4D to 3D reduction of Seiberg duality for $SU(N)$ susy gauge theories with adjoint matter: a partition function approach

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Chapter 1

Introduction

After the triumph of the Standard Model in the seventies, many theoretical physicist shifted their attention to the understanding of the dynamics of strongly coupled quantum field theories.
In that period, in fact, quantum field theories were properly understood only by using perturbation theory, which is only applicable if the theory is weakly coupled such as QED or QCD at high energies. However, QCD is strongly coupled at energies lower than a few GeVs, thus making perturbative methods unavailable in this regime.
In this context, dualities between theories at strong and weak coupling gained a lot of attention because by using them, we are able to map observables from the perturbative weakly-coupled theory to non-perturbative theory at strong coupling.

The first example of strong-weak duality is the electric-magnetic duality of Maxwell's equations discovered by Dirac in 1931. He discovered that the equations of motion are invariant if electric and magnetic field are exchanged. In the presence of charges, the duality exchanges electric and magnetic charges.
For a quantum system be invariant under such duality he demonstrated that electric and magnetic charge need to be inversely proportional to each other. Dirac's quantization condition for particles that posses either electric or magnetic charge reads $eg = 2n\pi\hbar$, where e is the electric charge and g the magnetic charge.

A generalization of Dirac's electric-magnetic duality is the Montonen-Olive duality. It states that a supersymmetric $\mathcal{N} = 4$ quantum field theory in four dimensions with gauge group G is equivalent to another theory with gauge group $\tilde{G}$ and with complexified gauge coupling $\tau' = -\frac{1}{\tau}$ and with electric and magnetic degrees of freedom exchanged. The dual gauge group G' in some cases is different from the gauge group G of the original theory.

The most important duality in modern physics is the AdS/CFT correspondence. It is another example of strong-weak duality between a gravitational theory in an Anti-deSitter space (AdS) and a conformal gauge theory (CFT) which lives on the boundary of the AdS space.
It was first conjectured by Maldacena between II_B string theory on $AdS_5 \times S^5$ and $\mathcal{N} = 4$ Super Yang-Mills in four dimension. It is an example of the holographic principle, which states that gravitational theories have the same number of degrees of freedom as gauge theories in a lower number of dimensions.
As we anticipated, the correspondence can be used to map observables in the strongly coupled field theory to observables in the weakly coupled gravitational theory in Anti-deSitter. Many quantities that were computable on both sides were calculated as a check of the duality, but the correspondence has been used in many different situations to improve our understanding of strongly coupled field theories and, at the same time, of gravity.
It is now one of the more active areas of research and it was generalized to Anti-deSitter spaces in other dimensions and to non-conformal field theories by considering spacetime configurations that are only asymptotically Anti-deSitter. As a result, the AdS/CFT correspondence is more generally called gauge/gravity duality.
More practical applications of the correspondence ranged from the calculation of the viscosity of the quark-gluon plasma to phenomenological models of high temperature superconductors in condensed matter physics. Even if these line of research are relatively new, gauge/gravity duality can provide some insights on the behavior of strongly coupled systems in other branches of physics.

Seiberg duality is another example of electric-magnetic duality. It relates the infrared behavior of two different supersymmetric theories, the electric and the magnetic theory.
The original Seiberg features as the electric theory a $\mathcal{N} = 1$ four dimensional gauge theory with $SU(N_c)$ gauge group and with N_f quarks in the fundamental and antifundamental representation. The magnetic theory is a $SU(N_f - N_c)$ gauge theory with N_f fundamental and antifundamental quarks and with N_f^2 singlets fields.
They are called mesons because they have the same quantum numbers of the mesons that can be constructed in the electric theory by combining a quark with an antiquark. However, in the magnetic theory they are fundamental fields and they are not composite degrees of freedom as in the electric theory.
Both the electric and magnetic theories feature a strongly-interacting superconformal fixed point for a particular range of the number of flavours and colours. The duality maps a strongly coupled theory into a weakly coupled one as the other

dualities discussed above. Therefore, we can use the magnetic degrees of freedom to analyse the properties of the electric theory where it is strongly coupled.
Many generalizations of Seiberg duality exist and they can be constructed by starting with a theory with a different gauge group or by adding matter fields in other representations.
Seiberg duality was initially discovered in four dimensional field theories but only a dew years later, a three dimensional analogue of Seiberg duality with $U(N_c)$ gauge group was found by Aharony.

For fifteen years it was believed that dualities in four and three dimensions were not linked to each other even if they had some striking similarities. A naive dimensional reduction, which is obtained by compactfiying a direction to a circle and then let the radius of the circle go to zero, is not compatible with the low-energy limit required by Seiberg duality in both teories. However, it is possible to relate four dimensional dualities to three dimensional ones if we keep the radius of the circle finite. If we flow to energies much lower than the inverse of the radius, the behavior of the theory will be effectively three dimensional, since the modes on the circle are too heavy to be excited.
The compactification results in an additional term in the superpotential that breaks the symmetries that are allowed in three dimensions but are forbidden in four, such as the axial symmetry.
As a result, by keeping the radius finite and flowing to low-energy we obtained a three dimensional duality with some of the symmetries broken by the superpotential term generated by the compactification. To restore those symmetries we can integrate out some quarks with large real masses. During this flow the superpotential term generated by the compactification goes to zero and we find a duality between three dimensional theories without additional the superpotential term.
We can apply this reasoning to various four dimensional dualities. In some cases we find already known dualities, while in other situations we discover unkown three dimensional dualities. Given that most of the three dimensional dualities have been derived from four dimensional dualities, it is conjectured that this is true for every three dimensional duality.

The process of dimensional reduction is understood in field theory in most cases, but some difficulties could arise anyway because of the non-perturbative dynamics of the theory.
However, there is an independent way to check the results obtained with these methods.
It is the partition function approach we will use in our work. It consists in the calculation of the superconformal index in four dimensions, for both dual theories.

The index is defined for supersymmetric field theories in four dimensions with one of the directions compactified into a circle. Mathematical identities guarantees that the indices of dual theories match.
The key property of the index is that if we shrink the radius of the circle to zero it reduces to the partition function of the theory in three dimensions with the superpotential term due to the compactification. At this point, we can perform the large mass flow we discussed previously in order to remove the superpotential. In this way we obtain the partition function of a three dimensional theory with all the symmetries that were broken by the superpotential term.
If we perform this procedure for both theories we obtain the expression of the partition functions, which are equal because of the identity of the indices in four dimensions. Moreover, the field content of the theory can be read easily from the expression of the partition function, since the symmetries are explicit and the superpotential can be read off from the constraints it imposes to the field charges.

Our work focuses on the dimensional reduction of Seiberg duality with $SU(N_c)$ and with an additional chiral field in the adjoint representation by using the partition function approach discussed earlier. This duality is known in literature as the Kutasov-Schwimmer-Seiberg (KSS) duality. Its three dimensional analogue is called Kim-Park duality and it has a similar matter content to the four dimensional duality with the addition of singlet fields in the magnetic theory that have the same charges of the monopole operators of the electric theory. The electric theory of Kim-Park is either $U(N_c)$ or $SU(N_c)$ which correspond to different magnetic theories.
The dimensional reduction of the KSS duality to the Kim-Park duality with $U(N_c)$ or $SU(N_c)$ gauge groups has been performed by Nii with field theory techniques. The scope of our work is to provide an independent check of his work by performing the dimensional reduction of KSS duality to $SU(N_c)$ Kim-Park duality on the partition function.
The advantage of this approach is that it is easier to work on the partition function rather than in field theory for theories with adjoint matter because their non-perturbative dynamics is not as well understood as SQCD theories yet . Whereas the reduction on the partition function to $U(N_c)$ Kim-Park duality has already been done in literature, the reduction of the $SU(N_c)$ case wasn't performed yet.
Our work showed that the KSS duality with $SU(N_c)$ gauge group reduced exactly to the Kim-Park duality with $SU(N_c)$ gauge group.
The identity between the electric and magnetic partition function is not yet demonstrated as a mathematical identity and it could be one of the situations where physics can show new non-trivial mathematical relations.

We begin our discussion in section 2 with a brief review of some of the properties of supersymmetric quantum field theories and the features of $SU(N_c)$ SQCD that will be needed in order to understand Seiberg duality. After these consideration we illustrate Seiberg and KSS duality.

In section 3 we consider the different properties of supersymmetric theories in three dimensions with respect to the four dimensional ones. Then, we introduce Aharony duality for $U(N_c)$ gauge group and Kim-Park duality for $U(N_c)$ or $SU(N_c)$ gauge groups.

In the fourth section we review the process of dimensional reduction in field theory, applying it to Seiberg duality, which results in a three dimensional duality that wasn't known before the introduction of this method. We consider also the flow to Aharony duality, obtained by gauging the baryonic symmetry. Then, we consider the reduction of KSS duality to Kim-Park with $U(N_c)$ and with $SU(N_c)$ gauge group.

In section 5 we give a brief introduction of the superconformal index, highlighting the key features that we will need in our analysis and we will also review the method of localization, which allows to compute partition functions exactly. Moreover, we give an overview of the dimensional reduction procedure on the index to the partition function.

The last section is dedicated to our work and contains the key steps of the dimensional reduction with the partition function approach. Most of the explicit calculations can be found in the appendix.

Chapter 2

Four dimensional dualities

2.1 Introduction

Supersymmetric quantum field theories enjoy an enlarged group of symmetries compared to other field theories. Since the symmetry group is a non-trivial combination of internal and spacetime symmetries, they have many unexpected features and new techniques were found to study them. Almost all of the new tools found are available only for supersymmetric field theories, making them the theatre for many advances in physics.

A more technical introduction on supersymmetry and its representation on fields can be found in appendix A.

In this section we will analyse more advanced features of supersymmetric field theories that have been used intensively in the discovery and in the analysis of electric magnetic duality and its generalisations.

2.1.1 General renormalization properties

A remarkable feature of supersymmetry is the constraint that the additional symmetry imposes on the renormalization properties of the theories.

One of the first aspects that brought attention to supersymmetry was that divergences of loop diagrams were milder because of the cancellation between diagrams with bosons and fermions running in the loops.

Nowadays we know powerful theorems that restrict the behaviour of supersymmetric field theories under renormalization. In order to preserve supersymmetry, the renormalization process has to preserve the Hilbert space structure. For example the wave function renormalization of different *particles* inside a multiplet must be the same, otherwise the renormalized lagrangian is not supersymmetric invariant anymore.

Moreover, in the supersymmetry algebra P^2 is still a Casimir operator i.e. it commutes with every operator in the algebra: particles in the same multiplet must have the same mass. Renormalization cannot break this condition, otherwise it would break supersymmetry.

For a *Super Yang Mills* theory with $\mathcal{N} = 1$ symmetry considerations lead to the additional requirement that gV, where g is the coupling and V is the vector superfield, cannot be renormalized.

Adding more supersymmetry the wave function renormalization of the various fields are even more constrained by symmetry. For example, for $\mathcal{N} = 4$ *SYM* the coupling constant is not renormalized at all.

Beta function for SYM and SQCD

Another nice feature of supersymmetric field theories is that some quantities can be calculated exactly. The first object of this kind that we encounter is the β-function of four dimensional $\mathcal{N} = 1$ *Super Yang Mills* theories with matter fields in representations R_i.

It is given by the exact *NSVZ β-function*

$$\beta(g) = \mu \frac{\mathrm{d}\, g}{\mathrm{d}\, \mu} = -\frac{g^3}{16\pi^2}\left[3\, T(Adj) - \sum_i T(R_i)(1-\gamma_i)\right]\left(1 - \frac{g^2\, T(Adj)}{8\pi^2}\right)^{-1} \tag{2.1}$$

where γ_i are the anomalous dimensions of the matter fields and $T(R_i)$ are the Dynkin indices[1] of their representation.
The anomalous dimensions are defined as

$$\gamma_i = -\mu \frac{\mathrm{d}\, \log(Z_i)}{\mathrm{d}\, \mu} \tag{2.2}$$

where Z_i is the wave function renormalization coefficient. The Dynkin indices of the gauge group $SU(N)$ for the fundamental and adjoint representation are

$$T(\mathrm{N}) = \frac{1}{2} \qquad T(\mathrm{Adj}) = N \tag{2.3}$$

The *NSVZ β-function* was first calculated using instanton methods in [1].

2.1.2 Superpotential: holomorphy and non-renormalization

Other than renormalization constraints, supersymmetry provides non-renormalization theorems for certain objects, such as the superpotential.

[1] The Dynkin index $T(R)$ of a representation R is defined as $\mathrm{Tr}\,(T^a T^b) = T(R)\delta^{ab}$ where T^a , T^b are the generators of the algebra in the representation R.

In [2] it has been demonstrated that the superpotential is tree-level exact, i.e. it does not receive correction in perturbation theory. However it usually receive contributions from nonperturbative dynamics. The superpotential features this property in theories with at least four supercharges and can be demonstrated independently using perturbative calculations or its holomorphic properties.
It was first demonstrated perturbatively, using the fact that for general supersymmetric field theories, supergraph loops diagrams with n external leg yield a term that can be written in the form

$$\int d^4x_1 \dots d^4x_n d^2\theta d^2\bar{\theta}\; G(x_1, \dots, x_n)\, F_1(x_1, \theta, \bar{\theta}) \dots F_n(x_n, \theta, \bar{\theta}) \tag{2.4}$$

where F_i are given by products of superfields and their covariant derivatives and $G(x_1, \dots, x_n)$ is translationally invariant function.
The importance of this result is that all contribution from Feynman diagrams are given by a single integral over full superspace ($d^2\theta d^2\bar{\theta}$) whereas the superpotential must be written as an integral in half-superspace($d^2\theta$ only) of chiral fields. Exploiting the fact that a product of chiral fields is a chiral field, the most general form of a superpotential is

$$W(\lambda, \Phi) = \sum_{n=1}^{\infty} \left(\int d^2\theta\, \lambda_n \Phi^n \;+\; \int d^2\bar{\theta}\, \lambda_n^\dagger \bar{\Phi}^n \right) \tag{2.5}$$

The second term of the superpotential is added in order to give a real lagrangian after the integration in superspace. From the definition, we can see that the superpotential is holomorphic in the fields and in the coupling constants.
An alternative proof of this theorem was provided by Seiberg [3] using a different approach. He noted that the coupling constants λ_n can be treated as background chiral superfields with no dynamics.
Using this observation we can assign transformation laws to the coupling constants, making the lagrangian invariant under a larger symmetry. Fields and coupling constants are charged under this symmetry and only certain combinations of them can appear in the superpotential. In addition, in a suitable weak coupling limit the effective superpotential must be identical with the tree-level one. These conditions, taken together, fix the expansion of the superpotential to the expression of the tree-level potential. A more detailed discussion can be found in [4] and [5].

2.1.3 Moduli space

The *classical moduli space* is the set of vacuum expectation values of the scalar fields that correspond to a zero of the scalar potential. The zeroes of the scalar potential can be found by solving to two different sets of equations, called $D{-}term$

and $F - term$ equations

$$\bar{F}^i(\phi) = 0 \qquad D^a(\phi, \bar{\phi}) = 0 \tag{2.6}$$

F-term equations are present only if there is a superpotential while the *D-term* equations are associated to the kinetic part of the lagrangian.
If the minimum of the scalar potential is different from zero the vacuum is not supersymmetric. In this case supersymmetry is spontaneously broken. Another possible situation is that the scalar potential has no minimum at all: the theory does not have any stable vacua.
If it is not possible to find vacuum expectation values that correspond to a zero in the scalar potential, then the theory does not posses a moduli space.
Gauge transformations should be taken into account in order to avoid redundance in the description. The moduli space describes physically inequivalent vacua, since the mass spectrum of the theory depends on the *VEVs* of the scalar fields, which differ in every point of the moduli space.
Because of supersymmetry, radiative corrections do not lift the energy of the ground state and the vacuum remains supersymmetric. As a result, only superpotentials generated from nonperturbative dynamics can lift the moduli space. We will see examples of this phenomenon in the analysis of SQCD.
An alternative description of the space of classical *D-flat* directions is given by the space of all holomorphic gauge invariant polynomials of scalar fields modulo classical relations between them [6]. As a result, gauge invariant polynomials of operators parametrize the classical moduli space of the theory. Using this description it's easier to find the moduli space of the theory in consideration. If a superpotential is present, *F-term* equations should be imposed on the gauge invariant polynomial used to describe *D-flat* direction. We will use this convenient description in the next chapters.

2.1.4 Phases of gauge theories

The dynamics of gauge theories can be classified according to the low-energy effective potential $V(R)$ between two test charges separated by a large distance R.

The possible forms of the potential, up to addictive constant, are

$$\text{Coulomb} \qquad V(R) \sim \frac{1}{R} \tag{2.7}$$

$$\text{free electric} \qquad V(R) \sim \frac{1}{R\,\log(R\Lambda)} \tag{2.8}$$

$$\text{free magnetic} \qquad V(R) \sim \frac{\log(R\Lambda)}{R} \tag{2.9}$$

$$\text{Higgs} \qquad V(R) \sim \text{constant} \tag{2.10}$$

$$\text{confining} \qquad V(R) \sim \sigma\, R \tag{2.11}$$

The first three phases feature massless gauge fields and their potential is $V(R) \sim g^2(R)/R$ and they differ because of the renormalization of the charge in the infrared. In the Coulomb phase, g^2_{IR} = constant, while in the free abelian/non-Abelian phase the coupling constant goes to zero as $g^2(R) \sim 1/\log(R\,\Lambda)$. The free electric phases is possible for abelian or non-Abelian theories. In the latter case for asymptotically free theories it is necessary that the renormalization group has a non-trivial infrared fixed point. The free magnetic phases is generated by magnetic monopoles acting as source of the field. Since magnetic and electric charges are related by Dirac quantization condition, the running of the coupling constant for magnetic monopoles is the inverse of electric charges.
The situation is completely different in the last two cases. In the Higgs phase gauge fields are massive and the potential is given by a Yukawa potential, exponentially suppressed at long distances that results in a constant value. The confining phase can be described by tube of confined gauge flux between the charges which, at large distances, acts as a string with costant tension, yielding a linear potential.

2.1.5 't Hooft anomaly matching conditions

The 't Hooft anomaly matching conditions are a great tool to investigate the global symmetries of the low-energy degrees of freedom of the theory.
Let's consider an asymptotically free gauge theory with global symmetry group G. While gauge symmetries cannot be anomalous because that would spoil the unitarity of the theory, this condition does not apply to anomalies involving only global symmetries.
We can compute the triangle anomaly for the global symmetry group in the ultraviolet and we will call it A_{UV}. After weakly gauging G we introduce additional fermions that are charged only under G in order to cancel the anomaly, since now it is a gauge symmetry.
Flowing towards the infrared, the anomaly is still zero if the global symmetry group is not broken. After constructing the low-energy effective field theory, we

can calculate the triangle anomalies for the group G involving the composite low energy fields, which results in the term A_{IR}. The contribution of the additional fermions remains identical since they are massless.

$$0 = A_{IR} + A_F = A_{UV} + A_F \quad \rightarrow \quad A_{IR} = A_{UV} \tag{2.12}$$

The anomaly coefficient can be easily computed since it is proportional to the group theoretical factor

$$A = \text{Tr }(T^a\{T^b, T^c\}) \tag{2.13}$$

Summarizing the result, we found that if the global symmetry group is not broken by the strong dynamics, triangle anomalies involving only the global symmetry group should be equal in the ultraviolet and in the infrared.
We will use these anomaly matching conditions to find if two dual theories are invariant under the same global symmetries in the IR as an additional check of electric-magnetic duality.

2.2 Seiberg duality

Electric magnetic duality relates the dynamics of two different gauge theories in their infrared fixed point. Even though the dual theories have different particle content, they describe the same IR physics. Moreover, whenever one of the dual theories gets more strongly coupled, the other becomes more weakly coupled. This is particularly useful because it provides an alternative, weakly coupled description of the original theory.

2.2.1 Electric theory: $SU(N)$ SQCD with N_f flavours

We will start our analysis on eletric-magnetic duality studying the first pair of theories that were discovered to be dual in [7]. We are going to analyse the properties of these theory in order to better understand the features of the duality. The electric theory is a $SU(N_c)$ supersymmetric gauge theory with N_f massless flavours. Its anomaly-free global symmetry group is

$$SU(N_f)_L \times SU(N_f)_R \times U(1)_B \times U(1)_R \tag{2.14}$$

The axial symmetry gives rise to an anomaly through the triangular graph $U(1)_A \times SU(N_c)^2$, hence it is not a symmetry of the theory.
The classical lagrangian in superspace language is given by

$$\mathcal{L} = \tau \int d^2\theta \, Tr(W_\alpha W^\alpha) + \text{h.c.} + \int d^2\theta \, d^2\bar{\theta} \, (Q^i)^\dagger e^V Q_i + \int d^2\theta \, d^2\bar{\theta} \, (\tilde{Q}^{\tilde{i}})^\dagger e^{-V} \tilde{Q}_{\tilde{i}} \tag{2.15}$$

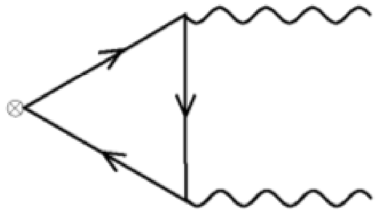


Figure 2.1: Feynman graphs contributing to the R-symmetry anomaly

Q and $\tilde{Q}$ represent left and right quark superfield respectively. The charges of the fields are given in the table below.

	$SU(N_c)$	$SU(N_f)_L$	$SU(N_f)_R$	$U(1)_B$	$U(1)_R$
Q	N_c	N_F	1	1	$\frac{N_f-N_c}{N_f}$
$\tilde{Q}$	$\overline{N_c}$	1	$\overline{N_F}$	-1	$\frac{N_f-N_c}{N_f}$

(2.16)

The value of the R-charge is fixed by the triangle anomaly $SU(N_c)^2\, U(1)_R$, which can be calculated by considering diagrams with two exiting gluons and the R-symmetry current inserted in the other vertex. Each fermion in the theory contributes to the anomaly which, as a result, is proportional to the R-charge of the fermions running in the loop and the Dynkin index of its representation. Requiring the anomaly to vanish, we find

$$R_{gaugino}T(\text{Ad}) + \sum_f (R_f - 1)T(r) = 0 \tag{2.17}$$

$$N_c + \frac{1}{2}2N_f(R_Q - 1) = 0 \quad \rightarrow \quad R_Q = \frac{N_f - N_c}{N_f} \tag{2.18}$$

where we set the gaugino R-charge to 1 in order to have gluons not charged under R-symmetry.
The anomaly free condition for the R-symmetry leads to a unique set of R-charges. As a result, we found the R-charges at the superconformal infrared point of the theory.

Classical moduli space

Consdering that the theory has no superpotential, the classical moduli space of the theory is given by *D-terms* only. They can be read from the on-shell lagrangian and are given by

$$D^a = g\left(Q^{*i}T^aQ_i - \tilde{Q}^{*i}T^a\tilde{Q}_i\right) = 0 \tag{2.19}$$

where T^a are the gauge group generators in fundamental or antifundamental representation and i is a flavour index.
After considering gauge and global symmetries, the squark *VEVs*, represented as $N_f \times N_c$ matrices, which satisfy the D-term equation are given by

- for $N_f \leq N_c$ and a_i generic

$$Q = \tilde{Q} = \begin{pmatrix} a_1 & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & a_2 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & \ddots & 0 & \vdots & \dots & \vdots \\ 0 & 0 & 0 & a_{N_f} & 0 & \dots & 0 \end{pmatrix} \tag{2.20}$$

- for $N_f \geq N_c$ and $|a_i|^2 - |\tilde{a}_i|^2 = a$ independent of i.

$$Q = \begin{pmatrix} a_1 & 0 & 0 & 0 \\ 0 & a_2 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & a_{N_c} \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix} \quad \tilde{Q} = \begin{pmatrix} \tilde{a}_1 & 0 & 0 & 0 \\ 0 & \tilde{a}_2 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \tilde{a}_{N_c} \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix} \tag{2.21}$$

For $N_f \leq N_c$ in a generic point of the moduli space the gauge group is broken to $SU(N_c - N_f)$ while for $N_f \geq N_c$ is broken completely. The gauge group breaks through the super Higgs mechanism: every broken generator is absorbed by the associated massless vector superfield. This process gives mass to the vector superfield. [2] The mass of the gauge superfield is given by the VEVs of the squarks.
As we said in section 2.1.3, we can study the classical moduli space by finding holomorphic gauge invariant polynomials in the operators and modding out classical relations between them. For $N_f < N_c$ we can only construct *mesons* out of squarks

$$M^i_j = Q^i \tilde{Q}_j \tag{2.22}$$

where color indices are summed. Mesons have maximal rank because $N_f < N_c$ and there are no classical constraints to impose on them.
When $N_f \geq N_c$ the mesons cannot have maximal rank anymore, it can be at most N_c. There are additional gauge invariant operators that can be constructed:

[2] Remember that massive representation of supersymmetry have twice the degrees of freedom of massless ones, because in the latter half of the supercharges are represented trivially.

baryons, that are defined as

$$B_{i_1,\ldots,i_{N_f-N_c}} = \epsilon_{i_1,\ldots,i_{N_f-N_c},j_1,\ldots,j_{N_c}} \epsilon^{a_1,\ldots,a_{N_c}} Q^{j_1}_{a_1} \ldots Q^{j_{N_c}}_{a_{N_c}} \tag{2.23}$$
$$\tilde{B}^{i_1,\ldots,i_{N_f-N_c}} = \epsilon^{i_1,\ldots,i_{N_f-N_c},j_1,\ldots,j_{N_c}} \epsilon_{a_1,\ldots,a_{N_c}} \tilde{Q}^{a_1}_{j_1} \ldots \tilde{Q}^{a_{N_c}}_{j_{N_c}} \tag{2.24}$$

Mesons and baryons can be written down using the *VEVs* we found solving the *D-term* equations (ignoring null components for baryons)

$$M = \begin{pmatrix} a_1\tilde{a}_1 & 0 & \ldots & 0 & 0 & \ldots & 0 \\ 0 & a_2\tilde{a}_2 & 0 & \ldots & 0 & \ldots & 0 \\ \vdots & 0 & \ddots & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & 0 & a_{N_c}\tilde{a_{N_c}} & 0 & \ldots & 0 \\ 0 & 0 & 0 & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{2.25}$$
$$B \simeq a_1 a_2 \ldots a_{N_c} \tag{2.26}$$
$$\tilde{B} \simeq \tilde{a}_1 \tilde{a}_2 \ldots \tilde{a_{N_c}} \tag{2.27}$$

We can see that if the mesons have rank less than N_c, then B or $\tilde{B}$ (or both) has to vanish and the other has rank one. If the mesons' rank is N_c both B and $\tilde{B}$ have rank one.
There are classical constraints that should be imposed between mesons and baryons, but depend on the number of flavours . For example for $N_f = N_c$ the classical constraint is $det\,(M) - B\tilde{B} = 0$.
Singularities of the moduli space can be investigated using the gauge invariant description we have just introduced. The part of the lagrangian that describes flat directions can be written in terms of mesons and baryons. The lagrangian involving mesons features a non-trivial Kahler potential that reads

$$K = 2\mathrm{Tr}\sqrt{M^\dagger M} \tag{2.28}$$

that generates a singular metric whenever the meson matrix is not invertible. This happens when some of the VEVs are zero, i.e. in points of the moduli space of enhanced gauge symmetry. The appearance of this singularities is related to the fact that some (or all) gluons are now massless and should be included in the low-energy description.

Quantum moduli space

Quantum dynamics modifies the structure of the moduli space of the theory in a different way depending on the number of colours and flavours.

Pure SYM For pure *Super Yang Mills*, i.e. no quarks, the theory exihibit a discrete set of N_c vacua. Without quarks a non-anomalous R-symmetry cannot be found, and the R-symmetry is broken down to the discreet symmetry $\mathbb{Z}_{2N_c}$. Using holomorphy and symmetry arguments, the form of the non-perturbative potential can be found and it can be shown that it induces the gaugino to condensate [8], meaning that

$$\langle \lambda\lambda \rangle = -\frac{32\pi^2}{N_c} a\, \Lambda^3 \tag{2.29}$$

where Λ is the dynamically generated scale of the theory defined as

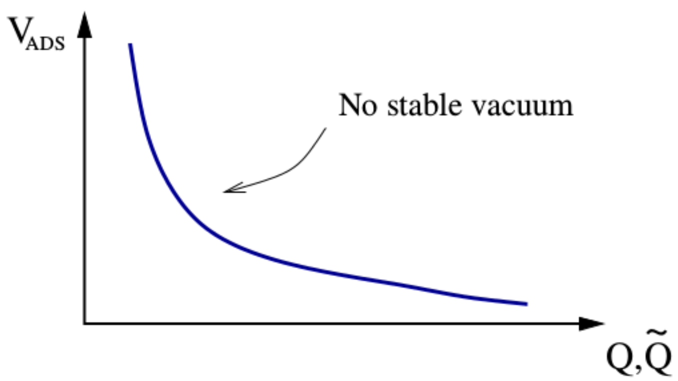
$$\Lambda = \mu e^{-\frac{2\pi i \tau}{b_0}} \qquad \tau = \frac{4\pi i}{g^2(\mu)} + \frac{\theta_{YM}}{2\pi} \qquad b_0 = 3N_c - N_f \tag{2.30}$$

where τ is the complexified gauge coupling. $|\Lambda|$ is defined as the scale at which the coupling constant diverges.
The gaugino condensation breaks R-symmetry to $\mathbb{Z}_2$ and in fact there are N_c physically different vacua labelled by different phases of the gaugino condensate.

$\mathbf{N_f < N_c}$ The quantum corrections for *SQCD* with $N_f < N_c$ flavours completely lift the moduli space through the Affleck-Dine-Seiberg (ADS) superpotential ([9][10]) which reads

$$W_{eff} = (N_c - N_f) \left(\frac{\Lambda^{3N_c - N_f}}{\det M} \right)^{\frac{1}{N_c - N_f}} \tag{2.31}$$

It is the only superpotential that is compatible with the symmetries of the theory and with the other properties of the superpotential we introduced in section 2.1.2. We can see that the ADS superpotential does not exist for $N_f \geq N_c$ because the exponent diverges for $N_f = N_c$ or the determinant vanishes for $N_f \geq N_c$ because mesons do not have maximal rank. Note that this superpotential is non-perturbative and thus it is not in contrast with the renormalization theorem of section 2.1.2. The effect of this superpotential is that the theory does not have ground state. The slope of the potential goes to zero only for $\det M \to \infty$.



This situation is the perfect example where, unlike the classical moduli space, quantum corrections lift completely the moduli space and the theory does not posseses a vacuum anymore.

$\mathbf{N_f = N_c}$ When the number of flavours is equal the number of colours of the theory, the classical moduli space was subject to the constraint

$$\det M - B\tilde{B} = 0 \tag{2.32}$$

In the quantum corrected moduli space mesons and baryons satisfy [11]

$$\det M - B\tilde{B} = \Lambda^{2N_c} \tag{2.33}$$

which flows to the classical constraint in the classical limit ($\Lambda \to 0$).

The effect of this relation is that the origin does not belong to the moduli space anymore and the moduli space is now smooth. For large expectation values of M, B and $\tilde{B}$ the classical and the quantum moduli space look similar, while in the origin of the moduli space quantum corrections modify drastically the structure of the moduli space. Moreover, the subspace with B or $\tilde{B}$ equal to zero, is not

Figure 2.2: Classical and quantum 2D slice of the moduli space near the origin for $N_f = N_c$

singular anymore while classically the meson matrix was constrained to have zero determinant.
The fact that the origin of moduli space is not in the quantum-corrected moduli space, implies that the global symmetry group is necessarily broken in some way, depending on the position of the moduli space

$$M^i_j = \Lambda^2 \delta^i_j \quad B = \tilde{B} = 0 \quad \rightarrow \quad SU(N_f)_V \times U(1)_B \times U(1)_R \tag{2.34}$$

$$M^i_j = 0 \quad B = -\tilde{B} = \Lambda^{N_c} \quad \rightarrow \quad SU(N_f)_L \times SU(N_f)_R \times U(1)_R \tag{2.35}$$

where $SU(N_f)_V$ is the diagonal vector subgroup.

$\mathbf{N_f = N_c + 1}$ In the case $N_f = N_c + 1$ the classical moduli space is constrained by

$$\det M \left(\frac{1}{M}\right)^j_i - B_i \tilde{B}^j = 0 \qquad M^i_j B_i = M^i_j \tilde{B}^j = 0 \tag{2.36}$$

and quantum corrections do not modify it. In the previous section we noted that the singularities in the classical moduli space are associated to the appearance of massless gluons. In the quantum picture, the interpretation of the singularities is different: they are associated with additional massless mesons and baryons. At the origin of the moduli space the theory is strongly coupled and the global symmetry (2.14) is unbroken and it can be checked using 't Hooft anomalies [11]. Far from the origin, the mesons and baryons interact with the potential

$$W = \frac{1}{\Lambda^{2N_c - 1}} \left(M^i_j B_i \tilde{B}^j - \det M \right) \tag{2.37}$$

that enforce the classical constraints (2.36) through the equations of motion. For large VEVs of the fields, mesons and baryons acquire large mass through the superpotential.

