Networks, Flows and Cuts

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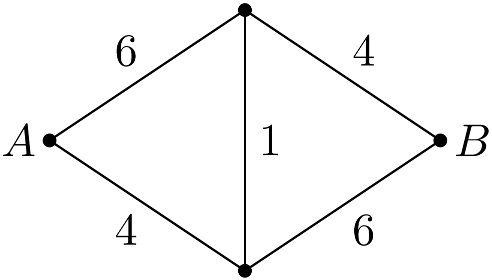
Outline

Network: Pipes and Capacities



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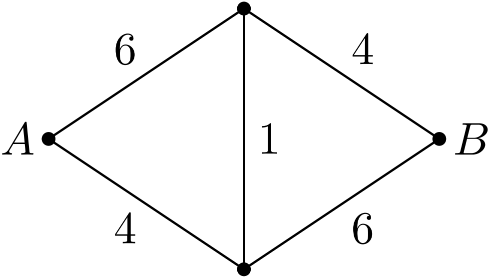

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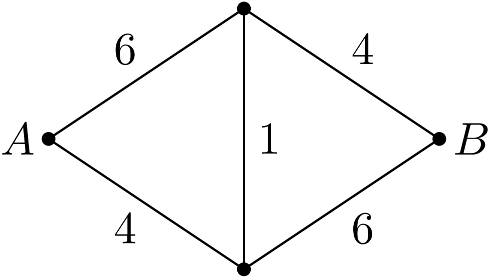

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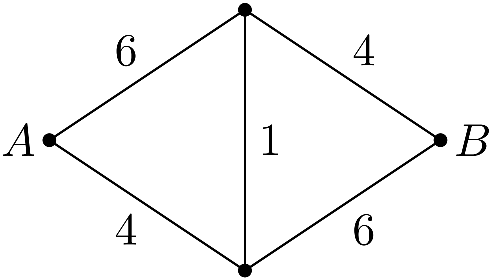

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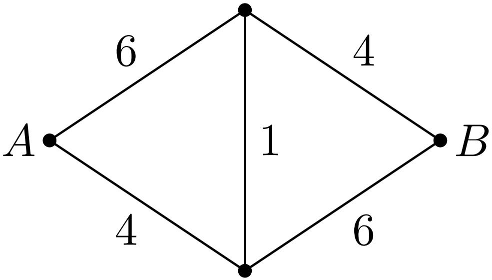

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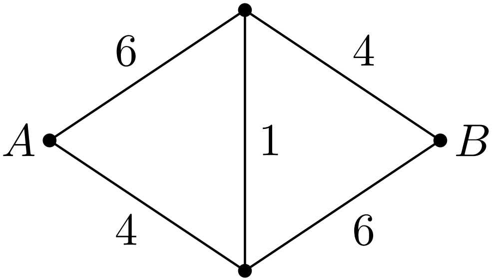

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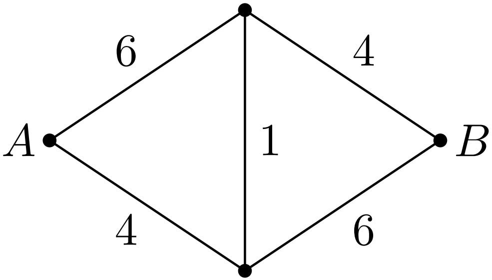

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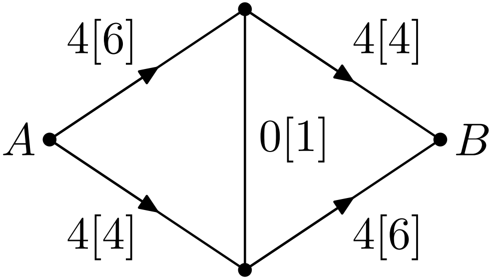

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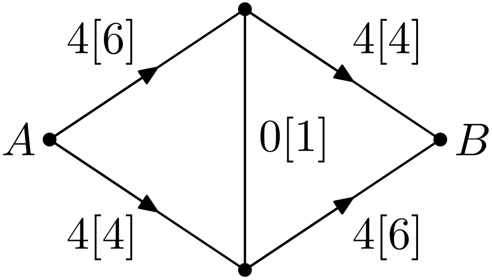

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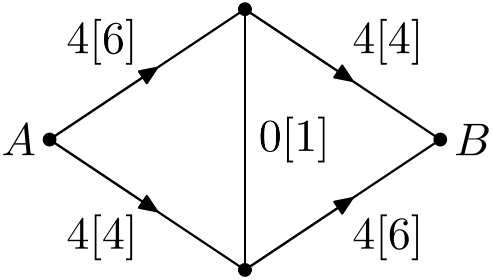

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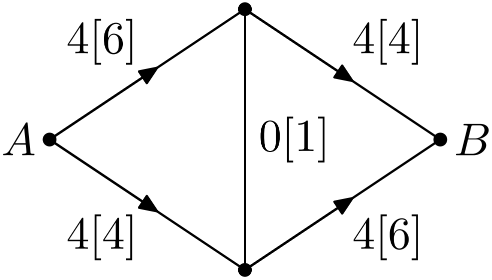

