4D to 3D reduction of Seiberg duality for $SU(N)$ susy gauge theories with adjoint matter: a partition function approach

Carlo Sana

1 Four dimensional dualities

——— INTRODUCTION OUTLINE ———
~ More symmetry = more tools for studying t\heories
~ Milder divergences
~ Renormalization constraints
~ Non renormalization theorems (perturbative)
~ Holomorphicity, couplings as background fields
~ Exact results (superpotential, exact beta function)
~ Moduli space

1.1 Introduction

Supersymmetric quantum field theories enjoy an enlarged group of symmetries compared to other field theories. Since the symmetry group is a non trivial combination of internal and spacetime symmetries, they have many unexpected features and new techniques were found to study them. Almost all of the new tools found are available only for supersymmetric field theories, making them the theatre for many advances in physics.

A more technical introduction on supersymmetry and its representation on fields can be found in appendix A.

In this section we will analyse more advanced features of supersymmetric field theories that has been used intensively in the discovery and in the analysis of electric magnetic duality and its generalisations.

1.1.1 General renormalization properties

A remarkable feature of supersymmetry is the constraint that the additional symmetry imposes on the renormalization properties of the theories.

One of the first aspects that brought attention to supersymmetry was that divergences of loop diagrams were milder because of the cancellation between diagrams with bosons and fermions running in the loops.

Nowadays we know powerful theorems that restrict the behaviour of supersymmetric field theories during renormalization. In order to preserve supersymmetry, the renormalization process has to preserve the Hilbert space structure. For example the wave function renormalization of different *particles* inside a multiplet must be the same, otherwise the renormalized lagrangian is not supersymmetric invariant anymore.

Moreover, in the supersymmetry algebra P^2 is still a Casimir operator i.e. it commutes with every operator in the algebra: particles in the same multiplet must have the same mass. Renormalization cannot break this condition, otherwise it would break supersymmetry.

For a *Super Yang Mills* theory with $\mathcal{N} = 1$ we have the additional requirement that gV, where g is the coupling and V is the vector superfield, cannot be renormalized by symmetry considerations.

Adding more supersymmetry the wave function renormalization of the various field are even more constrained by symmetry. For example, for $\mathcal{N} = 4$ *SYM* the fields and the coupling are not renormalized at all.

Beta function for SYM and SQCD

Another nice feature of supersymmetric field theories is that some quantities can be calculated exactly. The first object of this kind that we encounter is the β function of four dimensional $\mathcal{N} = 1$ *Super Yang Mills* theories with matter fields in representations R_i.

It is given by the *NSVZ β function*

$$\beta(g) = \mu \frac{\mathrm{d}\, g}{\mathrm{d}\, \mu} = -\frac{g^3}{16\pi^2}\left[3\, T(Adj) - \sum_i T(R_i)(1-\gamma_i)\right]\left(1 - \frac{g^2\, T(Adj)}{8\pi^2}\right)^{-1} \qquad \alpha = \frac{g^2}{4\pi} \tag{1.1}$$

where γ_i are the anomalous dimensions of the matter fields and $T(R_i)$ are the Dynkin indices of their representation.
The anomalous dimensions are defined as

$$\gamma_i = -\mu \frac{\mathrm{d}\, \log(Z_i)}{\mathrm{d}\, \mu} \tag{1.2}$$

where Z_i is the wave function renormalization coefficient. The Dynkin indices [1] of the gauge group $SU(N)$ for the fundamental and adjoint representation are

$$T(\mathrm{N}) = \frac{1}{2} \qquad T(\mathrm{Adj}) = N \tag{1.3}$$

[1] The Dynkin index $T(R)$ of a representation R is defined as $\mathrm{Tr}\,(T^a T^b) = T(R)\delta^{ab}$ where T^a , T^b are the generators of the algebra in the representation R.

The *NSVZ β function* was first calculated using instanton methods in [1]. Over the years it has been calculated in other ways using the fact that the action is holomorphic in the complexified coupling

$$\tau = \frac{4\pi i}{g_c^2} + \frac{\theta_{YM}}{2\pi} \tag{1.4}$$

Using the holomorphic coupling the action for the vector field is written as

$$\mathcal{L}_h(V_h) = \frac{1}{16\pi i} \int d^2\theta \; \tau \; W^a(V_h) W^a(V_h) + h.c. \tag{1.5}$$

whereas with the canonical normalization for the vector field is

$$\mathcal{L}_c(V_c) = \frac{1}{16\pi i} \int d^2\theta \left(\frac{4\pi i}{g_c^2} + \frac{\theta_{YM}}{2\pi} \right) W^a(g_c V_c) W^a(g_c V_c) + h.c. \tag{1.6}$$

Using the canonical normalization $g_c V_c$ is a real superfield, imposing that g_c is real. For this reason with the canonical normalization the lagrangian is not holomorphic in τ. Thanks to holomorphicity, the holomorphic coupling is only renormalized at one-loop and the β function can be computed exactly at one loop but its expression is different from *NSVZ β function*. The cause of this mismatch is that the *NSVZ β function* is defined using the canonical (or physical) coupling constant and receives contribution from all orders in perturbation theory.

At first sight, one should expect that the expressions should match since the first two orders in α of the β function are scheme independent. The reason why the two expressions differ is that the Jacobian of the transformation between canonical and holomorphic normalization is anomalous. Once the anomaly is taken into account the two expressions for the β function agree.
An explicit calculation relating the comparison between the two different approach can be found in [2].

1.1.2 Superpotential: holomorphy and non-renormalization

Other than renormalization constraints, supersymmetry provides non-renormalization theorems for certain objects, such as the superpotential.

In [3] it has been demonstrated that the superpotential is tree-level exact, i.e. it does not receive correction in perturbation theory. However it usually receive contributions from non perturbative dynamics. The superpotential features this property in theories with at least four supercharges and can be demonstrated independently using perturbative calculations or its holomorphic properties.

It was first demonstrated perturbatively, using the fact that for general supersymmetric field theories, supergraph loops diagrams with n external leg yield a term that can be written in the form

$$\int d^4x_1 \dots d^4x_n d^2\theta d^2\bar{\theta}\; G(x_1,\dots,x_n)\, F_1(x_1,\theta,\bar{\theta})\dots F_n(x_n,\theta,\bar{\theta}) \tag{1.7}$$

where $G(x_1,\dots,x_n)$ is translationally invariant function.
The importance of this result is that all contribution from Feynman diagrams are given by a single integral over full superspace ($d^2\theta d^2\bar{\theta}$) whereas the superpotential must be written as an integral in half-superspace($d^2\theta$ only) of chiral fields. Exploiting the fact that a product of chiral fields is a chiral field, the most general form of a superpotential is

$$W(\lambda,\Phi) = \sum_{n=1}^{\infty}\left(\int d^2\theta\,\lambda_n\Phi^n \;+\; \int d^2\bar{\theta}\,\lambda_n^{\dagger}\bar{\Phi}^n\right) \tag{1.8}$$

The second term of the superpotential is added in order to give a real lagrangian after the integration in superspace. From the definition, we can see that the superpotential is holomorphic in the fields and in the coupling constants.

Fifteen years later, Seiberg [4] provided a proof of this theorem using a different approach. He noted that the coupling constants λ_n can be treated as background fields, i.e. chiral superfields with no dynamics.

Using this observation we can assign transformation laws to the coupling constants, making the lagrangian invariant under a larger symmetry. Fields and coupling constants are charged under this symmetry and only certain combinations of them can appear in the superpotential. In addition, in a suitable weak coupling limit the effective superpotential must be identical with the tree-level one. These conditions, taken together, fix the expansion of the superpotential to the expression of the tree-level potential. A more detailed discussion can be found in [5] and [6].

1.1.3 Moduli space

Supersymmetric field theories have a larger set of vacua compared to ordinary field theories because of the presence of many scalar fields in the supermultiplets.

Lorentz invariance of the vacuum forbids fields with spin different from zero to acquire a vacuum expectation value. With the same reasoning, derivatives of scalar fields must be set to zero because of translational invariance of the vacuum. The scalar potential is the only term in the Lagrangian and in the Hamiltonian that can differ from zero. As a result, the minimums of the scalar potential are in one-to-one correspondence with the states of minimal energy of the theory.

For $4D$ $\mathcal{N}=1$ gauge theories with matter, the scalar potential for the squarks reads

$$V(\phi_i,\bar{\phi}_j) = F\bar{F} + \frac{1}{2}D^2 \overset{on-shell}{=} \frac{\partial W}{\partial \phi_i}F^i\frac{\partial \bar{W}}{\partial \bar{\phi}_i}\bar{F}^i + \frac{g^2}{2}\sum_a |\bar{\phi}_j (T^a)^j_i \phi^j + \xi^a|^2 \geq 0 \tag{1.9}$$

ξ^a is the Fayet-Iliopulos coefficient and differs from zero only for abelian factors of the gauge group. The last equality is valid since D and F are auxiliary fields with no dynamics. Their value is set by their equations of motion

$$\bar{F}_i = \frac{\partial \bar{W}}{\partial \bar{\phi}_i} \qquad D^a = -g\bar{\phi}T^a\phi - g\xi^a \tag{1.10}$$

Supersymmetric vacua are described by the sets of values of the scalar *VEVs* that give a zero scalar potential. This requirement is equivalent to two different sets of equations, called $D - term$ and $F - term$ equations

$$\bar{F}^i(\phi) = 0 \qquad D^a(\phi, \bar{\phi}) = 0 \tag{1.11}$$

F-term equations are present only if there is a superpotential while the *D-term* equations are always present.
If the minimum of the scalar potential is different from zero the vacuum is not supersymmetric. In this case supersymmetry is spontaneously broken. Another possible situation is that the scalar potential has no minimum at all: the theory does not have any stable vacua.

The *classical moduli space* is the set of solution of these equations for scalar *VEVs* and represents the classical supersymmetric vacua of the theory. Gauge transformation should be taken into account in order to avoid redundance in the description. The moduli space describe physically inequivalent vacua, since the mass spectrum of the theory depends on the *VEVs* of the scalar fields, that differ in every point of the moduli space.

Because of supersymmetry radiative corrections do not lift the energy of the ground state and the vacuum remains supersymmetric. As a result, only superpotentials generated from non perturbative dynamics can lift the moduli space. We will see examples of this phenomenon in the analysis of SQCD.

An alternative description of the space of classical *D-flat* directions is given by the space of all holomorphic gauge invariant polynomials of scalar fields modulo classical relations between them [7]. As a result, gauge invariant polynomials of operators parametrize the classical moduli space of the theory. Using this description it's easier to find the moduli space of the theory in consideration. If a superpotential is present, *F-term* equations should be imposed on the gauge invariant polynomial used to describe *D-flat* direction. We will use this convenient description in the next chapters.

1.1.4 Phases of gauge theories

The dynamics of gauge theories can be classified according to the low-energy effective potential $V(R)$ between two test charges separated by a large distance R.

The possible forms of the potential, up to addictive constant, are

$$\begin{aligned} &\text{Coulomb} && V(R) \sim \frac{1}{R} && (1.12)\\ &\text{free electric} && V(R) \sim \frac{1}{R\,\log(R\Lambda)} && (1.13)\\ &\text{free magnetic} && V(R) \sim \frac{\log(R\Lambda)}{R} && (1.14)\\ &\text{Higgs} && V(R) \sim \text{constant} && (1.15)\\ &\text{confining} && V(R) \sim \sigma\, R && (1.16) \end{aligned}$$

The first three phases feature massless gauge fields and their potential is $V(R) \sim g^2(R)/R$ and they differ because of the renormalization of the charge in the IR. In the Coulomb phase, $g_{IR}^2 = \text{constant}$, while in the free abelian/non-Abelian phase the coupling constant goes to zero as $g^2(R) \sim 1/\log(R\,\Lambda)$. The free electric phases is possible for abelian or non-Abelian theories.In the latter case for asymptotically free theories it's necessary that the renormalization group has a non trivial infrared fixed point. The free magnetic phases is generated by magnetic monopoles acting as source of the field. Since magnetic and electric charges are related by Dirac quantization condition, the running of the coupling constant for magnetic monopoles is the inverse of electric charges.
The situation is completely different in the last two cases. In the Higgs phase gauge fields are massive and the potential is given by a Yukawa potential, exponential suppressed at long distances that results in a constant value. The confining phase can be described by tube of confined gauge flux between the charges which, at large distances, acts as a string with costant tension, yielding a linear potential.

1.1.5 't Hooft anomaly matching conditions

The 't Hooft anomaly matching conditions are a great tool to investigate the global symmetries of the low-energy degrees of freedom of the theory.
Let's consider an asymptotically free gauge theory with global symmetry group G. Gauge symmetries can't be anomalous because that would spoil the unitarity of the theory but there's nothing wrong with the global symmetries in being anomalous.
We can compute the triangle anomaly for the global symmetry group in the ultraviolet and we will call it A_{UV}. After weakly gauging G we introduce additional fermions that are charged only under G in order to cancel the anomaly, since now it is a gauge symmetry.
Flowing towards the infrared, the anomaly is still zero if the global symmetry group is not broken. After constructing the low-energy effective field theory, we

can calculate the triangle anomalies for the group G involving the composite low energy fields which results in the term A_{IR}. Since the fermions we added contribute to the anomaly with the same term we have that

$$0 = A_{IR} + A_F = A_{UV} + A_F \quad \rightarrow \quad A_{IR} = A_{UV} \tag{1.17}$$

The anomaly coefficient can be easily computed since it is proportional to the group theoretic factor

$$A = \mathrm{Tr}\ (T^a\{T^b, T^c\}) \tag{1.18}$$

Summarizing the result, we found that if the global symmetry group is not broken by the strong dynamics, triangle anomalies involving only the global symmetry group should be equal in the ultraviolet and in the infrared.
Moreover, we will use these anomaly matching conditions to find if two dual theories are invariant under the same global symmetries in the IR as an additional check of electric magnetic duality.

