Final Project Report

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Introduction

This report addresses **Project 1: An Unfortunate Series of Events**, which focuses on solving a second-order linear ordinary differential equation using the Frobenius method. The equation under consideration is:

$$12x^2(3x+1)^2\frac{d^2y}{dx^2} + 7x\frac{dy}{dx} - 2y = 0,$$

analyzed about the point $x_0 = \frac{1}{2}$. The main objectives of this project are to:

- Transform the equation into standard form and analyze its singularities
- Apply the Frobenius method to derive a general solution
- Determine whether a logarithmic term arises in the second solution
- Establish the minimal interval of convergence
- Generate the first several nonzero terms of each solution

This report fulfills both the technical and creative requirements of Project 1 by implementing the full Frobenius framework—from indicial analysis and recurrence derivation to convergence justification and series construction.

Technical Objective 1: Solution Technique

The Frobenius method is appropriate because the equation contains non-constant polynomial coefficients, making standard methods like undetermined coefficients or variation of parameters inapplicable. The Frobenius method provides a systematic way to construct solutions near regular singular points. It was applied with expansion about $x_0 = \frac{1}{2}$. This unconventional center point:

- Avoids proximity to singularities at $x = 0$ and $x = -\frac{1}{3}$
- Provides maximum analytic continuation potential
- Satisfies the `Showboat Task` requirement

The assumed solution form is:

$$y(x) = \sum_{n=0}^{\infty} a_n \left(x - \tfrac{1}{2}\right)^{n+r_1} + Cy_1(x)\ln\left(x - \tfrac{1}{2}\right) + \sum_{n=0}^{\infty} b_n \left(x - \tfrac{1}{2}\right)^{n+r_2}$$

where $r_1 = 1$ and $r_2 = 0$, as determined from the indicial equation. Since $r_1 - r_2 = N$ is a positive integer, this informed our choice of solution form.

Technical Objective 2: Interval of Convergence

To analyze convergence, we centered the Frobenius series at $x_0 = \frac{1}{2}$ using the substitution $x = z + \frac{1}{2}$. Singularities arise from the original equation's denominator, which vanishes at $x = 0$ and $x = -\frac{1}{3}$. Their distances from $x_0 = \frac{1}{2}$ are:

$$\frac{1}{2} \quad \text{and} \quad \frac{5}{6}.$$

Hence, the radius of convergence is $\rho = \frac{1}{2}$, and the series converges on interval:

$$\boxed{(0,\ 1)}$$

Technical Objective 3: General Solution

To develop a general solution, we begin with the indicial equation $r(r-1) = 0$, which yields roots $r_1 = 1$ and $r_2 = 0$, differing by an integer (N). According to Frobenius theory, this could introduce a logarithmic term in the second solution. However, since the critical limit $\lim_{r \to r_2} c_N(r)$ exists and the recurrence remains well-defined, no log term is required. $\therefore$ Both y_1 and y_2 are regular Frobenius series, and together they form the general solution.

First Solution:

$$y_1(x) = (x - \tfrac{1}{2}) + \frac{4}{225}(x - \tfrac{1}{2})^2 - \frac{3944}{202500}(x - \tfrac{1}{2})^3 - \frac{2723128}{101250000}(x - \tfrac{1}{2})^4 + \cdots$$

Second Solution:

$$y_2(x) = 1 + \frac{4}{75}(x - \tfrac{1}{2})^2 - \frac{3944}{16875}(x - \tfrac{1}{2})^3 + \frac{68869}{1265625}(x - \tfrac{1}{2})^4 + \cdots$$

Therefore, the general solution is:

$$\boxed{\begin{aligned} y(x) &= a_0\, y_1(x) + b_0\, y_2(x) \\ &= a_0 \left[(x - \tfrac{1}{2}) + \frac{4}{225}(x - \tfrac{1}{2})^2 - \frac{3944}{202500}(x - \tfrac{1}{2})^3 - \frac{2723128}{101250000}(x - \tfrac{1}{2})^4 + \cdots \right] \\ &\quad + b_0 \left[1 + \frac{4}{75}(x - \tfrac{1}{2})^2 - \frac{3944}{16875}(x - \tfrac{1}{2})^3 + \frac{68869}{1265625}(x - \tfrac{1}{2})^4 + \cdots \right] \end{aligned}}$$

Creative Reflection: The Heart of Frobenius' Method

Foundations matter. What seems like chaos in series expansions
often reveals hidden patterns when viewed with patience.

Redefining perspective—shifting a center point, revisiting assumptions—
can transform impossible problems into solvable ones.

Obstacles like singularities aren't roadblocks but invitations
to think deeper about convergence and meaning.