$\mathbf{N_f > N_c + 1}$ Starting from $N_f = N_c + 2$ it is not possible to construct a physical superpotential out of gauge invariant operators, in analogy to the previous cases. The only $SU(N_f)_L \times SU(N_f)_R$ invariant superpotential that can be written is given by

$$W_{eff} \sim \det M - B_{ij} M^i_k M^j_l \tilde{B}^{kl} \tag{2.38}$$

because baryons have two flavour indices. However this superpotential does not have R-charge equal to two and if we add more flavours we should add other mesons to the superpotential. Therefore the classical moduli space is not modified by quantum corrections. As a result, near the origin the quantum corrected moduli space looks identical to the classical one. Unlike the case with $N_f = N_c + 1$ the

singularities in the moduli space cannot be interpreted as massless mesons and baryons and an effective description of these operator is singular [11]. Considering that the 't Hooft anomaly matching conditions are not satisfied in the singular points it is clear that a description using mesons and baryons is not correct.
To find a description of the low-energy degrees of freedom of the theory we will use Seiberg duality, which provides an alternative description of the theory.

Conformal window Thetheory is not asympotically free in the range $\frac{3}{2}N_c < N_f < 3N_c$. This can be seen by using the *NSVZ* β function(2.3), which reads

$$\beta(g) = -\frac{g^3}{16\pi^2}\frac{3N_c - N_f + N_f\gamma(g^2)}{1 - N_c\frac{g^2}{8\pi^2}} \tag{2.39}$$

$$\gamma(g^2) = -\frac{g^2}{8\pi^2}\frac{N_c^2 - 1}{N_c} + \mathcal{O}(g^4) \tag{2.40}$$

The β function is known to have a Banks-Zaks fixed point [12] in the 't Hooft limit with $\frac{N_f}{N_c} = 3 - \epsilon$ held fixed and $\epsilon \ll 1$. However, the fixed point exists in the range of values $\frac{3}{2}N_c \geq N_f \geq 3N_c$ with N_f and N_c finite. This is possible because one loop and two loop contributions to the beta function have opposite signs. As a result, the infrared theory is a non-trivial four dimensional superconformal theory. The infrared degrees of freedom are quarks and gluons that are not confining but are interacting as massless particles. The theory is in a free non-Abelian Coulomb phase.
The fact that the theory is superconformal implies that we have further restrictions on the algebra of operators [3]. Superconformal algebra imposes that the dimension of every operator satisfy this inequality involving the R-charge

$$D \geq \frac{3}{2}\,|R| \tag{2.41}$$

where the bound is satured for chiral fields.
The product of two chiral operator is constrained by this fact. Consider that for two operators O_1, O_2 we have $R(O_1O_2) = R(O_1) + R(O_2)$. Because of the bound (2.41), the scaling dimension of chiral operators is additive $D(O_1O_2) = D(O_1) + D(O_2) = \frac{3}{2}R(O_1) + \frac{3}{2}R(O_2)$. Note that the dimension of the operator is quantum corrected, i.e. contains the anomalous dimension of the operator.
Given that the superconformal R-symmetry is not anomalous and commutes with the global symmetry group of the theory, then it must be the R-symmetry that

[3] R-symmetry is part of the superconformal algebra instead of being an automorphism of the algebra, as in superPoincarè algebra.

appears in table 2.16. Because of (2.41), the gauge invariant operators we defined previously must have the following dimensions

$$D(Q\tilde{Q}) = \frac{3}{2}R(Q\tilde{Q}) = 3\frac{N_f - N_c}{N_f} \tag{2.42}$$

$$D(B) = D(\tilde{B}) = \frac{3}{2}N_c\frac{N_f - N_c}{N_c} \tag{2.43}$$

Gauge invariant operators should be in unitary representation of the superconformal algebra. Unitarity imposes that in general $D \geq 1$ and the equality holds for singlet fields. From the previous equation we can verify that $D(M) \geq 1$ for $N_f \geq \frac{3}{2}N_c$ and it becomes a free field for $N_f = \frac{3}{2}N_c$.
For fewer number of flavours, the meson field is inconsistent with the unitarity bound. The theory is conjectured to flow to a different phase.

$\mathbf{N_f > 3N_c}$ In this range, quarks prevail on gluons and change the sign of the β function. This is caused by the *charge screening* effect of quarks, that make the coupling constant smaller at larger distances.
The theory is in a free non-Abelian electric phase. Its behaviour is not well defined at high energies because of the presence of a Landau pole at $R \sim \Lambda^{-1}$, although the theory can be a good description of a low-energy limit of another theory.

2.2.2 Magnetic theory

The magnetic theory is a *SQCD* theory with the same global symmetries as the electric theory, but with gauge group $SU(\tilde{N}_c = N_f - N_c)$. In addition there are N_f^2 color singlets, that we will call mesons, Considering that they have the same properties as the mesons we can construct in the electric theory. In the magnetic theory they are fundamental fields i.e. they are not written as gauge invariant operators from quarks. Given that they are gauge invariant, they interact only through the superpotential

$$W = M^i_{\tilde{j}}\, q_i\, \tilde{q}^{\tilde{j}} \tag{2.44}$$

where we represented dual quarks as $q, \tilde{q}$ and mesons as $M^i_{\tilde{j}}$.
The charges of fields of the magnetic theory are

	$SU(N_c)$	$SU(N_f)_L$	$SU(N_f)_R$	$U(1)_B$	$U(1)_R$
q	N_c	$\overline{N_F}$	1	$\frac{N_c}{N_f - N_c}$	$\frac{N_c}{N_f}$
$\tilde{q}$	$\overline{N_c}$	1	N_F	$-\frac{N_c}{N_f - N_c}$	$\frac{N_c}{N_f}$
$M^i_{\tilde{j}}$	1	N_f	$\overline{N_f}$	0	$2\frac{N_f - N_c}{N_f}$

(2.45)

Dual quarks are in opposite representations of flavour symmetries.

Mesons in the magnetic theory have the same charges of the mesons constructed from electric quarks. Baryons constructed from dual quarks have the same baryonic charge as the electric baryons and are mapped into each other by the duality. It can be demonstrated that they are proportional to each other.
Similarly to the electric theory, the R-charges can be chosen in order for the R-symmetry to be non-anomalous.
However, the R-charges can be found by imposing the duality. If we use the fact that the mesons are built from electric quarks, their R-charge is twice the R-charge of electric quarks. The R-charges of the dual quarks can be found by imposing that the term $Mq\tilde{q}$ of (2.44) has R-charge two. In this way, we found the R-charges at the superconformal infrared fixed point.

Phases of the magnetic theory

In order to study the phases of the dual theory we can map the various ranges of flavours of the electric theory into ranges for the magnetic theory by considering the relation $\tilde{N}_c = N_f - N_c$

$$\begin{array}{cccc} \text{Electric} & N_c + 2 \leq N_f \leq \frac{3}{2}N_c & \frac{3}{2}N_c < N_f < 3N_c & N_f \geq 3N_f \\ \text{Magnetic} & N_f \geq 3N_c & \frac{3}{2}N_c < N_f < 3N_c & N_c + 2 \leq N_f \leq \frac{3}{2}N_c \end{array} \tag{2.46}$$

As we can see from the table, both the electric and magnetic theories are in the conformal windows for the same values of N_f, N_c. In the other two intervals, the dual theories are in opposite phases.
For $N_c + 2 \leq N_f \leq \frac{3}{2}N_c$ the electric theory is strongly coupled and we can't describe its dynamics. On the other hand, the magnetic theory is free and can be used to describe the physics for this range of flavours.
When the electric theory is in a free non-Abelian phases ($N_f \geq 3N_c$), the magnetic theory is strongly coupled.
This feature of Seiberg duality is particularly useful, because it gives us a weakly coupled description of a strongly-coupled theory.

Duality

In the *conformal window* the electric and magnetic theories we introduced previously give an equivalent description of the same physics in the infrared. In this range, the magnetic theory has a non-trivial infrared fixed point too. At this fixed point, the superpotential (2.44) is a relevant perturbation because it has dimension $D = 1 + 3\frac{N_c}{N_f} < 3$ and it drives the theory to a new fixed point.
Electric mesons have different dimension in the UV from the magnetic ones because they are constructed from a pair of electric quarks. For this reason it is

necessary to introduce an energy scale μ in order to match their dimension in the UV: $M = \mu M_m$, where M_m are the magnetic mesons.
The matching of the scales between the electric and mangetic theory results in the following relation between them [7]

$$\Lambda^{3N_c-N_f}\tilde{\Lambda}^{3(N_f-N_c)-N_f} = (-1)^{N_f-N_c}\mu^{N_f} \tag{2.47}$$

The consequences of this equation are that when one theory is strongly coupled, the other is weakly coupled, as we noted in the previous section. Moreover, it ensures that the dual of the dual theory is the electric theory itself. The dual of the dual magnetic theory is a $SU(N_c)$ $SQCD$ theory with scale Λ, d^i and $\tilde{d}_{\tilde{j}}$ quarks and additional singlets $M^i_{\tilde{j}}$ and $N^{\tilde{j}}_i = q_i q^{\tilde{j}}$ with superpotential

$$W = \frac{1}{\tilde{\mu}} N^{\tilde{j}}_i d^i \tilde{d}_{\tilde{j}} + \frac{1}{\mu} M^i_{\tilde{j}} N^{\tilde{j}}_i = \frac{1}{\mu} N^{\tilde{j}}_i \left(-d^i d_{\tilde{j}} + M^i_{\tilde{j}} \right) \tag{2.48}$$

considering that from the previous relation we have $\tilde{\mu} = -\mu$.
Meson fields are massive and can be integrated out by using their equation of motion,which result in

$$N^{\tilde{j}}_i = 0 \qquad M^i_{\tilde{j}} = d^i d_{\tilde{j}} \tag{2.49}$$

Considering that the dual theories describe the same physics, there should be a mapping of gauge invariant operators between them. We already saw that electric and magnetic mesons match in the infrared. A mapping should exists also for baryonic operators. Indeed it does and it is given by

$$\begin{aligned} B^{i_1 \dots i_{N_c}} &= C\ \epsilon^{i_1 \dots i_{N_c} j_1 \dots i_{N_f - N_c}} b_{j_1 \dots j_{N_f - N_C}} \\ \tilde{B}^{i_1 \dots i_{N_c}} &= C\ \epsilon_{\tilde{i}_1 \dots \tilde{i}_{N_c} \tilde{j}_1 \dots \tilde{i}_{N_f - N_c}} b_{\tilde{j}_1 \dots \tilde{j}_{N_f - N_C}} \\ \text{where} \qquad C &= \sqrt{-(-\mu)^{N_c - N_f} \Lambda^{3N_c - N_f}} \end{aligned} \tag{2.50}$$

using (2.47) it can be shown that these mappings preserve the involutive action of the duality.

As an additional check of the duality, 't Hooft anomaly matching conditions have been calculated in [7] for the electric and magnetic theories for the various global symmetries and in both theories they are given by

$$\begin{aligned}
SU(N_f)^3 &\longrightarrow N_c \\
U(1)_R\, SU(N_f)^2 &\longrightarrow -\frac{N_c^2}{2N_f} \\
U(1)_B\, SU(N_f)^2 &\longrightarrow \frac{N_c}{2} \\
U(1)_R &\longrightarrow -N_c^2-1 \\
U(1)_R^3 &\longrightarrow -N_c^2-1-2\frac{N_c^4}{N_f^2} \\
U(1)_B^2 U(1)_R &\longrightarrow -2N_c^2
\end{aligned} \tag{2.51}$$

Another important check of the duality is that it remains valid under mass perturbations of the electric theory. Suppose to add a superpotential term that gives mass to the quark in the last flavour and is equal to

$$W_{mass}^{el} = m\, Q_{N_f}\tilde{Q}^{N_f} \tag{2.52}$$

Flowing to the IR the number of quarks is decreased by one, driving the theory to a more strongly coupled fixed point[4]. The new scale of the theory is given in terms of the old one by

$$\Lambda_L^{3N_c-(N_f-1)} = m\, \Lambda^{3N_c-N_f} \tag{2.53}$$

In the magnetic theory the mass perturbation is mapped in the term

$$W_{mass}^{mag} = m\, M_{N_f}^{N_f} \tag{2.54}$$

Because of this term, the gauge group gets higgsed to $SU(N_f-1-N_c)$ and only N_f-1 light quarks remain in the theory.
The scale of the magnetic theory Λ_L is modified and reads

$$\tilde{\Lambda}^{3(N_f-N_c-1)-(N_f-1)} = -\frac{\tilde{\Lambda}^{3(N_f-N_c)-N_f}}{< q_{N_f}\tilde{q}^{N_f} >} \tag{2.55}$$

where $< q_{N_f}\tilde{q}^{N_f} >= -\mu m$ because of the equation of motion for the massive flavour.
As expected the magnetic theory becomes more weakly coupled.
We conclude that the duality is preserved under massive deformations.

[4]Considering that matter in the fundamental representation contributes with a positve term it is easy to see that this is true.

2.3 Kutasov-Schwimmer-Seiberg duality

A possible generalization of Seiberg duality can be found by adding matter fields in different representations of the gauge group. Kutasov, Schwimmer and Seiberg ([13], [14], [15]) considered $SU(N)$ SQCD with the addition of a matter field in the adjoint representation of the gauge group and found that it admitted a magnetic dual.

2.3.1 Electric theory

The classical electric theory can be summarized by the following charge table of charges under the global symmetry group $SU(N_f)_L \times SU(N_f)_R \times U(1)_B \times U(1)_R$

	$SU(N_c)$	$SU(N_f)_L$	$SU(N_f)_R$	$U(1)_B$	$U(1)_R$
Q	N_c	N_f	1	1	$1-\frac{2}{k+1}\frac{N_c}{N_f}$
$\tilde{Q}$	$\overline{N_c}$	1	$\overline{N_f}$	-1	$1-\frac{2}{k+1}\frac{N_c}{N_f}$
X	N_c^2-1	1	1	0	$\frac{2}{k+1}$

(2.56)

and by the addition of the superpotential

$$W_{Adj} = g_k \mathrm{Tr}\, X^{k+1} \tag{2.57}$$

The superpotential (2.57) drives the theory to a new infrared fixed point. It is either relevant or dangerously irrelevant, depending on the value of k [14].
Being dangerously irrelevant means that in the ultraviolet it is an irrelevant term, whose coupling gets weaker along the renormalization group flow. However, at some point in the flow, the fields acquire a dimension such that it becomes a relevant perturbation, getting stronger along the flow and driving the theory to a different fixed point in the infrared.

The theory posseses two different R-symmetries but the superpotential breaks explicitly one of them and imposes that the adjoint matter field has R-charge $\frac{2}{k+1}$.

The remaining R-charges can be fixed by imposing that the R-symmetry is anomaly free as we did previously. Using formula (2.17), considering also the fermion in the adjoint matter multiplet we have

$$\begin{gathered} N_c + (R_Q - 1)\frac{1}{2}2N_f + (R_X - 1)N_c = 0 \\ (R_Q - 1)N_f = -R_X N_c \quad \longrightarrow \quad R_Q = 1 - R_X \frac{N_c}{N_f} \end{gathered} \tag{2.58}$$

Imposing this condition the R-charges of the fields were fixed completely, as in Seiberg duality. This has been possible because the superpotential (2.57) fixed independently the R-charge R_X. Otherwise the condition (2.58) fixes R_Q as a

function of R_X with R_X generic.
The gauge invariant operators that can be constructed are mesons and baryons multiplied with powers of the adjoint field. Meson operators are given by

$$(M_j)^i_{\tilde{i}} = \tilde{Q}_{\tilde{i}} X^j Q^i \qquad j = 0, 1, \dots, k-1 \tag{2.59}$$

Baryons are more easily introduced by first defining "dressed quarks"

$$Q_{(l)} = X^l Q \qquad l = 0, \dots, k-1 \tag{2.60}$$

Baryons are defined as

$$B^{i_1, i_2, \dots, i_k} = Q^{i_1}_{(0)} \dots Q^{i_k}_{(k-1)} \qquad \text{with } \sum_{l=1}^{k} i_l = N_c \tag{2.61}$$

with color indices contracted with an ϵ tensor.

Vacuum structure

In analogy with the condition $N_f \geq N_c$ in *SQCD* we would like to find a range of values of N_f, N_c and k such that the theory admits stable vacua. We can add a weak deformation to the superpotential (2.57), by adding terms with lower order powers in X

$$W(X) = \sum_{l=1}^{k} g_l \mathrm{Tr}\, X^{l+1} + \lambda \mathrm{Tr}\, X \qquad \text{with } g_l \ll 1 \tag{2.62}$$

where we introduced λ as a Lagrange multiplier to enforce the tracelessness of X. Considering that it is a weak perturbation, the large field behavior of the superpotential is not modified. Hence, if we cannot find stable vacua with the weak perturbation, the original theory doesn't have any vacua too.

The theory has a large set of multiple vacua for $Q = \tilde{Q} = 0$ and $X \neq 0$. X can be diagonalized with eigenvalues x_i. The F-terms are given by setting $W'(x_i) = 0$. Now, $W'(x_i)$ is a polynomial of degree k in the eigenvalues x_i admitting k distinct solutions in general. As a result, ground states are labeled by a set of k integers $(i_1, \dots, i_k)$, describing how many eigenvalues are in the l-th minimum. Clearly, considering that X has N_c eigenvalues we have

$$\sum_{l=1}^{k} i_l = N_c \tag{2.63}$$

In every vacuum, X has a quadratic potential, which corresponds to a mass term and can be integrated out. The X expectation values break the gauge group in the following way

$$SU(N_c) \longrightarrow SU(i_1) \times SU(i_2) \times \dots \times SU(i_k) \times U(1)^{k-1} \tag{2.64}$$

Each $SU(i_l)$ sector describes a decoupled *SQCD* model which has stable vacua only if $N_f \geq N_c$, hence considering every sector we have

$$i_l \leq N_f \qquad \forall 1 \leq l \leq k \tag{2.65}$$

Taking the limit $g_l \to 0$ we find that we must have

$$N_f \geq \frac{N_c}{k} \tag{2.66}$$

For every choice of i_l there is a moduli space obtained by giving expectation values to the quarks. Hence, the moduli space of the theory consists of different disconnected components, associated to different choice of i_l .

2.3.2 Magnetic theory

The magnetic theory is constructed in a similar way as the magnetic theory in Seiberg duality. The dual theory has gauge group $SU(\tilde{N}_c) = SU(kN_f - N_c)$. The baryonic charge of the dual quarks is found by imposing that baryons in the electric theory are proportional to the baryons constructed from dual quarks.

	$SU(\tilde{N}_c)$	$SU(N_f)_L$	$SU(N_f)_R$	$U(1)_B$	$U(1)_R$
q	$\tilde{N}_c$	$\overline{N_F}$	1	$\frac{N_c}{kN_f-N_c}$	$1-\frac{2}{k+1}\frac{\tilde{N}_c}{N_f}$
$\tilde{q}$	$\overline{\tilde{N}_c}$	1	N_F	$-\frac{N_c}{kN_f-N_c}$	$1-\frac{2}{k+1}\frac{\tilde{N}_c}{N_f}$
Y	1	$\tilde{N}_c^{\,2}-1$	1	0	$\frac{2}{k+1}$
M_j	1	N_f	$\overline{N_f}$	0	$2-\frac{4}{k+1}\frac{N_c}{N_f}+j\frac{2}{k+1}$

(2.67)

The magnetic theory has a superpotential

$$W = \operatorname{Tr} Y^{k+1} + \sum_{j=0}^{k-1} M_j q Y^{k-j-1} \tilde{q} \qquad \text{dove } M_j = QY^j\tilde{Q} \tag{2.68}$$

The charges of the fields are easily found by requiring duality for the two theories. In this way, the charges of the mesons are given in terms of the electric quarks, which are fixed, and the superpotential fixes the charges for the remaining fields. Using this method, the R-charges of the dual quarks are given by

$$R_q = R_X - R_Q \tag{2.69}$$

where we used that $R_X = R_Y = \frac{2}{k+1}$ because of (2.68) and (2.57).

Duality and mass deformations

Similarly to Seiberg duality, the duality is valid in a range of (N_f, N_c) in which both the electric and magnetic theory are asymptotically free. In that window, the superpotentials $\mathrm{Tr}\, X^{k+1}$ and $\mathrm{Tr}\, Y^{k+1}$ are both relevant in the infrared. An estimate of the window is given in [16] through a-maximization.
The mapping of gauge invariant operators between the two theories is given by

$$\begin{aligned} X &\longleftrightarrow Y \\ QX^j\tilde{Q} &\longleftrightarrow M_j \\ B_{el}^{(i_1,i_2,\dots,i_k)} &\longleftrightarrow B_{mag}^{(j_1,j_2,\dots,j_k)} \end{aligned} \tag{2.70}$$

where $j_l = N_f - i_{k+1-l}$ and $l = 1, \dots, k$.

The charge assignment necessary for the mapping of the operators is consistent with 't Hooft anomaly matching conditions which are given by

$$\begin{aligned} SU(N_f)^3 &\longrightarrow N_c d^{(3)}(N_f) \\ U(1)_R\, SU(N_f)^2 &\longrightarrow -\frac{2}{k+1}\frac{N_c^2}{N_f} d^{(2)}(N_f) \\ SU(N_f)^2 U(1)_B &\longrightarrow N_c d^{(2)}(N_f) \\ U(1)_R &\longrightarrow -\frac{2}{k+1}(N_c^2+1) \\ U(1)_R^3 &\longrightarrow \left(\left(\frac{2}{k+1}-1\right)^3+1\right)(N_c^2-1) - \frac{16}{(k+1)^3}\frac{N_c^4}{N_f^2} \\ U(1)_B^2 U(1)_R &\longrightarrow -\frac{4}{k+1}N_c^2 \end{aligned} \tag{2.71}$$

We consider now mass deformations of the electric theory in order to understand if duality is preserved under such deformations. Let's modify the electric superpotential by adding a mass term

$$W_{el} = g_k \mathrm{Tr}\, X^{k+1} + m\tilde{Q}_{N_f} Q^{N_f} \tag{2.72}$$

The number of flavours in the infrared is reduced by one unit. In order to preserve the duality, the magnetic theory should have gauge group $SU(k(N_f-1)-N_c) = SU(kN_f - N_c - k)$. Let's see if this is the case. The dual potential reads

$$W_{mag} = g_k \mathrm{Tr}\, Y^{k+1} + \sum_{j=1}^{k} M_j \tilde{q} Y^{k-j-1} q + m(M_0)_{N_f}^{N_f} \tag{2.73}$$

Integrating out the massive fields we find

$$q_{N_f} Y^l \tilde{q}^{N_f} = -\delta_{l,k} m \quad l = 0, \dots, k-1 \tag{2.74}$$

which fixes the expectation values to

$$\begin{aligned} \tilde{q}_{\alpha}^{N_f} =& \delta_{\alpha,1} \\ q_{\alpha}^{N_f} =& \delta^{\alpha,k} \\ Y_{\beta}^{\alpha} =& \begin{cases} \delta_{\beta+1}^{\alpha} & \beta = 1,\dots,k-1 \\ 0 & \text{otherwise} \end{cases} \end{aligned} \tag{2.75}$$

These expectation values break the gauge group to $SU(kN_f - N_c) \to SU(k(N_f - 1) - N_c)$ through the Higgs mechanism and reduces the number of flavours by one unit as required by duality. As a result, duality is preserved under mass deformations.

Chapter 3

Three dimensional dualities

3.1 Supersymmetric field theories in three dimensions

Spinors in three dimensions have different properties than their four dimensional counterpart.
The dimension of the representation in an arbitrary dimension D is given by $2^{\frac{D}{2}}$ for D even, while $2^{\frac{D-1}{2}}$ for D odd. Hence, in three dimension we have a two dimensional representation. In odd dimensions representations are irreducible and Weyl spinors do not exist: the chirality operator (γ_{D+1} or γ^*) is proportional to the identity.
Gamma matrices can be chosen real and the Majorana condition can be imposed, lowering the degress of freedom of the representation from four to two.
Since $3d$ $\mathcal{N}=2$ theories have four supercharges, we can use the $4d$ $\mathcal{N}=1$ superspace formalism.
The supersymmetry algebra and its representations can be found by dimensional reduction from four dimensions. The reduced algebra reads

$$\{Q_\alpha, Q_\beta\} = \{\bar{Q}_\alpha, \bar{Q}_\beta\} = 0 \qquad \{Q_\alpha, \bar{Q}_\beta\} = 2\sigma^\mu_{\alpha\beta}P_\mu + 2i\epsilon_{\alpha\beta}Z \tag{3.1}$$

The central charge Z is the component of the momentum along the reduced direction. Because of the presence of the central charge in the algebra, now states must satisfy a BPS bound of the form $M \geq Z$, which imply that massless representations have null central charge.
The automorphism of this algebra is $U(1)_R \simeq SO(2)_R$, as in four dimensions.
Superspace formalism is similar to what we introduced previously, with proper changes due to the different spinor representations in three dimension such as gamma matrices.

Chiral superfields contain *on-shell* one complex scalar and a complex spinor as in four dimensions.
Vector superfield contains an additional real scalar field with respect to the four dimensional superfield. The scalar σ comes from the component of the vector field A_μ along the reduced direction.

Coulomb branch and dualized photon

In three dimensions a free photon can be dualized to a scalar γ since it has only one polarization. Instead of defining a potential A in order to solve the Bianchi identity for the potential $dF = d^2A = 0$, we can define a scalar such that

$$*F = \frac{g^2}{\pi} d\gamma \quad \rightarrow \quad d*F = 0 \tag{3.2}$$

where $*$ is the Hodge operator and g is the gauge coupling.
The Bianchi identity $dF = 0$ can be seen as a conservation law for the topological current, which is defined as

$$J_\mu^{top} = \frac{1}{2\pi} * F \ = \ \frac{1}{2\pi} d\gamma \qquad \partial^\mu J_\mu \tag{3.3}$$

The topological current acts as shifts of the dualized photon $\gamma \rightarrow \gamma + \alpha$.
The quantization of the magnetic flux implies that the dualised photon is periodic and is normalized in such a way that $\gamma \sim \gamma + 2\pi$.
From a vector superfield V we can define a linear multiplet Σ, defined as

$$\Sigma = \epsilon^{\alpha\beta} D_\alpha D_\beta V \tag{3.4}$$

It satisfies $D^2\Sigma = 0$ and is gauge invariant under a transformation $V \rightarrow V + i(\Phi - \Phi^\dagger)$. The lowest component of Σ is the scalar in the vector multiplet and it contains also a term $\bar{\theta}\sigma_\rho\theta\, \epsilon^{\mu\nu\rho} F_{\nu\rho}$, whose bosonic part is proportional to the topological current. The super Yang-Mills action can be written also as

$$S \sim \int d^4\theta \, \Sigma^2 \tag{3.5}$$

The vacuum expectation values of scalar fields in the vector multiplet parametrize a subset of the moduli space of the theory which is called the *Coulomb branch*. Using the dualized photon we can turn the vector multiplet into a chiral multiplet, whose lowest component is a good parametrization of the Coulomb branch

$$Y = \exp\left(\frac{2\pi\sigma}{g^2} + i\gamma\right) \tag{3.6}$$

because of its periodicty, it's natural to assign γ to a phase factor.
The dualization of the photon was possible because there wasn't matter in the theory. However, matter fields couple with σ with mass terms. As a result, in a generic point of the Coulomb branch, all charged matter fields are massive and can be integrated out and in a low-energy description we can still dualize the gauge field.
A similar reasoning can be applied to non-Abelian gauge theories. In fact, in a generic point of the Coulomb branch, the VEV of σ breaks the gauge group to its maximal torus $U(1)^{r_G}$, where r_G is the rank of the gauge group. We now have r_G massless vector multiplets that can be dualized into chiral multiplets Y_i with $i = 1, \dots, r_G$.
Note that since the definition of Y depends on the gauge coupling, it will be modified by quantum corrections.
For $U(N)$ or $SU(N)$ theories it's better to use the following coordinates on the Coulomb branch, because they are related to the simple roots of the algebra

$$Y_k \sim \exp\left(\frac{\sigma_j - \sigma_{j+1}}{\hat{g}^2} + i(\gamma_j - \gamma_{j+1})\right) \qquad \hat{g}^2 = \frac{g^2}{4\pi} \tag{3.7}$$

with $j = 1, \dots, N-1$ for $SU(N)$ or with $j = 1, \dots, N$ for $U(N)$. From now on we will fix to a Weyl chamber by setting $\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_N$ with a Weyl transformation.
For $SU(N)$ gauge theories is useful to define

$$Y = \prod_{j=1}^{r_G} Y_k \tag{3.8}$$

since it's gonna be a coordinate of the unlifted Coulomb branch for such theories.

Real masses

In $3D\,\mathcal{N} = 2$ theories there's another way of giving a mass to a chiral multiplet other than with the superpotential term $W_{mass} = m\Phi^2$, which correspond to a holomorphic mass.
In these theories we can couple a global symmetry, which is not a R-symmetry, to a background vector multiplet. If we give a vacuum expectation value $\hat{m}$ to the scalar in the multiplet each field charged under that symmetry will receive a mass $q\hat{m}$, where q is the charge of the field under that symmetry. If a chiral field is charged under different global symmetry, its mass mass is a sum of the real masses for every global symmetry that we gauged. Real masses are parity odd and belong to vector multiplets rather than chiral fields. For this reason, they cannot appear in holomorphic objects such as the superpotential.

In addition, real masses relative to global abelian symmetries contribute to the central charge of the supersymmetry algebra Z through

$$Z = \sum q_i m_i \tag{3.9}$$

where q_i is the charge of the field and m_i the real masses under a global symmetry $U(1)_i$.
Moreover, the central charge Z can be promoted to a background linear superfield whose lowest component is Z.
Another symmetry that can contribute to the central charge is the topological symmetry $U(1)_J$, whose current is given by the linear multiplet Σ we defined in (3.4).
The added term is given by $\int d^4\theta \; V_b\Sigma$, where V_b is a background vector multiplet. Integrating by parts and defining $\Sigma_b = \epsilon^{\alpha\beta} D_\alpha D_\beta V_b$ we obtain $\int d^4\theta \; \Sigma_b V$.
Thus, the scalar component of the background vector field ξ is a Fayet-Iliopulos term for the vector superfield . It contributes to the central charge with a mass $m_J = \xi$.

Monopoles

In three dimensions there exist finite energy solutions that can be understood in terms of four dimensional monopoles after compactifying one direction. The scalar σ will play the role of the scalar Higgs field in the adjoint representation typically used to introduce monopoles [17]. Different topological solutions can be found since the scalar field at infinity must approach a vaccum solution. Such solution doesn't have to be the same in every direction and as a result we find different topological solutions depending on how we choose to map the two-sphere at infinity to the gauge group G. The mapping relates S^2_∞ to the gauge group rather than S^3_∞ since monopoles are localized in space rather than in space-time.
For a generic vacuum expectation value of σ, with $\sigma_i \neq \sigma_j$, the gauge group breaks to its maximal torus $U(1)^{r_G}$. The possible windings around S^2_∞ are then characterised by

$$\Pi_2\left(G/U(1)^{r_G}\right) = \Pi_1(U(1)^{r_G}) = \mathbb{Z}^{r_G} \tag{3.10}$$

The equality holds if $\Pi_2(G) = 0$, which is true for every semisimple group and if $\Pi_1(G) = 0$, which is satisfied if the group is the covering group of the Lie algebra. As a result there are r_G different topological solutions. Each of these solutions carry a magnetic charge for each of the $U(1)^{r_G}$ unbroken gauge fields. For example, for $G = SU(N_c)$ we have $N_c - 1$ different topological solutions.
For $SU(2)$ gauge group one solution to the equation is given by the 't Hooft-Polyakov monopole, which in singular gauge reads [18]

$$\sigma = \left(v \coth(vr) - \frac{1}{r}\right)\tau^3 \qquad A_\mu = \epsilon_{ij3}\hat{r}^i\left(\frac{1}{r} + \frac{v}{\sinh(vr)}\right) \tag{3.11}$$

This solution can be easily extended to $SU(N)$ gauge group by embedding the above solutions in a $SU(2)$ subgroup of the gauge group. Since the gauge group is broken in the vacuum, we cannot use $SU(N)$ gauge transformation to generate other solutions. As a results, each embedding of $SU(2)$ will result in a monopole charged under different $U(1)$ factors of the Cartan. Note that there are $N-1$ special embeddings of $SU(2)$ into $SU(N)$ which result in monopoles charged only under one factor of the $U(1)$ in the Cartan subalgebra. They are given by the N-1 contiguous 2x2 blocks in the diagonal.
The importance of monopole operators in three dimensions is due to the fact that flowing to the infrared, they flow to the coordinates on the Coulomb branch we introduced in (3.7) for $SU(N)$ theories [19].
The charges of the monopoles can be written as

$$\vec{g} = 2\pi \sum_a n_a \vec{\alpha_a} \tag{3.12}$$

where $\vec{\alpha_a}$ are the simple roots of the $su(N)$ Lie algebra. Dirac quantization condition imposes that $n_a \in \mathbb{Z}$.
Moreover, the mass of the monopoles is subject to the BPS bound which is given by

$$M_{monopole} \geq \frac{|\vec{g}\cdot\sigma|}{e^2} = \frac{2\pi}{e^2}\sum_i n_i \sigma_i \tag{3.13}$$

where σ_i are the vaccum expectation values of σ. Monopoles that sature the bound are called BPS monopoles.

