General soln:-

A soln of DE is said to be general soln if It consist of the sum of complementary and particular solns. i.e

$$y = y_c + y_p.$$

Complementary soln — which is not free of arbitrary constants

particular soln. — A soln which is free of arbitrary constants is known as particular soln.

For example: The soln of linear-first order DE i.e

$$\frac{x\,dy}{dx} - y = x^2 \sin x$$

is

$$y = \underbrace{cx}_{y_c} + \underbrace{x\cos x}_{y_p}$$

General soln.

* **Singular soln** :- A soln that cannot be obtained by specializing any of the parameters in the family of solns. Such in extra soln is called a singular solns. For example.

$$y = \frac{1}{16}x^4 \quad \text{and} \quad y = 0 \quad (\text{Trivial soln}).$$

are the soln of $\frac{dy}{dx} = xy^{1/2}$

By solving separating of variable

$y^{1/2} = -x^2 \cancel{+ \frac{c}{2}}$

Example: 3 (linear DE)

$$x\frac{dy}{dx} - 4y = x^6 e^x \longrightarrow (1)$$

Eq(1) in standard form is

$$\frac{dy}{dx} - \frac{4y}{x} = x^5 e^x \longrightarrow (II)$$

comparing with general form we have

$$\frac{dy}{dx} + p(x)\, y = f(x).$$

$p(x) = \frac{-4}{x}$ and $f(x) = x^5 e^x$

The above ftns are contineous in ~~the~~ $(0,\infty)$.

The $I.F = e^{\int p(x)\,dx} = e^{\int \frac{-4}{x}\,dx}$

$\Rightarrow I.F = e^{-4\int \frac{1}{x}\,dx} = e^{-4\ln x} = e^{\ln x^{-4}}$

$\Rightarrow$ $\boxed{I.F = x^{-4}}$

Now xing Eq (ii) ~~By~~ $I.F = x^{-4}$ we get

$$x^{-4}\frac{dy}{dx} - x^{-4}\frac{4}{x}y = x^{-4}x^5 e^x$$

$$\Rightarrow x^{-4}\frac{dy}{dx} - 4x^{-5}y = x e^x.$$

$$\Rightarrow \frac{d}{dx}\left(x^{-4}y\right) = xe^x$$

$$\Rightarrow \int d(x^{-4}y) = \int \underline{x}\,\underline{e^x}\,dx$$

$$\Rightarrow x^{-4}y = x\int e^x dx - \int\left(\frac{d}{dx}(x)\int e^x dx\right)dx$$

$$\Rightarrow x^{-4}y = xe^x - \int 1\cdot e^x\,dx$$

$$\Rightarrow x^{-4} y = x e^{x} - e^{x} + c.$$

$$\Rightarrow y = \frac{x e^{x} - e^{x} + c}{x^{-4}}$$

$$\Rightarrow \boxed{y = x^{5} e^{x} - x^{4} e^{x} + x^{4} c.} \longrightarrow (\text{iii})$$

The above soln is known as general soln.

Note: Noted that ~~in~~ the DE - ~~in~~ example A.

for $a_1(x) \frac{dy}{dx} + a_0(x) y = g(x)$.

The value of x for which $a_1(x) = 0$ are called singular point.

Singular point are potentially troublesome. ~~Esp.~~ specially, if $p(x) = \frac{a_0(x)}{a_1(x)}$ is discontinuous at a point, the discontinuity may carry over to soln of DE.

Example 4:- Solve $(x^2-9)\dfrac{dy}{dx} + xy = 0 \longrightarrow (1)$

In standard form Eq(1) $\Rightarrow$

$$\frac{dy}{dx} + \frac{x}{(x^2-9)}\,y = 0 \longrightarrow (11)$$

Here $p(x) = \dfrac{x}{x^2-9}$ and $f(x) = 0$

Here $a_1(x) = x^2-9$

if $a_1(x) = 0$

$\Rightarrow x^2-9 = 0$

$\Rightarrow x = \pm 3$

are the Singular points

$p(x)$ is continuous in $(-\infty,-3)$ and $(-3,3), (3,\infty)$

$$I.F = e^{\int p(x)\,dx}$$

$$= e^{\int \frac{x}{x^2-9}\,dx}$$

$$= e^{\frac{1}{2}\int \frac{2x}{x^2-9}\,dx}$$

$$= e^{\frac{1}{2}\ln|x^2-9|}$$

$$= e^{\ln|x^2-9|^{1/2}}.$$

$$\boxed{I.F = \sqrt{x^2-9}}$$

So the soln is

$$\frac{d}{dx}\left(\sqrt{x^2-9}\;y\right) = 0$$

$$\Rightarrow (x^2-9)\,y = \int 0\,dx$$

$$\Rightarrow (x^2-9)\,y = c$$

$\Rightarrow \boxed{y = \frac{c}{\sqrt{x^2-9}}}$ This is general soln for either $x<-3$ or $x>3$.

