

Eulerian and Hamiltonian Cycles

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Toy Genome Assembly Problem

Find a string whose all substrings of length 3 are

AGC, ATC, CAG, CAT, CCA, GCA, TCA, TCC.

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How is it related to cycles in graphs?..

Outline

Eulerian Cycles

Hamiltonian Cycles

Genome Assembly

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- A cycle must have the same starting and ending nodes

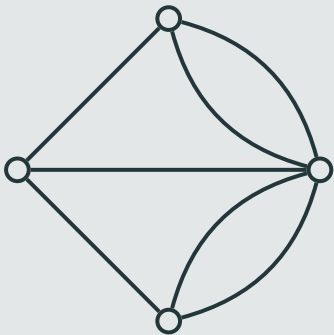
Eulerian Cycle

An **Eulerian cycle (or path)** visits every edge exactly once.

- The definition works for both directed and undirected graphs
- A cycle must have the same starting and ending nodes
- While in a path the starting and ending node should not necessarily be equal

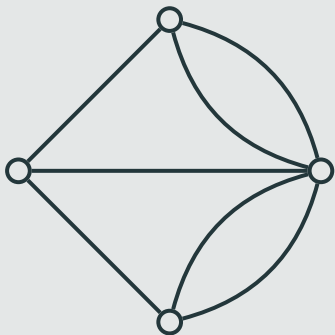
Example

Non-Eulerian graph

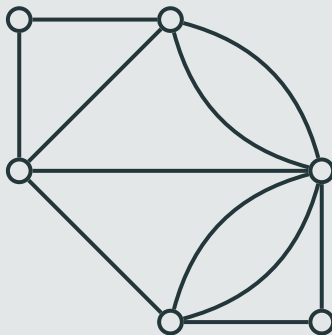


Example

Non-Eulerian graph

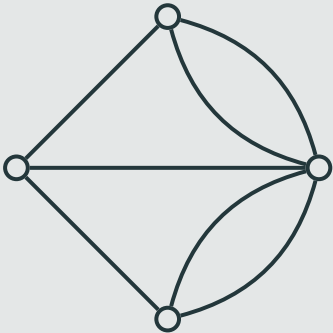


Eulerian graph

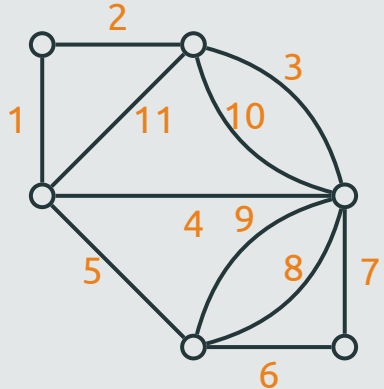


Example

Non-Eulerian graph



Eulerian graph



Criteria

Theorem

A connected *undirected* graph contains an Eulerian cycle, if and only if the degree of every node is even.

Criteria

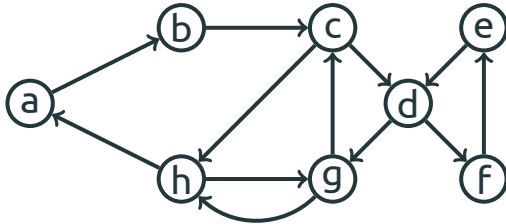
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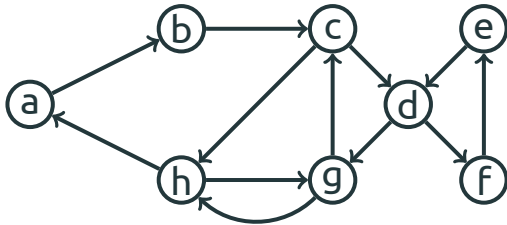
A strongly connected *directed* graph contains an Eulerian cycle, if and only if, for every node, its in-degree is equal to its out-degree.

