x, y), namely $n(x, y) = 1$.*

Proof. By hypothesis $\mathscr{X}^+$ is a semistable log-regular log scheme over S^+, hence the saturation index of $\mathscr{X}^+ \to S^+$ is 1. Thus, by Lemma 2.5.3 the morphism of log schemes $\mathscr{X}^+ \to S^+$ induces a saturated morphism of characteristic monoids at every point of $\mathscr{X}^+$. The saturation condition implies that the fibred product in the category of log schemes coincides with the fibred product in the category of fs log schemes. In particular, the underlying scheme of $\mathscr{Z}^+$ coincides with the usual schematic fibred product, hence its points are characterized as follows:

$$z = (x, y, s, \mathfrak{p}) \text{ and } \mathcal{O}_{\mathscr{Z},z} = (\mathcal{O}_{\mathscr{X},x} \otimes_R \mathcal{O}_{\mathscr{Y},y})_{\mathfrak{p}}$$

where x and y are points of $\mathscr{X}^+$ and $\mathscr{Y}^+$ both mapped to the same point s of S, while $\mathfrak{p}$ is a prime ideal of the tensor product of residue fields $\kappa(x) \otimes_{\kappa(s)} \kappa(y)$. We look for a characterization of points z in $\mathscr{Z}^+$ that lie in the Kato fan $F_{\mathscr{Z}}$.

If the point z lies in $F_{\mathscr{Z}}$, then the maximal ideal $\mathfrak{m}_z$ is equal to the ideal $\mathcal{I}_{\mathscr{Z},z}$ by definition. By the flatness of the models $\mathscr{X}^+$ and $\mathscr{Y}^+$ over S^+, the morphisms of local rings $\mathcal{O}_{\mathscr{X},x} \to \mathcal{O}_{\mathscr{Z},z}$ and $\mathcal{O}_{\mathscr{Y},y} \to \mathcal{O}_{\mathscr{Z},z}$ are injective. Hence, the equalities $\mathfrak{m}_x = \mathcal{I}_{\mathscr{X},x}$ and $\mathfrak{m}_y = \mathcal{I}_{\mathscr{Y},y}$ hold. Thus, the points z in $\mathscr{Z}^+$ that lie in the Kato fan $F_{\mathscr{Z}}$ are necessarily points such that the projections x and y to $\mathscr{X}^+$ and $\mathscr{Y}^+$ lie in their associated Kato fans. Therefore, we may assume $x \in F_{\mathscr{X}}$, $y \in F_{\mathscr{Y}}$, and it remains to characterize the prime ideals $\mathfrak{p}$ such that $z = (x, y, s, \mathfrak{p}) \in F_{\mathscr{Z}}$.

By log-regularity of $\mathscr{Z}^+$, the point z lies in the associated Kato fan if and only if $\dim \mathcal{O}_{\mathscr{Z},z} = \mathrm{rank} \mathcal{C}^{\mathrm{gp}}_{\mathscr{Z},z}$. At the level of characteristic sheaves it holds that

$$\mathrm{rank} \mathcal{C}^{\mathrm{gp}}_{\mathscr{Z},z} = \mathrm{rank} \mathcal{C}^{\mathrm{gp}}_{\mathscr{X},x} + \mathrm{rank} \mathcal{C}^{\mathrm{gp}}_{\mathscr{Y},y} - 1.$$

Since x and y are both assumed to be points in the associated Kato fans, the equality between dimension of local rings and rank of the groupifications of characteristic sheaves lead to the equivalence

$$\begin{aligned} z \in F_{\mathscr{Z}} \Leftrightarrow \dim \mathcal{O}_{\mathscr{Z},z} &= \mathrm{rank}\mathcal{C}^{\mathrm{gp}}_{\mathscr{Z},z} = \mathrm{rank}\mathcal{C}^{\mathrm{gp}}_{\mathscr{X},x} + \mathrm{rank}\mathcal{C}^{\mathrm{gp}}_{\mathscr{Y},y} - 1 \\ &= \dim \mathcal{O}_{\mathscr{X},x} + \dim \mathcal{O}_{\mathscr{Y},y} - 1. \end{aligned}$$

By log-regularity of $\mathscr{Z}^+$, it holds that $\dim \mathcal{O}_{\mathscr{Z},z} \geqslant \mathrm{rank}\mathcal{C}^{\mathrm{gp}}_{\mathscr{Z},z}$, thus the inequality

$$\dim \mathcal{O}_{\mathscr{Z},z} \geqslant \dim \mathcal{O}_{\mathscr{X},x} + \dim \mathcal{O}_{\mathscr{Y},y} - 1$$

is always true and equality holds only for minimal prime ideals $\mathfrak{p}$ of $\kappa(x) \otimes_{\kappa(s)} \kappa(y)$. Therefore, in order to determine the number $n(x, y)$ of preimages of (x, y) in $F_{\mathscr{Z}}$, we need to study the number of minimal prime ideals of the tensor product $\kappa(x) \otimes_{\kappa(s)} \kappa(y)$.

If s is the closed point of S, then its residue field is the algebraically closed field k. It follows that the tensor product $\kappa(x) \otimes_k \kappa(y)$ is a domanin, hence it has a unique minimal prime ideal, namely 0. We obtain that $n(x, y) = 1$ for any pair of points of $F_{\mathscr{X}} \times_{F_S} F_{\mathscr{Y}}$ that lie in the special fibres.

Let (x, y) be a pair of points in $F_{\mathscr{X}} \times_{F_S} F_{\mathscr{Y}}$ whose closures intersect the special fibres, namely there exist $x' \in F_{\mathscr{X}} \cap \mathscr{X}_k$ and $y' \in F_{\mathscr{Y}} \cap \mathscr{Y}_k$ such that x' is in the closure of x and y' in the closure of y. Then, by the previous part of the proof and by Lemma 2.4.11, we have $n(x, y) \leqslant n(x', y') = 1$. □

3. The skeleton of a log-regular log scheme

3.1. Construction of the skeleton of a log-regular log scheme.

(3.1.1) Let $\mathscr{X}^+$ be a log-regular log scheme over S^+. Let x be a point of the associated Kato fan F. Denote by $F(x)$ the set of points y of F such that x lies in the closure of $\{y\}$, and by $\mathcal{C}_{F(x)}$ the restriction of $\mathcal{C}_F$ to $F(x)$. Denote by $\mathrm{Spec}\, \mathcal{C}_{\mathscr{X},x}$ the spectrum of the monoid $\mathcal{C}_{\mathscr{X},x} = \mathcal{C}_{F,x}$. Then there exists a canonical isomorphism of monoid spaces

$$(F(x), \mathcal{C}_{F(x)}) \to \mathrm{Spec}\, \mathcal{C}_{\mathscr{X},x} : y \mapsto \{s \in \mathcal{C}_{\mathscr{X},x} \,|\, s(y) = 0\}$$

where the expression $s(y) = 0$ means that $s'(y) = 0$ for any representative s' of s in $\mathcal{M}_{\mathscr{X},x}$. In particular, we obtain a bijective correspondence between the faces of the monoid $\mathcal{C}_{\mathscr{X},x}$ and the points of $F(x)$, and for every point y of $F(x)$, a surjective cospecialization morphism of monoids

$$\tau_{x,y} : \mathcal{C}_{\mathscr{X},x} \to \mathcal{C}_{\mathscr{X},y}$$

which induces an isomorphism of monoids

$$S^{-1}\mathcal{C}_{\mathscr{X},x}/(S^{-1}\mathcal{C}_{\mathscr{X},x})^{\times} \cong \mathcal{C}_{\mathscr{X},x}/S \xrightarrow{\sim} \mathcal{C}_{\mathscr{X},y}$$

where S denotes the monoid of elements s in $\mathcal{C}_{\mathscr{X},x}$ such that $s(y) \neq 0$.

(3.1.2) For each point x in F, we denote by σ_x the set of morphisms of monoids

$$\alpha : \mathcal{C}_{\mathscr{X},x} \to (\mathbb{R}_{\geq 0}, +)$$

such that $\alpha(\pi) = 1$ for every uniformizer π in R. We endow σ_x with the topology of pointwise covergence, where $\mathbb{R}_{\geq 0}$ carries the usual Euclidean topology. Note that σ_x is a polygon, possibly semiopen, in the real affine space

$$\{\alpha : \mathcal{C}^{\mathrm{gp}}_{\mathscr{X},x} \to (\mathbb{R}, +) \,|\, \alpha(\pi) = 1 \text{ for every uniformizer } \pi \text{ in } R\}.$$

If y is a point of $F(x)$, then the surjective cospecialization morphism $\tau_{x,y}$ induces a topological embedding $\sigma_y \to \sigma_x$ that identifies σ_y with a face of σ_x.

(3.1.3) We denote by T the disjoint union of the topological spaces σ_x with x in F. On the topological space T, we consider the equivalence relation $\sim$ generated by couples of the form $(\alpha, \alpha \circ \tau_{x,y})$ where x and y are points in F such that x lies in the closure of $\{y\}$ and α is a point of σ_y.

The skeleton of $\mathscr{X}^+$ is defined as the quotient of the topological space T by the equivalence relation $\sim$. We denote this skeleton by $\mathrm{Sk}(\mathscr{X}^+)$. It is clear that $\mathrm{Sk}(\mathscr{X}^+)$ has the structure of a polyhedral complex with cells $\{\sigma_x,\, x \in F\}$ and that the faces of a cell σ_x are precisely the cells σ_y with y in $F(x)$.

(3.1.4) We note that σ_x is empty for any point x that does not lie in the special fibre $\mathscr{X}_k$: indeed, outside the special fibre any uniformizer is an invertible element, so it is trivial in $\mathcal{C}_{\mathscr{X},x}$ and is mapped to 0 by any morphism of monoids. Therefore, the construction of the skeleton associated to $\mathscr{X}^+$ only concerns the points in the Kato fans F that lie in the special fibre. In particular, given a generic point $x \in \mathscr{X}_k$ of an intersection of components of $D_{\mathscr{X}}$, where at least one component is not in the special fibre, the corresponding face σ_x is unbounded.

In other words, the skeleton associated to a log-regular scheme $\mathscr{X}^+$, where $D_{\mathscr{X}}$ allows horizontal components, generalizes Berkovich's skeleta by admitting unbounded faces in the direction of the horizontal components. It also generalizes the construction performed by Gubler, Rabinoff and Werner in [GRW16] of a skeleton associated to a strictly semistable pair.

3.2. Embedding the skeleton in the non-archimedean generic fiber.

(3.2.1) Let $\mathscr{X}^+$ be a log-regular log scheme over S^+. Let x be a point of the associated Kato fan F. As the log structure on $\mathscr{X}^+$ is of finite type, the characteristic monoid $\mathcal{C}_{\mathscr{X},x}$ is of finite type too, and thus $\mathcal{C}^{\mathrm{gp}}_{\mathscr{X},x}$ is a free abelian group of finite rank. Hence there exists a section

$$\zeta : \mathcal{M}^{\mathrm{gp}}_{\mathscr{X},x}/\mathcal{M}^{\times}_{\mathscr{X},x} \to \mathcal{M}^{\mathrm{gp}}_{\mathscr{X},x}.$$

The section ζ restricts to $\mathcal{C}_{\mathscr{X},x} \to \mathcal{M}_{\mathscr{X},x}$; indeed, if $x \in \mathcal{M}_{\mathscr{X},x}$ then $\zeta(\overline{x}) - x \in \mathcal{M}^{\times}_{\mathscr{X},x}$. Therefore we may choose a section

$$\mathcal{C}_{\mathscr{X},x} \to \mathcal{M}_{\mathscr{X},x} \tag{3.2.2}$$

of the projection homomorphism

$$\mathcal{M}_{\mathscr{X},x} \to \mathcal{C}_{\mathscr{X},x}$$

and use this section to view $\mathcal{C}_{\mathscr{X},x}$ as a submonoid of $\mathcal{M}_{\mathscr{X},x}$. Note that $\mathcal{C}_{\mathscr{X},x} \setminus \{0\}$ generates the ideal $\mathcal{I}_{\mathscr{X},x}$ of $\mathcal{O}_{\mathscr{X},x}$.

We propose a generalisation of [MN15], Lemma 2.4.4.

Lemma 3.2.3. *Let A be a Noetherian ring, let I be an ideal of A and let $(y_1, \dots, y_m)$ be a system of generators for I. We denote by $\hat{A}$ the I-adic completion of A. Let B be a subring of A such that the elements $y_1, \dots, y_m$ belong to B and generate the ideal $B \cap I$ in B. Then, in the ring $\hat{A}$, every element f of B can be written as*

$$f = \sum_{\beta \in \mathbb{Z}^m_{\geqslant 0}} c_\beta y^\beta \tag{3.2.4}$$

where the coefficients c_β belong to $((A \setminus I) \cap B) \cup \{0\}$.

Proof. Let f be an element of B, we construct an expansion for f of the form (3.2.4) by induction. If f belongs to the complement of I, the conclusion trivially holds. Otherwise, f belongs to I and we can write f as a linear combination of the elements $y_1, \ldots, y_m$ with coefficients in B:

$$f = \sum_{j=1}^{m} b_j y_j, \quad b_j \in B.$$

By induction hypothesis, we suppose that i is a positive integer and that we can write every f in B as a sum of an element f_i of the form (3.2.4) and a linear combination of degree i monomials in the elements $y_1, \ldots, y_m$ with coefficients in B. We apply this assumption to the coefficients b_j, hence

$$b_j = b_{j,i} + \sum_{\substack{\beta \in \mathbb{Z}^m_{\geqslant 0} \\ |\beta| = i}} b_{j,\beta} y^\beta, \quad b_{j,\beta} \in B.$$

Then we can write f as a sum of an element f_{i+1} of the form (3.2.4) and a linear combination of degree $i+1$ monomials in the elements $y_1, \ldots, y_m$ with coefficients in B

$$f = \underbrace{\sum_{j=1}^{m} b_{j,i} y_j}_{f_{i+1}} + \sum_{j=1}^{m} \Big(\sum_{\substack{\beta \in \mathbb{Z}^m_{\geqslant 0} \\ |\beta| = i}} b_{j,\beta} y^\beta \Big) y_j$$

such that f_i and f_{i+1} have the same coefficients in degree less or equal to i. Iterating this construction we finally find an expansion of f of the required form. $\square$

(3.2.5) Let f be an element of $\mathcal{O}_{\mathscr{X},x}$. Considering $A = B = \mathcal{O}_{\mathscr{X},x}$, $I = \mathfrak{m}_x$ and a system of generators for $\mathfrak{m}_x$ in $\mathcal{C}_{\mathscr{X},x} \setminus \{1\}$, by Lemma 3.2.3 we can write f as a formal power series

$$f = \sum_{\gamma \in \mathcal{C}_{\mathscr{X},x}} c_\gamma \gamma \tag{3.2.6}$$

in $\widehat{\mathcal{O}}_{\mathscr{X},x}$, where each coefficient c_γ is either zero or a unit in $\mathcal{O}_{\mathscr{X},x}$. We call this formal series an *admissible expansion* of f. We set

$$S = \{\gamma \in \mathcal{C}_{\mathscr{X},x} \,|\, c_\gamma \neq 0\} \tag{3.2.7}$$

and we denote by $\Gamma_x(f)$ the set of elements of S that lie on a compact face of the convex hull of $S + \mathcal{C}_{\mathscr{X},x}$ in $\mathcal{C}^{\mathrm{gp}}_{\mathscr{X},x} \otimes_{\mathbb{Z}} \mathbb{R}$. We call $\Gamma_x(f)$ the *initial support* of f at x and we remark that the notation makes sense in view of the next proposition.

Proposition 3.2.8.