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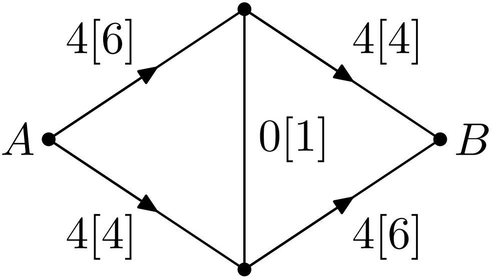

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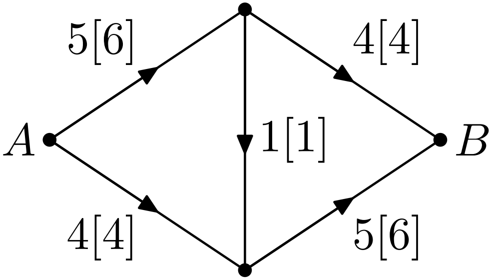

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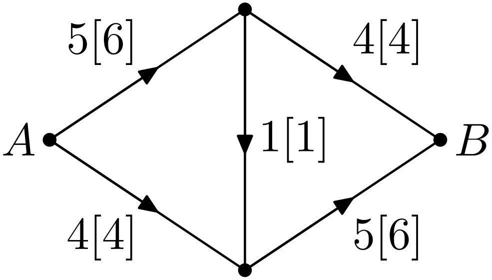

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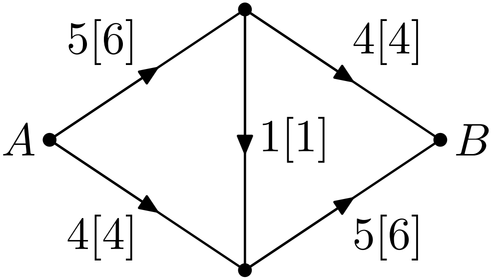

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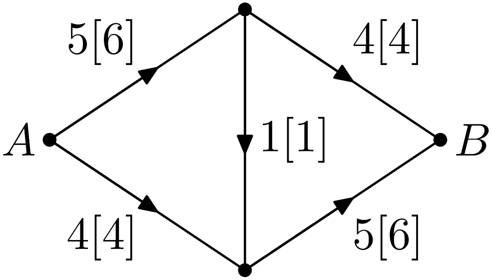
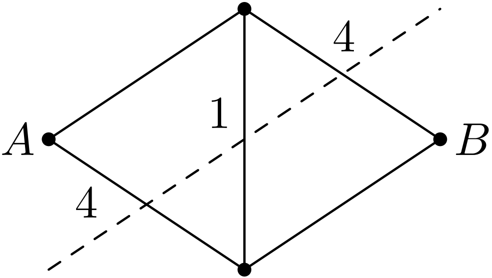


- flow (from A to B): 9
- maximum flow?
- 9 is possible, not more than 10

Why 9 Is Maximal



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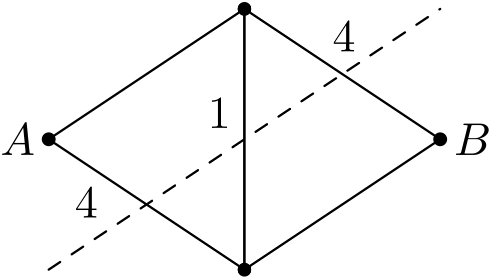

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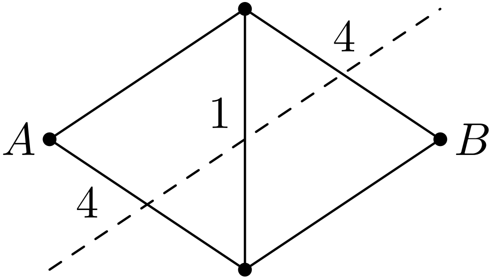

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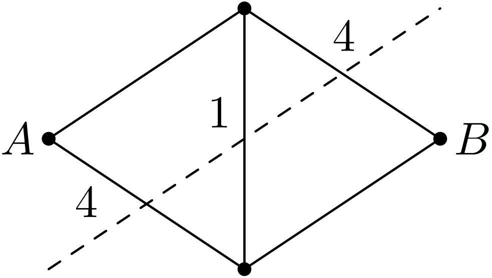

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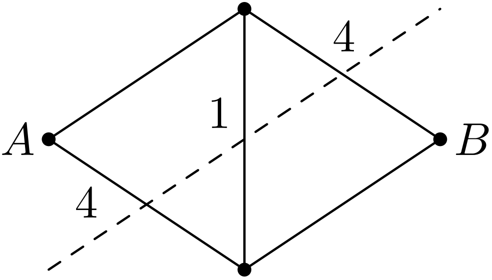

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- total flow:
  $\sum_j f[A,j] = \sum_i f[i,B]$

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- Ford–Fulkerson: the equality happens for some flow and some cut

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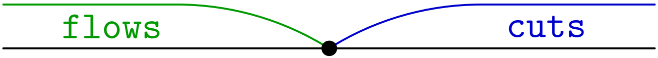

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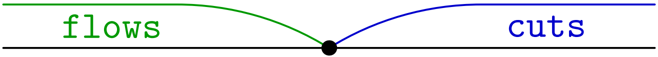

Theorem: maximal flow = minimal cut