ALGEBRAIC AND SYMPLECTIC CURVES OF DEGREE 8

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Abstract. We study the existence of some irreducible projective plane curves of degree 8 with some prescribed topological type of singularities in the algebraic and symplectic worlds.

Introduction

Since the eighties the study of the theory of complex analytic and algebraic varieties have been enriched by the study of pseudo-holomorphic and complex symplectic varieties. In the case of curves in the projective plane these new objects are strongly related to braid monodromy, see [16, 13, 12], and can be constructed by suitable deformations of some arrangements of algebraic plane curves.

The starting point of this paper is an unpublished idea of S.Yu. Orevkov, which is explained in [11] and outlined in §2. The main idea is to deform symplectically a tricuspidal quartic (or deltoid) such that the tangent lines to the cusps are not concurrent. The idea of Orevkov, performed in detail by M. Golla and L. Starkston is to apply a standard Cremona transformation in order to obtain an irreducible symplectic curve with a configuration of topological type of singularities which does not exist in the algebraic category.

In this work, we replace the Cremona transformation by a Kummer cover, in order to compare the symplectic and algebraic structures of curves of degree $4n$ with $3n$ singularities having the topological type of $u^{2n}-v^3=0$. The case $n=2$ offers significant interesting properties and we focus our attention in this case since the study of algebraic structures seems to be cumbersome for $n>2$. We prove the existence of symplectic curves C_{symp} of degree 8 with 6 singular points of type $\mathbb{E}_6$.

For $n=2$ the primary goal is to determine all algebraic curves of degree 8 with 6 singular points with the topological type of $\mathbb{E}_6$. Unfortunately, the goal was too ambitious and has not been reached. As it usually happens, the existence of symmetries is helpful and in this paper we determine all such curves fixed by a non-trivial projective automorphism. There is exactly one such curve $C_{8,2}$ (up to projective automorphism, of course) fixed by an involution and four such curves $C_{8,3}^i$, $i=1,\dots,r$, invariant by an automorphism of order 3; there are no more curves invariant~~s~~ by automorphism. These four curves have equations in conjugate number fields $\mathbb{K}_i \subset \mathbb{C}$ isomorphic to $\mathbb{Q}[t]/p(t)$ where $p(t)$ is an irreducible polynomial of degree 4. A main question is if they share topological properties. Two of the roots of $p(t)$ are real and two complex conjugate; in this last case, complex conjugation is a homeomorphism of $\mathbb{P}^2$ reversing orientations on the curves. In the general case, most likely these curves are rigid by dimension arguments.

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Another result in this paper is that there is no homeomorphism of $\mathbb{P}^2$ sending C_{symp} to an algebraic symmetric curve, but it may be isotopic to a non symmetric one (if such a curve exists). There is also no homeomorphism of $\mathbb{P}^2$ sending $C_{8,2}$ to one of the $C^i_{8,3}$, and, besides complex conjugation, we do not know the existence of homeomorphism of $\mathbb{P}^2$ exchanging the curves $C^i_{8,3}$ (respecting or reversing orientations).

Some proofs need non-straightforward computer algebra steps and rely heavily on computations in `Sagemath` [18]. The steps are described in several notebooks located in `https://github.com/enriqueartal/SymplecticOctics` which can be executed either in a computer with the last version of `Sagemath` or online using `Binder` [17].

In §1 we describe some known properties of the deltoid and compute a special presentation of the fundamental group of the complement of the deltoid and the tangent lines at the cusps. In §2 we study the topology of a symplectic deformation of the previous arrangement of curves.

This paper is inspirated in the ideas fruitfully discussed in the workshop *Complex and symplectic curve configurations* organized by M. Golla, P. Pokorav and L. Starkston in Nantes, December 2022, and in particular in [11]. I would like to thank the organizers, the attendants, and specially the team work on braid monodromy and symplectic curves.

1. The deltoid and its tangents at the singular points

The deltoid (or tricuspidal quartic, i.e, plane quartic with three ordinary cusps) is an important plane projective curve. It is rigid, in the sense, that two deltoids are projectively isomorphic. As it is the dual of a nodal cubic, it has the following well-known property.

Property 1.1. *The three tangent lines at the cusps of a deltoid are concurrent lines.*

A symmetric equation of the deltoid is

$$y^2z^2 + z^2x^2 + x^2y^2 - 2xyz(x + y + z) = 0.$$

The equation of the curve in the right-hand side of Figure 1 is:

$$v^4 + 4(1 + u)v^3 + 18uv^2 - 27u^2 = 0.$$

We are going to use this property to have a topological model of the curve and its tangents.

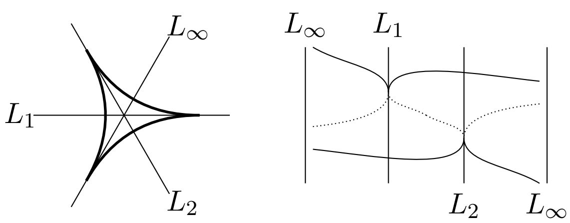

Figure 1. Left: usual deltoid. Right: real picture with real parts of non real branches.

Using the techniques of [3], applied to the right-hand side of Figure 1, we obtain the following result. The justification of this figure can be found in the notebook `ConstructionSymplecticGroup`.

