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SECTION : 2A

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X ——————————————— X

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Ex 4.1

Q 19 $f_1(x) = x$, $f_2(x) = x - 1$, $f_3(x) = x + 3$

Sol

Using Wronskiam

$$W(f_1(x), f_2(x), f_3(x)) = \begin{vmatrix} f_1(x) & f_2(x) & f_3(x) \\ f_1'(x) & f_2'(x) & f_3'(x) \\ f_1''(x) & f_2''(x) & f_3''(x) \end{vmatrix}$$

$\Rightarrow f_1(x) = x$, $f_1'(x) = 1$, $f_1''(x) = 0$

$\Rightarrow f_2(x) = x - 1$, $f_2'(x) = 1$, $f_2''(x) = 0$

$\Rightarrow f_3(x) = x + 3$, $f_3'(x) = 1$, $f_3''(x) = 0$

Now,

$$W(x, x-1, x+3) = \begin{vmatrix} x & x-1 & x+3 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{vmatrix}$$

$$\Rightarrow x \begin{vmatrix} 1 & 1 \\ 0 & 0 \end{vmatrix} - x-1 \begin{vmatrix} 1 & 1 \\ 0 & 0 \end{vmatrix} + x+3 \begin{vmatrix} 1 & 1 \\ 0 & 0 \end{vmatrix}$$

$$\Rightarrow x(0) - (x-1)(0) + (x+3)(0)$$

$$\Rightarrow 0$$

$W = 0$

So Solution are Linear Dependent

X ——————————————————————— X

20. $f_1(x) = 2+x$, $f_2(x) = 2+|x|$

Solution:

$f_1(x) = 2+x$, $f_2(x) = 2+x$, $f_3(x) = 2-x$

$f_1(x) = 2+x$, $f_2'(x) = 1$, $f_2''(x) = 0$

$f_2(x) = 2+x$, $f_2'(x) = 1$, $f_2''(x) = 0$

$f_3(x) = 2-x$, $f_2'(x) = -1$, $f_2''(x) = 0$

$$W(f_1(x), f_2(x), f_3(x)) = \begin{vmatrix} f_1(x) & f_2(x) & f_3(x) \\ f_1'(x) & f_2'(x) & f_3'(x) \\ f_1''(x) & f_2''(x) & f_3''(x) \end{vmatrix}$$

$$= \begin{vmatrix} 2+x & 2+x & 2-x \\ 1 & 1 & -1 \\ 0 & 0 & 0 \end{vmatrix}$$

$$\Rightarrow 2+x \begin{vmatrix} 1 & -1 \\ 0 & 0 \end{vmatrix} - 2+x \begin{vmatrix} 1 & -1 \\ 0 & 0 \end{vmatrix} + 2-x \begin{vmatrix} 1 & 1 \\ 0 & 0 \end{vmatrix}$$

$$\Rightarrow 2+x(0) - 2+x(0) + 2-x(0)$$

$$\Rightarrow 0$$

$W = 0$

SOLUTIONS are Linear Dependent

X———X

21. $f_1(x) = 1+x$, $f_2(x) = x$, $f_3(x) = x^2$

<u>Sol</u>

$f_1(x) = 1+x$, $f_1'(x) = 1$, $f_1''(x) = 0$

$f_2(x) = x$, $f_2'(x) = 1$, $f_2''(x) = 0$

$f_3(x) = x^2$, $f_2'(x) = 2x$, $f_2''(x) = 2$

$$W(f_1(x), f_2(x), f_3(x)) = \begin{vmatrix} f_1(x) & f_2(x) & f_3(x) \\ f_1'(x) & f_2'(x) & f_3'(x) \\ f_1''(x) & f_2''(x) & f_3''(x) \end{vmatrix}$$

$$W = \begin{vmatrix} 1+x & x & x^2 \\ 1 & 1 & 2x \\ 0 & 0 & 2 \end{vmatrix}$$

$$\Rightarrow 1+x \begin{vmatrix} 1 & 2x \\ 0 & 2 \end{vmatrix} - x \begin{vmatrix} 1 & 2x \\ 0 & 2 \end{vmatrix} + x^2 \begin{vmatrix} 1 & 1 \\ 0 & 0 \end{vmatrix}$$

$\Rightarrow (1+x)[2] - x(2) + 0$

$\Rightarrow \cancel{2x} + 2 - \cancel{2x}$

$W \Rightarrow 2 \neq 0$ So Linear Independent.

22. $f_1(x) = e^{x}$, $f_2(x) = e^{-x}$, $f_3(x) = \sinh x$

Sol

$f_1(x) = e^{x}$, $f_1'(x) = e^{x}$, $f_1''(x) = e^{x}$

$f_2(x) = e^{-x}$, $f_2'(x) = -e^{-x}$, $f_2''(x) = e^{-x}$

$f_3(x) = \sinh x$, $f_3'(x) = \cosh x$, $f_3''(x) = \sinh x$

$$W(f_1(x), f_2(x), f_3(x)) = \begin{vmatrix} f_1(x) & f_2(x) & f_3(x) \\ f_1'(x) & f_2'(x) & f_3'(x) \\ f_1''(x) & f_2''(x) & f_3''(x) \end{vmatrix}$$

$$\Rightarrow \begin{vmatrix} e^{x} & e^{-x} & \sinh x \\ e^{x} & -e^{-x} & \cosh x \\ e^{x} & e^{-x} & \sinh x \end{vmatrix}$$

$$\Rightarrow e^{x} \begin{vmatrix} -e^{-x} & \cosh x \\ e^{-x} & \sinh x \end{vmatrix} - e^{-x} \begin{vmatrix} e^{x} & \cosh x \\ e^{x} & \sinh x \end{vmatrix}$$

$$+ \sinh x \begin{vmatrix} e^{x} & -e^{-x} \\ e^{x} & e^{-x} \end{vmatrix}$$

$$\Rightarrow e^{x}\left(-e^{-x}\sinh x - e^{-x}\cosh x\right) - e^{-x}\left(e^{x}\sinh x + e^{x}\cosh x\right)$$

$$+ \sinh x\left(e^{x}e^{-x} - e^{x}\left(-e^{-x}\right)\right)\Big)$$

$$= -e^{-x+x}\sinh x - e^{x-x}\cosh x - e^{-x+x}\sinh x + e^{-x+x}\cosh x$$

$$+ \sinh x\,(1+1)$$

$$\Rightarrow -\sinh x - \cancel{\cosh x} - \sinh x + \cancel{\cosh x}$$

$$+ 2\sinh x$$

$$\Rightarrow -\cancel{2\sinh x} + \cancel{2\sinh x}$$

$$W = 0$$

So Given Solution are Linearly Dependent.