(1) The element

$$f_x = \sum_{\gamma \in \Gamma} c_\gamma(x)\gamma \quad \in k(x)[\mathcal{C}_{\mathscr{X},x}]$$

depends on the choice of the section (3.2.2), but not on the expansion (3.2.6).

(2) The subset Γ of $\mathcal{C}_{\mathscr{X},x}$ only depends on f and x, and not on the choice of the section (3.2.2) or the expansion (3.2.6).

Proof. If we denote by I the ideal of $k(x)[\mathcal{C}_{\mathscr{X},x}]$ generated by $\mathcal{C}_{\mathscr{X},x} \setminus \{1\}$, then it follows from [Kat94] that there exists an isomorphism of $k(x)$-algebras

$$\operatorname{gr}_I k(x)[\mathcal{C}_{\mathscr{X},x}] \to \operatorname{gr}_{\mathfrak{m}_x} \mathcal{O}_{\mathscr{X},x}. \tag{3.2.9}$$

Using this result and following the argument of [MN15] Proposition 2.4.6, we show that f_x does not depend on the expansion of f. Let

$$f = \sum_{\gamma \in \mathcal{C}_{\mathscr{X},x}} c'_\gamma \gamma$$

be another admissible expansion of f with associated set Γ' and element f'_x. Then

$$0 = \sum_{\gamma \in \mathcal{C}_{\mathscr{X},x}} (c_\gamma - c'_\gamma)\gamma = \sum_{\gamma \in \mathcal{C}_{\mathscr{X},x}} d_\gamma \gamma$$

where the right hand side is an admissible expansion obtained by choosing admissible expansions for the elements $c_\gamma - c'_\gamma$ that do not lie in $\mathcal{O}^\times_{\mathscr{X},x} \cup \{0\}$. In particular $d_\gamma(x) = c_\gamma(x) - c'_\gamma(x)$ for any γ in $\Gamma_x \cup \Gamma'_x$. The isomorphism of graded algebras in (3.2.9) implies that the elements d_γ must all vanish, hence $\Gamma_x = \Gamma'_x$ and $f_x = f'_x$.

Point (2) follows from the fact that the coefficients c_γ of f_x are independent of the chosen section up to multiplication by a unit in $\mathcal{O}_{\mathscr{X},x}$, so that the support Γ of f_x only depends on f and x. □

Proposition 3.2.10. *Let x be a point of F and let*

$$\alpha : \mathcal{C}_{\mathscr{X},x} \to (\mathbb{R}_{\geq 0}, +)$$

be an element of σ_x.Then there exists a unique minimal real valuation

$$v : \mathcal{O}_{\mathscr{X},x} \setminus \{0\} \to \mathbb{R}_{\geq 0}$$

such that $v(m) = \alpha(\overline{m})$ for each element m of $\mathcal{M}_{\mathscr{X},x}$.

Proof. We will prove that the map

$$v : \mathcal{O}_{\mathscr{X},x} \setminus \{0\} \to \mathbb{R} : f \mapsto \min\{\alpha(\gamma) \,|\, \gamma \in \Gamma_x(f)\} \tag{3.2.11}$$

satisfies the requirements in the statement. We fix a section

$$\mathcal{C}_{\mathscr{X},x} \to \mathcal{M}_{\mathscr{X},x}.$$

It is straightforward to check that $(f \cdot g)_x = f_x \cdot g_x$ for all f and g in $\mathcal{O}_{\mathscr{X},x}$. This implies that v is a valuation. It is obvious that $v(m) = \alpha(\overline{m})$ for all m in $\mathcal{M}_{\mathscr{X},x}$, since we can write m as the product of an element of $\mathcal{C}_{\mathscr{X},x}$ and a unit in $\mathcal{O}_{\mathscr{X},x}$.

Now we prove minimality.Consider any real valuation

$$w : \mathcal{O}_{\mathscr{X},x} \to \mathbb{R}$$

such that $w(f) = \alpha(\overline{m})$ for each element m of $\mathcal{M}_{\mathscr{X},x}$, and let f be an element of $\mathcal{O}_{\mathscr{X},x}$. We must show that $w(f) \geq v(f)$.

We set

$$C_\alpha = \mathcal{C}_{\mathscr{X},x} \setminus \alpha^{-1}(0).$$

We denote by I the ideal in $\mathcal{O}_{\mathscr{X},x}$ generated by C_α and by A the I-adic completion of $\mathcal{O}_{\mathscr{X},x}$. By Lemma 3.2.3, we see that we can write f in A as

$$\sum_{\beta \in C_\alpha \cup \{1\}} d_\beta \beta \tag{3.2.12}$$

where d_β is either zero or contained in the complement of I in $\mathcal{O}_{\mathscr{X},x}$.

Since $\alpha(\beta) > 0$ for every $\beta \in C_\alpha$, we can find an integer $N > 0$ such that $w(g) > w(f)$ for every element g in I^N. Since $w(\beta) = \alpha(\beta)$ for all β in $\mathcal{C}_{\mathscr{X},x}$, we can write

$$w(f) \geq \min\{\alpha(\beta) \,|\, d_\beta \neq 0\}.$$

We consider the coefficients in the expansion (3.2.12) of f. Applying Lemma 3.2.3 as in paragraph (3.2.5), we can write admissible expansions of these coefficients in $\widehat{\mathcal{O}}_{\mathscr{X},x}$ as

$$d_\beta = \sum_{\gamma \in \mathcal{C}_{\mathscr{X},x}} c_{\gamma,\beta}\gamma, \quad c_{\gamma,\beta} \in \mathcal{O}^\times_{\mathscr{X},x} \cup \{0\},$$

with $\alpha(\gamma) = 0$ in the expansions of d_β that belong to $\mathfrak{m}_x \setminus I$.

Therefore we obtain an admissible expansion of f

$$f = \sum_{\substack{\beta \in C_\alpha \cup \{1\} \\ \gamma \in \mathcal{C}_{\mathscr{X},x}}} c_{\gamma,\beta}\, \gamma\beta$$

and we have $v(f) = \min\{\alpha(\gamma\beta) \,|\, c_{\gamma,\beta} \neq 0\} = \min\{\alpha(\beta) \,|\, d_\beta \neq 0\} \geq w(f)$. □

Remark 3.2.13. In the definition (3.2.11) of the valuation v, we compute the minimum over the terms in the initial support of f: these elements are a finite number and they only depends on x and f by Proposition 3.2.8. Therefore, this minimum provides a well-defined function on $\mathcal{O}_{\mathscr{X},x} \setminus \{0\}$. Nevertheless, it is equivalent to consider the minimum over all the terms of an admissible expansion of f, i.e. for any admissible expansion $f = \sum_{\gamma \in \mathcal{C}_{\mathscr{X},x}} c_\gamma \gamma$

$$\min\{\alpha(\gamma) \,|\, \gamma \in \Gamma_x(f)\} = \min\{\alpha(\gamma) \,|\, \gamma \in S\},$$

where $S = \{\gamma \in \mathcal{C}_{\mathscr{X},x} \,|\, c_\gamma \neq 0\}$ as in (3.2.7). Indeed, any element that belongs to S can be written as a sum of an element of the initial support of f and an element of $\mathcal{C}_{\mathscr{X},x}$. Since the morphism α is additive and takes positive real values, then the minimum is necessarily attained by the elements in the initial support.

(3.2.14) We will denote the valuation v from Proposition 3.2.10 by $v_{x,\alpha}$. Since $v_{x,\alpha}$ induces a real valuation on the function field of $\mathscr{X}_K$ that extends the discrete valuation v_K on K, it defines a point of the K-analytic space $\widehat{\mathscr{X}_\eta}$, which we will denote by the same symbol $v_{x,\alpha}$. We now show that the characterization of $v_{x,\alpha}$ in Proposition 3.2.10 implies that

$$v_{y,\alpha'} = v_{x,\alpha' \circ \tau_{x,y}}$$

for every y in $F(x)$ and every α' in σ_y.

Firstly we note that $\mathcal{O}_{\mathscr{X},y}$ is the localization of $\mathcal{O}_{\mathscr{X},x}$ with respect to the elements of $m \in \mathcal{M}_{\mathscr{X},x}$ in the kernel of $\tau_{x,y}$. Indeed, by construction of $\tau_{x,y}$, the kernel is given by

$$\ker(\tau_{x,y}) = \{s \in \mathcal{C}_{\mathscr{X},x} | s(y) \neq 0\};$$

to obtain $\mathcal{O}_{\mathscr{X},y}$ from $\mathcal{O}_{\mathscr{X},x}$, we localize by

$$S = \{a \in \mathcal{O}_{\mathscr{X},x} | a(y) \neq 0\};$$

therefore we can identify the set of elements in $\mathcal{M}_{\mathscr{X},x}$ in $\ker(\tau_{x,y})$ with the set S, recalling that for points in the Kato fan $\mathcal{C}_{\mathscr{X},x} \setminus \{1\}$ generates the maximal ideal of $\mathcal{O}_{\mathscr{X},x}$. Therefore we are dealing with these two morphisms:

$$\mathcal{O}_{\mathscr{X},x} \hookrightarrow S^{-1}\mathcal{O}_{\mathscr{X},x} = \mathcal{O}_{\mathscr{X},y},$$

$$\mathcal{C}_{\mathscr{X},x} \twoheadrightarrow \mathcal{C}_{\mathscr{X},x}/S = \mathcal{C}_{\mathscr{X},y}.$$

Let f be an element of $\mathcal{O}_{\mathscr{X},x}$. Under the notations of Lemma 3.2.3, we apply the lemma to $A = \mathcal{O}_{\mathscr{X},y}$ and $B = \mathcal{O}_{\mathscr{X},x}$, choosing a system of generators of $\mathfrak{m}_y$ in $\mathcal{C}_{\mathscr{X},x}$: we can find an admissible expansion of f of the form

$$f = \sum_{\delta \in \mathcal{C}_{\mathscr{X},y}} d_\delta \delta \quad \text{with } d_\delta \in (\mathcal{O}_{\mathscr{X},x} \cap \mathcal{O}^\times_{\mathscr{X},y}) \cup \{0\}.$$

Admissible expansions of coefficients d_δ induce an admissible expansion for f by

$$f = \sum_{\delta \in \mathcal{C}_{\mathscr{X},y}} \big(\sum_{\gamma \in S} c_{\gamma\delta}\gamma\big)\delta \quad \text{with } c_{\gamma\delta} \in \mathcal{O}^\times_{\mathscr{X},x} \cup \{0\},$$

where γ runs through the set S since $d_\delta \in \mathcal{O}^\times_{\mathscr{X},y}$. Thus we have

$$\begin{aligned} v_{y,\alpha'}(f) &= \min\{\alpha'(\delta) \mid \delta \in \Gamma_y(f)\} \\ &= \min\{\alpha' \circ \tau_{x,y}(\gamma\delta) \mid \delta \in \Gamma_y(f), \gamma \in S\} \\ &= \min\{\alpha' \circ \tau_{x,y}(\gamma\delta) \mid \gamma\delta \in \Gamma_x(f)\} \\ &= v_{x,\alpha' \circ \tau_{x,y}}(f). \end{aligned}$$

Hence, we obtain a well-defined map

$$\iota : \mathrm{Sk}(\mathscr{X}^+) \to \widehat{\mathscr{X}_\eta}$$

by sending α to $v_{x,\alpha}$ for every point x of F and every $\alpha \in \sigma_x$.

Proposition 3.2.15. *The map*

$$\iota : \mathrm{Sk}(\mathscr{X}^+) \to \widehat{\mathscr{X}_\eta}$$

is a topological embedding.

Proof. First, we show that ι is injective. Let x be a point of F and α an element of σ_x. Let y be the point of $F(x)$ corresponding to the face $\mathcal{C}_{\mathscr{X},x} \setminus \alpha^{-1}(0)$ of $\mathcal{C}_{\mathscr{X},x}$. Then α factors through an element

$$\alpha' : \mathcal{C}_{\mathscr{X},y} \to \mathbb{R}_{\geq 0}$$

of σ_y. Note that $\alpha = \alpha'$ in $\mathrm{Sk}(\mathscr{X}^+)$ because $\alpha = \alpha' \circ \tau_{x,y}$. Moreover, since $(\alpha')^{-1}(0) = \{1\}$, the center of the valuation $v_{y,\alpha'}$ is the point y, so that $\mathrm{red}_{\mathscr{X}}(v_{y,\alpha'}) = y$. Thus we can recover y from $v_{y,\alpha'}$. Then we can also reconstruct α' by looking at the values of $v_{y,\alpha'}$ at the elements of $\mathcal{M}_{\mathscr{X},y}$. We conclude that ι is injective.

Now, we show that ι is a homeomorphism onto its image. For any valuation v in $\mathrm{Sk}(\mathscr{X}^+)$ and any small open neighbourhood U of $\iota(v)$ in $\mathscr{X}_K^{\mathrm{an}}$, there exists a closed subset C in $\mathrm{Sk}(\mathscr{X}^+)$ such that $U \cap \iota(\mathrm{Sk}(\mathscr{X}^+)) \subseteq \iota(C)$ and, up to subdivisions, we can assume that the C is a closed cell of $\mathrm{Sk}(\mathscr{X}^+)$. Therefore, it suffices to prove that the restriction of ι to any closed cell σ_x of $\mathrm{Sk}(\mathscr{X}^+)$ is an homeomorphism. The restriction $\iota_{|\sigma_x}$ is an injective map from a compact set to the Hausdorff space $\widehat{\mathscr{X}_\eta}$, so we reduce to show that $\iota_{|\sigma_x}$ is continuous, to conclude that $\iota_{|\sigma_x}$ is a homeomorphism. By definition of the Berkovich topology, it is enough to prove that the map

$$\sigma_x \to \mathbb{R} : \alpha \mapsto v_{x,\alpha}(f)$$

is continuous for every f in $\mathcal{O}_{\mathscr{X},x}$. This is obvious from the formula (3.2.11).

check if argument is enough clear ∎

(3.2.16) From now on, we will view $\mathrm{Sk}(\mathscr{X}^+)$ as a topological subspace of $\mathscr{X}_K^{\mathrm{an}}$ by means of the embedding ι in Proposition 3.2.15. If $\mathscr{X}$ is regular over R and $\mathscr{X}_k$ is a divisor with strict normal crossings, the skeleton $\mathrm{Sk}(\mathscr{X}^+)$ was described in [MN15], Section 3.1.

3.3. **Contracting the generic fibre to the skeleton.**

(3.3.1) We denote by $D_{\mathscr{X},\mathrm{hor}}$ the component of $D_{\mathscr{X}}$ not contained in the special fibre $\mathscr{X}_k$. The inclusion $\iota : \mathrm{Sk}(\mathscr{X}^+) \to \widehat{\mathscr{X}_\eta} \cap (\mathscr{X}_K \setminus D_{\mathscr{X},\mathrm{hor}})^{\mathrm{an}}$ admits a continuous retraction

$$\rho_{\mathscr{X}} : \widehat{\mathscr{X}_\eta} \cap (\mathscr{X}_K \setminus D_{\mathscr{X},\mathrm{hor}})^{\mathrm{an}} \to \mathrm{Sk}(\mathscr{X}^+)$$

constructed as follows. Let x be a point of $\widehat{\mathscr{X}_\eta}$ and consider the reduction map

$$\mathrm{red}_{\mathscr{X}} : \widehat{\mathscr{X}_\eta} \to \mathscr{X}_k.$$

Let $E_1, \ldots, E_r$ be the irreducible components of $D_{\mathscr{X}}$ passing through the point $\mathrm{red}_{\mathscr{X}}(x)$. We denote by ξ the generic point of the connected component of $E_1 \cap \ldots \cap E_r$ that contains $\mathrm{red}_{\mathscr{X}}(x)$. By Lemma 2.2.3, ξ is a point in the associated Kato fan F. We set α to be the morphism of monoids

$$\alpha : \mathcal{C}_{\mathscr{X},\xi} \to \mathbb{R}_{\geqslant 0}$$

such that $\alpha(\overline{m}) = v_x(m)$ for any element m of $\mathcal{M}_{\mathscr{X},\xi}$. In particular $\alpha(\pi) = v_x(\pi) = 1$ as we assumed the normalization of all valuations in the Berkovich space. Then $\rho_{\mathscr{X}}(x)$ is the point of $\mathrm{Sk}(\mathscr{X}^+)$ corresponding to the couple (ξ, α). By construction $\rho_{\mathscr{X}}$ is continuous and right inverse to the inclusion ι.

