# 4D to 3D reduction of Seiberg duality for $SU(N)$ susy gauge theories with adjoint matter: a partition function approach

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# 1 Four dimensional dualities

—— INTRODUCTION OUTLINE ———
~ More symmetry = more tools for studying t\heories
~ Milder divergences
~ Renormalization constraints
~ Non renormalization theorems (perturbative)
~ Holomorphicity, couplings as background fields
~ Exact results (superpotential, exact beta function )
~ Moduli space

## 1.1 Introduction

Supersymmetric quantum field theories enjoy an enlarged group of symmetries compared to other field theories. Since the symmetry group is a non trivial combination of internal and spacetime symmetries, they have many unexpected features and new techniques were found to study them. Almost all of the new tools found are available only for supersymmetric field theories, making them the theatre for many advances in physics.

A more technical introduction on supersymmetry and its representation on fields can be found in appendix A.

In this section we will analyse more advanced features of supersymmetric field theories that has been used intensively in the discovery and in the analysis of electric magnetic duality and its generalisations.

### 1.1.1 General renormalization properties

A remarkable feature of supersymmetry is the constraint that the additional symmetry imposes on the renormalization properties of the theories.

One of the first aspects that brought attention to supersymmetry was that divergences of loop diagrams were milder because of the cancellation between diagrams with bosons and fermions running in the loops.

Nowadays we know powerful theorems that restrict the behaviour of supersymmetric field theories during renormalization. In order to preserve supersymmetry, the renormalization process has to preserve the Hilbert space structure. For example the wave function renormalization of different *particles* inside a multiplet must be the same, otherwise the renormalized lagrangian is not supersymmetric invariant anymore.

Moreover, in the supersymmetry algebra $P^2$ is still a Casimir operator i.e. it commutes with every operator in the algebra: particles in the same multiplet must have the same mass. Renormalization cannot break this condition, otherwise it would break supersymmetry.

For a *Super Yang Mills* theory with $\mathcal{N} = 1$ we have the additional requirement that $gV$, where $g$ is the coupling and $V$ is the vector superfield, cannot be renormalized by symmetry considerations.

Adding more supersymmetry the wave function renormalization of the various field are even more constrained by symmetry. For example, for $\mathcal{N} = 4$ *SYM* the fields and the coupling are not renormalized at all.

### Beta function for SYM and SQCD

Another nice feature of supersymmetric field theories is that some quantities can be calculated exactly. The first object of this kind that we encounter is the $\beta$ function of four dimensional $\mathcal{N} = 1$ *Super Yang Mills* theories with matter fields in representations $R_i$.

It is given by the *NSVZ $\beta$ function*

$$\beta(g) = \mu \frac{\mathrm{d}\, g}{\mathrm{d}\, \mu} = -\frac{g^3}{16\pi^2}\left[3\, T(Adj) - \sum_i T(R_i)(1-\gamma_i)\right]\left(1 - \frac{g^2\, T(Adj)}{8\pi^2}\right)^{-1} \qquad \alpha = \frac{g^2}{4\pi} \tag{1.1}$$

where $\gamma_i$ are the anomalous dimensions of the matter fields and $T(R_i)$ are the Dynkin indices of their representation.
The anomalous dimensions are defined as

$$\gamma_i = -\mu \frac{\mathrm{d}\, \log(Z_i)}{\mathrm{d}\, \mu} \tag{1.2}$$

where $Z_i$ is the wave function renormalization coefficient. The Dynkin indices [1] of the gauge group $SU(N)$ for the fundamental and adjoint representation are

$$T(\mathrm{N}) = \frac{1}{2} \qquad T(\mathrm{Adj}) = N \tag{1.3}$$

The *NSVZ $\beta$ function* was first calculated using instanton methods in [1]. Over the years it has been calculated in other ways using the fact that the action is holomorphic in the complexified coupling

$$\tau = \frac{4\pi i}{g_c^2} + \frac{\theta_{YM}}{2\pi} \tag{1.4}$$

Using the holomorphic coupling the action for the vector field is written as

$$\mathcal{L}_h(V_h) = \frac{1}{16\pi i}\int d^2\theta\, \tau\, W^a(V_h)W^a(V_h) + h.c. \tag{1.5}$$

whereas with the canonical normalization for the vector field is

$$\mathcal{L}_c(V_c) = \frac{1}{16\pi i}\int d^2\theta \left(\frac{4\pi i}{g_c^2} + \frac{\theta_{YM}}{2\pi}\right) W^a(g_c V_c)W^a(g_c V_c) + h.c. \tag{1.6}$$