X ——————————— X

Q 27 - 30

27. $x^2y'' - 6xy' + 12y = 0$, x^3 , x^4

$y_1 = x^3$

$y_1' = 3x^2$, $y_1'' = 6x$

$\Rightarrow x^2(6x) - 6x(3x^2) + 12(x^3) = 0$

$\Rightarrow 6x^3 - 18x^3 + 12x^3 = 0$

$0 = 0$

So $y_1 = x^3$ is a sol.

Now

$y_2 = x^4$

$y_2' = 4x^3$, $y_2'' = 12x^2$

Putting value of y, y', y''

$x^2(12x^2) - 6x(4x^3) + 12x^4 = 0$

$12x^4 - 24x^4 + 12x^4 = 0$

$0 = 0.$

So $y_2 = x^4$ is also a sol.

Using Wronskiam

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

$$= \begin{vmatrix} x^3 & x^4 \\ 3x^2 & 4x^3 \end{vmatrix}$$

$$= 4x^6 - 3x^6$$

$W \Rightarrow x^6 \neq 0$

So y_1 and y_2 are Independent sol are also a Fundamental set of Sols.

$$\boxed{y_c = c_1 x^3 + c_2 x^4}$$

28. $x^2y'' + xy' + y = 0$, $\cos(\ln x)$. $\sin(\ln x)$

Sol

$$y_1 = \cos(\ln x)$$

$$y_1' = -\sin(\ln x)\frac{1}{x}$$

$$y_1'' = -\sin(\ln x)(-x^{-2}) + -\cos(\ln x)\left(\frac{1}{x}\right)\left(\frac{1}{x}\right)$$

$$y_1'' = \sin(\ln x)(x^{-2}) - \cos(\ln x)\left(\frac{1}{x^2}\right)$$

$$\boxed{y_1'' = \frac{1}{x^2}\left[\sin(\ln x) - \cos(\ln x)\right]}$$

$$\cancel{x^2}\left(\frac{1}{\cancel{x^2}}\left[\sin(\cancel{l}nx) - \cos(\cancel{l}nx)\right] + \cancel{x}\left(\frac{1}{\cancel{x}}\left(-\sin(\cancel{l}nx)\right)\right)\right.$$

$$+ \cos(\cancel{l}nx)$$

$$0 = 0$$

Now, $y_2 = \sin(\ln x)$

$$y_2' = \cos(\ln x)\left(\frac{1}{x}\right)$$

$$y_2'' = \cos(\ln x) - x^{-2} + \left(-\sin(\ln x)\left(\frac{1}{x}\right)\left(\frac{1}{x}\right)\right.$$

$$y_2'' = -\cos(\ln x)\, x^{-2} - \sin(\ln x)\, x^{-2}$$

$$\boxed{y_2'' = \frac{1}{x^2}\left[-\cos(\ln x) - \sin(\ln x)\right]}$$

$$\cancel{x^2}\left[\frac{1}{\cancel{x^2}}\left(\cancel{-\cos(\ln x)} - \cancel{\sin(\ln x)}\right)\right] + \cancel{x}\left(\frac{1}{\cancel{x}}\cancel{\cos(\ln x)}\right) + \cancel{\cos(\ln x)}$$

$$0 = 0$$

Now using Wronskian

$$W\left(\cos(\ln x), \sin(\ln x)\right) = \begin{vmatrix} \cos(\ln x) & \sin(\ln x) \\ -\sin(\ln x)\frac{1}{x} & \cos(\ln x)\frac{1}{x} \end{vmatrix}$$

$$\Rightarrow \cos(\ln x) \cdot \cos(\ln x) \cdot \frac{1}{x} - \left(-\sin(\ln x)\frac{1}{x} \cdot \sin(\ln x)\right)$$

$$= \frac{1}{x}\left[\cos^2(\ln x) + \sin^2(\ln x)\right]$$

$\Rightarrow \frac{1}{x} \neq 0 \Rightarrow$ So It forms Fundamental Set of solution.

$$y_c = c_1 \cos(\ln x) + c_2 \sin(\ln x)$$

29. $x^3y''' + 6x^2y'' + 4xy' - 4y = 0$; $x, x^{-2}, x^{-2}\ln x$

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$\hookrightarrow$ $y_1 = x$, $y_1' = 1$, $y_1'' = 0$, $y_1''' = 0$

$4x - 4x = 0 \Rightarrow 0 = 0$

$\hookrightarrow$ $y_2 = x^{-2}$, $y_2' = -2x^{-3}$, $y_2'' = +6x^{-4}$

$y_2''' = -24x^{-5}$

$\Rightarrow x^3(-24x^{-5}) + 6x^2(6x^{-4}) + 4x(-2x^{-3}) - 4x^{-2} = 0$

$\Rightarrow -24x^{-2} + 36x^{-2} - 8x^{-2} - 4x^{-2} = 0$

$\Rightarrow 0 = 0$

$\hookrightarrow$ $y_3 = x^{-2}\ln x$

$y_3' = x^{-2}\left(\frac{1}{x}\right) + (-2)(x)^{-3}\ln x$

$y_3' = x^{-3} - 2x^{-3}\ln x$

$y_3'' = -3x^{-4} - 2\left[-3x^{-4}\ln x + x^{-3}\left(\frac{1}{x}\right)\right]$

$y_3'' = -3x^{-4} + 6x^{-4}\ln x - 2x^{-4}$

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$$y_3''' = 12x^{-5} - 24x^{-5}\ln x + 8x^{-5}$$