(3.3.2) Given a dominant morphism $f : \mathscr{X}^+ \to \mathscr{Y}^+$ of integral flat separated log-regular S-schemes, it induces a map between the set of birational points $\mathrm{Bir}(\mathscr{X}_K) \to \mathrm{Bir}(\mathscr{Y}_K)$. As $\mathrm{Bir}(\mathscr{X}_K) \subseteq (\mathscr{X}_K \setminus D_{\mathscr{X},\mathrm{hor}})^{\mathrm{an}}$, we can employ the retraction ρ to define a map of skeleta as follows

$$\begin{tikzcd}
\widehat{\mathscr{X}_\eta} \cap \mathrm{Bir}(\mathscr{X}_K) \arrow[r, "\widehat{f}"] \arrow[d, shift right, "\rho_{\mathscr{X}}"'] & \widehat{\mathscr{Y}_\eta} \cap \mathrm{Bir}(\mathscr{Y}_K) \arrow[d, "\rho_{\mathscr{Y}}"] \\
\mathrm{Sk}(\mathscr{X}^+) \arrow[u, shift right, "\iota_{\mathscr{Y}}"'] \arrow[r, dotted] & \mathrm{Sk}(\mathscr{Y}^+).
\end{tikzcd}$$

This association makes the skeleton construction $\mathrm{Sk}(\mathscr{X}^+)$ functorial in $\mathscr{X}^+$.

3.4. **Skeleton of a fs fibred product.**

(3.4.1) Let $\mathscr{X}^+$ and $\mathscr{Y}^+$ be log-smooth log schemes over S^+, let $\mathscr{Z}^+$ be their fs fibred product. Let

$$\mathrm{Sk}(\mathscr{Z}^+) \to \mathrm{Sk}(\mathscr{X}^+) \times \mathrm{Sk}(\mathscr{Y}^+)$$

be the continuous map of skeleta functorially associated to the projections $\mathrm{pr}_{\mathscr{X}} : \mathscr{Z}^+ \to \mathscr{X}^+$ and $\mathrm{pr}_{\mathscr{Y}} : \mathscr{Z}^+ \to \mathscr{Y}^+$. We denote this map by $\big(\mathrm{pr}_{\mathrm{Sk}(\mathscr{X})}, \mathrm{pr}_{\mathrm{Sk}(\mathscr{Y})}\big)$ and we recall that it is constructed considering the diagram

$$\begin{tikzcd}
\widehat{\mathscr{Z}_\eta} \arrow[r, "(\widehat{\mathrm{pr}_{\mathscr{X}}}, \widehat{\mathrm{pr}_{\mathscr{Y}}})"] \arrow[d, shift right, "\rho_{\mathscr{Z}}"'] & \widehat{\mathscr{X}_\eta} \times \widehat{\mathscr{Y}_\eta} \arrow[d, "(\rho_{\mathscr{X}}, \rho_{\mathscr{X}})"] \\
\mathrm{Sk}(\mathscr{Z}^+) \arrow[u, shift right, "\iota_{\mathscr{Z}}"'] \arrow[r, "(\mathrm{pr}_{\mathrm{Sk}(\mathscr{X})}, \mathrm{pr}_{\mathrm{Sk}(\mathscr{Y})})"] & \mathrm{Sk}(\mathscr{X}^+) \times \mathrm{Sk}(\mathscr{Y}^+).
\end{tikzcd} \tag{3.4.2}$$

Proposition 3.4.3. *Assume that the residue field k is algebraically closed. If $\mathscr{X}^+$ is semistable, then the map $\left(\mathrm{pr}_{\mathrm{Sk}(\mathscr{X})}, \mathrm{pr}_{\mathrm{Sk}(\mathscr{Y})}\right)$ is a homeomorphism.*

Proof. The surjectivity of the map $\left(\mathrm{pr}_{\mathrm{Sk}(\mathscr{X})}, \mathrm{pr}_{\mathrm{Sk}(\mathscr{Y})}\right)$ follows from the commutativity of the diagram (3.4.2) and the surjectivity of $(\rho_{\mathscr{X}}, \rho_{\mathscr{X}}) \circ (\widehat{\mathrm{pr}_{\mathscr{X}}}, \widehat{\mathrm{pr}_{\mathscr{Y}}})$. To prove the injectivity of $\left(\mathrm{pr}_{\mathrm{Sk}(\mathscr{X})}, \mathrm{pr}_{\mathrm{Sk}(\mathscr{Y})}\right)$, we provide an explicit description of the map $\mathrm{pr}_{\mathrm{Sk}(\mathscr{X})}$.

We recall that the projection $\widehat{\mathrm{pr}_{\mathscr{X}}}$ is such that a valuation v on the function field $K(\mathscr{Z}_K)$ maps to the composition $v \circ i$ where $i : K(\mathscr{X}_K) \hookrightarrow K(\mathscr{Z}_K)$.

Let $v_{z,\varepsilon}$ be the valuation in $\mathrm{Sk}(\mathscr{Z}^+)$ corresponding to a couple (z, ε) with $z \in F_{\mathscr{Z}}$ and $\varepsilon \in \sigma_z$. We consider the morphism of associated Kato fans

$$F_{\mathscr{Z}} \to F_{\mathscr{X}} \times F_{\mathscr{Y}}$$

as established in Proposition 2.4.8. We denote respectively by $\mathrm{pr}_{F_{\mathscr{X}}}$ and $\mathrm{pr}_{F_{\mathscr{Y}}}$ the projection to the first and second factor. Then $\mathrm{pr}_{F_{\mathscr{X}}}(z)$ is a point in the associated Kato fan $F_{\mathscr{X}}$, that we denote by x. We consider the morphism of monoids

$$i_x : \mathcal{C}_{\mathscr{X},x} \to \mathcal{C}_{\mathscr{Z},z}$$

and the composition

$$\begin{aligned} \mathrm{pr}_{\mathscr{X}}(\varepsilon) : \qquad \mathcal{C}_{\mathscr{X},x} &\xrightarrow{i_x} \mathcal{C}_{\mathscr{Z},z} = (\mathcal{C}_{\mathscr{X},x} \oplus_{\mathbb{N}} \mathcal{C}_{\mathscr{Y},y})^{\mathrm{sat}} \xrightarrow{\varepsilon} \mathbb{R}_{\geq 0} \\ a &\longmapsto [a, 1] \longmapsto \varepsilon([a, 1]). \end{aligned}$$

It trivially satisfies $\varepsilon \circ i_x(\pi) = 1$. In order to conclude that it correctly defines a point in the skeleton $\mathrm{Sk}(\mathscr{X}^+)$, we need to check the compatibility with respect to the equivalence relation $\sim$. Indeed, suppose that $\varepsilon = \varepsilon' \circ \tau_{z,z'}$ for some $z' \in \overline{\{z\}}$. We denote by x' the projection of z' under the local isomorphism of associated Kato fans. The diagram

$$\begin{tikzcd}
\mathcal{C}_{\mathscr{X},x} \arrow[r, "i_x"] \arrow[d, "\tau_{x,x'}"] & \mathcal{C}_{\mathscr{Z},z} \arrow[d, "\tau_{z,z'}"] \arrow[dr, "\varepsilon"] & \\
\mathcal{C}_{\mathscr{X},x'} \arrow[r, "i_{x'}"] & \mathcal{C}_{\mathscr{Z},z'} \arrow[r, "\varepsilon'"'] & \mathbb{R}_{\geq 0}
\end{tikzcd}$$

is commutative as made up by a commutative square and a commutative triangle of arrows. Therefore, by commutativity

$$\mathrm{pr}_{\mathscr{X}}(\varepsilon) = \mathrm{pr}_{\mathscr{X}}(\varepsilon') \circ \tau_{x,x'}$$

and this implies that $\mathrm{pr}_{\mathscr{X}}(\varepsilon)$ defines a well-defined point $v_{x,\mathrm{pr}_{\mathscr{X}}(\varepsilon)}$ of $\mathrm{Sk}(\mathscr{X}^+)$.

We claim that $v_{x,\mathrm{pr}_{\mathscr{X}}(\varepsilon)}$ is indeed the image of $v_{z,\varepsilon}$ under the map $\mathrm{pr}_{\mathrm{Sk}(\mathscr{X})}$, hence that the equality in the following inner diagram holds

$$\begin{tikzcd}
\widehat{\mathscr{Z}}_\eta \arrow[rrr, "\widehat{\mathrm{pr}_{\mathscr{X}}}"] \arrow[ddd, shift right, "\rho_{\mathscr{Z}}"'] & & & \widehat{\mathscr{X}}_\eta \arrow[ddd, "\rho_{\mathscr{X}}"] \\
& v_{z,\varepsilon} \arrow[r, mapsto] & v_{z,\varepsilon} \circ i \arrow[d, mapsto] & \\
& v_{z,\varepsilon} \arrow[u, mapsto] \arrow[r, mapsto] & \rho_{\mathscr{X}}(v_{z,\varepsilon} \circ i) = v_{x,\mathrm{pr}_{\mathscr{X}}(\varepsilon)} & \\
\mathrm{Sk}(\mathscr{Z}^+) \arrow[uuu, shift right, "\iota_{\mathscr{Z}}"'] \arrow[rrr, "\mathrm{pr}_{\mathrm{Sk}(\mathscr{X})}"] & & & \mathrm{Sk}(\mathscr{X}^+).
\end{tikzcd}$$

We denote $\rho_{\mathscr{X}}(v_{z,\varepsilon} \circ i)$ by (x, α) as a point of $\mathrm{Sk}(\mathscr{X}^+)$. By definition of the retraction $\rho_{\mathscr{X}}$, the morphism α is characterized by the fact that $\alpha(\overline{m}) = (v_{z,\varepsilon} \circ i)(m)$ for any m in $\mathcal{M}_{\mathscr{X},x}$ and then we have

$$\alpha(\overline{m}) = (v_{z,\varepsilon} \circ i)(m) = v_{z,\varepsilon}(m) = \varepsilon(\overline{m}).$$

On the other hand, for any m in $\mathcal{M}_{\mathscr{X},x}$

$$v_{x,\mathrm{pr}_{\mathscr{X}}(\varepsilon)}(m) = \mathrm{pr}_{\mathscr{X}}(\varepsilon)(\overline{m}) = \varepsilon(\overline{m})$$

hence we obtain that α coincide with the morphism $\mathrm{pr}_{\mathscr{X}}(\varepsilon)$. It means that their associated points $\rho_{\mathscr{X}}(v_{z,\varepsilon} \circ i)$ and $v_{x,\mathrm{pr}_{\mathscr{X}}(\varepsilon)}$ coincide in $\mathrm{Sk}(\mathscr{X}^+)$.

Given a pair of points in $\mathrm{Sk}(\mathscr{X}^+) \times \mathrm{Sk}(\mathscr{Y}^+)$, we know by surjectivity of $\left(\mathrm{pr}_{\mathrm{Sk}(\mathscr{X})}, \mathrm{pr}_{\mathrm{Sk}(\mathscr{Y})}\right)$ that they are of the form

$$\left(v_{x,\mathrm{pr}_{\mathscr{X}}(\varepsilon)}, v_{y,\mathrm{pr}_{\mathscr{Y}}(\varepsilon)}\right).$$

The assumptions of semistability of $\mathscr{X}^+$ and algebraic closedness of k guarantee that there is a unique z in $F_{\mathscr{Z}}$ in the fibre of x and y, by Proposition 2.5.4 and Remark 3.1.4. Moreover, we can uniquely reconstruct ε by looking at the values of $v_{x,\mathrm{pr}_{\mathscr{X}}(\varepsilon)}$ at the elements of $\mathcal{M}_{\mathscr{X},x}$ and respectively of $v_{y,\mathrm{pr}_{\mathscr{Y}}(\varepsilon)}$ at the elements of $\mathcal{M}_{\mathscr{Y},y}$. We conclude that $\left(\mathrm{pr}_{\mathrm{Sk}(\mathscr{X})}, \mathrm{pr}_{\mathrm{Sk}(\mathscr{Y})}\right)$ is injective. $\square$

The assumption of semistability is crucial in the result of Proposition 3.4.3. To see this, it is helpful to consider an example.

Example 3.4.4. Let q be the equation a generic quartic curve in $\mathbb{P}^2_{\mathbb{C}((t))}$. Then $\mathscr{X} : tq + x^2y^2 = 0$ gives the equation of a family of genus 3 curves, degenerating to two double lines. The dual complex $\mathcal{D}(\mathscr{X}_{\mathbb{C}})$ of the special fibre $\mathscr{X}_{\mathbb{C}}$ is a line segment and $\mathscr{X}$ has 4 singularities of type A_1 in each component of the special fibre, corresponding to the base points of the family. In this case taking a semistable model of $\mathscr{X}_{\mathbb{C}((t))}$ requires an order two base change, which induces coverings branched at each of these singular points (see [HM98], p.133 for details). Let $\mathscr{Y}$ be such a semistable reduction. Thus the special fibre of $\mathscr{Y}$ consists of two elliptic curves, call them E_1 and E_2, which intersect in two points, p_A and p_B, which are the preimages of the point $(0:0:1)$. The dual complex $\mathcal{D}(\mathscr{Y}_{\mathbb{C}})$ of the special fibre $\mathscr{Y}_{\mathbb{C}}$ is isomorphic to S^1.

We will compare the dual complex of $(\mathscr{X} \times_R \mathscr{X})_{\mathbb{C}}$ with that of $(\mathscr{Y} \times_R \mathscr{Y})_{\mathbb{C}}$. The models $\mathscr{X}$ and $\mathscr{Y}$ are not log-regular at every point, but from our prospective it is enough that they are log-regular at the generic point of each stratum. For the product with a semistable model, the dual complex is the product of the dual complexes, and $\mathcal{D}((\mathscr{Y} \times_R \mathscr{Y})_{\mathbb{C}})$ is therefore a real 2-torus $S^1 \times S^1$.

On the other hand, the dual complex of the product $(\mathscr{X} \times_R \mathscr{X})_{\mathbb{C}}$ is given by a quotient of $S^1 \times S^1$ by the action of $\mathbb{Z}/2\mathbb{Z}$. $\mathcal{D}((\mathscr{Y} \times_R \mathscr{Y})_{\mathbb{C}})$ has the structure of a cell complex, whose cells correspond to ordered pairs of strata in $\mathcal{D}(\mathscr{Y}_{\mathbb{C}})$, so

zero-dimensional strata:	$(E_1, E_2), (E_1, E_1), (E_2, E_1), (E_2, E_2)$
one-dimensional strata:	$(E_1, p_A), (E_1, p_B), (E_2, p_A), (E_2, p_B)$
	$(p_A, E_1), (p_B, E_1), (p_A, E_2), (p_B, E_2)$
two-dimensional strata:	$(p_A, p_A), (p_A, p_B), (p_B, p_A), (p_B, p_B)$.