Proposition 1.2. *The braid monodromy of the deltoid projecting from the intersection point of the tangent line to the cusps, when one of these tangents is the line at infinity (as in Figure 1, right) is given by* $(\sigma_2 \cdot \sigma_1)^2$ *(for* L_1*) and* $(\sigma_2 \cdot \sigma_3)^2$ *(for* L_2*).*

As a consequence, the fundamental group $G_{\Delta L}$ *of the complement of the deltoid and the lines* L_1, L_2, L_∞ *is generated by* $c_1, \ldots, c_4, \ell_1, \ell_2, \ell_\infty$ *with the relations*

(R1) $[\ell_2, c_1] = 1$
(R2) $\ell_2^{-1} \cdot c_2 \cdot \ell_2 = (c_2 \cdot c_3 \cdot c_4) \cdot c_3 \cdot (c_2 \cdot c_3 \cdot c_4)^{-1}$
(R3) $\ell_2^{-1} \cdot c_3 \cdot \ell_2 = (c_2 \cdot c_3) \cdot c_4 \cdot (c_2 \cdot c_3)^{-1}$
(R4) $\ell_2^{-1} \cdot c_4 \cdot \ell_2 = c_2$
(R5) $\ell_1^{-1} \cdot c_1 \cdot \ell_1 = (c_1 \cdot c_2) \cdot c_3 \cdot (c_1 \cdot c_2)^{-1}$
(R6) $\ell_1^{-1} \cdot c_2 \cdot \ell_1 = (c_1 \cdot c_2) \cdot c_1 \cdot (c_1 \cdot c_2)^{-1}$
(R7) $\ell_1^{-1} \cdot c_3 \cdot \ell_1 = c_1 \cdot c_2 \cdot c_1^{-1}$
(R8) $[\ell_1, c_4] = 1$
(R9) $c \cdot \ell_1 \cdot \ell_2 \cdot \ell_\infty = 1$

where $c = c_1 \cdot \ldots \cdot c_4$.

This monodromy can be also computed using `Sagemath` [18] with the optional package `Sirocco` [14], but in this case it can be done directly.

In the presentation of $G_{\Delta L}$ we may omit the generator ℓ_∞ using (R9) which comes from the situation at infinity. Actually, Zariski-van Kampen method can be thought to happen in the blow-up of the projection point (the point at infinity of the vertical lines), see Figure 2. Then (R9) comes from the boundary of a neighbourhood of the exceptional divisor E, see [15, 6]. Note that the *natural* meridian e of E is the inverse of c. The normal crossing situation implies that e (and hence c and $\ell_1 \cdot \ell_2 \cdot \ell_\infty$) commute with $\ell_1, \ell_2, \ell_\infty$. This is a consequence of (R2)-(R7): ℓ_1, ℓ_2 commute with c as their conjugation action comes from braids.

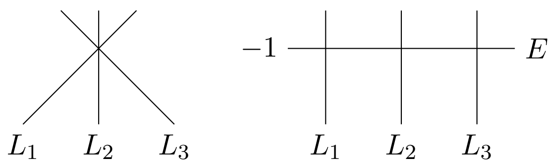

FIGURE 2. Blow-up of the projection point.

Remark 1.3. Note that $G_{\Delta L}$ is a semidirect product $\mathbb{F}_4 \rtimes \mathbb{F}_2$ where $c_1, \ldots, c_4$ are the generators of normal subgroup $\mathbb{F}_4$, ℓ_1, ℓ_2 are the generators of $\mathbb{F}_{\cancel{4}}$ 2 and (R2)-(R7), determine the conjugation action.

This group has been computed in [1], but we need the above computation both for completeness and to deal with the symplectic deformations.

2. Symplectic deformations

In the context of symplectic geometry, it is possible to construct a *deltoid* for which the pseudo-holomorphic tangent lines at the cusps are not concurrent. This was communicated long time ago to the author by S.Yu. Orevkov and was formally written in [11, § 8]. Moreover it can be done as a deformation of the algebraic curve which is an isotopy outside a neighbourhood of the triple point.

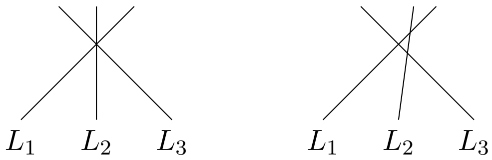

FIGURE 3. Symplectic deformation of an ordinary triple point.

Using the classical Seifert-van Kampen theorem, the fundamental group of the complement of this symplectic curve has the same presentation of the algebraic one, adding the relations from the situation in the right-hand side of Figure 3, i.e., $[\ell_1, \ell_2] = [\ell_1, \ell_\infty] = [\ell_2, \ell_\infty] = 1$. This technique has been used in [5, 2, 8] Actually, the following holds.

Corollary 2.1. *The fundamental group $G_{s\Delta L}$ of the complement of the* symplectic *deltoid and the tangent lines at the cusps has the generators and relators of Proposition 1.2 plus the relation*

(R10) $[\ell_1, \ell_2] = 1$.

It is useful to have a semidirect presentation of this group.

Corollary 2.2. *The group $G_{s\Delta L}$ is a semidirect product $G_0 \rtimes \mathbb{Z}^2$ where the action is as in the algebraic case and*

$$G_0 = \langle c_1, \dots, c_4 \mid c_3 \cdot c_4 \cdot c_3 = c_4 \cdot c_3 \cdot c_4,\ c_1 \cdot c_2 \cdot c_1 = c_2 \cdot c_1 \cdot c_2 \rangle.$$

Proof. We start with the semidirect product structure $G_{\Delta L} = \mathbb{F}_4 \rtimes \mathbb{F}_2$ and the natural epimorphism $G_{\Delta L} \twoheadrightarrow G_{s\Delta L}$:

$$\begin{tikzcd}
1 \arrow[r] & \mathbb{F}_4 \arrow[r] \arrow[d] & G_{\Delta L} \arrow[r] \arrow[d] & \mathbb{F}_2 \arrow[r] \arrow[d] \arrow[l, bend right] & 1 \\
1 \arrow[r] & G_0 \arrow[r] & G_{s\Delta L} \arrow[r] & \mathbb{Z}^2 \arrow[r] \arrow[l, bend left] & 1.
\end{tikzcd}$$

The semidirect structure comes from the fact that the below exact sequence splits as seen in the relations. The conjugation in the first exact sequence is given by the action of the braids $\tau_1 := (\sigma_2 \cdot \sigma_1)^2$ (for ℓ_1) and $\tau_2 := (\sigma_2 \cdot \sigma_3)^2$ (for ℓ_2). Since ℓ_1, ℓ_2 commute in $G_{s\Delta L}$, then in G_0 we have the relations (checked in `ConstructionSymplecticGroup`)

$$c_i^{\tau_1 \cdot \tau_2} = c_i^{\tau_2 \cdot \tau_1}, \qquad i = 1, \dots, 4,$$

which translates into the relations in the statement. $\square$

3. ALGEBRAIC AND SYMPLECTIC CREMONA TRANSFORMATIONS

Following the ideas of Orevkov, Golla and Starkston formalized in [11, § 8] an example of rational singular curves which exist in the symplectic category and not in the algebraic one.