Putting values of y_3, y_3', y_3'', y_3'''

$$\Rightarrow x^3(12x^{-5} - 24x^{-5}\ln x + 8x^{-5}) + 6x^2(-3x^{-4} + 6x^{-4}\ln x - 2x^{-4})$$
$$+4x(x^{-3} - 2x^{-3}\ln x) - 4(x^{-2}\ln x)$$

$$\cancel{12x^{-2}} - \cancel{24x^{-2}\ln x} + \cancel{8x^{-2}} - \cancel{18x^{-2}} + \cancel{36x^{-2}\ln x}$$
$$-\cancel{12x^{-2}} + \cancel{4x^{-2}} - \cancel{8x^{-2}\ln x} - \cancel{4x^{-2}\ln x}$$

$$0 = 0$$

Using Wronskian

$$W(y_1, y_2, y_3) = \begin{vmatrix} x & x^{-2} & x^{-2}\ln x \\ 1 & -2x^{-3} & x^{-3} - 2x^{-3}\ln x \\ 0 & 6x^{-4} & -3x^{-4} + 6x^{-4}\ln x - 2x^{-4} \end{vmatrix}$$

$$\Rightarrow x \begin{vmatrix} -2x^{-3} & x^{-3} - 2x^{-3}\ln x \\ 6x^{-4} & -3x^{-4} + 6x^{-4}\ln x - 2x^{-4} \end{vmatrix} - x^{-2} \begin{vmatrix} 1 & x^{-3} - 2x^{-3}\ln x \\ 0 & -3x^{-4} + 6x^{-4}\ln x - 2x^{-4} \end{vmatrix}$$

$$+ x^{-2}\ln x \begin{vmatrix} 1 & -2x^{-3} \\ 0 & 6x^{-4} \end{vmatrix}$$

$$x\left(+6x^{-7} - 12x^{-7}\ln x + 4x^{-7}\right) - \left(6x^{-7} - 12x^{-7}\ln x\right)$$

$$- x^{-2}\left[+3x^{-4} \mp 6x^{-4}\ln x + 2x^{-4}\right)$$

$$\mp x^{-2}\ln x(6x^{-4})$$

$$\Rightarrow 4x^{-6} - 3x^{-6} + \cancel{6x^{-6}\ln x} - 2x^{-6} \mp \cancel{x^{-2}\ln x(6x^{-4})}$$

$\Rightarrow x^{-6} \neq 0$ {Linearly Independent}

So y_1, y_2, y_3 forma Fundamental Set of solution.

So

$$\boxed{y_c = c_1 x + c_2 x^{-2} + c_3 x^{-2}\ln x}$$

X ———————————— X

30. $y^{(4)} + y'' = 0; \quad 1, x, \cos x, \sin x$

Sol

$\to y_1 = 1, \; y_1' = 0, \; y_1'' = 0, \; y''' = 0, \; y^{(4)} = 0$

$0 = 0$

$\to y_2 = x, \; y_2' = 1, \; y_2'' = 0, \; y_2''' = 0, \; y_2^{(4)} = 0$

$0 = 0$

$\to y_3 = \cos x, \; y_3' = -\sin x, \; y_3'' = -\cos x, \; y_3''' = +\sin x$

$y_3^{(4)} = \cos x$

$\cos x - \cos x = 0$

$0 = 0$

$\to y_4 = \sin x, \; y_4' = \cos x, \; y_4'' = -\sin x, \; y_4''' = -\cos x$

$y_4^{(4)} = \sin x$

$\sin x - \sin x = 0$

$0 = 0$

$$W(y_1, y_2, y_3, y_4) = \begin{vmatrix} y_1 & y_2 & y_3 & y_4 \\ y_1' & y_2' & y_3' & y_4' \\ y_1'' & y_2'' & y_3'' & y_4'' \\ y_1''' & y_2''' & y_3''' & y_4''' \end{vmatrix}$$

$$\Rightarrow \begin{vmatrix} 1 & x & \cos x & \sin x \\ 0 & 1 & -\sin x & \cos x \\ 0 & 0 & -\cos x & -\sin x \\ 0 & 0 & \sin x & -\cos x \end{vmatrix}$$

$$= \begin{vmatrix} 1 & -\sin x & \cos x \\ 0 & -\cos x & -\sin x \\ 0 & \sin x & -\cos x \end{vmatrix}$$

$$= (-\cos x)(-\cos x) - (-\sin x)(\sin x)$$

$$= 1 \neq 0$$

So, L.I.D so form Fundamental Set of Sol.

$$\boxed{y = c_1 + c_2 x + c_3 \cos x + c_4 \sin x}$$

General Solution

Q# 39 $y'' - 4y' + 4y = 2e^{2x} + 4x - 12$

$y = c_1 e^{2x} + c_2 x e^{2x} + x^2 e^{2x} + x - 2 \; , (-\infty, \infty)$

Sol $y'_{c_1} = 2e^{2x} \; , \; y_{c_1}(x) = e^{2x}$

$4e^{2x} - 4(2e^{2x}) + 4e^{2x} = 0$

$4e^{2x} - 8e^{2x} + 4e^{2x} = 0$

$0 = 0$

$y_{c_2}(x) = xe^{2x}$

$= x\,2e^{2x} + e^{2x}$

$\boxed{y'_{c_2}(x) = e^{2x}(1 + 2x)}$

$4e^{2x}(1+x) - 4e^{2x}(1+2x) + 4xe^{2x} = 0$

$4e^{2x}(1 + x - 1 - 2x + x) = 0$

$0 = 0$

Using Wronskian

$W(e^{2x}, xe^{2x}) = \begin{vmatrix} e^{2x} & xe^{2x} \\ 2e^{2x} & e^{2x}(1+2x) \end{vmatrix}$

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$\Rightarrow e^{2x} \cdot e^{2x}(1+2x) - xe^{2x} \cdot 2e^{2x}$

