# Concrete Math: Homework 4

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# Problem 1

证明 $\binom{-1/3}{n}\binom{-2/3}{n} = \binom{3n}{2n}\binom{2n}{n}/3^{3n}$ , 其中 $n \in \mathbb{Z}$.

# Solution

根据定义有：

$$\begin{aligned}
\binom{-1/3}{n} &= \frac{(-\frac{1}{3})^{\underline{n}}}{n!} = \frac{(-\frac{1}{3})(-\frac{1}{3}-1)\cdots(-\frac{1}{3}-n+1)}{n!} \\
\binom{-2/3}{n} &= \frac{(-\frac{2}{3})^{\underline{n}}}{n!} = \frac{(-\frac{2}{3})(-\frac{2}{3}-1)\cdots(-\frac{2}{3}-n+1)}{n!}
\end{aligned}$$

则：

$$\begin{aligned}
\binom{-1/3}{n}\binom{-2/3}{n} &= \frac{(-\frac{1}{3})(-\frac{1}{3}-1)\cdots(-\frac{1}{3}-n+1)}{n!} \cdot \frac{(-\frac{2}{3})(-\frac{2}{3}-1)\cdots(-\frac{2}{3}-n+1)}{n!} \\
&= \frac{(n-1+\frac{1}{3})(n-2+\frac{1}{3})\cdots(\frac{1}{3}) \cdot (n-1+\frac{2}{3})(n-2+\frac{2}{3})\cdots(\frac{2}{3})}{(n!)^2} \\
&= \frac{(3n-2)(3n-5)(3n-8)\cdot 4 \cdot 1 \cdot (3n-1)(3n-4)(3n-7)\cdot 5 \cdot 2}{(n!)^2 \cdot 3^{2n}} \\
&= \frac{1 \cdot 2 \cdot 4 \cdot 5 \cdots (3n-2)(3n-1)}{(n!)^2 \cdot 3^{2n}} \\
&= \frac{(3n)!}{(n!)^2 \cdot 3^{2n} \cdot 3 \cdot 6 \cdots 3n} \\
&= \frac{(3n)!}{(n!)^2 \cdot 3^{2n} \cdot 3^n \cdot n!} \\
&= \frac{1}{3^{3n}} \frac{(3n)!}{(2n)! \cdot n!} \frac{(2n)!}{(n!)^2} \\
&= \frac{1}{3^{3n}} \binom{3n}{2n} \binom{2n}{n}
\end{aligned}$$

# Problem 2

用超几何级数方法求 $\sum_k \binom{m}{k+n}\binom{k+n}{2k}4^k$.

# Solution

设 $t_k = \binom{m}{k+n}\binom{k+n}{2k}4^k$, 原式为 $S(m,n) = \sum_k t_k$. 当 $k < 0$ 时，$\binom{k+n}{2k} = 0$，所以求和项 $t_k = 0$, 所以下面只考虑 $k >= 0$ 的情况.
首先

$$\begin{aligned}
t_0 &= \binom{m}{n} \\
t_k &= \binom{m}{k+n}\binom{k+n}{2k}4^k = \frac{m!(k+n)!4^k}{(k+n)!(m-k-n)!(2k)!(n-k)!} = \frac{m!4^k}{(m-k-n)!(2k)!(n-k)!}
\end{aligned}$$

那么

$$\begin{aligned}
\frac{t_{k+1}}{t_k} &= \frac{4(m-k-n)!(2k)!(n-k)!}{(m-k-1-n)!(2k+2)!(n-k-1)!} \\
&= \frac{4(m-k-n)(n-k)}{(2k+1)(2k+2)} \\
&= \frac{(k+n-m)(k-n)(1)}{(k+\frac{1}{2})(k+1)}
\end{aligned}$$

