Concrete Math: Homework 6

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Problem 1

求概率生成函数是 $G(z) = \frac{z^3}{z^3-8(z-1)}$ 的离散随机变量的 3 阶累积量.

Solution

如果概率生成函数可以分解成两个概率生成函数的比值:

$$G(z) = \frac{H(z)}{F(z)}$$

则

$$\ln G(z) = \ln H(z) - \ln F(z)$$
$$\ln G(e^t) = \ln H(e^t) - \ln F(e^t)$$

所以此时 $G(z)$ 的各阶累积量为 $H(z)$ 和 $F(z)$ 的对应累积量的差.
本题中伪概率生成函数为:

$$H(z) = z^3,\ F(z) = z^3 - 8(z-1)$$

所以

$$H(e^t) = e^{3t} = 1 + \frac{3}{1!}t + \frac{9}{2!}t^2 + \frac{27}{3!}t^3 + \cdots$$
$$F(e^t) = e^{3t} - 8(e^t - 1) = 1 + \frac{3-8}{1!}t + \frac{9-8}{2!}t^2 + \frac{27-8}{3!}t^3 + \cdots$$

对于 $H(z)$ 而言:

$$\mu_1 = 3,\ \mu_2 = 9,\ \mu_3 = 27$$
$$\Rightarrow \begin{cases} \kappa_1 = \mu_1 = 3 \\ \kappa_2 = \mu_2 - \mu_1^2 = 0 \\ \kappa_3 = \mu_3 - 3\mu_1\mu_2 + 2\mu_1^3 = 0 \end{cases}$$

对于 $F(z)$ 而言:

$$\mu_1 = -5,\ \mu_2 = 1,\ \mu_3 = 19$$
$$\Rightarrow \begin{cases} \kappa_1 = \mu_1 = -5 \\ \kappa_2 = \mu_2 - \mu_1^2 = -24 \\ \kappa_3 = \mu_3 - 3\mu_1\mu_2 + 2\mu_1^3 = -216 \end{cases}$$

则 $G(z)$ 的 3 阶累积量为 $\kappa_3 = 0 - (-216) = 216$.

Problem 2

连续抛掷一枚硬币直到第一次出现 HTHTH 时停止 (H 表示正面, T 表示反面), 其中每次抛掷是独立的, 且每次正面出现的概率是 p. 求抛掷硬币次数的期望和方差.

Solution

设所有不包含 HTHTH 模式的序列集合是 N, 目标序集合是 S, 则有

$$\begin{aligned}1+N(H+T)&=N+S\\N(HTHTH)&=S(1+TH+THTH)\end{aligned}$$

当 $k=5,3,1$ 时, 有 $A_{(k)}=A^{(k)}$, 所以

$$\begin{aligned}EX&=\sum_{k=1}^{m}\widetilde{A}_{(k)}[A_{(k)}=A^{(k)}]=\widetilde{A}_{(1)}+\widetilde{A}_{(3)}+\widetilde{A}_{(5)}=\frac{1}{p}+\frac{1}{p^2(1-p)}+\frac{1}{p^3(1-p)^2}\\VX&=(EX)^2-\sum_{k=1}^{m}(2k-1)\widetilde{A}_{(k)}[A_{(k)}=A^{(k)}]=(EX)^2-\widetilde{A}_{(1)}-5\widetilde{A}_{(3)}-9\widetilde{A}_{(5)}\\&=\left(\frac{1}{p}+\frac{1}{p^2(1-p)}+\frac{1}{p^3(1-p)^2}\right)^2-\frac{1}{p}-\frac{5}{p^2(1-p)}-\frac{9}{p^3(1-p)^2}\end{aligned}$$

Problem 3

求 $(n+2+\frac{3}{n+1})^n$ 精确到相对误差 $O(n^{-2})$ 的渐近值.