$\Rightarrow e^{4x}(1+2x) - 2xe^{4x}$

$W \Rightarrow e^{4x}$

$W \neq 0$

So,

$\boxed{y = c_1 e^{2x} + c_2 x e^{2x}}$ → is the general solution

X ———————————— X

34. $2x^2 y'' + 5xy' + y = x^2 - x$

$y = c_1 x^{-1/2} + c_2 x^{-1} + \frac{1}{15}x^2 - \frac{1}{6}x$

Sol

$y_{c_1} = x^{-1/2}$

$y_{c_1}' = -\frac{1}{2} x^{-1/2+1}$

$y_{c_1}' = -\frac{1}{2} x^{1/2}$

$y_{c_1}'' = -\frac{1}{2}\left(\frac{1}{2}\right) x^{1/2+1}$

$\boxed{y_{c_1}'' = -\frac{1}{4} x^{3/2}}$

$2x^2\left(\cancel{\frac{1}{4}x^{3/2}}\right) \cancel{+5x\left(\frac{-1}{2}x^{1/2}\right)} + x^{-1/2} = 0$

$\frac{-1}{2}x^{5/2} +$

Q#34 $2x^2y'' + 5xy' + y = x^2 - x$

$y = c_1x^{-1/2} + c_2x^{-1} + \frac{1}{15}x^2 - \frac{1}{6}x$

Sol

$\hookrightarrow y_1 = x^{-1/2}$

$y_1' = \frac{-1}{2}x^{-3/2}, \quad y_1'' = \frac{3}{4}x^{-5/2}$

Putting value y_1, y_1', y_1''

$2x^2 \times \left(\frac{3}{4}\right)x^{-5/2} + 5x \times -\frac{1}{2}x^{-3/2} + x^{-1/2} = 0$

$\frac{3}{2}x^{-1/2} - \frac{5}{2}x^{-1/2} + x^{-1/2} = 0$

$-x^{-1/2} + x^{-1/2} = 0$

$0 = 0$

$\hookrightarrow y_2 = x^{-1}, \; y_2' = -x^{-2}, \; y_2'' = 2x^{-3}$

$2x^2 \times 2x^{-3} + 5x \times -x^{-2} + x^{-1} = 0$

$4x^{-1} - 5x^{-1} + x^{-1} = 0$

$5\cancel{x^{-1}} - \cancel{5x^{-1}} = 0 \Rightarrow 0 = 0.$

Using Wronskian

$$W(x^{-1/2}, x^{-1}) = \begin{vmatrix} x^{-1/2} & x^{-1} \\ -\frac{1}{2}x^{-3/2} & -x^{-2} \end{vmatrix}$$

$$W(x^{-1/2}, x^{-1}) = x^{-1/2} \times -x^{-2} - x^{-1} \times -\frac{1}{2}x^{-3/2}$$

$$\Rightarrow x^{-5/2} + \frac{1}{2x}x^{-5}$$

$$W = -\frac{1}{2}x^{-\frac{5}{2}} \neq 0$$

So y_{c_1} and y_{c_2} are Linearly Independent

$$\boxed{y_c = c_1 x^{-1/2} + c_2 x^{-1}}$$

$$\Rightarrow y_p = \frac{1}{15}x^2 - \frac{1}{6}x$$

$$\Rightarrow y_p' = \frac{2}{15}x - \frac{1}{6} \quad \text{and} \quad y_p'' = \frac{2}{15}$$

$$\Rightarrow 2x^2 \times \frac{2}{15} + 5x\left(\frac{2}{15}x - \frac{1}{6}\right) + \frac{1}{15}x^2 - \frac{1}{6}x^2 - x$$

$$= \frac{4}{15}x^2 + \frac{2}{3}x^2 - \frac{5}{6}x + \frac{1}{15}x^2 - \frac{1}{6} = 24e^{x}$$

$$x^2 - x = x^2 - x$$

So

$$y_p = \frac{1}{15}x^2 - \frac{1}{6}x$$

$$\boxed{y = c_1 x^{-1/2} + c_2 x^{-1} + \frac{1}{15}x^2 - \frac{1}{6}x}$$

X ———————— X

Q# 35 $y'' - 6y' + 5y = 5x^2 + 3x - 16 - 9e^{2x}$

(b)

$$y' = 6e^{2x} \quad , \quad y'' = 12e^{2x}$$

$$y'' - 6y' + 5y = 12e^{2x} - 6(6e^{2x}) + 5(3e^{2x})$$

$$= 12e^{2x} - 36e^{2x} + 15e^{2x}$$

$$= -9e^{2x}$$

So $y_{p_1} = 3e^{2x}$ gives

$$\boxed{y'' - 6y' + 5y = -9e^{2x}}$$

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$y_{p_2}(x) = x^2 + 3x$

$= 2x + 3, \quad y'' = 2$

$+ 5y = 2 - 6(2x + 3) + 5(x^2 + 3x)$

$= 2 - 12x - 18 + 5x^2 + 15x$

$\Rightarrow 5x^2 + 3x - 16$

$= x^2 + 3x$ gives

$\boxed{- 6y' + 5y = 5x^2 + 3x - 16}$

$y_{p_1} = 3e^{2x}$ is the y_p of

$y'' - 6y' + 5y = -9e^{2x}$

$_2 = x^2 + 3x$ is the y_p of

$y'' - 6y' + 5y = 5x^2 + 3x - 16$

$- 6y' + 5y = \underbrace{5x^2 + 3x - 16}_{g_2(x)} \underbrace{+ -9x^2}_{g_1(x)}$