Proof (Directed Case)



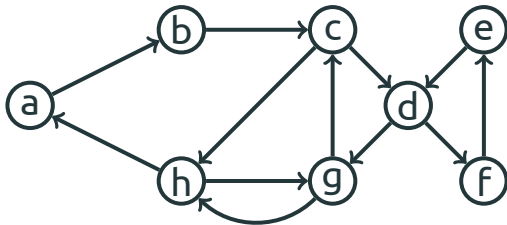
if some node is imbalanced,
there is clearly no Eulerian cycle

Proof (Directed Case)



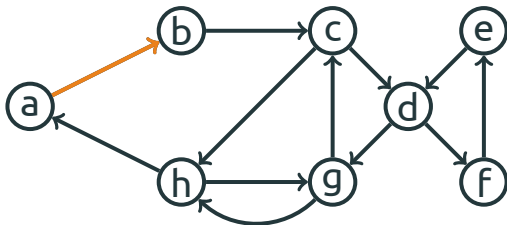
thus, assume that the graph is balanced

Proof (Directed Case)

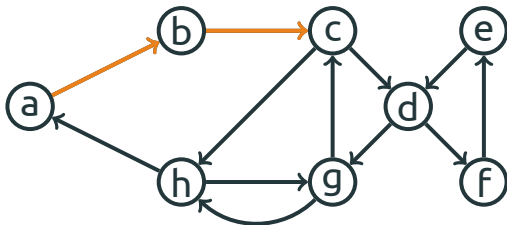


start walking from some node

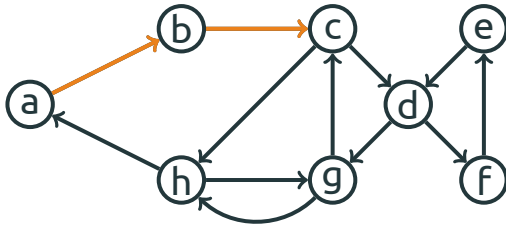
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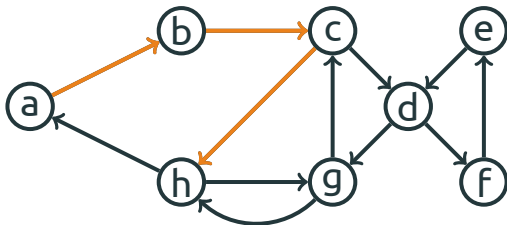


Proof (Directed Case)

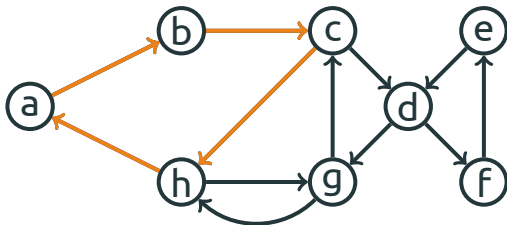


since the graph is balanced, at some point
we'll return back to the starting node

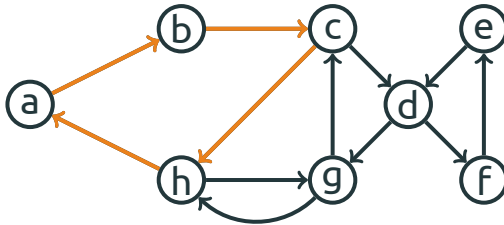
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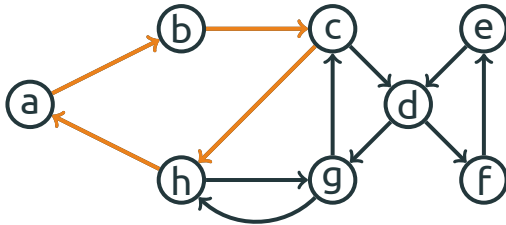


Proof (Directed Case)



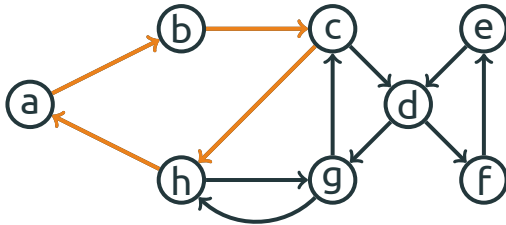
OK, what's next? we haven't traversed all the edges and now we are stuck

Proof (Directed Case)



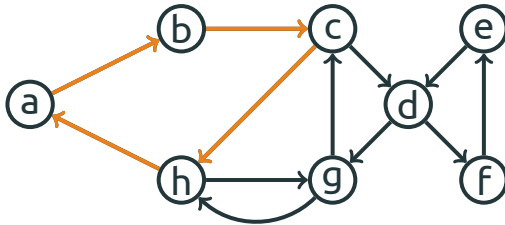
since the graph is strongly connected,
there must be untraversed edges going
out of a node from the current cycle

Proof (Directed Case)