The action of $\mathbb{Z}/2$ fixes E_1 and E_2, while switching p_A and p_B. Therefore it fixes exactly the zero dimensional strata while acting freely on the other points. The quotient, the complex $\mathcal{D}((\mathscr{X} \times_R \mathscr{X})_{\mathbb{C}})$, is piecewise-linearly homeomorphic to the sphere S^2. In particular, it is not isomorphic to the product of two line segments.

4. The weight function for pairs

4.1. Weight function associated to a logarithmic pluricanonical form.

(4.1.1) Let X be a connected, smooth and proper K-variety of dimension n. We introduce the following notation: for any log-regular model $\mathscr{X}^+$ of X, for any point $x = (\xi_x, |\cdot|_x) \in \widehat{\mathscr{X}_\eta}$ and for any divisor D on $\mathscr{X}^+$ whose support does not contain ξ_x, we set

$$v_x(D) = -\ln|f(x)|$$

where f is any element of $K(X)^\times$ such that $D = \mathrm{div}(f)$ locally at $\mathrm{red}_{\mathscr{X}}(x)$.

(4.1.2) Let (X, Δ_X) be a pair where Δ_X is an effective $\mathbb{Q}$-divisor such that $\Delta_X = \sum a_i \Delta_{X,i}$ has $0 \leqslant a_i \leqslant 1$, and $X^+ = (X, \lceil \Delta_X \rceil)$ is a log-regular log scheme over K. Let ω be a regular m-pluricanonical form on X^+ with poles of order at most ma_i along $\Delta_{X,i}$, for some m such that $ma_i \in \mathbb{N}$ for any i. We call such forms Δ_X-logarithmic m-pluricanonical forms.

Moreover, ω is a regular section of the logarithmic m-pluricanonical bundle of X^+ and for each log-regular model $\mathscr{X}^+$ of X^+, the form ω defines a divisor $\mathrm{div}_{\mathscr{X}^+}(\omega)$ on $\mathscr{X}^+$. Note that the multiplicity in $\mathrm{div}_{\mathscr{X}^+}(\omega)$ of the closure $\overline{\Delta_{X,i}}$ in $\mathscr{X}$ of $\Delta_{X,i}$ is at least $m(1-a_i)$.

(4.1.3) For any point x in $\mathrm{Sk}(\mathscr{X}^+)$ that is associated to an irreducible component of $\mathscr{X}_k$, the value

$$v_x(\mathrm{div}_{\mathscr{X}^+}(\omega)) + m \tag{4.1.4}$$

does not depend on the choice of the model $\mathscr{X}^+$; indeed, this follows from the same arguments of [MN15], Proposition 4.2.4. We denote this value by $\mathrm{wt}_\omega(x)$ and call it the weight of x with respect to ω. Since any snc model $\mathscr{X}$ of X can be turned by resolution of singularities (check right assumption, [MN15], 4.4.1?) into a log-regular model of X^+, the formula 4.1.4 computes the weight with respect to ω at every divisorial point of X^{an}. Thus, we obtain a function

$$\mathrm{wt}_\omega : \mathrm{Div}(X) \to \mathbb{Q}, \quad x \mapsto \mathrm{wt}_\omega(x)$$

on the set of divisorial points, called the weight function associated to ω. From the definition, we remark that for every integer $d > 0$

$$\mathrm{wt}_{\omega^{\otimes d}}(x) = d \cdot \mathrm{wt}_\omega(x),$$

and for every non-zero rational function f on X

$$\mathrm{wt}_{f\omega}(x) = \mathrm{wt}_\omega + v_x(f).$$

We prove that the formula 4.1.4 actually expresses the weight of every divisorial point in $\mathrm{Sk}(\mathscr{X}^+)$.

Proposition 4.1.5. *If x is a divisorial point in $\mathrm{Sk}(\mathscr{X}^+)$, then*

$$\mathrm{wt}_\omega(x) = v_x(\mathrm{div}_{\mathscr{X}^+}(\omega)) + m.$$

Proof. If the point x is associated to an irreducible component of the special fibre, then the equality holds by definition. As in [MN15], Proposition 2.4.11, any divisorial point can be reduced to such a representation by a finite sequence of blow-ups of strata of $D_{\mathscr{X}}$. Therefore, it suffices to consider one blow-up morphism $h : \mathscr{Y} \to \mathscr{X}$ of a stratum Z of $D_{\mathscr{X}}$ and check that

$$v_x(\operatorname{div}_{\mathscr{X}^+}(\omega)) + m = v_x(\operatorname{div}_{\mathscr{Y}^+}(\omega)) + m,$$

where $\mathscr{Y}^+$ is the log-regular log scheme with $D_{\mathscr{Y}}$ equals to the sum of the closure of the strict trasform $\lceil \Delta_Y \rceil$ of $\lceil \Delta_X \rceil$ and the special fibre $\mathscr{Y}_k$.

We denote by E the exceptional divisor of the blow-up h, by $r = r_h + r_v$ the codimension of Z in $\mathscr{X}$, where r_h and r_v are the number of irreducible components of $\overline{\lceil \Delta_X \rceil}$ and respectively of the special fibre $\mathscr{X}_k$, containing Z. We denote the projections onto S^+ by $s_{\mathscr{X}} : \mathscr{X}^+ \to S^+$ and $s_{\mathscr{Y}} : \mathscr{Y}^+ \to S^+$ and by π a uniformizer in R. Then we have that

$$\begin{aligned} h^*(\omega^{\log}_{\mathscr{X}^+/S^+}) &= h^*\big(\omega_{\mathscr{X}/R} \otimes \mathcal{O}_{\mathscr{X}}(\overline{\lceil \Delta_X \rceil}_{,\mathrm{red}} + \mathscr{X}_{k,\mathrm{red}} - s^*_{\mathscr{X}}(\pi))\big) \\ &= \omega_{\mathscr{Y}/R} \otimes \mathcal{O}_{\mathscr{Y}}((1-r)E) \otimes \mathcal{O}_{\mathscr{Y}}(\overline{\lceil \Delta_Y \rceil}_{,\mathrm{red}} + r_h E + \mathscr{Y}_{k,\mathrm{red}} + (r_v - 1)E - s^*_{\mathscr{Y}}(\pi)) \\ &= \omega_{\mathscr{Y}/R} \otimes \mathcal{O}_{\mathscr{Y}}(\overline{\lceil \Delta_Y \rceil}_{,\mathrm{red}} + \mathscr{Y}_{k,\mathrm{red}} - s^*_{\mathscr{Y}}(\pi)) = \omega^{\log}_{\mathscr{Y}^+/S^+}. \end{aligned}$$

This implies that

$$v_x(\operatorname{div}_{\mathscr{X}^+}(\omega)) + m = v_x\big(h^*(\operatorname{div}_{\mathscr{X}^+}(\omega))\big) + m = v_x(\operatorname{div}_{\mathscr{Y}^+}(\omega)) + m$$

and concludes the proof. ☐

(4.1.6) We define a function

$$\operatorname{wt}_{\mathscr{X}^+,\omega} : \operatorname{Sk}(\mathscr{X}^+) \to \mathbb{R}, \quad x \mapsto v_x(\operatorname{div}_{\mathscr{X}^+}(\omega)) + m$$

on the skeleton associated to a log-regular model $\mathscr{X}^+$ and we call it the weight function associated to ω and $\mathscr{X}^+$. By Proposition 4.1.5, if x is a divisorial point of $\operatorname{Sk}(\mathscr{X}^+)$, then the weight at x associated to ω and $\mathscr{X}^+$ is actually independent on the choice of the model and equal to $\operatorname{wt}_\omega(x)$. From the definition, we remark that for each point x in $\operatorname{Sk}(\mathscr{X}^+)$, for every integer $d > 0$

$$\operatorname{wt}_{\omega^{\otimes d}}(x) = d \cdot \operatorname{wt}_\omega(x),$$

and for every non-zero rational function f on X

$$\operatorname{wt}_{f\omega}(x) = \operatorname{wt}_\omega + v_x(f).$$

Following the arguments of Lemma 4.2.8 and Proposition 4.3.4 in [MN15], we can prove that there exists a unique function on the set of birational points of X^{an}

$$\operatorname{wt}_\omega : \operatorname{Bir}(X) \to \mathbb{R}, \text{ such that } \operatorname{wt}_\omega(x) = \operatorname{wt}_{\mathscr{X}^+,\omega}(x)$$

for every birational point x and every log-regular model $\mathscr{X}^+$ such that $x \in \operatorname{Sk}(\mathscr{X}^+)$. Moreover, after the arguments of Proposition 4.4.5 in [MN15], we obtain a function on the Berkovich space X^{an} by setting

$$\operatorname{wt}_\omega(x) = \sup_{\mathscr{X}^+}\{\operatorname{wt}_\omega(\rho_{\mathscr{X}}(x))\} \in \mathbb{R} \cup \{+\infty\}$$

where $\mathscr{X}^+$ runs through the set of all log-regular model $\mathscr{X}^+$ of X^+. We have that

$$\operatorname{wt}_\omega(x) = v_x(\operatorname{div}_{\mathscr{X}^+}(\omega)) + m \tag{4.1.7}$$

for every birational point x and every log-regular model $\mathscr{X}^+$ such that $x \in \operatorname{Sk}(\mathscr{X}^+)$. We call it the weight function associated to ω.

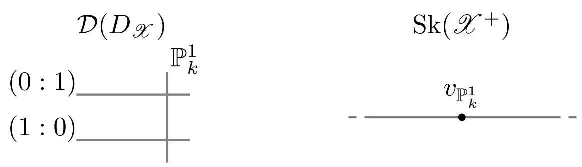
(4.1.8) Similarly to [MN15], Section 4.5, we define the Kontsevich-Soibelman skeleton $\mathrm{Sk}(X, \Delta_X, \omega)$ as the closure of the set of divisorial points of X^{an} where the weight function wt_ω reaches its minimal value, denoted by $\mathrm{wt}_\omega(X^+)$.

The two following examples show that the generalization of the weight function to a variety with boundary may allow to construct Kontsevich-Soibelman skeleta for varieties with Kodaira dimension $-\infty$. Furthermore, we will notice that only the components of Δ_X with coefficient $a_i = 1$ determine strata that are contained in the Kontsevich-Soibelma skeleta.

Example 4.1.9. Let X be the projective line $\mathbb{P}^1_K$ with affine coordinates x and y, and $\Delta_X = (0:1) + (1:0)$. Then $a_i = 1$ for any i and there exist Δ_X-logarithmic canonical forms. For example, we consider

$$\omega = \frac{dx}{x} = \frac{dy}{y}.$$

Let $\mathscr{X} = \mathbb{P}^1_R$ and $D_{\mathscr{X}} = (0:1) + (1:0) + \mathbb{P}^1_k$. The log scheme $\mathscr{X}^+ = (\mathscr{X}, D_{\mathscr{X}})$ is a log-regular model of $X^+ = (X, \lceil \Delta_X \rceil)$ and the associated skeleton $\mathrm{Sk}(\mathscr{X}^+)$ looks like this:

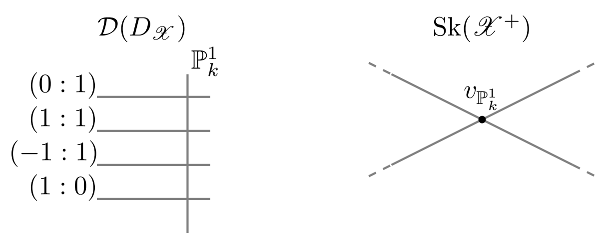


Since $\mathrm{div}_{\mathscr{X}^+}(\omega) = 0$, the weight associated to ω is minimal at any point of the skeleton $\mathrm{Sk}(\mathscr{X}^+)$. Thus $\mathrm{Sk}(X, \Delta_X, \omega) = \mathrm{Sk}(\mathscr{X}^+)$.

Example 4.1.10. Let $X = \mathbb{P}^1_K$ and $\Delta_X = \frac{1}{2}(0:1) + \frac{1}{2}(1:0) + \frac{1}{2}(1:1) + \frac{1}{2}(-1:1)$. So $a_i = \frac{1}{2}$ for any i and there exist Δ_X-logarithmic 2-pluricanonical forms. We set

$$\omega = \frac{1}{x+1} \cdot \frac{1}{x-1} \cdot \frac{1}{x}(dx)^2 = \frac{1}{y+1} \cdot \frac{1}{1-y} \cdot \frac{1}{y}(dy)^2.$$

We consider $\mathscr{X} = \mathbb{P}^1_R$ and $D_{\mathscr{X}} = (0:1) + (1:0) + (1:1) + (-1:1) + \mathbb{P}^1_k$, then $\mathscr{X}^+ = (\mathscr{X}, D_{\mathscr{X}})$ is a log-regular model of $X^+ = (X, \lceil \Delta_X \rceil)$ and $\mathrm{Sk}(\mathscr{X}^+)$ is



Since $\mathrm{div}_{\mathscr{X}^+}(\omega) = (0:1) + (1:0) + (1:1) + (-1:1)$, the weight associated to ω is minimal at the divisorial point $v_{\mathbb{P}^1_k}$ corresponding to $\mathbb{P}^1_k$ and is strictly increasing with slope 1 along the unbounded edges, when we move away from the point $v_{\mathbb{P}^1_k}$. Therefore, $\mathrm{Sk}(X, \Delta_X, \omega) = \{v_{\mathbb{P}^1_k}\}$.

4.2. **Weight function for log-regular models.**

(4.2.1) In case the divisor Δ_X is empty, the forms we considered in the previous paragraph to construct the weight functions are simply the sections of the m-pluricanonical line bundle $\omega_{X/K}^{\otimes m}$ on X, for some $m > 0$. Given a m-pluricanonical form ω on X, we check that the weight function wt_ω defined as in (4.1.7) coincides with the definition of the weight function associated to ω according to [MN15]. This results in a generalized formula for the usual weight of ω at the birational points, in terms of their representations in skeleta associated to log-regular models of X.

Proposition 4.2.2. *Let $\mathscr{X}$ be a model of X over R such that $\mathscr{X}^+$ is log-regular over S^+. If x is a point of $\mathrm{Sk}(\mathscr{X}^+)$, then the weight of ω at x as in [MN15] is given by*

$$\mathrm{wt}_\omega(x) = v_x(\mathrm{div}_{\mathscr{X}^+}(\omega)) + m.$$

Proof. In order to compute the weight of ω at x according to the definition in [MN15], we can consider any snc model $\mathscr{Y}$ of X such that x is a point of $\mathrm{Sk}(\mathscr{Y})$. Then, by [NX16], Section 3.2.2, the weight is given by

$$v_x(\mathrm{div}_{\mathscr{Y}^+}(\omega)) + m.$$

As we noticed in Remark 2.3.2, we can obtain an snc model $\mathscr{Y}$ adapted to x by means of a log blow-up $h : \mathscr{Y}^+ \to \mathscr{X}^+$ of $\mathscr{X}^+$ (Propositions 2.2.6 and 2.3.1). Moreover, the corresponding skeleton $\mathrm{Sk}(\mathscr{Y}^+)$ is given by a subdivision of $\mathrm{Sk}(\mathscr{X}^+)$ (Proposition 2.1.6) and coincides with $\mathrm{Sk}(\mathscr{Y})$. Therefore it suffices to prove that $v_x(\mathrm{div}_{\mathscr{Y}^+}(\omega)) = v_x(\mathrm{div}_{\mathscr{X}^+}(\omega))$ for such a model $\mathscr{Y}^+$.