1.2 Seiberg duality

Electric magnetic duality relates the dynamics of two different gauge theories in their infrared fixed point. Even though the dual theories have different particle content, they describe the same IR physics. Moreover, whenever one of the dual theories gets more strongly coupled, the other become more weakly coupled. This is particularly useful since it provides an alternative, weakly coupled description of the original theory.

1.2.1 Electric theory: $SU(N)$ SQCD with N_f flavours

We will start our analysis on eletric-magnetic duality studying the first pair of theories that were discovered to be dual in [8]. We are gonna analyse the properties of these theory in order to better understand the features of the duality.

The electric theory is a $SU(N_c)$ supersymmetric gauge theory with N_f flavours. Its non anomalous global symmetry group is

$$SU(N_f)_L \times SU(N_f)_R \times U(1)_B \times U(1)_R$$

. Since the axial symmetry $U(1)_A$ is anomalous it cannot be part of the global symmetries.

The classical lagrangian is written in terms of superfield as

$$\mathcal{L} = \tau \int d^2\theta\ Tr(W_\alpha W^\alpha) + \text{h.c.} + \int d^2\theta\, d^2\bar{\theta}\ Q^\dagger e^V Q + \int d^2\theta\, d^2\bar{\theta}\ \tilde{Q}^\dagger e^{-V} \tilde{Q} \tag{1.19}$$

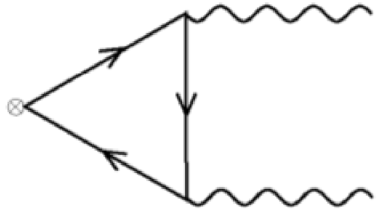


Figure 1.1: Feynman graphs contributing to the R-symmetry anomaly

Q and $\tilde{Q}$ represent left and right quark superfield respectively.

	$SU(N_c)$	$SU(N_f)_L$	$SU(N_f)_R$	$U(1)_B$	$U(1)_R$
Q	N_c	N_F	1	1	$\frac{N_f-N_c}{N_f}$
$\tilde{Q}$	$\overline{N_c}$	1	$\overline{N_F}$	-1	$\frac{N_f-N_c}{N_f}$

(1.20)

The charges of the fields are given in the table below. The value of the R-charge is fixed by the triangle anomaly $SU(N_c)^2\,U(1)_R$, given by diagrams with two exiting gluons and R-symmetry current inserted in the cross. Every fermion in the theory contributes to the anomaly which, as a result, is proportional to the R-charge of the fermion running in the loop and the Dynkin index of its representation

$$R_{gaugino}T(\mathrm{Ad})+\sum_f(R_f-1)T(r)=0 \tag{1.21}$$

$$N_c+\frac{1}{2}2N_f(R_Q-1)=0 \quad\rightarrow\quad R_Q=\frac{N_f-N_c}{N_f} \tag{1.22}$$

where we set the gaugino R-charge to 1 in order to have gluons not charged under R-symmetry.

Since the non anomalous R-symmetry condition leads to a unique set of R-charges, we found the R-charges at the superconformal infrared point of the theory. This is peculiar to *SQCD* with matter in the fundamental representation.

Classical moduli space

Since there is no superpotential, the classical moduli space of the theory is given by *D-terms* only. They can be read from the on-shell lagrangian and are given by

$$D^a=g\left(Q^{*i}T^aQ_i-\tilde{Q}^{*i}T^a\tilde{Q}_i\right)=0 \tag{1.23}$$

where T^a are the gauge group generators in fundamental or antifundamental representation and i is a flavour index.

After considering gauge and global symmetries, the squark *VEVs*, represented as $N_f \times N_c$ matrices, that satisfy the D-term equation are for $N_f \le N_c$ and a_i generic

$$Q = \tilde{Q} = \begin{pmatrix} a_1 & & & & \vdots \\ & a_2 & & & \vdots \\ & & \ddots & & \vdots \\ & & & a_{N_f} & \vdots \end{pmatrix} \tag{1.24}$$

for $N_f \ge N_c$ and $|a_i|^2 - |\tilde{a}_i|^2 = a$ independent of i.

$$Q = \begin{pmatrix} a_1 & & & \\ & a_2 & & \\ & & \ddots & \\ & & & a_{N_c} \\ \dots & \dots & \dots & \dots \end{pmatrix} \quad \tilde{Q} = \begin{pmatrix} \tilde{a}_1 & & & \\ & \tilde{a}_2 & & \\ & & \ddots & \\ & & & \tilde{a}_{N_c} \\ \dots & \dots & \dots & \dots \end{pmatrix} \tag{1.25}$$

For $N_f \le N_c$ in a generic point of the moduli space the gauge group is broken to $SU(N_c - N_f)$ while for $N_f \ge N_c$ is broken completely. The gauge group breaks through the super Higgs mechanism, where every broken generator gets absorbed by the (originally) massless vector superfield in order to make a massive vector superfield [2] The mass of the gauge superfield is given by the VEVs of the squarks.

As we said in the previous section, we can study the classical moduli space by finding holomorphic gauge invariant polynomial in the operators and modding out classical relations between them. For $N_f \le N_c$ we can only construct *mesons* out of squarks

$$M^i_j = Q^i \tilde{Q}_j \tag{1.26}$$

where color indices are summed. Mesons have maximal rank since $N_f \le N_c$ and there are no classical constraints to impose on them.
When $N_f \ge N_c$ the mesons cannot have maximal rank anymore, it can be at most N_c. There are additional gauge invariant operators that can be constructed: *baryons*, that are defined as

$$B_{i_1,\dots,i_{N_f-N_c}} = \epsilon_{i_1,\dots,i_{N_f-N_c},j_1,\dots,j_{N_c}} \epsilon^{a_1,\dots,a_{N_c}} Q^{j_1}_{a_1} \dots Q^{j_{N_c}}_{a_{N_c}} \tag{1.27}$$

$$\tilde{B}_{i_1,\dots,i_{N_f-N_c}} = \epsilon^{i_1,\dots,i_{N_f-N_c},j_1,\dots,j_{N_c}} \epsilon_{a_1,\dots,a_{N_c}} \tilde{Q}^{a_1}_{j_1} \dots \tilde{Q}^{a_{N_c}}_{j_{N_c}} \tag{1.28}$$

Mesons and baryons can be written down using the *VEVs* we found solving the

[2]Remember that massive representation of supersymmetry have twice the degrees of freedom of massless ones, because in the latter half of the supercharges are represented trivially.

D-term equations (ignoring null components for baryons)

$$M = \begin{pmatrix} a_1\tilde{a_1} & & & \\ & a_2\tilde{a_2} & & \\ & & \ddots & \\ & & & a_{N_c}\tilde{a_{N_c}} \\ & & & \end{pmatrix} \tag{1.29}$$

$$B \simeq a_1 a_2 \dots a_{N_c} \tag{1.30}$$

$$\tilde{B} \simeq \tilde{a_1}\tilde{a_2} \dots \tilde{a_{N_c}} \tag{1.31}$$

We can see that if the mesons have rank less than N_c, then B or $\tilde{B}$ (or both) has to vanish and the other has rank one. If the mesons' rank is N_c both B and $\tilde{B}$ have rank one.

There are classical constraints that should be imposed between mesons and baryons, but depend on the number of flavours . For example for $N_f = N_c$ the classical constraint is $det\,(M) - B\tilde{B} = 0$.

Singularities of the moduli space can be investigated using the gauge invariant description we just introduced. The part of the lagrangian that describes flat directions can be written in terms of mesons and baryons. The lagrangian involving mesons features a non trivial Kahler potential that reads

$$K = 2\mathrm{Tr}\,\sqrt{M^\dagger M} \tag{1.32}$$

that generate a singular metric whenever the meson matrix is not invertible. This happens when some of the VEVs are zero, i.e. in points of the moduli space of enhanced gauge symmetry. The appearance of this singularities is related to the fact that some (or all) gluons are now massless and should be included in the low-energy description.

Quantum moduli space

Quantum dynamics modifies the structure of the moduli space of the theory in a different way depending on the number of flavours.

$\mathbf{N_f = 0}$

For pure *Super Yang Mills*, i.e. no quarks, the theory exihibit a discrete set of N_c vacua. Without quarks a non anomalous R-symmetry cannot be found, and the R-symmetry is broken down to the discreet symmetry $\mathbb{Z}_{2N_c}$. Using holomorphy and symmetry arguments, the form of the non perturbative potential can be found and it can be shown that induces the gaugino to condensate [9], meaning that

$$\langle\lambda\lambda\rangle = -\frac{32\pi^2}{N_c} a\,\Lambda^3 \tag{1.33}$$

where Λ is the dynamically generated scale of the theory defined as

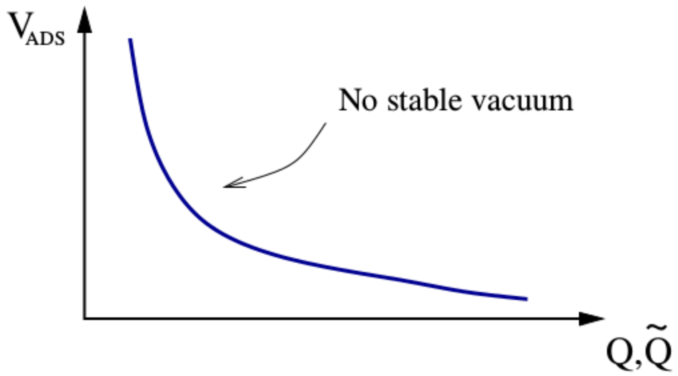
$$\Lambda = \mu e^{-\frac{2\pi i \tau}{b_0}} \qquad \tau = \frac{4\pi i}{g^2(\mu)} + \frac{\theta_{YM}}{2\pi} \qquad b_0 = 3N_c - N_f \tag{1.34}$$

where τ is the complexified gauge coupling. $|\Lambda|$ is defined as the scale at which the coupling constant blows up.
The gaugino condensation breaks R-symmetry to $\mathbb{Z}_2$ and in fact there are N_c physically different vacua labelled by different phases of the gaugino condensate.

$\mathbf{N_f < N_c}$
The quantum corrections for *SQCD* with $N_f < N_c$ flavours completely lift the moduli space through the Affleck-Dine-Seiberg (ADS) superpotential ([10][11]) which reads

$$W_{eff} = (N_c - N_f)\left(\frac{\Lambda^{3N_c-N_f}}{\det M}\right)^{\frac{1}{N_c-N_f}} \tag{1.35}$$

It is the only superpotential that is compatible with the symmetries of the theory and with the other properties of the superpotential we introduced in section 1.1.2. We can see that the ADS superpotential do not exist for $N_F \geq N_c$ since the exponent diverges for $N_f = N_c$ or the determinant vanishes for $N_f \geq N_c$ since the mesons do not have maximal rank. Note that this superpotential is non perturbative and thus it is not in contrast with the renormalization theorem of section 1.1.2. The effect of this superpotential is that the theory does not have ground state. The slope of the potential goes to zero only for $\det M \to \infty$.



This situation is the perfect example when, unlike the classical moduli space, quantum corrections lift completely the moduli space and the theory does not posses a vacuum anymore.

$\mathbf{N_f = N_c}$

When the number of flavours is equal the number of colours of the theory, the classical moduli space was subject to the constraint

$$\det M - B\tilde{B} = 0 \tag{1.36}$$

in the quantum corrected moduli space mesons and baryons satisfy [12]

$$\det M - B\tilde{B} = \Lambda^{2N_c} \tag{1.37}$$

which flows to the classical constraint in the classical limit ($\Lambda \rightarrow 0$).

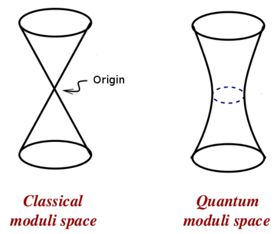


Figure 1.2: Schematical representation of the quantum and classical moduli space near the origin

The effect of this relation is that the origin does not belong to the moduli space anymore and the moduli space is now smooth. For large expectation values of M, B and $\tilde{B}$ the classical and the quantum moduli space look similar, while in the origin quantum correction modify drastically the structure of the moduli space. Moreover, the subspace with B or $\tilde{B}$ is zero, is not singular anymore while classical the meson matrix was constrained to have zero determinant.

Since the origin is not in the quantum moduli space the global symmetry group (1.2.1) is necessarily broken in some way, depending on the position of the moduli space

$$M^i_j = \Lambda^2 \delta^i_j \quad B = \tilde{B} = 0 \quad \rightarrow \quad SU(N_f)_V \times U(1)_B \times U(1)_R \tag{1.38}$$

$$M^i_j = 0 \quad B = -\tilde{B} = \Lambda^{N_c} \quad \rightarrow \quad SU(N_f)_L \times SU(N_f)_R \times U(1)_R \tag{1.39}$$

$\mathbf{N_f = N_c + 1}$

In the case $N_f = N_c + 1$ the classical moduli space is constrained by

$$\det M \left(\frac{1}{M}\right)^j_i - B_i \tilde{B}^j = 0 \qquad M^i_j B_i = M^i_j \tilde{B}^j = 0 \tag{1.40}$$

and quantum corrections do not modify it. In the previous section we noted that the singularities in the classical moduli space are associated to the appearance of massless gluons. In the quantum picture, the interpretation of the singularities is different: they are associated with additional massless mesons and baryons. At the origin of the moduli space the theory is strongly coupled and the global symmetry (1.2.1) is unbroken and it can be checked using 't Hooft anomalies [12]. Far from the origin, the mesons and baryons interact with the potential

$$W = \frac{1}{\Lambda^{2N_c-1}}\left(M^i_j B_i \tilde{B}^j - \det M\right) \tag{1.41}$$

that enforce the classical constraints (1.40) through the equations of motion. For large VEVs of the fields, mesons and baryons acquire large mass through the superpotential.