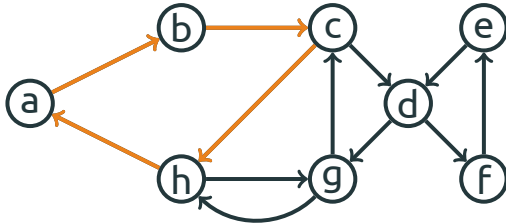
the current cycle has neither beginning nor end

Proof (Directed Case)



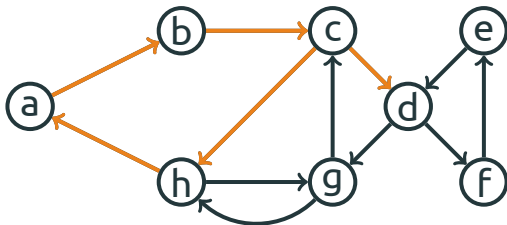
so, we may assume that its
starting and ending point is the node c

Proof (Directed Case)

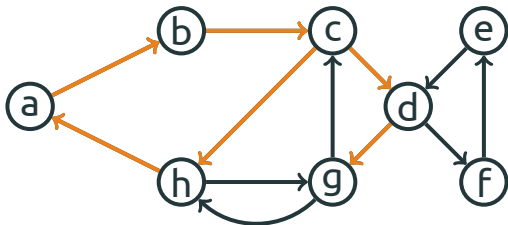


we then continue exploring
the graph from the node c

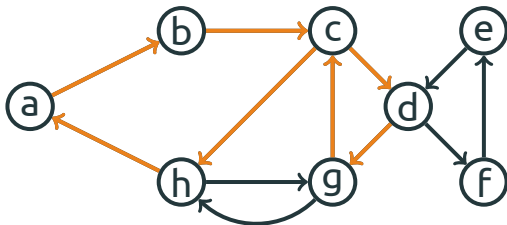
Proof (Directed Case)



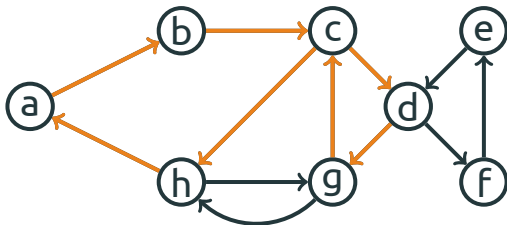
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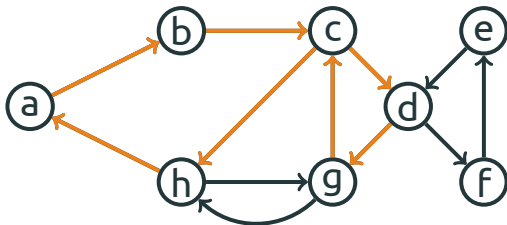


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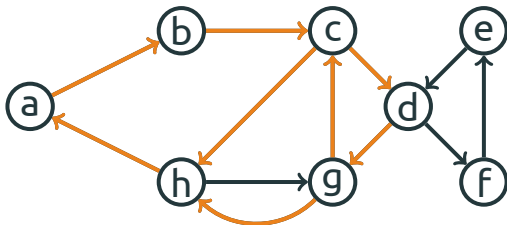
we got larger cycle

Proof (Directed Case)

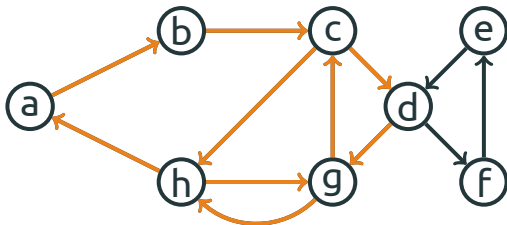


continue from g

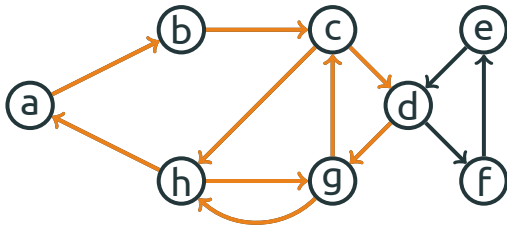
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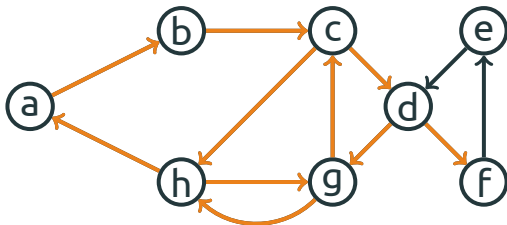