[1] The Dynkin index $T(R)$ of a representation $R$ is defined as $\mathrm{Tr}\,(T^a T^b) = T(R)\delta^{ab}$ where $T^a$ , $T^b$ are the generators of the algebra in the representation $R$.

Using the canonical normalization $g_c V_c$ is a real superfield, imposing that $g_c$ is real. For this reason with the canonical normalization the lagrangian is not holomorphic in $\tau$. Thanks to holomorphicity, the holomorphic coupling is only renormalized at one-loop and the $\beta$ function can be computed exactly at one loop but its expression is different from *NSVZ* $\beta$ *function*. The cause of this mismatch is that the *NSVZ* $\beta$ *function* is defined using the canonical (or physical) coupling constant and receives contribution from all orders in perturbation theory.

At first sight, one should expect that the expressions should match since the first two orders in $\alpha$ of the $\beta$ function are scheme independent. The reason why the two expressions differ is that the Jacobian of the transformation between canonical and holomorphic normalization is anomalous. Once the anomaly is taken into account the two expressions for the $\beta$ function agree.
An explicit calculation relating the comparison between the two different approach can be found in [2].

### 1.1.2 Superpotential: holomorphy and non-renormalization

Other than renormalization constraints, supersymmetry provides non-renormalization theorems for certain objects, such as the superpotential.

In [3] it has been demonstrated that the superpotential is tree-level exact, i.e. it does not receive correction in perturbation theory. However it usually receive contributions from non perturbative dynamics. The superpotential features this property in theories with at least four supercharges and can be demonstrated independently using perturbative calculations or its holomorphic properties.

It was first demonstrated perturbatively, using the fact that for general supersymmetric field theories, supergraph loops diagrams with $n$ external leg yield a term that can be written in the form

$$\int d^4x_1 \dots d^4x_n d^2\theta d^2\bar{\theta}\, G(x_1, \dots, x_n)\, F_1(x_1, \theta, \bar{\theta}) \dots F_n(x_n, \theta, \bar{\theta}) \tag{1.7}$$

where $G(x_1, \dots, x_n)$ is translationally invariant function.
The importance of this result is that all contribution from Feynman diagrams are given by a single integral over full superspace ($d^2\theta d^2\bar{\theta}$) whereas the superpotential must be written as an integral in half-superspace($d^2\theta$ only) of chiral fields. Exploiting the fact that a product of chiral fields is a chiral field, the most general form of a superpotential is

$$W(\lambda, \Phi) = \sum_{n=1}^{\infty} \left( \int d^2\theta\, \lambda_n \Phi^n \; + \; \int d^2\bar{\theta}\, \lambda_n^\dagger \bar{\Phi}^n \right) \tag{1.8}$$

The second term of the superpotential is added in order to give a real lagrangian after the integration in superspace. From the definition, we can see that the superpotential is holomorphic in the fields and in the coupling constants.

Fifteen years later, Seiberg [4] provided a proof of this theorem using a different approach. He noted that the coupling constants $\lambda_n$ can be treated as background fields, i.e. chiral superfields with no dynamics.

Using this observation we can assign transformation laws to the coupling constants, making the lagrangian invariant under a larger symmetry. Fields and coupling constants are charged under this symmetry and only certain combinations of them can appear in the superpotential. In addition, in a suitable weak coupling limit the effective superpotential must be identical with the tree-level one. These conditions, taken together, fix the expansion of the superpotential to the expression of the tree-level potential. A more detailed discussion can be found in [5] and [6].

### 1.1.3 Moduli space

Supersymmetric field theories have a larger set of vacua compared to ordinary field theories because of the presence of many scalar fields in the supermultiplets.