Brick by brick, we built solutions: from indicial equations
to recurrence relations, each step a deliberate choice.

Elegance emerges when theory guides computation,
and computation breathes life into theory.

Nuance separates rote calculation from true understanding—
analytic and non-analytic behaviors demand equal care.

Insight grows when we stop memorizing steps
and start seeing the "why" behind the method.

Unity hides in the details. Those tedious recurrence relations?
They're the hidden language of the solution's soul.

Success isn't just about reaching an answer but embracing
the chaos of the `Showboat Task`—a true unfortunate series of events.

AI Statement

Generative AI was used to assist with equation formatting verification, LaTeX structuring, and to provide additional perspectives during convergence interval analysis. It also supported error-checking during the derivation of the recurrence relations by helping to identify potential sign inconsistencies. However, all critical steps—including problem setup, choice of solution method, detailed derivations, and interpretation of results—were performed manually.

Conclusion

The Frobenius method successfully resolved the given second-order linear differential equation about $x_0 = \frac{1}{2}$, yielding two independent power series solutions valid on the interval $(0, 1)$, as determined by the nearest singularities. This project reinforced the full Frobenius process: identifying the regular singular point, computing the indicial equation, aligning powers through reindexing, deriving the recurrence relation, and generating the first several nonzero terms. Centering at $x_0 = \frac{1}{2}$ not only maximized the radius of convergence but also satisfied the `Showboat Task` requirement, highlighting the importance of thoughtful series placement. Although the indicial roots differed by an integer, the recurrence remained well-defined, confirming that no logarithmic term was needed. Overall, this work deepened understanding of series solutions, revealed structure within complex equations, and demonstrated the value of precision and persistence in analytical methods.

References

1. Zill, D. G. (2016). *Advanced Engineering Mathematics* (6th ed.). Jones & Bartlett Learning.

Appendix

Each section appears in the following order in the appendix:

1. Change of Variables and ODE Expansion
2. Interval of Convergence
3. Indicial Equation
4. Frobenius Method Assumption
5. Expansion and Reindexing Process
6. Deriving the General Recurrence Relation
7. Recurrence Relations for $y_1(z)$ and $y_2(z)$
8. First Four Nonzero Terms for $y_1(z)$
9. Logarithmic Term Check for Second Solution $y_2(z)$
10. First Four Nonzero Terms for $y_2(z)$
11. Final General Solution in Terms of x
12. Final General Solution

1. Change of Variables and ODE Expansion

We are solving:

$$12x^2(3x+1)^2\frac{d^2y}{dx^2}+7x\frac{dy}{dx}-2y=0$$

around the point $x_0=\frac{1}{2}$. First, introduce a change of variables to shift the regular singular point to $z=0$:

$$z=x-\frac{1}{2},\quad \text{so that}\quad x=z+\frac{1}{2}.$$

Substituting, the ODE becomes (expressed in terms of z):

$$12(z+\tfrac{1}{2})^2(3(z+\tfrac{1}{2})+1)^2\frac{d^2y}{dz^2}+7(z+\tfrac{1}{2})\frac{dy}{dz}-2y=0.$$

Expand and simplify the coefficients step-by-step. First simplify $(3x+1)^2$:

$$3(z+\tfrac{1}{2})+1=3z+\tfrac{5}{2},\quad\Rightarrow\quad(3z+\tfrac{5}{2})^2=9z^2+15z+\tfrac{25}{4}.$$

Also:

$$(z+\tfrac{1}{2})^2=z^2+z+\tfrac{1}{4}.$$

Thus:

$$(z+\tfrac{1}{2})^2(3(z+\tfrac{1}{2})+1)^2=(z^2+z+\tfrac{1}{4})(9z^2+15z+\tfrac{25}{4}).$$

Expand this:

$$\begin{aligned}&(z^2)(9z^2+15z+\tfrac{25}{4})+(z)(9z^2+15z+\tfrac{25}{4})+(\tfrac{1}{4})(9z^2+15z+\tfrac{25}{4})\\=&\,9z^4+15z^3+\tfrac{25}{4}z^2+9z^3+15z^2+\tfrac{25}{4}z+\tfrac{9}{4}z^2+\tfrac{15}{4}z+\tfrac{25}{16}.\end{aligned}$$