Piecewise - linear Differential Equation: —

When either $p(x)$ or $f(x)$ in first order linear DE is a piecewise defined ftn is then referred as a piecewise - linear DE. For example the below problem is piecewise - linear DE.

Example: 6: Solve the IVP

$$\frac{dy}{dx} + y = f(x) \qquad y(0)=0 \quad \text{and} \quad f(x) = \begin{cases} 1 & 0 \le x \le 1 \\ 0 & x > 1 \end{cases}$$

Soln:— First solve the DE for the interval i.e $0 \le x \le 1$

$\therefore \frac{dy}{dx} + y = 1 \longrightarrow ①$ (which is already in the standrad form)

$p(x) = 1. \quad \therefore I.F = e^{\int dx} = e^x$

$\therefore \frac{d}{dx}(e^x y) = e^x$

Integrating the above Eq.

$\Rightarrow \int d(e^x y) = \int e^x dx$

$\Rightarrow e^x y = e^x + c_1 \quad \Rightarrow y = e^x e^{-x} + e^{-x} c_1$

$\Rightarrow \boxed{y = 1 + C_1 e^{-x}}$ Now $y(0) = 0 \Rightarrow \boxed{C_1 = -1}$

$\Rightarrow \boxed{y = 1 - e^{-x}}$

Now Solve for $x > 1$ or $(1, \infty)$.

$$\frac{dy}{dx} + y = 0 \qquad \therefore f(x) = 0 \text{ for } x > 1$$

$$p(x) = 1 \qquad \text{i.F} = e^{\int dx} = e^{x}$$

$$\Rightarrow \frac{d}{dx}(e^{x} y) = e^{x} \cdot (0)$$

$$\Rightarrow \frac{d}{dx}(e^{x} y) = 0$$

$$\Rightarrow \int d e^{x} y = \int 0 \, dx$$

$$\Rightarrow e^{x} y = C_2$$

$$\Rightarrow \boxed{y = e^{-x} C_2}$$

Hence we can write

$$y = \begin{cases} 1 - e^{-x} & 0 \le x \le 1 \\ C_2 e^{-x} & x > 1 \end{cases} \longrightarrow (*)$$

To find the value of C_2? we appling the defination of continuity at a point i.e $x = 1$.

$\therefore$ $$\lim_{x \to 1^+} y(x) = y(1)$$

$$\Rightarrow \lim_{x \to 1^+} (C_2 e^{-x}) = 1 - e^{-1}$$

$$\Rightarrow C_2 e^{-1} = 1 - e^{-1}$$

$$\Rightarrow \boxed{C_2 = e - 1}$$ using this value in eq (*)

we have

$$y = \begin{cases} 1 - e^{x} & 0 \leq x \leq 1 \\ (e-1) e^{-x} & x > 1 \end{cases}$$

x ———— x ———— x ———— x

Error function :- In mathematic and engineering some fns are defined in terms of non-elementary integrals. i.e $\int e^{-x^2} dx$ or $\int \sin x^2$

Defination : A simple elementary fns that do not possess antiderivatives, then that integral of called these fns are called non-elementary. Two special fns are the error fn and complementry error fn. i.e

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt \quad \text{and} \quad \operatorname{erfc}(x) = \frac{2}{\sqrt{2}} \int_x^\infty e^{-t^2} dt$$

As we know that $\int_0^\infty e^{-t^2}dt = \frac{\sqrt{\pi}}{2}$ and defined the property of integral

$$\int_0^\infty = \int_0^x + \int_x^\infty$$

we can write

$$\frac{2}{\sqrt{\pi}}\int_0^\infty e^{-t^2}dt = \underbrace{\frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt}_{\text{erf}(x)} + \underbrace{\frac{2}{\sqrt{\pi}}\int_x^\infty e^{-t^2}dt}_{\text{erfc}(x)} = 1$$

$$\Rightarrow \boxed{\text{erf}(x) + \text{erfc}(x) = 1}$$

Noted that $\text{erf}(0) = 0$

Example 7: $\frac{dy}{dx} - 2xy = 2 \quad y(0) = 1 \longrightarrow (1)$

Soln: Here $p(x) = -2x$

$$I.F = e^{\int -2x\,dx} = e^{-\frac{2x^2}{2}} = e^{-x^2}$$

$$e^{-x^2}\frac{dy}{dx} - 2e^{-x^2}xy = 2e^{-x^2}$$

$$\Rightarrow \frac{d}{dx}\left(e^{-x^2}y\right) = 2e^{-x^2} \rightarrow$$

$$\Rightarrow \int \frac{d}{dx}\left(e^{-x^2}y\right)dx = 2\underbrace{\int e^{-x^2}dx}_{\text{non-elementary Integral}}$$

we identify $x_0 = 0$ ad use definit integration over the interval $[0, x]$.