$$y'' - 6y' + 5y = -10x^2 - 6x + 32 + e^{2x}$$

$$= -2(5x^2 + 3x - 16) + \left(\frac{-1}{9}\right)(-9e^{2x})$$

↳ $y_{p1} = 3e^{2x}$ give

$$y'' - 6y' + 5y = -9e^{2x}$$

↳ $y_{p2} = x^2 + 3x$ give

$$y'' - 6y' + 5y = 5x^2 + 3x - 16$$

$$y'' - 6y' + 5y = -2(5x^2 + 3x - 16) + \left(\frac{-1}{9}\right)(-9e^{2x})$$

$$y = \frac{-1}{9} y_{p1} - 2y_{p2}$$

$$= \frac{-1}{\cancel{9}_3}(\cancel{3}e^{2x}) - 2(x^2 + 3x)$$

$$\boxed{y = -\frac{1}{3}e^{2x} - 2x^2 - 6x}$$

and

$$\boxed{y = \frac{-1}{3}e^{2x} - 2x^2 - 6x}$$

Q#2

Q10 $x^2y'' + 2xy' - 6y = 0$, $y_1 = x^2$

Sol

$$y'' + \frac{2y'}{x} - \frac{6}{x^2}y = 0$$

$$P(x) = \frac{2}{x}$$

Using Formula

$$y_2 = y_1 \int \left(\frac{e^{-\int p(x)\,dx}}{y_1^2} \right) dx$$

$$y_2 = x^2 \int \frac{e^{-\int \frac{2}{x}\,dx}}{x^4}\,dx$$

$$y_2 = x^2 \int \frac{|x|^{-2}}{x^4}\,dx$$

$$y_2 = x^2 \int x^{-6}\,dx$$

$$y_2 = x^2 \cdot \frac{x^{-5}}{-5}$$

$$y_2 = \frac{-1}{5} x^{-3}$$

let $c = \frac{-1}{5}$

$$\boxed{y_2 = c x^{-3}}$$

X———————X

~~11. $xy'' + y' = 0$; $y_1 = \ln x$~~

~~Sol~~

~~$p(x) = 1$~~

$$\cancel{y_2 = y_1 \int \frac{e^{-\int p(x)\,dx}}{y_1^2}\,dx}$$

$$\cancel{y_2 = \ln x \int \frac{e^{-x}}{(\ln x)^2}\,dx}$$

$$\cancel{y_2 = \ln x \int \frac{dx}{e^{x}(\ln x)^2}}$$

11. $xy'' + y' = 0$, $y_1 = \ln x$

Sol

Dividing by x

$$y'' + \frac{1}{x} y' = 0$$

$$\boxed{P(x) = \frac{1}{x}}$$

$$y_2 = y_1 \int \frac{e^{-\int p(x)\,dx}}{(y_1)^2}\,dx$$

$$y_2 = \ln x \int \frac{e^{-\int \frac{1}{x}\,dx}}{(y_1)^2}\,dx$$

$$y_2 = \ln x \int \frac{x^{-1}}{(\ln x)^2}\,dx$$

$$y_2 = \ln x \int \frac{1}{x(\ln x)^2}\,dx$$

let $u = \ln x$, $du = \frac{1}{x}\,dx$

$$y_2 = \ln x \int \frac{1}{u^2}\,du$$

$y_2 = \ln u \, \frac{1}{u}$

Back Substitution

$y_2 = \cancel{\ln x} \, \frac{1}{\cancel{\ln x}}$

$$\boxed{y_2 = 1}$$

X —————————— X

12. $4x^2 y'' + y = 0 \quad , \quad y_1 = x^{1/2} \ln x$

Sol

$p(x) = 0$

$y_2 = y_1 \int \frac{e^{-\int p(x)dx}}{(y_1)^2} dx$

$y_2 = x^{1/2} \ln x \int \frac{e^{-\int 0 dx}}{(x^{1/2} \ln x)^2} dx$

$y_2 = x^{1/2} \ln x \int \frac{C}{x (\ln x)^2} dx$

let $v = \ln x$

$dv = \frac{1}{x} dx$

$$y_2 = x^{1/2} \ln x \int \frac{1}{v^2} dv$$

$$y_2 = x^{1/2} \ln x \times \frac{1}{v}$$

Back substitution

$$y_2 = x^{1/2} \cancel{\ln x} \times \frac{1}{\cancel{\ln x}}$$

$$\boxed{y_2 = x^{1/2}}$$

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13. $x^2 y'' - x y' + 2y = 0$; $y_1 = x \sin(\ln x)$

Sol $y'' - \frac{x y'}{x^2} + \frac{2y}{x^2} = 0$

$$\Rightarrow y'' - \frac{1}{x} y' + \frac{2}{x^2} y = 0$$

$$\boxed{p(x) = -\frac{1}{x}}$$

6

$$y_2 = y_1 \int \frac{e^{+\int p(x)dx}}{y_1^2} dx$$

$$y_2 = x\sin(\ln x) \int \frac{e^{+\int \frac{1}{x} dx}}{\left(x\sin(\ln x)\right)^2} dx$$

$$= x\sin(\ln x) \int \frac{e^{\ln|x|^{+1}}}{x^2 (\sin(\ln x))^2} dx$$

$$= x\sin(\ln x) \int \frac{\cancel{x}\,1}{x^{\cancel{2}}(\sin(\ln x))^2} dx$$

$$= x\sin(\ln x) \int \frac{1}{\sin^2(\ln x)} \times \frac{1}{x} dx$$

let $u = \ln x$, $\boxed{du = \frac{1}{x} dx}$

$$= x\sin(\ln x) \int \frac{1}{\sin^2 u} du$$

$$= x\sin(\ln x) \int \csc^2 u \, du$$

$y_2 = x \sin(\ln x) \times -\cot(\ln x)$

$y_2 = x \sin(\cancel{\ln x}) \times \dfrac{-\cos(\ln x)}{-\sin(\cancel{\ln x})}$

$y_2 = -\cos(\ln x)$

$\boxed{\text{let } c_2 = -1}$

$y_2 = c_2 \, x \cos(\ln x)$

$$\boxed{y_2 = x \cos(\ln x)}$$

X ———————————— X

14. $x^2 y'' - 3xy' + 5y = 0 \; ; \; y_1 = x^2 \cos(\ln x)$

Sol:

$\dfrac{\cancel{x^2}}{\cancel{x^2}} y'' - \dfrac{3x}{x^2} y' + \dfrac{5}{x^2} y = 0$