Lorentz invariance of the vacuum forbids fields with spin different from zero to acquire a vacuum expectation value. With the same reasoning, derivatives of scalar fields must be set to zero because of translational invariance of the vacuum. The scalar potential is the only term in the Lagrangian and in the Hamiltonian that can differ from zero. As a result, the minimums of the scalar potential are in one-to-one correspondence with the states of minimal energy of the theory.

For $4D$ $\mathcal{N}=1$ gauge theories with matter, the scalar potential for the squarks reads

$$V(\phi_i,\bar{\phi}_j) = F\bar{F} + \frac{1}{2}D^2 \stackrel{on-shell}{=} \frac{\partial W}{\partial \phi_i}F^i \frac{\partial \bar{W}}{\partial \bar{\phi}_i}\bar{F}^i + \frac{g^2}{2}\sum_a |\bar{\phi}_j (T^a)^j_i \phi^j + \xi^a|^2 \geq 0 \tag{1.9}$$

$\xi^a$ is the Fayet-Iliopulos coefficient and differs from zero only for abelian factors of the gauge group. The last equality is valid since $D$ and $F$ are auxiliary fields with no dynamics. Their value is set by their equations of motion

$$\bar{F}_i = \frac{\partial \bar{W}}{\partial \bar{\phi}_i} \qquad D^a = -g\bar{\phi}T^a\phi - g\xi^a \tag{1.10}$$

Supersymmetric vacua are described by the sets of values of the scalar *VEVs* that give a zero scalar potential. This requirement is equivalent to two different sets of equations, called $D-term$ and $F-term$ equations

$$\bar{F}^i(\phi) = 0 \qquad D^a(\phi,\bar{\phi}) = 0 \tag{1.11}$$

*F-term* equations are present only if there is a superpotential while the *D-term* equations are always present.
If the minimum of the scalar potential is different from zero the vacuum is not supersymmetric. In this case supersymmetry is spontaneously broken. Another possible situation is that the scalar potential has no minimum at all: the theory does not have any stable vacua.

The *classical moduli space* is the set of solution of these equations for scalar *VEVs* and represents the classical supersymmetric vacua of the theory. Gauge transformation should be taken into account in order to avoid redundance in the description. The moduli space describe physically inequivalent vacua, since the mass spectrum of the theory depends on the *VEVs* of the scalar fields, that differ in every point of the moduli space.

Because of supersymmetry radiative corrections do not lift the energy of the ground state and the vacuum remains supersymmetric. As a result, only superpotentials generated from non perturbative dynamics can lift the moduli space. We will see examples of this phenomenon in the analysis of SQCD.

An alternative description of the space of classical *D-flat* directions is given by the space of all holomorphic gauge invariant polynomials of scalar fields modulo classical relations between them [7]. As a result, gauge invariant polynomials of operators parametrize the classical moduli space of the theory. Using this description it's easier to find the moduli space of the theory in consideration. If a superpotential is present, *F-term* equations should be imposed on the gauge invariant polynomial used to describe *D-flat* direction. We will use this convenient description in the next chapters.

### 1.1.4 Phases of gauge theories

The dynamics of gauge theories can be classified according to the low-energy effective potential $V(R)$ between two test charges separated by a large distance $R$. The possible forms of the potential, up to addictive constant, are