The most well known birational is the map

$$\begin{aligned} \mathbb{P}^2 &\dashrightarrow \mathbb{P}^2 \\ [x : y : z] &\longmapsto [yz : zx : xy]. \end{aligned}$$

Geometrically is obtained by the blow-up of the points $[1 : 0 : 0], [0 : 1 : 0], [0 : 0 : 1]$ and the blow-down of the strict transforms of the lines $x = 0, y = 0, z = 0$ which are pairwise disjoint smoot rational (-1)-curves. We can also consider a *symplectic* Cremona transformation which gives the following result.

Proposition 3.1 ([11, § 8]). *Let Σ_{alg} (resp. Σ_{symp}) be the space of algebraic (resp. symplectic) irreducible curves of degree 8 in $\mathbb{P}^2$ having three singular points with the topological type of $u(v^3+u^5)=0$.*

(1) *The space Σ_{alg} is empty.*
(2) *The space Σ_{symp} is non-empty and it can be embedded in the space of symplectic deltoids such that their tangent lines to the cuspidal points are not concurrent.*

We can go further and compute some topological invariants of this curve, in particular the fundamental group of its complement.

Corollary 3.2. *If $C\in\Sigma_{\text{symp}}$ comes from a Cremona transformation associated to the tangent lines of a symplectic deltoid (isotopic to an algebraic one), then its fundamental group is the non-abelian semidirect product $\mathbb{Z}/6\rtimes\mathbb{Z}/4$.*

Proof. If P is an ordinary double point and two commuting meridians of the branches, a meridian of the exceptional component of the blow-up of the P is the product of the meridians, see e.g. [4, Lemma 3.6] (probably well-known result).

The complement of C is homeomorphic to the complement of the strict transform of the deltoid and the tangent lines by the blow-ups. For the total transform, we have to add the exceptional components. From the deformation in Figure 3, we see that these meridians are $\ell_1\cdot\ell_2$, $\ell_1\cdot\ell_\infty$, and $\ell_2\cdot\ell_\infty$.

From [9, Lemma 4.18], the fundamental group of the complement of the strict transform is obtained by *killing* this meridian. An easy computation gives the result. □

Remark 3.3. This group is also the fundamental group of the complement of an algebraic curve, as it is shown using similar techniques in [19].

4. Kummer covers

With the same ideas of §3, we are going to construct new examples replacing the standard Cremona transformation by Kummer covers, i.e., Galois covers

$$\begin{aligned}\mathbb{P}^2 &\longrightarrow \mathbb{P}^2\\ [x:y:z] &\longmapsto [x^n:y^n:z^n].\end{aligned}$$

Starting from three pseudo-holomorphic non-concurrent lines there is a symplectic counterpart.

Proposition 4.1. *There are irreducible symplectic curves C_{symp} of degree 8 in $\mathbb{P}^2$ with 6 singular points of type $\mathbb{E}_6$ for which the fundamental group G_{symp} of their complement is generated by c_1', c_2, c_3, c_4, $c_1' = c_2^{-1}\cdot c_1\cdot c_2$, with relations*

$$[c_2,c_4]=[c_1',c_3]=1,\ c_1'\cdot c_2\cdot c_1' = c_2\cdot c_1'\cdot c_2,\ c_3\cdot c_2\cdot c_3 = c_2\cdot c_3\cdot c_2,$$
$$c_3\cdot c_4\cdot c_3 = c_4\cdot c_3\cdot c_4,\ (c_2\cdot c_1'\cdot c_3\cdot c_4)^2=1,$$

and the conjugation action is derived from the action in Remark 1.3.

As in §3, we denote by Λ_{symp} and Λ_{alg} the spaces of symplectic or algebraic curves of degree 8 having 6 singular points of type $\mathbb{E}_6$.

Proof. The existence of such a curve C_{symp} comes from a symplectic Kummer cover for $n=2$, starting from a symplectic deltoid as in §2, taking the tangent lines to the cusps for the ramification lines of the Kummer cover. The degree of the preimage of the deltoid is 8 and each cusp produces two $\mathbb{E}_6$ points.

For the fundamental group, let G_{orb22} be the orbifold fundamental group of the complement of the deltoid where the orbifold structure comes from the action of $\mathbb{Z}/2\times\mathbb{Z}/2$ as deck group of the Kummer cover.

Hence, G is the quotient of the group in Corollary 2.1 with some extra relations

(R11) $\ell_1^2=1$
(R12) $\ell_2^2=1$
(R13) $\ell_\infty^2=1$

As in the proof of Corollary 2.2, this group G is a semidirect product $G_{\text{symp}}\rtimes\mathbb{Z}/2\times\mathbb{Z}/2$. In order to find G_{symp} we consider the relations $c_i^{\tau_j}=c_i^{\tau_j^{-1}}$, for $i=1,\dots,4$ and $j=1,2$. Moreover we can combine the relations (R9) and (R13) to rewrite them in terms only of $c_1,\dots,c_4$. Replacing c_1 by c_1' we obtain the relation of the statement. Details can be found in `ConstructionSymplecticGroup`.

Since the fundamental group of the complement of C_{symp} is the kernel of the epimorphism $G\to\mathbb{Z}/2\times\mathbb{Z}/2$ given by

$$c_i\mapsto(0,0),\quad \ell_1\mapsto(1,0),\quad \ell_2\mapsto(0,1),\quad \ell_\infty\mapsto(1,1),$$

we obtain that this group is G_{symp}. □

Remark 4.2. Using `GAP4` [10] via `Sagemath` [18] we have:

$$G/G'\cong\mathbb{Z}/8,\quad G'/G''\cong\mathbb{Z}/3,\quad G''/G'''\cong(\mathbb{Z}/2)^6,\quad G'''/G^{(4)}\cong\mathbb{Z}^9\oplus(\mathbb{Z}/2)^5\oplus\mathbb{Z}/4.$$

We need to understand Λ_{alg} in order to check if the elements found in Λ_{symp} are isotopic to algebraic curves. Unfortunately computations are cumbersome and our attempts failed. Most probably this space is discrete, and we have been able to obtain some particular elements. Some geometric properties of these curves are presented in the following section.