3.1.1 Moduli spaces of gauge theories

Moduli space of $U(1)$ gauge theory with N_f massless flavours

For large values of σ the Coulomb branch of the theory can be parametrized by the chiral superfield we defined before $X = \exp\left(\frac{2\pi\sigma}{g^2} + i\gamma\right)$, which correspond to a cylinder. However, the metric for γ receives quantum correction and thus the topology of moduli space is changed by perturbation theory. The Higgs branch intersects the Coulomb branch for $\sigma = 0$ and since it is invariant under $U(1)_J$, the radius of the circle must shrink to zero where they meet, since the topological symmetry acts as shifts on the circle γ.
Therefore, near the origin the moduli space looks like the intersection of three cones: the Higgs branch and two cones corresponding to two distinct parts of the Coulomb branch as in figure 3.1. Half of the Coulomb branch is parametrized semi-classically by the field $V_+ \sim e^{\frac{\Phi}{g^2}}$, while the other half by $V_- \sim e^{-\frac{\Phi}{g^2}}$. Two different chiral fields are needed since near the origin the moduli space shrinks to

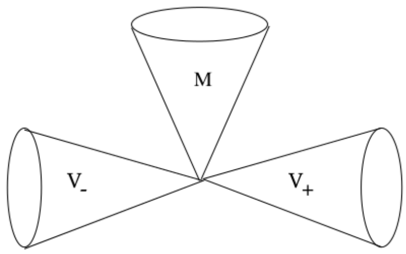


Figure 3.1: Schematic picture of the origin of the moduli space for a $U(1)$ gauge theory.

a point and $V_+ \to 0$. The Higgs branch is not modified by quantum corrections and the mesons can still be used to parametrize it.
The charges of the fields under the symmetries are given by

	$U(1)_R$	$U(1)_J$	$U(1)_A$	$SU(N_f)_R$	$SU(N_f)_R$
Q	0	0	1	N_f	1
$\tilde{Q}$	0	0	1	1	$\overline{N_f}$
M	0	0	2	N_f	$\overline{N_f}$
$V_\pm$	N_f	± 1	$-N_f$	1	1

(3.14)

The symmetries force the superpotential to have the form

$$W = -N_f \left(V_+ V_- \det(M)\right)^{\frac{1}{N_f}} \tag{3.15}$$

The superpotential is singular at the origin of the moduli space, indicating that there are massless degrees of freedom that need to be taken into account.
We can give real masses $\bar{m}_i$ ($-\bar{m}_i$) to the quarks Q_i ($\tilde{Q}_i$). The Higgs branch is parametrised by the diagonal elements of M_i^i which intersects the Coulomb branch at $\sigma = \bar{m}_i$. At every intersection, we have $U(1)$ theory with one flavour. The coulomb branch is parametrised, at every intersection, by $V_{i,\pm} = e^{\pm\frac{\Phi - \bar{m}_i}{g^2}}$ with a superpotential given by

$$W = -\sum_{i=1}^{N_f} M_i^i V_{i,+} V_{i,-} + \sum_{i=1}^{N_f - 1} \lambda_i \left(V_{i,+} V_{i+1,-} - 1\right) \tag{3.16}$$

where λ_i are Lagrange multipliers in order the enforce the semiclassical identification $V_{i,+} V_{i+1,-} = 1$.

Moduli spaces for $SU(N_c)$ theories with $N_c > 2$ and with N_f flavours

We will consider a $SU(N_c)$ gauge theory with equal number of Q and $\tilde{Q}$ in order to avoid Chern-Simons term. The simmetries of the theory are given by the following

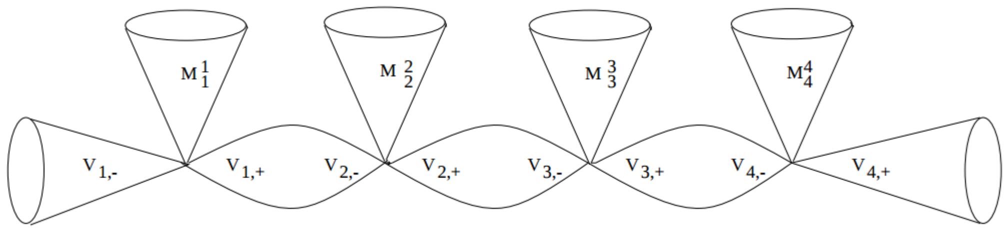


Figure 3.2: Moduli space of vacua for SQED with four flavours

table

	$U(1)_R$	$U(1)_B$	$U(1)_A$	$SU(N_f)_L$	$SU(N_f)_R$
Q	0	1	1	N_f	1
$\tilde{Q}$	0	-1	1	1	$\overline{N_f}$
M	0	0	2	N_f	$\overline{N_f}$
$Y_{j\neq K}$	-2	0	0	1	1
Y_K	$2(N_f-1)$	0	$-2N_f$	1	1
Y	$2(N_f-N_c+1)$	0	$-2N_f$	1	1

(3.17)

where Y_j are defined as in (3.7) and Y is defined as $Y=\prod_{j=1}^{N_c-1} Y_j \sim e^{\frac{\Phi_1-\Phi_{N_c}}{g^2}}$. The adjoint scalar VEV is ordered in such a way that $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_{N_c}$. The value of j for which σ changes sign is called K and the associated instanton is Y_K. For $N_f>1$ the theory has a Higgs branch, which break the gauge group in a generic point to $SU(N_f-N_c)$ for $N_f<N_c-1$ and completely for $N_f \geq N_c-1$. The Higgs branch is parametrised as in four dimensions by mesons and for $N_f \geq N_c$ also by baryons.

For theory withouth quarks, instantons generate the Affleck-Harvey-Witten [20] superpotential

$$W=\sum_{j=1}^{N_c-1} \frac{1}{Y_j} \tag{3.18}$$

which prohibits the theory to have stable supersymmetric vacuas.

The theory with massless quarks features a superpotential generated by N_c-2 instantons

$$W_{inst}=\sum_{j\neq K} \frac{1}{Y_j} \tag{3.19}$$

which doesn't completely lift the Coulomb branch. In fact, the following subspace is classically unlifted

$$\sigma_1>\sigma_2=\cdots=0>\sigma_{N_c}=-\sigma_1 \tag{3.20}$$

The last equality holds because we have $\sum_i \sigma_i = 0$ for $SU(N_c)$ gauge group. The quantum corrected moduli space is different from the classical one and for $N_f < N_c - 1$ the effective superpotential is given by

$$W_{eff} = (N_c - N_f - 1)(Y \det(M))^{\frac{1}{N_f - N_c + 1}} \tag{3.21}$$

which admits no stable vacuum.
For $N_f = N_c - 1$ we obtain a constraint on the moduli space given by

$$Y \det(M) = 1 \tag{3.22}$$

which results in a merging between the Higgs and the Coulomb branch.

For $N_f \geq N_c$, the low-energy description contains baryons. The quantum moduli space is the same as the semi-classical moduli space and there is a superconformal fixed point at the origin. In the $N_f = N_c$ case an effective description of the moduli space can be given and the superpotential is

$$W = -Y(\det(M) - B\tilde{B}) \tag{3.23}$$

which forces the theory to flow to a non-trivial fixed point in the infrared.
For $N_f > N_c$ an effective description is not known.

The above analysis can be extended to theories with $U(N_c)$ gauge group with $N_f \leq N_c$. We will discuss the theory with $N_f = N_c$ because the results with lower number of flavours can be obtained by integrating out quarks.
The theory with $U(N_c)$ is obtained by gauging the global $U(1)_B$ factor.
Given that we have $N_f = N_c$, the superpotential (3.23) from the $SU(N_c)$ part of the group is present and for the $U(1)$ factor we have one flavour given by $X = B\tilde{B}$ and chiral fields V_+ , V_-, defined as

$$V_+ \sim \exp\left(\frac{\Phi_1}{g^2} + i\gamma_1\right) \qquad\qquad V_- \sim \exp\left(\frac{\Phi_{N_c}}{g^2} + i\gamma_{N_c}\right) \tag{3.24}$$

because the null trace condition does not hold anymore.
The $U(1)$ sector has a superpotential given by

$$W = -XV_+V_-$$

As a result, the superpotential for the theory is given by

$$W = -Y(\det(M) - X) - XV_+V_- \tag{3.25}$$

Integrating out the massive field Y we obtain a superpotential

$$W = -V_+V_- \det(M) \tag{3.26}$$

3.2 Aharony duality

One of the first three dimensional dualities was found by Aharony [21] and Karch [22] for theories with $U(N_c)$ as gauge group and with non-chiral matter in the fundamental representation. The charges of the fields are given by the following table

	$U(1)_R$	$U(1)_A$	$U(1)_J$	$SU(N_f)_L$	$SU(N_f)_R$
Q	0	1	0	N_f	1
$\tilde{Q}$	0	1	0	$\overline{N}_f$	1
M	0	2	0	1	1
$V_\pm$	$N_f - N_c + 1$	$-N_f$	± 1	1	1

(3.27)

where $V_\pm$ are defined as in the previous section and parametrize the unlifted Coulomb branch. The R-charges of the quarks have been fixed to zero in order to have the fermion in the multiplet with R-charge -1. The moduli space of this theory was described in the previous section.
The dual theory is a $U(\tilde{N}_c) = U(N_f - N_c)$ gauge theory with N_f flavours q , $\tilde{q}$ and with singlet fields M , V_+ , V_- with the same quantum number of the electric theory. The charges are chosen in order to be consistent with a potential $W = M\, q\tilde{q}$ as in four dimensional dualities.
The global charges of the theory are given by

	$U(1)_R$	$U(1)_A$	$U(1)_J$	$SU(N_f)_L$	$SU(N_f)_R$
q	1	-1	0	$\overline{N}_f$	1
$\tilde{q}$	1	-1	0	1	N_f
$\tilde{V}_\pm$	$N_c - N_f - 1$	N_f	± 1	1	1

(3.28)

The magnetic theory features a superpotential of the form

$$W = M_i^{\tilde{i}} q^i \tilde{q}_{\tilde{i}} + V_+ \tilde{V}_- + V_- \tilde{V}_+ \tag{3.29}$$

The dual theory now contains the fields V_+ , V_- which are matched with the coordinate on the Coulomb branch of the original theory. Remember that in three dimensions the moduli space contains the Coulomb branch, which wasn't present in four dimensional theories. The introduction of the monopoles is crucial in order to have the same moduli space between the two theories.
The dual theory exists only for $N_f > N_c$. Starting with $N_f = N_c + 1$ we obtain a dual theory with $U(1)$ gauge group. Adding a mass term of the form mM in the dual theory breaks the gauge group. After integrating out quarks and $\tilde{V}_\pm$ fields we obtain an effective description given only by mesons and $V_\pm$ with no other fields. The superpotential is constrained by symmetries to be of the form

$$W = -V_+ V_- \det(M) \tag{3.30}$$

We found exactly the dual description of the theory we introduced in (3.26).
We can compare the moduli space between the dual theories by looking at specific vacua. For example, let's consider a point on the Higgs branch in the electric theory with vacuum expectation values of the mesons of rank N_c. The low-energy dual theory associated to such VEV is a $U(N_f - N_c)$ theory with $N_f - N_c$ massless flavours whose moduli space can be described by a superpotential of the form $W = \tilde{V}_+ \tilde{V}_- \det(q\tilde{q})$ since $N_f = \tilde{N}_c$.
The equation of motion resulting from the addition of this term to the superpotential (3.29) set $V_+ = V_- = 0$, as in the electric theory.

3.3 Kim-Park duality

Aharony duality can be generalised with the addition of a field in the adjoint representation [23].
The electric theory contains N_f flavours and a chiral multiplet in the adjoint representation. The theory has a superpotential given by $W = trX^{k+1}$ that fixes the R-charge of the field to $\Delta_X = \frac{2}{k+1}$. The R-charges of the quarks have been left unconstrained to the generic value Δ_Q.
The charges of the fields are given by

	$SU(N_f)_L$	$SU(N_f)_R$	$U(1)_A$	$U(1)_J$	$U(1)_R$
Q	N_f	1	1	0	Δ_Q
$\tilde{Q}$	1	$\overline{N_f}$	1	0	Δ_Q
X	1	1	0	0	$\frac{2}{k+1}$
$v_{j,\pm}$	1	1	$-N_f$	± 1	$-N_f(\Delta_Q - 1) - \frac{2}{k+1}(N_c - 1 - j)$

(3.31)

where $v_{j,\pm}$ with $j = 0, \ldots, k-1$ are the monopole operators dressed with powers of the adjoint field. They are defined as [23]

$$v_{j,\pm} = \mathrm{Tr}\,(v_{0,\pm} X^j) \tag{3.32}$$

where $v_{0,\pm}$ is the bare monopole operator we defined in the previous section.
To see the reason why there are k different monopoles in this theory we can weakly deform the superpotential to a generic polynomial, as we did in (2.62). The effect of this deformation is the breaking of the gauge group $U(N_c)$ in

$$U(N_c) \;\rightarrow\; U(r_1) \times U(r_2) \times \cdots \times U(r_k) \quad \text{with } \sum r_i = N_c \tag{3.33}$$

Following the same reasoning, in the infrared we find k distinct SQCD sectors with gauge group $U(r_i)$ without adjoint matter, since it has been integrated out.

Every sector must have a pair of monopoles since it is a $U(r_i)$ theory.
Going in the limit of vanishing coupling we find that the original theory have k pairs of monopole operators.

The magnetic theory is a $U(kN_f - N_c)$ gauge theory with N_f dual quarks $(q, \tilde{q})$, adjoint matter Y and singlet fields $(M_j)^i_{\tilde{i}}$ and $v_{j,\pm}$. Mesons are constructed from the quarks and the adjoint matter in the electric theory and the $v_{j,\pm}$ are the electric monopole operators.
The charges of the fields are given by

	$SU(N_f)_L$	$SU(N_f)_R$	$U(1)_A$	$U(1)_J$	$U(1)_R$
q	N_f	1	1	0	Δ_q
$\tilde{q}$	1	$\overline{N_f}$	1	0	Δ_q
Y	1	1	0	0	$\frac{2}{k+1}$
M_j	N_f	$\overline{N_f}$	2	0	$2\Delta_Q + j\frac{2}{k+1}$
$v_{j,\pm}$	1	1	$-N_f$	± 1	$-N_f(\Delta_Q - 1) - \frac{2}{k+1}(N_c - 1 - j)$
$\tilde{v}_{j,\pm}$	1	1	N_f	± 1	$N_f(\Delta_Q - 1) + \frac{2}{k+1}(N_c + 1 + j)$

(3.34)

The fields $\tilde{v}_{j,\pm}$ are the monopole operators of the magnetic theory and are constructed in the same way of the electric ones. The charges of the dual quarks and of the adjoint matter are chosen in order to be compatible with a superpotential

$$W = \mathrm{Tr}\, Y^{k+1} + \sum_{j=0}^{k-1} M_j\, \tilde{q}\, Y^{k-j-1} q \tag{3.35}$$

as Kutasov-Schwimmer duality in four dimensions.
The charges of the monopole operators can be calculated by counting their fermion zero modes and are compatible with a superpotential of the form

$$W_{monopoles} = \sum_{j=0}^{k-1} (v_{i,+}\, \tilde{v}_{k-j-1,-} + v_{j,-}\, \tilde{v}_{k-j-1,+}) \tag{3.36}$$

The duality maps the electric adjoint field with the magnetic one. Mesons and monopoles operators in the electric theory are mapped as singlets in the magnetic one.

Chapter 4

Reduction of 4D dualities to 3D

In the previous chapter we introduced few examples of electric-magnetic dualities in four and three spacetime dimensions. It is natural to wonder what is the relation between them since they show some similiraties but at the same time the dynamics of quantum field theories is different between different spacetime dimensionalities. For example, in three dimensions there are no anomalies and the moduli space features the Coulomb branch. Moreover, in three dimensions the axial symmetry is not broken by anomalies and theories may have an additional topological symmetry, associated to abelian factors in the gauge group, which is enforced by a Bianchi identity.
The presence of a scalar field in every vector multiplet allows real mass terms for fields charged under global symmetries, whereas in four dimensions we could only generate holomorphic mass terms through superpotential terms.
Despite these differences four dimensional dualities can be dimensionally reduced to three dimensions in general and in some cases the reduction process will unveil new dualities in three dimensions.

4.1 General procedure of reducing dualities

The näive dimensional reduction of four dimensional dualities doesn't result in a three dimensional duality in general. It can be achieved compactifying a spatial dimension into S^1 with radius r and then going in the limit $r \to 0$.
Let's analyse the behaviour of the the theory at finite radius to understand the reasons why the näive reduction doesn't work.

In four dimensional theories, the strong coupling scale is given by

$$\Lambda^b = \exp\left(-\frac{8\pi^2}{g_4^2}\right) \tag{4.1}$$

where b is the one-loop β function coefficient and we set the renormalization scale μ to 1 for convenience. Recall the following relation for strong coupling scales between dual theories

$$\Lambda^b \tilde{\Lambda}^b = (-1)^{N_f - N_c} \tag{4.2}$$

After compactifying a dimension, the strong coupling scale is modified [19] since $g_4^2 = 2\pi r g_3^2$ and as a result

$$\Lambda^b = \exp\left(-\frac{4\pi}{r g_3^2}\right) \tag{4.3}$$

The consequence of the compactification is that if we take the limit $r \to 0$, the strong coupling scale goes to zero. Moreover, it's clear that the relation between strong coupling scales of dual theories (4.2) is incompatible with this limit, since both Λ and $\tilde{\Lambda}$ go to zero.
Thus, the $r \to 0$ limit does not commute with the infrared limit at fixed coupling needed in the duality.
We can take the limit in which we keep Λ , $\tilde{\Lambda}$ and r fixed and look at energies $E \ll \Lambda\,,\, \tilde{\Lambda}\,,\, \frac{1}{r}$. In this limit, we are in the infrared, at fixed coupling but with a finite radius. However, this is not a problem since the effective theory at low energy Edoesn't see the compactified dimension. As a result, the dynamics of the theory is effectively three dimensional.
The theories we find with this procedure have few properties that differentiate them to purely three-dimensional theories. The scalar associated to the compactified direction of A_3 is periodic, since the holonomy of the gauge field on the circle is gauge invariant

$$P\left(\exp\left(i \oint A_3\right)\right) \tag{4.4}$$

As a result the scalar is compact whereas the three dimensional one is not. The period of the compactified scalar depends on the global properties of the gauge group. For $SU(N)$ and $U(N)$ gauge groups the period is $\sim \frac{1}{r}$ while for other groups such as $SP(2N)$ or $SO(N)$ its expression is more involved.
Four dimensional theories on the circle have an additional non-perturbative superpotential generated by a Kaluza-Klein monopole. This particular type of instanton/monopole configuration [1] of the gauge field is generated through a non periodic gauge transformations on an instanton configuration of the gauge field.

[1] Using this expression we refer to the fact that 't Hooft/Polyakov monopoles in four dimension can be interpreted as three-dimensional instantons after ignoring the time dimension.

Even though the transformation is not periodic around the circle, the gauge field obtained acting with it remains periodic. For this reason it's associated to a different topological sector with respect to standard instanton/monopole configurations [24].
The superpotential generated reads

$$W = \eta Y_{low} \tag{4.5}$$

where $\eta = \Lambda^b$ and Y_{low} is the Coulomb branch coordinate we defined in (3.8).
Theories obtained with this method are dual to each other and they differ from truly three-dimensional theories for the reasons we explained above. For example, the introduction of the η superpotential breaks the axial symmetry that is anomalous in four dimensions, but is allowed in three-dimensional theories.
After these consideration we would like to flow from these dualities with finite radius r into truly three-dimensional dualities, i.e. dualities for theories well-defined at high energies. The compactness of the Coulomb branch is not really a problem since for $SU(N)$ gauge theories the η superpotential lifts completely the Coulomb branch together with the Affleck-Harvey-Witten superpotential (3.19). Regarding the η superpotential, in the case we will analyse it's possible to find monopole operators Y_{high}, well-defined at high energies, that flow to the Coulomb branch coordinates Y_{low} at low energies. Using these operators it is possible to define the theories in the ultraviolet, thus turning the dualities we find by reduction as standalone three-dimensional dualities.

4.2 Dimensional reduction of $SU(N_c)$ Seiberg duality

We introduced Seiberg duality in four dimensions in section 2.2 but its analog in three dimensions was discovered only by the procedure of dimensional reduction that we will introduce more precisely in this section.
The superpotential that is generated for four dimensional theories compactified is given by the sum of the AHW superpotential (3.18) and the η superpotential (4.5) and together they lift the Coulomb branch completely.

Comparison of moduli spaces

From the previous sections we know that three-dimensional theories perturbed by ηY should be the same at low energies as four dimensional theories on the circle. From the four dimensional duality we know that the compactified theory is dual to a four dimensional magnetic theory. Such theory can be mapped into a three

dimensional theories by the same arguments we used for the electric theory. We will compare the moduli spaces between four and three dimensional theories to see if this argument is valid in this case.
For $N_f = N_c - 1$ the four dimensional theory does not admit a vacuum but the unperturbed three dimensional theory has a moduli space subject to the constraint (3.22). Adding the η superpotential result in a theory without a stable vacuum, as the four dimensional theory.
For $N_f = N_c$ the four dimensional theory has an effective description that can be implemented by a Lagrange multiplier that reads

$$W_{3d}^{S^1} = \lambda \left(B\tilde{B} - \det(M) + \Lambda^{2N_c} \right) = \lambda \left(B\tilde{B} - \det(M) + \eta \right) \tag{4.6}$$

since $b = 3N_c - N_f = 2N_c$ for these particular values.
The three-dimensional theory with the additiona η superpotential has an effective description given by

$$W_{3d} = Y \left(B\tilde{B} - \det(M) + \eta \right) \tag{4.7}$$

which suggests the identification between λ and Y.
For $N_f = N_c + 1$ it can be demonstrated that the moduli spaces match but the situation is a little more involved and we refer to [19] for a more complete discussion.

4.2.1 Flow to a pure three-dimensional duality

Let's consider a theory with $N_f + 1$ flavours and let's give a real mass $\hat{m}$ to the last flavour. We will set in the limit in which $\hat{m} \to \infty$ in order to integrate out the last flavour. This can be achieved by giving expectation values to the fields associated to the diagonal $SU(N_f + 1) \times U(1)_B$ flavour symmetry given by

$$\text{diag}(0, \ldots, m) = \text{diag}(m_B - M, \ldots, m_B + N_f M) \quad \text{with } m_B = M \tag{4.8}$$

where M is associated to the diagonal $SU(N_f + 1)$ and m_B to $U(1)_B$.
The real masses are easily mapped to the dual theory by considering the charges of the quarks under the global symmetries in question (cfr. tables (2.16) and (2.45)).

$$M \;\to\; -M \qquad m_B \;\to\; \frac{N_c}{N_f - N_c + 1} m_B \tag{4.9}$$

As a result the first N_f flavours get a mass $\hat{m}_1$ and the last one a mass of $\hat{m}_2$

$$\hat{m}_1 = \frac{\hat{m}}{N_f - N_c + 1} \qquad \hat{m}_2 = \frac{\hat{m}(N_c - N_f)}{N_f - N_c + 1} \tag{4.10}$$

In the electric theory we will be interested in configrations that remain at a finite distance from the origin in the Coulomb branch in the limit $\hat{m} \to 0$. For these type of vacua, the last flavour is massive and can be integrated out at low energies, resulting in an effective $SU(N_c)$ theory with N_f flavours. Since we integrated out an equal number of left and right fermions, no Chern-Simons are generated [25]. The monopole operator Y_{high} is related to the low-energy coordinate Y_{low} by the relation $Y_{low} = Y_{high}/m$, where m is the complex mass of quark [19]. Since the real mass is given by a component of a vector multiplet, the superpotential $W = \eta Y_{high}$ vanishes, since we didn't assign any complex mass to the quark.
As a result, the electric theory we obtain is a $SU(N_c)$ gauge theory with N_f massless flavours and no η-superpotential. The absence of the η-superpotential results in a theory with axial symmetry, since it was the only term that was breaking it. The charges of the fields are given by

	$SU(N_c)$	$U(1)_R$	$U(1)_A$	$U(1)_B$	$SU(N_f)_L$	$SU(N_f)_R$
Q	N_c	1	-1	1	$\overline{N_f}$	1
$\tilde{Q}$	$\overline{N_c}$	1	-1	-1	1	N_f
Y	1	$2(N_c - N_f - 1)$	$-2N_f$	0	1	1

(4.11)

The magnetic theory does not have vacuum at the origin of the Coulomb branch since each flavour acquire a mass proportional to $\hat{m}$. The vacuum configuration with the great number of massless quarks is given by following the vacuum expectation value of $\tilde{\sigma}$

$$\tilde{\sigma} = \mathrm{diag}\left(\overbrace{-\tilde{m}_1, \ldots, -\tilde{m}_1}^{N_f - N_c \text{ values}}, -\tilde{m}_2\right) \tag{4.12}$$

Note that this expectation value is traceless, as it should be.
This vacuum breaks the gauge symmetry $SU(N_f - N_c + 1) \to SU(N_f - N_c) \times U(1)$. We can view it as $U(N_f - N_c)$ with quarks in the representations

component	quark	$U(N_f - N_c)$
$1, \ldots, N_f$	q	$(N_f - N_c)_{\,1}$
$1, \ldots, N_f$	$\tilde{q}$	$(\overline{N_f - N_c})_{\,1}$
$N_f + 1$	p	$1_{-(N_f - N_c)}$
$N_f + 1$	$\tilde{p}$	$1_{\,(N_f - N_c)}$

(4.13)

The mesons' components created from quarks with with the same real mass remain light, since left and right quarks have opposite charges and global symmetries. Thus, the massless mesons are a $N_f \times N_f$ matrix, that can be identified with the mesons M of the electric theory, and a single singlet field $M_{N_f+1}^{N_f+1}$ which we will call Y, since we will identify later with the monopole operator Y.

The off-diagonal components, i.e $M^i_{N_f+1}$ and $M_i^{N_f+1}$ with $i = 1, \ldots, N_f$ have a mass proportional to $\hat{m}$ and can be integrated out.
The η superpotential can be written in terms of $U(N_f - N_c)$ Coulomb branch variables that we defined in (3.24). Since the gauge group comes from the breaking of $SU(N_f - N_c + 1)$ the superpotential is written as

$$W = \tilde{\eta}\tilde{V}_- \tag{4.14}$$

In addition to this superpotential, there's a AHW contribution to the superpotential, related to the breaking of the gauge group from $SU(N_f - N_c + 1)$ to $SU(N_f - N_c) \times U(1)$ which is given by a term

$$W = \tilde{V}_+ \tag{4.15}$$

The complete superpotential of the magnetic theory is given by

$$W_{mag} = M\, q\tilde{q} + Y\, p\tilde{p} + \tilde{\eta}\tilde{V}_- + \tilde{V}_+ \tag{4.16}$$

At this point, $\tilde{\eta}$ does not play any role anymore and it can be absorbed in a redefinition of $\tilde{V_-}$. The Coulomb branch of this theory is completely lifted by the last two terms of the superpotential.
The global symmetries of the theory consist of the $SU(N_f)_L \times SU(N_f)_R \times U(1)_R$ symmetries inherited from four dimensions, the topological symmetry $U(1)_J$ associated to the abelian factor in the gauge group and two abelian factors given by $U(1)_B$ and $U(1)_A$, that are not broken by the superpotential.
From the matching of the gauge invariant operators between the two theories we can fix most of the quantum numbers. The baryons of the electric theory are matched in the magnetic theory to $b = q^{N_f - N_c}p$ and $\tilde{b} = \tilde{q}^{N_f - N_c}\tilde{b}$.
The $U(1)_B$ symmetry mixes with abelian term of the gauge group and it can be chosen arbitrarily.
The charges are the following

	$U(N_f - N_c)$	$SU(N_f)_L$	$SU(N_f)_R$	$U(1)_B$	$U(1)_A$	$U(1)_R$
q	$(N_f - N_c)_{\,1}$	$\overline{N_f}$	1	0	-1	1
$\tilde{q}$	$(\overline{N_f - N_c})_{\,-1}$	1	$\overline{N_f}$	0	-1	1
p	$1_{-(N_f - N_c)}$	1	1	N_c	N_f	$-(N_f - N_c)$
$\tilde{p}$	$1_{(N_f - N_c)}$	1	1	$-N_c$	N_f	$-(N_f - N_c)$
M	1	1	1	0	2	0
Y	1	1	1	0	$-2N_f$	$2(N_f - N_c + 1)$
$\tilde{V}_\pm$	1	1	1	0	0	2

(4.17)

The meson field Y has the same quantum numbers of the Y field in the electric and for this reason is natural to identify them. The quantum numbers of $V_\pm$

are calculated by counting zero modes and the fact that are consistent with the superpotential (4.16) is a check of the result.
As a result of this process we found a three-dimensional duality that wasn't known as a standalone duality before the method of dimensional reduction was discovered in [19]. The duality can be checked by the matching of the partition functions between the two theories. We will focus on this approach in the next chapters.
As an additional check of this duality we can use it to the derive Aharony duality. Since it is a duality involving $U(N)$ gauge theories, we can achieve this by gauging the baryonic symmetry, since it has opposite charges for left and right quarks.
The Coulomb branch coordinates are now given by the independent operators $V_{\pm}$. The $U(N_c)$ theory has an additional global symmetry $U(1)_J$ associated to the $U(1)$ factor in the gauge group. The duality maps the electric $U(N_c)$ theory with N_f flavours with superpotential

$$W_{el} = \eta V_+ V_- \tag{4.18}$$

to a $U(N_f - N_c)$ theory with N_f flavours and mesons M with

$$W_{mag} = \tilde{\eta} Y_+ Y_- + M q\tilde{q} \tag{4.19}$$

Both theories have a one-dimensional Coulomb branch which is not lifted by the superpotential, since it only affects the $SU(N)$ factor of the group and a matching between the two is possible.
The same procedure can be applied to the duality we found without the η superpotential. The steps on the electric theory are already described in the previous discussion, while the charges of the magnetic theory we used in (4.17) was chosen in such a way that the only fields charged under the baryonic symmetry were $p, \tilde{p}$. As a result, the new sector of the theory is a $U(1)$ gauge theory with only one flavour. Its low-energy description is given by a superpotential, relating its monopole operators V_+, V_- with the meson $N = p\tilde{p}$ by

$$W = -V_+ V_- N \tag{4.20}$$

The superpotential we obtained by the reduction (4.16) contains a term given by $W = YN$. The equation of motion of N sets $Y = \tilde{V}_+ \tilde{V}_-$. Integrating out the massive fields the monopole operators $\hat{X}_{\pm}$ of the low-energy theory $U(N_f - N_c)$ can be related to the high-energy monopoles$Y_{\pm}$ of $U(N_f - N_c) \times U(1)_B$ by the relation [19]

$$Y_{\pm} \simeq \hat{X}_{\pm} \tilde{V}_{\mp} \tag{4.21}$$

As a result the superpotential [2] is given by

$$W_{mag} = M q\tilde{q} + \hat{X}_- \tilde{V}_+ + \hat{X}_+ \tilde{V}_- \tag{4.22}$$

[2] We rescaled the field Y_- by a factor $\tilde{\eta}$ since it does not play any role in the duality anymore.

which is the same superpotential of the magnetic theory in Aharony duality, after the identification of the singlets $\tilde{V}_\pm$ with the fields $V_\pm$ of the electric theory.