$$\text{Coulomb} \qquad V(R) \sim \frac{1}{R} \tag{1.12}$$

$$\text{free electric} \qquad V(R) \sim \frac{1}{R\,\log(R\Lambda)} \tag{1.13}$$

$$\text{free magnetic} \qquad V(R) \sim \frac{\log(R\Lambda)}{R} \tag{1.14}$$

$$\text{Higgs} \qquad V(R) \sim \text{constant} \tag{1.15}$$

$$\text{confining} \qquad V(R) \sim \sigma\, R \tag{1.16}$$

The first three phases feature massless gauge fields and their potential is $V(R) \sim g^2(R)/R$ and they differ because of the renormalization of the charge in the IR. In the Coulomb phase, $g^2_{IR} =$ constant, while in the free abelian/non-Abelian phase the coupling constant goes to zero as $g^2(R) \sim 1/\log(R\,\Lambda)$. The free electric phases is possible for abelian or non-Abelian theories.In the latter case for asymptotically free theories it's necessary that the renormalization group has a non trivial infrared fixed point. The free magnetic phases is generated by magnetic monopoles acting as source of the field. Since magnetic and electric charges are related by Dirac quantization condition, the running of the coupling constant for magnetic monopoles is the inverse of electric charges.
The situation is completely different in the last two cases. In the Higgs phase gauge fields are massive and the potential is given by a Yukawa potential, exponential suppressed at long distances that results in a constant value. The confining phase can be described by tube of confined gauge flux between the charges which, at large distances, acts as a string with costant tension, yielding a linear potential.

### 1.1.5 't Hooft anomaly matching conditions

The 't Hooft anomaly matching conditions are a great tool to investigate the global symmetries of the low-energy degrees of freedom of the theory.
Let's consider an asymptotically free gauge theory with global symmetry group $G$. Gauge symmetries can't be anomalous because that would spoil the unitarity of the theory but there's nothing wrong with the global symmetries in being anomalous.
We can compute the triangle anomaly for the global symmetry group in the ultraviolet and we will call it $A_{UV}$. After weakly gauging $G$ we introduce additional fermions that are

charged only under $G$ in order to cancel the anomaly, since now it is a gauge symmetry. Flowing towards the infrared, the anomaly is still zero if the global symmetry group is not broken. After constructing the low-energy effective field theory, we can calculate the triangle anomalies for the group $G$ involving the composite low energy fields which results in the term $A_{IR}$. Since the fermions we added contribute to the anomaly with the same term we have that

$$0 = A_{IR} + A_F = A_{UV} + A_F \quad \rightarrow \quad A_{IR} = A_{UV} \tag{1.17}$$

The anomaly coefficient can be easily computed since it is proportional to the group theoretic factor

$$A = \mathrm{Tr}\ (T^a\{T^b, T^c\}) \tag{1.18}$$

Summarizing the result, we found that if the global symmetry group is not broken by the strong dynamics, triangle anomalies involving only the global symmetry group should be equal in the ultraviolet and in the infrared.
Moreover, we will use these anomaly matching conditions to find if two dual theories are invariant under the same global symmetries in the IR as an additional check of electric magnetic duality.

## 1.2 Seiberg duality

Electric magnetic duality relates the dynamics of two different gauge theories in their infrared fixed point. Even though the dual theories have different particle content, they describe the same IR physics. Moreover, whenever one of the dual theories gets more strongly coupled, the other become more weakly coupled. This is particularly useful since it provides an alternative, weakly coupled description of the original theory.

### 1.2.1 Electric theory: $SU(N)$ SQCD with $N_f$ flavours

We will start our analysis on eletric-magnetic duality studying the first pair of theories that were discovered to be dual in [8]. We are gonna analyse the properties of these theory in order to better understand the features of the duality.

The electric theory is a $SU(N_c)$ supersymmetric gauge theory with $N_f$ flavours. Its non anomalous global symmetry group is

$$SU(N_f)_L \times SU(N_f)_R \times U(1)_B \times U(1)_R$$

. Since the axial symmetry $U(1)_A$ is anomalous it cannot be part of the global symmetries.

The classical lagrangian is written in terms of superfield as

$$\mathcal{L} = \tau \int d^2\theta\, Tr(W_\alpha W^\alpha) + \text{h.c.} + \int d^2\theta\, d^2\bar{\theta}\ Q^\dagger e^V Q + \int d^2\theta\, d^2\bar{\theta}\ \tilde{Q}^\dagger e^{-V} \tilde{Q} \tag{1.19}$$

$Q$ and $\tilde{Q}$ represent left and right quark superfield respectively.
The charges of the fields are given in the table below. The value of the R-charge is fixed by the triangle anomaly $SU(N_c)^2\, U(1)_R$, given by diagrams with two exiting gluons and R-symmetry current inserted in the cross. Every fermion in the theory contributes to the

| | $SU(N_c)$ | $SU(N_f)_L$ | $SU(N_f)_R$ | $U(1)_B$ | $U(1)_R$ |
|---|---|---|---|---|---|
| $Q$ | $N_c$ | $N_F$ | $1$ | $1$ | $\frac{N_f-N_c}{N_f}$ |
| $\tilde{Q}$ | $\overline{N_c}$ | $1$ | $\overline{N_F}$ | $-1$ | $\frac{N_f-N_c}{N_f}$ |