$y'' - \dfrac{3}{x} y' + \dfrac{5}{x^2} y = 0$

$$p(x) = -\frac{3}{x}$$

$$y_2 = y_1 \int \frac{e^{-\int p(x)dx}}{y_1^2} dx$$

$$y_2 = x^2 \cos(\ln x) \int \frac{e^{+3 \ln|x|}}{(x^2 \cos(\ln x))^2} dx$$

$$y_2 = x^2 \cos(\ln x) \int \frac{e^{\ln|x|^3}}{x^4 \cos^2(\ln x)} dx$$

$$y_2 = x^2 \cos(\ln x) \int \frac{x^{-1}}{\cos^2(\ln x)} dx$$

$$y_2 = x^2 \cos(\ln x) \int \frac{1}{\cos^2(\ln x)} \cdot \frac{1}{x} dx$$

let $\cancel{x}$ $u = \ln x$

$du = \frac{1}{x} dx$

$$y_2 = x^2 \cos(\ln x) \times \int \frac{1}{\cos^2 u} du$$

$$\boxed{\because \frac{1}{\cos^2 x} = \sec^2 x}$$

$$y_2 = x^2 \cos(\ln x) \times (\tan u)$$

$$y_2 = x^2 \cancel{\cos(\ln x)} \times \frac{\sin(\ln x)}{\cancel{\cos(\ln x)}}$$

$$\boxed{y_2 = x^2 \sin(\ln x)} \rightarrow \text{Second solution}$$

15. $(1-2x-x^2)y'' + 2(1+x)y' - 2y = 0 \;,\; y_1 = x+1$

$$\frac{\cancel{(-x^2-2x+1)}y''}{\cancel{-x^2-2x+1}} + \frac{2(1+x)y'}{-x^2-2x+1} \frac{-2y=0}{-x^2-2x+1}$$

$$y'' + \left(\frac{2+2x}{-x^2-2x+1}\right)y' - \frac{2y}{-x^2-2x+1} = 0$$

$$p(x) = \frac{2+2x}{-x^2-2x+1}$$

$$y_2 = y_1 \int \frac{e^{-\int p(x)\,dx}}{y_1^2}\,dx$$

$$y_2 = (x+1)\int\left(\frac{e^{-\int \frac{2+2x}{-x^2-2x+1}dx}}{(x+1)^2}\right)dx$$

$$y_2 = (x+1)\int \frac{e^{\ln(1-2x-x^2)}}{(x+1)^2}\,dx$$

$$y_2 = (x+1)\int \frac{1-2x-x^2}{x^2+2x+1}\,dx$$

$$y_2 = (x+1)\left[\int \frac{2}{(x+1)^2}\,dx - \int 1\,dx\right]$$

$$y_2 = (x+1)\left[\int 2(x+1)^{-2}\,dx - \int 1\,dx\right]$$

$$y_2 = (x+1)\left[-2(x+1)^{-1} - x\right]$$

$$y_2 = (x+1)\left(\frac{-2}{x+1} - x\right]$$

$$\boxed{y_2 = -x^2 - x - 2}$$

21. $x^2y'' + (x^2 - x)y' + (1-x)y = 0, \quad y_1 = x$

Sol

$$y'' + \left(1 - \frac{1}{x}\right)y' + \left(\frac{1}{x^2} - \frac{1}{x}\right)y = 0$$

$$p(x) = \left(1 - \frac{1}{x}\right)$$

Using Formula

$$y_2 = y_1 \int \frac{e^{-\int p(x)dx}}{y_1^2} dx$$

$$y_2 = x \int \frac{e^{-(1-\frac{1}{x})dx}}{y_1^2} dx$$

$$y_2 = x \int \frac{e^{\ln x - x}}{x^2} dx$$

$$y_2 = x \int \frac{\cancel{e}^{\ln x} \times e^{-x}}{x^2} dx$$

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$$y_2 = x\int \frac{\cancel{x}e^{-x}}{\cancel{x}^{2}}$$

$$y_2 = x\int \frac{e^{-x}}{x}$$

$$\boxed{y_2 = x\int \frac{1}{e^{x}x}}$$

22. $2xy'' - (2x+1)y' + y = 0 \; ; \; y_1 = e^{x}$

Sol

$$y'' - \left(\frac{2x+1}{2x}\right)y' + \frac{y}{2x} = 0$$

$$y'' - \left(1 + \frac{1}{2x}\right)y' + \frac{1}{2x}y = 0$$

$$p(x) = -\left(1 + \frac{1}{2x}\right)$$

$$y_2 = y_1 \int \left(\frac{e^{-\int p(x)\,dx}}{y_1^2} \right) dx$$

$$y_2 = e^x \int \frac{e^{\int \left(1+\frac{1}{2x}\right) dx}}{(e^x)^2}\, dx$$

$$y_2 = e^x \int \frac{e^{x + 2\ln|x|}}{(e^x)^2}\, dx$$

$$y_2 = e^x \int \frac{e^{\ln|x|^2} \cdot \cancel{e^x}}{(e^x)^{\cancel{2}}}\, dx$$

$$\boxed{y_2 = e^x \int \frac{x^2}{e^x}\, dx}$$

Excercise 4.3

Q5 $y'' + 8y' + 16y = 0$

Sol

$$m^2 + 8m + 16 = 0$$

$$m^2 + 4m + 4m + 16 = 0$$

$$m(m+4) + 4(m+4) = 0$$

$$(m+4)^2 = 0$$

$$m_1 = -4 \;,\; m_2 = -4$$

$$\boxed{y = c_1 e^{-4x} + c_2 e^{-4x} x}$$

Q#6 $y'' - 10y' + 25y = 0$

$$m^2 - 10m + 25 = 0$$

$$m^2 - 5m - 5m + 25 = 0$$

$$m(m-5) - 5(m-5) = 0$$

$$(m-5)^2 = 0$$

$m_1 = 5, \; m_2 = 5$

$$\boxed{y = c_1 e^{5x} + c_2 x e^{5x}}$$

X————X

Q#7 $12y'' - 5y' - 2y = 0$

$12m^2 - 5m - 2 = 0$

$12m^2 - 8m + 3m - 2 = 0$

$(3m - 2)(4m + 1) = 0$

$3m = 2 \quad , \quad 4m_2 = -1$

$m_1 = \frac{2}{3} \quad , \quad m_2 = -\frac{1}{4}$

Solution will be;