4.3 Reduction of KSS duality

4.3.1 $U(N_c)$ duality

We will first focus on the reduction of KSS duality introduced in section 2.3 but with $U(N_c)$ gauge group which is obtained by gauging the baryonic $U(1)_B$ symmetry. The dynamics of the $U(1)$ factor in the gauge group is infrared free so it does not affect the duality.
The reduction of the theory to $\mathbb{S}^3 \times \mathbb{S}^1$ features an η superpotential that is generated by Kaluza-Klein monopoles as in the previous section.
Because of the superpotential $\operatorname{Tr} X^{k+1}$ there are $2k$ unlifted directions in the moduli space that are parametrised by $t_i, \pm$ as in (3.32). The counting of the fermion zero modes for the KK-monopoles shows that they generate a superpotential [26]

$$W_\eta = \eta \sum_{j=0}^{k-1} t_{j,+} \, t_{k-1-j,-} \tag{4.23}$$

In the magnetic theory a similar superpotential is generated, involving magnetic monopoles $\tilde{t}_{i,\pm}$ of the dual theory. The addition of these superpotential guarantees that the duality is consistent with the compactification.
Since we are interested in the reduction to the Kim-Park duality without η-superpotential, we will add real masses to some quarks as we did in section 4.2.1. Since the baryonic symmetry is gauged, we need to start from $N_f + 2$ flavours and assign masses $(m, \tilde{m})$ associated to the background gauging of the $SU(N_f+2)_L \times SU(N_f+2)_R$ global symmetry

$$m_{el} = \operatorname{diag}\left(\underbrace{m_A, \ldots, m_A}_{N_f}, M - \frac{m_A N_f}{2}, -M + \frac{m_A N_f}{2}\right) \tag{4.24}$$

$$\tilde{m}_{el} = \operatorname{diag}\left(\underbrace{m_A, \ldots, m_A}_{N_f}, -M + \frac{m_A N_f}{2}, M - \frac{m_A N_f}{2}\right) \tag{4.25}$$

This mass assignment breaks the symmetry group to $SU(N_f)_L \times SU(N_f)_R \times U(1)_A$, where $U(1)_A$ is the axial symmetry that the η-superpotential breaks. However, since we are in the limit in which M is large, the η-superpotential vanish in this flow since the quarks don't have complex masses as in[19]. As a result, the theory

flows to the electric theory of Kim-Park duality.
The magnetic theory is now a $U(k(N_f+2)-N_c)$ gauge theory with N_f+2 flavours. The real masses of the electric theory are easily mapped to the magnetic theory by comparing the global symmetries

$$m_{mag} = \text{diag}\left(\underbrace{m_A, \dots, m_A}_{N_f}, M - \frac{m_A N_f}{2}, -M - \frac{m_A N_f}{2}\right) \tag{4.26}$$

$$\tilde{m}_{mag} = \text{diag}\left(\underbrace{m_A, \dots, m_A}_{N_f}, -M - \frac{m_A N_f}{2}, M - \frac{m_A N_f}{2}\right) \tag{4.27}$$

The magnetic theory can be weakly perturbed by a polynomial superpotential in the adjoint matter field

$$W = \sum_{j=2}^{k+1} s_j \,\text{Tr}\, \tilde{X}^j \tag{4.28}$$

which in this case doesn't contain a term proportional to $\text{Tr}\,\tilde{X}$ as in the $SU(N)$ case since it would force $\tilde{X}$ to be traceless.
The superpotential triggers a breaking of the gauge group

$$U(k(N_f+2)-N_c) \longrightarrow U(r_1) \times \dots \times U(r_k) \quad \sum_{j=1}^{k} r_i = k(N_f+2) - N_c \tag{4.29}$$

As in section 2.3.1, in every vacuum the adjoint matter is massive and is integrated out, resulting in k different SQCD sectors without adjoint matter. The vacuum configuration for the adjoint scalar such that we have the largest number of massless particles is

$$\tilde{\sigma}_{U(r_i)} = \text{diag}\ (0, \dots, -M, M) \tag{4.30}$$

that results in the breaking pattern

$$U(r_1) \times \dots U(r_k) \longrightarrow \left(U(r_1-2) \times U(1)^2\right) \times \dots \times \left(U(r_k-2) \times U(1)^2\right) \tag{4.31}$$

The fields that remain massless in this vacuum are the following

- $\left(q^i_j, \tilde{q}^i_j\right)$ dual quarks with $i = 1, \dots, k$ and$j = 1, \dots, N_f$ which corresponds to the first N_f flavours of quarks for every $U(r_i - 2)$ sector tha remain massless
- $(p^i_a, \tilde{p}^i_a)$ with $a = 1, 2$. They are the last two flavours of quarks charged under one of the two $U(1)$ sector for each value of i

- $(M_j)^i_i$ mesons with $j = 0, \ldots, k-1$ coming from the first N_f flavours of electric quarks.
- $(M_j)^a$ mesons associated to $M^{N_f+1}_{N_f+1}$ and $M^{N_f+2}_{N_f+2}$

and the other fields obtain a large mass in this limit.
We can define the following composite operators out of these fields where for $\tilde{X}$ we intend its value in the vacuum since it is massive.

$$(N_j)^i = q^i_j \tilde{X}^j \tilde{q}^i_j \tag{4.32}$$

$$(\hat{N}_j)^i_a = p^i_a \tilde{X}^j \tilde{p}^i_a \tag{4.33}$$

The superpotential can be rewritten and is given by

$$W = \sum_{j=0}^{k-1} \sum_{i=1}^{k} M_j N^i_{k-j-1} + \sum_{a=1}^{2} \sum_{i=1}^{k} \sum_{j=0}^{k-1} M^a_j (\hat{N}_j)^i_a \tag{4.34}$$

The two $U(1)$ sectors can be given an effective description using (3.15) which results in

$$W_{U(1)} = \sum_{a=1,2} \sum_{i=1}^{k} (\hat{N}_0)^i_a v^{i,a}_+ v^{i,a}_- \tag{4.35}$$

where $v^{i,a}_\pm$ are the Coulomb branch coordinates of the i-th $U(1)$ sector.
The breaking of each gauge group factor from $U(r_i) \to U(r_i - 2) \times U(1)^2$ generates three Affleck-Harvey-Witten terms in the superpotential

$$W_{AHW} = \sum_{i=1}^{k} \left(\tilde{V}_{i,+} v^{i,1}_- + \tilde{V}_{i,-} v^{i,2}_+ \right) \tag{4.36}$$

where $V_{i,\pm}$ are the Coulomb branch coordinates for the $U(r_i - 2)$ sectors.
The equations of motion set the terms with $(\hat{N_j})^i_a$. After sending the mass M to infinity, we can weakly switch off the deformation in the adjoint field. As a result, the $U(r_i - 2)$ factors recombine to $U(kN_f - N_c)$ with adjoint matter Y.
While removing the deformation, the 2k $U(1)$ factors become $U(k) \times U(k)$ each with one flavour and one adjoint matter field. Such a theory should not have a vacuum (2.66) but the additional terms in the superpotential should modify this condition. We assume that this enhancement to $U(k)$ does not modify the results and we will see in the next chapter with an independent argument on the partition function that our assumption is indeed correct.
The Coulomb branch coordinates of the $U(r_i - 2)$ terms give the Coulomb branch coordinates of the gauge group $U(kN_f - N_c)$ while the coordinates of the $U(1)^2$ factors are identified with the chiral fields $v_{j,\pm}$ (6.41) that are mapped to the

Coulomb branch coordinates of the electric theory. The superpotential is then given by

$$W = \mathrm{Tr}\,\tilde{X}^{k+1} + \sum_{j=0}^{k-1} M_j\, qY^{k-j-1}\tilde{q} + \sum_{j=0}^{k-1} (v_{j,+}\widetilde{v_{k-1-j,-}} + v_{j,-}\tilde{v}_{k-1-j,+}) \tag{4.37}$$

which is the same as the one appearing in the magnetic theory of Kim-Park duality.

4.3.2 $SU(N_c)$ duality

This duality first appeared in [27] and it was obtained from the duality of the previous section by ungauging the $U(1)$ factor. In order to derive the duality with $SU(N_c)$ gauge group we do not gauge the baryonic symmetry as in the previous case and we start directly from KSS duality in four dimensions.
The Coulomb branch of this theory is different from a $U(N_c)$ theory and the monopole operators are now given by [26]

$$Y_i \sim \exp\left(\frac{\Phi_i - \Phi_{i+1}}{g^2}\right) \qquad Y = \prod_i Y_i \tag{4.38}$$

$$V_{i,j} \sim (X_{11})^i (X_{N_cN_c})^j\, Y \tag{4.39}$$

These monopole operators are related to the $U(N_c)$ ones by the relations

$$V = V_+V_- \qquad V_{i,j} = V_{i,+}V_{-,j} \tag{4.40}$$

After the compactification on the circle, Kaluza-Klein monopoles generate the terms [26]

$$W_{\eta,el} = \sum_{j=0}^{k-1} \eta\, V_{j,k-1-j} \qquad W_{\tilde{\eta},mag} = \sum_{j=0}^{k-1} \tilde{\eta}\, \tilde{V}_{j,k-1-j} \tag{4.41}$$

which can be verified by the counting of the zero modes.
The electric theory is the reduction of the electric theory in four dimensions with the addition of the η superpotential defined above. The same is true for the magnetic theory too.
As in the previous cases, we can flow to a duality withouth these terms in the superpotential by a suitable assignment of real masses.
We start our reduction with $N_f + 1$ flavours. In the electric theory we assign the following values for real masses associated to the $SU(N_f + 1)_L \times SU(N_f + 1)_R \times$

$U(1)_B$ global symmetry

$$m_{el} = \begin{pmatrix} (M_B - M) + m_A & & & \\ & \ddots & & \\ & & (M_B - M) + m_A & \\ & & & (M_B + N_f M) - m_A N_f \end{pmatrix} \tag{4.42}$$

$$= \begin{pmatrix} m_A & & & \\ & \ddots & & \\ & & m_A & \\ & & & -m_A N_f + m \end{pmatrix} \qquad m = M_B + N_f M \tag{4.43}$$

$$\tilde{m}_{el} = \begin{pmatrix} M - M_B + m_A & & & \\ & \ddots & & \\ & & M - M_B + m_A & \\ & & & -M_B - N_f M - m_A N_f \end{pmatrix} \tag{4.44}$$

$$= \begin{pmatrix} m_A & & & \\ & \ddots & & \\ & & m_A & \\ & & & -m - m_A N_f \end{pmatrix} \qquad m = M_B + N_f M \tag{4.45}$$

This mass assignment breaks the global symmetry group to $SU(N_f)_L \times SU(N_f)_R \times U(1)_A \times U(1)_B$.
By comparison of the charges of the fields under the global symmetries, we can map the masses to the magnetic theory

$$\tilde{M} \ = \ -M \qquad \tilde{M}_B \ \to \ \frac{N_c}{k(N_f+1) - N_c} M_B \qquad \tilde{m}_A = m_A \tag{4.46}$$

which results in

$$m_{mag} = \begin{pmatrix} m_1 - m_A & & & \\ & \ddots & & \\ & & m_1 - m_A & \\ & & & m_2 + m_A N_f \end{pmatrix} \tag{4.47}$$

$$\tilde{m}_{mag} = \begin{pmatrix} -m_1 - m_A & & & \\ & \ddots & & \\ & & -m_1 - m_A & \\ & & & -m_2 + m_A N_f \end{pmatrix} \tag{4.48}$$

where we defined

$$m_1 = \tilde{M}_B - \tilde{M} = \frac{k}{k(N_f+1)-N_c}m \tag{4.49}$$

$$m_2 = \tilde{M}_B + N_f\tilde{M} = -\frac{kN_f - N_c}{k(N_f+1)-N_c}m \tag{4.50}$$

The global symmetries in the magnetic theory are broken in the same way as in the electric one.
We can introduce a deformation to the superpotential for both theories

$$W_{el} = \sum_{j=1}^{k} s_j \mathrm{Tr}\, X^j \qquad W_{mag} = \sum_{j=1}^{k} \tilde{s}_j \mathrm{Tr}\, Y^j \tag{4.51}$$

Where the terms proportional to $\mathrm{Tr}\, X, \mathrm{Tr}\, Y$ are Lagrange multipliers to enforce the traceless of X and Y.
By giving a large value to m we flow to a theory withouth η superpotential, as in the previous cases. In the electric side, the deformation breaks the gauge group $SU(N_c) \to SU(i_1) \times \cdots \times SU(i_k) \times U(1)^{k-1}$ with $\sum_j i_j = N_c$.
Taking the vacuum expectation value for σ at the origin, the last quark flavour gets integrated out and we find a theory with N_f flavours and no η superpotential.
In the magnetic side, the perturbation breaks the gauge group in a similar way, from $SU(k(N_f+1)-N_c)$ to

$$U(j_1) \times \cdots \times U(j_k)/U(1) \simeq SU(j_1) \times \cdots \times SU(j_k) \times U(1)^k/U(1) \tag{4.52}$$

$$\simeq SU(j_1) \times \cdots \times SU(j_k) \times U(1)^{k-1} \tag{4.53}$$

$$\text{with} \quad \sum_l j_l = k(N_f+1) - N_c \tag{4.54}$$

However, since each quark has a large real mass, the vacuum cannot be chosen at the origin and we need to choose the vacuum in order to have the largest number of massless fields. As a result, the gauge group is broken further into

$$SU(i_1-1) \times \cdots \times SU(i_k-1) \times U(1)^k \times U(1)^{k-1} \tag{4.55}$$

since the vacuum of $\tilde{\sigma}$ in every $SU(i_k)$ sector is given by

$$\tilde{\sigma}_{i_1} = \begin{pmatrix} -m_1 & & & \\ & \ddots & & \\ & & -m_1 & \\ & & & -m_2 \end{pmatrix}, \ldots, \; \tilde{\sigma}_{i_k} = \begin{pmatrix} -m_1 & & & \\ & \ddots & & \\ & & -m_1 & \\ & & & -m_2 \end{pmatrix} \tag{4.56}$$

The $U(1)^{k-1}$ factor is associated to the breaking of the group from the vacuum expectation values of Y while the $U(1)^k$ factor results from the breaking caused

by the vacuum choice of $\tilde{\sigma}$.
Before doing so, we are going to use $3D\,\mathcal{N}=2$ mirror symmetry [25] in the $U(1)^{k-1}$ sector of the deformed theory. As a result, this sector will be described by a $U(1)$ gauge theory with k flavours $(b_i, \tilde{b}_i)$.
We need to stress that we used mirror symmetry applied to the $U(1)^{k-1}$ gauge sector in the deformed theory but after switching off the deformation, the gauge group becomes $U(k)$ and a mirror dual is not known.

Sistemare qui, aggiungi che è caso limite di dualità note, abeliani duali a singoletti?

We assume that this enhancement to $U(k)$ in the undeformed theory does not modify the results we obtained. We will show an independent method to reduce the duality through the partition function which does not rely on the deformation of the superpotential. This method will yield the same result we found using this assumption.
In order to obtain Kim-Park duality, we will need to switch off the deformation on both sides of the duality.
In the electric side, the gauge group recovers the original $SU(N_c)$ and we find the electric theory of Kim-Park duality, without η superpotential. The charges of the fields are given by

Fields	$SU(N_f)_L$	$SU(N_f)_R$	$U(1)_B$	$U(1)_A$	$U(1)_J$	$U(1)_R$
Q	N_f	0	N_c	1	0	Δ_Q
$\tilde{Q}$	0	$\overline{N_f}$	$-N_c$	1	0	Δ_Q
X	0	0	0	0	0	Δ_X

(4.57)

In the magnetic side, we recover $SU(kN_f - N_c) \times U(1) \times U(1)_{mirror} \simeq U(kN_f - N_c) \times U(1)_{mirror}$ with the following massless fields

- $U(kN_f - N_c)$ quarks in the (anti)fundamental representation $(q, \tilde{q}$
- $U(kN_f - N_c)$ adjoint matter Y
- $U(kN_f - N_c)$ singlets M_j with $N_f \times N_f$ flavours indices.
- $U(1)$ quarks with k flavours $(b, \tilde{b})$, obtained by mirror symmetry

The quantum numbers of these fields are obtained by the matching of the mesons and baryons.

The charges of the fields are the following

	$SU(N_f)_L$	$SU(N_f)_R$	$U(1)_B$	$U(1)_A$	$U(1)_J$	$U(1)_R$
q	$\overline{N_f}$	0	0	-1	0	$\Delta_Y - \Delta_Q$
$\tilde{q}$	0	N_f	0	-1	0	$\Delta_Y - \Delta_Q$
Y	0	0	0	0	0	Δ_Y
M_j	N_f	$\overline{N_f}$	0	2	0	$2\Delta_Q + j\Delta_Y$
b_j	0	0	0	$-N_f$	1	$N_f(1-\Delta_Q) + \Delta_Y(-N_c+j+1)$
$\tilde{b}_j$	0	0	0	$-N_f$	-1	$N_f(1-\Delta_Q) + \Delta_Y(-N_c+j+1)$
$\tilde{v}_{j,\pm}$	1	1	0	N_f	± 1	$-(1-\Delta_Q)N_f + \Delta_Y(N_c+1+j)$

(4.58)

where $\tilde{v}_{i,\pm}$ is one of the two Coulomb branch coordinates of the $U(kN_f - N_c)$ gauge group.
Note that the fields $(b_i, \tilde{b}_i)$ are charged under the topological symmetry and have the right quantum numbers to couple with the monopole operators $\tilde{v}_{i,\pm}$.
The baryonic symmetry can mix with the $U(1)$ factor in the $U(kN_f - N_c)$ gauge group and as a result its definition is ambiguous. We chose to remove this ambiguity by leaving all the fields uncharged under it. Another possibility is to give opposite charges to the quarks $(q, \tilde{q})$.
The η superpotential can be rewritten using monopole operators at high energy through the identification [26]

$$\tilde{V}_{j,k-1-j} = b_j \tilde{v}_{k-1-j,-} \tag{4.59}$$

As in the previous cases, there is a Affleck-Harvey-Witten superpotential associated to the gauge symmetry breaking $SU(k(N_f+1) - N_c) \to U(kN_f - N_c) \times U(1)^{mirror}$ given by

$$W_{AHW} = \sum_{j=0}^{k-1} \tilde{b}_{j+1} \tilde{v}_{k-1-j,+} \tag{4.60}$$

Adding all these terms we obtain the superpotential of the theory which reads

$$W_{mag} = \mathrm{Tr}\, Y^{k+1} + \sum_{j=0}^{k-1} M_j\, \tilde{q} Y^{k-1-j} q + \sum_{j=0}^{k-1} \left(b_j \tilde{v}_{k-j-1,-} + \tilde{b}_j \tilde{v}_{k-j-1,+} \right) \tag{4.61}$$

which is consistent with what was found with ungauging technique from the $U(N_c)$ duality in [27].

Chapter 5

Superconformal index and partition functions

5.1 Superconformal index

The superconformal index is a quantity that can be calculated in superconformal field theories that counts the number of short multiplets in the theory, which satisfy a BPS bound. The scaling dimensions of these states saturate a unitarity bound. For $\mathcal{N}=4$ field theories it was first introduced in [28] and it was later modified by Romelsberger for $\mathcal{N}=1$ field theories in [29].
The superconformal index can be introduced as a generalization of the Witten index, which is defined as

$$I = \mathrm{Tr}\left[(-1)^F e^{-\beta E}\right] \tag{5.1}$$

where F is the fermion number and E the energy of the state.
The exponential term in the energy can be seen as a regulator, otherwise the counting would not be well defined. However, the energy E can be substituted with any operator with positive eigenvalues that commutes with the supercharges. There can be added other operators that commute with the supercharges in order to resolve the degeneracies.
We are interested in four dimensional superconformal field theories which are naturally quantized in radial quantization on $S^3 \times \mathbb{R}$. As a result, the isometry group is now $SU(2)_L \times SU(2)_R \times \mathbb{R}$ and the eight supercharges split into doublets of $SU(2)_L$ for Q_α and of $SU(2)_R$ for S_α.
We can pick one of the supercharges, e.g. Q_1, and from the superconformal algebra we have the relation

$$\{Q_1, Q_1^\dagger\} = H - \frac{3}{2}R - 2J_1 = \Delta \tag{5.2}$$

where H is the Hamiltonian in radial quantization, R is the R-charge generator and J_1 is one of the generators of $SU(2)_L$. However, in a free theory Δ has an infinite number of ground states. For this reason the operator used as a regulator is [29]

$$\Xi = H - \frac{1}{2}R \geq \frac{2}{3}H \tag{5.3}$$

which damps as well as the Hamiltonian.
As we said before, we can add other terms in order to differentiate the various states. Such terms are given by the Cartan generators J_1, J_2 of the $SU(2)_L \times SU(2)_R$ isometry of the three-sphere and by the generators e_a of the internal symmetry group.
The superconformal index on the sphere $S^3 \times S^1$ is then defined as [30]

$$i(x, h_a) = \mathrm{Tr}\left[(-1)^F e^{-\mu\Xi} x^{2J_2} \prod_a h_a^{e_a}\right] \tag{5.4}$$

$$= \mathrm{Tr}\left[(-1)^F t^{\Delta+2J_1-R} x^{2J_2} \prod_i h_a^{e_a}\right] \tag{5.5}$$

where we defined $t = e^{-\mu}$. WIth a change of variables, the superconformal index can be written as

$$i(p, q, h_a) = \mathrm{Tr}\left[(-1)^F (pq)^{\frac{\Delta}{2}} p^{J_1+J_2-\frac{R}{2}} q^{J_1-J_2-\frac{R}{2}} \prod_a u_a^{e_a}\right] \tag{5.6}$$

where p, q are defined by

$$p = tx \quad q = tx^{-1} \tag{5.7}$$

The values of p, q must satisfy the following conditions in order for the index to be real and well-defined.

$$\mathrm{Im}(pq) = 0 \qquad |p/q| = 1 \qquad |pq| < 1 \tag{5.8}$$

The superconformal index receives contribution only from states with $\Delta = 0$, since $J_1 \pm J_2 - \frac{R}{2}$ and the internal generators commute with Δ. The superconformal index can be calculated in two steps. First, we need to obtain the index on *single particle states*.
The single particle index for chiral multiples Φ_i with R-charge r_i, with flavour symmetry group F_i and gauge group G_i reads [31]

$$i_\Phi(t, x, h, g) = \sum_i \frac{t^{r_i}\chi_{F_i}(h)\,\chi_{G_i}(z) - t^{2-r_i}\chi_{\bar{F}_i}(h)\chi_{\bar{G}_i}(z)}{(1-tx)(1-tx^{-1})} = \tag{5.9}$$

$$= \sum_i \frac{(pq)^{\frac{r_i}{2}}\chi_{F_i}(h)\,\chi_{G_i}(z) - (pq)^{1-\frac{r_i}{2}}\chi_{\bar{F}_i}(h)\chi_{\bar{G}_i}(z)}{(1-p)(1-q)} \tag{5.10}$$

where h and z are the chemical potential for the global and gauge symmetry group respectively. $\chi_{F_i}(h)$ and $\chi_{G_i}(z)$ are the characters of the representation of Φ_i.
The index for a vector superfield in the adjoint representation of a gauge group G is given by [31]

$$i_V(t,x,g) = \frac{2t^2 - t(x+x^{-1})}{(1-tx)(1-tx^{-1})}\chi_{adj}(z) = \frac{2(pq)-(p+q)}{(1-p)(1-q)}\chi_{adj}(z) \tag{5.11}$$

The index on the single particle states for the complete theory is given by the sum of indices for every field. Then, the superconformal index can be calculated by taking the Plethystic exponential [32] of the full single particle index

$$I(t,x,h) = \int_G \mathrm{d}\mu(g)\, \exp\left(\sum_{n=1}^{\infty} i(t^n, x^n, h^n, z^n)\right) \tag{5.12}$$

where $\mathrm{d}\mu(g)$ is a G invariant measure.
The definition of the superconformal index can be extended to the manifold $S^3_b \times S^1$, where S^3_b is the squashed three-sphere whose isometry group is only $U(1)_L \times U(1)_R$ [33]. The squashed sphere is defined by considering the metric

$$ds^2 = l^2(dx_0^2 + dx_1^2) + \bar{l}^2(dx_2^2 + dx_3^2) \tag{5.13}$$
$$\text{with} \qquad x_0^2 + x_1^2 + x_2^2 + x_3^2 = 1 \tag{5.14}$$

With the change of coordinates

$$x_0 = \cos\theta\cos\phi \tag{5.15}$$
$$x_1 = \cos\theta\sin\phi \tag{5.16}$$
$$x_2 = \sin\theta\cos\chi \tag{5.17}$$
$$x_3 = \sin\theta\sin\chi \tag{5.18}$$

we obtain the metric of the squashed sphere with squashing parameter $b = l/\bar{l}$

$$ds^2 = f(\theta)^2 d\theta^2 + l^2\cos^2\theta d\phi^2 + \bar{l}^2\sin^2\theta d\chi^2 \tag{5.19}$$
$$\text{with} \quad f(\theta) = \sqrt{l^2\sin^2\theta + \bar{l}^2\cos^2\theta} \tag{5.20}$$

The index on the squashed sphere $S^3_b \times S^1$ should coincide with the index on the round sphere $S^3 \times S^1$ since they have the same topology and the indices receive contributions from the same BPS states. The indices match up to a rescaling of the fugacities $(p,q) \to (p^{\frac{1}{a}}, q^{\frac{1}{b}})$ [34] and a factor given by the Casimir energy [35]. The rescaling of the fugacities does not modify the physical content of the index because (p,q) are arbitrary parameters and they still satisfy the conditions (5.8).

An exact calculation of the index can be performed using the following mathematical identities involving the elliptic gamma function Γ_e. The index of a chiral field in general can be written as

$$i_\Phi(p,q,y) = \frac{y - pq/y}{(1-p)(1-q)} \qquad \text{with} \quad p = tx \; q = tx^{-1} \; y = t^R \tag{5.21}$$

where z is the fugacity associated to gauge and/or global symmetries. It can be written as an elliptic gamma function through

$$\exp\left(\sum_{n=1}^{\infty} \frac{1}{n} i_S(p^n, q^n, y^n)\right) = \prod_{j,k\geq 0} \frac{1 - y^{-1}p^{j+1}q^{k+1}}{1 - yp^jq^k} \stackrel{def}{=} \Gamma_e(y; p, q) \tag{5.22}$$

$$\text{where} \quad i_S(p,q,y) = \frac{y - pq/y}{(1-p)(1-q)} \tag{5.23}$$

The contribution to the index of chiral field in the fundamental or antifundamental representation of global and gauge symmetries can be generally written as

$$\Gamma_e\left((pq)^{\frac{R_i}{2}} \prod_a u_a^{e_a}; p, q\right) \tag{5.24}$$

Using the same variables we can write the *single particle index* of a vector multiplet into

$$i_V(p,q) = -\left(\frac{p}{1-p} + \frac{q}{1-q}\right) \tag{5.25}$$

and use the following identities, where z is the chemical potential associated to the gauge group

$$\exp\left(\sum_{n=1}^{\infty} \frac{1}{n} i_V(p^n, q^n)(z^n + z^{-n})\right) = \frac{1}{(1-z)(1-z^{-1})\Gamma_e(z;p,q)\Gamma_e(z^{-1};p,q)} \tag{5.26}$$

$$\exp\left(\sum_{n=1}^{\infty} \frac{1}{n} i_V(p^n, q^n)\right) = (p;p)(q;q) \tag{5.27}$$

and the *Q-Pochhammer symbol* is defined as

$$(x;p) = \prod_{j\geq 0}(1 - xp^j) \tag{5.28}$$

The contribution of a vector field in the adjoint representation reads

$$(p;p)^{r_G}(q;q)^{r_G} \prod_{1\leq i<j\leq N_c} \frac{1}{(1-\frac{z_i}{z_j})(1-\frac{z_j}{z_i})\Gamma_e(\frac{z_i}{z_j};p,q)\Gamma_e(\frac{z_j}{z_i};p,q)} \tag{5.29}$$

The calculation of the superconformal index is now reduced to a integral over the gauge group of products of elliptic gamma functions.
We will provide a more detailed calculation for the electric and magnetic theories of Kutasov-Schwimmer duality in the next chapter.
The equality of the superconformal index for pairs of dual theories can be demonstrated using integral identities of elliptic Gamma functions that were first demonstrated by Rains in [36]. We will use this fact as a starting point in our discussion on the dimensional reduction of dualities to three dimensions.

5.2 Localization

Localization was first introduced by Witten [37] and Nekrasov [38] and later introduced for $\mathcal{N} = 2, 4$ theories on S^4 by Pestun in [39] and then applied to S^3 and S^3_b for $\mathcal{N} = 2$ theories in [40], [41] and [33]. It can be used to calculate exactly the functional integral on a curved manifold.
The basic principle about localization is the following. Suppose that the theory in question admits a *fermionic* symmetry δ which leaves the measure of the path integral invariant and that it squares to a *bosonic* symmetry Δ_B, which can be a Lorentz or gauge transformation.
Consider the following modified partition function, which correspond to the partition function we want to calculate for $t = 0$

$$Z(t) = \int \mathcal{D}\phi \; e^{-S(\phi)-t(\delta V)} \tag{5.30}$$

where V is a fermionic operator invariant under Δ_B, i.e. $\Delta_B V = 0$.
This partition function is independent of t since

$$\frac{dZ}{dt} = -\int \mathcal{D}\phi \; \delta V \, e^{-S(\phi)-t(\delta V)} = -\int \mathcal{D}\phi \; \delta \left(e^{-S(\phi)-t(\delta V)}\right) = 0 \tag{5.31}$$

Where the last equality holds if the fields decay sufficiently fast at infinity.
Since $Z(t)$ is independent of t we can calculate it at $t = 0$, which is the original partition function or at $t = \infty$, where the integral localizes on the saddle points of the integral if the bosonic part of δV is positive definite.
In three dimensions, these saddle points are parametrized by the values of the scalar field in the vector multiplet and through gauge symmetry the integral can be reduced to the Cartan subalgebra. The saddle point conditions for the other fields set their value on the saddle points to zero.
The partition function receives contribution from the path integral only at one-loop and it can be computed exactly. The one-loop term is given by a determinant of Laplace and Dirac operators for every field appearing in the lagrangian.

$$Z \sim \int [d\sigma] \; e^{S_{saddle}(\sigma)} \, \frac{\Delta_\lambda}{\Delta_{A_\mu}} \prod_i \frac{\Delta_{\psi_i}}{\Delta_{\phi_i}} \tag{5.32}$$

The partition function can be written

$$Z(\mu_a, \nu_b) \sim \int [d\sigma]\, e^{S_{saddle}(\sigma)}\, Z^{1-loop}_{gauge}(\sigma)\, Z^{1-loop}_{matter}(\sigma, \mu_a, \nu_b) \tag{5.33}$$

where μ_a, ν_b are the real masses associated to global symmetries for left and right quarks. The one loop contribution for matter fields is given by a hyperbolic gamma function, sometimes called *double-sine* function [33] which depends on the charges of the field under the global and gauge symmetries.

$$Z^{1-loop}_{\Phi_i} = \prod_i \Gamma_h(\omega \Delta_i + \sum_a \mu_a e_a; \omega_1, \omega_2) \tag{5.34}$$

where $\omega_1 = ib$, $\omega_2 = ib^{-1}$ and $\omega = \frac{\omega_1+\omega_2}{2}$. An overview on the properties of the hyperbolic gamma function can be found in appendix B.
For the vector multiplet the result is similar and since it is in the adjoint representation of the gauge group it is given by

$$Z^{1-loop}_{V} = \prod_{1 \leq i < j \leq dim(G)} \frac{1}{\Gamma_h(\pm(\sigma_i - \sigma_j))} \tag{5.35}$$

As a result, the partition function can be calculated by an integral on the Cartan subalgebra of the gauge group of products of hyperbolic gamma functions and exponential factors that are associate to Fayet-Iliopulos and Chern-Simons terms.

5.3 Reduction of the superconformal index to the partition function

The superconformal index calculated on $S^3_b \times S^1$ reduces to the partition function of the dimensionally reduced theory on S^3_b with η superpotential in the limit $S^1 \to 0$ [42] [43].
A heuristic argument to support this claim is that the superconformal index receives contributions only from BPS states in four dimensions which are in one-to-one correspondence with three dimensional states with non zero eigenvalue of the operators appearing in (5.32).
Moreover, once the BPS condition in four dimensions is dimensionally reduced to three dimensions, it is equivalent to the saddle point conditions of the partition function [34].
In order to obtain the partition function, it is necessary that all the fugacities appearing in the index flow to one [43]. They must be parametrized as function of the radius r of S^1

$$p = e^{2\pi i r \omega_1} \quad q = e^{2\pi i r \omega_2} \quad u_a = e^{2\pi i r \mu_a} \quad z_i = e^{2\pi i r \sigma_i} \tag{5.36}$$

where σ_i is the scalar in the vector multiplet and μ_a are the scalars in the background gauge multiplet that become real masses .
Since the superconformal index is written in terms of elliptic gamma functions, we can use the following identity [44] to perform the limit $r \to 0$

$$\lim_{r\to 0^+} \Gamma_e(e^{2\pi i r z}; e^{2\pi i r \omega_1}, e^{2\pi i r \omega_2}) \sim e^{\frac{-i\pi^2}{6r\omega_1\omega_2}(z-\omega)} \Gamma_h(z;\omega_1,\omega_2) \tag{5.37}$$

Using this formula, we can reduce every elliptic gamma function in the index to a hyperbolic gamma function, which is the building block of the partition function in three dimension. The parametrization of the fugacities in (5.36) ensures that the arguments of the hyperbolic gamma functions obtained using this identity match the expressions present in the 1-loop corrections [43] that we introduced in (5.34) and (5.35).
Applying the formula for every factor in the index we obtain the expression for the partition function in three dimension multiplied by a divergent prefactor, which is given by the product of the exponential factors in (5.37) for every field.
It can be shown [19] that this prefactor is proportional to the $U(1)_R$-gravity-gravity and flavour-gravity-gravity anomalies which coincide for dual theories. As a result, this factor can be removed without problems since we are considering the reduction of the index for dual theories.
The advantage of obtaining the partition functions in this way is that the equality of the partition functions for dual theories is guaranteed by the matching of the corresponding four dimensional indices. The matching of the indices for dual theories is supported by mathematical identities between integrals of elliptic gamma functions [36].
Reducing the indices to three dimensions we obtained expressions involving integrals of hyperbolic gamma functions. Mathematical identities that support the equality of the partition functions [44] are not known for every three dimensional duality. In any case, when the reduction process from four dimensions is well defined, the identity between the indices guarantees that the partition functions are identical.

Chapter 6

Dimensional reduction of KSS duality

The scope of this chapter is to provide an independent check of the reduction of KSS duality with $SU(N_c)$ gauge group to the Kim-Park duality [26]. We introduced these dualities in previous sections and we reviewed the procedure of dimensional reduction in field theory of [26] in section 4.3.2.
We will reduce KSS duality to three dimensions first by calculating the superconformal index in four dimensions and then reducing it to the three dimensional partition function by shrinking the radius of the circle.
As we said before, the presence of the compactified dimension induces an additional η superpotential, which can be seen in the partition function from the constraints it imposes on the real masses in the partition function. We will flow to the Kim-Park duality by assigning real masses in a similar way to section 4.3.2.
As a result, we give an alternative derivation of the reduction of KSS duality without relying on some of the assumptions made in [26].