$\mathbf{N_f > N_c + 1}$
Starting from $N_f = N_c + 2$ it is not possible to construct a sensible physical superpotential out of gauge invariant operators, in analogy to the previous cases. The only $SU(N_f)_L \times SU(N_f)_R$ invariant superpotential that can be written is given by

$$W_{eff} \sim \det M - B_{ij} M^i_k M^j_l \tilde{B}^{kl} \tag{1.42}$$

since baryons have two flavour indices. However this superpotential does not have R-charge equal to two and if we add more flavours we should add other mesons to the superpotential. Therefore the classical moduli space is not modified by quantum corrections. As a result, near the origin the quantum corrected moduli space looks identical to the classical one. Unlike the case with $N_f = N_c + 1$ the singularities in the moduli space cannot be interpreted as massless mesons and baryons and an effective description of these operator is singular [12]. Since 't Hooft anomaly matching conditions are not satisfied in the singular points it is clear that a description using mesons and baryons is not correct.

To find a description of the low-energy degrees of freedom of the theory we will use Seiberg duality, which provides an alternative description of the theory,

$\mathbf{\frac{3}{2}N_c \geq N_f \geq 3N_c}$: the conformal window
In this range the theory is not asympotically free. This can be seen by using the $NSVZ$ β function, which, using (1.3), reads

$$\beta(g) = -\frac{g^3}{16\pi^2}\frac{3N_c - N_f + N_f\gamma(g^2)}{1 - N_c\frac{g^2}{8\pi^2}} \tag{1.43}$$

$$\gamma(g^2) = -\frac{g^2}{8\pi^2}\frac{N_c^2 - 1}{N_c} + \mathcal{O}(g^4) \tag{1.44}$$

The β function is known to have a Banks-Zacks fixed point [13] in the 't Hooft limit with $\frac{N_f}{N_c} = 3 - \epsilon$ held fixed and $\epsilon \ll 1$. However, the fixed point exists in the range of values $\frac{3}{2}N_c \geq N_f \geq 3N_c$ with N_f and N_c finite. This is possible because one loop and two loop contribution to the beta function have opposite signs. As a result, the infrared red theory is a non-trivial four dimensional superconformal theory. The infrared degrees of freedom are quarks and gluons that are not confining but are interacting as massless particles. The theory is in a free non-Abelian Coulomb phase.

Since the theory is superconformal, we have further restriction on the algebra of operators [3]. Superconformal algebra imposes that the dimension of every operator satisfy this inequality involving the R-charge

$$D \geq \frac{3}{2}\,|R| \tag{1.45}$$

where the bound is satured for chiral fields. The operator product expansion (*OPE*) of two chiral operator is constrained by this fact. Since $R(O_1O_2) = R(O_1) + R(O_2)$, we have that for chiral operators, $D(O_1O_2) = D(O_1) + D(O_2)$. Note that the dimension of the operator is quantum corrected, i.e. contains the anomalous dimension of the operator. Therefore, the OPE is not singular and the product of operators is well-defined. Because of this fact, chiral operators form the *chiral ring*.

Since the superconformal R-symmetry is not anomalous and commutes with the global symmetry group of the theory it must be the R-symmetry that appears in table 1.20. The gauge invariant operators we defined previously must have

$$D(Q\tilde{Q}) = \frac{3}{2}R(Q\tilde{Q}) = 3\frac{N_f - N_c}{N_f} \tag{1.46}$$

$$D(B) = D(\tilde{B}) = \frac{3}{2}N_c\frac{N_f - N_c}{N_c} \tag{1.47}$$

Gauge invariant operators should be in unitary representation of the superconformal algebra. Unitarity imposes that in general $D \geq 1$ and the equality holds for free fields. From the previous equation we can verify that $D(M) \geq 1$ for $N_f \geq \frac{3}{2}N_c$ and it becomes a free field for $N_f = \frac{3}{2}N_c$.

For fewer number of flavours, the meson field is inconsistend with the unitarity bound. The theory is conjecture to flow to a different phase.

$\mathbf{N_f > 3N_c}$

In this range, quarks prevails on glouns and change the sign of the β function. This is caused by the *charge screening* effect of quarks, that make the coupling constant smaller at larger distances.

The theory is in a free non-Abelian electric phase. Its behaviour is not very well

[3]R-symmetry is contained directly in the superconformal algebra instead of being an automorphism of the algebra, as in superPoincarè algebra.

defined at high energies because of the presence of a Landau pole at $R \sim \Lambda^{-1}$, although the theory can be a good description of a low-energy limit of another theory.

Magnetic theory

The magnetic theory is a *SQCD* theory with the same global symmetries as the electric theory, but with gauge group $SU(N_f - N_c)$. In addition there are N_f^2 color singlets, that we will call mesons, since they have the same properties as the mesons we can construct in the electric theory. In the magnetic theory they are fundamental fields i.e. they are not written as gauge invariant operators from quarks. Since they are gauge invariant, they interact only through the superpotential

$$W = M^i_{\tilde{j}}\, q_i\, \tilde{q}^{\tilde{j}} \tag{1.48}$$

where we represented dual quarks as $q, \tilde{q}$ and mesons as $M^i_{\tilde{j}}$.
The charges of fields of the magnetic theory are

	$SU(N_c)$	$SU(N_f)_L$	$SU(N_f)_R$	$U(1)_B$	$U(1)_R$
q	N_c	$\overline{N_F}$	1	$\frac{N_c}{N_f-N_c}$	$\frac{N_c}{N_f}$
$\tilde{q}$	$\overline{N_c}$	1	N_F	$-\frac{N_c}{N_f-N_c}$	$\frac{N_c}{N_f}$
$M^i_{\tilde{j}}$	1	N_f	$\overline{N_f}$	0	$2\frac{N_f-N_c}{N_f}$

(1.49)

Dual quarks sit in opposite representation of flavour symmetries.

Mesons in the magnetic theory have the same charges of the mesons constructed from electric quarks. Baryons constructed from dual quarks have the same baryonic charge as the electric baryons. Moreover, it can be demonstrated that they are proportional to each other.
Similarly to the electric theory, the R-charges can be chosen in order to be non anomalous.
However, the R-charges can be found using the duality. If we impose that the meson is built from electric quarks, its R-charge is twice the R-charge of electric quarks. Since the superpotential (1.48) must have R-charge two, we find the R-charges of the magnetic quarks. In this way, we found the R-charges at the superconformal infrared fixed point.

Duality

In the *conformal window* the electric and magnetic theories we introduced previously give an equivalent description of the same physics in the infrared. In this range, the magnetic theory has a non trivial infrared fixed point too. At this fixed

point, the superpotential (1.48) is a relevant perturbation since it has dimension $D = 1 + 3\frac{N_c}{N_f} < 3$ and it drives the theory in a new fixed point.
Electric mesons have different dimension in the UV from the magnetic ones since they are construct from a pair of quark. For this reason it's necessary to introduce a dimensionful scale μ in order to match their dimension in the UV: $M = \mu M_m$, where M_m are the magnetic mesons.
The strong coupling scale of the electric Λ and magnetic $\tilde{\Lambda}$ theories are related by the relation

$$\Lambda^{3N_c - N_f}\tilde{\Lambda}^{3(N_f - N_c) - N_f} = (-1)^{N_f - N_c}\mu^{N_f} \tag{1.50}$$

This relation shows that when one theory is strongly coupled, the other is weakly coupled. Moreover, it ensures that the dual of the dual theory is the electric theory itself. The dual of the dual magnetic theory is a $SU(N_c)$ $SQCD$ theory with scale Λ, d^i and $\tilde{d}_{\tilde{j}}$ quarks and additional singlets $M^i_{\tilde{j}}$ and $N^{\tilde{j}}_i = q_i q^{\tilde{j}}$ with superpotential

$$W = \frac{1}{\tilde{\mu}} N^{\tilde{j}}_i d^i \tilde{d}_{\tilde{j}} + \frac{1}{\mu} M^i_{\tilde{j}} N^{\tilde{j}}_i = \frac{1}{\mu} N^{\tilde{j}}_i \left(-d^i d_{\tilde{j}} + M^i_{\tilde{j}}\right) \tag{1.51}$$

since from the previous relation we have $\tilde{\mu} = -\mu$.
Meson fields are massive and can be integrated out by using their equation of motion which result in

$$N^{\tilde{j}}_i = 0 \qquad M^i_{\tilde{j}} = d^i d_{\tilde{j}} \tag{1.52}$$

Since the dual theories describe the same physics, there should be a mapping of gauge invariant operators between them. We already saw that electric and magnetic mesons match in the infrared. A mapping should exists also for baryonic operators. Indeed it does and it is given by

$$\begin{aligned} B^{i_1 \dots i_{N_c}} &= C\ \epsilon^{i_1 \dots i_{N_c} j_1 \dots i_{N_f - N_c}} b_{j_1 \dots j_{N_f - N_C}} \\ \tilde{B}^{i_1 \dots i_{N_c}} &= C\ \epsilon_{\tilde{i}_1 \dots \tilde{i}_{N_c} \tilde{j}_1 \dots \tilde{i}_{N_f - N_c}} b_{\tilde{j}_1 \dots \tilde{j}_{N_f - N_C}} \\ \text{where} \qquad C &= \sqrt{-(-\mu)^{N_c - N_f}\Lambda^{3N_c - N_f}} \end{aligned} \tag{1.53}$$

using (1.50) it can be shown that these mappings preserve the $\mathbb{Z}_2$ character of the duality.

As an additional check of the duality 't Hooft anomaly matching conditions have been calculated in [8] for the electric and magnetic theories for the various global symmetries and they are given by

$$\begin{array}{rcl} SU(N_f)^3 & \longrightarrow & N_c \\ U(1)_R\, SU(N_f)^2 & \longrightarrow & -\dfrac{N_c^2}{2N_f} \\ U(1)_B\, SU(N_f)^2 & \longrightarrow & \dfrac{N_c}{2} \\ U(1)_R & \longrightarrow & -N_c^2-1 \\ U(1)_R^3 & \longrightarrow & -N_c^2-1-2\dfrac{N_c^4}{N_f^2} \\ U(1)_B^2 U(1)_R & \longrightarrow & -2N_c^2 \end{array} \tag{1.54}$$

Another important check of the duality is that it remains valid under mass perturbations of the electric theory. Suppose to add a superpotential term that give mass to the quark in the last flavour and is equal to

$$W^{el}_{mass} = m\, Q_{N_f}\tilde{Q}^{N_f} \tag{1.55}$$

Flowing to the IR the number of quark is decreased by one, driving the theory to a more strongly coupled fixed point[4]. The new scale of the theory is given in terms of the old one by

$$\Lambda_L^{3N_c-(N_f-1)} = m\, \Lambda^{3N_c-N_f} \tag{1.56}$$

In the magnetic theory the mass perturbation is mapped in the term

$$W^{mag}_{mass} = m\, M^{N_f}_{N_f} \tag{1.57}$$

Because of this term, the gauge group gets higgsed to $SU(N_f-1-N_c)$ and only N_f-1 light quarks remain in the theory.
The scale of the magnetic theory Λ_L is modified and reads

$$\tilde{\Lambda}^{3(N_f-N_c-1)-(N_f-1)} = -\frac{\tilde{\Lambda}^{3(N_f-N_c)-N_f}}{< q_{N_f}\tilde{q}^{N_f} >} \tag{1.58}$$

where $< q_{N_f}\tilde{q}^{N_f} >= -\mu m$ because of the equation of motion for the massive flavour.
As expected the magnetic theory becomes more weakly coupled.
We conclude that the duality is preserved under massive deformations.

[4]Since matter in the fundamental representation contribute with a positve term it is easy to see that this is true.

1.3 Kutasov-Schwimmer duality

A possible generalization of Seiberg duality can be found by adding matter fields in different representations of the gauge group. Kutasov and Schwimmer ([14], [15]) considered $SU(N)$ SQCD with the addition of a matter fields in the adjoint representation of the gauge group and found that it admitted a magnetic dual.

1.3.1 Electric theory

The classical electric theory can be summarized by the following table of charges of the fields under the global symmetry group $SU(N_f)_L \times SU(N_f)_R \times U(1)_B \times U(1)_R$

	$SU(N_c)$	$SU(N_f)_L$	$SU(N_f)_R$	$U(1)_B$	$U(1)_R$
Q	N_c	N_F	1	1	$1-\frac{2}{k+1}\frac{N_c}{N_f}$
$\tilde{Q}$	$\overline{N_c}$	1	$\overline{N_F}$	-1	$1-\frac{2}{k+1}\frac{N_c}{N_f}$
X	N_c^2-1	1	1	0	$\frac{2}{k+1}$

(1.59)

and by the addition of the superpotential

$$W_{Adj} = g_k \mathrm{Tr}\, X^{k+1} \tag{1.60}$$

The superpotential (1.60) drives the theory to a new infrared fixed point. It is either relevant or dangerously irrelevant, depending on the value of k [15].
Being dangerously irrelevant means that in the ultraviolet it is an irrelevant term, whose coupling gets weaker and weaker along the renormalization group flow. At some point in the flow, the fields acquire dimensions such that it is a relevant perturbation, getting stronger along the flow and driving the theory to a different fixed point in the infrared.

The theory posses two different R-symmetries but the superpotential breaks explicitly one of them and imposes that the adjoint matter has R-charge $\frac{2}{k+1}$.

The remaining R-charges can be fixed by imposing that the R-symmetry is anomaly free as we did previously. Using formula (1.21), considering also the fermion in the adjoint matter multiplet we have

$$\begin{gathered} N_c + (R_Q-1)\frac{1}{2}2N_f + (R_X-1)N_c = 0 \\ (R_Q-1)N_f = -R_X N_c \quad \longrightarrow \quad R_Q = 1 - R_X\frac{N_c}{N_f} \end{gathered} \tag{1.61}$$

Imposing this condition the R-charges of the fields were fixed completely, as in Seiberg duality. This has been possible because the superpotential (1.60) fixed independently the R-charge R_X. Otherwise the condition (1.61) fixes R_Q as a function of R_X with R_X generic.