5. Properties of curves in Λ_{alg}

Let us state some lemmas about the symmetry properties of these curves.

Lemma 5.1. *Let $C\in\Lambda_{\text{alg}}$ symmetric by the action of a projective involution Φ_2. Then two of the singular points are in the line of fixed points in Φ_2 and the other ones form two orbits.*

Proof. An $\mathbb{E}_6$ point cannot be invariant by an involution with isolated fixed point. Let us assume that there is no fixed point. Then the quotient of C in $\mathbb{P}^2_{(1,1,2)}$ is a curve of degree 8 with three triple points. Using Bézout Theorem and the curve of degree 2 passing through the three singular points we get a contradiction. The only possible case is the one in the statement. □

Lemma 5.2. *Let $C\in\Lambda_{\text{alg}}$ symmetric by the action of a projective automorphism Φ_3 of order 3. Then Φ_3 has no line of fixed points, there are 2 orbits and the curve passes through two isolated fixed points of Φ_3 (tangent to the fixed lines not containing the two fixed points in the curve).*

Proof. By Bézout Theorem, there is no line of fixed points. Note that the singular points cannot be fixed points and in that case the orbits would be aligned.

Hence there are three fixed points and two orbits of three singular points. Let us consider the lines joining the fixed points. It is clear that the curve contains two of them and it is tangent to the line not joining them. □

Lemma 5.3. *There is no $C \in \Lambda_{\text{alg}}$ symmetric by the action of a projective automorphism Φ of order $n > 3$.*

Proof. Note first that we cannot have a line of fixed points, only isolated points. The case $n > 7$ is ruled out immediately.

For $n = 4$, we have one fixed point of order 4 and two fixed points of order 2 and no fixed point is singular. For the one of order 4 for the features of E_6 and for the other ones as they are in a line of fixed points for Φ^2. As we have six points, we also rule out this case.

fixed?

In the case $n = 5$, we need that one ~~singular~~ point is singular but the arithmetics of the intersection points with the lines joining the fixed points are not compatible.

For the case $n = 6$, the restrictions for $2, 3$ give only one possible case, a *rotation*. The fixed points cannot be singular points and the case of one orbit is not possible because of Lemma 5.1. □

These curves have interesting properties from the birational point of view. Let $C \in \Lambda_{\text{alg}}$ (though most of the following facts may be also valid in the symplectic case). Le $P_1, \ldots, P_6$ be the singular points. They are not in a conic (from Bézout Theorem). Let $\mathcal{C}_i$, $1 \leq i \leq 6$, be the unique conic passing through $P_1, \ldots, \widehat{P_i}, \ldots, P_6$. Again, by Bézout Theorem, these conics are irreducible.

Proposition 5.4. *Let $\Psi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ the birational map obtained by blowing-up the points $P_1, \ldots, P_6$ and blowing-down the strict transforms of $\mathcal{C}_1, \ldots, \mathcal{C}_6$. Let $Q_1, \ldots, Q_6$ be the* images *of the conics and* let *~~Left~~ $\mathcal{D}_1, \ldots, \mathcal{D}_6$ the* images *of the exceptional components.*

Then, the strict transform of C is a smooth quartic curve D passing through the points $Q_1, \ldots, Q_6$. There exist six points $R_1, \ldots, R_6$ such that as divisors

$$D \cdot \mathcal{D}_i = Q_1 + \cdots + \widehat{Q_i} + \cdots + Q_6 + 3R_i.$$

These twelve points are pairwise distinct.

Note that in particular, C is not hyperelliptic.

Proof. The map Ψ factors as in the following diagram

$$\begin{tikzcd} & X \arrow[dl, "\sigma_1"'] \arrow[dr, "\sigma_2"] & \\ \mathbb{P}^2 \arrow[rr, dashed, "\Psi"] & & \mathbb{P}^2 \end{tikzcd}$$

The map σ_1 is the composition of the blow-ups of the points $P_1, \ldots, P_6$. Under these blow-ups, let us denote by $\mathcal{D}_i$ the exceptional divisors, and denote also by $\mathcal{C}_i$ the strict transform of $\mathcal{C}_i$. As each conic has been affected by 5 blow-ups, $(\mathcal{C}_i)^2_X = -1$, and these strict transforms are pairwise disjoint. Hence the map σ_2 is the blow-down of the curves $\mathcal{C}_i$. Under these blow-downs, the images $\mathcal{D}_i = \sigma_2(\mathcal{D}_i)$ are conics; they pass through 5 of the six exceptional points $Q_{\cancel{i}} = \sigma_{\cancel{1}}(\mathcal{C}_{\cancel{i}})$.

j 2 j

The other interesection point is the strict transform of a singular point which becomes a smooth point after blowing-up having intersection number 3 with the exceptional divisor. ☐

Unfortunately, this description is not useful for the computations.

6. Algebraic curves with $\mathbb{Z}/2$-action

missing negative sign(s)

From the lemmas in §5, we can assume that $C_{8,2} \in \mathcal{A}_{\text{alg}}$ is fixed by the involution $\Phi_2 : \mathbb{P}^2 \to \mathbb{P}^2$ given by $\Phi_2([x:y:z]) = [x:y:z]$ and that two of the $\mathbb{E}_6$ points are $P_1 = [1:0:0]$ and $P_2 = [0:1:0]$. The isolated fixed point $[0:0:1]$ is not in the curve. The tangent lines to P_i must be fixed by the action and it is easily seen that they are not tangent to $z=0$, hence the tangent lines are $L_x : \{x=0\}$ and $L_y : \{y=0\}$.