6.1 Electric theory

The electric theory was introduced in section 2.3 and we report the charge table in order to fix our notation. For the four dimensional theory we have

	$SU(N_c)$	$SU(N_f)_L$	$SU(N_f)_R$	$U(1)_B$	$U(1)_R$
Q	N_c	N_F	1	1	$1 - R_X \frac{N_c}{N_f} = R_Q$
$\tilde{Q}$	$\overline{N_c}$	1	$\overline{N_F}$	-1	$1 - R_X \frac{N_c}{N_f} = R_Q$
X	$N_c^2 - 1$	1	1	0	$\frac{2}{k+1} = R_X$

(6.1)

and for the three dimensional theory

$$
\begin{array}{|c|c|cccccc|}
\hline
 & SU(N_c) & SU(N_f)_L & SU(N_f)_R & U(1)_B & U(1)_A & U(1)_J & U(1)_R \\
\hline
Q & N_c & N_f & 1 & 1 & 1 & 0 & \Delta_Q \\
\tilde{Q} & \overline{N_c} & 1 & \overline{N_f} & -1 & 1 & 0 & \Delta_Q \\
X & N_c^2-1 & 1 & 1 & 0 & 0 & 0 & \frac{2}{k+1}=\Delta_X \\
\hline
\end{array}
\tag{6.2}
$$

6.1.1 Calculation of the index

The single particle index can be calculated with equations (5.10) and (5.11)

$$
\begin{aligned}
&i_E(p,q,v,y,\tilde{y},z) = \\
&-\left(\frac{p}{1-p}+\frac{q}{1-q}-\frac{1}{(1-p)(1-q)}\left((pq)^{\frac{R_X}{2}}-(pq)^{1-\frac{R_X}{2}}\right)\right)\left(p_{N_c}(z)\,p_{N_c}(z^{-1})-1\right) \\
&+\frac{1}{(1-p)(1-q)}\Bigg((pq)^{\frac{1}{2}R_Q}\,v\,p_{N_f}(y)\,p_{N_c}(z)-(pq)^{1-\frac{1}{2}R_Q}\,\frac{1}{v}\,p_{N_f}(y^{-1})\,p_{N_c}(z^{-1}) \\
&\qquad +(pq)^{\frac{1}{2}R_Q}\,\frac{1}{v}\,p_{N_f}(\tilde{y}\,)\,p_{N_c}(z^{-1})-(pq)^{1-\frac{1}{2}R_Q}\,v\,p_{N_f}(\tilde{y}^{-1})\,p_{N_c}(z)\Bigg)
\end{aligned}
\tag{6.3}
$$

where p,q are defined as in the previous chapters and the other fugacities are associated to the symmetries

$$
v\to U(1)_B \quad y\to SU(N_F)_L \quad \tilde{y}\to SU(N_F)_R \quad z\to SU(N_c)
\tag{6.4}
$$

In (6.3) we have written explicitly the character for the adjoint, fundamental and antifundamental representations of $SU(N)$, which are given by

$$
\chi_{adj}(z)=p_N(z)\,p_N(z^{-1})-1 \quad \chi_N(y)=p_N(y) \quad \chi_{\overline{N}}(\tilde{y})=p_N(\tilde{y}^{-1})
\tag{6.5}
$$

$$
p_N(x)=\sum_{i=1}^{N}x_i \qquad p_N(x)=\sum_{i=1}^{N}\frac{1}{x_i}
\tag{6.6}
$$

Moreover, given that the fugacities are in the Cartan subgroup of $SU(N_c)$ or $SU(N_f)$ we have the following constraints

$$
\prod_i^{N_c} z_i=1 \qquad \prod_i^{N_f} y_i=1 \qquad \prod_i^{N_f}\tilde{y}_i=1
\tag{6.7}
$$

If we use the definition of the polynomials $p_N(x)$ we obtain

$$
\begin{aligned}
&i_E(p,q,v,y,\tilde{y},z) = \\
&-\left(\frac{p}{1-p}+\frac{q}{1-q}-\frac{1}{(1-p)(1-q)}\left((pq)^{\frac{R_X}{2}}-(pq)^{1-\frac{R_X}{2}}\right)\right)\left(\sum_{1\le i,j\le N_c}\frac{z_i}{z_j}-1\right) \\
&+\frac{1}{(1-p)(1-q)}\sum_{a=1}^{N_f}\sum_{j=1}^{N_c}\Bigg((pq)^{\frac{1}{2}R_Q}\,v\,y_a\,z_j-(pq)^{1-\frac{1}{2}R_Q}\,v^{-1}\,y_a^{-1}\,z_j^{-1} \\
&\qquad\qquad +(pq)^{\frac{1}{2}R_Q}\,v^{-1}\,\tilde{y}_a\,z_j^{-1}-(pq)^{1-\frac{1}{2}R_Q}\,v\,\tilde{y}_a^{-1}\,z_j\Bigg)
\end{aligned}
\tag{6.8}
$$

The index is given by the sum of contributions for every field in the theory

$$
i_E(p,q,v,y,\tilde{y},z)=i_E^V(p,q,z)+i_E^X(p,q,z)+i_E^{Q,\tilde{Q}}(p,q,v,y,z) \tag{6.9}
$$

and the complete index is given by the Plethystic exponential

$$
\begin{aligned}
I_E(p,q,v,y,\tilde{y}) &= \int_{SU(N_c)} d\mu(z)\ \exp\left(\sum_{n=1}^{\infty}\frac{1}{n}i_E(p^n,q^n,v^n,y^n,\tilde{y}^n,z^n)\right) \\
&= \int_{SU(N_c)} d\mu(z) \prod_{fields\ F} I_E^F(p,q,v,y,\tilde{y},z)
\end{aligned}
\tag{6.10}
$$

where I_E^F is given by the Plethystic exponential for the field F.
Given that the fugacity z is in the maximal torus of the gauge group, we can restrict the integral to T^{N_c-1}. The integration on the gauge group reduces to

$$
\int_{SU(N_c)} d\mu(z)\,f(z)=\frac{1}{N_c!}\int_{T^{N_c-1}}\prod_{i=1}^{N_c}\frac{dz_i}{2\pi i z_i}\Delta(z)\Delta(z^{-1})f(z)\Bigg|_{\prod_{i=1}^{N_c} z_i=1} \tag{6.11}
$$

where $\Delta(z)$ is the Vandermonde determinant, which is given by

$$
\Delta(z) = \prod_{\substack{1\le i,j<\le N_c\\ i\neq j}}^{N_c}(z_i-z_j) = \prod_{\substack{1\le i,j<\le N_c\\ i\neq j}}^{N_c}\left(1-\frac{z_i}{z_j}\right)z_j = \prod_{\substack{1\le i,j<\le N_c\\ i\neq j}}^{N_c}\left(1-\frac{z_i}{z_j}\right)
$$

where the last equality holds because $\prod_{i=1}^{N_c} z_i=1$.

We will calculate the superconformal index by considering separately the contributions from the various fields. The complete calculations for every field can be found in appendix C.

Vector field The vector field contributes to the index as

$$
\begin{aligned}
I_E^V(p,q,z) =& \exp\left(\sum_{n=1}^{\infty}\frac{1}{n} i_E^{Vett}(p^n,q^n,z^n)\right) = \\
& \exp\left(\sum_{n=1}^{\infty} -\frac{1}{n}\left(\frac{p^n}{1-p^n}+\frac{q^n}{1-q^n}\right)\left(\left(\sum_{1\le i,j\le N_c}\frac{z_i^n}{z_j^n}\right)-1\right)\right) = \\
=&(p;p)^{N_c-1}(q;q)^{N_c-1}\prod_{1\le i<j\le N_c}\frac{1}{\left(1-\frac{z_i}{z_j}\right)\left(1-\frac{z_j}{z_i}\right)\Gamma_e(\frac{z_i}{z_j};p,q)\Gamma_e(\frac{z_j}{z_i};p,q)} \\
=&(p;p)^{N_c-1}(q;q)^{N_c-1}\frac{1}{\Delta(z)\Delta(z^{-1})}\prod_{1\le i<j\le N_c}\frac{1}{\Gamma_e(\frac{z_i}{z_j};p,q)\Gamma_e(\frac{z_j}{z_i};p,q)}
\end{aligned}
\tag{6.12}
$$

Note that there is a term that cancels the Vandermonde determinant coming from the integration measure.

Adjoint matter The single particle index for this field reads

$$
i_E^X(p,q,z) = \frac{1}{(1-p)(1-q)}\left((p\,q)^{\frac{R_X}{2}}-(p\,q)^{1-\frac{R_X}{2}}\right)\left(\left(\sum_{1\le i,j\le N_c}\frac{z_i}{z_j}\right)-1\right) \tag{6.13}
$$

which results in

$$
I_E^X(p,q,z) = \Gamma_e\left((pq)^{\frac{R_X}{2}};p,q\right)^{N_c-1}\prod_{1\le i<j\le N_c}\Gamma_e\left((pq)^{\frac{R_X}{2}}\frac{z_i}{z_j}\right)\Gamma_e\left((pq)^{\frac{R_X}{2}}\frac{z_j}{z_i}\right) \tag{6.14}
$$

Quarks The quarks in the fundamental and antifundamental representation have a single particle index of the form

$$
\begin{aligned}
& i_E^{Q,\tilde{Q}}(p,q,v,y,\tilde{y}) = \\
& \frac{1}{(1-p)(1-q)}\sum_{j=1}^{N_c}\left(\sum_{a=1}^{N_f}\left((pq)^{\frac{1}{2}R_Q}\,v\,y_a\,z_j-(pq)^{1-\frac{1}{2}R_Q}\,v^{-1}\,y_a^{-1}\,z_j^{-1}\right)\right. \\
& \left. +\sum_{b=1}^{N_f}\left((pq)^{\frac{1}{2}R_Q}\,v^{-1}\,\tilde{y}_b\,z_j^{-1}-(pq)^{1-\frac{1}{2}R_Q}\,v\,\tilde{y}_b^{-1}\,z_j\right)\right)
\end{aligned}
\tag{6.15}
$$

where Q and $\tilde{Q}$ generate the first and second line of the index, respectively. Their contribution can be written as

$$
I_E^{Q,\tilde{Q}}(p,q,v,y,\tilde{y},z) = \prod_{j}^{N_c}\prod_{a,b}^{N_f}\Gamma_e\left((pq)^{\frac{R_Q}{2}}\,v\,y_a\,z_j\right)\Gamma_e\left((pq)^{\frac{R_Q}{2}}\,v^{-1}\,\tilde{y}_b^{-1}\,z_j^{-1}\right) \tag{6.16}
$$

Expression of the superconformal index

The formula for the superconformal index can be obtained by multiplying all the contributions calculated above and reads

$$I_E(p,q,v,y,\tilde{y}) = \frac{(p;p)^{N_c-1}(q;q)^{N_c-1}}{N_c!}\,\Gamma_e\left((pq)^{\frac{R_X}{2}};p,q\right)^{N_c-1}$$
$$\oint_{T^{N_c-1}}\left(\prod_{i=1}^{N_c}\frac{dz_i}{2\pi i z_i}\right)\delta\left(\prod_{i=1}^{N_c} z_i - 1\right)\prod_{1\le i<j\le N_c}\frac{\Gamma_e\left((pq)^{\frac{R_X}{2}}\frac{z_i}{z_j}\right)\Gamma_e\left((pq)^{\frac{R_X}{2}}\frac{z_j}{z_i}\right)}{\Gamma_e\left(\frac{z_i}{z_j};p,q\right)\Gamma_e\left(\frac{z_j}{z_i};p,q\right)}$$
$$\prod_{j}^{N_c}\prod_{a,b}^{N_f}\Gamma_e\left((pq)^{\frac{R_Q}{2}}\,v\,y_a\,z_j\right)\Gamma_e\left((pq)^{\frac{R_Q}{2}}\,v^{-1}\,\tilde{y}_b^{-1}\,z_j^{-1}\right) \quad (6.17)$$

The index is written as an elliptic integral and as a result it is well defined only if the fugacities satisfy a balancing condition. It is given by

$$(pq)^{N_c R_X + N_f R_Q}\prod_{a=1}^{N_f} y_a y_a^{-1} = (pq)^{N_f} \quad (6.18)$$

Physically, this constraint correspond to the anomaly-free condition of the R-symmetry in four dimensions and for the compactified theory is imposed by the η superpotential.

6.1.2 Reduction of the superconformal index to the partition function

We can now proceed to reduce the index to the partition function on the squashed three sphere by appliying the procedure we introduced in the previous chapter. We need to parametrize the fugacities as

$$p = e^{2\pi i r\omega_1} \quad q = e^{2\pi i r\omega_2} \quad z_i = e^{2\pi i r\sigma_i} \quad (6.19)$$
$$y_a = e^{2\pi i r m_a} \quad y_a = e^{2\pi i r\tilde{m}_a} \quad v = e^{2\pi i r m_B} \quad (6.20)$$

and we have the following conditions on the real masses

$$\sum_a m_a = 0 \qquad \sum_b m_b = 0 \qquad \sum_j \sigma_j = 0 \quad (6.21)$$

By using the identity (5.37) on each term we can obtain the partition function as an integral of products of hyperbolic gamma functions. An explicit calculation can be found in appendix C.2.
We will use the following notation for the hyperbolic gamma function

$$\Gamma_h(z;\omega_1,\omega_2) = \Gamma_h(z) \qquad \Gamma_h(u\pm z) = \Gamma_h(u+z)\Gamma_h(u-z) \quad (6.22)$$

From now on we will denote the three dimensional R-charges of quarks as Δ_Q and as Δ_X for the adjoint matter because they are going to be different from the four dimensional values, which are fixed by the R-symmetry anomaly.

Vector field For the vector field in the adjoint representation we can use the identity (B.8) applied to $SU(N_c)$ gauge group

$$\lim_{r\to 0}\frac{(p;p)^{N_c-1}(q;q)^{N_c-1}}{N_c!}\oint_{T^{N_c-1}}\prod_{j=1}^{N_c-1}\frac{dz_j}{2\pi i z_j}\prod_{1\le i<j\le N_c}\frac{1}{\Gamma_e(\frac{z_i}{z_j})\Gamma_e(\frac{z_j}{z_i})}\sim$$
$$\sim \exp\left(-\frac{i\pi\omega(N_c^2-1)}{6r\omega_1\omega_2}\right)\frac{1}{N_c!}\int\prod_{j=1}^{N_c-1}\frac{d\sigma_j}{\sqrt{-\omega_1\omega_2}}\prod_{1\le i<j\le N_c}\frac{1}{\Gamma_h(\pm(\sigma_i-\sigma_j)} \quad (6.23)$$

Adjoint matter

$$\Gamma_e\left((pq)^{\frac{\Delta_X}{2}}\right)^{N_c-1}\prod_{1\le i<j\le N_c}\Gamma_e\left((pq)^{\frac{\Delta_X}{2}}\left(\frac{z_i}{z_j}\right)\right)\Gamma_e\left((pq)^{\frac{\Delta_X}{2}}\left(\frac{z_j}{z_i}\right)\right)\sim$$
$$\exp\left(-\frac{i\pi}{6r\omega_1\omega_2}\left((N_c^2-1)\omega(\Delta_X-1)\right)\right)\prod_{1\le i<j\le N_c}\Gamma_h\left(\Delta_X\omega\pm(\sigma_i-\sigma_j)\right) \quad (6.24)$$

Quarks

$$\prod_{a,b}^{N_f}\prod_{j}^{N_c}\Gamma_e\left((pq)^{\Delta_Q v y_a z_j}\right)\Gamma_e\left((pq)^{\Delta_Q v^{-1}\tilde{y}_b^{-1}z_j^{-1}}\right)=$$
$$=\exp\left(-\frac{i\pi}{6r\omega_1\omega_2}(2\omega N_cN_f(R_Q-1)+N_c\sum_a m_a-N_c\sum_b\tilde{m}_b)\right)\times$$
$$\times\prod_{a,b}^{N_f}\prod_{j}^{N_c}\Gamma_h(\mu_a+\sigma_j)\Gamma_h(\nu_b-\sigma_j) \quad (6.25)$$

Where we defined the real masses μ_a,ν_b

$$\mu_a=\omega\Delta_Q+m_a+m_B \quad \nu_b=\omega\Delta_Q-\tilde{m}_b-m_B \quad (6.26)$$

Divergent contribution

Summing up all the exponential terms from the previous limits we obtain

$$\exp\Bigg[-\frac{i\pi}{6r\omega_1\omega_2}\Bigg(\omega\Bigg((N_c^2-1)+(N_c^2-1)(\Delta_X-1)+$$
$$+2N_cN_f(R_Q-1)\Bigg)+N_c\sum_a m_a-N_c\sum_b\tilde{m}_b\Bigg)\Bigg] \quad (6.27)$$

which is compatible with the following general formula [19]

$$\exp\left[-\frac{i\pi\,\omega}{6r\omega_1\omega_2}\left(\omega\left(|G|+\sum_\alpha(R_\alpha-1)\right)+\sum_a m_a\sum_\alpha e^{(\alpha)_a}\right)\right] \tag{6.28}$$

which is proportional to the gravitational anomaly [19] which is identical between the electric and the magnetic theory because of the KSS duality.

Partition function

We can write down the formula for the partition function of the electric theory by putting all these pieces together and ignoring the divergent prefactor we calculated previously. The partition function is then given by

$$\begin{aligned} Z_{el}(\mu_a,\nu_b) &= \frac{1}{N_c!}\,\Gamma_h(\Delta_X\omega;\omega_1,\omega_2)^{N_c-1}\int\prod_{i=1}^{N_c}\frac{d\sigma_i}{\sqrt{-\omega_1\omega_2}}\,\delta(\sum_i\sigma_i) \\ &\prod_{1\le i<j\le N_c}\frac{\Gamma_h(\Delta_X\omega\pm(\sigma_i-\sigma_j))}{\Gamma_h(\pm(\sigma_i-\sigma_j))}\prod_{a,b}^{N_f}\prod_{j=1}^{N_c}\Gamma_h(\mu_a+\sigma_j)\Gamma_h(\nu_b-\sigma_j) \end{aligned} \tag{6.29}$$

with

$$\mu_a=\omega\Delta_Q+m_a+m_B \quad \nu_b=\omega\Delta_Q-\tilde{m}_b-m_B \tag{6.30}$$

Consider that we obtained the partition function by compactifying on S^1 we have the following condition on the real masses, obtained from (6.18)

$$\frac{1}{2}\left(\sum_a\mu_a+\nu_a\right)=\omega\left(-N_c+N_c(1-\Delta_X)+N_f\right) \tag{6.31}$$

Note that this condition is equivalent to require that the R-symmetry in four dimension is anomaly free. As a result the R-charges of the fields for the theory with η superpotential are identical to the four dimensional R-charges, as we noted in the field theory side of the duality in the previous chapters.

6.1.3 Flow to a duality without η superpotential

In order to obtain the duality without η superpotential we start with N_f+1 flavours and we assign real masses in the same way as (4.45).
The real mass assignment breaks the gauge group $SU(N_f+1)_L\times SU(N_f+1)_R\times U(1)_B\to SU(N_f)_L\times SU(N_f)_R\times U(1)_A\times U(1)_B$.

We will assign a large value to the mass m. The real masses are given by

$$\mu_a = \begin{cases} \omega\,\Delta_Q + m_a + m_A + m_B & a = 1,\ldots,N_f \\ \omega\,\Delta_M + m - m_A N_f + m_B & a = N_f + 1 \end{cases}$$
$$\nu_b = \begin{cases} \omega\,\Delta_Q + \tilde{m}_b + m_A - m_B & b = 1,\ldots,N_f \\ \omega\,\Delta_M - m - m_A N_f - m_B & b = N_f + 1 \end{cases} \tag{6.32}$$

The R-charge of the last flavour is different from the other quarks because the axial symmetry $U(1)_A$ mixes with the R-symmetry.
With this real mass assignment we can now perform the limit $m \to \infty$ which can be done on the partition function using the following mathematical identity for the hyperbolic gamma function [44]

$$\lim_{t\to\infty} \Gamma_h(z+t;\omega_1,\omega_2) = \exp\left[\mathrm{sign}(t)\frac{\pi i}{2\omega_1\omega_2}\left((z+t-\omega)^2 - \frac{\omega_1^2+\omega_2^2}{12}\right)\right] \tag{6.33}$$

Applying this identity to the terms associated to the last flavour of quarks that have mass m we obtain

$$\begin{aligned} \Gamma_h(\mu_{N_f+1}(m)+\sigma_i) &= \exp\Bigg(\mathrm{sign}(m)\frac{\pi i}{2\omega_1\omega_2}\Big[[\omega(\Delta_M-1)+\sigma_i+ \\ &\qquad +(m+m_B-N_f m_A)]^2 - \frac{\omega_1^2+\omega_2^2}{12}\Big]\Bigg) \\ \Gamma_h(\nu_{N_f+1}(m)-\sigma_i) &= \exp\Bigg(\mathrm{sign}(-m)\frac{\pi i}{2\omega_1\omega_2}\Big[[\omega(\Delta_M-1)-\sigma_i+ \\ &\qquad (-m-m_B-N_f m_A)]^2 - \frac{\omega_1^2+\omega_2^2}{12}\Big]\Bigg) \end{aligned} \tag{6.34}$$

Inserting these expressions in the partition function we obtain

$$\begin{aligned} &\prod_{i=1}^{N_c} \exp\left[\frac{\pi i}{2\omega_1\omega_2}\left[4(m+m_B)(\omega(\Delta_M-1)-m_A N_f)+4\sigma_i(\omega(\Delta_M-1)-m_A N_f)\right]\right] = \\ &\exp\left[\frac{\pi i}{2\omega_1\omega_2}\left[4N_c(m+m_B)(\omega(\Delta_M-1)-m_A N_f)+4\Big(\sum_{i=1}^{N_c}\sigma_i\Big)\Big(\omega(\Delta_M-1)-m_A N_f\Big)\right]\right] \end{aligned} \tag{6.35}$$

We can define the $c(x) = e^{\frac{i\pi x}{2\omega_1\omega_2}}$ in order to ease the notation. The exponential factor then becomes

$$c(4N_c(m+m_B)(\omega(\Delta_M-1)-m_A N_f)\; c(4\Big(\sum_{i=1}^{N_c}\sigma_i\Big)\Big(\omega(\Delta_M-1)-m_A N_f\Big) \tag{6.36}$$

We can now use the condition on the real masses that is generated by the η superpotential (6.31) which reads

$$\frac{1}{2}\sum_a \mu_a + \nu_a = \omega(-N_c\Delta_X + N_f + 1) = \omega(N_f\Delta_Q + \Delta_M) \tag{6.37}$$

Using this condition we can write the R-charge of last flavour Δ_M as a function of the real masses of the light quarks and the exponential factor reads

$$\begin{aligned} &c\,(4N_c(m+m_B)(\omega(N_f(1-\Delta_Q)-N_c\Delta_X)-m_AN_f)) \\ &\qquad\qquad c\left(4\Big(\sum_{i=1}^{N_c}\sigma_i\Big)\Big(\omega(N_f(1-\Delta_Q)-N_c\Delta_X)-m_AN_f)\Big)\right) \end{aligned} \tag{6.38}$$

The terms on the first line can be taken outside the integral and we will found an identical term in the magnetic theory. The second term will be set to one by the delta function $\delta(\sum_i \sigma_i)$ upon integration.
The partition function for the electric theory after integrating out the N_f+1-th flavour is then given by

$$\begin{aligned} Z_{el}(\mu_i,\nu_i) \;=&\; \\ & c\,(4N_c(m+m_B)(-m_AN_f+\omega(N_f(1-\Delta)-N_c\Delta_X)))\,\frac{1}{N_c!}\Gamma_h(\Delta_X\omega)^{N_c-1} \\ & \int\prod_{i=1}^{N_c}\frac{d\sigma_i}{\sqrt{-\omega_1\omega_2}}\,\delta(\sum_i\sigma_i)\prod_{1\le i<j\le N_c}\frac{\Gamma_h(\Delta_X\omega\pm(\sigma_i-\sigma_j))}{\Gamma_h(\pm(\sigma_i-\sigma_j))} \\ & \prod_{a,b=1}^{N_f}\prod_{j=1}^{N_c}\Gamma_h(m_a+m_B+m_A+\sigma_j)\Gamma_h(-\tilde{m}_a-m_B+m_A-\sigma_j) \end{aligned} \tag{6.39}$$

Note that except for the exponential factors discussed above, it corresponds to the partition function for a pure three-dimensional theory, without η superpotential. The charges can be easily read off from the partition function and are given by

Fields	$SU(N_f)_L$	$SU(N_f)_R$	$U(1)_B$	$U(1)_A$	$U(1)_J$	$U(1)_R$
Q	N_f	0	N_c	1	0	Δ_Q
$\tilde{Q}$	0	$\overline{N_f}$	$-N_c$	1	0	Δ_Q
X	0	0	0	0	0	Δ_X

6.2 Magnetic theory

The four dimensional theory of Kutasov-Schwimmer duality was introduced in section 2.3. It is a $SU(\tilde{N}_c) = SU(kN_f - N_c)$ SQCD theory with N_f flavours $q, \tilde{q}$, a matter field Y in the adjoint representation and k mesons M_j that are identified with the electric mesons. There is a superpotential for the adjoint field $\mathrm{Tr}\, Y^{k+1}$ that results in the condition $R_X = R_Y$. The four dimensional magnetic theory can be summarized by

	$SU(\tilde{N}_c)$	$SU(N_f)_L$	$SU(N_f)_R$	$U(1)_B$	$U(1)_R$
q	$\tilde{N}_c$	$\overline{N_F}$	1	$\frac{N_c}{kN_f-N_c}$	$1-\frac{2}{k+1}\frac{\tilde{N}_c}{N_f}=R_q$
$\tilde{q}$	$\overline{\tilde{N}_c}$	1	N_F	$-\frac{N_c}{kN_f-N_c}$	$1-\frac{2}{k+1}\frac{\tilde{N}_c}{N_f}=R_q$
Y	1	$\tilde{N}_c^{\,2}-1$	1	0	$\frac{2}{k+1}=R_X$
M_j	1	N_f	$\overline{N_f}$	0	$2-\frac{4}{k+1}\frac{N_c}{N_f}+j\frac{2}{k+1}$

(6.40)

and the three dimensional one by

	$SU(N_f)_L$	$SU(N_f)_R$	$U(1)_B$	$U(1)_A$	$U(1)_J$	$U(1)_R$
q	N_f	1	$\frac{N_c}{kN_f-N_c}$	1	0	Δ_Q
$\tilde{q}$	1	$\overline{N_f}$	$-\frac{N_c}{kN_f-N_c}$	1	0	Δ_Q
Y	1	1	0	0	0	$\frac{2}{k+1}=\Delta_Y$
M_j	N_f	$\overline{N_f}$	0	2	0	$2\Delta_Q+j\frac{2}{k+1}$

(6.41)

6.2.1 Superconformal index

We will now repeat the process that was applied to the electric theory for the magnetic side. The single particle index is given by

$$
\begin{aligned}
&i_M(p,q,\tilde{v},y,\tilde{y},\tilde{z}) = \\
&-\left(\frac{p}{1-p}+\frac{q}{1-q}-\frac{1}{(1-p)(1-q)}\Big((pq)^{\frac{R_Y}{2}}-(pq)^{1-\frac{R_Y}{2}}\Big)\right)\Big(p_{\tilde{N}_c}(\tilde{z})\,p_{\tilde{N}_c}(\tilde{z}^{-1})-1\Big)+\\
&+\frac{1}{(1-p)(1-q)}\left((pq)^{\frac{1}{2}R_q}\,\tilde{v}\,p_{N_f}(y^{-1})\,p_{\tilde{N}_c}(\tilde{z})-(pq)^{1-\frac{1}{2}R_q}\,\frac{1}{\tilde{v}}\,p_{N_f}(y)\,p_{\tilde{N}_c}(\tilde{z}^{-1})+\right.\\
&\qquad\left.+(pq)^{\frac{1}{2}R_q}\,\frac{1}{\tilde{v}}\,p_{N_f}(\tilde{y}\,)\,p_{\tilde{N}_c}(\tilde{z}^{-1})-(pq)^{1-\frac{1}{2}R_q}\,\tilde{v}\,p_{N_f}(\tilde{y}^{-1})\,p_{\tilde{N}_c}(\tilde{z})\right)+\\
&\qquad\sum_{l=0}^{k-1}\left((pq)^{R_Q+l\frac{R_Y}{2}}p_{N_f}(y)p_{N_f}(\tilde{y}^{-1})-(pq)^{1-(R_Q+\frac{R_Y}{2}l)}p_{N_f}(y^{-1}p_{N_f}\tilde{y}\right)
\end{aligned}
\tag{6.42}
$$

where R_q is the R-charge of the dual quark which is fixed in term of the charge R_Q of the electric quark by requiring the validity of the duality. The last line is the contribution from the k different mesons.

Using the explicit form of the polynomials $p_N(x)$ we have

$$
\begin{aligned}
& i_M(p,q,\tilde{v},y,\tilde{y},\tilde{z}) = \\
& -\left(\frac{p}{1-p}+\frac{q}{1-q}-\frac{1}{(1-p)(1-q)}\Big((p\,q)^{\frac{R_Y}{2}}-(p\,q)^{1-\frac{R_Y}{2}}\Big)\right)\Big(\sum_{i,j}^{\tilde{N}_c}\tilde{z}_i\tilde{z}_j^{-1}-1\Big)+ \\
& +\frac{1}{(1-p)(1-q)}\quad\Bigg[\sum_j^{\tilde{N}_c}\Bigg(\sum_a^{N_f}\Big((p\,q)^{\frac{1}{2}R_q}\,\tilde{v}\,y_a^{-1}\,\tilde{z}_j\Big)-(p\,q)^{1-\frac{1}{2}R_q}\,\frac{1}{\tilde{v}}\,y_a\,\tilde{z}_j^{-1}+ \\
& \qquad +\sum_b^{N_f}\Bigg((p\,q)^{\frac{1}{2}R_q}\,\frac{1}{\tilde{v}}\,(\tilde{y}_b\,)\,(\tilde{z}_j^{-1})-(p\,q)^{1-\frac{1}{2}R_q}\,\tilde{v}\,\tilde{y}_b^{-1}\,\tilde{z}_j\Bigg)\Bigg)+ \\
& \qquad \sum_a^{N_f}\sum_b^{N_f}\sum_{l=0}^{k-1}\Bigg((pq)^{R_Q+l\frac{R_Y}{2}}y_a\tilde{y}_b^{-1}-(pq)^{1-(R_Q+l\frac{R_Y}{2})}y_a^{-1}\tilde{y}_b\Bigg)\Bigg]
\end{aligned}
\tag{6.43}
$$

The index is now formally similar to the electric one except for the contribution of the mesons which can be calculated by identifying the $(pq)^{R_q+l\frac{R_Y}{2}}y_a\tilde{y}_b^{-1}$ with the argument of the elliptic gamma function. Their contribution reads

$$
\exp\left(\sum_n^{\infty}\frac{1}{n}i_M^{Meson}(p^n,q^n,\tilde{v}^n,y^n,\tilde{y}^n)\right)=\prod_a^{N_f}\prod_b^{N_f}\prod_{l=0}^{k-1}\Gamma_e((pq)^{R_q+l\frac{R_Y}{2}}y_a\tilde{y}_b^{-1};p,q) \tag{6.44}
$$

The complete expression for the superconformal index for the magnetic theory is given by

$$
\begin{aligned}
& I_{Mag}(p,q,y,\tilde{v},\tilde{y}) = \\
& \frac{1}{\tilde{N}_c!}(p;p)^{\tilde{N}_c-1}(q;q)^{\tilde{N}_c-1}\,\Gamma_e((pq)^{\frac{R_Y}{2}};p,q)^{\tilde{N}_c-1}\left(\prod_a^{N_f}\prod_b^{N_f}\prod_{l=0}^{k-1}\Gamma_e((pq)^{R_Q+l\frac{R_Y}{2}}y_a\tilde{y}_b{}^{-1};p,q)\right) \\
& \qquad \int\left(\prod_{i=1}^{\tilde{N}_c}\frac{dz_i}{2\pi i z_i}\right)\delta\left(\prod_{i=1}^{\tilde{N}_c}z_i-1\right)\prod_{1\le i<j\le\tilde{N}_c}\frac{\Gamma_e\Big((pq)^{\frac{R_Y}{2}}\frac{z_i}{z_j}\Big)\Gamma_e\Big((pq)^{\frac{R_Y}{2}}\frac{z_j}{z_i}\Big)}{\Gamma_e(\frac{z_i}{z_j};p,q)\Gamma_e(\frac{z_j}{z_i};p,q)} \\
& \qquad\qquad \prod_j^{\tilde{N}_c}\prod_{a,b}^{N_f}\Gamma_e((pq)^{\frac{1}{2}R_q}\tilde{v}\,y_a^{-1}\tilde{z}_j;p,q)\;\Gamma_e((pq)^{\frac{1}{2}R_q}\tilde{v}^{-1}\tilde{y}_b\tilde{z}_j{}^{-1};p,q)
\end{aligned}
\tag{6.45}
$$

6.2.2 Reduction to the partition function

We reduce the index to the partition function using the same procedure employed for the electric theory.