The gauge invariant operators that can be constructed are mesons and baryons multiplied with powers of the adjoint field. Mesons operators are given by

$$(M_j)_{\tilde{i}}^{i} = \tilde{Q}_{\tilde{i}} X^j Q^i \qquad j = 0, 1, \ldots, k-1 \tag{1.62}$$

Baryons are more easily introduced by first defining "dressed quarks"

$$Q_{(l)} = X^l Q \qquad l = 0, \ldots, k-1 \tag{1.63}$$

Baryons are defined as

$$B^{i_1, i_2, \ldots, i_k} = Q_{(0)}^{i_1} \ldots Q_{(k-1)}^{i_k} \qquad \text{with } \sum_{l=1}^{k} i_l = N_c \tag{1.64}$$

with color index contracted with an ϵ tensor.

Vacuum structure

In analogy to the condition $N_f \geq N_c$ in *SQCD* we would like to find a range of values of N_f, N_c and k such that the theory admits stable vacua. We can add a weak deformation to the superpotential (1.60), by adding terms with lower order powers in X

$$W(X) = \sum_{l=1}^{k} g_l \mathrm{Tr}\, X^{l+1} + \lambda \mathrm{Tr}\, X \qquad \text{with } g_l \ll 1 \tag{1.65}$$

where we introduced λ as a Lagrange multiplier to enforce the tracelessness of X. Since it is a weak perturbation, the large field behavior of W is not modified. Hence, if we can't find stable vacua with the weak perturbation, the original theory doesn't have any vacua too.

The theory has a large sets of multiple vacua for $Q = \tilde{Q} = 0$ and $X \neq 0$. X can be diagonalized with eigenvalues x_i. The F-terms are given by setting $W'(x_i) = 0$. Now, $W'(x_i)$ is a polynomial of degree k in the eigenvalues x_i admitting k distinct solutions in general. As a result, ground states are labeled by a set of k integers $(i_1, \ldots, i_k)$, describing how many eigenvalues are sitting in the l-th minimum. Clearly, since X has N_c eigenvalues we have

$$\sum_{l=1}^{k} i_l = N_c \tag{1.66}$$

In every vacuum, X has a quadratic potential, which correspond to a mass term and can be integrated out. The X expectation values break the gauge group in the following way

$$SU(N_c) \longrightarrow SU(i_1) \times SU(i_2) \times \cdots \times SU(i_k) \times U(1)^{k-1} \tag{1.67}$$

Each $SU(i_l)$ sector describes a decoupled $SQCD$ model which has stable vacua only if $N_f \geq N_c$, hence considering every sector we have

$$i_l \leq N_f \qquad \forall 1 \leq l \leq k \tag{1.68}$$

Taking the limit $g_l \to 0$ we find that we must have

$$N_f \geq \frac{N_c}{k} \tag{1.69}$$

For every choice of i_l there is a moduli space obtained by giving expectation values of the quarks. Hence, the moduli space of the theory consists of different disconnected components, associated to different choice of i_l .

1.3.2 Magnetic theory

The magnetic theory is constructed in a similar way as the magnetic theory in Seiberg duality. The dual theory has gauge group $SU(\tilde{N}_c) = SU(kN_F - N_c)$. The baryonic charge of the dual quarks is found by imposing that baryons in the electric theory are proportional to the baryons constructed from dual quark.

	$SU(\tilde{N}_c)$	$SU(N_f)_L$	$SU(N_f)_R$	$U(1)_B$	$U(1)_R$
q	$\tilde{N}_c$	$\overline{N_F}$	1	$\frac{N_c}{kN_f - N_c}$	$1 - \frac{2}{k+1}\frac{\tilde{N}_c}{N_f}$
$\tilde{q}$	$\overline{\tilde{N}_c}$	1	N_F	$-\frac{N_c}{kN_f - N_c}$	$1 - \frac{2}{k+1}\frac{\tilde{N}_c}{N_f}$
Y	1	$\tilde{N}_c^{\,2} - 1$	1	0	$\frac{2}{k+1}$
M_j	1	N_f	$\overline{N_f}$	0	$2 - \frac{4}{k+1}\frac{N_c}{N_f} + j\frac{2}{k+1}$

(1.70)

The magnetic theory has a superpotential

$$W = \operatorname{Tr} Y^{k+1} + \sum_{j=0}^{k-1} M_j q Y^{k-j-1} \tilde{q} \qquad \text{dove } M_j = Q Y^j \tilde{Q} \tag{1.71}$$

The charges of the fields are easily found by requiring duality for the two theories. In this way, the charges of the mesons are given in terms of the electric quarks, which are fixed, and the superpotential fixes the charges for the remaining fields. Using this method, the R-charges of the dual quarks are given by

$$R_q = R_X - R_Q \tag{1.72}$$

where we used that $R_X = R_Y = \frac{2}{k+1}$ because of (1.71) and (1.60).

Duality and mass deformations

Similarly to Seiberg duality, the duality is valid in a range of (N_f, N_c) in which both the electric and magnetic theory are asymptotically free. In that window, the superpotentials $\mathrm{Tr}\, X^{k+1}$ and $\mathrm{Tr}\, Y^{k+1}$ are both relevant in the infrared. An estimate of the window is given in [16] through a-maximization.

The mapping of gauge invariant operators between the two theories is given by

$$\begin{aligned} X &\longleftrightarrow Y \\ QX^j\tilde{Q} &\longleftrightarrow M_j \\ B_{el}^{(i_1,i_2,\dots,i_k)} &\longleftrightarrow B_{mag}^{(j_1,j_2,\dots,j_k)} \;\; j_l = N_f - i_{k+1-l} \; l = 1,2,\dots,k \end{aligned} \tag{1.73}$$

The charge assignment necessary for the mapping of the operators is consistent with 't Hooft anomaly matching conditions which are given by

$$\begin{aligned} SU(N_f)^3 &\longrightarrow N_c d^{(3)}(N_f) \\ U(1)_R\, SU(N_f)^2 &\longrightarrow -\frac{2}{k+1}\frac{N_c^2}{N_f} d^{(2)}(N_f) \\ SU(N_f)^2 U(1)_B &\longrightarrow N_c d^{(2)}(N_f) \\ U(1)_R &\longrightarrow -\frac{2}{k+1}(N_c^2+1) \\ U(1)_R^3 &\longrightarrow \left(\left(\frac{2}{k+1}-1\right)^3+1\right)(N_c^2-1) - \frac{16}{(k+1)^3}\frac{N_c^4}{N_f^2} \\ U(1)_B^2 U(1)_R &\longrightarrow -\frac{4}{k+1}N_c^2 \end{aligned} \tag{1.74}$$

We consider now mass deformations of the electric theory in order to understand if duality is preserved under such deformations. Let's modify the electric superpotential by adding a mass term

$$W_{el} = g_k \mathrm{Tr}\, X^{k+1} + m\tilde{Q}_{N_f} Q^{N_f} \tag{1.75}$$

The number of flavours in the IR is reduced by one unit keeping the number of colours the same. In order to preserve the duality, the magnetic theory should have gauge group $SU(k(N_f-1)-N_c) = SU(kN_f - N_c - k)$. Let's see if this happens. The dual potential reads

$$W_{mag} = g_k \mathrm{Tr}\, Y^{k+1} + \sum_{j=1}^{k} M_j \tilde{q} Y^{k-j-1} q + m(M_0)_{N_f}^{N_f} \tag{1.76}$$

Integrating out the massive fields we find

$$q_{N_f} Y^l \tilde{q}^{N_f} = -\delta_{l,k} m \quad l = 0,\dots,k-1 \tag{1.77}$$

which fixes the expectation values to

$$
\begin{aligned}
\tilde{q}_{\alpha}^{N_f} =& \delta_{\alpha,1} \\
q_{\alpha}^{N_f} =& \delta^{\alpha,k} \\
Y_{\beta}^{\alpha} =& \begin{cases} \delta_{\beta+1}^{\alpha} & \beta = 1, \ldots, k-1 \\ 0 & \text{otherwise} \end{cases}
\end{aligned}
\tag{1.78}
$$

These expectation values break the gauge group to $SU(kN_f - N_c) \to SU(k(N_f - 1) - N_c)$ through the Higgs mechanism and reduces the number of flavours by one unit as required by duality. As a result duality is preserved under mass deformations.

2 | Three dimensional dualities

2.1 Supersymmetry and field theories in three dimensions

Spinors in three dimensions have different properties than their four dimension counterpart.
The dimension of the representation in an arbitrary dimension D is given by $2^{\frac{D}{2}}$ for D even, while $2^{\frac{D-1}{2}}$ for D odd. Hence, in three dimension we have a two dimensional representation. In odd dimensions representations are irreducible and Weyl spinors do not exist: the chirality operator (γ_{D+1} or γ^*) is proportional to the identity.
Gamma matrices can be chosen real and the Majorana condition can be imposed, lowering the degress of freedom of the representation from four to two.
Since $3d$ $\mathcal{N} = 2$ theories have four supercharges, we can use the $4d$ $\mathcal{N} = 1$ superspace formalism.
The supersymmetry algebra and its representations can be found by dimensional reduction from four dimensions. The reduced algebra reads

$$\{Q_\alpha, Q_\beta\} = \{\bar{Q}_\alpha, \bar{Q}_\beta\} = 0 \qquad \{Q_\alpha, \bar{Q}_\beta\} = 2\sigma^\mu_{\alpha\beta} P_\mu + 2i\epsilon_{\alpha\beta} Z \tag{2.1}$$

The central charge Z is the component of the momentum along the reduced direction. Because of the presence of the central charge in the algebra, now states must satisfy a BPS bound of the form $M \geq Z$, which imply that massless representations have null central charge.
The automorphism of this algebra is $U(1)_R \simeq SO(2)_R$, as in four dimensions.
Superspace formalism is similar to what we introduced previously, with proper changes due to the different spinor representation in three dimension such as gamma matrices.
Chiral superfields contain *on-shell* one complex scalar and a complex spinor as in four dimensions.
Vector superfield contains an additional real scalar field with respect to the four dimensional superfield. The scalar σ comes from the component of the vector field

A_μ along the reduced direction.

Coulomb branch and dualized photon

In three dimensions a free photon can be dualized to a scalar γ since it has only one polarization. Instead of defining a potential A in order to solve the Bianchi identity for the potential $dF = d^2A = 0$, we can define a scalar such that

$$*F = \frac{g^2}{\pi} d\gamma \quad \rightarrow \quad d * F = 0 \tag{2.2}$$

where $*$ is the Hodge operator and g is the gauge coupling.
The Bianchi identity $dF = 0$ can be seen as a conservation law for the topological current, which is defined as

$$J_\mu^{top} = \frac{1}{2\pi} * F \;=\; \frac{1}{2\pi} d\gamma \qquad \partial^\mu J_\mu \tag{2.3}$$

The topological current acts as shifts of the dualized photon $\gamma \rightarrow \gamma + \alpha$.
The quantization of the magnetic flux implies that the dualised photon is periodic and is normalized in such a way that $\gamma \sim \gamma + 2\pi$.

From a vector superfield V we can define a linear multiplet Σ, defined as

$$\Sigma = \epsilon^{\alpha\beta} D_\alpha D_\beta V \tag{2.4}$$

It satisfies $D^2\Sigma = 0$ and is gauge invariant under a transformation $V \rightarrow V + i(\Phi - \Phi^\dagger)$. The lowest component of Σ is the scalar in the vector multiplet and it contains also a term $\bar{\theta}\sigma_\rho\theta\, \epsilon^{\mu\nu\rho} F_{\nu\rho}$, whose bosonic part is proportional to the topological current. The super Yang-Mills action can be written also as

$$S \sim \int d^4\theta \; \Sigma^2 \tag{2.5}$$

The vacuum expectation values of scalar fields in the vector multiplet parametrize a subset of the moduli space of the theory which is called the *Coulomb branch*. Using the dualized photon we can turn the vector multiplet into a chiral multiplet, whose lowest component is a good parametrization of the Coulomb branch

$$Y = \exp\left(\frac{2\pi\sigma}{g^2} + i\gamma\right) \tag{2.6}$$

because of its periodicty, it's natural to assign γ to a phase factor.
The dualization of the photon was possible because there wasn't matter in the theory. However, matter fields couple with σ with mass terms. As a result, in a generic point of the Coulomb branch, all charged matter fields are massive and can

be integrated out and in a low-energy description we can still dualize the gauge field.
We can generalize the dualization to non-Abelian gauge theories with a similar reasoning. In a generic point of the Coulomb branch, the VEV of σ breaks the gauge group to its maximal torus $U(1)^{r_G}$, where r_G is the rank of the gauge group. We now have r_G massless vector multiplets that can be dualized into chiral multiplets Y_i with $i = 1, \ldots, r_G$.
Note that since the definition of Y depends on the gauge coupling, it will be modified by quantum corrections.

For $U(N)$ or $SU(N)$ theories it's better to use the following coordinates on the Coulomb branch, since they are related to the simple roots of the algebra

$$Y_k \sim \exp\left(\frac{\sigma_j - \sigma_{j+1}}{\hat{g}^2} + i(\gamma_j - \gamma_{j+1})\right) \qquad \hat{g}^2 = \frac{g^2}{4\pi} \tag{2.7}$$

with $j = 1, \ldots, N-1$ for $SU(N)$ or with $j = 1, \ldots, N$ for $U(N)$. From now on we will fix to a Weyl chamber by setting $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_N$ with a Weyl transformation.
For $SU(N)$ gauge theories is useful to define

$$Y = \prod_{j=1}^{r_G} Y_k \tag{2.8}$$

since it's gonna be a coordinate of the unlifted Coulomb branch for such theories.