$$\int_0^x \frac{d}{dt}(e^{-t^2} y)\, dt = 2\int_0^x e^{-t^2} dt.$$

$$\Rightarrow e^{-x^2} y = 2\int_0^x e^{-t^2} dt + C. \longrightarrow (11)$$

using IC $y(0) = 0 \Rightarrow$

$$e^{0} y(0) = 2\int_0^0 e^{-t^2} dt + C$$

$$1 \cdot (1) = 0 + C.$$

$\Rightarrow \boxed{C = 1}$ put in Eq (11) we hence

$$e^{-x^2} y = 1 + 2\int_0^x e^{-t^2} dt.$$

$$\Rightarrow y = e^{x^2} + 2e^{x^2}\int_0^x e^{-t^2} dt.$$

$$= e^{x^2}\left[1 + \sqrt{\pi}\underbrace{\left(\frac{2}{\sqrt{\pi}}\right)\int_0^x e^{-t^2} dt}_{erf(x)}\right].$$

$$\boxed{y = e^{x}\left[1 + \sqrt{\pi}\, erf(x)\right]}$$

x ——— x ——— x ——— x

Remarks: Occasionally the 1^{st}-order DE is not linear in one variable but it is linear in the other variable. i.e

$$\frac{dy}{dx} = \frac{1}{x+y^2}$$ is not linear in y.

But its reciprocal $\frac{dx}{dy} = x+y^2$ is linear in x.

$$\Rightarrow \frac{dx}{dy} - x = y^2$$

Here $p(y) = -1$ $\qquad$ $I.F = e^{-\int dy} = e^{-y}$.

$$\Rightarrow \boxed{I.F = e^{-y}}$$

$$e^{-y}\left(\frac{dx}{dy} - x\right) = e^{-y} y^2$$

$$\Rightarrow \frac{d}{dy}\left(e^{-y} x\right) = e^{-y} y^2$$

$$\Rightarrow e^{-y} x = \int \underset{II}{e^{-y}} \underset{I}{y^2}\, dy.$$

$$\Rightarrow \boxed{x = -y^2 - 2y + 2 + Ce^{y}}$$ → Implicit soln.

X ——— X ——— X ——— X

Solve → Ex: 2.3

Exact Equation:

Differential of a fn of two variable:-

If $z = f(x,y)$ is a function of two variable with continuous first partial derivatives in a region R of the xy plane then recall from the calculus, the differential is defined

$$dz = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy \longrightarrow (i)$$

In Special case if $f(x,y) = c$ then

$$\boxed{\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = 0} \longrightarrow (ii)$$

Exact differential:- A differential expression $M(x,y)\,dx + N(x,y)\,dy = 0$ is an exact differential in the region R of xy-plane if it corresponds to the differential of some fn $f(x,y)$ in R.

Exact Equation:- A first-order differential Eq of the form $M(x,y)dx + N(x,y)\,dy = 0$ is said to be exact equation if the expression on the left-hand side is an exact differential.

For example $x^2y^3dx + x^3y^2dy = 0$ is exact Eq. b/c

$$d\left(\tfrac{1}{3}x^3y^3\right) = x^2y^3dx + x^3y^2dy$$

Criterion for an exact differential:

Let $M(x,y)$ and $N(x,y)$ are continuous and have continuous 1^{st}-order partial derivatives in a Rectangular xy region R defined $a<x<b$, $c<y<d$. then a necessary and sufficient condition that $M(x,y)dx + N(x,y)dy = 0$ be an exact differential is

$$\boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}} \longrightarrow (*)$$

Note:

$M(x,y)dx + N(x,y)dy = 0$

or $\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = 0$

$\Rightarrow \boxed{M(x,y) = \frac{\partial f}{\partial x}, \ N(x,y) = \frac{\partial f}{\partial y}}$