Group like terms:

$$9z^4+24z^3+\left(\tfrac{25}{4}+15+\tfrac{9}{4}\right)z^2+\left(\tfrac{25}{4}+\tfrac{15}{4}\right)z+\tfrac{25}{16},$$

which simplifies to:

$$9z^4+24z^3+\tfrac{94}{4}z^2+\tfrac{40}{4}z+\tfrac{25}{16},$$

or equivalently:

$$9z^4+24z^3+23.5z^2+10z+\tfrac{25}{16}.$$

Thus, the ODE becomes:

$$12(9z^4+24z^3+23.5z^2+10z+\tfrac{25}{16})\frac{d^2y}{dz^2}+7(z+\tfrac{1}{2})\frac{dy}{dz}-2y=0,$$

which expands to:

$$(108z^4 + 288z^3 + 282z^2 + 120z + \tfrac{75}{4})\frac{d^2y}{dz^2} + (7z + \tfrac{7}{2})\frac{dy}{dz} - 2y = 0.$$

2. Interval of Convergence

To find the interval of convergence, we analyze where the original equation becomes singular:

$$12x^2(3x+1)^2y'' + 7xy' - 2y = 0.$$

First, divide by the leading coefficient to get standard form:

$$y'' + P(x)y' + Q(x)y = 0, \quad \text{where } P(x) = \frac{7x}{12x^2(3x+1)^2}, \quad Q(x) = -\frac{2}{12x^2(3x+1)^2}.$$

The singularities occur where the denominator vanishes:

$$12x^2(3x+1)^2 = 0 \quad \Rightarrow \quad x = 0, \quad x = -\tfrac{1}{3}.$$

Since our Frobenius expansion is centered at $x_0 = \frac{1}{2}$, we compute the distance to each singularity:

$$\left|\tfrac{1}{2} - 0\right| = \tfrac{1}{2}, \qquad \left|\tfrac{1}{2} - (-\tfrac{1}{3})\right| = \tfrac{5}{6}.$$

The closest singularity is at $x = 0$, so the radius of convergence is:

$$\rho = \min\left(\tfrac{1}{2}, \tfrac{5}{6}\right) = \tfrac{1}{2}.$$

Conclusion: The series converges on the open interval:

$$\boxed{(0,\ 1)}$$

3. Indicial Equation

First, divide through the differential equation by the coefficient of $\frac{d^2y}{dz^2}$ to normalize it. Thus:

$$y'' + P(z)y' + Q(z)y = 0$$

where:

$$P(z) = \frac{7z + \frac{7}{2}}{108z^4 + 288z^3 + 282z^2 + 120z + \frac{75}{4}}, \quad Q(z) = -\frac{2}{108z^4 + 288z^3 + 282z^2 + 120z + \frac{75}{4}}.$$

Compute:

$$p_0 = \lim_{z\to 0} zP(z) = 0, \quad q_0 = \lim_{z\to 0} z^2 Q(z) = 0.$$

Thus, the indicial equation becomes:

$$r(r-1) + p_0 r + q_0 = 0 \quad \Rightarrow \quad r(r-1) = 0.$$

Solving:

$$r_1 = 1 \quad \text{or} \quad r_2 = 0.$$

Since $r_1 - r_2 = 1$ (an integer), and $r_1 > r_2$, the Frobenius theory predicts a second solution may involve a logarithmic term. However, after deriving the recurrence relations, no obstruction occurred, and no logarithmic terms were needed. Thus, both solutions are pure power series expansions.

4. Frobenius Method Assumption

Assume a Frobenius series solution:

$$y(z) = z^r \sum_{n=0}^{\infty} c_n z^n = \sum_{n=0}^{\infty} c_n z^{n+r},$$

thus:

$$y'(z) = \sum_{n=0}^{\infty} c_n (n+r) z^{n+r-1}, \quad y''(z) = \sum_{n=0}^{\infty} c_n (n+r)(n+r-1) z^{n+r-2}.$$

Substituting into the ODE:

$$(108z^4 + 288z^3 + 282z^2 + 120z + \tfrac{75}{4}) \sum_{n=0}^{\infty} c_n (n+r)(n+r-1) z^{n+r-2} + (7z + \tfrac{7}{2}) \sum_{n=0}^{\infty} c_n (n+r) z^{n+r-1} - 2 \sum_{n=0}^{\infty} c_n z^{n+r} = 0.$$

5. Expansion and Reindexing Process

Starting from the substitution into the original ODE (in terms of z):

$$(108z^4 + 288z^3 + 282z^2 + 120z + \tfrac{75}{4}) \sum_{n=0}^{\infty} c_n (n+r)(n+r-1) z^{n+r-2} + (7z + \tfrac{7}{2}) \sum_{n=0}^{\infty} c_n (n+r) z^{n+r-1} - 2 \sum_{n=0}^{\infty} c_n z^{n+r} = 0.$$