Table 1.1: Charges of the electric theory

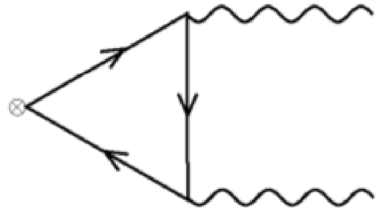


Figure 1.1: Feynman graphs contributing to the R-symmetry anomaly

anomaly which, as a result, is proportional to the R-charge of the fermion running in the loop and the Dynkin index of its representation

$$R_{gaugino}T(\mathrm{Ad}) + \sum_f (R_f - 1)T(r) = 0 \tag{1.20}$$

$$N_c + \frac{1}{2}2N_f(R_Q - 1) = 0 \quad \rightarrow \quad R_Q = \frac{N_f - N_c}{N_f} \tag{1.21}$$

where we set the gaugino R-charge to 1 in order to have gluons not charged under R-symmetry.

Since the non anomalous R-symmetry condition leads to a unique set of R-charges, we found the R-charges at the superconformal infrared point of the theory. This is peculiar to *SQCD* with matter in the fundamental representation.

### Classical moduli space

Since there is no superpotential, the classical moduli space of the theory is given by *D-terms* only. They can be read from the on-shell lagrangian and are given by

$$D^a = g\left(Q^{*i}T^aQ_i - \tilde{Q}^{*i}T^a\tilde{Q}_i\right) = 0 \tag{1.22}$$

where $T^a$ are the gauge group generators in fundamental or antifundamental representation and $i$ is a flavour index.

After considering gauge and global symmetries, the squark *VEVs*, represented as $N_f \times N_c$ matrices, that satisfy the D-term equation are for $N_f \leq N_c$ and $a_i$ generic

$$Q = \tilde{Q} = \begin{pmatrix} a_1 & & & & \vdots \\ & a_2 & & & \vdots \\ & & \ddots & & \vdots \\ & & & a_{N_f} & \vdots \end{pmatrix} \tag{1.23}$$

for $N_f \geq N_c$ and $|a_i|^2 - |\tilde{a}_i|^2 = a$ independent of $i$.

$$Q = \begin{pmatrix} a_1 & & & \\ & a_2 & & \\ & & \ddots & \\ & & & a_{N_c} \\ \dots & \dots & \dots & \dots \end{pmatrix} \quad \tilde{Q} = \begin{pmatrix} \tilde{a}_1 & & & \\ & \tilde{a}_2 & & \\ & & \ddots & \\ & & & \tilde{a}_{N_c} \\ \dots & \dots & \dots & \dots \end{pmatrix} \tag{1.24}$$

For $N_f \leq N_c$ in a generic point of the moduli space the gauge group is broken to $SU(N_c - N_f)$ while for $N_f \geq N_c$ is broken completely. The gauge group breaks through the super Higgs mechanism, where every broken generator gets absorbed by the (originally) massless vector superfield in order to make a massive vector superfield [2] The mass of the gauge superfield is given by the VEVs of the squarks.

As we said in the previous section, we can study the classical moduli space by finding holomorphic gauge invariant polynomial in the operators and modding out classical relations between them. For $N_f \leq N_c$ we can only construct *mesons* out of squarks

$$M^i_j = Q^i \tilde{Q}_j \tag{1.25}$$

where color indices are summed. Mesons have maximal rank since $N_f \leq N_c$ and there are no classical constraints to impose on them.
When $N_f \geq N_c$ the mesons cannot have maximal rank anymore, it can be at most $N_c$. There are additional gauge invariant operators that can be constructed: *baryons*, that are defined as

$$B_{i_1,\dots,i_{N_f-N_c}} = \epsilon_{i_1,\dots,i_{N_f-N_c},j_1,\dots,j_{N_c}} \epsilon^{a_1,\dots,a_{N_c}} Q^{j_1}_{a_1} \dots Q^{j_{N_c}}_{a_{N_c}} \tag{1.26}$$

$$\tilde{B}_{i_1,\dots,i_{N_f-N_c}} = \epsilon^{i_1,\dots,i_{N_f-N_c},j_1,\dots,j_{N_c}} \epsilon_{a_1,\dots,a_{N_c}} \tilde{Q}^{a_1}_{j_1} \dots \tilde{Q}^{a_{N_c}}_{j_{N_c}} \tag{1.27}$$