The quotient $\mathbb{P}^2/\Phi_2$ is isomorphic to the weighted projective plane $\mathbb{P}^2_\omega$, $\omega = (1,1,2)$, and the map is $\pi : \mathbb{P}^2 \to \mathbb{P}^2_\omega$ where $\pi([x:y:z]) = [x:y:z^2]_\omega$. From the orbifold point of view there is an orbifold X_2 constructed on $\mathbb{P}^2_\omega$ with the usual orbifold structure around $[0:0:1]_\omega$ and also on the line $L_z : \{z=0\}$, with an action of the cyclic group of order 2.

Lemma 6.1. *Let $\tilde{C}_{8,2} := \Phi_2(C_{8,2})$. Then $\tilde{C}_{8,2}$ is a curve of ω-degree 8, with two singular points $\mathbb{E}_6$ and two ordinary cusps (not two of them in the same curve of ω-degree 1). Moreover there is a curve of ω-degree 2 tangent t~~w~~o the two cusps.*

This is obvious from the description of $C_{8,2}$. The way to compute the space of all such curves (up to automorphism) is the following one. We start with a polynomial

$$f(x,y,z) = \sum_{i+j+2k=8} a_{ijk} x^i y^j z^k.$$

Since it does not pass through $[0:0:1]_\omega$, we may assume that $a_{004} = 1$. Recall that

$$\operatorname{Aut} \mathbb{P}^2_\omega = \{\Phi_{B,c} \mid \big| B \in \mathrm{GL}(2;\mathbb{C}), \quad c \in \mathbb{C}^3\}$$

where $\Phi_{B,d}([x:y:z]_\omega) = [b_{11}x + b_{12}y : b_{21}x + b_{22}y : z + c_{xx}x^2 + c_{xy}xy + c_{yy}y^2]_\omega$ for

$$B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}, \quad c = (c_{xx}, c_{xy}, c_{yy}).$$

Note that $\Phi_{B,c} = \Phi_{-B,c}$. Using this group we can assume that the $\mathbb{E}_6$ points are at $[1:0:0]_\omega$ and $[0:1:0]_\omega$, while the cusps are at $[1:1:0]_\omega$ and $[a_1:1:1]_\omega$. Moreover we
+2? fix that the -2-curve tangent to the cusps is $z = bxy$. Note that $[x:y:z]_\omega \mapsto [y:x:z]_\omega$ is the only automorphism fixing this family of curves.

Remark 6.2. It seems more natural to take these curve to be $z=0$. It can be done but the computations are more complicated.

The conditions about the singular points give a system of equations. Direct attempts failed and in the notebook `OcticInvolution` of `Sagemath` we obtain the existence of a unique solution up to automorphism. We have normalized this solution to have a simpler form.

Theorem 6.3. *Let $C_{8,2}$ be a projective plane curve of degree 8 having 6 singular points of type $\mathbb{E}_6$ and fixed by an involution. Then it is projectively equivalent to the curve of equation*

$$\begin{gathered}-\frac{11}{3}x^5y^3 - \frac{407}{16}x^4y^4 - 44x^3y^5 - \frac{11}{8}x^4y^2z^2 + \frac{33}{2}x^2y^4z^2 + \frac{27}{176}x^4z^4 \\ -\frac{4}{11}x^3yz^4 - \frac{49}{11}x^2y^2z^4 - \frac{48}{11}xy^3z^4 + \frac{243}{11}y^4z^4 - \frac{5}{6}x^2z^6 + 10y^2z^6 + z^8 = 0\end{gathered}$$

This curve is not fixed by any other automorphism.

The proof of the unicity relies on the `Sagemath` worksheet, but the fact that this equation satisfies the condition is much easier, see `CheckCurveInvolution`.

Theorem 6.4. *The fundamental group of the complement of $C_{8,2}$ is*

$$G_2 = \langle x, y, z \mid [x, z] = 1,\ xyx = yxy,\ yzy = zyz,\ (xy^2z)^2 = 1\rangle,$$
$$G_2/G_2' \cong \mathbb{Z}/8, \quad G_2'/G_2'' \cong \mathbb{Z}/3, \quad G_2''/G_2''' \cong (\mathbb{Z}/2)^4 \quad G_2''' \cong \mathbb{Z}^3 \times \mathbb{Z}/2.$$

In particular it is not isomorphic to the fundamental group in Proposition 4.1, and hence $C_{8,2}$ is not isotopic to the symplectic curve in §4.

This theorem has been proved using `Sagemath` and `Sirocco`, see the details in the notebook `FundamentalGroupInvolution`. Note that `Sirocco` uses interval arithmetic which certifies the results.

7. Algebraic curves with $\mathbb{Z}/3$-action

From the lemmas of §5 we may assume that the automorphism of order 3 is $\Phi_3 : \mathbb{P}^2 \to \mathbb{P}^2$, $\Phi_3([x : y; z]) = [\zeta x : \bar{\zeta} y : z]$ where ζ is a primitive cubic root of unity. Let $C_{8,3}$ $\cancel{\in}$ $\in \Lambda_{\text{alg}}$ fixed by the Φ_3. Let $X_3 := \mathbb{P}^2/\Phi_3$ its quotient and let $D_{8,3} \subset X_3$ be the image of $C_{8,3}$. X_3 ~~It~~ is a normal surface with three isolated cyclic points of type $\frac{1}{3}(1, -1)$. This notation stands for the following. Let μ_d be the group of d-roots of unity in $\mathbb{C}$. Then $\frac{1}{d}(a, b)$ is the quotient of $\mathbb{C}^2$ by the action of μ_d defined by $\zeta \cdot (x, y) = (\zeta^a x, \zeta^b y)$.

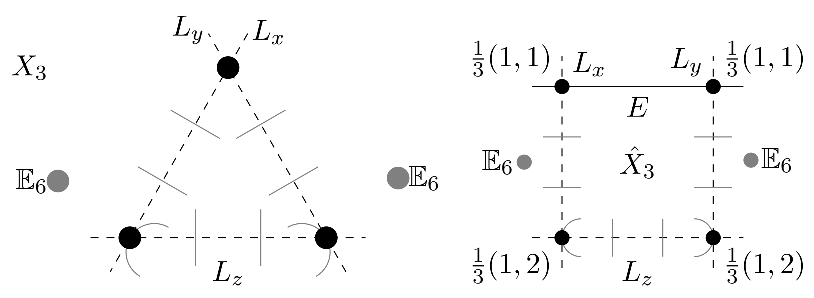

Figure 4. Surface X_3 with the image of the curve and $(1,1)-$blow-up of (B1).