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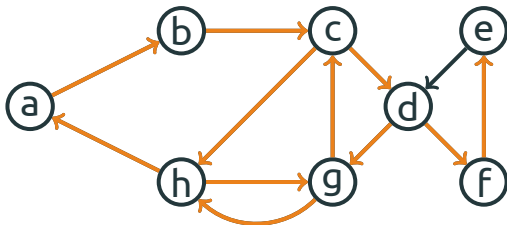


continue from d

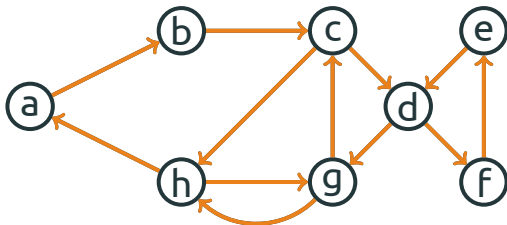
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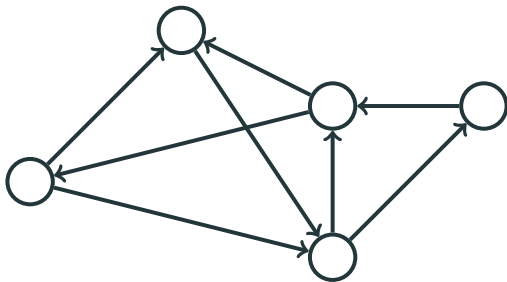
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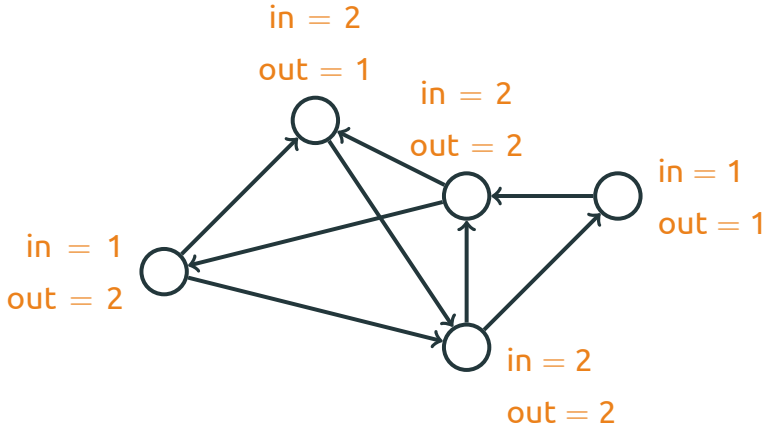
Path Instead of Cycle

- The criteria for a path is similar
- A graph is allowed to contain two imbalanced nodes: one for the starting point and one for the ending point of a path
- By adding an edge between these two nodes, one gets a graph with an Eulerian cycle

