Lines on cubic surfaces, elliptic surfaces and the E_6 lattice

T. Shioda, *Weierstrass Transformations and Cubic Surfaces*

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Cubic Surfaces

Theorem

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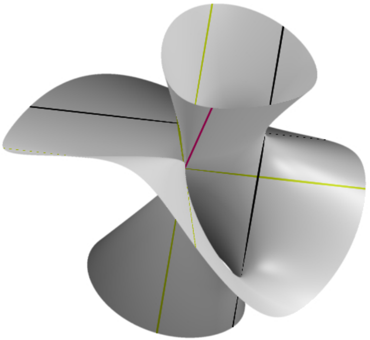
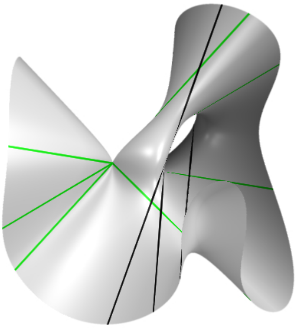
However...the lines are usually not defined over the reals.

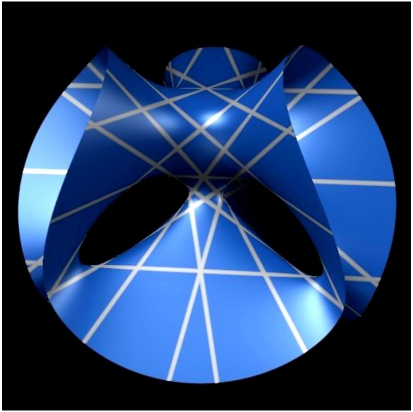


5 real lines



7 real lines

Cubic Surfaces

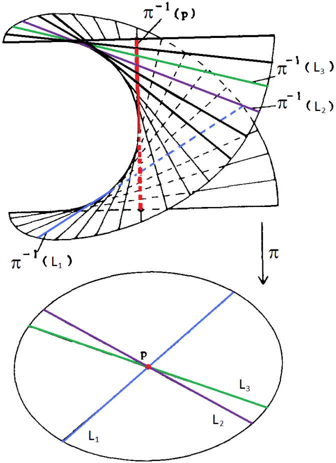


The Clebsch Diagonal Cubic

$$x^3 + y^3 + z^3 + w^3 - (x + y + z + w)^3 = 0$$

Blowing up

- Given any smooth algebraic surface, replace a point by a line (the Exceptional Line): a copy of $\mathbb{P}^1$.
- Each point on this line corresponds a tangent direction at the point.



Cubic Surfaces and blow up's

- Every smooth cubic in $\mathbb{P}^3$ is isomorphic to $\mathbb{P}^2$ blown up at six points (not all on a conic and no three on a line).
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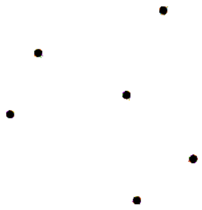

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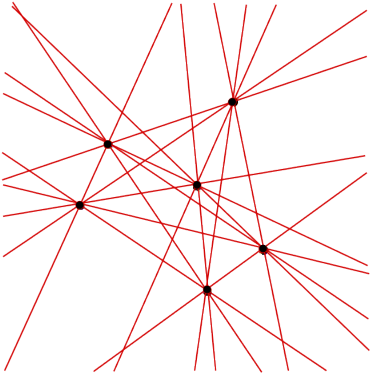

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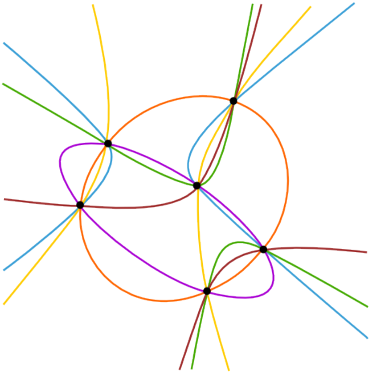

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Given six points in $\mathbb{P}^2$, ¿Can find the equation of the corresponding smooth cubic surface and the 27 lines in it?

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Example (Shioda)

The minumum field extension of $\mathbb{Q}$ where all the 27 lines of the cubic surface

$$y^2 + 2yz = x^3 + x + xz^2 + z + z^2 + 1$$

are defined is the splitting field of a polynomial of degree 27. The degree of this extension is $51,840$.

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More Generally (Harris)

There are no explicit equations for the 27 lines of a general cubic surface.

Question 2:

Given the six points you blow up, ¿Can you find the equation of the resulting smooth cubic surface and the 27 lines in it?

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- Let f_0, f_1, f_2, f_3 be a basis. The equation of any plane cubic through the six points is a linear combination of the f_i.
- Define the rational map $\mathbb{P}^2 \to \mathbb{P}^3$

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- The map is not defined at the six points since all the f_i give zero, but it extends to a map from the blowup of $\mathbb{P}^2$ at those points to $\mathbb{P}^3$ which is in fact an ISOMORPHISM.

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The Article

T. Shioda, *Weierstrass Transformations and Cubic Surfaces*, 1994.

- Explicit equation for the cubic in terms of the equations of the blown-up points.
- The construction involves the E_6 lattice and its dual E_6^*

The answer

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- The condition on their configuration is equivalent to:
  - $u_i \neq u_j$ if $i \neq j$.
  - $\sum u_i \neq 0$
  - $u_i + u_j + u_k \neq 0$ for i, j, k distinct.

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  - $\sum u_i \neq 0$
  - $u_i + u_j + u_k \neq 0$ for i, j, k distinct.
- Let c_n be the n-th symmetric function in the u_i: $\prod(x - u_i) = x^6 - c_1 x^5 + c_2 x^4 + \ldots + c_6$

The answer

Let ε_n is the n-th symmetric function in the 27 forms:

$$a_i = \frac{c_1}{3} - u_i \qquad a'_i = \frac{-2c_1}{3} - u_i \qquad a''_{ij} = \frac{c_1}{3} - u_i - u_j$$

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The equation of the cubic surface obtained by blowing up $P_1, \ldots, P_6$ is

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The answer

The equations for the 27 lines are also explicit:

$$x = az + b \qquad \bigcap \qquad y = dz + e$$

L_i: $a = a_i$.

L_i': $a = a_i'$.

L_{ij}': $a = a_{ij}'$.

$$b = \text{complicated expression in } c_1, c_2, c_3, c_4, u_i$$

In all cases:

$$d = (a^3 + ap_2)/2$$

$$e = (3a^2b - d^2 + ap_1 + bp_2 + q_2)/2$$

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The Mordell-Weil lattice of the elliptic curve

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- The 54 minimal roots are given by 27 points $P = (x, y)$ of the form

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- These roots generate the Mordell-Weil group, and one can give six explicit points which generate the Mordell-Weil group.

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- So we have a cubic surface and the 27 lines on it.

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- So we have a cubic surface and the 27 lines on it.
- Now relate this somehow to the blowup of six points in the plane....

The 28 bitangents