Mesons and baryons can be written down using the *VEVs* we found solving the *D-term* equations (ignoring null components for baryons)

$$M = \begin{pmatrix} a_1\tilde{a_1} & & & \\ & a_2\tilde{a_2} & & \\ & & \ddots & \\ & & & a_{N_c}\tilde{a_{N_c}} \\ & & & \end{pmatrix} \tag{1.28}$$

$$B \simeq a_1 a_2 \dots a_{N_c} \tag{1.29}$$

$$\tilde{B} \simeq \tilde{a_1}\tilde{a_2} \dots \tilde{a_{N_c}} \tag{1.30}$$

We can see that if the mesons have rank less than $N_c$, then $B$ or $\tilde{B}$ (or both) has to vanish and the other has rank one. If the mesons' rank is $N_c$ both $B$ and $\tilde{B}$ have rank one.

There are classical constraints that should be imposed between mesons and baryons, but depend on the number of flavours . For example for $N_f = N_c$ the classical constraint is $det\,(M) - B\tilde{B} = 0$.

[2] Remember that massive representation of supersymmetry have twice the degrees of freedom of massless ones, because in the latter half of the supercharges are represented trivially.

Singularities of the moduli space can be investigated using the gauge invariant description we just introduced. The part of the lagrangian that describes flat directions can be written in terms of mesons and baryons. The lagrangian involving mesons features a non trivial Kahler potential that reads

$$K = 2\mathrm{Tr}\sqrt{M^\dagger M} \tag{1.31}$$

that generate a singular metric whenever the meson matrix is not invertible. This happens when some of the VEVs are zero, i.e. in points of the moduli space of enhanced gauge symmetry. The appearance of this singularities is related to the fact that some (or all) gluons are now massless and should be included in the low-energy description.

### Quantum moduli space

Quantum dynamics modifies the structure of the moduli space of the theory in a different way depending on the number of flavours.

$\mathbf{N_f = 0}$
For pure *Super Yang Mills*, i.e. no quarks, the theory exihibit a discrete set of $N_c$ vacua. Without quarks a non anomalous R-symmetry cannot be found, and the R-symmetry is broken down to the discreet symmetry $\mathbb{Z}_{2N_c}$. Using holomorphy and symmetry arguments, the form of the non perturbative potential can be found and it can be shown that induces the gaugino to condensate [9], meaning that

$$\langle \lambda\lambda \rangle = -\frac{32\pi^2}{N_c} a\,\Lambda^3 \tag{1.32}$$

where $\Lambda$ is the dynamically generated scale of the theory defined as

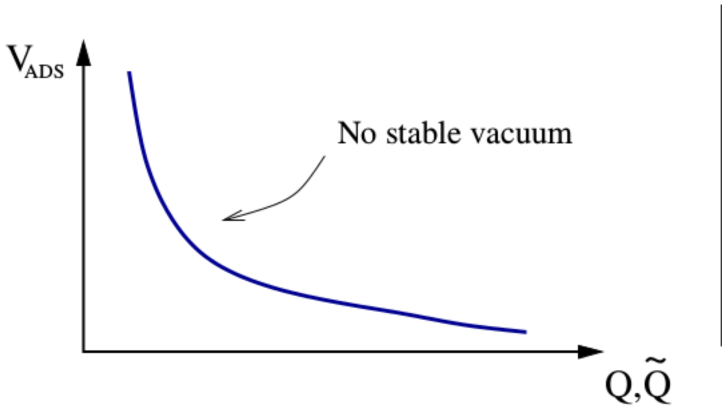
$$\Lambda = \mu e^{-\frac{2\pi i \tau}{b_0}} \qquad \tau = \frac{4\pi i}{g^2(\mu)} + \frac{\theta_{YM}}{2\pi} \qquad b_0 = 3N_c - N_f \tag{1.33}$$

where $\tau$ is the complexified gauge coupling. $|\Lambda|$ is defined as the scale at which the coupling constant blows up.
The gaugino condensation breaks R-symmetry to $\mathbb{Z}_2$ and in fact there are $N_c$ physically different vacua labelled by different phases of the gaugino condensate.