Solution

$$
\begin{aligned}
&\left(n+2+\frac{3}{n+1}\right)^n = n^n\left(1+\frac{2}{n}+\frac{3}{n(n+1)}\right)^n \\
&= n^n\left(1+2n^{-1}+3n^{-2}\frac{1}{1+n^{-1}}\right)^n \\
&= n^n\left(1+2n^{-1}+3n^{-2}(1+O(n^{-1}))\right)^n \\
&= n^n\left(1+2n^{-1}+3n^{-2}+O(n^{-3})\right)^n \\
&= n^n\exp\left\{n\ln(1+2n^{-1}+3n^{-2}+O(n^{-3}))\right\} \\
&= n^n\exp\left\{n\left[2n^{-1}+3n^{-2}+O(n^{-3})-\frac{(2n^{-1}+3n^{-2}+O(n^{-3}))^2}{2}+O((2n^{-1}+3n^{-2}+O(n^{-3}))^3)\right]\right\} \\
&= n^n\exp\left\{n\left[2n^{-1}+3n^{-2}+O(n^{-3})-2n^{-2}\right]\right\} \\
&= n^n\exp\left\{n\left[2n^{-1}+n^{-2}+O(n^{-3})\right]\right\} \\
&= n^n\exp\left\{2+n^{-1}+O(n^{-2})\right\} \\
&= n^ne^2\left(1+n^{-1}+O(n^{-2})\right)\left(1+O(n^{-2})\right) \\
&= n^ne^2\left(1+n^{-1}+O(n^{-2})\right) \\
&= n^ne^2(1+n^{-1})\left(1+O(n^{-2})\frac{1}{1+n^{-1}}\right) \\
&= n^ne^2(1+n^{-1})\left(1+O(n^{-2})(1+O(n^{-1}))\right) \\
&= n^ne^2(1+n^{-1})\left(1+O(n^{-2})\right)
\end{aligned}
$$

Problem 4

求 $\sum_{k=1}^{n} \frac{1}{n^2+k^2}$ 精确到绝对误差 $O(n^{-5})$ 的渐近值.

Solution

设 $f(x) = \frac{1}{n^2+x^2}$, 则

$$f^{(1)}(x) = \frac{-2x}{(n^2+x^2)^2}$$
$$f^{(2)}(x) = \frac{6x^2-2n^2}{(n^2+x^2)^3}$$
$$f^{(3)}(x) = \frac{24x(n^2-x^2)}{(n^2+x^2)^4} \Rightarrow f^{(3)}(n)\Big|_0^n = 0$$
$$f^{(4)}(x) = \frac{24(5(2n^2-x^2)^2-19n^4)}{(n^2+x^2)^5} \Rightarrow \begin{cases} f^{(4)}(x) \geq 0, & 0 \leq x \leq \sqrt{2-\sqrt{\frac{19}{5}}}n \\ f^{(4)}(x) < 0, & \sqrt{2-\sqrt{\frac{19}{5}}}n < x \leq n \end{cases}$$

为方便计算, 设 $f^{(4)}(x)$ 的零点为 $cn = \sqrt{2-\sqrt{\frac{19}{5}}}n$, 其中 c 为常数系数. 取欧拉求和公式中的 $m = 2$