则 $t_k$ 是超几何项, 原式可以用超几何级数表示为:

$$S(m,n) = \sum_k t_k = \sum_{k\geq 0} t_k = \binom{m}{n} F\left(\left.{n-m,\ -n \atop \frac{1}{2}}\right| 1\right) \tag{1}$$

根据书上公式 (5.93) 有:

$$F\left(\left.{a,\ -n \atop c}\right| 1\right) = \frac{(a-c)^{\underline{n}}}{(-c)^{\underline{n}}} \tag{2}$$

代入公式 (1) 可得:

$$\begin{aligned}
S(m,n) &= \binom{m}{n}\frac{(n-m-\frac{1}{2})^{\underline{n}}}{(-\frac{1}{2})^{\underline{n}}} \\
&= \binom{m}{n}\frac{(n-m-\frac{1}{2})(n-m-\frac{3}{2})\cdots(-m+\frac{1}{2})}{(-\frac{1}{2})(-\frac{3}{2})\cdots(-\frac{1}{2}-n+1)} \\
&= \binom{m}{n}\frac{(2m-1)(2m-3)\cdots(2m-2n+1)}{1\cdot 3\cdots(2n-1)} \\
&= \binom{m}{n}\frac{\frac{(2m)!}{(2m-2n)!\cdot 2^n\cdot\frac{m!}{(m-n)!}}}{\frac{(2n)!}{2^n\cdot n!}} \\
&= \binom{m}{n}\frac{(2m)!}{(2n!)(2m-2n)!}\cdot\frac{n!(m-n)!}{m!} \\
&= \binom{2m}{2n}
\end{aligned}$$

# Problem 3

用 Gosper 方法求

1. $\sum \frac{\delta k}{k^3-k}$
2. $\sum \binom{-3}{2k} 2^k \delta k$

# Solution

## 1.

设 $t(k) = \frac{1}{k^3-k}$,$\sum \frac{\delta k}{k^3-k} = T(k) + C$.
首先

$$\frac{t(k+1)}{t(k)} = \frac{(k+1)k(k-1)}{(k+2)(k+1)k} = \frac{k-1}{k+2}$$

所以

$$\begin{aligned} &p(k) = 1,\ q(k) = k-1,\ r(k) = k+1 \\ &Q(k) = -2,\ R(k) = 2 \\ &1 = (k-1)s(k+1) - (k+1)s(k) \end{aligned}$$

所以 $deg(Q) = deg(R)$, 则 $deg(s) = d = deg(p) - deg(Q) = 0$, 所以 $s(k) = \alpha_0$, 代入可得:

$$\begin{aligned} &1 = \alpha_0(k-1) - \alpha_0(k+1) \\ &\Rightarrow s(k) = \alpha_0 = -\frac{1}{2} \end{aligned}$$

可得:

$$T(k) = \frac{-\frac{1}{2}(k+1)\frac{1}{k^3-k}}{1} = -\frac{1}{2k(k-1)}$$

## 2.

首先 $t_k = \binom{-3}{2k}2^k$, 有:

$$\frac{t(k+1)}{t(k)} = \frac{\binom{-3}{2k+2}2^{k+1}}{\binom{-3}{2k}2^k} = \frac{2(k+2)(k+\frac{3}{2})}{(k+1)(k+\frac{1}{2})}$$

所以

$$\begin{aligned} &p(k) = (k+1)(k+\frac{1}{2}),\ q(k) = 2,\ r(k) = 1 \\ &Q(k) = 1,\ R(k) = 3 \\ &(k+1)(k+\frac{1}{2}) = 2s(k+1) - s(k) \end{aligned}$$