$y = c_1 e^{m_1 x} + c_2 e^{m_2 x}$

$$\boxed{y = c_1 e^{\frac{2}{3}x} + c_2 e^{-\frac{1}{4}x}}$$

8. $y'' + 4y' - y = 0$

Sol

$$m^2 + 4m - 1 = 0$$

$$m = \frac{-(+4) \pm \sqrt{(4)^2 - 4(1)(-1)}}{2(1)}$$

$$m = \frac{-4 \pm \sqrt{16+4}}{2}$$

$$m = \frac{-4 \pm \sqrt{20}}{2}$$

$$m = \frac{-4 \pm 2\sqrt{5}}{2}$$

$$m = \frac{\cancel{-4}^{-2}}{\cancel{2}} \pm \frac{\cancel{2}\sqrt{5}}{\cancel{2}}$$

$$m = -2 \pm \sqrt{5}$$

$$\boxed{y = c_1 e^{(-2+\sqrt{5})x} + c_2 e^{(-2-\sqrt{5})x}}$$

24. $y^{(4)} - 2y'' + y = 0$

Sol

$y = e^{mx}$, $y' = me^{mx}$, $y'' = m^2 e^{mx}$, $y''' = m^3 e^{mx}$

$y^{(4)} = m^4 e^{mx}$

$m^4 - 2m^2 + 1 = 0$

$(m^2 - 1)(m^2 - 1) = 0$

$(m+1)(m-1)(m+1)(m-1) = 0$

$m_1, m_2 = 1$ and $m_3, m_4 = -1$

$$\boxed{y = c_1 e^{1x} + c_2 x e^{x} + c_3 e^{-x} + c_4 x e^{-x}}$$

X ———————— X

25. $16 \dfrac{d^4y}{dx^4} + 24 \dfrac{d^2y}{dx^2} + 9y = 0$

Sol Put $y = e^{mx}$

$\Rightarrow 16m^4 e^{mx} + 24m^2 e^{mx} + 9e^{mx} = 0$

$e^{mx}(16m^4 + 24m^2 + 9) = 0$

$$\Rightarrow 16m^4 + 24m^2 + 9 = 0$$

$$(4m^2+3)(4m^2+3) = 0$$

$$4m^2 = -3$$

$$\sqrt{m^2} = \sqrt{\frac{-3}{4}}$$

$$m_{1,2} = \pm\frac{\sqrt{3}}{2} i \quad , \quad m_{3,4} = \pm\frac{\sqrt{3}}{2} i$$

$$\boxed{y = c_1 \cos\frac{\sqrt{3}}{2}x + c_2 \sin\frac{\sqrt{3}}{2}x + c_3 x \cos\frac{\sqrt{3}}{2}x + c_4 \sin\frac{\sqrt{3}}{2}x}$$

X————————X

26. $\frac{d^4y}{dx^4} - 7\frac{d^2y}{dx^2} - 18y = 0$

Sol

Put $y = e^{mx}$, $y' = me^{mx}$, $y'' = m^2 e^{mx}$

$y''' = m^3 e^{mx}$, $y^{(4)} = m^4 e^{mx}$

$$m^4 e^{mx} - 7m^2 e^{mx} - 18e^{mx} = 0$$

$$e^{mx}(m^4 - 7m^2 - 18) = 0$$

$$e^{mx} \neq 0$$

$$m^4 - 7m^2 - 18 = 0$$

$$\cancel{m^4 - 9m + 2m - 18 = 0}$$

$$\cancel{m(m^3 - 9) + 2t}$$

$$\Rightarrow (m-3)(m^3 + 3m^2 + 2m + 6)$$

$$\Rightarrow (m-3)(m+3)(m^2+2) = 0$$

$$m_1 = 3,\ m_2 = -3,\ m_{3,4} = \sqrt{-2} \Rightarrow \pm\sqrt{2}i$$

General Sol

$$\boxed{y = c_1 e^{3x} + c_2 e^{-3x} + c_3 \cos\sqrt{2}x + c_4 \sin\sqrt{2}x}$$

X———X

29. $y'' + 16y = 0$, $y(0) = 2$, $y'(0) = -2$

Sol

Put $y = e^{mx}$, $y' = me^{mx}$, $y'' = m^2 e^{mx}$

$$m^2 e^{mx} + 16 e^{mx} = 0$$

$$e^{mx}(m^2 + 16) = 0$$

$$m^2 + 16 = 0$$

$$m^2 = -16$$

$$m_1 = \pm 4i$$

$$\cancel{y = c_1 e^{4x} + c_2 e^{-4x}}$$

$$\boxed{y = c_1 \cos 4x + c_2 \sin 4x}$$

$y(0) = 2$, $x = 0$, $y = 2$

$$2 = c_1 (1)$$

$$\boxed{c_1 = 2}$$

$$y' = c_1(-\sin 4x)4 + c_2 \cos 4x(4)$$

$$y' = 4c_2 \cos 4x$$

$y'(0) = -2$

$-2 = 4 c_2 \cos 4(0)$

$c_2 4 = -2$

$c_2 = -\frac{2}{4}$ X ———— X

$\boxed{c_2 = \frac{-1}{2}}$

$$\boxed{y_c = 2\cos 4x - \frac{1}{2}\sin 4x}$$

X ———— X

27. $\frac{d^5 u}{dr^5} + 5\frac{d^4 u}{dr^4} - 2\frac{d^3 u}{dr^3} - 10\frac{d^2 u}{dr^2} + \frac{du}{dx}$

$+5u = 0$

Sol Put $y = e^{mx}$, $y' = me^{mx}$, $y'' = m^2 e^{mx}$, $y''' = m^3 e^{mx}$

$y^{(4)} = m^4 e^{mx}$, $y^{(5)} = m^5 e^{mx}$

$e^{mx}(m^5 + 5m^4 - 2m^3 - 10m^2 + m + 5) = 0$

$m^5 + 5m^4 - 2m^3 - 10m^2 + m + 5 = 0$

$(m+5)(m^2-1)^2 = 0$

$m_1 = -5,\ m_2, m_3 = 1,\ m_{4,5} = -1$

$$\boxed{y_c = c_1 e^{-5x} + c_2 e^{x} + c_3 x e^{x} + c_4 e^{-x} + c_5 x e^{-x}}$$