There is a birational transformation to pass from X_3. to $\mathbb{P}^2$. These are the steps:

(B1) $(1,1)$-blow-up of the image of $[0 : 0 : 1]$ in X_3, with exceptional component E. We obtain a *singular ruled surface* with four singular points in two fibers, the strict transforms of L_x, L_y. The new ones are of type $\frac{1}{3}(1,1)$. The two sections in the right-hand side of Figure 4 have self-interesection $\frac{1}{3}$ (L_z below) and $-\frac{1}{3}$ (E above), see [7] for details on weighted blow-ups.

(B2) $(1,1)$-blow-up of the two points of type $\frac{1}{3}(1,1)$, with exceptional components E_x, E_y of self-interesection -3. The self-interesection of the strict transforms of L_x, L_y is $-\frac{1}{3}$.
$-\frac{1}{3}$?

(B3) Blow-down of the strict transforms of the images of the lines $x=0$, $y=0$; it is the inverse of a $(1,3)$-blow-up of a smooth points. The result is a smooth surface, actually the Hirzebruch ruled surface Σ_1 where the (-1)-curve is E.

(B4) Contract the (-1)-curve E.

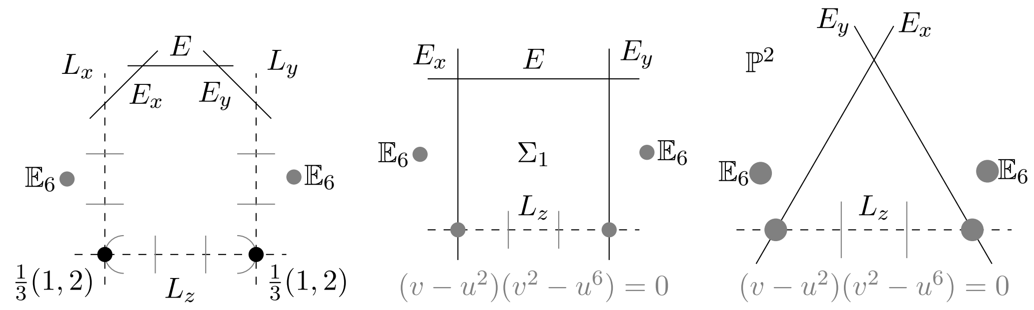

FIGURE 5. $(1,1)$-Blow-ups of (B2), blow-downs of (B3) and blow-down of (B4).

Actually all this operation has simple coordinates. The composition of the quotient and the birational map is a rational map $\Theta : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ given by

$$\Theta([x:y:z]) = [x^3 : y^3 : xyz].$$

Lemma 7.1. *Let C the image of $C_{8,3}$ by Θ. Then C is a curve of degree 8, with two singular points $\mathbb{E}_6$ and two singularities with the topological type of $(u-v^2)(u^2-v^6)=0$ having maximal contact with the tangent line.*

We proceed as in §6. Let us take

$$f(x,y,z) = \sum_{i+j+k=8} a_{ijk} x^i y^j z^k.$$

with $a_{008}=1$ since $[0:0:1] \notin C$. We place the reducible singular points at $[1:0:0]$ and $[0:1:0]$ with respective tangent lines $y=0$ and $z=0$. One of the $\mathbb{E}_6$ points is at $[1:1:1]$ and for the other one we use two new variables. The only automorphism fixing this family of curves is $[x:y:z] \mapsto [y:x:z]$.

The system of equations is more complicated that the one in §6, but we managed to obtain the solutions using `Sagemath`, see the notebook `OcticAuto3`. In Appendix A, the common procedure is explained. To describe the solution we need to introduce the number field $\mathbb{K} := \mathbb{Q}[\eta]$, where η is a solution of $p(t) := t^4 - 2t^3 + t^2 - 2t - 2$. This polynomial has two real roots η_1, η_2 and two complex conjugate roots η_3, η_4.

Theorem 7.2. *Let $C_{8,3}$ be a projective plane curve of degree 8 having 6 singular points of type $\mathbb{E}_6$ and fixed by an automorphism of order ~~4~~ 3. Then it is projectively equivalent to a curve $C_{8,3}^{\eta_i}$ whose equation is obtained as follows. Let*

$$G_0(x,y,z) = \frac{F(x^3, y^3, xyz)}{x^8 y^8},$$

where F is the equation in the Appendix B. Then, $G(x,y,z) := G_0(x+\zeta y, x+\bar{\zeta} y, z)$ with coefficients in $\mathbb{K} = \mathbb{Q}[\eta_i]$. This curve is not fixed by any other automorphism.

The fundamental group of the complement of any such curve is cyclic of order 8.

The proof of this theorem can be checked in `OcticAuto3`. The computation of the fundamental group takes much longer than it took in the case of §6 and it has been done with `Sagemath` and `Sirocco`, see the notebook `FundamentalGroupAuto3`.

As for the other type of curves, the long computation is only needed to prove that these curves are the only ones. It is easier to prove that the satisfy the required condition, see `CheckCurveAuto3`.

8. Alternative way to compute the fundamental groups

There is an alternative way to compute this fundamental group. We can compute $G_3^{\mathrm{orb}} := \pi_1^{\mathrm{orb}}(X_3 \setminus D_{8,3}^{\eta_i})$ and $G_2^{\mathrm{orb}} := \pi_1^{\mathrm{orb}}(X_2 \setminus \tilde{C}_{8,2})$. In this particular situation it does not really save computation time but in other cases it allows to obtain a faster and computer-free approach.

The orbifold fundamental group $\pi_1^{\mathrm{orb}}(X_2 \setminus \tilde{C}_{8,2})$ is computed following several steps, see `Alternatives2`:

(Orb^{2}1) Blow up $[0:0:1]_\omega$; we obtain a surface Σ_2 (a ruled Hirzebruch surface) with an exceptional component E, with self-interesection -2. We compute the group $\pi_1(\Sigma_2 \setminus (\tilde{C}_{8,2} \cup L_z \cup E))$.