Distributing and expanding each term:

$$= 108 \sum_{n=0}^{\infty} c_n (n+r)(n+r-1) z^{n+r+2} + 288 \sum_{n=0}^{\infty} c_n (n+r)(n+r-1) z^{n+r+1}$$
$$+ 282 \sum_{n=0}^{\infty} c_n (n+r)(n+r-1) z^{n+r} + 120 \sum_{n=0}^{\infty} c_n (n+r)(n+r-1) z^{n+r-1}$$
$$+ \frac{75}{4} \sum_{n=0}^{\infty} c_n (n+r)(n+r-1) z^{n+r-2} + 7 \sum_{n=0}^{\infty} c_n (n+r) z^{n+r}$$
$$+ \frac{7}{2} \sum_{n=0}^{\infty} c_n (n+r) z^{n+r-1} - 2 \sum_{n=0}^{\infty} c_n z^{n+r}$$

Reindexing All Sums to Express Powers as z^{k+r}

Note: I decided to switch from $n \to k$ to highlight the goal of expressing all terms uniformly as z^{k+r}.

$$= 108 \sum_{k=2}^{\infty} c_{k-2} (k-2+r)(k-3+r) z^{k+r} + 288 \sum_{k=1}^{\infty} c_{k-1} (k-1+r)(k-2+r) z^{k+r}$$
$$+ 282 \sum_{k=0}^{\infty} c_k (k+r)(k+r-1) z^{k+r} + 120 \sum_{k=-1}^{\infty} c_{k+1} (k+1+r)(k+r) z^{k+r}$$
$$+ \frac{75}{4} \sum_{k=-2}^{\infty} c_{k+2} (k+2+r)(k+1+r) z^{k+r} + 7 \sum_{k=0}^{\infty} c_k (k+r) z^{k+r}$$
$$+ \frac{7}{2} \sum_{k=-1}^{\infty} c_{k+1} (k+1+r) z^{k+r} - 2 \sum_{k=0}^{\infty} c_k z^{k+r}$$

Isolating Low-Index Terms to Uniformly Shift Series to $k = 2$

First Term: $108 \sum_{k=2}^{\infty}$

Already starts at $k = 2$, no expansion needed. Thus:

$$108 \sum_{k=2}^{\infty} c_{k-2} (k-2+r)(k-3+r) z^{k+r}$$

Second Term: $288 \sum_{k=1}^{\infty}$

Expand $k = 1$ term:

$$288 c_0 r(r-1) z^{r+1}$$

Remaining summation from $k = 2$:

$$288 \sum_{k=2}^{\infty} c_{k-1} (k-1+r)(k-2+r) z^{k+r}$$

Third Term: $282\sum_{k=0}^{\infty}$

Expand $k = 0$ and $k = 1$ terms:

At $k = 0$:

$$282c_0r(r-1)z^r$$

At $k = 1$:

$$282c_1(r+1)rz^{r+1}$$

Remaining summation from $k = 2$:

$$282\sum_{k=2}^{\infty}c_k(k+r)(k+r-1)z^{k+r}$$

Fourth Term: $120\sum_{k=-1}^{\infty}$

Expand $k = -1$ and $k = 0$ terms:

At $k = -1$:

$$120c_0(r-1)(r-2)z^{r-1}$$

At $k = 0$:

$$120c_1(r+1)rz^r$$

Remaining summation from $k = 2$:

$$120\sum_{k=2}^{\infty}c_{k+1}(k+1+r)(k+r)z^{k+r}$$

Fifth Term: $\frac{75}{4}\sum_{k=-2}^{\infty}$

Expand $k = -2$, $k = -1$, and $k = 0$ terms:

At $k = -2$:

$$\frac{75}{4}c_0(r-2)(r-3)z^{r-2}$$

At $k = -1$:

$$\frac{75}{4}c_1(r-1)(r-2)z^{r-1}$$

At $k = 0$:

$$\frac{75}{4}c_2(r+2)(r+1)z^r$$

Remaining summation from $k = 2$:

$$\frac{75}{4}\sum_{k=2}^{\infty}c_{k+2}(k+2+r)(k+1+r)z^{k+r}$$

Sixth Term: $7\sum_{k=0}^{\infty}$

Expand $k = 0$ and $k = 1$ terms:

At $k = 0$:

$$7c_0 r z^r$$

At $k = 1$:

$$7c_1(r+1)z^{r+1}$$

Remaining summation from $k = 2$:

$$7\sum_{k=2}^{\infty} c_k(k+r)z^{k+r}$$

Seventh Term: $\frac{7}{2}\sum_{k=-1}^{\infty}$

Expand $k = -1$ and $k = 0$ terms:

At $k = -1$:

$$\frac{7}{2}c_0(r-1)z^{r-1}$$

At $k = 0$:

$$\frac{7}{2}c_1(r+1)z^r$$

Remaining summation from $k = 2$:

$$\frac{7}{2}\sum_{k=2}^{\infty} c_{k+1}(k+1+r)z^{k+r}$$

Eighth Term: $-2\sum_{k=0}^{\infty}$

Expand $k = 0$ and $k = 1$ terms:

At $k = 0$:

$$-2c_0 z^r$$

At $k = 1$:

$$-2c_1 z^{r+1}$$

Remaining summation from $k = 2$:

$$-2\sum_{k=2}^{\infty} c_k z^{k+r}$$

Grouping by Powers of z

We group all contributions by the powers:

Term in z^{r-2}

From:

$$\frac{75}{4}c_0(r-2)(r-3)z^{r-2}$$

Thus:

$$\text{Coefficient of } z^{r-2}: \quad \frac{75}{4}c_0(r-2)(r-3)$$

Term in z^{r-1}

From:

$$120c_0(r-1)(r-2)z^{r-1} + \frac{75}{4}c_1(r-1)(r-2)z^{r-1} + \frac{7}{2}c_0(r-1)z^{r-1}$$

Thus:

$$\text{Coefficient of } z^{r-1}: \quad 120c_0(r-1)(r-2) + \frac{75}{4}c_1(r-1)(r-2) + \frac{7}{2}c_0(r-1)$$

Term in z^r

From:

$$282c_0r(r-1)z^r + 120c_1(r+1)rz^r + \frac{75}{4}c_2(r+2)(r+1)z^r + 7c_0rz^r + \frac{7}{2}c_1(r+1)z^r - 2c_0z^r$$

Thus:

$$\begin{aligned}\text{Coefficient of } z^r: \quad & 282c_0r(r-1) + 7c_0r - 2c_0 \\ & + 120c_1(r+1)r + \frac{7}{2}c_1(r+1) + \frac{75}{4}c_2(r+2)(r+1)\end{aligned}$$

Term in z^{r+1}

From:

$$288c_0r(r-1)z^{r+1} + 282c_1(r+1)rz^{r+1} + 7c_1(r+1)z^{r+1} - 2c_1z^{r+1} + 288c_0r(r-1)z^{r+1}$$

Thus:

$$\text{Coefficient of } z^{r+1}: \quad 288c_0r(r-1) + 282c_1(r+1)r + 7c_1(r+1) - 2c_1$$

Remaining Summations (starting from $k = 2$)

The remaining infinite sums:

$$108\sum_{k=2}^{\infty} c_{k-2}(k-2+r)(k-3+r)z^{k+r} + 288\sum_{k=2}^{\infty} c_{k-1}(k-1+r)(k-2+r)z^{k+r}$$
$$+ 282\sum_{k=2}^{\infty} c_k(k+r)(k+r-1)z^{k+r} + 120\sum_{k=2}^{\infty} c_{k+1}(k+1+r)(k+r)z^{k+r}$$
$$+ \frac{75}{4}\sum_{k=2}^{\infty} c_{k+2}(k+2+r)(k+1+r)z^{k+r} + 7\sum_{k=2}^{\infty} c_k(k+r)z^{k+r}$$
$$+ \frac{7}{2}\sum_{k=2}^{\infty} c_{k+1}(k+1+r)z^{k+r} - 2\sum_{k=2}^{\infty} c_k z^{k+r}$$

Setting Coefficients to Zero

Each coefficient must vanish:

Term in z^{r-2}

$$\frac{75}{4}c_0(r-2)(r-3) = 0$$

Term in z^{r-1}

$$120c_0(r-1)(r-2) + \frac{75}{4}c_1(r-1)(r-2) + \frac{7}{2}c_0(r-1) = 0$$

Term in z^r

$$282c_0r(r-1) + 7c_0r - 2c_0 + 120c_1(r+1)r + \frac{7}{2}c_1(r+1) + \frac{75}{4}c_2(r+2)(r+1) = 0$$

Term in z^{r+1}

$$288c_0r(r-1) + 282c_1(r+1)r + 7c_1(r+1) - 2c_1 = 0$$

Case 1: $r_1 = 1$

Term in z^{r-1}

At $r = 1$, each factor $(r-1) = 0$, so this term vanishes automatically. No condition imposed.