Log blow-ups are log-étale morphisms ([Sai04], Section 2.1) and for log-étale morphisms the sheaf of log differentials is stable under pullback ([Kat94], Proposition 3.12), therefore

$$h^*\omega^{\log}_{\mathscr{X}^+/S^+} \simeq \omega^{\log}_{\mathscr{Y}^+/S^+}.$$

Then $\mathrm{div}_{\mathscr{Y}^+}(\omega) = h^*\mathrm{div}_{\mathscr{X}^+}(\omega)$ and in particular for points x of the skeleton $\mathrm{Sk}(\mathscr{X}^+) = \mathrm{Sk}(\mathscr{Y})$, it holds that

$$v_x(\mathrm{div}_{\mathscr{Y}^+}(\omega)) = v_x(h^*\mathrm{div}_{\mathscr{X}^+}(\omega)) = v_x(\mathrm{div}_{\mathscr{X}^+}(\omega)).$$

□

4.3. Weight function and Kontsevich-Soibelman skeleton for products.

(4.3.1) Let $\mathscr{X}^+$ and $\mathscr{Y}^+$ be log-smooth models over S^+ of $X^+ = (X, \lceil\Delta_X\rceil)$ and $Y^+ = (Y, \lceil\Delta_Y\rceil)$ respectively. Then, the fs fibred product $\mathscr{Z}^+ = \mathscr{X}^+ \times^{\mathrm{fs}}_{S^+} \mathscr{Y}^+$ is a log-regular model of $Z^+ := X^+ \times^{\mathrm{fs}}_K Y^+$. Therefore, given ω_{X^+} and ω_{Y^+} Δ_X-logarithmic and Δ_Y-logarithmic m-pluricanonical forms on (X, Δ_X) and (Y, Δ_Y) respectively, the form

$$\varpi = \mathrm{pr}^*_{X^+}\omega_{X^+} \otimes \mathrm{pr}^*_{Y^+}\omega_{Y^+}$$

is a Δ_Z-logarithmic m-pluricanonical form on (Z, Δ_Z), where $\Delta_Z = X \times_K \Delta_Y + \Delta_X \times_K Y$. Viewing these forms as rational sections of logarithmic m-pluricanonical bundles, we see that $\mathrm{div}_{\mathscr{Z}^+}(\varpi) = \mathrm{div}_{\mathscr{Z}^+}(\mathrm{pr}^*_{X^+}\omega_{X^+} \otimes \mathrm{pr}^*_{Y^+}\omega_{Y^+})$ according to (2.4.6).

(4.3.2) Let z be a point of $F_{\mathscr{Z}}$; as before, we denote by x and y the images of z under the local isomorphism $F_{\mathscr{Z}} \to F_{\mathscr{X}} \times F_{\mathscr{Y}}$. Any morphism $\varepsilon \in \sigma_z$ defines a point $v_{z,\varepsilon}$ in $\mathrm{Sk}(\mathscr{Z}^+)$. For the sake of convenience, we simply denote the valuations by the corresponding morphisms and we denote $\alpha = \mathrm{pr}_{\mathscr{X}}(\varepsilon)$ and $\beta = \mathrm{pr}_{\mathscr{Y}}(\varepsilon)$. We aim to relate the valuation $v_\varepsilon\big(\mathrm{div}_{\mathscr{Z}^+}(\varpi)\big)$ to the values

$$v_\alpha\big(\mathrm{div}_{\mathscr{X}^+}(\omega_{X^+})\big)\ ,\ v_\beta\big(\mathrm{div}_{\mathscr{Y}^+}(\omega_{Y^+})\big).$$

(4.3.3) Let $f_x \in \mathcal{O}_{\mathscr{X},x}$ be a local equation of $\operatorname{div}_{\mathscr{X}^+}(\omega_{X^+})$ around x. In order to evaluate $v_{x,\alpha}$ on f_x, we consider an admissible expansion of f_x as in (3.2.6)

$$f_x = \sum_{\gamma \in \mathcal{C}_{\mathscr{X},x}} c_\gamma \gamma.$$

Furthermore, this expansion induces also an expansion of $\mathrm{pr}_{\mathscr{X}}^*(f_x)$ by

$$\mathrm{pr}_{\mathscr{X}}^*(f_x) = \sum_{\gamma \in \mathcal{C}_{\mathscr{X},x}} \mathrm{pr}_{\mathscr{X}}^*(c_\gamma)\gamma$$

as formal power series in $\widehat{\mathcal{O}}_{\mathscr{Z},z}$, since the morphism of characteristic sheaves $\mathcal{C}_{\mathscr{X},x} \hookrightarrow \mathcal{C}_{\mathscr{Z},z}$ is injective. Following the same procedure for a local equation $f_y \in \mathcal{O}_{\mathscr{Y},y}$ of $\operatorname{div}_{\mathscr{Y}^+}(\omega_{Y^+})$ around y, we get an expansion of f_y that extends to $\mathrm{pr}_{\mathscr{Y}}^*(f_y)$:

$$f_y = \sum_{\delta \in \mathcal{C}_{\mathscr{Y},y}} d_\delta \delta.$$

(4.3.4) A local equation of ϖ around z is determined by $\mathrm{pr}_{\mathscr{X}}^*(f_x)\ \mathrm{pr}_{\mathscr{Y}}^*(f_y)$. Thus

$$v_\varepsilon(\operatorname{div}_{\mathscr{Z}^+}(\varpi)) = v_\varepsilon\big(\mathrm{pr}_{\mathscr{X}}^*(f_x)\ \mathrm{pr}_{\mathscr{Y}}^*(f_y)\big)$$

and by multiplicativity of the valuation v_ε

$$v_\varepsilon\big(\mathrm{pr}_{\mathscr{X}}^*(f_x)\ \mathrm{pr}_{\mathscr{Y}}^*(f_y)\big) = v_\varepsilon\big(\mathrm{pr}_{\mathscr{X}}^*(f_x)\big) + v_\varepsilon\big(\mathrm{pr}_{\mathscr{Y}}^*(f_y)\big).$$

Recalling Remark 3.2.13, the valuation can be computed as follows

$$v_\varepsilon\big(\mathrm{pr}_{\mathscr{X}}^*(f_x)\big) = \min\{\varepsilon(\gamma) \,|\, c_\gamma \neq 0\};$$

as the elements γ belong to $\mathcal{C}_{\mathscr{X},x}$ and α is defined to be $\mathrm{pr}_{\mathscr{X}}(\varepsilon)$

$$\min\{\varepsilon(\gamma) \,|\, c_\gamma \neq 0\} = \min\{\alpha(\gamma) \,|\, c_\gamma \neq 0\} = v_{x,\alpha}(f_x).$$

Hence, we conclude that

$$\begin{aligned} v_\varepsilon\big(\operatorname{div}_{\mathscr{Z}^+}(\varpi)\big) &= v_\varepsilon\big(\mathrm{pr}_{\mathscr{X}}^*(f_x)\big) + v_\varepsilon\big(\mathrm{pr}_{\mathscr{Y}}^*(f_y)\big) \\ &= v_\alpha(f_x) + v_\beta(f_y) \\ &= v_\alpha\big(\operatorname{div}_{\mathscr{X}^+}(\omega_{X^+})\big) + v_\beta\big(\operatorname{div}_{\mathscr{Y}^+}(\omega_{Y^+})\big). \end{aligned} \tag{4.3.5}$$

(4.3.6) This result turns out to be advantageous to compute the weight function wt_ϖ on divisorial points of $\mathrm{Sk}(\mathscr{Z}^+)$:

$$\begin{aligned} \mathrm{wt}_\varpi(\varepsilon) &= v_\varepsilon\big(\operatorname{div}_{\mathscr{Z}^+}(\varpi)\big) + m \\ &= v_\alpha\big(\operatorname{div}_{\mathscr{X}^+}(\omega_{X^+})\big) + v_\beta\big(\operatorname{div}_{\mathscr{Y}^+}(\omega_{Y^+})\big) + m \\ &= \mathrm{wt}_{\omega_{X^+}}(\alpha) + \mathrm{wt}_{\omega_{Y^+}}(\beta) - m. \end{aligned} \tag{4.3.7}$$

(4.3.8) Under the notations of the previous paragraphs, our computations lead to the following result.

Theorem 4.3.9. *Suppose that the residue field k is algebraically closed and that $\mathscr{X}^+$ is semistable. Then, the homeomorphism of skeleta*

$$\mathrm{Sk}(\mathscr{Z}^+) \xrightarrow{\sim} \mathrm{Sk}(\mathscr{X}^+) \times \mathrm{Sk}(\mathscr{Y}^+)$$

given in Proposition 3.4.3 restricts to a homeomorphism of Kontsevich-Soibelman skeleta

$$\mathrm{Sk}(Z, \Delta_Z, \varpi) \xrightarrow{\sim} \mathrm{Sk}(X, \Delta_X, \omega_{X^+}) \times \mathrm{Sk}(Y, \Delta_Y, \omega_{Y^+}).$$

Proof. This follows immediately from the equality (4.3.7) that shows that a point in $\mathrm{Sk}(Z,\Delta_Z)$ has minimal value $\mathrm{wt}_{\varpi}(Z,\Delta_Z)$ if and only if its projections have minimal value $\mathrm{wt}_{\omega_{X^+}}(X,\Delta_X)$ and $\mathrm{wt}_{\omega_{Y^+}}(Y,\Delta_Y)$. □

5. The essential skeleton of a product

We will need a few notions from birational geometry, see [KM08]. Let (X,Δ_X) be a pair such that X is normal, Δ_X is effective, and $K_X+\Delta_X$ is $\mathbb{Q}$-Cartier. Then we say that (X,Δ_X) is log canonical if for every log resolution $f\colon Z\to X$, in the formula

$$K_Z+\Delta_Z = f^*(K_X+\Delta_X)+\sum a_D D,$$

where Δ_Z is the strict transform of Δ_X plus the reduced exceptional divisor, all the a_D are nonnegative. The sum ranges over all components of Δ_Z. In fact the quantity a_D depends only on the valuation corresponding to D, and this condition needs only be tested on a single log resolution.

We will be most interested in the case where (X,Δ_X) satisfies the stronger condtion of being divisorially log terminal, or dlt. A closed subset $Y\subset X$ is called a log canonical center if for some (respectively any) log resolution, Y is the image of a divisor D with $a_D=0$. The pair (X,Δ_X) is said to be dlt if for every log canonical center Y, there is a neighborhood of the generic point of Y where (X,Δ_X) is snc.

5.1. **Essential skeleton of a pair.**

(5.1.1) Let (X,Δ_X) be a pair such that $X^+=(X,\lceil\Delta_X\rceil)$ is a log-regular log scheme. Let ω be a non-zero regular Δ_X-logarithmic m-pluricanonical form on (X,Δ_X). Let $\mathscr{X}^+$ be a log-smooth model of X^+ over S^+. Then, the associated divisor $\mathrm{div}_{\mathscr{X}^+}(\omega)$ is effective. There exist unique positive integers d and n such that the divisor $\mathrm{div}_{\mathscr{X}^+}(\omega^{\otimes d}\pi^{-n})$ is effective and the multiplicity of some component of the special fibre is zero: we denote this divisor by $D_{\min}(\mathscr{X},\omega)$. It follows from the properties of the weight function (Paragraph 4.1.6) that for any $x\in\mathrm{Sk}(\mathscr{X}^+)$

$$v_x(D_{\min}(\mathscr{X},\omega))+dm=\mathrm{wt}_{\omega^{\otimes d}\pi^{-n}}(x)=d\cdot\mathrm{wt}_\omega+v_x(\pi^{-n})=d\cdot\mathrm{wt}_\omega(x)-n.$$

Lemma 5.1.2. *Let $v_{x,\alpha}$ be a divisorial point in $\mathrm{Sk}(\mathscr{X}^+)$, then $v_{x,\alpha}\in\mathrm{Sk}(X,\Delta_X,\omega)$ if and only if $D_{\min}(\mathscr{X},\omega)$ does not contain x.*

Proof. We denote $v_{x,\alpha}$ simply by α. By the above series of equalities, the weight function wt_ω reaches its minimum at α if and only if $v_\alpha(D_{\min}(\mathscr{X},\omega))$ is minimal, hence in particular equal to zero.

Let $h:\mathscr{Y}^+\to\mathscr{X}^+$ be a sequence of blow-up morphisms of strata of $D_{\mathscr{X}}$ such that α corresponds to an irreducible component E of $D_{\mathscr{Y}}$. As in the proof of Proposition 4.1.5

$$h^*(D_{\min}(\mathscr{X},\omega))=h^*(\mathrm{div}_{\mathscr{X}^+}(\omega^{\otimes d}\pi^{-n}))=\mathrm{div}_{\mathscr{Y}^+}(\omega^{\otimes d}\pi^{-n}).$$

Therefore we have that $v_\alpha(D_{\min}(\mathscr{X},\omega))>0$ if and only if $E\subseteq\mathrm{div}_{\mathscr{Y}^+}(\omega^{\otimes d}\pi^{-n})$, and this holds if and only if $x\in D_{\min}(\mathscr{X},\omega)$. □

(5.1.3) We define the essential skeleton $\mathrm{Sk}(X,\Delta_X)$ of (X,Δ_X) as the union of all Kontsevich-Soibelman skeleta $\mathrm{Sk}(X,\Delta_X,\omega)$, where ω ranges over all regular Δ_X-logarithmic pluricanonical forms.

The reason to define the essential skeleton this way is that it behaves nicely under birational morphisms. Let $f\colon X' \to X$ be a log resolution. Then there is a $\mathbb{Q}$-divisor Γ with snc support, and no coefficient exceeding 1, such that $K_{X'} + \Gamma' = f^*(K_X + \Delta_X)$. Take $\Delta_{X'}$ to be the positive part of Γ'.

Proposition 5.1.4. *Under the identification of the birational points of X with those of X', $\mathrm{Sk}(X, \Delta_X)$ is identified with $\mathrm{Sk}(X', \Delta_{X'})$.*

Proof. It suffices to check the proposition for divisorial valuations. By Lemma 5.1.2 a divisorial valuation v is in the skeleton $\mathrm{Sk}(X, \Delta_X)$ if and only if there exists a log-regular model $\mathscr{X}^+$ for $(X, \lceil \Delta_X \rceil)$ such that v is a log canonical center of $(\mathscr{X}, \Delta_X + \mathscr{X}_{k,\mathrm{red}})$, and there is a regular Δ_X-logarithmic pluricanonical form ω whose divisor $D_{\min}(\mathscr{X}, \omega)$ does not contain the center of v. As the essential skeleton $\mathrm{Sk}(X, \Delta_X)$ is a union of cells closed inside the skeleton associated to any log-regular model, we choose log-regular models $\mathscr{X}^+$ and $\mathscr{X}'^+$ so that f extends to a log resolution $f_R\colon \mathscr{X}' \to \mathscr{X}$. We denote by $D = \overline{\Delta_X} + \mathscr{X}_{k,\mathrm{red}}$ and $D' = \overline{\Delta_{X'}} + \mathscr{X}'_{k,\mathrm{red}}$.

Suppose v is a divisorial point in $\mathrm{Sk}(X, \Delta_X) \cap \mathrm{Sk}(\mathscr{X}^+)$. Then let ω be a Δ_X-logarithmic m-pluricanonical form on (X, Δ_X) whose associated divisor $\mathrm{div}_{\mathscr{X}^+}(\omega)$ in $\mathscr{X}^+$ does not contain the center of v. As $(\mathscr{X}, D)$ is a dlt pair, we have that

$$K_{\mathscr{X}'} + D' = f_R^*(K_{\mathscr{X}} + D) + M$$

where M is effective. Thus there is a $\Delta_{X'}$-logarithmic m-pluricanonical form ω' on $\mathscr{X}'$, whose associated divisor $\mathrm{div}_{\mathscr{X}'^+}(\omega') = f_R^*(\mathrm{div}_{\mathscr{X}^+}(\omega)) + mM$. As v is a log canonical center of $(\mathscr{X}, D)$, M does not vanish along v, so neither does $\mathrm{div}_{\mathscr{X}'^+}(\omega')$. Likewise v is a log canonical center of $(\mathscr{X}', D')$. It follows that $v \in \mathrm{Sk}(X', \Delta_{X'}, \omega')$.