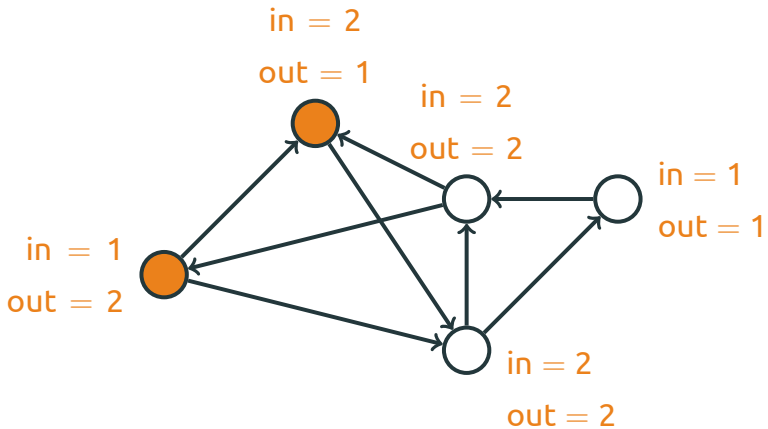
Path to Cycle



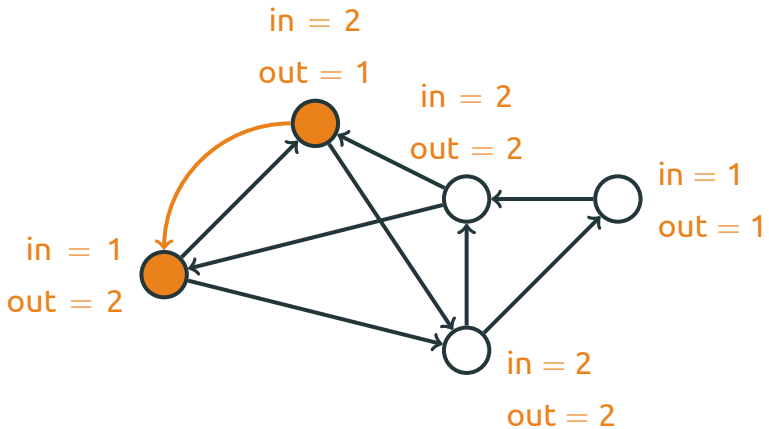
Path to Cycle



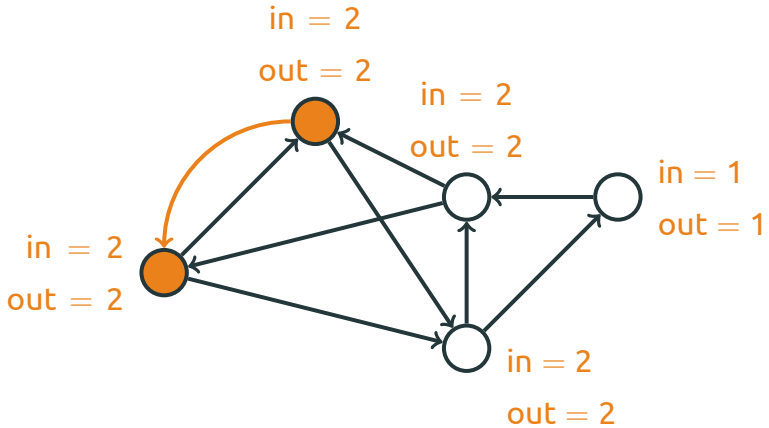
Path to Cycle



Path to Cycle



Path to Cycle



Efficient Algorithms

The proof of existence of an Eulerian cycle can be transformed into an efficient algorithm for constructing an Eulerian cycle

Outline

Eulerian Cycles

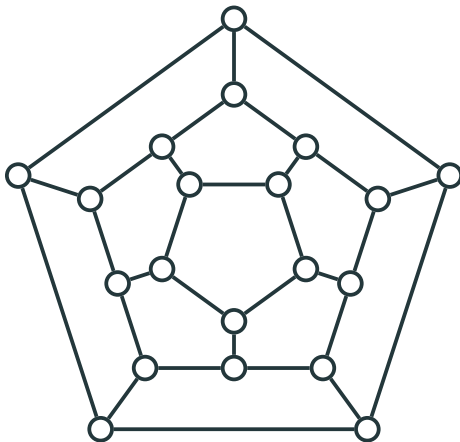
Hamiltonian Cycles

Genome Assembly

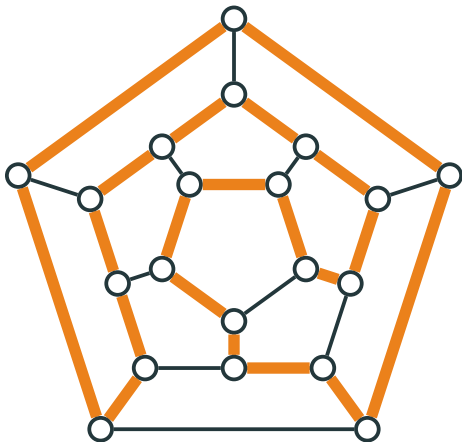
Hamiltonian Cycle

A **Hamiltonian cycle** visits every node of a graph exactly once.

Example



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Simple Criteria?

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- No polynomial time algorithm known
- The question whether there is a polynomial time algorithm for the Hamiltonian cycle problem is the P versus NP problem, the most important open problem in Computer Science, with a prize of \$1M from the Clay Mathematics Institute (<http://www.claymath.org/millennium-problems>)

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Every two neighbor 3-substrings have
a common part, called an overlap, of length 2