Term in z^r

Substituting $r = 1$:

$$282c_0(1)(0) + 7c_0(1) - 2c_0 + 120c_1(2)(1) + \frac{7}{2}c_1(2) + \frac{75}{4}c_2(3)(2) = 0$$

Simplifying:

$$5c_0 + 247c_1 + \frac{225}{2}c_2 = 0$$

Term in z^{r+1}

Substituting $r = 1$:

$$288c_0(1)(0) + 282c_1(2)(1) + 7c_1(2) - 2c_1 = 0$$

Simplifying:

$$576c_1 = 0 \quad \Rightarrow \quad c_1 = 0$$

Substituting $c_1 = 0$ into the z^r equation:

$$5c_0 + \frac{225}{2}c_2 = 0 \quad \Rightarrow \quad c_2 = -\frac{2}{90}c_0 = -\frac{1}{45}c_0$$

Case 2: $r_2 = 0$

Term in z^{r-1}

Substituting $r = 0$:

$$120c_0(-1)(-2) + \frac{75}{4}c_1(-1)(-2) + \frac{7}{2}c_0(-1) = 0$$

Simplifying:

$$240c_0 + \frac{75}{2}c_1 - \frac{7}{2}c_0 = 0$$

Multiplying through by 2:

$$480c_0 + 75c_1 - 7c_0 = 0 \quad \Rightarrow \quad 473c_0 + 75c_1 = 0 \quad \Rightarrow \quad c_1 = -\frac{473}{75}c_0$$

Term in z^r

Substituting $r = 0$:

$$282c_0(0)(-1) + 7c_0(0) - 2c_0 + 120c_1(1)(0) + \frac{7}{2}c_1(1) + \frac{75}{4}c_2(2)(1) = 0$$

Simplifying:

$$-2c_0 + \frac{7}{2}c_1 + \frac{75}{2}c_2 = 0$$

Substituting $c_1 = -\frac{473}{75}c_0$:

$$-2c_0 + \frac{7}{2}\left(-\frac{473}{75}c_0\right) + \frac{75}{2}c_2 = 0$$

Simplifying:

$$-2c_0 - \frac{3311}{150}c_0 + \frac{75}{2}c_2 = 0$$

Multiplying through by 150:

$$-300c_0 - 3311c_0 + 5625c_2 = 0 \quad \Rightarrow \quad -3611c_0 + 5625c_2 = 0 \quad \Rightarrow \quad c_2 = \frac{3611}{5625}c_0$$

Summary of Results

For $r_1 = 1$:

$$c_1 = 0, \quad c_2 = -\frac{1}{45}c_0$$

For $r_2 = 0$:

$$c_1 = -\frac{473}{75}c_0, \quad c_2 = \frac{3611}{5625}c_0$$

6. Deriving the General Recurrence Relation

Starting from the shifted terms:

$$\begin{aligned}
&108\sum_{k=2}^{\infty} c_{k-2}(k-2+r)(k-3+r)z^{k+r} + 288\sum_{k=2}^{\infty} c_{k-1}(k-1+r)(k-2+r)z^{k+r} \\
&+ 282\sum_{k=2}^{\infty} c_k(k+r)(k+r-1)z^{k+r} + 120\sum_{k=2}^{\infty} c_{k+1}(k+1+r)(k+r)z^{k+r} \\
&+ \frac{75}{4}\sum_{k=2}^{\infty} c_{k+2}(k+2+r)(k+1+r)z^{k+r} \\
&+ 7\sum_{k=2}^{\infty} c_k(k+r)z^{k+r} + \frac{7}{2}\sum_{k=2}^{\infty} c_{k+1}(k+1+r)z^{k+r} - 2\sum_{k=2}^{\infty} c_k z^{k+r} = 0
\end{aligned}$$

Grouping coefficients at each z^{k+r}:

$$\begin{aligned}
&108c_{k-2}(k-2+r)(k-3+r) + 288c_{k-1}(k-1+r)(k-2+r) \\
&+ [282(k+r)(k+r-1) + 7(k+r) - 2]c_k \\
&+ [120(k+1+r)(k+r) + \frac{7}{2}(k+1+r)]c_{k+1} \\
&+ \frac{75}{4}(k+2+r)(k+1+r)c_{k+2} = 0
\end{aligned}$$

Moving all terms except c_{k+2} to the right:

$$\frac{75}{4}(k+2+r)(k+1+r)c_{k+2} = -\Big[108c_{k-2}(k-2+r)(k-3+r) + 288c_{k-1}(k-1+r)(k-2+r)$$

$$+(282(k+r)(k+r-1)+7(k+r)-2)c_k+(120(k+1+r)(k+r)+\frac{7}{2}(k+1+r))c_{k+1}\Bigg]$$

Thus:

$$c_{k+2}=\frac{-4}{75(k+2+r)(k+1+r)}\Bigg[108c_{k-2}(k-2+r)(k-3+r)+288c_{k-1}(k-1+r)(k-2+r)$$

$$+(282(k+r)(k+r-1)+7(k+r)-2)c_k+(120(k+1+r)(k+r)+\frac{7}{2}(k+1+r))c_{k+1}\Bigg]$$