Conversely, for any m, pullback along with multiplication by the divisor of discrepancies M induces an isomorphism of vector spaces

$$H^0(\mathscr{X}, mK_{\mathscr{X}} + mD) \cong H^0(\mathscr{X}', mK_{\mathscr{X}'} + mD').$$

So if v is a divisorial point in $\mathrm{Sk}(X', \Delta_{X'})$, it is a log canonical center of $(\mathscr{X}, D')$ and there is a regular $\Delta_{X'}$-logarithmic m-pluricanonical form ω' not containing the center of v, for some $m > 0$. As a result v is also a log canonical center of $(\mathscr{X}, D)$, so we need only to check the existence of a regular Δ_X-logarithmic pluricanonical form on $\mathscr{X}$ not vanishing at the center of v. We take the preimage of ω' in $H^0(\mathscr{X}, mK_{\mathscr{X}} + mD)$, and the associated divisor in $\mathscr{X}$ must not contain the center of v because its pullback does not. □

(5.1.5) Suppose that (X, Δ_X) is a dlt pair over K, such that $K_X + \Delta_X$ is semiample. Suppose thar $\mathscr{X}$ is a good dlt minimal model of (X, Δ_X) over R and let $D = \overline{\Delta_X} + \mathscr{X}_{k,\mathrm{red}}$. We consider a log resolution $f : X' \to X$ that extends to a log resolution of $(\mathscr{X}, D)$. We write $K_{X'} + \Gamma' = f^*(K_X + \Delta_X)$. Let $\Delta_{X'}$ be the positive part of Γ'. We may embed the open dual complex $\mathcal{D}_0^{=1}(\mathscr{X}, D)$ into the birational points of X.

Proposition 5.1.6. *This embedding identifies $\mathcal{D}_0^{=1}(\mathscr{X}, D)$ with $\mathrm{Sk}(X', \Delta_{X'})$.*

Proof. Choose a regular R-model $\mathscr{X}'$ for X' which is a log resolution of $(\mathscr{X}, D)$ extending f and let $D' = \overline{\Delta_{X'}} + \mathscr{X}'_{k,\mathrm{red}}$, namely we have the log resolutions

$(X', \Delta_{X'}) \to (X, \Delta_X)$ where (X, Δ_X) dlt and $\Delta_{X'}$ is the positive part of Γ'

$(\mathscr{X}', D') \to (\mathscr{X}, D)$ where $D = \overline{\Delta_X} + \mathscr{X}_{k,\mathrm{red}}$ and $D' = \overline{\Delta_{X'}} + \mathscr{X}'_{k,\mathrm{red}}$.

As in the previous proof, it suffices to check the proposition for divisorial valuations. Let v be a divisorial valuation, and suppose $v \in \mathcal{D}_0^{=1}(\mathscr{X}, D)$. Then v is a log canonical center for $(\mathscr{X}, D)$, so v is also a log canonical center of $(\mathscr{X}', D')$. For a sufficiently divisible index, we may find a Δ_X-logarithmic pluricanonical form on (X, Δ_X) whose associated divisor in $\mathscr{X}$ has vanishing locus C such that C is a divisor not containing the center of v. After pullback, we get a $\Delta_{X'}$-logarithmic pluricanonical form ω' whose associated divisor in $\mathscr{X}'$ is supported on the strict transform of C and the exceptional divisors of positive log discrepancy. But none of these contain v. Thus, $v \in \mathrm{Sk}(X', \Delta_{X'}, \omega')$.

Conversely, if v is a divisorial point in $\mathrm{Sk}(X', \Delta_{X'})$, then v is a log canonical center of $(\mathscr{X}', D')$, so v is a log canonical center of $(\mathscr{X}, D)$, hence an element of the open dual complex $\mathcal{D}_0^{=1}(\mathscr{X}, D)$. □

(5.1.7) We define the essential skeleton $\mathrm{Sk}(X, \Delta_X)$ of a dlt pair (X, Δ_X) as the essential skeleton of any log-resolution of (X, Δ_X): Proposition 5.1.4 guarantees that this is well-defined. Thus, Proposition 5.1.6 compares the essential skeleton to the skeleton of a good minimal dlt model of (X, Δ_X). the result can be restated as follows: if (X, Δ_X) is a dlt pair with $K_X + \Delta_X$ semiample and $(\mathscr{X}, D)$ is a good dlt minimal model of (X, Δ_X) over R, then $\mathcal{D}_0^{=1}(\mathscr{X}, D) = \mathrm{Sk}(X, \Delta_X)$. This generalizes [NX16], Theorem 3.3.3 to dlt pairs.

5.2. **Essential skeleta and products of log-regular models.**

(5.2.1) We say that a pair (X, Δ_X) has non-negative Kodaira-Iitaka dimension if some multiple of the line bundle $K_X + \Delta_X$ has a regular section.

Theorem 5.2.2. *Assume that the residue field k is algebraically closed. Let (X, Δ_X) and (Y, Δ_Y) be pairs such that $X^+ = (X, \lceil \Delta_X \rceil)$ and $Y^+ = (Y, \lceil \Delta_Y \rceil)$ are log-regular log scheme over K. Suppose that both pairs have non-negative Kodaira-Iitaka dimension and both admit log-regular models $\mathscr{X}^+$ and $\mathscr{Y}^+$ over S^+, such that $\mathscr{X}^+$ is semistable. Then, the homeomorphism of skeleta*

$$\mathrm{Sk}(\mathscr{Z}^+) \xrightarrow{\sim} \mathrm{Sk}(\mathscr{X}^+) \times \mathrm{Sk}(\mathscr{Y}^+)$$

of Proposition 3.4.3 induces a homeomorphism of essential skeleta

$$\mathrm{Sk}(Z, \Delta_Z) \xrightarrow{\sim} \mathrm{Sk}(X, \Delta_X) \times \mathrm{Sk}(Y, \Delta_Y)$$

where $\mathscr{Z}^+$, Z^+ and Δ_Z are the respective products.

Proof. It follows immediately from Theorem 4.3.9 that we have the inclusion $\mathrm{Sk}(X, \Delta_X) \times \mathrm{Sk}(Y, \Delta_Y) \subseteq \mathrm{Sk}(Z, \Delta_Z)$. Thus, we reduce to prove the following statement. Let $v_{z,\varepsilon}$ be a divisorial point in $\mathrm{Sk}(\mathscr{Z}^+)$ and $(v_{x,\alpha}, v_{y,\beta})$ be the corresponding pair in $\mathrm{Sk}(\mathscr{X}^+) \times \mathrm{Sk}(\mathscr{Y}^+)$ under the isomorphism of Proposition 3.4.3; if $v_{z,\varepsilon}$ lies in the essential skeleton $\mathrm{Sk}(Z, \Delta_Z)$, then $v_{x,\alpha}$ lies in $\mathrm{Sk}(X, \Delta_X)$.

Assume that $v_{z,\varepsilon}$ lies in the essential skeleton $\mathrm{Sk}(Z, \Delta_Z)$. Then there exists a non-zero regular Δ_Z-logarithmic m-pluricanonical form ω on Z^+, such that $v_{z,\varepsilon} \in \mathrm{Sk}(Z, \Delta_Z, \omega)$. By Lemma 5.1.2, $D_{\min}(\mathscr{Z}, \omega)$ does not contain z.

Let E be an irreducible component of $\mathscr{Y}_k$ containing y and denote by ξ_E the generic point of E. Then the point in the Kato fan of $\mathscr{Z}^+$ corresponding to (x, ξ_E) is not contained in $D_{\min}(\mathscr{Z}, \omega)$, as otherwise z would be contained in it.

As k is algebraically closed, we can choose a k-rational point p in E such that p is contained in no other components of $D_{\mathscr{Y}}$ and $D_{\min}(\mathscr{Z}, \omega)$ does not contained the locus $\overline{\{x\}} \times_R \{p\}$. By Hensel's Lemma, p can be lifted to an R-rational point of $\mathscr{Y}$.

The pull-back of $\mathscr{Z}^+$ along this R-rational point is an embedding $i : \mathscr{X}^+ \to \mathscr{Z}^+$, so we have the diagram

$$\begin{tikzcd} \mathscr{X}^+ \arrow[r, "i"] \arrow[d] & \mathscr{Z}^+ \arrow[r, "\mathrm{pr}_{\mathscr{X}}"] \arrow[d, "\mathrm{pr}_{\mathscr{Y}}"] & \mathscr{X}^+ \arrow[d] \\ S \arrow[r] & \mathscr{Y}^+ \arrow[r] & S. \end{tikzcd}$$

Since S has trivial normal bundle in $\mathscr{Y}$, we have that

$$\omega^{\log}_{X^+/K} = i^*\big(\omega^{\log}_{Z^+/K}\big),$$

so $i^*(\omega)$ is a non-zero regular logarithmic m-pluricanonical form on X^+ and in particular is a regular Δ_X-logarithmic m-pluricanonical form. Moreover, $D_{\min}(\mathscr{X}, i^*(\omega)) = i^*(D_{\min}(\mathscr{Z}, \omega))$. Finally, x is not contained in $D_{\min}(\mathscr{X}, i^*(\omega))$, as otherwise $i(x) = \{x\} \times_R \{p\}$ would be contained in $D_{\min}(\mathscr{Z}, \omega)$. By Lemma 5.1.2, x is a point of $\mathrm{Sk}(X, \Delta_X, i^*(\omega))$ and this concludes the proof. □

Remark 5.2.3. Consider the case where the line bundles $K_X + \Delta_X$ and $K_Y + \Delta_Y$ are semi-ample, i.e. some multiple of them is base point free. It follows from the arguments of [NX16], Theorem 3.3.3 that the essential skeleton of (Z, Δ_Z) is a finite union of Kontsevich-Soibelman skeleta where the union runs through a generating set of global sections of a sufficiently large multiple of $K_Z + \Delta_Z$. We can construct such a set from generating sets of global sections of multiples of $K_X + \Delta_X$ and $K_Y + \Delta_Y$ respectively, via tensor product. Then, in this case, the result of Theorem 5.2.2 follows directly from Theorem 4.3.9.

5.3. **Essential skeleta and products of dlt models.**

Lemma 5.3.1. *Let M be the monoid generated by $r_1 \dots r_{n_1}$, $s_1 \dots s_{n_2}$ with the single relation $\sum_{i=1}^{n_1} r_i = \sum_{j=1}^{n_2} s_j$. Then any small $\mathbb{Q}$-factorialization of the affine toric variety $W = \mathrm{Spec}\,(k[M])$ associated to M is a log resolution.*

Proof. We calculate the fan of W. Let N be the dual lattice of M. The fan associated to W is the cone of elements of $N \otimes \mathbb{R}$ which are non-negative on M. We consider these as linear functions l on the vector space spanned by the r_i and s_i, subject to the restriction that $l(\sum_{i=1}^{n_1} r_i) = l(\sum_{j=1}^{n_2} s_j)$. Let x_{ij} be the function which is 1 on r_i and s_j and 0 on all others. Then the fan of W is given by the single cone C_W spanned by the x_{ij}.

Any $\mathbb{Q}$-factorialization $\widetilde{W}$ corresponds to a simplicial subdivision of the cone C_W (see [Ful93], p.65). We now check that every choice of $\widetilde{W}$ is non-singular.

A maximal cone of $\widetilde{W}$ is spanned by $n = n_1 + n_2 - 1$ independent rays of C_W. Each ray of C_W corresponds to a choice of x_{ij}, and we can index these by edges of the complete bipartite graph B on the r_i and s_j. These x_{ij} are independent if and only if the corresponding edges form a spanning tree. Let $w_1 \dots w_n$ span a maximal cone of $\widetilde{W}$. On this affine chart, $\widetilde{W}$ is smooth if and only if the w_i generate N as a lattice. We have shown already that the x_{ij} generate N. But every x_{ij} is either one of the w_i, or it completes a cycle in B, so that it is a $\mathbb{Z}$-linear combination of the w_i. □

Proposition 5.3.2. *Let $(\mathscr{X}, \Delta_{\mathscr{X}})$ and $(\mathscr{Y}, \Delta_{\mathscr{Y}})$ be semistable good projective dlt minimal pairs over the germ of a pointed curve C. The product $(\mathscr{Z}, \Delta_{\mathscr{Z}})$ is a log* *canonical pair,* $K_{\mathscr{Z}} + \Delta_{\mathscr{Z}}$ *is semiample, and the log canonical centers of* $(\mathscr{Z}, \Delta_{\mathscr{Z}})$ *are strata of the coefficient* 1 *part of* $\Delta_{\mathscr{Z}}$*.*

Proof. The product $\mathscr{Z}$ is normal as $\mathscr{X}$ and $\mathscr{Y}$ are semistable. The divisor $K_{\mathscr{Z}}+\Delta_{\mathscr{Z}}$ is semiample by pullback of semiample divisors.

Let $\widetilde{\mathscr{X}}$ and $\widetilde{\mathscr{Y}}$ be log resolutions of $\mathscr{X}$ and $\mathscr{Y}$. Then, we have

$$K_{\widetilde{\mathscr{X}}} + \Delta_{\widetilde{\mathscr{X}}} = f^*_{\mathscr{X}}(K_{\mathscr{X}} + \Delta_{\mathscr{X}}) + \sum a_i E_{\mathscr{X},i}$$
$$K_{\widetilde{\mathscr{Y}}} + \Delta_{\widetilde{\mathscr{Y}}} = f^*_{\mathscr{Y}}(K_{\mathscr{Y}} + \Delta_{\mathscr{Y}}) + \sum b_j E_{\mathscr{Y},j}$$

where the coefficients a_i and b_j are non-negative. Let $\widetilde{\mathscr{Z}}$ be a toroidal log resolution of the fs product $\widetilde{\mathscr{X}} \times^{\mathrm{fs}} \widetilde{\mathscr{Y}}$. In particular, $\widetilde{\mathscr{Z}}$ is a log resolution of $\mathscr{Z}$ and we can write

$$K_{\widetilde{\mathscr{Z}}} + \Delta_{\widetilde{\mathscr{Z}}} = f^*_{\mathscr{Z}}(K_{\mathscr{Z}} + \Delta_{\mathscr{Z}}) + \sum c_h E_{\mathscr{Z},h}$$

where $\Delta_{\widetilde{\mathscr{Z}}}$ is effective. Over the generic fibre, $(\mathscr{Z}, \Delta_{\mathscr{Z}})$ is dlt, so we need only compute discrepancies over the special fibre, namely study the positivity of the coefficients c_h.

Let Γ be a divisor of $\widetilde{\mathscr{Z}}$ over the special fibre, denote by v_Γ the corresponding divisorial valuation in $\mathrm{Sk}(\widetilde{\mathscr{Z}^+})$, by $\Gamma_{\mathscr{X}}$, $\Gamma_{\mathscr{Y}}$ and $\Gamma_{\mathscr{Z}}$ its images in $\mathscr{X}$, $\mathscr{Y}$ and $\mathscr{Z}$. The projections of v_Γ in $\mathrm{Sk}(\widetilde{\mathscr{X}^+})$ and $\mathrm{Sk}(\widetilde{\mathscr{Y}^+})$ are divisorial valuations. Up to subdivisions of the skeleta, we can assume without loss of generality that the projections correspond to divisors $\Gamma_{\widetilde{\mathscr{X}}}$ and $\Gamma_{\widetilde{\mathscr{Y}}}$.