所以 $deg(Q) = deg(R)$, 则 $deg(s) = d = deg(p) - deg(Q) = 2$, 所以 $s(k) = \alpha_2 k^2 + \alpha_1 k + \alpha_0$, 代入可得:

$$(k+1)(k+\frac{1}{2}) = 2\alpha_2(k+1)^2 + 2\alpha_1(k+1) + 2\alpha_0 - \alpha_2 k^2 - \alpha_1 k - \alpha_0$$

$$\Rightarrow \begin{cases} \alpha_2 = 1 \\ \alpha_1 = -\frac{5}{2} \\ \alpha_0 = \frac{7}{2} \end{cases} \Rightarrow\ s(k) = k^2 - \frac{5}{2}k + \frac{7}{2} \ \Rightarrow\ T(k) = \frac{(k^2 - \frac{5}{2}k + \frac{7}{2})\binom{-3}{2k}2^k}{(k+1)(k+\frac{1}{2})} = (k^2 - \frac{5}{2}k + \frac{7}{2}) \cdot 2^{k+1}$$

# Problem 4

用 Gosper-Zeilberger 方法求 $S(n) = \sum_k \binom{n}{2k}$ 的递归式

# Solution

先假定 $l = 1$, 有:

$$\begin{aligned}
&\hat{t}(n,k) = \beta_0(n)t(n,k) + \beta_1(n)t(n+1,k) \\
&\frac{t(n+1,k)}{t(n,k)} = \frac{\binom{n+1}{2k}}{\binom{n}{2k}} = \frac{n+1}{n+1-2k} \\
&p(n,k) = \beta_0(n)(n+1-2k) + \beta_1(n)(n+1) \\
&\hat{t}(n,k) = p(n,k)\frac{t(n,k)}{n+1-2k} \\
&\bar{t}(n,k) = \frac{t(n,k)}{n+1-2k} = \frac{\binom{n}{2k}}{n+1-2k} \\
&\frac{\bar{t}(n,k+1)}{\bar{t}(n,k)} = \frac{(n-2k+1)(n-2k)}{(2k+2)(2k+1)}
\end{aligned}$$

可得:

$$\begin{cases} p(n,k) = 1 \\ q(n,k) = 4k^2 - (4n+2)k + (n^2+n) \\ r(n,k) = 4k^2 - 2k \end{cases} \Rightarrow \begin{cases} Q(n,k) = -4nk + (n^2+n) \\ R(n,k) = 8k^2 - (4n+4)k + (n^2+n) \end{cases}$$

所以:

$$\begin{cases} deg(Q) < deg(R) \\ \lambda' = -4n,\ \lambda = 8 \end{cases} \Rightarrow deg(s) = deg(p) - deg(R) + 1 = 0 \Rightarrow s = \alpha_0(n)$$

代入可得:

$$\begin{aligned}
&\beta_0(n)(n+1-2k) + \beta_1(n)(n+1) = (4k^2 - (4n+2)k + (n^2+n))\alpha_0(n) - (4k^2 - 2k)\alpha_0(n) \\
&-2\beta_0(n)k + (\beta_0(n) + \beta_1(n))(n+1) = -4n\alpha_0(n)k + (n^2+n)\alpha_0(n)
\end{aligned}$$

比较等式两边关于 k 的系数可得:

$$\begin{cases} -2\beta_0(n) = -4n\alpha_0(n) \\ \beta_0(n) + \beta_1(n) = n\alpha_0(n) \end{cases} \Rightarrow \begin{cases} \alpha_0(n) = 1 \\ \beta_0(n) = 2n \\ \beta_1(n) = -n \end{cases}$$

所以:

$$\begin{aligned}
&\hat{t}(n,k) = 2n \cdot t(n,k) - n \cdot t(n+1,k) \\
&\sum_k [2n \cdot t(n,k) - n \cdot t(n+1,k)] = \sum_k \hat{t}(n,k) = 0 \\
&2nS(n) - nS(n+1) = 0
\end{aligned}$$

也即

$$\begin{cases} S(0) = 1,\ S(1) = 1 \\ S(n+1) = 2S(n),\ (n \geq 1) \end{cases} \Rightarrow \begin{cases} S(0) = 1 \\ S(n) = 2^{n-1}\ (n \geq 1) \end{cases}$$