Vector field We can use the identity (B.8) as in the electric theory with $\tilde{N}_c$ and $\tilde{\sigma}$ instead of N_c and σ. The result is

$$\lim_{r\to 0} \frac{(p;p)^{\tilde{N}_c-1}(q;q)^{\tilde{N}_c-1}}{\tilde{N}_c!} \oint_{T^{\tilde{N}_c-1}} \prod_{j=1}^{\tilde{N}_c-1} \frac{d\tilde{z}_j}{2\pi i \tilde{z}_j} \prod_{1\leq i<j\leq \tilde{N}_c} \frac{1}{\Gamma_e(\frac{\tilde{z}_i}{\tilde{z}_j})\Gamma_e(\frac{\tilde{z}_j}{\tilde{z}_i})} \sim$$

$$\sim \exp\left(-\frac{i\pi\omega(\tilde{N}_c^2-1)}{6r\omega_1\omega_2}\right)\frac{1}{\tilde{N}_c!}\int \prod_{j=1}^{\tilde{N}_c-1}\frac{d\tilde{\sigma}_j}{\sqrt{-\omega_1\omega_2}} \prod_{1\leq i<j\leq \tilde{N}_c} \frac{1}{\Gamma_h(\pm(\tilde{\sigma}_i-\tilde{\sigma}_j)} \tag{6.46}$$

Adjoint matter The reduction of the adjoint matter is identical to the electric theory (using (5.37)) which results in

$$\Gamma_e\left((pq)^{\frac{\Delta_Y}{2}}\right)^{\tilde{N}_c-1} \prod_{1\leq i<j\leq \tilde{N}_c} \Gamma_e\left((pq)^{\frac{\Delta_Y}{2}}\left(\frac{\tilde{z}_i}{\tilde{z}_j}\right)\right)\Gamma_e\left((pq)^{\frac{\Delta_Y}{2}}\left(\frac{\tilde{z}_j}{\tilde{z}_i}\right)\right) \sim$$

$$\exp\left(-\frac{i\pi}{6r\omega_1\omega_2}\left((\tilde{N}_c^2-1)\omega(\Delta_Y-1)\right)\right) \prod_{1\leq i<j\leq \tilde{N}_c} \Gamma_h\left(\Delta_Y\omega \pm (\tilde{\sigma}_i-\tilde{\sigma}_j)\right) \tag{6.47}$$

Quarks

$$\prod_{j}^{\tilde{N}_c}\prod_{a,b}^{N_f} \Gamma_e((pq)^{\frac{1}{2}\Delta_q}\,\tilde{v}\, y_i^{-1}\,\tilde{z}_j; p, q)\Gamma_e((pq)^{\frac{1}{2}\Delta_q}\,\tilde{v}^{-1}\tilde{y}_b\,\tilde{z}_j^{-1}; p, q) =$$

$$= \exp\left(\frac{-i\pi}{6r\omega_1\omega_2}\Big(2N_f\tilde{N}_c\omega(\Delta_q-1)+\tilde{N}_c\Big(\sum_{a=1}^{N_f}-m_a+\tilde{m}_a\Big)\Big)\right)\Gamma_h\Big(\tilde{\mu}_a+\tilde{\sigma}_j\Big)\Gamma_h\Big(\tilde{\nu}_b-\tilde{\sigma}_j\Big) \tag{6.48}$$

where we defined the real masses $\tilde{\mu}_a, \tilde{\nu}_b$

$$\tilde{\mu}_a = -m_a + \tilde{m}_B + \omega\Delta_q \qquad \tilde{\nu}_b = +\tilde{m}_b - \tilde{m}_B + \omega\Delta_q \tag{6.49}$$

Note that the signs for the masses associated to the flavour group are inverted with respect to the electric theory (6.32), because the quarks are in opposite representations. Moreover, we denoted $\tilde{m}_B$ as the baryonic mass in the magnetic theory. It can be related to the electric baryonic mass by

$$\tilde{m}_B = \frac{N_c}{kN_f - N_c} m_B \tag{6.50}$$

In addition, the R-charges of the dual quarks are fixed by the superpotential interaction with the mesons which results in the condition

$$R_q = R_Y - R_Q \qquad \xrightarrow{3D} \qquad \Delta_q = \Delta_Y - \Delta_Q \tag{6.51}$$

The real masses are then

$$\tilde{\mu}_a = -m_a + \frac{N_c}{kN_f - N_c} m_B + \omega(\Delta_X - \Delta_Q) \quad \tilde{\nu}_b = +\tilde{m}_b - \frac{N_c}{kN_f - N_c} m_B + \omega(\Delta_X - \Delta_Q) \tag{6.52}$$

Mesons Applying the same procedure to the mesons we obtain

$$\begin{aligned}
&\prod_a^{N_f}\prod_b^{N_f}\prod_{l=0}^{k-1}\Gamma_e((pq)^{\frac{1}{2}\left(2\Delta_Q+l\Delta_X\right)}\, y_a\tilde{y_b}^{-1};p,q) = \\
&\quad \exp\left(\frac{-i\pi}{6r\omega_1\omega_2}\left(\omega N_f^2\sum_{l=0}^{k-1}\left(2\left(\Delta_Q-\frac{1}{2}\right)+l\Delta_X\right)+N_f\left(\sum_{l=0}^{k-1}\sum_i^{N_f} m_i-\tilde{m}_i\right)\right)\right) \\
&\qquad \prod_a^{N_f}\prod_b^{N_f}\prod_{l=0}^{k-1}\Gamma_h\left(\omega(2\Delta_Q+l\Delta_X)+m_a-\tilde{m}_b\right)
\end{aligned} \tag{6.53}$$

Divergent contributions

Summing up all the contributions from the various fields we obtain the following expression

$$\begin{aligned}
&\omega(\tilde{N}_c^2-1)+\left(\omega N_f^2\sum_{l=0}^{k-1}\left(2\left(\Delta_Q-\frac{1}{2}\right)+l\Delta_X\right)+N_f\left(\sum_{l=0}^{k-1}\sum_i^{N_f} m_i-\tilde{m}_i\right)\right)+ \\
&\quad +\left((\tilde{N}_c^2-1)\omega(\Delta_Y-1)\right)+\left(2N_f\tilde{N}_c\omega(\Delta_q-1)+\tilde{N}_c\sum_{a=1}^{N_f}-m_a+\tilde{m}_a\right)
\end{aligned} \tag{6.54}$$

We can verify that it is identical to the divergent contribution that we obtained in the electric theory by using the explicit value of $\tilde{N}_c = kN_f - N_c$, $\Delta_Y = \Delta_X = \frac{2}{k+1}$ and $\Delta_q = \Delta_X - \Delta_Q$. Note that this matching is a non trivial check of the duality and it corresponds to the 't Hooft anomaly matching for the gravitational anomalies. The complete calculation can be found in appendix D.

The final expression is identical to the electric one and is given by

$$\omega(N_c^2-1)+(N_c^2-1)\omega(\Delta_X-1)+2N_fN_c\omega(\Delta_Q-1)+N_c\Big(\sum_a^{N_f} m_a-\tilde{m}_a\Big) \tag{6.55}$$

6.2.3 Partition function

The partition function for the magnetic theory without the divergent factor is given by

$$\begin{aligned} Z_{mag}(\mu_a, \nu_b, \tilde{\mu}_a, \tilde{\nu}_b) &= \frac{1}{\tilde{N}_c!} \Gamma_h(\Delta_X \omega; \omega_1, \omega_2)^{\tilde{N}_c - 1} \left(\prod_{a,b=1}^{N_f} \prod_{l=0}^{k-1} \Gamma_h\left(\mu_a + \nu_b + l\omega\Delta_X\right) \right) \\ & \int \prod_{i=1}^{kN_f - N_c} d\tilde{\sigma}_i \, \delta(\sum_i \tilde{\sigma}_i) \prod_{i<j}^{kN_f - N_c} \frac{\Gamma_h(\Delta_Y \omega \pm (\tilde{\sigma}_i - \tilde{\sigma}_j))}{\Gamma_h(\pm(\tilde{\sigma}_i - \tilde{\sigma}_j))} \\ & \prod_{a,b=1}^{N_f} \prod_{j=1}^{kN_f - N_c} \Gamma_h\Big(\tilde{\mu}_a + \tilde{\sigma}_j\Big) \Gamma_h\Big(\tilde{\nu}_b - \tilde{\sigma}_j\Big) \end{aligned} \tag{6.56}$$

where we cefined

$$\begin{aligned} \mu_a &= \omega\Delta_Q + m_a \quad \nu_a = \omega\Delta_Q - \tilde{m}_a \\ \tilde{\mu}_a = \omega(\Delta_X - \Delta_Q) + \frac{N_c}{kN_f - N_c} m_B - m_a &\quad \tilde{\nu}_a = \omega(\Delta_X - \Delta_Q) - \frac{N_c}{kN_f - N_c} m_B + \tilde{m}_a \end{aligned} \tag{6.57}$$

where the same condition (6.31) on the R-charges and real masses holds.

6.2.4 Flow to a theory withouth η superpotential

We need to perform the same procedure we already applied to the electric theory in order to flow to a pure three dimensional theory.
We start from a $SU(k(N_f + 1) - N_c)$ gauge theory with $N_f + 1$ flavours, adjoint matter and $k(N_f + 1)^2$ mesons singlets. We break the global symmetry group in the same way as we did in the electric side.
Considering that we have already chosen a vaccum on the electric side, we need to find the vaccum on the magnetic side that can be matched with the electric one.
The mapping of the real masses, considering the divergent terms is the following

$$\begin{aligned} \tilde{\mu_a} &= \begin{cases} \tilde{\mu_a} = \omega(\Delta_X - \Delta_Q) + m_B \frac{N_c}{k(N_f+1) - N_c} - m_a - m_A + m_1 & a = 1 \dots N_f \\ \tilde{\mu}_{N_f+1} = \omega(\Delta_X - \Delta_M) + m_B \frac{N_c}{k(N_f+1) - N_c} + m_A N_f + m_2 \end{cases} \\ \tilde{\nu_a} &= \begin{cases} \tilde{\nu}_b = \omega(\Delta_X - \Delta_Q) - m_B \frac{N_c}{k(N_f+1) - N_c} + \tilde{m}_b - m_A - m_1 & a = 1 \dots N_f \\ \tilde{\nu}_{N_f+1} = \omega(\Delta_X - \Delta_M) - m_B \frac{N_c}{k(N_f+1) - N_c} + m_A N_f - m_2 \end{cases} \end{aligned} \tag{6.58}$$

where we set

$$m_1 = \frac{k}{k(N_f + 1) - N_c} m \qquad m_2 = \frac{N_c - kN_f}{k(N_f + 1) - N_c} m \tag{6.59}$$

The masses for the mesons are different, because they are not charged under the baryonic symmetry and the mass m in the electric theory is the result of the sum of flavour and baryonic divergent mass terms. As a result they are given by

$$\begin{aligned}
\mu_a &= \begin{cases} m_a + m_A - M + \omega\Delta \\ MN_f - m_A N_f + \omega\Delta_M \end{cases} \\
\nu_a &= \begin{cases} -\tilde{m}_a + m_A + M + \omega\Delta \\ -MN_f - m_A N_f + \omega\Delta_M \end{cases}
\end{aligned} \quad \text{with} \quad M = \frac{m}{N_f + 1} \tag{6.60}$$

The mapping of real masses between electric and magnetic theory is identical to the one discussed in section 4.3.2. As in section 4.3.2 we need to find the vacuum for $\tilde{\sigma}$ such that we have the most number of massless degress of freedom. However, considering that we don't have to introduce the polynomial deformation in the adjoint matter, the vacuum will be different. It is given by

$$\tilde{\sigma}_{VEV} = \begin{pmatrix} -m_1 \mathbf{1}_{kN_f - N_c} & 0 \\ 0 & -m_2 \mathbf{1}_k \end{pmatrix} \tag{6.61}$$

Note that this vacuum choice is traceless.
This vacuum expectation value for σ is more symmetric than (4.56) and breaks the gauge group only as $U(kN_f - N_c) \times U(k)/U(1)$. From now on, with $\tilde{\sigma}$ we will refer first $kN_f - N_c$ entries and with ρ with the other k entries of the original gauge scalar.
We can now perform the limit $m \to \infty$ on the partition by using the identity (6.33).

Mesons Most of the components of the mesons remains massless, because only the non diagonal $M_i^{N_f+1}$ and $M^i_{N_f+1}$ terms are massive.

$$\begin{aligned}
&\prod_a^{N_f+1} \prod_b^{N_f+1} \prod_{l=0}^{k-1} \Gamma_h\Big(\mu_a + \nu_b + l\omega\Delta_X)\Big) = \\
&\left(\prod_{l=0}^{k-1} \left(\prod_a^{N_f} \prod_b^{N_f} \Gamma_h\Big(\mu_a + \nu_b + l\omega\Delta_X)\Big) \right) \Gamma_h(-2m_A N_f + \omega(2\Delta_M + l\Delta_X) \right) \\
&\left(\prod_{l=0}^{k-1} \prod_a^{N_f} \Gamma_h(m_a - m_A(N_f - 1) - \underbrace{M(N_f + 1)}_{m} + \omega(\Delta_Q + \Delta_M + l\Delta_X)) \right. \\
&\left. \Gamma_h(-\tilde{m}_a + \underbrace{M(N_f + 1)}_{m} - m_A(N_f - 1) + \omega(\Delta_Q + \Delta_M + l\Delta_X)) \right)
\end{aligned} \tag{6.62}$$

Using the identity for the terms in the last two rows we obtain an exponential factor given by

$$\exp\left\{\frac{\pi i}{2\omega_1\omega_2}\sum_{l=0}^{k-1}\sum_{a}^{N_f}\left[-m_a^2+\tilde{m}_a^2+4m\left(m_A(N_f-1)+\omega\left(\Delta_Q+\Delta_M+l\Delta_X-1\right)\right)\right]\right\} \tag{6.63}$$

Quarks The vacuum expectation value for $\tilde{\sigma}$ was chosen such that only the off-diagonal components would become massive, similarly to the mesons.

$$\begin{aligned}&\left(\prod_{i=1}^{N_f}\prod_{j=1}^{kN_f-N_c}\Gamma_h\Big(\tilde{\mu}_i+\tilde{\sigma_j}\Big)\Gamma_h\Big(\tilde{\nu}_i-\tilde{\sigma_j}\Big)\right)\left(\prod_{j=1}^{k}\Gamma_h\Big(\tilde{\mu}_{N_f+1}+\rho_j\Big)\Gamma_h\Big(\tilde{\nu}_{N_f+1}-\rho_j\Big)\right)\\&\left(\prod_{j=1}^{kN_f-N_c}\Gamma_h\Big(\tilde{\mu}_{N_f+1}+\tilde{\sigma_j}\Big)\Gamma_h\Big(\tilde{\nu}_{N_f+1}-\tilde{\sigma_j}\Big)\right)\left(\prod_{i=1}^{N_f}\prod_{j=1}^{k}\Gamma_h\Big(\tilde{\mu}_i+\rho_j\Big)\Gamma_h\Big(\tilde{\nu}_i-\rho_j\Big)\right)\end{aligned} \tag{6.64}$$

The terms from the first line corresponds to massless quarks, while the second contains massless components that will generate exponential terms.
The result of the limit $m\to\infty$ on the first of these terms gives

$$\begin{aligned}\lim_{m\to\infty}\prod_{j=1}^{kN_f-N_c}&\Gamma_h\Big(\tilde{\mu}_{N_f+1}+\tilde{\sigma_j}\Big)\Gamma_h\Big(\tilde{\nu}_{N_f+1}-\tilde{\sigma_j}\Big)=\\&\exp\left\{\frac{i\pi}{2\omega_1\omega_2}\left[\left(\sum_{j}^{kN_f-N_c}4\sigma_j'\left(-m_AN_f-\omega(\Delta_X-\Delta_M-1)\right)\right)+\right.\right.\\&4(kN_f-N_c)\left(m_B\frac{N_c}{k(N_f+1)-N_c}\left(-m_AN_f-\omega(\Delta_X-\Delta_M-1)\right)+\right.\\&\left.\left.\left.m\left(m_AN_f+\omega(\Delta_X-\Delta_M-1)\right)\right)\right]\right\}\end{aligned} \tag{6.65}$$

While the other term gives

$$\begin{aligned}&\left(\prod_{i=1}^{N_f}\prod_{j=1}^{k}\Gamma_h\Big(\tilde{\mu}_i+\rho_j\Big)\Gamma_h\Big(\tilde{\nu}_i-\rho_j\Big)\right)=\\&\qquad\exp\left\{\left[\frac{i\pi}{2\omega_1\omega_2}\left(4N_f\sum_{j=1}^{k}\rho_j'\left(-m_A+\omega(\Delta_X-\Delta_Q-1)\right)\right)\right.\right.\\&+k\sum_{a}^{N_f}(m_a^2-\tilde{m}_a^2)+4kN_f\left(m_B\frac{N_c}{k(N_f+1)-N_c}\left(-m_A+\omega(\Delta_X-\Delta_Q-1)\right)\right.\\&\left.\left.\left.+m(-m_A+\omega(\Delta_X-\Delta_Q-1)\right)\right]\right\}\end{aligned} \tag{6.66}$$

Adjoint matter The term associated to the adjoint matter can be written as

$$\prod_{i<j}^{k(N_f+1)-N_c} \frac{\Gamma_h(\Delta_X\omega \pm (\tilde{\sigma}_i - \tilde{\sigma}_j))}{\Gamma_h(\pm(\tilde{\sigma}_i - \tilde{\sigma}_j))} = \prod_{i<j}^{kN_f-N_c} \frac{\Gamma_h(\Delta_X\omega \pm (\tilde{\sigma}_i - \tilde{\sigma}_j))}{\Gamma_h(\pm(\tilde{\sigma}_i - \tilde{\sigma}_j))}$$
$$\prod_{i<j}^{k} \frac{\Gamma_h(\Delta_X\omega \pm (\rho_i - \rho_j))}{\Gamma_h(\pm(\rho_i - \rho_j))} \prod_{i\leq kN_f-N_c} \prod_{j\leq k} \frac{\Gamma_h(\Delta_X\omega \pm (\tilde{\sigma}_i - \rho_j))}{\Gamma_h(\pm(\tilde{\sigma}_i - \rho_j))} \tag{6.67}$$

where only the last term represent massive contributions. The exponential term associated to this factor is

$$\exp\left\{\frac{i\pi}{2\omega_1\omega_2}\left[4\omega\Delta_X\left(k(kN_f - N_c)(m) + \sum_{i=1}^{kN_f-N_c}\sum_{j=1}^{k}(\sigma'_i + \rho'_j)\right)\right]\right\} \tag{6.68}$$

Summation of the contribution We expect that the sum of the contributions from the various term will be similar to the divergent factors we found in the electric theory in order to the remaining partition functions to be equal. By an explicit calculation we can see that this is indeed true (appendix D). In fact, we have

$$\exp\Bigg\{\frac{i\pi}{2\omega_1\omega_2}\Bigg(4\Bigg(N_c(m+m_B) + \Bigg(\sum_{i=1}^{kN_f-N_c}\tilde{\sigma}_i\Bigg) + \Bigg(\sum_{j=1}^{k}\rho_j\Bigg)\Bigg)\Bigg)$$
$$\Bigg(-m_A N_f + \omega\Big(-N_c\Delta_X + N_f(1-\Delta_Q)\Big)\Bigg)\Bigg)\Bigg\} \tag{6.69}$$

The term proportional to $m + m_B$ are identical to what we found in the electric theory while we will analyze the other two terms once we have write down the partition function.

Partition function

The partition function is given by multiplying the terms we found in the previous sections. We wrote down explicitly the δ-function as $e^{2\pi i\xi\sum_i\sigma_i}$. This is equivalent to the gauging of the topological symmetry $U(1)_J$, because a term like this is generated by a Fayet-Iliopulos term, which corresponds to the charge of the topological symmetry. This term will be crucial in the following considerations on the partition function.

$$
\begin{aligned}
Z_{mag} = & \frac{1}{(kN_f-N_c)!\,k!}\Gamma_h(\Delta_X\omega;\omega_1,\omega_2)^{(k(N_f+1)-N_c)-1} \\
& c\,(4N_c(m_B+m)(-m_AN_f+\omega(-N_c\Delta_X+N_f(1-\Delta_Q)))) \\
& \left(\prod_{j=0}^{k-1}\left(\prod_{a=1}^{N_f}\prod_{b=1}^{N_f}\Gamma_h\Big(\mu_a+\nu_b+j\omega\Delta_X)\Big)\right)\Gamma_h\Big(-2m_AN_f+\omega(2\Delta_M+j\Delta_X)\Big)\right) \\
& \int\prod_{i=1}^{kN_f-N_c}\frac{d\sigma_i}{\sqrt{-\omega_1\omega_2}}\prod_{i=1}^{k}\frac{d\rho_i}{\sqrt{-\omega_1\omega_2}}\int d\xi\, e^{2\pi i\xi(\sum\sigma_i+\sum\rho_i)} \\
& \prod_{i<j}^{kN_f-N_c}\frac{\Gamma_h(\Delta_X\omega\pm(\sigma_i-\sigma_j))}{\Gamma_h(\pm(\sigma_i-\sigma_j))}\prod_{i<j}^{k}\frac{\Gamma_h(\Delta_X\omega\pm(\rho_i-\rho_j))}{\Gamma_h(\pm(\rho_i-\rho_j))} \\
& \left(\prod_{a,b=1}^{N_f}\prod_{j=1}^{kN_f-N_c}\quad\Gamma_h\left(\left(\tilde\sigma_j+m_B\frac{N_c}{k(N_f+1)-N_c}\right)-m_a-m_A+\omega(\Delta_X-\Delta_Q)\right)\right. \\
& \qquad\qquad\left.\Gamma_h\left(-\left(\tilde\sigma_j+m_B\frac{N_c}{k(N_f+1)-N_c}\right)+\tilde m_b-m_A+\omega(\Delta_X-\Delta_Q)\right)\right) \\
& \left(\prod_{i=1}^{k}\Gamma_h\left(\pm(\rho_i+m_B\frac{N_c}{k(N_f+1)-N_c})+m_AN_f+\omega(\Delta_X-\Delta_M)\right)\right) \\
& c\left(4\left(\sum_{i=1}^{kN_f-N_c}\sigma_i+\sum_{j=1}^{k}\rho_j\right)(-m_AN_f+\omega(-N_c\Delta_X+N_f(1-\Delta_Q))\right)
\end{aligned}
\tag{6.70}
$$

We need to rescale ξ in order to normalize its contribution to the partition function as a standard Fayet-Iliopulos term. This is achieved by the following rescaling

$$
\xi \;\longrightarrow\; \frac{\xi}{\omega_1\omega_2} \qquad d\xi \;\longrightarrow\; |\omega_1\omega_2|\, d\xi \tag{6.71}
$$

We can remove the term, proportional to $\sum_i\tilde\sigma_i+\sum_j\rho_j$ in the last line by a finite shift in ξ.

With a shift in σ_i and ρ_j we can renormalize the baryonic charges such that the baryonic real mass appear with a prefactor of $\frac{N_c}{kN_f-N_c}$, which is the standard charge associated to N_f flavours. The (traceless) shift is given by

$$
\begin{aligned}
\sigma_i &\quad\longrightarrow\quad \sigma_i+\frac{kN_c}{(kN_f-N_c)(k(N_f+1)-N_c)}m_B \\
\rho_j &\quad\longrightarrow\quad \rho_j-\frac{N_c}{(k(N_f+1)-N_c)}m_B
\end{aligned}
\tag{6.72}
$$

Moreover, this shift makes the $U(k)$ quarks uncharged under the baryonic symmetry. The gamma function associated to them is not function of the baryonic mass

anymore.
After these consideration the partition function can be rewritten as

$$
\begin{aligned}
Z_{mag} =& \frac{1}{(kN_f - N_c)!\, k!} \Gamma_h(\Delta_X \omega; \omega_1, \omega_2)^{(k(N_f+1)-N_c)-1} \\
& \left(\prod_{j=0}^{k-1} \left(\prod_{a=1}^{N_f} \prod_{b=1}^{N_f} \Gamma_h \left(\mu_a + \nu_b + j\omega\Delta_X \right) \right) \Gamma_h \left(-2m_A N_f + 2\omega(\Delta_M + j\Delta_X) \right) \right) \\
& \int \prod_{i=1}^{kN_f - N_c} \frac{d\sigma_i}{\sqrt{-\omega_1\omega_2}} \prod_{i=1}^{k} \frac{d\rho_i}{\sqrt{-\omega_1\omega_2}} \int |\omega_1\omega_2| d\xi \, e^{\frac{2\pi i \xi}{\omega_1\omega_2}(\sum \sigma_i + \sum \rho_i)} \\
& \prod_{i<j}^{kN_f - N_c} \frac{\Gamma_h(\Delta_X \omega \pm (\sigma_i - \sigma_j))}{\Gamma_h(\pm(\sigma_i - \sigma_j))} \prod_{i<j}^{k} \frac{\Gamma_h(\Delta_X \omega \pm (\rho_i - \rho_j))}{\Gamma_h(\pm(\rho_i - \rho_j))} \\
& \left(\prod_{a,b=1}^{N_f} \prod_{j=1}^{kN_f - N_c} \quad \Gamma_h \left(\left(\tilde{\sigma}_j + m_B \frac{N_c}{kN_f - N_c} \right) - m_a - m_A + \omega(\Delta_X - \Delta_Q) \right) \right. \\
& \qquad \left. \Gamma_h \left(- \left(\tilde{\sigma}_j + m_B \frac{N_c}{kN_f - N_c} \right) + \tilde{m}_b - m_A + \omega(\Delta_X - \Delta_Q) \right) \right) \\
& \left(\prod_{i=1}^{k} \Gamma_h \left(\pm \rho_i + m_A N_f + \omega(\Delta_X - \Delta_M) \right) \right)
\end{aligned}
\tag{6.73}
$$

We can now proceed to the dualization of the $U(k)$ sector as we did in the field theory analysis of the duality. However in that case, we had to introduce a perturbation that broke $U(k)$ into $U(1)^k$ and use field theory arguments to dualize the broken sector. Removing the perturbation in the superpotential, the gauge group recovers to $U(k)$ and we had to assume that this enhancement doesn't change the results obtained.
On the other hand, on the partition function we still have the unbroken sector $U(k)$ and we can use the following mathematical identity [44] to dualize it into a theory with chiral fields and no gauge group.
Let's define the partition function for a $U(N_c)$ theory with adjoint matter as

$$
\begin{aligned}
W_{N_c,K}\left(\mu_a, \nu_b, \tau, \lambda\right) =& \frac{\Gamma_h(\tau)^{N_c}}{N_c!} \int \prod_{i=1}^{N_c} \frac{d\sigma_i}{\sqrt{-\omega_1\omega_2}} e^{\frac{i\pi}{\omega_1\omega_2}\left(2\lambda \sum_i \sigma - 2K \sum_i \sigma_i^2\right)} \prod_{i<j}^{N_c} \frac{\Gamma_h(\tau \pm (\sigma_i - \sigma_j)}{\Gamma_h\left(\pm(\sigma_i - \sigma_j\right)} \\
& \prod_{i=1}^{N_c} \prod_{a,b=1}^{N_f} \Gamma_h(\mu_a + \sigma_i) \Gamma_h(\nu_b - \sigma_i)
\end{aligned}
\tag{6.74}
$$

where λ and K are the Fayet-Iliopulos and Chern-Simons couplings respectively. The theory can than be dualized using the following identity [44]

$$W_{N_c,0}\left(\mu_a,\nu_b,\tau,\lambda\right)=\prod_{j=0}^{N_c-1}\Gamma_h\left(\omega-\frac{\mu+\nu}{2}-j\tau\pm\lambda\right)\Gamma_h\left(j\tau+\mu+\nu\right)$$
$$\Gamma_h\left((j+1)\tau\right)\exp\left(\frac{2\pi i}{\omega_1\omega_2}N_c\frac{\lambda}{2}(\mu-\nu)\right) \quad (6.75)$$

Thus, the $U(k)$ sector is dual to a theory without gauge group and with $4k$ singlets, where $2k$ of them are charged under the topological symmetry. This is easily understood because the Fayet-Iliopulos term λ is the charge of this symmetry.
We can apply this identity with the following values for the real masses

$$\begin{aligned}\mu_a &= m_A N_f+\omega(\Delta_X-\Delta_M)\\ \nu_a &= m_A N_f+\omega(\Delta_X-\Delta_M)\\ \tau &= \omega\Delta_X\\ \lambda &= \xi\\ N_c &= k\end{aligned} \quad (6.76)$$

We obtain the expression

$$\prod_{j=0}^{k-1}\Gamma_h\left(\omega-j\omega\Delta_X-(m_AN_f+\omega(\Delta_X-\Delta_M))\pm\xi\right)$$
$$\Gamma_h\left((j+1)\omega\Delta_X\right)\Gamma_h\left(2m_AN_f+2\omega(\Delta_X-\Delta_M)+j\omega\Delta_X\right) \quad (6.77)$$

The terms on the first line can be rewritten as

$$\begin{aligned}&\prod_{j=0}^{k-1}\Gamma_h\left(\omega-j\omega\Delta_X-(m_AN_f+\omega((N_c+1)\Delta_X-N_f(1-\Delta_Q)-1)\pm\xi\right)=\\ &=\prod_{j=0}^{k-1}\Gamma_h\left(\pm\xi+\omega\left(2+N_f(1-\Delta_Q)-\Delta_X(N_c+1+j)\right)-m_AN_f\right)=\\ &=\prod_{j=0}^{k-1}\Gamma_h\left(\pm\xi+\omega\left(2+N_f(1-\Delta_Q)-\Delta_X(N_c+1+(k-1-j)\right)-m_AN_f\ \right)\\ &=\prod_{j'=1}^{k}\Gamma_h\left(\pm\xi+\omega\left(N_f(1-\Delta_Q)-\Delta_X(N_c-j-1)\right)-m_AN_f\right)\end{aligned} \quad (6.78)$$

using the explicit value of Δ_X.
These singlet fields $(b_j,\tilde{b}_j)$ are mapped to the magnetic monopoles of the electric

theory which have the following charges

	$U(kN_f - N_c) \times U(1)_{mir}$	$U(1)_B$	$U(1)_A$	$U(1)_J$	$U(1)_R$
b_j	1_1	0	$-N_f$	1	$N_f(1-\Delta_Q) + \Delta_X(j+1-N_c)$
$\tilde{b}_j$	1_{-1}	0	$-N_f$	-1	$N_f(1-\Delta_Q) + \Delta_X(j+1-N_c)$

(6.79)

They are exactly the same fields we introduced in the field theory analysis of the duality in section 4.3.2.
We can repeat the procedure for one of the other singlets of (6.77) and we obtain

$$\begin{aligned}&\prod_{j=0}^{k-1} \Gamma_h\left(2m_A N_f + 2\omega(\Delta_X - \Delta_M) + j\omega\Delta_X\right) = \\ &= \prod_{j=0}^{k-1} \Gamma_h\left(2m_A N_f + \omega(2 - j\Delta_X - 2\Delta_M)\right)\end{aligned} \tag{6.80}$$

After inserting this term back in the partition function we can use the mathematical identity $\Gamma_h(2\omega - x)\Gamma_h(x) = 1$ which is equivalent to the fact that if two chiral fields acquire a holomorphic mass they are integrated out and disappear from the partition function. Applying the identity to the following terms, they disappear from the partition function

$$\prod_{j=0}^{k-1} \Gamma_h\left(2m_A N_f + \omega(2 - j\Delta_X - 2\Delta_M)\right)\Gamma_h\left(-2m_A N_f + 2\omega(\Delta_M + j\Delta_X)\right) \tag{6.81}$$

There is an additional term that can be demonstrated that is equal to one. First suppose k is even

$$\begin{aligned}\prod_{j=1}^{k} \Gamma_h\left(j\frac{2\omega}{k+1}\right) = \prod_{j=1}^{k/2} \Gamma_h\left(j\frac{2\omega}{k+1}\right)\Gamma_h\left((k+1-j)\frac{2\omega}{k+1}\right) = \\ \prod_{j=1}^{k/2} \Gamma_h\left(j\frac{2\omega}{k+1}\right)\Gamma_h\left(2\omega - j\frac{2\omega}{k+1}\right) = 1\end{aligned} \tag{6.82}$$

where in the last line we used the property $\Gamma_h(x)\Gamma_h(2\omega - x) = 1$.
In case k is odd, we have

$$\begin{aligned}\prod_{j=1}^{k} \Gamma_h\left(j\frac{2\omega}{k+1}\right) = \prod_{j=1}^{\frac{k-1}{2}} \Gamma_h\left(j\frac{2\omega}{k+1}\right)\Gamma_h\left((k+1-j)\frac{2\omega}{k+1}\right)\Gamma_h\left(\frac{k+1}{2}\frac{2\omega}{k+1}\right) = \\ \prod_{j=1}^{n} \Gamma_h\left(j\frac{2\omega}{k+1}\right)\Gamma_h\left(2\omega - j\frac{2\omega}{k+1}\right)\Gamma_h(\omega) = 1\end{aligned} \tag{6.83}$$

where we used also the property $\Gamma_h(\omega) = 1$

6.2.5 Resulting partition function

$$
\begin{aligned}
Z_{mag} =& \frac{1}{(kN_f-N_c)!}\Gamma_h(\Delta_X\omega;\omega_1,\omega_2)^{kN_f-N_c-1} \\
&\left(\prod_{j=0}^{k-1}\prod_{a}^{N_f}\prod_{b}^{N_f}\Gamma_h\Big(\mu_a+\nu_b+j\omega\Delta_X\Big)\right) \\
&\int\prod_{i=1}^{kN_f-N_c}\frac{d\sigma_i}{\omega_1\omega_2}\int d\xi\, e^{\frac{\pi i}{\omega_1\omega_2}2\xi\sum\sigma_i}\prod_{i<j}^{kN_f-N_c}\frac{\Gamma_h(\Delta_X\omega\pm(\sigma_i-\sigma_j))}{\Gamma_h(\pm(\sigma_i-\sigma_j))} \\
&\left(\prod_{a,b}^{N_f}\prod_{j=1}^{kN_f-N_c}\Gamma_h\Big(m_B\frac{N_c}{kN_f-N_c}-m_a-m_A+\omega(\Delta_X-\Delta_Q)+\tilde{\sigma}_j\Big)\right. \\
&\qquad\qquad\left.\Gamma_h\Big(-m_B\frac{N_c}{kN_f-N_c}+\tilde{m}_b-m_A+\omega(\Delta_X-\Delta_Q)-\tilde{\sigma}_j\Big)\right) \\
&\prod_{j'=0}^{k-1}\Gamma_h\left(\pm\xi+\omega\left(N_f(1-\Delta_Q)-\Delta_X(N_c-j')\right)-m_AN_f\right)
\end{aligned}
\tag{6.84}
$$

The charges of the fields can be easily read from the arguments of the gamma functions and differ from the assignment of [26] only because of the bayonic charge of the quarks

	$SU(N_f)_L$	$SU(N_f)_R$	$U(1)_B$	$U(1)_A$	$U(1)_J$	$U(1)_R$
q	$\overline{N_f}$	0	$\frac{N_c}{kN_f-N_c}$	-1	0	$\Delta_X-\Delta_Q$
$\tilde{q}$	0	N_f	$-\frac{N_c}{kN_f-N_c}$	-1	0	$\Delta_X-\Delta_Q$
X	0	0	0	0	0	Δ_X
M_j	N_f	$\overline{N_f}$	0	2	0	$2\Delta_Q+j\Delta_X$
b_j	0	0	0	$-N_f$	1	$N_f(1-\Delta_Q)+\Delta_X(j+1-N_c)$
$\tilde{b}_j$	0	0	0	$-N_f$	-1	$N_f(1-\Delta_Q)+\Delta_X(j+1-N_c)$

(6.85)

We can modify baryonic charge of the quarks by mixing the baryonic symmetry with the $U(1)$ of the $U(kN_f-N_c)$ by the following shift in $\tilde{\sigma}$

$$
\tilde{\sigma}_i \;\longrightarrow\; \tilde{\sigma}_i-\frac{N_c}{kN_f-N_c}m_B \tag{6.86}
$$

This shift is not traceless and in fact generates a mixed Chern-Simons term k_{BJ} with the topological symmetry which contributes to the partition function as

$$
\int d\xi\; e^{\frac{2\pi i}{\omega_1\omega_2}\xi\left(N_cm_B+\sum_i\tilde{\sigma}_i\right)} \tag{6.87}
$$

After this shift, the charge of the theory are identical to the charges in [26]. Moreover, the appearance of the mixed Chern-Simons is expected because the $U(1)_{mirror}$

is the gauging of the topological symmetry which has the Fayet-Iliopulos term ξ as a charge. The FI-term is associated to the $U(1)$ factor in the $U(kN_f - N_c)$ gauge group and as a result, we can understand the appearance of the mixed Chern-Simons term [26].