Choose a $\Delta_{\mathscr{X}}$-logarithmic and a $\Delta_{\mathscr{Y}}$-logarithmic pluricanonical forms $\omega_{\mathscr{X}}$ on $\mathscr{X}$ and $\omega_{\mathscr{Y}}$ on $\mathscr{Y}$ respectively, such that the divisors $\mathrm{div}_{\mathscr{X}^+}(\omega_{\mathscr{X}})$ and $\mathrm{div}_{\mathscr{Y}^+}(\omega_{\mathscr{Y}})$ do not contain $\Gamma_{\mathscr{X}}$ and $\Gamma_{\mathscr{Y}}$ respectively, where $\mathscr{X}^+ = (\mathscr{X}, \lceil \Delta_{\mathscr{X}} \rceil)$ and $\mathscr{Y}^+ = (\mathscr{Y}, \lceil \Delta_{\mathscr{Y}} \rceil)$. Then, the divisor $\mathrm{div}_{\mathscr{Z}^+}(\omega_{\mathscr{Z}})$, associated to the wedge product $\omega_{\mathscr{Z}}$ of the pullbacks $\omega_{\mathscr{X}}$ and $\omega_{\mathscr{Y}}$ to $\mathscr{Z}$, does not contain $\Gamma_{\mathscr{Z}}$. Denote by $\omega_{\widetilde{\mathscr{X}}}$, $\omega_{\widetilde{\mathscr{Y}}}$ and $\omega_{\widetilde{\mathscr{Z}}}$ the pullback of the respective forms to $\widetilde{\mathscr{X}}$, $\widetilde{\mathscr{Y}}$ and $\widetilde{\mathscr{Z}}$. Then, we have

$$\mathrm{div}_{\widetilde{\mathscr{X}}^+}(\omega_{\widetilde{\mathscr{X}}}) = f^*_{\mathscr{X}}(\mathrm{div}_{\mathscr{X}^+}(\omega_{\mathscr{X}})) + \sum a_i E_{\mathscr{X},i}$$
$$\mathrm{div}_{\widetilde{\mathscr{Y}}^+}(\omega_{\widetilde{\mathscr{Y}}}) = f^*_{\mathscr{Y}}(\mathrm{div}_{\mathscr{Y}^+}(\omega_{\mathscr{Y}})) + \sum b_j E_{\mathscr{Y},j}$$
$$\mathrm{div}_{\widetilde{\mathscr{Z}}^+}(\omega_{\widetilde{\mathscr{Z}}}) = f^*_{\mathscr{Z}}(\mathrm{div}_{\mathscr{Z}^+}(\omega_{\mathscr{Z}})) + \sum c_h E_{\mathscr{Z},h}.$$

As $\mathrm{div}_{\mathscr{X}^+}(\omega_{\mathscr{X}})$, $\mathrm{div}_{\mathscr{Y}^+}(\omega_{\mathscr{Y}})$ and $\mathrm{div}_{\mathscr{Z}^+}(\omega_{\mathscr{Z}})$ do not contain $\Gamma_{\mathscr{X}}$, $\Gamma_{\mathscr{Y}}$ and $\Gamma_{\mathscr{Z}}$ respectively, we have

$$v_{\Gamma_{\widetilde{\mathscr{X}}}}(\mathrm{div}_{\widetilde{\mathscr{X}}^+}(\omega_{\widetilde{\mathscr{X}}})) = \sum a_i v_{\Gamma_{\widetilde{\mathscr{X}}}}(E_{\mathscr{X},i}) \geqslant 0$$
$$v_{\Gamma_{\widetilde{\mathscr{Y}}}}(\mathrm{div}_{\widetilde{\mathscr{Y}}^+}(\omega_{\widetilde{\mathscr{Y}}})) = \sum b_j v_{\Gamma_{\widetilde{\mathscr{Y}}}}(E_{\mathscr{Y},j}) \geqslant 0$$
$$v_{\Gamma}(\mathrm{div}_{\widetilde{\mathscr{Z}}^+}(\omega_{\widetilde{\mathscr{Z}}})) = \sum c_h v_{\Gamma}(E_{\mathscr{Z},h}).$$

From the formula 4.3.5, $v_\Gamma(\mathrm{div}_{\widetilde{\mathscr{Z}}^+}(\omega_{\widetilde{\mathscr{Z}}})) = v_{\Gamma_{\widetilde{\mathscr{Y}}}}(\mathrm{div}_{\widetilde{\mathscr{Y}}^+}(\omega_{\widetilde{\mathscr{Y}}})) + v_{\Gamma_{\widetilde{\mathscr{X}}}}(\mathrm{div}_{\widetilde{\mathscr{X}}^+}(\omega_{\widetilde{\mathscr{X}}}))$, hence we obtain that the log discrepancy of Γ with respect to the pair $(\mathscr{Z}, \Delta_{\mathscr{Z}})$ is non-negative. Moreover, it is zero if and only if the log discrepancies of $\Gamma_{\widetilde{\mathscr{X}}}$ and $\Gamma_{\widetilde{\mathscr{Y}}}$ are both zero, namely if and only if $\Gamma_{\mathscr{X}}$ and $\Gamma_{\mathscr{Y}}$ are log canonical centres of $(\mathscr{X}, \Delta_{\mathscr{X}})$ and $(\mathscr{Y}, \Delta_{\mathscr{Y}})$ respectively. Since for dlt pairs the log canonical centres are the strata of the coefficient 1 part of the boundary, it follows that any log canonical centre of $(\mathscr{X}, \Delta_{\mathscr{X}})$ is a product of such strata, hence a stratum of the coefficient part 1 of $(\mathscr{X}, \Delta_{\mathscr{X}})$. $\square$

6. Applications

6.1. **Weight functions and skeleta for finite quotients.**

(6.1.1) Let X be a connected, smooth and proper K-variety and let G be a finite group acting on X. Let X^{an} be the analytification of X. We recall that any point of X^{an} is a pair $(x, |\cdot|_x)$ with $x \in X$ and $|\cdot|_x$ an absolute value on the residue field $\kappa(x)$ that extends the absolute value on K. For any point x of X, an element g of the group G induces an isomorphism between the residue fields $\kappa(x)$ and $\kappa(g.\, x)$, that we still denote by g. Then, the action of G extends to X^{an} in the following way

$$g.\,(x, |\cdot|_x) = (g.\,x, |\cdot|_x \circ g^{-1}).$$

In particular the action preserves the sets of divisorial and birational points of X.

Let $f : X \to Y = X/G$ be the quotient map of K-schemes, let $f^{\mathrm{an}} : X^{\mathrm{an}} \to Y^{\mathrm{an}}$ be the map of Berkovich spaces induced by functoriality and let $\tilde{f} : X^{\mathrm{an}} \to X^{\mathrm{an}}/G$ be the quotient map of topological spaces.

Proposition 6.1.2. *([Ber95], Corollary 5) Under the above notations, there is a canonical homeomorphism between X^{an}/G and Y^{an} such that $\tilde{f}$ and f^{an} are identified.*

(6.1.3) Let y be a divisorial point of Y^{an} and consider a regular snc R-model $\mathscr{Y}$ of Y adapted to y, i.e. such that y is the divisorial point associated to $(\mathscr{Y}, E)$ for some irreducible component E of $\mathscr{Y}_k$. We denote by $\mathscr{X}$ the normalization of $\mathscr{Y}$ inside $K(X)$, where $K(\mathscr{Y}) = K(Y) = K(X)^G \hookrightarrow K(X)$.

We check that $\mathscr{X}$ is an R-model of X; it is enough to show that the base change $\mathscr{X}_K$ is isomorphic to X. We consider the following commutative diagram

$$\begin{tikzcd}
\mathscr{X}_K \arrow[rrd] \arrow[rddd] & & \\
& X \arrow[lu, dotted] \arrow[r, dotted] \arrow[d, "f"] & \mathscr{X} \arrow[d] \\
& Y \arrow[r] \arrow[d] & \mathscr{Y} \arrow[d] \\
& \operatorname{Spec} K \arrow[r] & S
\end{tikzcd}$$

As the X is a normal variety endowed with a morphism $X \to \mathscr{Y}$, by universal property of normalization, it factors uniquely through $\mathscr{X}$ and the diagram is still commutative. Then by universal property of fibred product, there exists a morphism $X \to \mathscr{X}_K$. Therefore, it suffices to prove that

$$[K(X) : K(\mathscr{X}_K)] = 1.$$

Indeed, if this is the case, then $X \to \mathscr{X}_K$ is a finite birational morphism between normal varietes, hence an isomorphism.

The degree of the extension $[K(X) : K(\mathscr{X}_K)]$ may be computed on an affine open, so we assume that $\mathscr{Y}$ is an affine scheme with associated ring $K[\mathscr{Y}]$. Then we consider the diagram of inclusions

$$\begin{tikzcd}
 & K(\mathscr{X}) = K(\mathscr{X}_K) \arrow[dr] & \\
\widehat{K[\mathscr{Y}]} = K[\mathscr{X}] \arrow[ur] \arrow[rr] & & K(X) \\
K[\mathscr{Y}] \arrow[u] \arrow[rr] & & K(\mathscr{Y}) = K(Y) = K(X)^G \arrow[u, "\text{finite deg}"']
\end{tikzcd}$$

As $K(X)$ is finite field extension of $K(\mathscr{Y})$ and $K[\mathscr{X}]$ the integral closure of $K[\mathscr{Y}]$ in $K(X)$, then $K(X)$ is the fraction field of $K[\mathscr{X}]$. Thus, $K(\mathscr{X}) = \mathrm{Frac}(K[\mathscr{X}]) = K(X)$ and in particular we conclude that $[K(X) : K(\mathscr{X}_K)] = 1$.

Remark 6.1.4. This procedure of normalization illustrates a way to start with a regular snc R-model $\mathscr{Y}$ of Y adapted to a point $y \in \mathrm{Div}(Y)$ and construct an R-model $\mathscr{X}$ of X that, by normality, is regular at generic points of the special fibre $\mathscr{X}_k$.

Lemma 6.1.5. *Let ω be a m-pluricanonical form on X. If ω is G-invariant, then the weight function associated to ω on the set of birational points is stable under the action of G.*

Proof. Let x be a birational point of X and g an element of G. There exist snc models $\mathscr{X}$ and $\mathscr{X}'$ over R such that $x \in \mathrm{Sk}(\mathscr{X})$ and $g.\,x \in \mathrm{Sk}(\mathscr{X}')$. By replacing them by an snc model $\mathscr{Y}$ that dominates both $\mathscr{X}$ and $\mathscr{X}'$, we can assume that both points lies in $\mathrm{Sk}(\mathscr{Y})$. The weights of ω at x and $g.\,x$ can be computed using the formula 4.1.7, so

$$\begin{aligned}
\mathrm{wt}_\omega(g.\,x) &= v_{g.x}(\mathrm{div}_{\mathscr{Y}^+}(\omega)) + m = v_x((g^{-1})^*\mathrm{div}_{\mathscr{Y}^+}(\omega)) + m \\
&= v_x(\mathrm{div}_{\mathscr{Y}^+}(\omega)) + m = \mathrm{wt}_\omega(x)
\end{aligned}$$

as ω is a G-invariant form. Thus we see that birational points in the same G-orbit have the same weight with respect to ω. □

Corollary 6.1.6. *Let ω be a G-invariant pluricanonical form on X. Then the Kontsevich-Soibelman skeleton* $\mathrm{Sk}(X, \omega)$ *is stable under the action of G.*

Proof. This follows immediately from Lemma 6.1.5. □

Proposition 6.1.7. *Let ω be a G-invariant m-pluricanonical form on X. Let y be a divisorial point of Y^{an}. Then, for any divisorial point $x \in (f^{\mathrm{an}})^{-1}(y)$, the weights of ω at x and y coincide.*

Proof. Let $\mathscr{Y}$ be a regular snc R-model such that y has divisorial representation $(\mathscr{Y}, E)$. Let $\mathscr{X}$ be the normalization of $\mathscr{Y}$ in $K(X)$: as we observed in Remark 6.1.4, it is an R-model of X, regular at generic points of the special fibre $\mathscr{X}_k$. The preimage of E coincides with the pull-back of the Cartier divisor E on $\mathscr{X}$, hence $f^{-1}(E)$ still defines a codimension one subset on $\mathscr{X}$. We denote by F_i the irreducible components of $f^{-1}(E)$ and we associate to F_i's their corresponding divisorial valuations $x_i = (\mathscr{X}, F_i)$. By Lemma 6.1.5, it is enough to prove the result for one of the x_i's. We denote it by $x = (\mathscr{X}, F)$ and we compare the weights of ω at y and x, namely

$$\mathrm{wt}_\omega(x) = v_x(\mathrm{div}_{\mathscr{X}^+}(\omega)) + m \quad \text{and} \quad \mathrm{wt}_\omega(y) = v_y(\mathrm{div}_{\mathscr{Y}^+}(\omega)) + m.$$

We recall that for log-étale morphisms the sheaves of logarithmic differentials are stable under pull-back ([Kat94], Proposition 3.12). Furthermore, it suffices to check that, locally around the generic point of F_i, the morphism $\mathscr{X}^+ \to \mathscr{Y}^+$ is a log-étale morphism of divisorial log structures, to conclude that the weights coincide. To this purpose, we will apply Kato's criterion for log étaleness ([Kat89], Theorem 3.5) to log schemes with respect to the étale topology.

We denote by ξ_F the generic point of F and by ξ_E the generic point of E. The divisorial log structures on $\mathscr{X}^+$ and $\mathscr{Y}^+$ have charts $\mathbb{N}$ at ξ_F and ξ_E. In the étale topology, the normalization morphism $\mathscr{X}^+ \to \mathscr{Y}^+$ admits a chart induced by $t : \mathbb{N} \to \mathbb{N}$ where $1 \mapsto m$ for some positive integer m:

$$\begin{tikzcd} \operatorname{Spec} \mathcal{O}_{\mathscr{X},\xi_F} \arrow[r] \arrow[d] & \operatorname{Spec} \mathbb{Z}[\mathbb{N}] \arrow[d] \\ \operatorname{Spec} \mathcal{O}_{\mathscr{Y},\xi_E} \arrow[r] & \operatorname{Spec} \mathbb{Z}[\mathbb{N}] \end{tikzcd}$$

Firstly, by the universal property of the fibre product, we have a morphism

$$\operatorname{Spec} \mathcal{O}_{\mathscr{X},\xi_F} \to \operatorname{Spec} \mathcal{O}_{\mathscr{Y},\xi_E} \times_{\operatorname{Spec} \mathbb{Z}[\mathbb{N}]} \operatorname{Spec} \mathbb{Z}[\mathbb{N}]$$

and it corresponds to

$$\mathcal{O}_{\mathscr{Y},\xi_E} \otimes_{\mathbb{Z}[\mathbb{N}]} \mathbb{Z}[\mathbb{N}] \to \mathcal{O}_{\mathscr{X},\xi_F}.$$

This is a morphism of finite type with finite fibres between regular rings and by [Liu02], Lemma 4.3.20 and [Now97] it is flat and unramified, hence étale. One of the two conditions in Kato's criterion for log étaleness is then fulfilled. Secondly, the chart $t : \mathbb{N} \mapsto \mathbb{N}$ induces a group homomorphism $t^{\mathrm{gp}} : \mathbb{Z} \mapsto \mathbb{Z}$; in particular, it is injective and it has finite cokernel. Then t satisfies the second condition of Kato's criterion for log étaleness. Therefore we conclude that $\mathrm{wt}_\omega(y) = \mathrm{wt}_\omega(x)$. □

Proposition 6.1.8. *Let ω be a G-invariant pluricanonical form on X. Then the canonical homeomorphism between X^{an}/G and Y^{an} of Proposition 6.1.2 induces the homeomorphism*

$$\mathrm{Sk}(X, \omega)/G \simeq \mathrm{Sk}(X/G, \omega).$$

Proof. This follows immediately from Corollary 6.1.6 and Proposition 6.1.7. □

(6.1.9) Let X be a smooth K-variety and let ω_X be a pluricanonical form on X. Let $\mathrm{pr}_j : X^n \to X$ be the j-th canonical projection. Then

$$\omega = \bigwedge_{1 \leqslant j \leqslant n} \mathrm{pr}_j^* \omega_X$$

is a pluricanonical form on X^n and moreover it is invariant under the action of $\mathfrak{S}_n$.