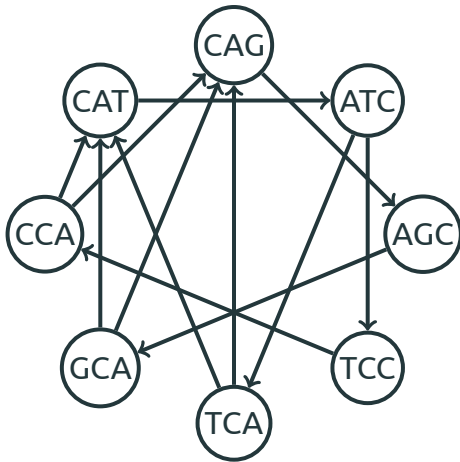
Finding a Permutation

- Goal: Find a string whose all substrings of length 3 are AGC, ATC, CAG, CAT, CCA, GCA, TCA, TCC

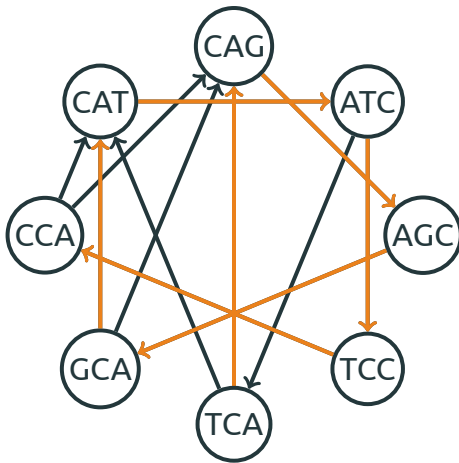
Finding a Permutation

- Goal: Find a string whose all substrings of length 3 are AGC, ATC, CAG, CAT, CCA, GCA, TCA, TCC
- Hence, we need to find an **order** of these 3-strings such that the overlap between any two consecutive strings is equal to 2

