# The 27 lines on a smooth cubic surface and the 28 bitangents on a smooth quartic curve.

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This comes from a worked out exercise in Beltrametti, Carletti, Gallarati, Monti 's book "Lectures on Curbes, Surfaces and Projective Varieties". It is exercise 5.8.13.

First let $S$ be a smooth cubic surface in $\mathbb{P}^3_{x_0,x_1,x_2,x_3}$ and assume that $p = [1 : 0 : 0 : 0] \in S$. The we can write the equation of $S$ in the form

$$f = x_0^2\phi_1(x_1, x_2, x_3) + 2x_0\phi_2(x_1, x_2, x_3) + \phi_3(x_1, x_2, x_3) = 0.$$

Let now $C$ be the curve in $\mathbb{P}^2_{x_1,x_2,x_3}$ with equation

$$\Delta = \phi_1\phi_3 - \phi_2^2$$

(which is the discriminant of $f$ when viewed as an element of $k[x_1, x_2, x_3][x_0]$!). Thus, a point $q \in C$ is a point for which $f(x_0, q)$ is a square in $k[x_0]$. Note that $\phi_i$ is homogeneous of degree $i$ and so $C$ is a quartic (degree 4) in $\mathbb{P}^2$.

Let now $V$ be the cone in $\mathbb{P}^3_{x_0,x_1,x_2,x_3}$ defined by $\Delta$. Note that the vertex of this cone is $p$. Another way to view $V$ is to consider $C$ inside $\{x_0 = 0\}$, and then take all the lines joining $C$ to $p$ ($V$ is the cone with vertex at $p$ over $C$).

Now, any line in $\mathbb{P}^3_{x_0,x_1,x_2,x_3}$ intersects $S$ at 3 points counting multiplicities, and the important fact about the lines generating $V$ is that the line joining a point $q \in C$ to $p$ will intersect $X$ at $p$ and at another point with multiplicity 2 (because $\Delta$ is a discriminant). Thus, all the lines that generate $V$ have a point of double intersection with $S$. To see this explicitly, let $q \in C$ and note let $L$ be the line joining $q$ and $p$. Parametrically it is given by

$$sp + tq = [s : tq_1 : tq_2 : tq_3]$$

and its intersection with $S$ is give by the roots of the equation $f(L) = 0$ which gives

$$s^2t\phi_1(q) + 2st^2\phi_2(q) + t^3\phi_3(q) = 0$$

(because $\phi_i$ is homogeneous of degree $i$). One root is $t = 0$ which corresponds to $p$, and the other is a double root because the discriminant of

$$\left(\frac{s}{t}\right)^2 \phi_1(q) + 2\left(\frac{s}{t}\right) \phi_2(q) + \phi_3(q) = 0$$

vanishes.

Now, we need to make sure that $C$ is smooth so that we know it has 28 bitangents, and this is where the 27 lines will come from. So assume $C$ is smooth, this will happen for general $S$, and so our argument will only work for general $S$.

Say $L$ is a bitangent to $C$ through the two points $q_1, q_2$. Then the cone over $L$ is a plane $\pi$ in $\mathbb{P}^3_{x_0,x_1,x_2,x_3}$ which intersects $S$ at two points with multiplicity 2 (the special points lying above $q_1$ and $q_2$), and so the intersection of $\pi$ with $S$ is a cubic curve with two double points. But a cubic curve in $\pi \cong \mathbb{P}^2$ can have two double points only if it is reducible, one component being a line, and the other being a conic, where the double points are the intersections of the two components. Thus, $\pi \cap S$ contains a line!

Now this argument can't work in general because $S$ would contain 28 lines and not 27. The point is that $\phi_i$ is homogeneous of degree $i$ and so $\phi_1 = 0$ is a bitangent to $C$ because its intersection with $C$ is given by the equation $\phi_2^2 = 0$ which will intersect $\phi_1 = 0$ at two double points, so it will intersect $C$ with multiplicity 2 at two points. The plane $\pi$ above $\phi_1 = 0$ does not satisfy what we were saying above for planes above bitangents to $C$ because if $q$ is one of points of tangency of $\phi_1 = 0$ with $C$, then the intersection of the line through $q$ and $p$ is given by the roots of

$$2st^2\phi_2(q) + t^3\phi_3(q) = 0$$

(since $\phi_1(q) = 0$) which implies that $t = 0$ corresponding to $p$ is a double root.

Apart from this special bitangent, the other 27 do each give a line on the cubic surface! One needs to argue that all the lines are distinct, and the point is that each line projects down to its the corresponding bitangent, i.e., if $L$ was cut by a plane $\pi$ above the bitangent to $C$ through the two points $q_1, q_2$, then $L$ contains the two double points, and so has points above both $q_1$ and $q_2$ which implies that it gets projected down to the whole bitangent. Since a bitangent only intersects the curve at its two points of tangency because the degree of $C$ is 4, then all 28 bitangents are distinct and so there are 27 lines!

Conversely, if $L$ is a line on $S$ and we assume that $p$ is not contained in any line, then the plane spanned by $p$ and $L$ intersects $S$ in a reducible cubic with two double points on $L$. Therefore, the lines joining $p$ to these double points will only intersect $S$ at two points, and so the projection of these lines will be two points in $C$. The line joining these points, which is the projection of the plane $pL$ will be a bitangent to $C$ because it only intersects $C$ at these two points (one needs to argue that the intersections are both double), and so this implies that the plane $pL$ was one of the 28 we considered before.

This concludes the proof of the existence of the 27 lines, and their relation to the 28 bitangents of plane quartics!