$$\begin{aligned}
&\sum_{k=1}^{n} \frac{1}{n^2+k^2} = \sum_{0\leq k<n}^{n} \frac{1}{n^2+k^2} + \frac{1}{2n^2} - \frac{1}{n^2} \\
&= \sum_{0\leq k<n}^{n} \frac{1}{n^2+k^2} - \frac{1}{2n^2} \\
&= -\frac{1}{2n^2} + \int_0^n \frac{1}{n^2+x^2}dx - \frac{1}{2}\frac{1}{n^2+x^2}\Big|_0^n + \frac{B_2}{2}f'(x)\Big|_0^n + \frac{B_4}{24}f^{(3)}(x)\Big|_0^n + O((2\pi)^{-2})\int_0^n |f^{(4)}(x)|dx \\
&= -\frac{1}{4n^2} + \frac{\pi}{4n} + \frac{1}{12}\frac{-2x}{(n^2+x^2)^2}\Big|_0^n + 0 + O((2\pi)^{-2})\left(\int_0^{\sqrt{2-\sqrt{\frac{19}{5}}}n} f^{(4)}(x)dx + \int_{\sqrt{2-\sqrt{\frac{19}{5}}}n}^{n} -f^{(4)}(x)dx\right) \\
&= -\frac{1}{4n^2} + \frac{\pi}{4n} - \frac{1}{24n^3} + O((2\pi)^{-2})\left(f^{(3)}(x)\Big|_0^{\sqrt{2-\sqrt{\frac{19}{5}}}n} - f^{(3)}(x)\Big|_{\sqrt{2-\sqrt{\frac{19}{5}}}n}^{n}\right) \\
&= -\frac{1}{4n^2} + \frac{\pi}{4n} - \frac{1}{24n^3} + 2O((2\pi)^{-2})f^{(3)}(\sqrt{2-\sqrt{\frac{19}{5}}}n) \\
&= -\frac{1}{4n^2} + \frac{\pi}{4n} - \frac{1}{24n^3} + O(\frac{24cn(n^2-c^2n^2)}{(n^2+c^2n^2)^4}) \\
&= -\frac{1}{4n^2} + \frac{\pi}{4n} - \frac{1}{24n^3} + O(\frac{24c(1-c^2)n^3}{(1+c^2)^4n^8}) \\
&= -\frac{1}{4n^2} + \frac{\pi}{4n} - \frac{1}{24n^3} + O(\frac{24c(1-c^2)}{(1+c^2)^4}n^{-5}) \\
&= -\frac{1}{4n^2} + \frac{\pi}{4n} - \frac{1}{24n^3} + O(n^{-5})
\end{aligned}$$

Problem 5

求 $\sum_k \binom{2n}{k}^3$ 精确到相对误差 $O(n^{-1/4})$ 的渐近值.

Solution 用 $n+k$ 代替 k

$$A_n = \sum_k \binom{2n}{n+k}^3 = \sum_k \left(\frac{(2n!)}{(n+k)!(n-k)!}\right)^3$$

使用尾部交换技巧

$$\left(\frac{(2n!)}{(n+k)!(n-k)!}\right)^3 = a_k(n) = b_k(n) + O(c_k(n)), \quad k \in D_n$$
$$A_n = \sum_k b_k(n) + O\left(\sum_{k\notin D_n} a_k(n)\right) + O\left(\sum_{k\notin D_n} b_k(n)\right) + \sum_{k\in D_n} O\left(c_k(n)\right)$$

对 $a_k(n)$ 使用对数形式的斯特林近似

$$\begin{aligned}
\ln a_k(n) &= 3\left(\ln(2n)! - \ln(n+k)! - \ln(n-k)!\right) \\
&= 3\left(\ln(2n)! - \ln(n+k)! - \ln(n-k)!\right) \\
&= 3\{2n\ln 2n - 2n + \frac{1}{2}\ln 2n + \sigma + O\left(n^{-1}\right) \\
&- (n+k)\ln(n+k) + n + k - \frac{1}{2}\ln(n+k) - \sigma + O\left((n+k)^{-1}\right) \\
&- (n-k)\ln(n-k) + n - k - \frac{1}{2}\ln(n-k) - \sigma + O\left((n-k)^{-1}\right)\}
\end{aligned}$$

定义 D_n

$$k \in D_n \Leftrightarrow |k| \le n^{1/2+\varepsilon}$$

在 D_n 内将 $a_K(n)$ 进一步化简

$$\begin{aligned}
\frac{1}{3}\ln a_k(n) = &\left(2n + \frac{1}{2}\right)\ln 2 - \sigma - \frac{1}{2}\ln n + O\left(n^{-1}\right) \\
&- \left(n + k + \frac{1}{2}\right)\ln(1 + k/n) - \left(n - k + \frac{1}{2}\right)\ln(1 - k/n)
\end{aligned}$$