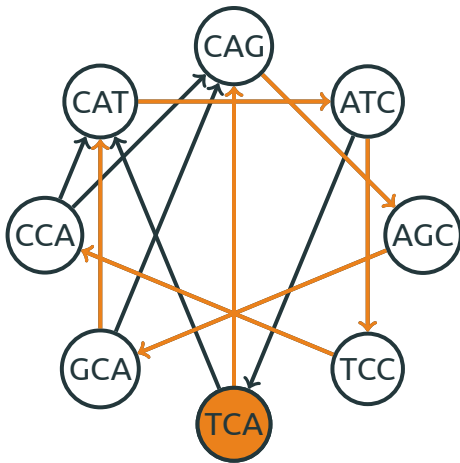
Overlap Graph



Overlap Graph

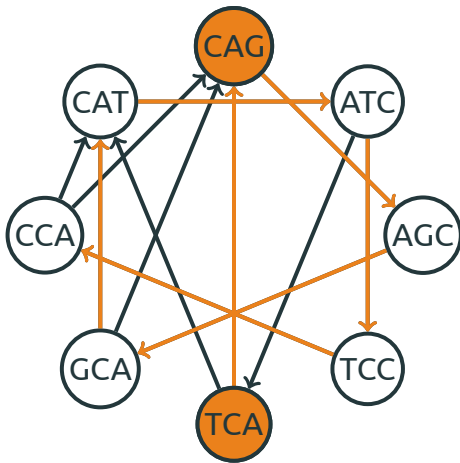


Overlap Graph



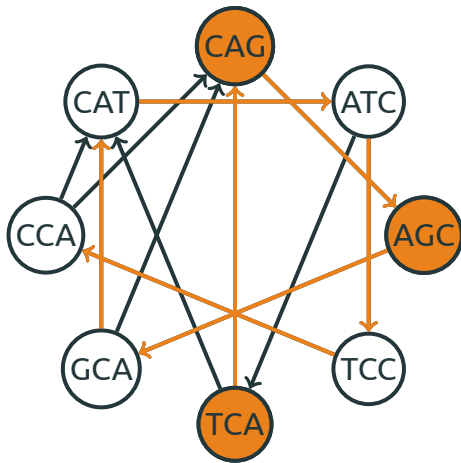
TCA

Overlap Graph



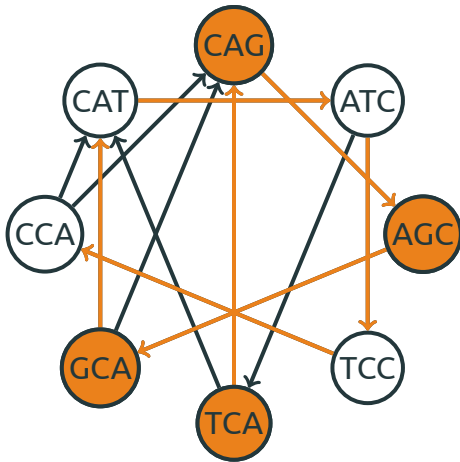
TCAG

Overlap Graph



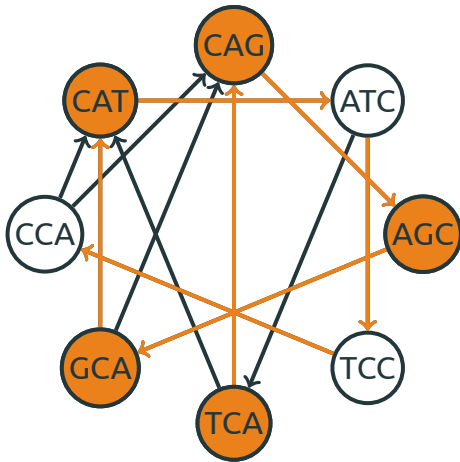
TCAGC

Overlap Graph



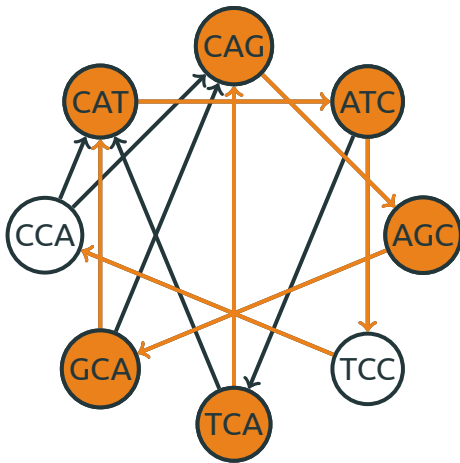
TCAGCA

Overlap Graph



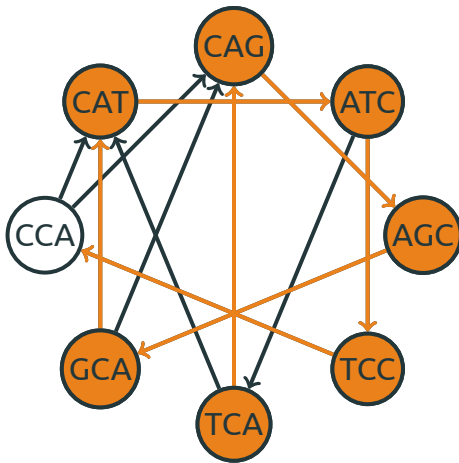
TCAGCAT

Overlap Graph



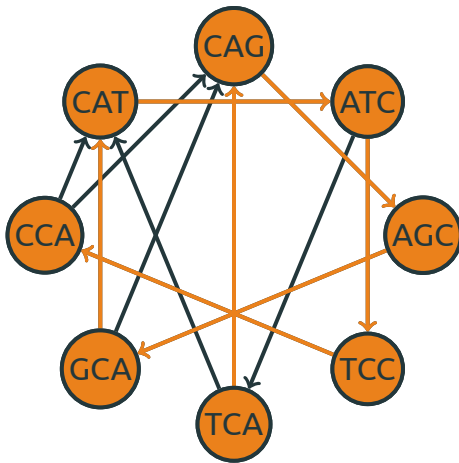
TCAGCATC

Overlap Graph



TCAGCATCC

Overlap Graph



TCAGCATCCA

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- The approach is useless for the case when there are thousands or millions of input strings

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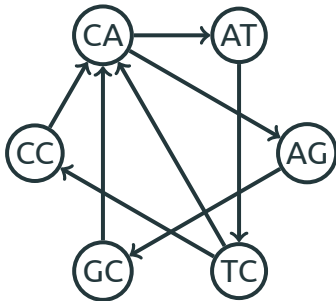
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- E.g., represent the string CAT as an edge
 $CA \rightarrow AT$

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- Let's instead try to represent each string by an **edge** (De Bruijn; Pevzner, Tang, Waterman)
- E.g., represent the string CAT as an edge $CA \rightarrow AT$
- Used in state-of-the-art genome assemblers