Real masses

In $3D\,\mathcal{N} = 2$ theories there's another way of giving a mass to a chiral multiplet other than the superpotential term $W_{mass} = m\Phi^2$, which correspond to a holomorphic mass.
In these theories we can couple a global symmetry, which is not a R-symmetry, to a background vector multiplet. We will give a vacuum expectation value $\hat{m}$ to the scalar in the multiplet. As a result, every field charged under that symmetry will receive a mass $q\,\hat{m}$, where q is the charge of the field under that symmetry. If a chiral field is charged under different global symmetry, its mass mass is a sum of the real masses for every global symmetry that we gauged. Real masses are parity odd and belong to vector multiplets rather than chiral fields. For this reason they cannot appear in holomorphic objects such as the superpotential.
In addition, real masses relative to global abelian symmetries contribute to the central charge of the supersymmetry algebra Zthrough

$$Z = \sum q_i m_i \tag{2.9}$$

where q_i is the charge of the field and m_i the real masses under a global symmetry $U(1)_i$.
Moreover, the central charge Z can be promoted to a background linear superfield whose lowest component is Z.
Another symmetry that can contribute to the central charge is $U(1)_J$, whose current is given by the linear multiplet we defined previously. The added term is given by $\int d^4\theta\ V_b\Sigma$, where V_b is a background vector multiplet. Integrating by parts and defining $\Sigma_b = \epsilon^{\alpha\beta}D_\alpha D_\beta V_b$ we obtain $\int d^4\theta\ \Sigma_b V$. Thus, the scalar component of the background vector field is a Fayet-Iliopulos term ξ, which contributes to the central charge $m_J = \xi$ [17].

Monopoles

In three dimensions there exist finite energy solutions that can be understood in terms of four dimensional monopoles after compactifying one direction. The scalar σ will play the role of the scalar Higgs field in the adjoint representation typically used to introduce monopoles [18]. Different topological solutions can be found since the scalar field at infinity must approach a vaccum solution. Such solution doesn't have to be the same in every direction and as a result we find different topological solutions depending on how we choose to map the two-sphere at infinity to the gauge group G. The mapping relates S^2_∞ to the gauge group rather than S^3_∞ since monopoles are localized in space rather than in space-time.
For a generic vacuum expectation value of σ, with $\sigma_i \neq \sigma_j$, the gauge group breaks to its maximal torus $U(1)^{r_G}$. The possible windings around S^2_∞ are then characterised by

$$\Pi_2\left(G/U(1)^{r_G}\right) = \Pi_1(U(1)^{r_G}) = \mathbb{Z}^{r_G} \tag{2.10}$$

The equality holds if $\Pi_2(G) = 0$, which is true for every semisimple gorup and if $\Pi_1(G) = 0$, which is satisfied if the group is the covering group of the Lie algebra. As a result there are r_G different topological solutions. Each of these solutions carry a magnetic charge for each of the $U(1)^{r_G}$ unbroken gauge fields. For example, for $G = SU(N_c)$ we have $N_c - 1$ different topological solutions.
For $SU(2)$ gauge group one solution to the equation is given by the 't Hooft-Polyakov monopole, which in singular gauge reads

$$\sigma = \left(v\ \coth(vr) - \frac{1}{r}\right)\tau^3 \qquad A_\mu = \epsilon_{ij3}\hat{r}^i\left(\frac{1}{r} + \frac{v}{\sinh(vr)}\right) \tag{2.11}$$

This solution can be easily extended to $SU(N)$ gauge group by embedding the above solutios in a $SU(2)$ subgroup of the gauge group. Since the gauge group is broken in the vacuum, we cannot use $SU(N)$ gauge transformation to generate other solutions. As a results, each embedding of $SU(2)$ will result in a monopole

charged under different $U(1)$ factors of the Cartan. Note that there are $N-1$ special embeddings of $SU(2)$ into $SU(N)$ which result in monopoles charged only under one factor of the $U(1)$ in the Cartan subalgebra. They are given by the N-1 contiguous 2x2 blocks in the diagonal.
The importance of monopole operators in three dimensions is due to the fact that flowing to the infrared, they flow to the coordinates on the Coulomb branch we introduced in (2.7) for $SU(N)$ theories [19].
The carges of the monopoles can be written as

$$\vec{g} = 2\pi \sum_a n_a \vec{\alpha_a} \tag{2.12}$$

where $\vec{\alpha_a}$ are the simple roots of the $su(N)$ Lie algebra. Dirac quantization condition imposes that $n_a \in \mathbb{Z}$.
Moreover, the mass of the monopoles is subject to the BPS bound which is given by

$$M_{monopole} \geq \frac{|\vec{g}\cdot\sigma|}{e^2} = \frac{2\pi}{e^2}\sum_i n_i \sigma_i \tag{2.13}$$

where σ_i are the vaccum expectation values of σ. Monopoles that sature the bound are called BPS monopoles.

Da finire

2.1.1 Moduli spaces of gauge theories

Moduli space of $U(1)$ gauge theory with N_f massless flavours

For large values of σ the Coulomb branch of the theory can be parametrized by the chiral superfield we defined before $X = e^{\frac{\Phi}{g^2}}$ which correspond to a cylinder. However, the metric for γ receives quantum correction and thus the topology of moduli space is changed perturbatively. The Higgs branch intersects the Coulomb branch for $\sigma = 0$ and since it is invariant under $U(1)_J$, the radius of the circle must shrink to zero where they meet, since the topological symmetry acts as shifts on the circle γ.
Therefore, near the origin the moduli space looks like the intersection of three cones: the Higgs branch and two cones corresponding to two distinct parts of the Coulomb branch as in figure 2.1. Half of the Coulomb branch is parametrized semi-classically by the field $V_+ \sim e^{\frac{\Phi}{g^2}}$, while the other half by $V_- \sim e^{-\frac{\Phi}{g^2}}$. Two different chiral fields are needed since near the origin the moduli space shrinks to a point and $V_+ \to 0$. The Higgs branch is not modified by quantum corrections

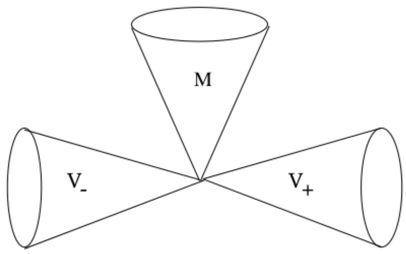


Figure 2.1: Schematic picture of the origin of the moduli space for a $U(1)$ gauge theory.

and the mesons can still be used to parametrize it.
The charges of the fields under the symmetries are given by

	$U(1)_R$	$U(1)_J$	$U(1)_A$	$SU(N_f)_R$	$SU(N_f)_R$
Q	0	0	1	N_f	1
$\tilde{Q}$	0	0	1	1	$\overline{N_f}$
M	0	0	2	N_f	$\overline{N_f}$
$V_\pm$	N_f	± 1	$-N_f$	1	1

(2.14)

The symmetries force the superpotential to have the form

$$W = -N_f \left(V_+ V_- \det\left(M\right)\right)^{\frac{1}{N_f}} \tag{2.15}$$

The superpotential is singular at the origin of the moduli space, indicating that there are massless degrees of freedom that need to be taken into account.
We can give real masses $\bar{m}_i$ ($-\bar{m}_i$) to the quarks Q_i ($\tilde{Q}_i$). The Higgs branch is parametrised by the diagonal elements of M_i^i which intersects the Coulomb branch at $\sigma = \bar{m}_i$. At every intersection, we have $U(1)$ theory with one flavour. The coulomb branch is parametrised, at every intersection, by $V_{i,\pm} = e^{\pm\frac{\Phi - \bar{m}_i}{g^2}}$ with a superpotential given by

$$W = -\sum_{i=1}^{N_f} M_i^i V_{i,+} V_{i,-} + \sum_{i=1}^{N_f - 1} \lambda_i \left(V_{i,+} V_{i+1,-} - 1\right) \tag{2.16}$$

where λ_i are Lagrange multipliers in order the enforce the semiclassical identification $V_{i,+} V_{i+1,-} = 1$.

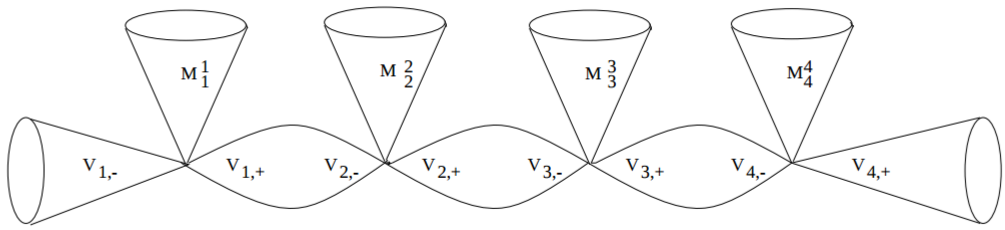


Figure 2.2: Moduli space of vacua for SQED with four flavours

Moduli space of $SU(2)$ gauge theories with flavours

We will consider a $SU(2)$ gauge theories with an even number of quarks, in order to avoid the addition of Chern Simons terms. The charges are given by

	$U(1)_R$	$U(1)_A$	$SU(2N_f)$
Q	0	1	$2N_f$
M	0	2	$N_f(2N_f-1)$
Y	$2N_f-2$	$-2N_f$	1

(2.17)

and Y is defined in (2.7) and parametrizes the Coulom branch. Mesons parametrises the Higgs branch and are subject to the condition rank$(M) \leq 2$.

A theory without quarks has a superpotential [20], generated by instantons, that lifts completely the Coulomb branch of the theory

$$W = \frac{1}{Y} \tag{2.18}$$

With $N_f = 1$ the quantum moduli space consists in a smooth merging between the Higgs and the Coulomb branch of the theory since the fields are subject to the constraint

$$MY = 1 \tag{2.19}$$

For $N_f \geq 2$, there is a unique superpotential consistent with the symmetries of the theory

$$W = -(N_f - 1)(Y\,\mathrm{Pf}M)^{\frac{1}{N_f-1}} \tag{2.20}$$

For $N_f = 2$ the Higgs and Coulomb branch are distinct and they touch at $M = 0\,,\, Y = 0$ instead of the point with $\sigma = 0$.
The superpotential is relevant and drives the theory to a interacting fixed point.

Moduli spaces for $SU(N_c)$ theories with $N_c > 2$ and with N_f flavours

We will consider a $SU(N_c)$ gauge theory with equal number of Q and $\tilde{Q}$ in order to avoid Chern-Simons term. The simmetries of the theory are given by the following table

	$U(1)_R$	$U(1)_B$	$U(1)_A$	$SU(N_f)_L$	$SU(N_f)_R$
Q	0	1	1	N_f	1
$\tilde{Q}$	0	-1	1	1	$\overline{N_f}$
M	0	0	2	N_f	$\overline{N_f}$
$Y_{j\neq K}$	-2	0	0	1	1
Y_K	$2(N_f-1)$	0	$-2N_f$	1	1
Y	$2(N_f-N_c+1)$	0	$-2N_f$	1	1

(2.21)

where Y_j are defined as in (2.7) and Y is defined as $Y = \prod_{j=1}^{N_c-1} Y_j \sim e^{\frac{\Phi_1-\Phi_{N_c}}{g^2}}$. The adjoint scalar VEV is ordered in such a way that $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_{N_c}$. The value of j for which σ changes sign is called K and the associated instanton is Y_K. For $N_f > 1$ the theory has a Higgs branch, which break the gauge group in a generic point to $SU(N_f - N_c)$ for $N_f < N_c - 1$ and completely for $N_f \geq N_c - 1$. The Higgs branch is parametrised as in four dimensions by mesons and for $N_f \geq N_c$ also by baryons.

For theory withouth quarks, instantons generate the Affleck-Harvey-Witten [20] superpotential

$$W = \sum_{j=1}^{N_c-1} \frac{1}{Y_j} \tag{2.22}$$

which prohibits the theory to have stable supersymmetric vacuas.

The theory with massless quarks features a superpotential generated by $N_c - 2$ instantons

$$W_{inst} = \sum_{j\neq K} \frac{1}{Y_j} \tag{2.23}$$

which doesn't completely lift the Coulomb branch. In fact, the following subspace is classically unlifted

$$\sigma_1 > \sigma_2 = \cdots = 0 > \sigma_{N_c} = -\sigma_1 \tag{2.24}$$

The last equality holds since for $SU(N_c)$ we have $\sum_i \sigma_i = 0$. The quantum corrected moduli space is different from the classical one and for $N_f < N_c - 1$ the effective superpotential is given by

$$W_{eff} = (N_c - N_f - 1)(Y \det(M))^{\frac{1}{N_f - N_c + 1}} \tag{2.25}$$

which admits no stable vacuum.
For $N_f = N_c - 1$ we obtain a constraint on the moduli space given by

$$Y \det(M) = 1 \tag{2.26}$$

which results in a merging between the Higgs and the Coulomb branch.

For $N_f \geq N_c$, the low-energy description contains baryons. The quantum moduli space is the same as the semi-classical moduli space and there is a superconformal fixed point at the origin. In the $N_f = N_c$ case an effective description of the moduli space can be given and the superpotential is

$$W = -Y(\det(M) - B\tilde{B}) \tag{2.27}$$

which forces the theory to flow to a non-trivial fixed point in the infrared.
For $N_f > N_c$ an effective description is not known.