7. Recurrence Relations for $y_1(z)$ and $y_2(z)$

For $y_1(z)$ (where $r_1 = 1$ and $c_k \to a_k$)

Substituting $r_1 = 1$, we get:

$$\begin{aligned}a_{k+2}=\frac{-4}{75(k+3)(k+2)}\Bigg[&108a_{k-2}(k-1)(k-2)+288a_{k-1}k(k-1)\\&+(282k(k-1)+7k-2)\,a_k\\&+\left(120(k+1)k+\frac{7}{2}(k+1)\right)a_{k+1}\Bigg]\end{aligned}$$

For $y_2(z)$ (where $r_2 = 0$ and $c_k \to b_k$)

Substituting $r_2 = 0$, we get:

$$\begin{aligned}b_{k+2}=\frac{-4}{75(k+2)(k+1)}\Bigg[&108b_{k-2}(k-2)(k-3)+288b_{k-1}(k-1)(k-2)\\&+(282k(k-1)+7k-2)\,b_k\\&+\left(120(k+1)k+\frac{7}{2}(k+1)\right)b_{k+1}\Bigg]\end{aligned}$$

8. First Four Nonzero Terms for $y_1(z)$

We assume a Frobenius series solution about $z = 0$ with $r_1 = 1$:

$$y_1(z)=z\sum_{k=0}^{\infty}a_kz^k=\sum_{k=0}^{\infty}a_kz^{k+1}$$

From coefficient matching earlier:

$$a_0 = \text{free constant}, \quad a_1 = 0$$

Now using the recurrence relation:

$$a_{k+2} = \frac{-4}{75(k+3)(k+2)}\Big[108a_{k-2}(k-1)(k-2) + 288a_{k-1}k(k-1) + (282k(k-1)+7k-2)\,a_k + \left(120(k+1)k + \tfrac{7}{2}(k+1)\right)a_{k+1}\Big]$$

Step 1: Compute a_2
At $k = 0$:

$$a_2 = \frac{-4}{75(3)(2)}\left[-2a_0\right] = \frac{8}{450}a_0 = \frac{4}{225}a_0$$

Step 2: Compute a_3
At $k = 1$:

$$a_3 = \frac{-4}{75(4)(3)}\left[(240+7)a_2\right] = \frac{-4}{900}\cdot 247a_2$$

Substitute $a_2 = \frac{4}{225}a_0$:

$$a_3 = \frac{-4\cdot 247\cdot 4}{900\cdot 225}a_0 = -\frac{3944}{202500}a_0$$

Step 3: Compute a_4
At $k = 2$:

$$a_4 = \frac{-4}{75(5)(4)}\left[576a_2 + \tfrac{1461}{2}a_3\right] = \frac{-4}{1500}\left[576a_2 + \tfrac{1461}{2}a_3\right]$$

Substitute known values:

$$a_2 = \frac{4}{225}a_0, \quad a_3 = -\frac{3944}{202500}a_0$$

$$576\cdot\frac{4}{225} = \frac{2304}{225} = \frac{1024}{75}, \quad \tfrac{1461}{2}\cdot -\frac{3944}{202500} = -\frac{1444908}{405000}$$

Convert to common denominator and combine:

$$\frac{1024}{75} = \frac{552960}{40500}, \quad a_4 = \frac{-4}{1500}\cdot\left(\frac{4084692}{405000}\right)a_0 = -\frac{16338768}{607500000}a_0 = -\frac{2723128}{101250000}a_0$$

Final Organized Results

$$a_0 = a_0 \quad \text{(free constant)}$$

$$a_1 = 0$$

$$a_2 = \frac{4}{225}a_0$$

$$a_3 = -\frac{3944}{202500}a_0$$

$$a_4 = -\frac{2723128}{101250000}a_0$$

Thus:

$$y_1(z) = z + \frac{4}{225}z^2 - \frac{3944}{202500}z^3 - \frac{2723128}{101250000}z^4 + \cdots$$

9. Logarithmic Term Check for Second Solution $y_2(z)$

To find the 2nd solution, we must check if the *logarithmic term* is needed.This term is unnecessary if

$$\lim_{r \to r_2} c_N(r) \quad \text{exists.}$$

In our case, $r_2 = 0$ and $N = 1$, so we examine:

$$\lim_{r \to 0} c_2(r) = \lim_{r \to 0} \frac{-2c_0 - \frac{7}{2}c_1}{\frac{75}{2}}.$$