(Orb^{2}2) To compute this group we consider an affine chart, say the complement of E and L_x, using the standard Zariski-van Kampen method. In `Alternatives2` we have a finitely presented group with five generators $x_0, \dots, x_4$, where x_2 is a meridian of L_z and $e := (x_0 \cdot \ldots \cdot x_4)^{-1}$ is a meridian of E. Following [12], a meridian of L_x is e^2.

(Orb^{2}3) The group G_{orb}^2 is obtained by adding the relations $x_2^2 = e^2 = 1$.

(Orb^{2}4) The group G_2 is the kernel of the map $G_2^{\mathrm{orb}} \twoheadrightarrow \mathbb{Z}/2$ defined by $x_i \mapsto 0$, $i \neq 2$, and $x_2 \mapsto 1$. In `Alternatives2`, we prove that x_2 is central and of order 2. Hence $G_2^{\mathrm{orb}} \cong G_2 \times \mathbb{Z}/2$.

(Orb^{2}5) Actually G_2 is the orbifold fundamental group of the complement of $\tilde{C}_{8,2}$, where the unique orbifold point is the singular one.

We follow a similar strategy to compute the orbifold fundamental groups $G_3^{\mathrm{orb}} = \pi_1^{\mathrm{orb}}(X_3 \setminus \tilde{D}_{8,3}^{\eta_i})$, see `Alternatives3`:

(Orb^{3}1) We start with the final birational model of the rational map and compute $\pi_1(\mathbb{P}^2 \setminus (\tilde{D}_{8,3}^{\eta_i} \cup E_x \cup E_y))$. Actually, we take the affine chart of the complement of E_x and compute the fundamental group of the complement of $\tilde{D}_{8,3}^{\eta_i}$ and E_y.

(Orb^{3}2) Using the standard Zariski-van Kampen method we obtain in `Alternatives3` a finitely presented group with nine generators $x_0, \dots, x_8, \ell_y$, where e_y is a meridian of E_y, the x_i's are meridians of $\tilde{D}_{8,3}^{\eta_i}$ $e := (x_0 \cdot \ldots \cdot x_8)^{-1}$ is a meridian of E, and $e_x := e_y^{-1} \cdot e$ is a meridian of E_x.

(Orb^{3}3) Following [15], we deduce that for the group G_3^{orb} we have to add the relations deduced from the divisor $E + E_x + E_y$ in Figure 6:

$$e_x \cdot e_y = e \text{ (known)}, \quad e = e_x^3 = e_y^3 \Rightarrow e = e_x \cdot e_y = e_x^3 = 1.$$

(Orb^{3}4) With this new relation we have computed in `Alternatives3` that all the groups are $\mathbb{Z}/24$ and hence we recover the abelianity of G_3.

In `Alternatives3` we have computed a simplified braid monodromy for $\tilde{D}_{8,3}^{\eta_i}$ which may give some hints about the topological equivalence of these curves.

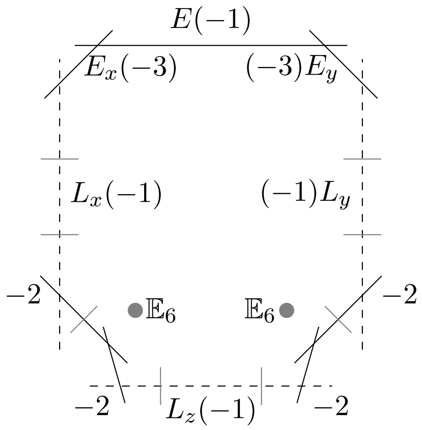

Figure 6. Minimal resolution of $\hat{X}_3$

9. Conclusions

We summarize the results an open questions.

(C1) There is no homeomorphism $\Phi_i : \mathbb{P}^2 \to \mathbb{P}^2$ such that $\Phi_i(C_{8,3}^{\eta_i}) = C_{8,2}$.
(C2) There is no homeomorphism $\Psi_i : \mathbb{P}^2 \to \mathbb{P}^2$ such that $\Psi_i(C_{8,3}^{\eta_i}) = C_{\text{symp}}$.
(C3) There is no homeomorphism $\Psi_i : \mathbb{P}^2 \to \mathbb{P}^2$ such that $\Psi_i(C_{8,2}) = C_{\text{symp}}$.
(C4) The complex conjugation is a homeomorphism $\Phi : \mathbb{P}^2 \to \mathbb{P}^2$ such that $\Phi(C_{8,3}^{\eta_3}) = C_{8,3}^{\eta_4}$.
(C5) The existence of homeomorphisms $\Phi_{i,j} : \mathbb{P}^2 \to \mathbb{P}^2$ such that $\Phi(C_{8,3}^{\eta_i}) = C_{8,3}^{\eta_j}$ is an open question, for $i \neq j$ and $\{i,j\} \neq \{3,4\}$.
(C6) The existence of a homeomorphism $\Phi_{3,4} : \mathbb{P}^2 \to \mathbb{P}^2$ such that $\Phi(C_{8,3}^{\eta_3}) = C_{8,3}^{\eta_4}$ which preserves the orientation of the curves is an open question.
(C7) The existence of other curves in Λ_{alg} is an open question.
(C8) The existence of curves in Λ_{alg} isotopic to C_{symp} is an open question.

10. Perspectives

A direct approach to compute Λ_{alg} seems to be hopeless. Isolating special properties for the known solutions would help to get new ideas that would allow either to discard new cases or to obtain some new ones. We know that there is no more curve in Λ_{alg} fixed by a non-trivial homeomorphism.

In particular we are going to compute the smooth quartics of §5. The Cremona transformation described in that section corresponds to the 2-dimensional projective system of quintics having double points at six fixed points. The computations leading to the following results are done in the notebooks `Birational2` and `Birational3`. Moreover the system of points described in Proposition 5.4 are also computed.