The above analysis can be extended to theories with $U(N_c)$ gauge group with $N_f \leq N_c$. We will discuss the theory with $N_f = N_c$ since the results with lower number of flavours can be obtained by integrating out quarks.
The theory with $U(N_c)$ is obtained by gauging the global $U(1)_B$ factor.
Since we have $N_f = N_c$ the superpotential (2.27) from the $SU(N_c)$ part of the group is presenet and for the $U(1)$ factor we have one flavour given by $X = B\tilde{B}$ and chiral fields V_+ , V_-, defined as

$$V_+ \sim \exp\left(\frac{\Phi_1}{g^2} + i\gamma_1\right) \qquad V_- \sim \exp\left(\frac{\Phi_{N_c}}{g^2} + i\gamma_{N_c}\right) \tag{2.28}$$

since the null trace condition does not hold anymore.
The $U(1)$ sector has a superpotential given by

$$W = -XV_+V_-$$

As a result, the superpotential for the theory is given by

$$W = -Y(\det(M) - X) - XV_+V_- \tag{2.29}$$

Integrating out the massive field Y we obtain a superpotential

$$W = -V_+V_-\det(M) \tag{2.30}$$

2.2 Aharony duality

One of the first three dimensional dualities was found by Aharony in [21] for theories with $U(N_c)$ as gauge group and with non chiral matter in the fundamental representation. The charges of the fields are given by the following table

	$U(1)_R$	$U(1)_A$	$U(1)_J$	$SU(N_f)_L$	$SU(N_f)_R$
Q	0	1	0	N_f	1
$\tilde{Q}$	0	1	0	$\overline{N_f}$	1
M	0	2	0	1	1
$V_\pm$	$N_f - N_c + 1$	$-N_f$	± 1	1	1

(2.31)

where $V_\pm$ are defined as in the previous section and parametrize the unlifted Coulomb branch. The R-charges of the quarks have been fixed to zero in order to have the fermion in the multiplet with R-charge -1. The moduli space of this theory was described in the previous section.

The dual theory is a $U(\tilde{N}_c) = U(N_f - N_c)$ gauge theory with N_f flavours $q\,,\,\tilde{q}$ and with singlet fields $M\,, V_+\,, V_-$ with the same quantum number of the electric theory. The charges are chosen in order to be consistent with a potential $W = M\,q\tilde{q}$ as in four dimensional dualities.
The global charges of the theory are given by

	$U(1)_R$	$U(1)_A$	$U(1)_J$	$SU(N_f)_L$	$SU(N_f)_R$
q	1	-1	0	$\overline{N_f}$	1
$\tilde{q}$	1	-1	0	1	N_f
$\tilde{V}_\pm$	$N_c - N_f - 1$	N_f	± 1	1	1

(2.32)

The magnetic theory features a superpotential of the form

$$W = M_i^{\tilde{i}} q^i \tilde{q}_{\tilde{i}} + V_+\,\tilde{V}_- + V_-\,\tilde{V}_+ \tag{2.33}$$

The dual theory now contains the fields $V_+\,,\,V_-$ which are matched with the coordinate on the Coulomb branch of the original theory. Remember that in three dimensions the moduli space contains the Coulomb branch, which wasn't present in four dimensional theories. The introduction of the monopoles is crucial in order to have the same moduli space between the two theories.
The dual theory exists only for $N_f > N_c$. Starting with $N_f = N_c + 1$ we obtain a dual theory with $U(1)$ gauge group. Adding a mass term of the form mM in the dual theory breaks the gauge group. After integrating out quarks and $\tilde{V}_\pm$ fields we obtain an effective description given only by mesons and $V_\pm$ with no other fields. The superpotential is constrained by symmetries to be of the form

$$W = -V_+ V_- \det{(M)} \tag{2.34}$$

We found exactly the dual description of the theory we introduced in (2.30).
We can compare the moduli space between the dual theories by looking at specific

vacua. For example, let's consider a point on the Higgs branch in the electric theory with VEV of the mesons of rank N_c. The low-energy dual theory associated to such VEV is a $U(N_f - N_c)$ theory with $N_f - N_c$ massless flavours whose moduli space can be described by a superpotential of the form $W = \tilde{V}_+ \tilde{V}_- \det(q\tilde{q})$ since $N_f = \tilde{N}_c$.
The equation of motion resulting from the addition of this term to the superpotential (2.33) set $V_+ = V_- = 0$, as in the electric theory.

2.3 Kim-Park duality

Aharony duality can be generalised with the addition of a field in the adjoint representation [22].
The electric theory contains N_f flavours and a multiplet in the adjoint representation. The theory has a superpotential given by $W = trX^{k+1}$ that fixes the R-charge of the field to $\Delta_X = \frac{2}{k+1}$. The R-charges of the quarks have been left unconstrained to generic the value Δ_Q.
The charges of the fields are given by

	$SU(N_f)_L$	$SU(N_f)_R$	$U(1)_A$	$U(1)_J$	$U(1)_R$
Q	N_f	1	1	0	Δ_Q
$\tilde{Q}$	1	$\overline{N_f}$	1	0	Δ_Q
X	1	1	0	0	$\frac{2}{k+1}$
$v_{j,\pm}$	1	1	$-N_f$	± 1	$-N_f(\Delta_Q - 1) - \frac{2}{k+1}(N_c - 1 - j)$

(2.35)

where $v_{j,\pm}$ with $j = 0, \ldots, k-1$ are the monopole operators dressed with powers of the adjoint field. They are defined as

$$v_{j,\pm} = \mathrm{Tr}\,(v_{0,\pm} X^j) \tag{2.36}$$

where $v_{0,\pm}$ is the bare monopole operator we defined in the previous section.
To see the reason why there are k different monopoles in this theory we can weakly deform the superpotential to a generic polynomial, as we did in (1.65). The effect of this deformation is the breaking of the gauge group $U(N_c)$ in

$$U(N_c) \;\rightarrow\; U(r_1) \times U(r_2) \times \cdots \times U(r_k) \quad \text{with} \sum r_i = N_c \tag{2.37}$$

Following the same reasoning, in the infrared we find k distinct SQCD sector with gauge group $U(r_i)$ without adjoint matter, since it has been integrated out.
Every sector must have a pair of monopoles since it is a $U(r_i)$ theory.
Going in the limit of vanishing coupling we find that the original theory have k

pairs of monopole operators.

Dici una ref che scrive perchè son vestiti in quel modo

The magnetic theory is a $U(kN_f - N_c)$ gauge theory with N_f dual quarks $(q, \tilde{q})$, adjoint matter Y and singlet fields $(M_j)^{i}_{i}$ and $v_{j,\pm}$. Mesons are constructed from the quarks and the adjoint matter in the electric theory and the $v_{j,\pm}$ are the electric monopole operators.

The charges of the fields are given by

	$SU(N_f)_L$	$SU(N_f)_R$	$U(1)_A$	$U(1)_J$	$U(1)_R$
q	N_f	1	1	0	Δ_Q
$\tilde{q}$	1	$\overline{N_f}$	1	0	Δ_Q
Y	1	1	0	0	$\frac{2}{k+1}$
M_j	N_f	$\overline{N_f}$	2	0	$2\Delta_Q + j\frac{2}{k+1}$
$v_{j,\pm}$	1	1	$-N_f$	± 1	$-N_f(\Delta_Q - 1) - \frac{2}{k+1}(N_c - 1 - j)$
$\tilde{v}_{j,\pm}$	1	1	N_f	± 1	$N_f(\Delta_Q - 1) + \frac{2}{k+1}(N_c + 1 + j)$

(2.38)

The fields $\tilde{v}_{j,\pm}$ are the monopole operators of the magnetic theory and are constructed in the same way of the electric ones. The charges of the dual quarks and of the adjoint matter are chosen in order to be compatible with a superpotential

$$W = \mathrm{Tr}\, Y^{k+1} + \sum_{j=0}^{k-1} M_j \, \tilde{q} \, Y^{k-j-1} q \tag{2.39}$$

as Kutasov-Schwimmer duality in four dimensions.

The charges of the monopole operators can be calculated by counting their fermion zero modes and are compatible with a superpotential of the form

$$W_{monopoles} = \sum_{j=0}^{k-1} \left(v_{i,+} \, \tilde{v}_{k-j-1,-} + v_{j,-} \, \tilde{v}_{k-j-1,+} \right) \tag{2.40}$$

The duality maps the electric adjoint field with the magnetic one. Mesons and monopoles operators in the electric theory are mapped as singlets in the magnetic one.

3 | Reduction of 4D dualities to 3D

In the previous chapter we introduced few examples of electric-magnetic dualities in four and three spacetime dimensions. It is natural to wonder what is the relation between them since they show some similirаties but at the same time the dynamics of field theories in three and four dimensions is different.
For example, in three dimensions there are no anomalies and the moduli space features the Coulomb branch. Moreover, in three dimensions the axial symmetry is not broken by anomalies and we have a topological symmetry, enforced by a Bianchi identity.
The presence of scalars in the vector multiplets allows real mass terms for fields charged under global symmetries, whereas in four dimensions we could only generate holomorphic mass terms.
Despite these differences we will be able to dimensionally reduce four dimensional dualities to three dimensional dualities and in some cases we will find dualities that weren't known before as stand-alone three dimensional dualities.

3.1 General procedure of reducing dualities

The näive dimensional reduction of the four dimensional dualities does not yield a three dimensional duality. The dimensional reduction of four dimensional theories can be achieved compactifying a spatial dimension into S^1 with radius r and then going in the limit $r \to 0$.
Let's analyse the behaviour of the the theory at finite radius. In four dimensional theories, the strong coupling scale is given by

$$\Lambda^b = \exp\left(-\frac{8\pi^2}{g_4^2}\right) \tag{3.1}$$

where b is the one-loop β function coefficient. Recall the following relation for strong coupling scales between dual theories

$$\Lambda^b \tilde{\Lambda}^b = (-1)^{N_f - N_c} \tag{3.2}$$

After compactifying a dimension, the strong coupling scale is modified [19] since

$$g_4^2 = 2\pi r g_3^2 \qquad \rightarrow \qquad \Lambda^b = \exp\left(-\frac{4\pi}{r g_3^2}\right) \tag{3.3}$$

The result of this relation is that if we take the limit $r \rightarrow 0$, the strong coupling scale goes to zero very fast. Moreover, it's clear that the relation (3.2) cannot hold anymore, since $\Lambda\,,\,\tilde{\Lambda} \rightarrow 0$ at the same time.
Thus, the $r \rightarrow 0$ limit does not commute with the infrared limit at fixed coupling needed in the duality.
As a result, we can invert the order of the limits. We can take the limit in which we keep $\Lambda\,,\,\tilde{\Lambda}$ and r fixed and look at energies $E \ll \Lambda\,,\,\tilde{\Lambda}\,,\,\frac{1}{r}$. In this limit, we are in the infrared, at fixed coupling but with a finite radius. However, this is not a problem since the theory doesn't see the compactified dimensions since it hasn't enough energy to excite the Kaluza-Klein modes associated to it. As a result, the dynamics of the theory is effectively three dimensional.
The theories we find with this procedure have few properties that differentiate them to purely three-dimensional theories. The scalar associated to the compactified direction of A_3 is periodic, with period $\frac{1}{r}$ since the holonomy of the gauge field on the circle is gauge invariant

$$P\left(\exp\left(i\oint A_3\right)\right) \tag{3.4}$$

As a result the scalar is compact with period $\sim \frac{1}{r}$ whereas the three dimensional one is not.
Four dimensional theories on the circle have an additional non-perturbative superpotential generated by a Kaluza-Klein monopole,alternatively called twisted instanton. This particular type of instanton/monopole configuration [1] of the gauge field is generated through a non periodic gauge transformations on an instanton configuration of the gauge field. Even though the transformation is not periodic around the circle, the gauge field obtained acting with it remains periodic. For this reason it's associated to a different topological sector with respect to standard instanton/monopole configurations [23].
The superpotential generated reads

$$W = \eta Y_{low} \tag{3.5}$$

[1] Using this expression we refer to the fact that 't Hooft/Polyakov monopoles in four dimension can be interpreted as three-dimensional instantons after ignoring the time dimension.

where $\eta = \Lambda^b$ and Y_{low} is the Coulomb branch coordinate we defined in .
Theories obtained with this method are dual to each other and they differ from truly three-dimensional theories for the reasons we explained above. For example, the introduction of the η superpotential breaks the axial symmetry that is anomalous in four dimensions, but is allowed in three-dimensional theories.
After these consideration we would like to transform these dualities with finite radius r into truly three-dimensional dualities, i.e. dualities for theories well-defined at high energies. The compactness of the Coulomb branch is not really a problem since for $SU(N)$ gauge theories the η superpotential lifts completely the Coulomb branch together with AHW superpotential (2.23). The Coulomb branch $U(N)$ gauge theories is not completely lifted, but we will see that we can take a limit in which we focus only in specific points of the Coulomb branch and the compactness of the Coulomb branch becomes irrelevant.
Regarding the η superpotential, it's possible to find monopole operators Y_{high}, well-defined at high energies, that flow to the Coulomb branch coordinates Y_{low} at low energies. Using these operators it is possible to define the theories in the ultraviolet, thus turning the dualities we find by reduction as standalone three-dimensional dualities.

3.2 Dimensional reduction of Seiberg duality with $SU(N_c)$ gauge group

We introduced Seiberg duality in four dimensions in section **??** but we were its analog in three dimensions was discovered only by the procedure of dimensional reduction that we just introduced.
The superpotential that is generated for four dimensional theories compactified is given by the sum of the AHW superpotential and the η superpotential and together they lift the Coulomb branch completely.

