Concrete Math: Homework 4

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Problem 1

Show that $2\sqrt{n+1} - 2 \leq \sum_{k=1}^{n} \frac{1}{\sqrt{k}} \leq 2\sqrt{n} - 1$.

Solution

用积分放缩法证明上述不等式, 采用的积分函数是 $f(x) = \frac{1}{\sqrt{x}}$:

$$\int_{1}^{n+1} \frac{1}{\sqrt{x}} dx \leq \sum_{k=1}^{n} \frac{1}{\sqrt{k}} \leq 1 + \int_{1}^{n} \frac{1}{\sqrt{x}} dx$$

$$2\sqrt{x} \Big|_{1}^{n+1} \leq \sum_{k=1}^{n} \frac{1}{\sqrt{k}} \leq 1 + 2\sqrt{x} \Big|_{1}^{n}$$

$$2\sqrt{n+1} - 2 \leq \sum_{k=1}^{n} \frac{1}{\sqrt{k}} \leq 2\sqrt{n} - 1$$

Problem 2

Using the idea of generating function, solve the recurrences:

1. $f_0 = 1, f_1 = 2, f_n = 2f_{n-1} - f_{n-2} + (-2)^n \ for \ n \geq 2$.

2. $g_0 = 0, h_0 = 1, g_1 = h_1 = 2, g_n = 2h_{n-1} - g_{n-2}, h_n = g_{n-1} - h_{n-2} \ for \ n \geq 2$.

Solution

1.
注意到这里 $f_1 = 2$, 不满足初始递推关系, 修正如下:

$$f_n = 2f_{n-1} - f_{n-2} + (-2)^n [n \geq 0] + 2[n = 1]$$

带入生成函数可得:

$$
\begin{aligned}
F(z) = \sum_n f_n z^n &= \sum_n \left(2f_{n-1} - f_{n-2} + (-2)^n[n \geq 0] + 2[n = 1]\right) z^n \\
&= \sum_n 2f_{n-1} z^n - \sum_n f_{n-2} z^n + \sum_n (-2)^n [n \geq 0] z^n + \sum_n 2[n = 1] z^n \\
&= 2zF(z) - z^2 F(z) + \frac{1}{1 + 2z} + 2z
\end{aligned}
$$

解得:

$$
\begin{aligned}
F(z) &= \frac{1}{(1 + 2z)(z - 1)^2} + \frac{2z}{(z - 1)^2} \\
&= \frac{4}{9} \cdot \frac{1}{1 + 2z} - \frac{16}{9} \cdot \frac{1}{1 - z} + \frac{7}{3} \cdot \frac{1}{(1 - z)^2}
\end{aligned}
$$

所以:

$$f_n = [z^n]F(z) = \frac{4}{9}(-2)^n - \frac{16}{9} + \frac{7}{3}(n + 1), \ for \ n \geq 0.$$

2.
注意到这里 $h_0 = 1, h_1 = 2$, 不满足初始递推关系, 对所有的 n 修正如下:

$$
\begin{cases}
h_n = g_{n-1} - h_{n-2} + [n = 0] + 2[n = 1] \\
g_n = 2h_{n-1} - g_{n-2}
\end{cases}
$$

从而

$$
\begin{cases}
H(z) = \sum_n h_n z^n = zG(z) - z^2 H(z) + 1 + 2z \\
G(z) = \sum_n g_n z^n = 2zH(z) - z^2 G(z)
\end{cases}
$$

解得:

$$
\begin{cases}
H(z) = \dfrac{(1 + z^2)(1 + 2z)}{z^4 + 1} = (2z^3 + z^2 + 2z + 1)\dfrac{1}{z^4 + 1} = (2z^3 + z^2 + 2z + 1)\sum_{n \geq 0}(-1)^n z^{4n} \\
G(z) = \dfrac{2z(1 + 2z)}{z^4 + 1} = (4z^2 + 2z)\dfrac{1}{z^4 + 1} = (4z^2 + 2z)\sum_{n \geq 0}(-1)^n z^{4n}
\end{cases}
$$