Theorem 10.1. *The curve $C_{8,2}$ is birationally equivalent to*

$$z^4 - 3x^2z^2 + y^2z^2 - 36x^3y + 45x^2y^2 - 12xy^3 = 0.$$

ηi?

Theorem 10.2. *The curve $C_{8,3}$ is birationally equivalent to*

$$z^4 + \frac{3}{38}b_{12}xyz^2 + \frac{1}{19}(2b_{01} + \zeta c_{01})x^3z + \frac{1}{19}(2b_{01} + \bar{\zeta}c_{01})y^3z + \frac{3}{19}b_{20}x^2y^2 = 0,$$

where

$$
\begin{aligned}
b_{12} &= -97\eta^3 - 23\eta^2 - 130\eta - 92 \\
b_{01} &= 74\eta^3 + 6\eta^2 + 109\eta + 75 \\
c_{01} &= -51\eta^3 + \eta^2 - 42\eta - 35 \\
b_{20} &= 3596\eta^3 + 585\eta^2 + 4862\eta + 3325.
\end{aligned}
$$

Appendix A. Strategy of the computations

In §6 and §7 we need to find the zero locus of an ideal J_0 in a ring $\mathbb{C}[a_1, \ldots, a_n]$. More precisely we look for *non-degenerate* solutions, since the conditions imposed are closed conditions and the space we are looking for is only locally-closed.

The existence of degenerate solutions is a big computational problem. The strategy followed consists to define a *tree* of ideals whose root is J_0. This tree has levels and at each level we eliminate a variable.

Let us assume that we have inductively constructed un ideal $J_{j,k} \subset \mathbb{C}[a_1, \ldots, a_{n-k}]$. Using heuristic arguments we choose a generator f_0 of the ideal and a variable, say a_{n-k}, and we compute the resultants with respect to a_{n-k} of f_0 with the other generators. We factorize each one of these resultants and we eliminate the factors which are known to provide degenerate solutions. With the remaining factors, we combine them to give a family of ideals $J_{j',k+1}$ in $\mathbb{C}[a_1, \ldots, a_{n-k}]$.

Some of the leaves of this tree will stop with no solution and we pay attention to the ones ending in prime ideals of $\mathbb{C}[a_1]$.

Fix one of these leaves. Actually these ideals have coefficients in $\mathbb{Q}$. A prime ideal $J_{i',n-1} \subset \mathbb{C}[a_1]$ determines an extension $\mathbb{L}_1$ of $\mathbb{Q}$ where a solution leaves. Replacing the value of a_1 by this solution in the ideal $J_{i,n-2}$ we obtain a new ideal in $\mathbb{L}_1[a_2]$. We factorize these principal ideals. Either some of these process stop with no solution or we end with one solution.

In both cases we end with only one algebraic solution. For the case of §6 the solution lives in a degree 2 extension of $\mathbb{Q}$ but the symmetry allows us to end with a rational solution. In the case of §7 the solution lives in $\mathbb{K}_1 = \mathbb{Q}[\eta, \zeta]$, extension of $\mathbb{Q}$ of degree 8. The symmetry allows us to end with a solution $\mathbb{K}_1 = \mathbb{Q}[\eta]$, extension of $\mathbb{Q}$ of degree 4.

Appendix B. Equations

Let $\mathbb{K}_1 := \mathbb{Q}[\eta, \zeta]$ and let σ be the non-trivial automorphism of $\mathbb{K}_1$; the field $\mathbb{Q}[\eta]$ is the fixed field by σ. When η is real, σ is the complex conjugation. A curve of Lemma 7.1 is of the form

$$
F(x,y,z) = F_0(xy,z) + 2xyz(xF_1(xy,z) + yF_1^\sigma(xy,z)) + x^2y^2(x^2F_2(xy,z) + y^2F_2^\sigma(xy,z)),
$$

where $F_0, F_1, F_2 \in \mathbb{K}[t,z]$. We have

$$
F_0(t,z) = z^8 + \frac{2r_{16}}{19} tz^6 + + \frac{3r_{24}}{19^2} t^2 z^4 + \frac{2r_{32}}{19^2} t^3 z^2 + \frac{4r_{40}}{19} t^4,
$$

$$r_{16} = 437\eta^3 - 1270\eta^2 + 1130\eta - 1696$$
$$r_{24} = -596956\eta^3 + 1619007\eta^2 - 1523682\eta + 2184414$$
$$s_{23} = -2064411\eta^3 + 5739587\eta^2 - 5326476\eta + 7777170$$
$$s_{40} = 11524593\eta^3 - 28834395\eta^2 + 28396048\eta - 38303610.$$

$$F_1(t,z) = \frac{r_{15} + \zeta s_{15}}{19^3} z^4 + \frac{r_{23} + \zeta s_{23}}{19^2} tz^2 + 6\frac{r_{31} + 4\zeta s_{31}}{19^2} t^2,$$

$$r_{15} = -157924\eta^3 + 308331\eta^2 - 356378\eta + 387894$$
$$s_{15} = 182695\eta^3 - 547611\eta^2 + 485700\eta - 752178$$
$$r_{23} = 1276065\eta^3 - 3104444\eta^2 + 3107094\eta - 4100620$$
$$s_{23} = -2064411\eta^3 + 5739587\eta^2 - 5326476\eta + 7777170$$
$$r_{31} = -5295773\eta^3 + 14400235\eta^2 - 13528408\eta + 19435018$$
$$s_{31} = 6353433\eta^3 - 16472958\eta^2 + 15895154\eta - 22035984$$

$$F_2(t,z) = \frac{r_{22} + 2\zeta s_{22}}{19} z^2 + 2\frac{r_{30} + 4\zeta s_{30}}{19} t.$$

$$r_{22} = 74354\eta^3 - 196839\eta^2 + 187718\eta - 264358$$
$$s_{22} = 138989\eta^3 - 356263\eta^2 + 346016\eta - 475522$$
$$r_{30} = -8288405\eta^3 + 21480135\eta^2 - 20732048\eta + 28731618$$
$$s_{30} = -2845567\eta^3 + 7360179\eta^2 - 7111716\eta + 9841218$$

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