Comparison of moduli spaces

From the previous sections we know that three-dimensional theories perturbed by ηY should be the same at low energies as four dimensional theories on the circle. The four dimensional duality grants that the compactified theory is dual to a four dimensional magnetic theory. Such theory can be mapped into a three dimensional theories by the same arguments we used for the electric theory. We will compare the moduli spaces between four and three dimensional theories to see if this argument is valid in this case.
For $N_f = N_c - 1$ the four dimensional theory does not admit a vacuum but the

unperturbed three dimensional theory has a moduli space subject to the constraint (2.26). Adding the η superpotential result in a theory without a stable vacuum, as the four dimensional theory.
For $N_f = N_c$ the four dimensional theory has an effective description that can be implemented by a Lagrange multiplier that reads

$$W_{4d} = \lambda \left(B\tilde{B} - \det(M) + \Lambda^{2N_c} \right) = \lambda \left(B\tilde{B} - \det(M) + \eta \right) \tag{3.6}$$

since $b = 3N_c - N_f = 2N_c$ for these particular values.
The perturbed three-dimensional theory has an effective description given by

$$W_{3d} = Y \left(B\tilde{B} - \det(M) + \eta \right) \tag{3.7}$$

which suggests the identification between λ and Y.
For $N_f = N_c + 1$ it can be demonstrated that the moduli spaces match but the situation is a little more involved and we refer to [19] for a more complete discussion.

3.2.1 Flow to a pure three-dimensional duality

It is possibile to flow to a duality between three-dimensional theories withouth the η-superpotential with a flow obtained by the assignment of real masses.
Let's consider a theory with $N_f + 1$ flavours and let's give a real mass $\hat{m}$ to the last flavour. We will set in the limit in which $\hat{m} \to \infty$ in order to integrate out the last flavour. This can be achieved by giving expectation values to the fields associated to the diagonal $SU(N_f + 1) \times U(1)_B$ flavour symmetry given by

$$\text{diag}(0, \ldots, m) = \text{diag}(m_B - M, \ldots, m_B + N_f M) \tag{3.8}$$

where M is associated to $SU(N_f + 1)$ and m_B to $U(1)_B$.
The real masses are easily mapped to the dual theory by considering the charges of the quarks under the global symmetries in question. As a result the first N_f flavours get a mass $\hat{m}_1$ and the last one a mass of $\hat{m}_2$

$$\hat{m}_1 = \frac{\hat{m}}{N_f - N_c + 1} \qquad \hat{m}_2 = \frac{\hat{m}(N_c - N_f)}{N_f - N_c + 1} \tag{3.9}$$

In the electric theory we will be interested in configrations that remain at a finite distance from the origin in the Coulomb branch , in the limit $\hat{m} \to 0$. For these type of vacua, the last flavour is massive and can be ignored at low energies, resulting in an effective $SU(N_c)$ theory with N_f flavours.
The monopole operator Y_{high} is related to the low-energy coordinate Y_{low} by the

relation $Y_{low} = Y_{high}/m$, where m is the complex mass of quark. Since the real mass is given by a component of a vector multiplet, the superpotential $W = \eta Y_{high}$ vanishes, since we didn't assign any complex mass to the quark.
As a result, we obtain a $SU(N_c)$ gauge theory with N_f massless flavours and no η-superpotential as the electric theory. The absence of the η-superpotential results in a theory with axial symmetry. The charges of the fields are given by

	$SU(N_c)$	$U(1)_R$	$U(1)_A$	$U(1)_B$	$SU(N_f)_L$	$SU(N_f)_R$
Q	N_c	1	-1	1	$\overline{N_f}$	1
$\tilde{Q}$	$\overline{N_c}$	1	-1	-1	1	N_f
Y	1	$2(N_c - N_f - 1)$	$-2N_f$	0	1	1

(3.10)

The magnetic theory does not have vacuum at the origin of the Coulomb branch since every flavour gets a mass proportional to $\hat{m}$. The vacuum configuration with most massless quarks is given by following the vacuum expectation value of $\tilde{\sigma}$

$$\tilde{\sigma} = \text{diag}\,(\overbrace{-\tilde{m}_1, \ldots, -\tilde{m}_1}^{N_f - N_c \text{ values}}, -\tilde{m}_2) \tag{3.11}$$

Note that this expectation value is traceless, as it should be.
This vacuum breaks the gauge symmetry $SU(N_f - N_c + 1) \rightarrow SU(N_f - N_c) \times U(1)$. We can view it as $U(N_f - N_c)$ with quarks in the representations

component	quark	$U(N_f - N_c)_{U(1)}$
$1, \ldots, N_f$	q	$(N_f - N_c)_{\,1}$
$1, \ldots, N_f$	$\tilde{q}$	$(\overline{N_f - N_c})_{\,1}$
$N_f + 1$	p	$1_{-(N_f - N_c)}$
$N_f + 1$	$\tilde{p}$	$1_{\,(N_f - N_c)}$

(3.12)

The component of the mesons created from quarks with with the same real mass remain light, since left and right quarks have opposite charges and global symmetries. The off-diagonal components, i.e $M^i_{N_f+1}$ and $M_i^{N_f+1}$ with $i = 1, \ldots, N_f$ have a mass proportional to $\hat{m}$ and can be ignored.
The remaining massless mesons are a $N_f \times N_f$ singlet matrix, that can be identified with the mesons M of the electric theory, and a single component, which is $M^{N_f+1}_{N_f+1}$ which we will call Y.
The η superpotential can be written in terms of $U(N_f - N_c)$ Coulomb branch variables that we defined in (2.28). Since the gauge group comes from the breaking of $SU(N_f - N_c + 1)$ the superpotential is written as

$$W = \tilde{\eta}\tilde{V}_- \tag{3.13}$$

In addition to this superpotential, there's a AHW contribution to the superpotential, related to the breaking of the gauge group which is given by a term

$$W = \tilde{V_+} \tag{3.14}$$

The complete superpotential of the magnetic theory is given by

$$W_{mag} = M\, q\tilde{q} + Y\, p\tilde{p} + \tilde{\eta}\tilde{V_-} + \tilde{V_+} \tag{3.15}$$

The global symmetries of the theory consist of the $SU(N_f)_L \times SU(N_f)_R \times U(1)_R$ symmetries inherited from four dimensions, the topological symmetry $U(1)_J$ associated to the abelian factor in the gauge group and two abelian factors given by $U(1)_B$ and $U(1)_A$, that are not broken by the superpotential.
From the matching of the gauge invariant operators between the two theories we can fix some of the quantum numbers. The $U(1)_B$ symmetry mixes with abelian term of the gauge group and it can be chosen arbitrarily.
The charges are the following

	$U(N_f-N_c)$	$SU(N_f)_L$	$SU(N_f)_R$	$U(1)_B$	$U(1)_A$	$U(1)_R$
q	$(N_f-N_c)_{\,1}$	$\overline{N_f}$	1	0	-1	1
$\tilde{q}$	$\overline{(N_f-N_c)}_{\,-1}$	1	$\overline{N_f}$	0	-1	1
p	$1_{-(N_f-N_c)}$	1	1	N_c	N_f	$-(N_f-N_c)$
$\tilde{p}$	$1_{(N_f-N_c)}$	1	1	$-N_c$	N_f	$-(N_f-N_c)$
M	1	1	1	0	2	0
Y	1	1	1	0	$-2N_f$	$2(N_f-N_c+1)$
$V_\pm$	1	1	1	0	0	2

(3.16)

The meson field Y has the same quantum numbers of the Y field in the electric and for this reason is natural to identify them. The quantum numbers of $V_\pm$ are calculated by counting zero modes and the fact that are consistent with the superpotential (3.15) is nice check of the result.
The Coulomb branch of this theory is completely lifted by the last two terms of the superpotential.
As a result of this process we found a three-dimensional duality that wasn't known as a standalone duality before the method of dimensional reduction was discovered in [19]. The duality can be checked by the matching of the partition functions between the two theories. We will focus on this approach in the next chapters.

Appendices

A | Supersymmetry and superfields

A.1 Supersymmetry algebra

The supersymmetry algebra is an extension of the Poincarè group involving anticommutators together with commutators. Since it is not a ordinary Lie algebra, Coleman-Mandula theorem does not apply for theories that are invariant under it.

The supersymmetry algebra is divided into two subalgebras, the bosonic and fermionic part. The bosonic part contains Poincarè Lie algebra $(M_{\mu\nu}, P_\mu)$ while fermionic subalgebra is generated by the *supercharges* $(Q^I_\alpha, \bar{Q}^I_{\dot{\alpha}})$ with $I = 1, \ldots, \mathcal{N}$. When more than one pair of supercharges is present we refer to extended supersymmetry.

The supercharges sit in spinorial representations of the Lorentz group, respectively $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$.

We will not repeat the bosonic subalgebra, since is given by the Poincarè Lie algebra. The fermionic generators satisfy anticommutation rules between themselves and commutation rules with bosonic generators. For this reason, the supersymmetry algebra is defined in mathematical literature as a graded Lie algebra with grade one.
The (anti)commutation rules in four dimensions are

$$[P_\mu, Q^I_\alpha] = 0 \tag{A.1}$$

$$[P_\mu, \bar{Q}^I_{\dot{\alpha}}] = 0 \tag{A.2}$$

$$[M_{\mu\nu}, Q^I_\alpha] = i(\sigma_{\mu\nu})_\alpha^\beta Q^I_\beta \tag{A.3}$$

$$[M_{\mu\nu}, \bar{Q}^I_{\dot{\alpha}}] = i(\bar{\sigma}_{\mu\nu})^{\dot{\alpha}}_{\dot{\beta}} \bar{Q}^I_{\dot{\beta}} \tag{A.4}$$

$$\{Q^I_\alpha, \bar{Q}^J_{\dot{\beta}}\} = 2\sigma^\mu_{\ \alpha\dot{\beta}} P_\mu \delta^{IJ} \tag{A.5}$$

$$\{Q^I_\alpha, Q^J_\beta\} = \epsilon_{\alpha\beta} Z^{IJ} \qquad Z^{IJ} = -Z^{JI} \tag{A.6}$$

$$\{Q^I_{\dot{\alpha}}, Q^J_{\dot{\beta}}\} = \epsilon_{\dot{\alpha}\dot{\beta}} \left(Z^{IJ}\right)^* \tag{A.7}$$

This set of commutation rules can be found using symmetry arguments and enforcing the consistency of the algebra using the graded Jacobi identity.

It is important to stress the fact that Z^{IJ} are operators that span an invariant subalgebra: they are *central charges*. They play an important role especially in massive representations.

There is an additional symmetry that is not present in the previous commutation rules: R-symmetry. It is an automorphism of the algebra that act on the supercharges. For generic $\mathcal{N}$ the R-Symmetry group is $U(\mathcal{N})$[1].

A.2 Representations

Since the supercharges do not commute with Lorentz generators, their action on a state will result in a state with different spin: they generate a symmetry between bosons and fermions.
Representations of supersymmetry contain particle with different spin but same mass and they are organized in supermultiplets. The mass of particles in the same multiplet must be the same because P^2 is still a Casimir operator of the supersymmetry algebra, while the Pauli-Lubanski operator W^2 isn't anymore.

Moreover, the supersymmetry algebra imposes that every state must have positive energy and that every supermultiplet must contain the same number of bosonic and fermionic degrees of freedom *on-shell* .

Various supermultiplets exist and their properties depend on the number of supercharges of the theory and on what they represent e.g matter, glue or gravity.

Massless supermultiplet are typically shorter than massive multiplet because in the massless case half of the supercharges are represented trivially. Massive representation of extended supersymmetry can be shortened in case some of the central charges of the algebra are equal to twice the mass of the multiplet. These states are usually called (ultra)short multiplet or BPS states.

We will introduce the multiplets that can be defined for $4d\,\mathcal{N}=1$ theories and only later we will explain the differences with $3d\,\mathcal{N}=2$ theories. Representations are similar because in both cases we have the same number of supercharges.

For four dimensional theories, we can define two different multiplet that are invariant under supersymmetry transformations. The matter or chiral multiplet contains a complex scalar (*squark*) and a Weyl fermion (*quark*). It identifies the matter content of the theory. The vector or gauge multiplet contains a Weyl fermion (*gaugino*) and a vector (*gluon or photon*). Particles in the same multiplet transform in the same representation of global or gauge symmetries. For this reason the gaugino cannot represent matter.

A representation of these multiplets on fields can be easily found using the *superspace* formalism that we will introduce in the next section. In this formalism it

[1]This is not always true. For example for $\mathcal{N}=4$ the R-symmetry group is given by $SU(4)$

is possible to represent fields that are *off-shell*, in contrast with multiplets that we introduced previously that are *on-shell* since they represent states in Hilbert space.

A.2.1 Superfields and superspace in four dimensions

Supersymmetry representations on fields can be found more sistematically using the formulation of *superspace* instead of acting directly with supercharges and verifying that the algebra closes.

A simple formulation of superspace exist for theories with 4 supercharges while for theories with a bigger number of supercharges its definition is much more complex. We will be interested only in theories with 4 supercharges such as theories in 4D with $\mathcal{N}=1$ or 3D with $\mathcal{N}=2$.

Superspace can be seen as the extension of Minkowsky space with *fermionic coordinates* i.e. *Grassmann numbers* θ^α , $\bar{\theta}^{\dot\alpha}$. They anticommute between themselves and commute with everything else.

$$\{\theta^\alpha,\theta^\beta\}=0 \quad \{\bar\theta^{\dot\alpha},\bar\theta^{\dot\beta}\}=0 \quad \{\theta^\alpha,\bar\theta^{\dot\beta}\}=0 \qquad \alpha,\dot\alpha=1,2 \tag{A.8}$$

Derivation and integration in Grassmann variables are summarized by these rules

$$\partial_\alpha=\frac{\partial}{\partial\theta_\alpha} \quad \partial^\alpha=-\epsilon^{\alpha\beta}\partial_\beta \quad \bar\partial_{\dot\alpha}=\frac{\partial}{\partial\bar\theta^{\dot\alpha}} \quad \bar\partial^{\dot\alpha}=-\epsilon^{\dot\alpha\dot\beta}\partial_{\dot\beta} \quad \partial_\alpha\theta^\beta=\delta^\beta_\alpha \quad \partial_\alpha\bar\theta^{\dot\alpha}=0 \tag{A.9}$$

$$\int d\theta=0 \quad \int d\theta\,\theta=1 \quad d^2\theta=\frac{1}{2}d\theta^1 d\theta^2 \quad \int d^2\theta=\frac{1}{4}\epsilon^{\alpha\beta}\partial_\alpha\partial\beta \tag{A.10}$$

For a more detailed introduction on Grassmann numbers and their properties see [24].