       3

所以:

$$
h_n = [z^n]H(z) = \begin{cases} (-1)^{\frac{n}{4}}, \ n\%4 = 0 \\ 2(-1)^{\frac{n-1}{4}}, \ n\%4 = 1 \\ (-1)^{\frac{n-2}{4}}, \ n\%4 = 2 \\ 2(-1)^{\frac{n-3}{4}}, \ n\%4 = 3 \end{cases} = \begin{cases} (-1)^{\lfloor \frac{n}{4} \rfloor}, \ n\%4 = 0 \text{ or } n\%4 = 2 \\ 2 \cdot (-1)^{\lfloor \frac{n}{4} \rfloor}, \ n\%4 = 1 \text{ or } n\%4 = 3 \end{cases}
$$

$$
g_n = [z^n]G(z) = \begin{cases} 0, \ n\%4 = 0 \text{ or } n\%4 = 3 \\ 2(-1)^{\frac{n-1}{4}}, \ n\%4 = 1 \\ 4(-1)^{\frac{n-2}{4}}, \ n\%4 = 2 \end{cases} = \begin{cases} 0, \ n\%4 = 0 \text{ or } n\%4 = 3 \\ 2 \cdot (-1)^{\lfloor \frac{n}{4} \rfloor}, \ n\%4 = 1 \\ 4 \cdot (-1)^{\lfloor \frac{n}{4} \rfloor}, \ n\%4 = 2 \end{cases}
$$

Problem 3

A random variable X is said to have the Poisson distribution with mean λ if $\Pr(X = k) = \frac{e^{-\lambda}\lambda^k}{k!}$ for all $k \in \mathbb{N}$. Let X_1 and X_2 be independent random Poisson variables both with variance t. Calculate the distribution of $X_1 + X_2$.

Solution
先证明服从泊松分布的均值为 λ 的随机变量 X 的方差也是 λ, 设 $X \sim Pois(\lambda)$:

$$VarX = EX^2 - (EX)^2 = EX^2 - \lambda^2$$
$$= \sum_{k \geq 0} \frac{e^{-\lambda}\lambda^k k^2}{k!} - \lambda^2$$
$$= e^{-\lambda}\lambda \sum_{k \geq 1} \frac{\lambda^{k-1} k}{(k-1)!} - \lambda^2$$
$$= e^{-\lambda}\lambda \sum_{k \geq 0} \frac{\lambda^k(k+1)}{k!} - \lambda^2$$
$$= e^{-\lambda}\lambda \sum_{k \geq 0} \frac{\frac{\mathrm{d}\lambda^{k+1}}{\mathrm{d}\lambda}}{k!} - \lambda^2$$
$$= e^{-\lambda}\lambda \frac{\mathrm{d}(\sum_{k \geq 0} \frac{\lambda^{k+1}}{k!})}{\mathrm{d}\lambda} - \lambda^2$$
$$= e^{-\lambda}\lambda \frac{\mathrm{d}(\lambda \sum_{k \geq 0} \frac{\lambda^k}{k!})}{\mathrm{d}\lambda} - \lambda^2$$
$$= e^{-\lambda}\lambda \frac{\mathrm{d}(\lambda e^{\lambda})}{\mathrm{d}\lambda} - \lambda^2$$
$$= e^{-\lambda}\lambda(e^{\lambda} + \lambda e^{\lambda}) - \lambda^2$$
$$= \lambda$$

所以 X_1 和 X_2 独立同分布于参数为 t 的泊松分布, 所以 $X_1 + X_2$ 也服从泊松分布, 即 $X_1 \sim Pois(t)$ 和

$X_2 \sim Pois(t)$, 设 $X = X_1 + X_2$, 则 X 的概率分布为:

$$
\begin{aligned}
\Pr(X = k) &= \Pr(X_1 + X_2 = k) \\
&= \sum_i \Pr(X_1 = i, X_2 = k - i) \\
&= \sum_i \left[\Pr(X_1 = i) \cdot \Pr(X_2 = k - i) \right] \\
&= \sum_{i=0}^{k} \left[\frac{e^{-t} t^i}{i!} \cdot \frac{e^{-t} t^{k-i}}{(k-i)!} \right] \\
&= \frac{e^{-2t} t^k}{k!} \sum_{i=0}^{k} \frac{k!}{i!(k-i)!} \\
&= \frac{e^{-2t} t^k}{k!} \sum_{i=0}^{k} \binom{k}{i} \\
&= \frac{e^{-2t} t^k}{k!} \cdot 2^k \\
&= \frac{e^{-2t} (2t)^k}{k!}
\end{aligned}
$$

即 $X_1 + X_2 \sim Pois(2t)$.
事实上利用指数生成函数的二项卷积公式可以得到: 若 X_1, X_2 独立, 且 $X_1 \sim Pois(t_1)$, $X_2 \sim Pois(t_2)$,
则 $X_1 + X_2 \sim Pois(t_1 + t_2)$.

Problem 4

Solution