Method of soln:-

Firstly check the criteria in $(*)$ for $M(x,y)dx + N(x,y)dy = 0$

i) it holds then

$$\frac{\partial f}{\partial x} = M(x,y)$$

arbitrary fn ~~constant~~ of integr

$$\Rightarrow f(x,y) = \int M(x,y)\,dx + g(y) \longrightarrow (1)$$

Diff. ~~f(x,y)~~ eq (1) w.r.t y

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\int M(x,y)\,dx + \frac{\partial}{\partial y}g(y)$$

$$\Rightarrow \frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\int M(x,y)\,dx + g'(y)$$

$\because \frac{\partial f}{\partial y} = N(x,y)$

$$\Rightarrow g'(y) = N(x,y) - \frac{\partial}{\partial y}\int M(x,y)\,dx \longrightarrow (11)$$

Integrate Eq (ii) w.r.t y and put result in Eq (i) we get $f(x,y) = c$

$\Rightarrow$ we just also start the foregoing procedure with the assumption that $\frac{\partial f}{\partial y} = N(x,y)$.

$$f(x,y) = \int N(x,y)\, dy + h(x).$$

and $h'(x) = M(x,y) - \frac{\partial}{\partial x}\int N(x,y)\, dy.$

the procedure is same.

Example 1. Solve $2xy\,dx + (x^2-1)\,dy = 0 \longrightarrow (1)$

Comparing with General form of exact Eq.

$$M(x,y) = 2xy \text{ and } N(x,y) = (x^2-1).$$

$$\frac{\partial M}{\partial y} = 2x = \frac{\partial N}{\partial x} \Rightarrow [\text{Eq. is exact}].$$

then there exists a ftn $f(x,y)$ such that

$$\frac{\partial f}{\partial x} = 2xy \rightarrow (ii) \quad \frac{\partial f}{\partial y} = x^2 - 1 \rightarrow (iii)$$

integ. Eq (ii) $f(x,y) = x^2y + g(y) \rightarrow (iv)$

$g(y) \rightarrow$ arbitrary ftn.

take the partial derivative of Eq (iv) w.r.t. y.

$\frac{\partial f}{\partial y} = x^2 + g'(y)$. but $\frac{\partial f}{\partial y} = x^2 - 1$

$\therefore\ \cancel{x^2} - 1 = \cancel{x^2} + g'(y)$

$\Rightarrow\ g'(y) = -1 \quad \Rightarrow \quad \boxed{g(y) = -y} \longrightarrow (v)$

put (v) in Eq (iv)

$f(x,y) = x^2y - y$

So soln of Eq in implicit form is $f(x,y) = C$

$\Rightarrow\ \boxed{x^2y - y = c}$

From this the explicit soln can be writen as

$\boxed{y = \frac{c}{x^2 - 1}}$

x ———— x

Integrating factor:- when the D.Eq.

$M(x,y)\,dx + N(x,y)\,dy = 0$ is not exact then Sometime it is possible to find the integrating factor $\mu(x,y)$. So that after multiplying, the left hand side of

$$\mu(x,y)\,M(x,y)\,dx + \mu(x,y)\,N(x,y)\,dy = 0$$

is an exact diff. Eq.

(i) * If $\dfrac{M_y - N_x}{N}$ is a fn of x-alone, then I.F is

$$\mu(x) = e^{\int \frac{M_y - N_x}{N}\,dx}$$

(ii) * If $\dfrac{N_x - M_y}{M}$ is a fn of y-alone then I.F

$$\mu(y) = e^{\int \frac{N_x - M_y}{M}\,dy}$$

Example: A Nonexact DE made exact:-

Sol. $xy\,dx + (2x^2 + 3y^2 - 20)\,dy = 0 \longrightarrow (1)$

$M = xy, \quad N(x,y) = 2x^2 + 3y^2$

$\Rightarrow$ $M_y = x$ and $N_x = 4x$

Since $M_y \neq N_x$ So the Eq is not exact

Now $\dfrac{M_y - N_x}{N} = \dfrac{x - 4x}{2x^2+3y^2-20} = \dfrac{-3x}{2x^2+3y^2-20}$ } which is ftn of x and y both

Consider $\dfrac{N_x - M_y}{M} = \dfrac{4x - x}{x \cdot y} = \dfrac{3\cancel{x}}{\cancel{x}y} = \dfrac{3}{y}$] ftn of y alone.

So $I.F = \mu(y) = e^{\int \frac{N_x - M_y}{M} dy}$

$= e^{\int \frac{3}{y} dy}$

$= e^{\ln y^3}$

$$\boxed{\mu(y) = y^3}$$

xing Eq(1) by I.F.

$xy^4 dx + (2x^2y^3 + 3y^2 - 20y^3) dy = 0$

which becom an exact

$M_y = 3xy^3 = N_x$