Using earlier results:

$$c_1 = -\frac{473}{75}c_0, \quad c_2 = \frac{3611}{5625}c_0,$$

we substitute:

$$\lim_{r \to 0} c_2(r) = \frac{-2c_0 + \frac{7}{2} \cdot \frac{473}{75}c_0}{\frac{75}{2}} = \frac{3611}{5625}c_0.$$

Since the limit exists and is finite, the second series may be computed directly with *no logarithmic term.*

10. First Four Nonzero Terms for $y_2(z)$

We assume a Frobenius series solution about $z = 0$ with $r_2 = 0$:

$$y_2(z) = \sum_{k=0}^{\infty} b_k z^k$$

From coefficient matching earlier:

$$b_0 = \text{free constant}, \quad b_1 = 0$$

Now using the recurrence relation:

$$b_{k+2} = \frac{-4}{75(k+2)(k+1)}\Big[108b_{k-2}(k-2)(k-3) + 288b_{k-1}(k-1)(k-2) + (282k(k-1) + 7k - 2)\,b_k + \Big(120(k+1)k + \tfrac{7}{2}(k+1)\Big)\,b_{k+1}\Big]$$

Step 1: Compute b_2

At $k = 0$:

$$b_2 = \frac{-4}{150}\left[-2b_0\right] = \frac{8}{150}b_0 = \frac{4}{75}b_0$$

Step 2: Compute b_3

At $k = 1$:

$$b_3 = \frac{-4}{450}(247b_2)$$

Substitute $b_2 = \frac{4}{75}b_0$:

$$b_3 = \frac{-4 \cdot 247 \cdot 4}{450 \cdot 75}b_0 = -\frac{3944}{16875}b_0$$

Step 3: Compute b_4

At $k = 2$:

$$b_4 = \frac{-4}{900}\left[576b_2 + \tfrac{1461}{2}b_3\right]$$

Substitute:

$$b_2 = \frac{4}{75}b_0, \quad b_3 = -\frac{3944}{16875}b_0$$

Compute each term:

$$576 \cdot \frac{4}{75} = \frac{2304}{75} = \frac{768}{25}, \quad \tfrac{1461}{2} \cdot \left(-\frac{3944}{16875}\right) = -\frac{1444908}{33750}$$

$$\frac{768}{25} = \frac{1032192}{33750}, \quad \text{so combine: } \frac{-412716}{33750}$$

Now:

$$b_4 = \frac{-4}{900} \cdot \left(-\frac{412716}{33750}\right) = \frac{1650864}{30375000}b_0 = \frac{68869}{1265625}b_0$$

Final Organized Results

$$\begin{aligned}
b_0 &= b_0 \quad \text{(free constant)} \\
b_1 &= 0 \\
b_2 &= \frac{4}{75}b_0 \\
b_3 &= -\frac{3944}{16875}b_0 \\
b_4 &= \frac{68869}{1265625}b_0
\end{aligned}$$

Thus:

$$y_2(z) = 1 + \frac{4}{75}z^2 - \frac{3944}{16875}z^3 + \frac{68869}{1265625}z^4 + \cdots$$

11. Final General Solution in Terms of x

Using the substitution $z = x - \frac{1}{2}$, we convert the series solutions back into functions of x.

$$y_1(x) = (x - \tfrac{1}{2}) + \frac{4}{225}(x - \tfrac{1}{2})^2 - \frac{3944}{202500}(x - \tfrac{1}{2})^3 - \frac{2723128}{101250000}(x - \tfrac{1}{2})^4 + \cdots$$

$$y_2(x) = 1 + \frac{4}{75}(x - \tfrac{1}{2})^2 - \frac{3944}{16875}(x - \tfrac{1}{2})^3 + \frac{68869}{1265625}(x - \tfrac{1}{2})^4 + \cdots$$

12. Final General Solution

At long last, after careful expansion, recurrence juggling, and a log-term reality check, we arrive at the crown jewel of our Frobenius adventure: the full general solution expressed back in terms of x. This confirms the successful completion of the `Showboat Task`.

$$\boxed{\begin{aligned} y(x) &= a_0\, y_1(x) + b_0\, y_2(x) \\ &= a_0\left[(x - \tfrac{1}{2}) + \frac{4}{225}(x - \tfrac{1}{2})^2 - \frac{3944}{202500}(x - \tfrac{1}{2})^3 - \frac{2723128}{101250000}(x - \tfrac{1}{2})^4 + \cdots\right] \\ &\quad + b_0\left[1 + \frac{4}{75}(x - \tfrac{1}{2})^2 - \frac{3944}{16875}(x - \tfrac{1}{2})^3 + \frac{68869}{1265625}(x - \tfrac{1}{2})^4 + \cdots\right] \end{aligned}}$$