X ———— X

28. $2\dfrac{d^5y}{dx^5} - 7\dfrac{d^4y}{dx^4} + 12\dfrac{d^3y}{dx^3} + 8\dfrac{d^2y}{dx^2} = 0$

Sol

~~Put $y = e^{mx}$, $y' = e^{mmx}$ ... $y^{(5)} = m^5 e^{mx}$~~

X ———— X

$\cancel{m^5 - 5m^4}$

$\Rightarrow 2m^5 - 7m^4 + 12m^3 + 8m^2 = 0$

$m^2(2m^3 - 7m^2 + 12m + 8) = 0$

$\boxed{m = -\frac{1}{2}}$ is the factor Now

Synthetic Divison

$-\frac{1}{2}$	2	-7	12	8
		-1	4	-8
	2	-8	16	0

$(2m^2 - 8m + 16) = 0$

$m_2 = \frac{1}{2}$

$2m^2 - 8m + 16 = 0$

$$m_3 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$m_3 = \frac{8 \pm \sqrt{64 - 4(2)(16)}}{4}$$

$$m_3 = 2 \pm 2i$$

$$\boxed{\alpha = 2, \ \beta = 2}$$

$$y = c_1 e^{m_1 x} + c_2 x e^{m_1 x} + c_3 e^{m_2 x} + c_4 e^{\alpha x}(\cos\beta x + c_5 \sin\beta x)$$

$$\boxed{y = c_1 + c_2 x + c_3 e^{-x/2} + e^{2x}(c_4 \cos 2x + c_5 \sin 2x)}$$

X ———— X

37. $y'' - 10y' + 25y = 0$, $y(0) = 1$, $y(1) = 0$

Sol

$$y = e^{mx}, \ y' = me^{mx}, \ y'' = m^2 e^{mx}$$

$$m^2 e^{mx} - 10me^{mx} + 25e^{mx} = 0$$

$$e^{mx}(m^2 - 10m + 25) = 0$$

$$e^{mx} \neq 0$$

$$m^2 - 10m + 25 = 0$$

$$(m-5)^2 = 0$$

$$\boxed{m_1, m_2 = 5}$$

Roots are Real and Repeated

$$\boxed{y = c_1 e^{5x} + c_2 x e^{5x}}$$

$$y(0) = 1$$

$$1 = c_1 e^{5(0)} + c_2 (0) e^{5(0)}$$

$$\boxed{c_1 = 1}$$

$$y(1) = 0$$

$$0 = c_1 e^{5(1)} + c_2 (1) e^{5(1)}$$

$$0 = c_1 e^5 + c_2 e^5$$

$$\boxed{-c_1 = c_2}$$

$c_1 = 1e^1$ and $c_2 = -1$

$$\boxed{y = e^{5x} - xe^{5x}}$$

is the solution of DE $y'' - 10y' + 25y = 0$.

X———X

38. $y'' + 4y = 0$, $y(0) = 0$, $y(\pi) = 0$

Put $y = e^{mx}$, $y' = me^{mx}$, $y'' = m^2 e^{mx}$

$m^2 e^{mx} + 4e^{mx} = 0$

$e^{mx}(m^2 + 4) = 0$

$m^2 + 4 = 0$

$m^2 = -4$

$$\boxed{m_{1,2} = \pm 2i}$$

~~$y = c_1 e^{2x} + c_2 e^{-2x}$~~

$$\boxed{y = c_1 \cos 2x + c_2 \sin 2x}$$

$y(0) = 0$	$y(\pi) = 0$
$0 = c_1 (1)$	$0 = c_1 \cos 2(\pi) + c_2 \sin 2(\pi)$
$\boxed{c_1 = 0}$	$\boxed{c_1 = 0}$

$$\boxed{y_c = c_2 \sin 2x}$$

X ———————— X

39. $y'' + y = 0, \quad y'(0) = 0, \quad y'(\pi/2) = 0$

Sol

$$m^2 e^{mx} + e^{mx} = 0$$

$$e^{mx}(m^2+1) = 0$$

$$m^2 = -1$$

$$m_1, m_2 = \pm i$$

$$\boxed{y = c_1 \cos x + c_2 \sin x}$$

$$y' = c_1 - \sin x + c_2 \cos x$$

$\Rightarrow y'(0) = 0$

$$0 = c_1 - \sin(0) + c_2 \cos(0)$$

$$\boxed{c_2 = 0}$$

$$\boxed{y = 0}$$

$y'(\pi/2) = 0$

$$0 = -c_1 \sin\left(\frac{\pi}{2}\right) + c_2 \cancel{\cos\left(\frac{\pi}{2}\right)}^{\,0}$$

$$\boxed{c_1 = 0}$$

40. $y'' - 2y' + 2y = 0$, $y(0) = 1$, $y(\pi) = 1$

Sol

$y = e^{mx}$, $y' = me^{mx}$, $y'' = m^2 e^{mx}$

$m^2 e^{mx} - 2me^{mx} + 2e^{mx} = 0$

$e^{mx}(m^2 - 2m + 2) = 0$

$m^2 - 2m + 2 = 0$

$$m_1, m_2 = \frac{2 \pm \sqrt{4 - 4 \times 1 \times 2}}{2}$$

$$\boxed{m_1, m_2 = 1 \pm i}$$

$$\boxed{y = c_1 e^{x} \cos x + c_2 e^{x} \sin x}$$

$y(0) = 1$

$1 = c_1 e^{0}(1)$

$\boxed{c_1 = 1}$

$y(\pi) = 1$

$1 = c_1 e^{\pi} \cos \pi + c_2 e^{\pi} \sin(\pi)$

$1 = -c_1 e^{\pi}$

$\boxed{c_1 = -\frac{1}{e^{\pi}}}$

Since two values of c_1 so no-solution.

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