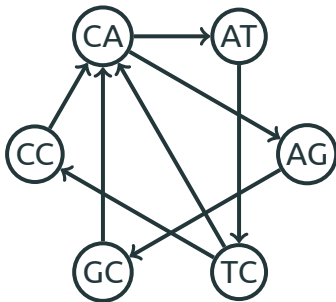
De Bruijn Graph

AGC, ATC, CAG, CAT, CCA, GCA, TCA, TCC



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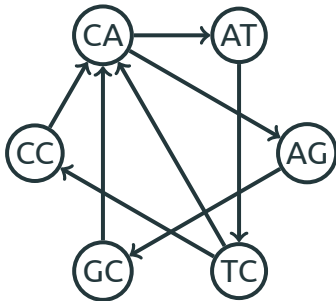
AGC, ATC, CAG, CAT, CCA, GCA, TCA, TCC



now, we need to find an order of edges

De Bruijn Graph

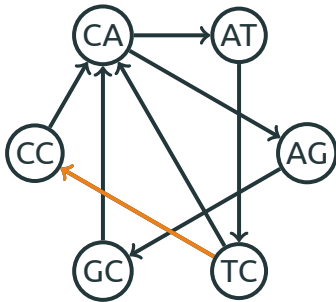
AGC, ATC, CAG, CAT, CCA, GCA, TCA, TCC



that is, an Eulerian path

De Bruijn Graph

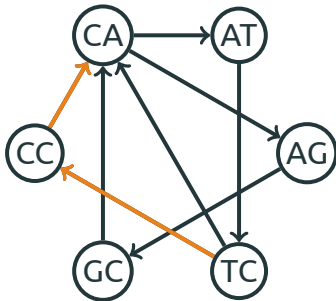
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TCC

De Bruijn Graph

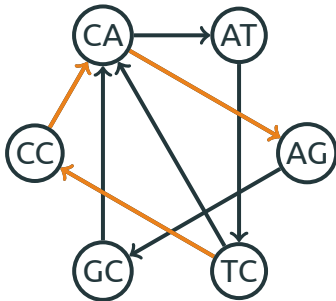
AGC, ATC, CAG, CAT, CCA, GCA, TCA, TCC



TCCA

De Bruijn Graph

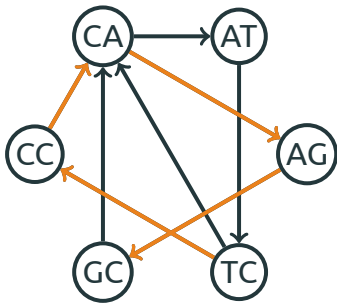
AGC, ATC, CAG, CAT, CCA, GCA, TCA, TCC



TCCAG

De Bruijn Graph

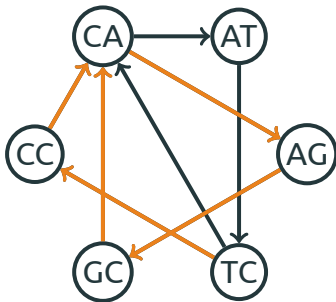
AGC, ATC, CAG, CAT, CCA, GCA, TCA, TCC



TCCAGC

De Bruijn Graph

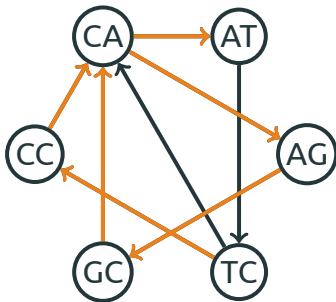
AGC, ATC, CAG, CAT, CCA, GCA, TCA, TCC



TCCAGCA

De Bruijn Graph

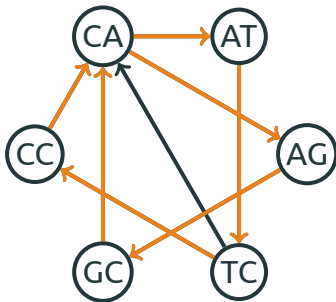
AGC, ATC, CAG, CAT, CCA, GCA, TCA, TCC



TCCAGCAT

De Bruijn Graph

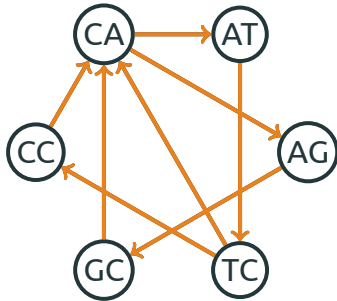
AGC, ATC, CAG, CAT, CCA, GCA, TCA, TCC



TCCAGCATC

De Bruijn Graph

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TCCAGCATCA

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- Hamiltonian cycle visits every **node** exactly once
- Look similar to each other, but differ drastically from the computational point of view
- Genome assembly: the right problem formulation makes all the difference!