在 D_n 内将 $\ln(1 \pm k/n)$ 展开

$$\begin{aligned}
\ln(1 \pm \frac{k}{n}) &= \pm\frac{k}{n} - \frac{k^2}{2n^2} + O(\frac{k^3}{n^3}) \\
&= \pm\frac{k}{n} - \frac{k^2}{2n^2} + O(\frac{n^{3/2+3\varepsilon}}{n^3}) \\
&= \pm\frac{k}{n} - \frac{k^2}{2n^2} + O(n^{-3/2+3\varepsilon})
\end{aligned}$$

将其与 $n \pm k + \frac{1}{2}$ 相乘得到

$$(n \pm k + \frac{1}{2})\ln(1 \pm \frac{k}{n}) = \pm k + \frac{k^2}{2n} + O(n^{-1/2+3\varepsilon})$$

Problem 5 continued on next page...

代入该结果进一步得到

$$\frac{1}{3}\ln a_k(n) = \left(2n+\frac{1}{2}\right)\ln 2 - \sigma - \frac{1}{2}\ln n - \frac{k^2}{n} + O(n^{-1/2+3\varepsilon})$$
$$a_k(n) = exp\left\{3\left(\left(2n+\frac{1}{2}\right)\ln 2 - \sigma - \frac{1}{2}\ln n - \frac{k^2}{n} + O(n^{-1/2+3\varepsilon})\right)\right\}$$
$$a_k(n) = \frac{2^{6n+3/2}}{e^{3\sigma}n^{3/2}}e^{-3k^2/n}\left(1+O(n^{-1/2+3\varepsilon})\right)$$

所以

$$b_k(n) = \frac{2^{6n+3/2}}{e^{3\sigma}n^{3/2}}e^{-3k^2/n}, \quad c_k(n) = 2^{6n}n^{-2+3\varepsilon}e^{-3k^2/n}$$

近似求和 $b_k(n)$, 并代入 $e^\sigma = \sqrt{2\pi}$

$$\begin{aligned}
\sum_k b_k(n) &= \sum_k \frac{2^{6n+3/2}}{e^{3\sigma}n^{3/2}}e^{-3k^2/n} = \frac{2^{6n+3/2}}{e^{3\sigma}n^{3/2}}\sum_k e^{-3k^2/n} \\
&= \frac{2^{6n+3/2}}{e^{3\sigma}n^{3/2}}\sum_k e^{-3k^2/n} = \frac{2^{6n+3/2}}{e^{3\sigma}n^{3/2}}\Theta_{\frac{n}{3}} \\
&= \frac{2^{6n+3/2}\sqrt{\pi}}{\sqrt{3}e^{3\sigma}n}(1+O(n^{-M})) \\
&= \frac{2^{6n}}{\sqrt{3}\pi n}(1+O(n^{-M}))
\end{aligned}$$

计算误差项

$$\sum_{(c)}(n) = \sum_{|k|\le n^{1/2+\varepsilon}} 2^{6n}n^{-2+3\varepsilon}e^{-3k^2/n} \le 2^{6n}n^{-2+3\varepsilon}\Theta_{\frac{n}{3}} = O(2^{6n}n^{-3/2+3\varepsilon})$$

其他的尾部经验证是可以忽略不计的, 所以

$$\sum_k \binom{2n}{k}^3 = \frac{2^{6n}}{\sqrt{3}\pi n} + O(2^{6n}n^{-3/2+3\varepsilon}) = \frac{2^{6n}}{\sqrt{3}\pi n}\left(1+O(n^{-1/2+3\varepsilon})\right)$$

可以取 $\varepsilon = \frac{1}{12}$ 得到

$$\sum_k \binom{2n}{k}^3 = \frac{2^{6n}}{\sqrt{3}\pi n}\left(1+O(n^{-1/4})\right)$$