Proposition 6.1.10. *Assume that the residue field k is algebraically closed. If X has semistable reduction write right assumption, then the Kontsevich-Soibelman skeleton of the n-th symmetric product of X associated to ω is isomorphic to the n-th symmetric product of the Kontsevich-Soibelman skeleton of X associated to ω_X.*

Proof. Iterating the result of Theorem reference, we have that the projection map defines an isomorphism of Kontsevich-Soibelman skeleta

$$\mathrm{Sk}(X^n, \omega) \xrightarrow{\sim} \mathrm{Sk}(X, \omega_X) \times \ldots \times \mathrm{Sk}(X, \omega_X).$$

Thus, applying Proposition 6.1.8 with the group $\mathfrak{S}_n$ acting on the product X^n, we obtain that

$$\mathrm{Sk}(X^n/\mathfrak{S}_n, \omega) \simeq \mathrm{Sk}(X^n, \omega)/\mathfrak{S}_n \simeq \mathrm{Sk}(X, \omega_X)^n/\mathfrak{S}_n$$

Since the action on the Kontsevich-Soibelman skeleton $\mathrm{Sk}(X^n, \omega)$ is induced from the symmetric action on X^n, and the projections $\mathrm{pr}_j : X^n \to X$ functorially induce the projections $\overline{\mathrm{pr}_j} : \mathrm{Sk}(X, \omega_X)^n \to \mathrm{Sk}(X, \omega)$, the action of $\mathfrak{S}_n$ on $\mathrm{Sk}(X, \omega)^n$ is exactly by permutations of the components. Thus,

$$\mathrm{Sk}(X, \omega_X)^n/\mathfrak{S}_n \simeq \mathrm{Sym}^n(\mathrm{Sk}(X, \omega_X)).$$

□

6.2. **The essential skeleton of Hilbert schemes of a K3 surface.**

(6.2.1) Let S be an irreducible regular surface. We consider $\mathrm{Hilb}^n(S)$ the Hilbert scheme of n points on S: by [Fog68] it is an irreducible regular variety of dimension $2n$. Moreover, the morphism

$$\rho_{HC} : \mathrm{Hilb}^n(S) \to S^n/\mathfrak{S}_n$$

that sends a zero-dimensional scheme $Z \subseteq S$ to its associated zero-cycle $\mathrm{supp}(Z)$ is a birational morphism, called the Hilbert-Chow morphism.

(6.2.2) Let S be a K3 surface over K, namely S is a complete non-singular variety of dimension two such that $\Omega^2_{S/K} \simeq \mathcal{O}_S$ and $H^1(S, \mathcal{O}_S) = 0$. In particular S is a variety with trivial canonical line bundle.

Corollary 6.2.3. *Assume that S has semistable reduction and the residue field k is algebraically closed. Then the essential skeleton of the Hilbert scheme of n points on S is isomorphic to the n-th symmetric product of the essential skeleton of S*

$$\mathrm{Sk}(\mathrm{Hilb}^n(S)) \xrightarrow{\sim} \mathit{Sym}^n(\mathrm{Sk}(S)).$$

Proof. This follows immediately from Corollary 6.1.10 and the birational invariance of the essential skeleton, [MN15], Proposition 4.6.3. □

Proposition 6.2.4. *If the essential skeleton of S is homeomorphic to a point, a closed interval or the 2-dimensional sphere, then the essential skeleton of* $\mathrm{Hilb}^n(S)$ *is homeomorphic to a point, the standard n-simplex or $\mathbb{CP}^n$ respectively.*

Proof. Applying Corollary 6.2.3, we reduce to the computation of the symmetric product of a point, a closed interval or the sphere S^2. Then, the result follows from [Hat02], Section 4K. □

6.3. **The essential skeleton of generalised Kummer varieties.**

(6.3.1) Let A be an abelian surface over K, namely a complete non-singular, connected group variety of dimension two. Since A is a group variety, the canonical line bundle is trivial and the group structure provides a multiplication morphism $m_{n+1} : A \times A \times \ldots \times A \to A$ that is invariant under the permutation action of $\mathfrak{S}_{n+1}$, hence it induces a morphism

$$\Sigma_{n+1} : \mathrm{Hilb}^{n+1}(A) \xrightarrow{\rho_{HC}} \mathrm{Sym}^{n+1}(A) \to A$$

by composition with the Hilbert-Chow morphism. Then $\mathrm{K}_n(A) = \Sigma^{-1}_{n+1}(1)$ is called the n-th generalised Kummer variety and is a hyper-Kähler manifold of dimension $2n$ ([Bea83]).

(6.3.2) In [HN17], Proposition 4.3.2, Halle and Nicaise, using Temkin's generalization of the weight function ([Tem16]), prove that the essential skeleton of an abelian variety A over K coincides with the construction of a skeleton of A done by Berkovich in [Ber90], Paragraph 6.5. It follows from this identification and [Ber90], Theorem 6.5.1 that the essential skeleton of A has a group structure, compatible with the group structure on A^{an} under the retraction ρ_A of A^{an} onto the essential skeleton, so the following diagram commutes

$$\begin{tikzcd} (A^{\mathrm{an}})^{n+1} \arrow[r, "m_{n+1}^{\mathrm{an}}"] \arrow[d, "(\rho_A)^{n+1}"'] & A^{\mathrm{an}} \arrow[d, "\rho_A"] \\ \mathrm{Sk}(A)^n \arrow[r, "\mu_{n+1}"] & \mathrm{Sk}(A) \end{tikzcd}$$

where μ denotes the multiplication of $\mathrm{Sk}(A)$.

Proposition 6.3.3. *Let A be an abelian surface over K. Assume that A has semistable reduction and the residue field k is algebraically closed. Then the essential skeleton of the n-th generalised Kummer variety is isomorphic to the symmetric quotient of the kernel of the morphism μ, namely*

$$\mathrm{Sk}(K_n(A)) \simeq \mathrm{Sk}\big(m_{n+1}^{-1}(1)/\mathfrak{S}_{n+1}\big) \simeq \mu_{n+1}^{-1}(1)/\mathfrak{S}_{n+1}.$$

Proof. The first homeomorphism follows from the birational invariance of the essential skeleton ([MN15], Proposition 4.6.3). We denote by

$$L = m_{n+1}^{-1}(1) \quad \text{and} \quad \Lambda = \mu_{n+1}^{-1}(1).$$

For any choice of an $\mathfrak{S}_{n+1}$-invariant generating canonical form on A, it follows from Proposition 6.1.8 that $\mathrm{Sk}(L/\mathfrak{S}_{n+1}) \simeq \mathrm{Sk}(L)/\mathfrak{S}_{n+1}$. We reduce to study the quotients $\mathrm{Sk}(L)/\mathfrak{S}_{n+1}$ and $\Lambda/\mathfrak{S}_{n+1}$.

Let $\mathfrak{S}'_n$ and $\mathfrak{S}''_n$ be the subgroups of $\mathfrak{S}_{n+1}$ of the permutations that fix n and $n+1$ respectively. Then $\mathfrak{S}_{n+1}$ is generated by the two subgroups, so its action on $\mathrm{Sk}(L)$ and Λ is completely determined by the actions of these subgroups. We consider the following isomorphisms

$$\begin{aligned} f_n &: L \xrightarrow{\sim} A^n \quad (z_1, \dots, z_{n+1}) \mapsto (z_1, \dots, z_{n-1}, z_{n+1}) \\ f_{n+1} &: L \xrightarrow{\sim} A^n \quad (z_1, \dots, z_{n+1}) \mapsto (z_1, \dots, z_{n-1}, z_n). \end{aligned}$$

Then f_n is $\mathfrak{S}'_n$-equivariant, f_{n+1} is $\mathfrak{S}''_n$-equivariant and the morphism ψ

$$\begin{tikzcd} & L \arrow[dl, "f_{n+1}"'] \arrow[dr, "f_n"] & \\ A^n \arrow[rr, "\psi"] & & A^n \\ (z_1, \dots, z_{n-1}, z_n) \arrow[rr, mapsto] & & (z_1, \dots, \prod_{i=1}^n z_i^{-1}). \end{tikzcd}$$

is equivariant with respect to the action of $\mathfrak{S}''_n$ on the source and of $\mathfrak{S}'_n$ on the target. Hence, we obtain a commutative diagram of equivariant isomorphisms. We denote by $\overline{f}_n$, $\overline{f}_{n+1}$ and $\overline{\psi}$ the isomorphisms induced on the essential skeleta. By Theorem reference we can identify $\mathrm{Sk}(A^n)$ with $\mathrm{Sk}(A)^n$. Thus, we have the commutative diagram

$$\begin{tikzcd} & \mathrm{Sk}(L) \arrow[dl, "\overline{f}_{n+1}"'] \arrow[dr, "\overline{f}_n"] & \\ \mathrm{Sk}(A)^n \arrow[rr, "\overline{\psi}"] & & \mathrm{Sk}(A)^n \\ (v_1, \dots, v_{n-1}, v_n) \arrow[rr, mapsto] & & (v_1, \dots, \prod_{i=1}^n v_i^{-1}). \end{tikzcd}$$

Then the action of $\mathfrak{S}_{n+1}$ on $\mathrm{Sk}(L)$ is induced by the isomorphisms $\overline{f}_n$ and $\overline{f}_{n+1}$ from the actions of $\mathfrak{S}''_n$ and $\mathfrak{S}'_n$ on $\mathrm{Sk}(A)^n$ and these actions are compatible as $\overline{\psi}$ is equivariant.

In a similar way, Λ is isomorphic to n copies of $\mathrm{Sk}(A)$ and comes equipped with an action of $\mathfrak{S}_{n+1}$. So, we have equivariant projections g_n and g_{n+1} with respect to $\mathfrak{S}'_n$ and $\mathfrak{S}''_n$. The equivariant morphism that completes and makes the diagram commutative is $\overline{\psi}$. Finally, we have the equivariant commutative diagram

$$\begin{tikzcd} & \mathrm{Sk}(L) \arrow[dl, "\overline{f}_{n+1}"'] \arrow[dr, "\overline{f}_n"] & \\ \mathfrak{S}''_n \circlearrowleft \mathrm{Sk}(A)^n \arrow[rr, "\overline{\psi}"] & & \mathrm{Sk}(A)^n \circlearrowright \mathfrak{S}'_n \\ & \Lambda \arrow[ul, "g_{n+1}"] \arrow[ur, "g_n"'] & \end{tikzcd}$$

and we conclude that the quotients $\mathrm{Sk}(Z)/\mathfrak{S}_{n+1}$ and $\Lambda/\mathfrak{S}_{n+1}$ are homeomorphic. □

Proposition 6.3.4. *If the essential skeleton of A is homeomorphic to a point, the circle S^1 or the torus $S^1 \times S^1$, then the essential skeleton of $K_n(A)$ is homeomorphic to a point, the standard n-simplex or $\mathbb{CP}^n$ respectively.*

Proof. The case of the point is trivial. For the circle S^1, it follows directly from [Mor67], Theorem. To prove the result for the torus $S^1 \times S^1$, we apply [Loo76], Theorem 3.4: the action of the symmetric group corresponds to the root system of A_n, the highest root is the sum of the simple roots, each with coefficient 1, and so the quotient is the complex projective space of dimension n.

add comment to the proof □

6.4. **Remarks.**

(6.4.1) The cases we consider in Proposition 6.2.4 and Proposition 6.3.4 are motivated by the work of Kulikov, Persson and Pinkham. In [Kul77] and [PP81], they consider degenerations over the unit complex disk, of surfaces such that some power of the canonical bundle is trivial. They prove that, after base change and birational transformations, any such degeneration can be arranged to be semistable with trivial canonical bundle, namely a *Kulikov degeneration.* Then, they classify the possible special fibres of Kulikov degenerations according to the type of the degeneration.

We recall that the monodromy operator T on $\mathrm{H}^2(X_t, \mathbb{Q})$ of the fibres X_t of a Kulikov degeneration is unipotent, so we denote by ν the nilpotency index of $\log(T)$, namely the positive integer such that $\log(T)^\nu = 0$ and $\log(T)^{(\nu-1)} \neq 0$. The type of the Kulikov degeneration is defined as the nilpotency index ν and called type I, II or III accordingly to it.

It follows from [Kul77], Theorem II, that the dual complex of the special fibre of a Kulikov degeneration of a K3 surface is a point, a closed interval or the sphere S^2

according to the respective type. For a degeneration of abelian surfaces, the dual complex of the special fibre is homeomorphic to a point, the circle S^1 or the torus $S^1 \times S^1$ according to the three types (see an overview of these results in [FM83]). In all cases, the dimension of the dual complex is equal $\nu - 1$, hence determined by the type.

(6.4.2) Hilbert schemes of K3 surfaces and generalised Kummer varieties represent two families of examples of hyper-Kähler varieties. For a semistable degeneration of hyper-Kähler manifolds over the unit disk, it is possible to define the type as the nilpotency index of the monodromy operator on the second cohomology group. It naturally extends the definition for Kulikov degenerations.

In [KLSV17], Kollár, Laza, Saccà and Voisin study the essential skeleton of a degeneration of hyper-Kähler manifolds in terms of the type. More precisely, in Theorem 0.10, given a minimal dlt degeneration of $2n$-dimensional hyper-Kähler manifolds, firstly they prove that the dual complex of the special fibre has dimension $(\nu - 1)n$, where ν denotes the type of the degeneration. Secondly, they prove that, in the type III case, the dual complex is a simply connected closed pseudo-manifold with the rational homology of $\mathbb{CP}^n$.

From this prospective, Proposition 6.2.4 and Proposition 6.3.4 confirm and strengthen their result for the specific cases of Hilbert schemes and generalized Kummer varieties. In particular, we turn the rational cohomological description of the essential skeleton (Theorem 0.10(ii)) into a topological characterization.

For Hilbert schemes associated to some type II degenerations of K3 surfaces, a complementary proof of our result is due to Gulbrandsen, Halle, Hulek and Zhang, see [GHH16] and [GHHZ17]. Their approach is based on the method of *expanded degenerations*, which first appeared in [Li01], and on the construction of suitable GIT quotients, in order to obtain an explicit minimal dlt degeneration for the associated family of Hilbert schemes.

(6.4.3) The structure of the essential skeleton of a degeneration of hyper-Kähler manifolds is relevant in the context of mirror symmetry and in view of the work of Kontsevich and Soibelman ([KS01], [KS06]). The SYZ fibration ([SYZ96]) is a conjectural geometric explanation for the phenomenon of mirror symmetry and, roughly speaking, asserts the existence of a special Lagrangian fibration, such that mirror pairs of manifolds with trivial canonical bundle should admit fibrewise dual special Lagrangian fibrations. Moreover, the expectation is that, for type III degenerations of $2n$-dimensional hyper-Kähler manifolds, the base of the SYZ fibration is $\mathbb{CP}^n$ (see for instance [Hwa08]).

The most relevant fact from our prospective is that Kontsevich and Soibelman predict that the base of the Lagrangian fibration of a type III degeneration is homeomorphic to the essential skeleton. So, the outcomes on the topology of the essential skeleton we obtain in Proposition 6.2.4 and Proposition 6.3.4 match the predictions of mirror symmetry about the occurrence of $\mathbb{CP}^n$ in the type III case.

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