Alternative derivation

We can find the same partition function by modifying the charge assignment of the quarks before dualizing the $U(k)$ sector. Consider the partition function (6.73) and apply a traceless shift of σ and ρ such that it makes the quarks $q, \tilde{q}$ uncharged under the baryonic symmetry and make the quarks $b, \tilde{b}$ charged under it. It can be achieved by

$$\tilde{\sigma} \longrightarrow \tilde{\sigma} - m_B \frac{N_c}{kN_f - N_c} \qquad \rho \longrightarrow \rho + m_B \frac{N_c}{k} \tag{6.88}$$

With this charge assignment the quark that will be dualized is different. The derivation after the dualization of the sector is identical except for the fact that with these real masses we have the additional mixed Chern-Simons term, now that $\mu - \nu \neq 0$

$$\exp\left(\frac{2\pi i}{\omega_1\omega_2} N_c \xi(\mu - \nu)\right) = \exp\left(\frac{2\pi i}{\omega_1\omega_2} \xi m_B N_c\right) \tag{6.89}$$

which match with the Fayet-Iliopulos term we obtained by the shift in $\tilde{\sigma}$ of the previous section. The quarks are uncharged under the baryonic symmetry as in the charge assignment of [26], while the other charges are identical to the table of the previous section. The charges are given by

	$SU(N_f)_L$	$SU(N_f)_R$	$U(1)_B$	$U(1)_A$	$U(1)_J$	$U(1)_R$
q	$\overline{N_f}$	0	0	-1	0	$\Delta_X - \Delta_Q$
$\tilde{q}$	0	N_f	0	-1	0	$\Delta_X - \Delta_Q$
X	0	0	0	0	0	Δ_X
M_j	N_f	$\overline{N_f}$	0	2	0	$2\Delta_Q + j\Delta_X$
b_j	0	0	0	$-N_f$	1	$N_f(1-\Delta_Q) + \Delta_X(j+1-N_c)$
$\tilde{b}_j$	0	0	0	$-N_f$	-1	$N_f(1-\Delta_Q) + \Delta_X(j+1-N_c)$

(6.90)

The partition function with these charges and with the mixed Chern-Simons term is

$$\begin{aligned}
Z_{mag} =& \frac{1}{(kN_f - N_c)!}\Gamma_h(\Delta_X\omega;\omega_1,\omega_2)^{kN_f-N_c-1} \\
&\left(\prod_{j=0}^{k-1}\prod_{a}^{N_f}\prod_{b}^{N_f}\Gamma_h\Big(\mu_a+\nu_b+j\omega\Delta_X)\Big)\right) \\
&\int\prod_{i=1}^{kN_f-N_c}\frac{d\sigma_i}{\omega_1\omega_2}\int d\xi\, e^{\frac{\pi i}{\omega_1\omega_2}2\xi\left(m_B N_c+\sum\sigma_i\right)}\prod_{i<j}^{kN_f-N_c}\frac{\Gamma_h(\Delta_X\omega\pm(\sigma_i-\sigma_j))}{\Gamma_h(\pm(\sigma_i-\sigma_j))} \\
&\left(\prod_{a,b}^{N_f}\prod_{j=1}^{kN_f-N_c}\quad\Gamma_h\Big(-m_a-m_A+\omega(\Delta_X-\Delta_Q)+\tilde{\sigma}_j\Big)\right. \\
&\qquad\qquad\left.\Gamma_h\Big(+\tilde{m}_b-m_A+\omega(\Delta_X-\Delta_Q)-\tilde{\sigma}_j\Big)\right) \\
&\prod_{j'=0}^{k-1}\Gamma_h\left(\pm\xi+\omega\left(N_f(1-\Delta_Q)-\Delta_X(N_c-j')\right)-m_A N_f\right)
\end{aligned} \tag{6.91}$$

Appendices

Appendix A

Supersymmetry and superfields

A.1 Supersymmetry algebra

The supersymmetry algebra is an extension of the Poincarè group involving anticommutators together with commutators. Since it is not a ordinary Lie algebra, Coleman-Mandula theorem does not apply for theories that are invariant under it.

The supersymmetry algebra is divided into two subalgebras, the bosonic and fermionic part. The bosonic part contains Poincarè Lie algebra $(M_{\mu\nu}, P_\mu)$ while fermionic subalgebra is generated by the *supercharges* $(Q^I_\alpha, \bar{Q}^I_{\dot{\alpha}})$ with $I = 1, \ldots, \mathcal{N}$. When more than one pair of supercharges is present we refer to extended supersymmetry.

The supercharges sit in spinorial representations of the Lorentz group, respectively $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$.

We will not repeat the bosonic subalgebra, since is given by the Poincarè Lie algebra. The fermionic generators satisfy anticommutation rules between themselves and commutation rules with bosonic generators. For this reason, the supersymmetry algebra is defined in mathematical literature as a graded Lie algebra with grade one.

The (anti)commutation rules in four dimensions are

$$[P_\mu, Q^I_\alpha] = 0 \tag{A.1}$$
$$[P_\mu, \bar{Q}^I_{\dot\alpha}] = 0 \tag{A.2}$$
$$[M_{\mu\nu}, Q^I_\alpha] = i(\sigma_{\mu\nu})^\beta_\alpha Q^I_\beta \tag{A.3}$$
$$[M_{\mu\nu}, \bar{Q}^I_{\dot\alpha}] = i(\bar\sigma_{\mu\nu})^{\dot\alpha}_{\dot\beta} \bar{Q}^I_{\dot\beta} \tag{A.4}$$
$$\{Q^I_\alpha, \bar{Q}^J_{\dot\beta}\} = 2\sigma^\mu_{\alpha\dot\beta} P_\mu \delta^{IJ} \tag{A.5}$$
$$\{Q^I_\alpha, Q^J_\beta\} = \epsilon_{\alpha\beta} Z^{IJ} \qquad Z^{IJ} = -Z^{JI} \tag{A.6}$$
$$\{Q^I_{\dot\alpha}, Q^J_{\dot\beta}\} = \epsilon_{\dot\alpha\dot\beta} \left(Z^{IJ}\right)^* \tag{A.7}$$

This set of commutation rules can be found using symmetry arguments and enforcing the consistency of the algebra using the graded Jacobi identity.
It is important to stress the fact that Z^{IJ} are operators that span an invariant subalgebra: they are *central charges*. They play an important role especially in massive representations.

There is an additional symmetry that is not present in the previous commutation rules: R-symmetry. It is an automorphism of the algebra that act on the supercharges. For generic $\mathcal{N}$ the R-Symmetry group is $U(\mathcal{N})$[1].

A.2 Representations

Since the supercharges do not commute with Lorentz generators, their action on a state will result in a state with different spin: they generate a symmetry between bosons and fermions.
Representations of supersymmetry contain particle with different spin but same mass and they are organized in supermultiplets. The mass of particles in the same multiplet must be the same because P^2 is still a Casimir operator of the supersymmetry algebra, while the Pauli-Lubanski operator W^2 isn't anymore.

Moreover, the supersymmetry algebra imposes that every state must have positive energy and that every supermultiplet must contain the same number of bosonic and fermionic degrees of freedom *on-shell* .

Various supermultiplets exist and their properties depend on the number of supercharges of the theory and on what they represent e.g matter, glue or gravity.

Massless supermultiplet are typically shorter than massive multiplet because in the massless case half of the supercharges are represented trivially. Massive representation of extended supersymmetry can be shortened in case some of the

[1]This is not always true. For example for $\mathcal{N} = 4$ the R-symmetry group is given by $SU(4)$

central charges of the algebra are equal to twice the mass of the multiplet. These states are usually called (ultra)short multiplet or BPS states.

We will introduce the multiplets that can be defined for $4d\,\mathcal{N}=1$ theories and only later we will explain the differences with $3d\,\mathcal{N}=2$ theories. Representations are similar because in both cases we have the same number of supercharges.

For four dimensional theories, we can define two different multiplet that are invariant under supersymmetry transformations. The matter or chiral multiplet contains a complex scalar (*squark*) and a Weyl fermion (*quark*). It identifies the matter content of the theory. The vector or gauge multiplet contains a Weyl fermion (*gaugino*) and a vector (*gluon or photon*). Particles in the same multiplet transform in the same representation of global or gauge symmetries. For this reason the gaugino cannot represent matter.

A representation of these multiplets on fields can be easily found using the *superspace* formalism that we will introduce in the next section. In this formalism it is possible to represent fields that are *off-shell*, in contrast with multiplets that we introduced previously that are *on-shell* since they represent states in Hilbert space.

A.2.1 Superfields and superspace in four dimensions

Supersymmetry representations on fields can be found more sistematically using the formulation of *superspace* instead of acting directly with supercharges and verifying that the algebra closes.

A simple formulation of superspace exist for theories with 4 supercharges while for theories with a bigger number of supercharges its definition is much more complex. We will be interested only in theories with 4 supercharges such as theories in 4D with $\mathcal{N}=1$ or 3D with $\mathcal{N}=2$.

Superspace can be seen as the extension of Minkowsky space with *fermionic coordinates* i.e. *Grassmann numbers* θ^α , $\bar{\theta}^{\dot{\alpha}}$. They anticommute between themselves and commute with everything else.

$$\{\theta^\alpha,\theta^\beta\}=0 \quad \{\bar{\theta}^{\dot{\alpha}},\bar{\theta}^{\dot{\beta}}\}=0 \quad \{\theta^\alpha,\bar{\theta}^{\dot{\beta}}\}=0 \qquad \alpha,\dot{\alpha}=1,2 \tag{A.8}$$

Derivation and integration in Grassmann variables are summarized by these rules

$$\partial_\alpha=\frac{\partial}{\partial\theta_\alpha} \quad \partial^\alpha=-\epsilon^{\alpha\beta}\partial_\beta \quad \bar{\partial}_{\dot{\alpha}}=\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}} \quad \bar{\partial}^{\dot{\alpha}}=-\epsilon^{\dot{\alpha}\dot{\beta}}\partial_{\dot{\beta}} \quad \partial_\alpha\theta^\beta=\delta_\alpha^\beta \quad \partial_\alpha\bar{\theta}^{\dot{\alpha}}=0 \tag{A.9}$$

$$\int d\theta=0 \quad \int d\theta\,\theta=1 \quad d^2\theta=\frac{1}{2}d\theta^1 d\theta^2 \quad \int d^2\theta=\frac{1}{4}\epsilon^{\alpha\beta}\partial_\alpha\partial\beta \tag{A.10}$$

For a more detailed introduction on Grassmann numbers and their properties see [45].

Using Grassmann numbers we can write the anticommutators in the supersymmetry algebra as commutators defining $\theta Q = \theta^\alpha Q_\alpha$ and $\bar\theta\bar Q = \bar\theta^{\dot\alpha}\bar Q_{\dot\alpha}$

$$[\theta Q, \bar\theta\bar Q] = 2\theta\sigma^\mu\bar\theta P_\mu \quad , \quad [\theta Q, \theta Q] = [\bar\theta\bar Q, \bar\theta\bar Q] = 0 \tag{A.11}$$

Using this trick we are able to represent the supersymmetry algebra as a Lie algebra. An element of the superPoincarè group can be found exponentiating the generators

$$G(x, \theta, \bar\theta, \omega) = \exp\left(ixP + i\theta Q + i\bar\theta\bar Q + \frac{1}{2}i\omega M\right) \tag{A.12}$$

The superspace is defined as the 4+4 dimensions group coset

$$M_{4|1} = \frac{\text{SuperPoincarè}}{\text{Lorentz}} \tag{A.13}$$

in analogy to Minkowsky space that can be defined as the coset between Poincarè and Lorentz groups.

A generic point in superspace is parametrized by $(x^\mu, \theta^\alpha, \bar\theta^{\dot\alpha})$. A superfields is a field in superspace i.e. function of the superspace coordinates. Since θ coordinates anticommute, the expansion of a superfield in fermionic coordinates stops at some point. The most general superfield $Y = Y(x, \theta, \bar\theta)$ is given by

$$\begin{aligned} Y(x, \theta, \bar\theta) = {} & f(x) + \theta\psi_1(x) + \bar\theta\bar\psi_2(x) + \theta\theta g_1(x) + \bar\theta\bar\theta g_2(x) \\ & + \theta\sigma^\mu\theta v_\mu(x) + \theta\theta\bar\theta\bar\lambda(x) + \bar\theta\bar\theta\theta\rho(x) + \theta\theta\bar\theta\bar\theta s(x) \end{aligned} \tag{A.14}$$

fields with uncontracted θ such as $\psi_1, \psi_2, \lambda, \rho$ are spinors while v_μ is a vector.

Supercharges can be represented as differential operators that act on superfield. Their expression is

$$\begin{cases} Q_\alpha &= -i\partial_\alpha - \sigma^\mu_{\alpha\dot\beta}\bar\theta^{\dot\beta}\partial_\mu \\ \bar Q_{\dot\alpha} &= +i\dot\partial_{\dot\alpha} + \theta^\beta\sigma^\mu_{\beta\dot\alpha}\bar\theta^{\dot\beta}\partial_\mu \end{cases} \tag{A.15}$$

An infinitesimal supersymmetry transformation on a superfield is defined by

$$\delta_{\epsilon,\bar\epsilon} Y = \left(i\epsilon Q + i\bar\epsilon\bar Q\right) Y \tag{A.16}$$

The powerfulness of the superfield formalism is due to the fact that an integral in full superspace coordinates of a superfield is supersymmetric invariant.

$$\delta_{\epsilon,\bar\epsilon} \int d^4x\, d^2\theta\, d^2\bar\theta\; Y = \int d^4x\, d^2\theta\, d^2\bar\theta\; \delta_{\epsilon,\bar\epsilon} Y = 0 \tag{A.17}$$

The first equality holds because the Grassmann measure is inviariant under translation while the second is true because we can see that the variation of the superfield is either killed by the integration in the θ variables or is proportional to a spacetime derivative that does not contribute after integration in space.

Using this fact we can construct supersymmetric invariant lagrangians by integrating superfields in superspace. Clearly, in order to find a physically significant lagrangian we should choose the superfield we wish to integrate wisely. More importantly, we want to use irreducible representation of supersymmetry i.e. the supermultiplets we introduced before. We need to find conditions that can be imposed on a general superfield that are invariant under a supersymmetry transformation.

Chiral superfield

One way to achieve this goal is to find an operator that commute with the supercharges and annihilate the superfield An example of such operator is the *covariant derivative*

$$\begin{cases} D_\alpha = & \partial_\alpha + i\sigma^\mu_{\alpha\dot{\beta}}\bar{\theta}^{\dot{\beta}}\partial_\mu \\ \bar{D}_{\dot{\alpha}} = & \dot{\partial}_{\dot{\alpha}} + i\theta^\beta\sigma^\mu_{\beta\dot{\alpha}}\bar{\theta}^{\dot{\beta}}\partial_\mu \end{cases} \tag{A.18}$$

We can define a (anti)chiral superfield Φ

$$\bar{D}_{\dot{\alpha}}\Phi = 0 \quad \text{chiral} \qquad\qquad D_\alpha\Psi = 0 \quad \text{anti-chiral} \tag{A.19}$$

This condition reduces the number of components of the superfield. It can be easily demonstrated that if Ψ is chiral, then $\bar{\Psi}$ is anti-chiral. As a result a chiral field cannot be real.

The expansion of a chiral fields in components is given by

$$\Phi(x,\theta,\bar{\theta}) = \phi(x)+\sqrt{2}\theta\psi(x)+i\theta\sigma^\mu\bar{\theta}\partial\mu\phi(x)-\theta\theta F(x)-\frac{i}{\sqrt{2}}\theta\theta\partial\mu\psi(x)\sigma^\mu\bar{\theta}-\frac{1}{4}\theta\theta\bar{\theta}\bar{\theta}(x) \tag{A.20}$$

We can see that a chiral field is composed by three fields: two complex scalars (ϕ and F) and a spinor (ψ).

The chiral superfield identifies the matter multiplet we introduced previously. It contains an additional bosonic field ($F(x)$) that is present because superfields provide an *off-shell* representation of supersymmetry and it is needed in order to close the algebra. It is called *auxiliary field* because it will not have kinetic terms in every Lagrangian that can be constructed.

Real or Vector Field

We can impose that the superfield is real. In this way we find the *real* or *vector* multiplet. Its general expression in component is messy and a simplification can be made noting that $\Phi+\bar{\Phi}$ is a vector superfield if Φ is chiral. Choosing an appropriate chiral field, the real superfield can be put in what is called Wess-Zumino gauge. We stress the fact that the Wess-Zumino gauge is not supersymmetric invariant: after a supersymmetry transformation the vector superfield acquire its general expression involving many other field components. In this gauge the vector superfield can be written as

$$V_{WZ}(x,\theta,\bar{\theta}) = \theta\sigma^{\mu}\bar{\theta}v_{\mu}(x) + i\theta\theta\bar{\theta}\bar{\lambda}(x) - i\bar{\theta}\bar{\theta}\theta\lambda(x) + \frac{1}{2}\theta\theta\bar{\theta}\bar{\theta}D(x) \tag{A.21}$$

The vector superfields represents the vector multiplet (which contains radiation) and similarly to the chiral superfields contains an auxiliary field ($D(x)$).

A.2.2 R-symmetry

R-symmetry was first introduced with the supersymmetry algebra. For the theories we will consider in superspace it is given by a global $U(1)_R$. It is defined by as a transformation of the Grassmann coordinates

$$\theta \;\rightarrow\; e^{i\alpha}\theta \qquad \bar{\theta} \;\rightarrow\; e^{-i\alpha}\bar{\theta} \tag{A.22}$$

α parametrizes the transformation. As a result supercharges transform under the transformation

$$Q \rightarrow\; e^{-i\alpha}Q \qquad \bar{Q} \;\rightarrow\; e^{+i\alpha}\bar{Q} \tag{A.23}$$

From this we find the commutator relations between supercharges and R-symmetry generator R

$$[R,Q] = -Q \qquad [R,\bar{Q}] = \bar{Q} \tag{A.24}$$

The R-charge of a superfield is defined by

$$Y(x,\theta,\bar{\theta}) \;\rightarrow\; e^{iR_Y\alpha}Y(x,\theta,\bar{\theta}) \tag{A.25}$$

Different component field in the superfield have different R-charge and are related because of A.24. For a chiral field we have

$$R[\phi] = R\,[\Phi] \qquad R[\psi] = R\,[\Phi]-1 \qquad R[F] = R\,[\Phi]-2 \tag{A.26}$$

The corresponding antichiral field carry opposite charges.

A.3 Supersymmetric actions

We will use the property we introduced in A.17 to generate supersymmetric invariant lagrangians. We start our analysis with chiral superfields. Since lagrangians are quadratic in the fields and must be real, the simplest kinetic term for a chiral superfield is given by $\bar{\Phi}\Phi$.

$$\mathcal{L}_{kin} = \int d^2\theta d^2\bar{\theta}\ \bar{\Phi}\Phi = \partial_\mu\bar{\phi}\partial^\mu\phi + \frac{i}{2}\left(\partial_\mu\psi\sigma^\mu\bar{\psi} - \psi\sigma^\mu\partial_\mu\bar{\psi}\right) + \bar{F}F + \text{total derivative} \tag{A.27}$$

which gives the correct kinetic terms for a scalar and a spinor. The auxiliary field doesn't have kinetic terms as predicted.
Many action can be find using a generalization of the equation above. It is called *Kahler* potential

$$K(\bar{\Phi}, \Phi) = \sum_{m,n=1}^{\infty} c_{m,n}\bar{\Phi}^m\Phi^n \qquad \text{where} \quad c_{m,n} = c_{n,m}* \tag{A.28}$$

The condition on the coefficient is imposed by the requirement of a real lagrangian.

Another way of finding supersymmetric actions is by integrating *chiral* superfields in half-superspace coordinates. We define the *superpotential* to be a holomorphic function of Φ

$$\mathcal{L}_{int} = \int d^2\theta d^2\ W(\Phi) + \int d^2\bar{\theta}\ \bar{W}(\bar{\Phi}) = \sum_{i=1}^{\infty} \int d^2\theta\ \lambda_n\Phi^n + \int d^2\bar{\theta}\ \lambda_n^\dagger\bar{\Phi}^n \tag{A.29}$$

The hermitian conjugate was added in order to have a real lagrangian.

Mixed terms with product of chiral and anti-chiral superfield are not present since they would be generic superfields and would not yield a supersymmetric lagrangian. In fact if $W(\Phi)$ is holomorphic and Φ is a chiral superfield, $W(\Phi)$ is a chiral superfield

$$\bar{D}_{\dot{\alpha}}W(\Phi) = \frac{\partial W}{\partial\Phi}\bar{D}_{\dot{\alpha}}\Phi + \frac{\partial W}{\partial\bar{\Phi}}\bar{D}_{\dot{\alpha}}\bar{\Phi} = 0 \tag{A.30}$$

and yield a proper lagrangian upon integration in $d^2\theta$.

Since the superpotential is integrated only in half superspace coordinates it need to be charged in an opposite way with respect to the integration measure under R-symmetry. Remembering that

$$R[\theta] = 1 \qquad [\bar{\theta}] = -1 \qquad R[d\theta] = -1 \qquad R[d\bar{\theta}] = 1 \tag{A.31}$$

It's easy to see that

$$R[W(\Phi)] = 2 \qquad R[\bar{W}(\bar{\Phi})] = -2 \tag{A.32}$$

For this reason in most situations the superpotential fix the supercharges of the fields.
The lagrangian of super Yang-Mills theories is given by

$$\mathcal{L}_{SYM} = \frac{1}{32\pi i}\left(\int d^2\theta \left(\frac{\theta_{YM}}{2\pi}+\frac{4\pi i}{g^2}\right) W_\alpha W^\alpha\right) = \tag{A.33}$$

$$= \mathrm{Tr}\left[-\frac{1}{4}F_{\mu\nu}F^{\mu nu} - i\lambda\sigma^\mu D_\mu\bar{\lambda}+\frac{1}{2}D^2\right]+\frac{\theta_{YM}}{32\pi^2}g^2\mathrm{Tr}F_{\mu\nu}\tilde{F}^{\mu\nu} \tag{A.34}$$

where we defined the chiral superfield W_α as

$$W_\alpha = -\frac{1}{4}\bar{D}\bar{D}\left(e^{-2gV}D_\alpha e^{2gV}\right) \qquad \bar{W}_{\dot{\alpha}} = -\frac{1}{4}DD\left(e^{2gV}\bar{D}_{\dot{\alpha}}Ve^{-2gV}\right) \tag{A.35}$$

It can be demonstrated that W_α is chiral and is invariant under the supergauge transformation $V \rightarrow V+\bar{\Phi}+\Phi$ while the vector superfield V was not.

From a perturbative point of view the inclusion of the term proportional to $\theta_{YM}TrF_{\mu\nu}\tilde{F}^{\mu nu}$ has no effect since it is proportional to a total derivative. It is a parity violating term that differs from zero only in non trivial topological configurations of the field (instantons).
The matter lagrangian we introduced is not invariant under gauge transformation. The correct gauge invariant lagrangian is given by

$$\mathcal{L}_{matter} = \int d^2\theta d^2\bar{\theta}\ \bar{\Phi}e^{2gV}\Phi \tag{A.36}$$

The superpotential is not automatically invariant under gauge transformation. As a result only certain expression are allowed.
There's an additional supersymmetric invariant lagrangian that can be constructed in a gauge theory when the gauge group contains abelian factors. It is called Fayet-Iliopulos term and can be present for every ideal A of the gauge group

$$\mathcal{L}_{FI} = \sum_A \xi_A \int d^2\theta d^2\bar{\theta}V^A = \frac{1}{2}\sum_A \xi_A D^A \tag{A.37}$$

Scalar potential

Supersymmetric field theories contain various scalar fields that can acquire non-zero vacuum expectation values. The scalar potential is the only term in the Lagrangian and in the Hamiltonian that can differ from zero since derivatives of the scalar fields have to be zero in order to preserve the Lorentz invariance of the vacuum. As a result, the minima of the scalar potential are in one-to-one correspondence with the states of minimal energy of the theory.
The scalar potential for $4D$ $\mathcal{N}=1$ gauge theories with matter reads

$$V(\phi_i,\bar{\phi}_j) = F\bar{F}+\frac{1}{2}D^2 \overset{on-shell}{=} \frac{\partial W}{\partial\phi_i}F^i\frac{\partial\bar{W}}{\partial\bar{\phi}_i}\bar{F}^i+\frac{g^2}{2}\sum_a |\bar{\phi}_j(T^a)^j_i\phi^j+\xi^a|^2 \geq 0 \tag{A.38}$$

ξ^a is the Fayet-Iliopulos coefficient and differs from zero only if the gauge group has abelian factors. The last equality is valid since D and F are auxiliary fields with no dynamics whose value is set by their equations of motion

$$\bar{F}_i = \frac{\partial \bar{W}}{\partial \bar{\phi}_i} \qquad D^a = -g\bar{\phi}T^a\phi - g\xi^a \tag{A.39}$$

It can be demonstrated that the scalar potential is positive definite function. The zeroes of the scalar potential can be found, if present, by solving separately the *F-term* and *D-term* equations

$$F = 0 \qquad D^a = 0 \tag{A.40}$$

Appendix B

Hypergeometric gamma functions

B.1 Elliptic hypergeometric functions

The elliptic functions are defined by

$$\begin{aligned} \Gamma_e(y;p,q) &= \prod_{j,k\geq 0} \frac{1-y^{-1}p^{j+1}q^{k+1}}{1-yp^jq^k} \\ \theta(z;p) &= \prod_{j\geq 0}(1-zp^j)(1-z^{-1}p^{j+1}) \\ (x;p) &= \prod_{j\geq 0}(1-xp^j) \end{aligned} \tag{B.1}$$

The single particle index for the vector field reads

$$i_E^V(p^n,q^n) \;=\; -\left(\frac{p^n}{1-p^n}+\frac{q^n}{1-q^n}\right) \tag{B.2}$$

The following identities are useful to calculate the Plethystic exponential for the vector field

$$\exp\left(\sum_{n=1}^{\infty}\frac{1}{n}i_E^V(p^n,q^n)\right) = (p;p)(q;q) \tag{B.3}$$

$$\begin{aligned} \exp\left(\sum_{n=1}^{\infty}\frac{1}{n}i_E^V(p^n,q^n)\right)\left(z^n+z^{-n}\right) \;&=\; \frac{\theta(z;p)\theta(z^{-1};q)}{1-z^2} \\ &=\; \frac{1}{(1-z)(1-z^{-1})\Gamma_e(z;p,q)\Gamma_e(z^{-1};p,q)} \end{aligned} \tag{B.4}$$

While the following is used for chiral fields

$$\Gamma_e(z;p,q) \;=\; \exp\left(\sum_{n=1}^{\infty}\frac{1}{n}\frac{z^n-\left(\frac{pq}{z}\right)^n}{(1-p^n)(1-q^n)}\right) \tag{B.5}$$

B.2 Hyperbolic gamma function

$$\Gamma_h(z;\omega_1,\omega_2)=\exp\left(\pi i\frac{(2z-\omega_1-\omega_2)^2}{8\omega_1\omega_2}-\pi i\frac{(\omega_1^2+\omega_2^2)}{24\omega_1\omega_2}\right) \\ \frac{(\exp(-2\pi i(z-\omega_2)/\omega_1);\exp(2\pi i\omega_2/\omega_1))_\infty}{(\exp(-2\pi z/\omega_2);\exp(-2\pi i\omega_1/\omega_2))_\infty} \tag{B.6}$$

Elliptic to hyperbolic The following mathematical identities will be used to find an expression in the limit of the shrinking circle. For chiral fields we will use [44]

$$\lim_{r\to 0^+}\Gamma_e(e^{2irz};e^{2\pi ir\omega_1},e^{2\pi ir\omega_2})\sim e^{-\frac{i\pi^2}{6r\omega_1\omega_2}(z-\omega)}\Gamma_h(z;\omega_1,\omega_2) \tag{B.7}$$

While for the vector field in the adjoint representation we will use this identity, which follows from the $SL(2,\mathbb{Z})$ properties of the $\theta(z;p)$ function [19]

$$\lim_{r\to 0}\frac{(p;p)^{r_G}(q;q)^{r_G}}{|W|}\oint_{T^{r_G}}\prod_{j=1}^{r_G}\frac{dz_j}{2\pi iz_j}\prod_{\alpha\in R_+}\theta(e^{\alpha};p)\theta(e^{-\alpha};q)\sim \\ \sim\exp\left(-\frac{i\,\pi\,\omega\,|G|}{6r\omega_1\omega_2}\right)\frac{1}{|W|}\int\prod_{j=1}^{r_G}\frac{d\sigma_j}{\sqrt{-\omega_1\omega_2}}\prod_{\alpha\in R_+}\frac{1}{\Gamma_h\left(\alpha(\sigma)\right)\Gamma_h\left(-\alpha(\sigma_j)\right)} \tag{B.8}$$

where $|G|$ is the number of generators of the group G, r_G its rank and $|W|$ is the order of the Weyl symmetry of the group. The roots α are taken only taken positive.