Using Grassmann numbers we can write the anticommutators in the supersymmetry algebra as commutators defining $\theta Q=\theta^\alpha Q_\alpha$ and $\bar\theta\bar Q=\bar\theta^{\dot\alpha}\bar Q_{\dot\alpha}$

$$[\theta Q,\bar\theta\bar Q]=2\theta\sigma^\mu\bar\theta P_\mu \quad , \quad [\theta Q,\theta Q]=[\bar\theta\bar Q,\bar\theta\bar Q]=0 \tag{A.11}$$

Using this trick we are able to represent the supersymmetry algebra as a Lie algebra. An element of the superPoincarè group can be found exponentiating the generators

$$G(x,\theta,\bar\theta,\omega)=\exp\left(ixP+i\theta Q+i\bar\theta\bar Q+\frac{1}{2}i\omega M\right) \tag{A.12}$$

The superspace is defined as the 4+4 dimensions group coset

$$M_{4|1}=\frac{\text{SuperPoincarè}}{\text{Lorentz}} \tag{A.13}$$

in analogy to Minkowsky space that can be defined as the coset between Poincarè and Lorentz groups.

A generic point in superspace is parametrized by $(x^\mu, \theta^\alpha, \bar{\theta}^{\dot{\alpha}})$. A superfields is a field in superspace i.e. function of the superspace coordinates. Since θ coordinates anticommute, the expansion of a superfield in fermionic coordinates stops at some point. The most general superfield $Y = Y(x, \theta, \bar{\theta})$ is given by

$$\begin{aligned} Y(x,\theta,\bar{\theta}) = & f(x) + \theta\psi_1(x) + \bar{\theta}\bar{\psi}_2(x) + \theta\theta g_1(x) + \bar{\theta}\bar{\theta}g_2(x) \\ & + \theta\sigma^\mu\theta v_\mu(x) + \theta\theta\bar{\theta}\bar{\lambda}(x) + \bar{\theta}\bar{\theta}\theta\rho(x) + \theta\theta\bar{\theta}\bar{\theta}s(x) \end{aligned} \tag{A.14}$$

fields with uncontracted θ such as $\psi_1, \psi_2, \lambda, \rho$ are spinors while v_μ is a vector.

Supercharges can be represented as differential operators that act on superfield. Their expression is

$$\begin{cases} Q_\alpha & = -i\partial_\alpha - \sigma^\mu_{\alpha\dot{\beta}}\bar{\theta}^{\dot{\beta}}\partial_\mu \\ \bar{Q}_{\dot{\alpha}} & = +i\dot{\partial}_{\dot{\alpha}} + \theta^\beta\sigma^\mu_{\beta\dot{\alpha}}\bar{\theta}^{\dot{\beta}}\partial_\mu \end{cases} \tag{A.15}$$

An infinitesimal supersymmetry transformation on a superfield is defined by

$$\delta_{\epsilon,\bar{\epsilon}}Y = \left(i\epsilon Q + i\bar{\epsilon}\bar{Q}\right)Y \tag{A.16}$$

The powerfulness of the superfield formalism is due to the fact that an integral in full superspace coordinates of a superfield is supersymmetric invariant.

$$\delta_{\epsilon,\bar{\epsilon}}\int d^4x\, d^2\theta\, d^2\bar{\theta}\; Y = \int d^4x\, d^2\theta\, d^2\bar{\theta}\; \delta_{\epsilon,\bar{\epsilon}}Y = 0 \tag{A.17}$$

The first equality holds because the Grassmann measure is inviariant under translation while the second is true because we can see that the variation of the superfield is either killed by the integration in the θ variables or is proportional to a spacetime derivative that does not contribute after integration in space.

Using this fact we can construct supersymmetric invariant lagrangians by integrating superfields in superspace. Clearly, in order to find a physically significant lagrangian we should choose the superfield we wish to integrate wisely. More importantly, we want to use irreducible representation of supersymmetry i.e. the supermultiplets we introduced before. We need to find conditions that can be imposed on a general superfield that are invariant under a supersymmetry transformation.

Chiral superfield

One way to achieve this goal is to find an operator that commute with the supercharges and annihilate the superfield An example of such operator is the *covariant* *derivative*

$$\begin{cases} D_\alpha = & \partial_\alpha + i\sigma^\mu_{\alpha\dot\beta}\bar\theta^{\dot\beta}\partial_\mu \\ \bar D_{\dot\alpha} = & \dot\partial_{\dot\alpha} + i\theta^\beta\sigma^\mu_{\beta\dot\alpha}\bar\theta^{\dot\beta}\partial_\mu \end{cases} \tag{A.18}$$

We can define a (anti)chiral superfield Φ

$$\bar D_{\dot\alpha}\Phi = 0 \quad \text{chiral} \qquad\qquad D_\alpha\Psi = 0 \quad \text{anti-chiral} \tag{A.19}$$

This condition reduces the number of components of the superfield. It can be easily demonstrated that if Ψ is chiral, then $\bar\Psi$ is anti-chiral. As a result a chiral field cannot be real.

The expansion of a chiral fields in components is given by

$$\Phi(x,\theta,\bar\theta) = \phi(x)+\sqrt{2}\theta\psi(x)+i\theta\sigma^\mu\bar\theta\partial\mu\phi(x)-\theta\theta F(x)-\frac{i}{\sqrt{2}}\theta\theta\partial\mu\psi(x)\sigma^\mu\bar\theta-\frac{1}{4}\theta\theta\bar\theta\bar\theta(x) \tag{A.20}$$

We can see that a chiral field is composed by three fields: two complex scalars (ϕ and F) and a spinor (ψ).

The chiral superfield identifies the matter multiplet we introduced previously. It contains an additional bosonic field ($F(x)$) that is present because superfields provide an *off-shell* representation of supersymmetry and it is needed in order to close the algebra. It is called *auxiliary field* because it will not have kinetic terms in every Lagrangian that can be constructed.

Real or Vector Field

We can impose that the superfield is real. In this way we find the *real* or *vector* multiplet. Its general expression in component is messy and a simplification can be made noting that $\Phi+\bar\Phi$ is a vector superfield if Φ is chiral. Choosing an appropriate chiral field, the real superfield can be put in what is called Wess-Zumino gauge. We stress the fact that the Wess-Zumino gauge is not supersymmetric invariant: after a supersymmetry transformation the vector superfield acquire its general expression involving many other field components. In this gauge the vector superfield can be written as

$$V_{WZ}(x,\theta,\bar\theta) = \theta\sigma^\mu\bar\theta v_\mu(x) + i\theta\theta\bar\theta\bar\lambda(x) - i\bar\theta\bar\theta\theta\lambda(x) + \frac{1}{2}\theta\theta\bar\theta\bar\theta D(x) \tag{A.21}$$

The vector superfields represents the vector multiplet (which contains radiation) and similarly to the chiral superfields contains an auxiliary field ($D(x)$).

A.2.2 R-symmetry

R-symmetry was first introduced with the supersymmetry algebra. For the theories we will consider in superspace it is given by a global $U(1)_R$. It is defined by as a transformation of the Grassmann coordinates

$$\theta \;\to\; e^{i\alpha}\theta \qquad \bar{\theta} \;\to\; e^{-i\alpha}\bar{\theta} \tag{A.22}$$

α parametrizes the transformation. As a result supercharges transform under the transformation

$$Q \to\; e^{-i\alpha}Q \qquad \bar{Q} \;\to\; e^{+i\alpha}\bar{Q} \tag{A.23}$$

From this we find the commutator relations between supercharges and R-symmetry generator R

$$[R, Q] = -Q \qquad [R, \bar{Q}] = \bar{Q} \tag{A.24}$$

The R-charge of a superfield is defined by

$$Y(x, \theta, \bar{\theta}) \;\to\; e^{iR_Y\alpha}Y(x, \theta, \bar{\theta}) \tag{A.25}$$

Different component field in the superfield have different R-charge and are related because of A.24. For a chiral field we have

$$R[\phi] = R\,[\Phi] \qquad R[\psi] = R\,[\Phi] - 1 \qquad R[F] = R\,[\Phi] - 2 \tag{A.26}$$

The corresponding antichiral field carry opposite charges.

A.3 Supersymmetric actions

We will use the property we introduced in A.17 to generate supersymmetric invariant lagrangians. We start our analysis with chiral superfields. Since lagrangians are quadratic in the fields and must be real, the simplest kinetic term for a chiral superfield is given by $\bar{\Phi}\Phi$.

$$\mathcal{L}_{kin} \;=\; \int d^2\theta d^2\bar{\theta}\;\bar{\Phi}\Phi = \partial_\mu\bar{\phi}\partial^\mu\phi + \frac{i}{2}\left(\partial_\mu\psi\sigma^\mu\bar{\psi} - \psi\sigma^\mu\partial_\mu\bar{\psi}\right) + \bar{F}F + \text{total derivative} \tag{A.27}$$

which gives the correct kinetic terms for a scalar and a spinor. The auxiliary field doesn't have kinetic terms as predicted.
Many action can be find using a generalization of the equation above. It is called *Kahler* potential

$$K(\bar{\Phi}, \Phi) = \sum_{m,n=1}^{\infty} c_{m,n}\bar{\Phi}^m\Phi^n \qquad \text{where} \quad c_{m,n} = c_{n,m}* \tag{A.28}$$

The condition on the coefficient is imposed by the requirement of a real lagrangian.

Another way of finding supersymmetric actions is by integrating *chiral* superfields in half-superspace coordinates. We define the *superpotential* to be a holomorphic function of Φ

$$\mathcal{L}_{int} = \int d^2\theta d^2\, W(\Phi) + \int d^2\bar{\theta}\, \bar{W}(\bar{\Phi}) = \sum_{i=1}^{\infty} \int d^2\theta\, \lambda_n \Phi^n + \int d^2\bar{\theta}\, \lambda_n^{\dagger} \bar{\Phi}^n \tag{A.29}$$

The hermitian conjugate was added in order to have a real lagrangian.

Mixed terms with product of chiral and anti-chiral superfield are not present since they would be generic superfields and would not yield a supersymmetric lagrangian. In fact if $W(\Phi)$ is holomorphic and Φ is a chiral superfield, $W(\Phi)$ is a chiral superfield

$$\bar{D}_{\dot{\alpha}} W(\Phi) = \frac{\partial W}{\partial \Phi} \bar{D}_{\dot{\alpha}} \Phi + \frac{\partial W}{\partial \bar{\Phi}} \bar{D}_{\dot{\alpha}} \bar{\Phi} = 0 \tag{A.30}$$

and yield a proper lagrangian upon integration in $d^2\theta$.

Since the superpotential is integrated only in half superspace coordinates it need to be charged in an opposite way with respect to the integration measure under R-symmetry. Remembering that

$$R[\theta] = 1 \qquad [\bar{\theta}] = -1 \qquad R[d\theta] = -1 \qquad R[d\bar{\theta}] = 1 \tag{A.31}$$

It's easy to see that

$$R[W(\Phi)] = 2 \qquad R[\bar{W}(\bar{\Phi})] = -2 \tag{A.32}$$

For this reason in most situations the superpotential fix the supercharges of the fields.

The lagrangian of super Yang-Mills theories is given by

$$\mathcal{L}_{SYM} = \frac{1}{32\pi i} \left(\int d^2\theta \left(\frac{\theta_{YM}}{2\pi} + \frac{4\pi i}{g^2} \right) W_\alpha W^\alpha \right) = \tag{A.33}$$

$$= \mathrm{Tr} \left[-\frac{1}{4} F_{\mu\nu} F^{\mu nu} - i\lambda \sigma^\mu D_\mu \bar{\lambda} + \frac{1}{2} D^2 \right] + \frac{\theta_{YM}}{32\pi^2} g^2 \mathrm{Tr} F_{\mu\nu} \tilde{F}^{\mu\nu} \tag{A.34}$$

where we defined the chiral superfield W_α as

$$W_\alpha = -\frac{1}{4} \bar{D}\bar{D} \left(e^{-2gV} D_\alpha e^{2gV} \right) \qquad \bar{W}_{\dot{\alpha}} = -\frac{1}{4} DD \left(e^{2gV} \bar{D}_{\dot{\alpha}} V e^{-2gV} \right) \tag{A.35}$$

It can be demonstrated that W_α is chiral and is invariant under the supergauge transformation $V \to V + \bar{\Phi} + \Phi$ while the vector superfield V was not.

From a perturbative point of view the inclusion of the term proportional to $\theta_{YM} Tr F_{\mu\nu} \tilde{F}^{\mu nu}$ has no effect since it is proportional to a total derivative. It is a

parity violating term that differs from zero only in non trivial topological configurations of the field (instantons).
The matter lagrangian we introduced is not invariant under gauge transformation. The correct gauge invariant lagrangian is given by

$$\mathcal{L}_{matter} = \int d^2\theta d^2\bar{\theta}\ \bar{\Phi} e^{2gV}\Phi \tag{A.36}$$

The superpotential is not automatically invariant under gauge transformation. As a result only certain expression are allowed.
There's an additional supersymmetric invariant lagrangian that can be constructed in a gauge theory when the gauge group contains abelian factors. It is called Fayet-Iliopulos term and can be present for every ideal A of the gauge group

$$\mathcal{L}_{FI} = \sum_A \xi_A \int d^2\theta d^2\bar{\theta} V^A = \frac{1}{2}\sum_A \xi_A D^A \tag{A.37}$$

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