Appendix C

Index and partition function for the electric theory

C.1 Calculation of the superconformal index

C.1.1 Contribution from the vector field

The Plethystic exponential for the index of the vector (6.12) is given by

$$
\begin{aligned}
I_E^V(p,q,z) = &\exp\left(\sum_{n=1}^{\infty}\frac{1}{n} i_E^{Vett}(p^n,q^n,z^n)\right) \stackrel{def}{=} \\
&\exp\left(\sum_{n=1}^{\infty} -\frac{1}{n}\left(\frac{p^n}{1-p^n}+\frac{q^n}{1-q^n}\right)\left(\left(\sum_{1\le i,j\le N_c}\frac{z_i^n}{z_j^n}\right)-1\right)\right) = \\
= &\exp\left(\sum_{n=1}^{\infty} -\frac{1}{n}\left(\frac{p^n}{1-p^n}+\frac{q^n}{1-q^n}\right)\left(\left(\sum_{\substack{1\le i,j\le N_c\\ i\neq j}}\frac{z_i^n}{z_j^n}\right)+\left(\sum_{i=1}^{N_c}1\right)-1\right)\right) = \\
= &\left[\exp\left(\sum_{n=1}^{\infty} -\frac{1}{n}\left(\frac{p^n}{1-p^n}+\frac{q^n}{1-q^n}\right)\left(\left(\sum_{\substack{1\le i,j\le N_c\\ i\neq j}}\frac{z_i^n}{z_j^n}\right)\right)\right)\right]\times \\
&\times \exp\left(\sum_{n=1}^{\infty} -\frac{1}{n}\left(\frac{p^n}{1-p^n}+\frac{q^n}{1-q^n}\right)\left(N_c-1\right)\right) =
\end{aligned} \tag{C.1}
$$

which can be rewritten more compactly as

$$
\left[\prod_{\substack{1\le i,j\le N_c\\ i\neq j}}\exp\left(\sum_{n=1}^{\infty} -\frac{1}{n}\, i_E^V(p^n,q^n)\frac{z_i^n}{z_j^n}\right)\right]\left[\exp\left(\sum_{n=1}^{\infty} -\frac{1}{n}\, i_E^V(p^n,q^n)\right)\right]^{N_c-1} \tag{C.2}
$$

We can apply directly the mathematical identity (B.3) to the last term obtaining

$$(p;p)^{N_c-1}(q;q)^{N_c-1}$$

Before applying the identity (B.4) to the first term of (C.2) we need to note that

$$\prod_{\substack{1\le i,j\le N_c\\ i\neq j}} \frac{z_i^n}{z_j^n} = \prod_{1\le i<j\le N_c}\left(\frac{z_i^n}{z_j^n}+\frac{z_j^n}{z_i^n}\right)$$

Using (B.4) we obtain

$$\begin{aligned}\prod_{1\le i<j\le N_c} \exp\left(\sum_{n=1}^{\infty} -\frac{1}{n}\left(\frac{p^n}{1-p^n}+\frac{q^n}{1-q^n}\right)\left(\frac{z_i^n}{z_j^n}+\frac{z_j^n}{z_i^n}\right)\right) = \\ \prod_{1\le i<j\le N_c} \exp\left(\sum_{n=1}^{\infty} -\frac{1}{n}\, i_E^V(p^n,q^n)\left(\frac{z_i^n}{z_j^n}+\frac{z_j^n}{z_i^n}\right)\right)\times \\ \times \prod_{1\le i<j\le N_c} \frac{1}{\left(1-\frac{z_i}{z_j}\right)\left(1-\frac{z_j}{z_i}\right)\Gamma_e(\frac{z_i}{z_j};p,q)\Gamma_e(\frac{z_j}{z_i};p,q)}\end{aligned}$$

Consider both terms, the full contribution to the index is given by

$$\begin{aligned}&(p;p)^{N_c-1}(q;q)^{N_c-1} \prod_{1\le i<j\le N_c} \frac{1}{\left(1-\frac{z_i}{z_j}\right)\left(1-\frac{z_j}{z_i}\right)\Gamma_e(\frac{z_i}{z_j};p,q)\Gamma_e(\frac{z_j}{z_i};p,q)}\\ &(p;p)^{N_c-1}(q;q)^{N_c-1}\frac{1}{\Delta(z)\Delta(z^{-1})} \prod_{1\le i<j\le N_c} \frac{1}{\Gamma_e(\frac{z_i}{z_j};p,q)\Gamma_e(\frac{z_j}{z_i};p,q)}\end{aligned}$$

C.1.2 Contribution from the adjoint matter

The single particle index of the adjoint matter is

$$i_E^X(p,q,z) = \frac{1}{(1-p)(1-q)}\left((p\,q)^{\frac{R_X}{2}} - (p\,q)^{1-\frac{R_X}{2}}\right)\left(\left(\sum_{1\le i,j\le N_c}\frac{z_i}{z_j}\right)-1\right) \tag{C.3}$$

similarly to the vector field index, we separe the terms depending on the presence of z_i

$$\begin{aligned}I_E^X(p,q,z) = &\left[\exp\left((N_c-1)\sum_{n=1}^{\infty}\frac{1}{n}\frac{(pq)^{\frac{R_X}{2}n}-(pq)^{\left(1-\frac{R_X}{2}\right)n}}{(1-p^n)(1-q^n)}\right)\right]\times\\ &\times\left[\prod_{1\le i<j\le N_c}\exp\left(\sum_{n=1}^{\infty}\frac{1}{n}\left((pq)^{\frac{R_X}{2}n}-(pq)^{\left(1-\frac{R_X}{2}\right)n}\right)\left(\frac{z_i^n}{z_j^n}+\frac{z_j^n}{z_i^n}\right)\right)\right]\end{aligned} \tag{C.4}$$

These terms can be written in terms of elliptic gamma functions using (B.5). In order to use it on the first factor, we define $y = (pq)^{\frac{R_X}{2}}$ and we find

$$\left[\exp\left(\sum_{n=1}^{\infty}\frac{1}{n}\frac{(y)^n - (\frac{pq}{y})^n}{(1-p^n)(1-q^n)}\right)\right]^{(N_c-1)} = \Gamma_e((pq)^{\frac{R_X}{2}}; p, q)^{(N_c-1)} \tag{C.5}$$

The second term is a little more involved since we have to rearrange the terms in the numerator of the fraction

$$\begin{aligned}\left((pq)^{\frac{R_X}{2}n} - (pq)^{\left(1-\frac{R_X}{2}\right)n}\right)\left(\frac{z_i^n}{z_j^n} + \frac{z_j^n}{z_i^n}\right) = \\ \left((pq)^{\frac{R_X}{2}n}\frac{z_i^n}{z_j^n} - (pq)^{\left(1-\frac{R_X}{2}\right)n}\frac{z_j^n}{z_i^n}\right) + \left((pq)^{\frac{R_X}{2}n}\frac{z_j^n}{z_i^n} - (pq)^{\left(1-\frac{R_X}{2}\right)n}\frac{z_i^n}{z_j^n}\right)\end{aligned} \tag{C.6}$$

in order to apply (B.5) we need to perform the following change of variables

$$y = (pq)^{\frac{R_X}{2}}\frac{z_i}{z_j} \qquad y' = (pq)^{\frac{R_X}{2}}\frac{z_j}{z_i} \tag{C.7}$$

which results in the expression

$$\prod_{1\le i<j\le N_c}\Gamma_e\left((pq)^{\frac{R_X}{2}}\frac{z_i}{z_j}\right)\Gamma_e\left((pq)^{\frac{R_X}{2}}\frac{z_j}{z_i}\right) \tag{C.8}$$

Combining the two factors, the contribution from the adjoint matter is

$$\Gamma_e((pq)^{\frac{R_X}{2}}; p, q)^{N_c-1}\prod_{1\le i<j\le N_c}\Gamma_e\left((pq)^{\frac{R_X}{2}}\frac{z_i}{z_j}\right)\Gamma_e\left((pq)^{s}\frac{z_j}{z_i}\right) \tag{C.9}$$

C.1.3 Contribution from the quarks

The single particle index for the quarks reads

$$\begin{aligned}&i_E^{Q,\tilde{Q}}(p, q, v, y, \tilde{y}) = \\ &\qquad\frac{1}{(1-p)(1-q)}\sum_{i=1}^{N_f}\sum_{j=1}^{N_c}\Bigg((pq)^{\frac{1}{2}R_Q}\,v\,y_i\,z_j - (pq)^{1-\frac{1}{2}R_Q}\,v^{-1}\,y_i^{-1}\,z_j^{-1} \\ &\qquad\qquad + (pq)^{\frac{1}{2}R_Q}\,v^{-1}\,\tilde{y}_i\,z_j^{-1} - (pq)^{1-\frac{1}{2}R_Q}\,v\,\tilde{y}_i^{-1}\,z_j\Bigg)\end{aligned} \tag{C.10}$$

which can be simplified by the following transformations

$$(pq)^{\frac{R_Q}{2}}\,v\,yz_j \to y_{i,j} \qquad \left((pq)^{-\frac{R_Q}{2}}\,v\,yz_j\right)^{-1} \to \tilde{y}_{i,j} \tag{C.11}$$

which results in

$$I_E^{Q,\tilde{Q}}(p,q,y,\tilde{y}) = \prod_{1\le j\le N_c}\prod_{1\le i\le N_f}\exp\left[\sum_{n=1}^{\infty} -\frac{1}{n}\frac{1}{(1-p^n)(1-q^n)}\right.$$
$$\left.\left(\left((y_i z_j)^n - \left(\frac{pq}{y_i z_j}\right)^n\right) + \left(\frac{1}{(\tilde{y}_i z_j)^n} - \left(pq\tilde{y}_i z_j\right)^n\right)\right)\right] \quad \text{(C.12)}$$

and using the identity (B.5) we obtain

$$I_E^{Q,\tilde{Q}}(p,q,v,y,\tilde{y},z) = \prod_{1\le j\le N_c}\prod_{1\le i\le N_f}\Gamma_e\left((pq)^{\frac{R_Q}{2}}\, v\, y_i\, z_j\right)\Gamma_e\left((pq)^{\frac{R_Q}{2}}\, v^{-1}\,\tilde{y}_i^{-1}\, z_j^{-1}\right) \quad \text{(C.13)}$$

C.2 Reduction of the index to the partition function

Vector field

$$\lim_{r\to 0}\frac{(p;p)^{N_c-1}(q;q)^{N_c-1}}{N_c!}\oint_{T^{N_c-1}}\prod_{j=1}^{N_c-1}\frac{dz_j}{2\pi i z_j}\prod_{1\le i<j\le N_c}\frac{1}{\Gamma_e(\frac{z_i}{z_j})\Gamma_e(\frac{z_j}{z_i})}\sim$$
$$\sim\exp\left(-\frac{i\pi\omega(N_c^2-1)}{6r\omega_1\omega_2}\right)\frac{1}{N_c!}\int\prod_{j=1}^{N_c-1}\frac{d\sigma_j}{\sqrt{-\omega_1\omega_2}}\prod_{1\le i<j\le N_c}\frac{1}{\Gamma_h(\pm(\sigma_i-\sigma_j)}\quad \text{(C.14)}$$

Adjoint matter

$$\Gamma_e((pq)^{\frac{\Delta_X}{2}})^{N_c-1} = \Gamma_e\left(e^{2\pi i r\frac{\Delta_X}{2}(\omega_1+\omega_2)}\right)^{N_c-1} = \Gamma_e\left(e^{2\pi i r\Delta_X\omega}\right)^{N_c-1} =$$
$$= \left[\exp\left(-\frac{i\pi}{6r\omega_1\omega_2}(\omega\Delta_X-\omega)\right)\right]^{N_c-1}\Gamma_h(\omega\Delta_X;\omega_1,\omega_2)^{N_c-1}$$

$$\Gamma_e\left((pq)^{\frac{\Delta_X}{2}}\left(\frac{z_i}{z_j}\right)\right)\Gamma_e\left((pq)^{\frac{\Delta_X}{2}}\left(\frac{z_j}{z_i}\right)\right) =$$
$$= \Gamma_e\left(e^{2\pi i r\frac{\Delta_X}{2}(\omega_1+\omega_2)}e^{2\pi i r(\sigma_i-\sigma_j)}\right)\Gamma_e\left(e^{2\pi i r\frac{\Delta_X}{2}(\omega_1+\omega_2)}e^{2\pi i r(\sigma_j-\sigma_i)}\right) =$$
$$= \Gamma_e\left(e^{2\pi i r(\Delta_X\omega+\sigma_i-\sigma_j)}\right)\Gamma_e\left(e^{2\pi i r(\Delta_X\omega+\sigma_j-\sigma_i)}\right)$$
$$\sim\exp\left(-\frac{i\pi}{6r\omega_1\omega_2}(2\omega\Delta_X+(\sigma_i-\sigma_j)+(\sigma_j-\sigma_i)-2\omega)\right)\ \Gamma_h(\Delta_X\omega\pm(\sigma_i-\sigma_j))$$
$$\sim\exp\left(-\frac{i\pi}{6r\omega_1\omega_2}(2\omega(\Delta_X-1)\right)\Gamma_h(\Delta_X\omega\pm(\sigma_i-\sigma_j)) \quad \text{(C.15)}$$

Quarks

$$
\begin{aligned}
&\prod_{a,b}^{N_f}\prod_{j}^{N_c}\Gamma_e\left((pq)^{\frac{R_Q}{2}}v\,y_a z_j\right)\Gamma_e\left((pq)^{\frac{R_Q}{2}}v^{-1}\tilde{y}_b^{-1}z_j^{-1}\right)= \\
&=\Gamma_e\left(e^{2\pi i r\left(\frac{R_Q}{2}(\omega_1+\omega_2)+(m_a+m_B+\sigma_j)\right)}\right)\Gamma_e\left(e^{2\pi i r\left(\frac{R_Q}{2}(\omega_1+\omega_2)+(-\tilde{m}_b-m_B-\sigma_j)\right)}\right)= \\
&=\,\exp\left(-\frac{i\pi}{6r\omega_1\omega_2}((\omega R_Q+m_a+m_B+\sigma_j-\omega)+(\omega R_Q-\tilde{m}_b-m_B-\sigma_j-\omega))\right) \\
&\qquad\qquad\qquad\qquad\times\Gamma_h(\omega R_Q+m_a+m_B+\sigma_j)\,\Gamma_h(\omega R_Q-\tilde{m}_b-m_B-\sigma_j)= \\
&=\,\exp\left(-\frac{i\pi}{6r\omega_1\omega_2}(2\omega(R_Q-1)+m_a-\tilde{m}_b)\right)\,\Gamma_h(\mu_a+\sigma_j)\Gamma_h(\nu_b-\sigma_j)
\end{aligned}
\tag{C.16}
$$

where we defined

$$
\mu_a=\omega\Delta+m_a+m_B\quad\nu_b=\omega\Delta-\tilde{m}_b-m_B \tag{C.17}
$$

Appendix D

Superconformal index and partition function for the magnetic theory

D.1 Reduction of the index to the partition function

Mesons

$$
\begin{aligned}
&\prod_{i}^{N_f}\prod_{j}^{N_f}\prod_{l=0}^{k-1}\Gamma_e((pq)^{r+ls}y_i\tilde{y_j}^{-1};p,q)=\\
&\prod_{i}^{N_f}\prod_{j}^{N_f}\prod_{l=0}^{k-1}\Gamma_e\Big(\exp\left(2\pi ir\left[(2\omega)(r+ls)+(m_i-\tilde{m}_j)\right]\right);p,q)\sim\\
&\overset{r\to 0}{\sim}\prod_{i}^{N_f}\prod_{j}^{N_f}\prod_{l=0}^{k-1}\exp\left(\frac{-i\pi}{6r\omega_1\omega_2}\Big(2\omega(r+ls)+m_i-\tilde{m}_j-\omega\Big)\right)\Gamma_h\Big(\omega(2\Delta_Q+l\Delta_X)+m_i-\tilde{m}_j\Big)=\\
&=\exp\left[\frac{-i\pi}{6r\omega_1\omega_2}\left(N_f^2\left(\sum_{l=0}^{k-1}2\omega(r+ls-\frac{1}{2})\right)+N_f\left(\sum_{l=0}^{k-1}\sum_{i}^{N_f}m_i-\tilde{m}_i\right)\right)\right]\\
&\qquad\qquad\prod_{i}^{N_f}\prod_{j}^{N_f}\prod_{l=0}^{k-1}\Gamma_h\Big(\omega(2\Delta_Q+l\Delta_X)+m_i-\tilde{m}_j\Big)\\
&=\exp\left[\frac{-i\pi}{6r\omega_1\omega_2}\left(N_f^2\left(\sum_{l=0}^{k-1}2\omega(\Delta_Q-\frac{1}{2})+\omega l\Delta_X\right)+N_f\left(\sum_{l=0}^{k-1}\sum_{i}^{N_f}m_i-\tilde{m}_i\right)\right)\right]\\
&\qquad\qquad\prod_{i}^{N_f}\prod_{j}^{N_f}\prod_{l=0}^{k-1}\Gamma_h\Big(\omega(2\Delta_Q+l\Delta_X)+m_i-\tilde{m}_j\Big)
\end{aligned}
\tag{D.1}
$$

Chiral fields

$$\prod_{1\le j\le \tilde{N}_c}\prod_{1\le i\le N_f}\Gamma_e((pq)^{\frac{1}{2}\Delta_q}\tilde{v}y_i^{-1}\tilde{z}_j;p,q)\Gamma_e\left((pq)^{\frac{1}{2}\Delta_q}\tilde{v}^{-1}\tilde{y}_i\tilde{z}_j^{-1};p,q\right)=$$
$$=\prod_{1\le j\le \tilde{N}_c}\prod_{1\le i\le N_f}\Gamma_e\left(\exp\left(2\omega\left(\frac{1}{2}\Delta_q\right)+\tilde{m_B}-m_i+\tilde{\sigma_j}\right);p,q\right)$$
$$\Gamma_e\left(\exp\left(2\omega\left(\frac{1}{2}\Delta_q\right)-\tilde{m_B}+\tilde{m}_i-\tilde{\sigma_j}\right);p,q\right)=$$
$$\Gamma_h\left(\omega\Delta_q+\tilde{m_B}-m_i+\tilde{\sigma_j}\right)\Gamma_h\left(\omega\Delta_q-\tilde{m_B}+\tilde{m}_i-\tilde{\sigma_j}\right)=$$
$$=\exp\left[\frac{-i\pi}{6r\omega_1\omega_2}\left(2N_f\tilde{N}_c\,\omega(\Delta_q-1)+\tilde{N}_c\left(\sum_{i=1}^{N_f}\tilde{m}_i-m_i\right)\right)\right]\Gamma_h\left(\mu_i+\tilde{\sigma_j}\right)\Gamma_h\left(\nu_i-\tilde{\sigma_j}\right) \tag{D.2}$$

where we defined the real masses

$$\mu_i=\omega\Delta_q+\tilde{m_B}-m_i \qquad \nu_i=\omega\Delta_q-\tilde{m_B}+\tilde{m}_i$$

Vector field

$$\lim_{r\to 0}\frac{(p;p)^{N_c-1}(q;q)^{N_c-1}}{N_c!}\oint_{T^{\tilde{N}_c-1}}\prod_{j=1}^{\tilde{N}_c-1}\frac{dz_j}{2\pi i z_j}\prod_{1\le i<j\le \tilde{N}_c}\frac{1}{\Gamma_e(\frac{z_i}{z_j})\Gamma_e(\frac{z_j}{z_i})}\sim$$
$$\sim\exp\left(-\frac{i\pi\omega(\tilde{N}_c^2-1)}{6r\omega_1\omega_2}\right)\frac{1}{\tilde{N}_c!}\int\prod_{j=1}^{\tilde{N}_c-1}\frac{d\sigma_j}{\sqrt{-\omega_1\omega_2}}\prod_{1\le i<j\le \tilde{N}_c}\frac{1}{\Gamma_h(\pm(\sigma_i-\sigma_j)} \tag{D.3}$$

Adjoint matter field

$$\Gamma_e((pq)^{\frac{\Delta_X}{2}})^{N_c-1}=\Gamma_e\left(\exp\left[2\pi i r\frac{\Delta_X}{2}(\omega_1+\omega_2)\right]\right)^{\tilde{N}_c-1}=\Gamma_e(e^{2\pi i r\Delta_X\omega})^{\tilde{N}_c-1}=$$
$$=\left\{\exp\left[-\frac{i\pi}{6r\omega_1\omega_2}(\omega\Delta_X-\omega)\right]\right\}^{\tilde{N}_c-1}\Gamma_h\left(\omega\Delta_X;\omega_1,\omega_2\right)^{\tilde{N}_c-1} \tag{D.4}$$

$$
\begin{aligned}
&\Gamma_e\left((pq)^{\frac{\Delta_X}{2}}\left(\frac{z_i}{z_j}\right)\right)\Gamma_e\left((pq)^{\frac{\Delta_X}{2}}\left(\frac{z_j}{z_i}\right)\right)= \\
&=\Gamma_e\left(\exp\left[2\pi i r\left(\frac{\Delta_X}{2}(\omega_1+\omega_2)+(\sigma_i-\sigma_j)\right)\right]\right)\Gamma_e\left(\exp\left[2\pi i r\left(\frac{\Delta_X}{2}(\omega_1+\omega_2)+(\sigma_j-\sigma_i)\right)\right]\right)= \\
&=\Gamma_e\left(\exp\left[2\pi i r(\Delta_X\omega+(\sigma_i-\sigma_j)\right]\right)\Gamma_e\left(\exp\left[2\pi i r(\Delta_X\omega+(\sigma_j-\sigma_i)\right]\right) \\
&=\exp\left(-\frac{i\pi}{6r\omega_1\omega_2}(2\omega\Delta_X+(\sigma_i-\sigma_j)+(\sigma_j-\sigma_i)-2\omega)\right)\ \Gamma_h(\Delta_X\omega+\sigma_i-\sigma_j)\Gamma_h(\Delta_X\omega+\sigma_j-\sigma i) \\
&=\exp\left(-\frac{i\pi}{6r\omega_1\omega_2}(2\omega(\Delta_X-1)\right)\ \Gamma_h(\Delta_X\omega+\sigma_i-\sigma j)\Gamma_h(\Delta_X\omega+\sigma_j-\sigma_i)
\end{aligned}
\tag{D.5}
$$

Sum of the divergent contributions The exponent of the diverging terms is given by

$$
\begin{aligned}
&\overbrace{(\omega(\Delta_x-1)(\tilde{N}_c-1)+(2\omega(\Delta_x-1)\frac{\tilde{N}_c(\tilde{N}_c-1)}{2})}^{\text{Adj Chiral}}+\overbrace{\omega(\tilde{N}_c^2-1)}^{\text{Vector}}+ \\
&\overbrace{2N_f\tilde{N}_c\omega(\Delta'-1)+\tilde{N}_c\Big(\sum_{i=1}^{N_f}-m_i+\tilde{m}_i\Big)}^{\text{Fond Chirals}}+\overbrace{N_f^2\Big(\sum_{l=0}^{k-1}2\omega(\Delta_Q+l\frac{\Delta_X}{2}-\frac{1}{2})\Big)+N_f\Big(\sum_{l=0}^{k-1}\sum_{i}^{N_f}m_i-\tilde{m}_i\Big)}^{\text{Mesons}}=
\end{aligned}
\tag{D.6}
$$

In order to simplify this expression we need to use the following identity

$$
\sum_{i=0}^{n} i=\frac{n(n+1)}{2}\qquad\longrightarrow\qquad\sum_{i=0}^{k-1} i=\frac{(k-1)k}{2}
\tag{D.7}
$$

which in the case of the meson becomes

$$
N_f^2\omega(2\Delta_Q-1)k+\omega\Delta_X\frac{(k-1)k}{2}+kN_f\Big(\sum_{i}^{N_f}m_i-\tilde{m}_i\Big)
\tag{D.8}
$$

The exponent is then

$$\begin{aligned}
&\omega(\tilde{N}_c^2-1)+(\omega(\Delta_X-1)(\tilde{N}_c^2-1)+2N_f\tilde{N}_c\omega(\Delta_q-1)+\tilde{N}_c\Big(\sum_{i=1}^{N_f}-m_i+\tilde{m}_i\Big)+\\
&+N_f^2\Big(\omega(2\Delta_Q-1)k+\omega\Delta_X\frac{(k-1)k}{2}\Big)+kN_f\Big(\sum_i^{N_f}m_i-\tilde{m}_i\Big)=\\
&=\omega((kN_f-N_c)^2-1)+(\omega(\Delta_X-1)((kN_f-N_c)^2-1)+\\
&\quad 2N_f(kN_f-N_c)\omega(\Delta_q-1)+(kN_f-N_c)\Big(\sum_{i=1}^{N_f}-m_i+\tilde{m}_i\Big)+\\
&+N_f^2\Big(\omega(2\Delta_Q-1)k+\omega\Delta_X\frac{(k-1)k}{2}\Big)+kN_f\Big(\sum_i^{N_f}m_i-\tilde{m}_i\Big)
\end{aligned} \tag{D.9}$$

Using the explicit value of the R-charge for the dual quark ($\Delta_q=\Delta_X-\Delta_Q$) and the explicit value of Δ_X we obtain

$$\begin{aligned}
&\omega\Delta_x(N_c^2-1)+\omega(-2N_fN_c)+\omega\Delta_Q(2N_fN_c)+N_c\Big(\sum_i^{N_f}m_i-\tilde{m}_i\Big)=\\
&=\omega\Delta_x(N_c^2-1)+(+\omega-\omega)(N_c^2-1)+2N_fN_c\omega(\Delta_Q-1)+N_c\Big(\sum_i^{N_f}m_i-\tilde{m}_i\Big)=\\
&=\omega(N_c^2-1)+(N_c^2-1)\omega(\Delta_x-1)+2N_fN_c\omega(\Delta_Q-1)+N_c\Big(\sum_i^{N_f}m_i-\tilde{m}_i\Big)
\end{aligned} \tag{D.10}$$

which coincides with the factor calculated in the electric theory.

Infinite mass limit The divergent contribution in the limit $m \to \infty$ is given by

$$\begin{aligned}
&\text{Mesons}\left\{\sum_{j=0}^{k-1} 4N_f m(-m_A(N_f-1)+\omega(\Delta_Q+\Delta_M+j\Delta_X-1))+\right.\\
&\text{Chiral 1}\begin{cases} +4(kN_f-N_c)\Big(m_B\frac{N_c}{k(N_f+1)-N_c}(-m_AN_f-\omega(\Delta_X-\Delta_M-1))+\\ +m(m_AN_f+\omega(\Delta_X-\Delta_M-1))+\\ +4\sum_j^{kN_f-N_c}\tilde{\sigma}_j\Big(-m_AN_f-\omega(\Delta_X-\Delta_M-1)\Big)\end{cases}\\
&\text{Chiral 2}\begin{cases} +4kN_f\Big(m_B\frac{N_c}{k(N_f+1)-N_c}(-m_A+\omega(\Delta_X-\Delta_Q-1))+\\ +m(-m_A+\omega(\Delta_X-\Delta_Q-1)+\\ +4N_f\sum_{j=1}^k\rho_j(-m_A+\omega(\Delta_X-\Delta_Q-1))\end{cases}\\
&\text{Adj matter}\begin{cases} +m(4\omega\Delta_Xk(kN_f-N_C)+\\ -4k\sum_i\tilde{\sigma}_i(\omega\Delta_X)+\\ +4(kN_f-N_c)\sum_i\rho_i(\omega\Delta_X)\end{cases}
\end{aligned} \tag{D.11}$$

Let us simplify this expression by separating it into various terms

Contributions proportional to m They are given by

$$\begin{aligned}
&m\bigg(-4kN_fm_A(N_f-1)-4N_fN_cm_A+4kN_fmm_A(N_f-1)\bigg)+\\
&4m\omega\bigg(kN_f(\Delta_Q+\Delta_M-1+\frac{(k-1)}{2}\Delta_X)+(kN_f-N_c)(\Delta_X-\Delta_M-1)+kN_f(\Delta_X-\Delta_Q-1)+\\
&\quad+\Delta_Xk(kN_f-N_C)\bigg)
\end{aligned} \tag{D.12}$$

Using the constraint on the real masses we obtain

$$\omega(-N_c\Delta_X+N_f+1)=\omega(N_f\Delta_Q+\Delta_M) \quad \to \quad \Delta_M=-N_c\Delta_X+N_f(1-\Delta_Q)+1 \tag{D.13}$$

Using the explici value of Δ_X and writing $\frac{k-1}{2}=\frac{k+1}{2}-1$

$$\begin{aligned}
&4m\bigg(-N_fN_cm_A+\omega\Big(kN_f(\Delta_Q-N_c\Delta_X+N_f(1-\Delta_Q)+1+\frac{k+1}{2}\Delta_X-\Delta_X-1\Big)+\\
&\quad+(kN_f-N_c)(\Delta_X-(-N_c\Delta_X+N_f(1-\Delta_Q)+1)-1)+kN_f(\Delta_X-\Delta_Q-1)\Big)+\\
&\quad+\Delta_Xk(kN_f-N_c)\bigg)
\end{aligned} \tag{D.14}$$

which leads to

$$\begin{aligned} &4m\omega N_c\left(-N_f m_A-\Delta_X N_c-(\Delta_Q-1)N_f\right)+\\ &+4m\omega(kN_f-N_c)\left((\Delta_X-2+k\Delta_X)\right)=\\ &=4m\omega N_c\left(-N_f m_A-\Delta_X N_c-(\Delta_Q-1)N_f\right)+\\ &+4m\omega(kN_f-N_c)\left(\Delta_X(k+1)-2)\right)=\\ &=4m\omega N_c\left(-N_f m_A+(1-\Delta_Q)N_f-\Delta_X N_c\right) \end{aligned} \tag{D.15}$$

which match with the result from the electric theory.

Contributions proportional to m_B They are given by

$$\begin{aligned} &4m_B\frac{N_c}{k(N_f+1)-N_c}\bigg((kN_f-N_c)(-m_AN_f-\omega(\Delta_X-\Delta_M-1)+\\ &\qquad+kN_f(-m_A+\omega(\Delta_X-\Delta_Q-1)\bigg)=\\ &=4m_B\frac{N_c}{k(N_f+1)-N_c}\bigg(kN_f(-m_A(N_f+1)+\omega(\Delta_M-\Delta_Q))+\\ &\qquad-N_c(-m_AN_f-\omega(\Delta_X-\Delta_M-1)\bigg)=\\ &=4m_B\frac{N_c}{k(N_f+1)-N_c}\bigg(kN_f(-m_A(N_f+1)+\omega(-N_c\Delta_X+N_f(1-\Delta_Q)+1-\Delta_Q))+\\ &\qquad+N_c(m_AN_f+\omega(\Delta_X-(-N_c\Delta_X+N_f(1-\Delta_Q)+1)-1))\bigg)=\\ &=4m_B\frac{N_c}{k(N_f+1)-N_c}\bigg(kN_f(-m_A(N_f+1)+\omega(-N_c\Delta_X+N_f-(N_f+1)\Delta_Q)+1))+\\ &\qquad+m_AN_fN_c+N_c\omega(\Delta_X(N_c+1)+N_f(\Delta_Q-1)-2)\bigg) \end{aligned} \tag{D.16}$$

Simplifying some fractions we get

$$\begin{aligned} &4m_BN_cN_f(-m_A)+4m_BN_cN_f(-\omega\Delta_Q)+4m_BN_c^2\omega\Delta_X\bigg(-1+\frac{k+1}{k(N_f+1)-N_c}\bigg)\\ &+4m_BN_cN_f-8m_BN_c^2\omega\frac{1}{k(N_f+1)-N_c} \end{aligned} \tag{D.17}$$

Expliciting the value of Δ_X we obtain

$$4m_BN_C(-m_AN_f+\omega(N_f(1-\Delta_Q)-N_c\Delta_X))+4m_BN_c^2\omega\frac{1}{k(N_f+1)-N_c}\bigg(\frac{2}{k+1}(k+1)-2\bigg) \tag{D.18}$$

which is the same as the contribution in the electric theory

$$4m_B N_C(-m_A N_f + \omega(N_f(1-\Delta_Q) - N_c\Delta_X)) \tag{D.19}$$

Contributions proportional to $\sum_i \sigma_i + \sum_j \rho_j$ They are given by

$$\begin{aligned}
&4\sum_{j}^{kN_f-N_c} \tilde{\sigma}_j\Big(-m_A N_f - \omega(\Delta_X(1+k) - \Delta_M - 1)\Big)+\\
&4N_f\sum_{j=1}^{k}\rho_j(-m_A + \omega(\Delta_X - \Delta_Q - 1)) + 4(kN_f - N_c)\sum_i \rho_i(\omega\Delta_X) =\\
=&4\sum_{j}^{kN_f-N_c} \tilde{\sigma}_j\Big(-m_A N_f - \omega(1-\Delta_M)\Big)+\\
+&4\sum_{j=1}^{k}\rho_j(-m_A N_f + \omega(\Delta_X(N_f + (kN_f - N_c)) - N_f\Delta_Q - N_f)) =\\
=&4\sum_{j}^{kN_f-N_c} \tilde{\sigma}_j\Big(-m_A N_f + \omega(\Delta_M - 1)\Big)+\\
+&4\sum_{j=1}^{k}\rho_j(-m_A N_f + \omega(\Delta_X N_f(k+1) - N_c\Delta_X) - N_f\Delta_Q - N_f)) =\\
=&4\sum_{j}^{kN_f-N_c} \tilde{\sigma}_j\Big(-m_A N_f + \omega(-N_c\Delta_X + N_f(1-\Delta_Q))\Big)++\\
+&4\sum_{j=1}^{k}\rho_j\,(-m_A N_f + \omega(-N_c\Delta_X + N_f(1-\Delta_Q)) =\\
&= 4\left(\sum_{i}^{kN_f-N_c}\sigma_i + \sum_{j}^{k}\rho_j\right)(-m_A N_f + \omega(-N_c\Delta_X + N_f(1-\Delta_Q))
\end{aligned} \tag{D.20}$$

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