$\mathbf{N_f < N_c}$
The quantum corrections for *SQCD* with $N_f < N_c$ flavours completely lift the moduli space through the Affleck-Dine-Seiberg (ADS) superpotential ([10][11]) which reads

$$W_{eff} = (N_c - N_f)\left(\frac{\Lambda^{3N_c - N_f}}{\det M}\right)^{\frac{1}{N_c - N_f}} \tag{1.34}$$

It is the only superpotential that is compatible with the symmetries of the theory and with the other properties of the superpotential we introduced in section 1.1.2. We can see that the ADS superpotential do not exist for $N_F \geq N_c$ since the exponent diverges for $N_f = N_c$ or the determinant vanishes for $N_f \geq N_c$ since the mesons do not have maximal rank. Note that this superpotential is non perturbative and thus it is not in contrast with the renormalization theorem of section 1.1.2. The effect of this superpotential is that the theory does not have ground state. The slope of the potential goes to zero only for $\det M \to \infty$.




This situation is the perfect example when, unlike the classical moduli space, quantum corrections lift completely the moduli space and the theory does not posses a vacuum anymore.

$\mathbf{N_f = N_c}$

When the number of flavours is equal the number of colours of the theory, the classical moduli space was subject to the constraint

$$\det M - B\tilde{B} = 0 \tag{1.35}$$

in the quantum corrected moduli space mesons and baryons satisfy [12]

$$\det M - B\tilde{B} = \Lambda^{2N_c} \tag{1.36}$$

which flows to the classical constraint in the classical limit ($\Lambda \to 0$).

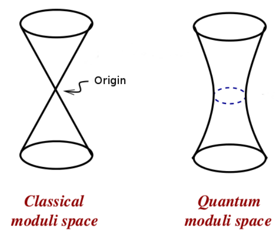



Figure 1.2: Schematical representation of the quantum and classical moduli space near the origin

The effect of this relation is that the origin does not belong to the moduli space anymore and the moduli space is now smooth. For large expectation values of $M$, $B$ and $\tilde{B}$ the classical and the quantum moduli space look similar, while in the origin quantum correction modify drastically the structure of the moduli space. Moreover, the subspace with $B$ or $\tilde{B}$ is zero, is not singular anymore while classical the meson matrix was constrained to have zero determinant.

Since the origin is not in the quantum moduli space the global symmetry group (1.2.1) is necessarily broken in some way, depending on the position of the moduli space

$$M^i_j = \Lambda^2 \delta^i_j \quad B = \tilde{B} = 0 \quad \rightarrow \quad SU(N_f)_V \times U(1)_B \times U(1)_R \tag{1.37}$$

$$M^i_j = 0 \quad B = -\tilde{B} = \Lambda^{N_c} \quad \rightarrow \quad SU(N_f)_L \times SU(N_f)_R \times U(1)_R \tag{1.38}$$

$\mathbf{N_f = N_c + 1}$

In the case $N_f = N_c + 1$ the classical moduli space is constrained by

$$\det M \left(\frac{1}{M}\right)^j_i - B_i \tilde{B}^j = 0 \qquad M^i_j B_i = M^i_j \tilde{B}^j = 0 \tag{1.39}$$

and quantum corrections do not modify it. In the previous section we noted that the singularities in the classical moduli space are associated to the appearance of massless gluons. In the quantum picture, the interpretation of the singularities is different: they are associated with additional massless mesons and baryons. At the origin of the moduli space the theory is strongly coupled and the global symmetry (1.2.1) is unbroken and it can be checked using 't Hooft anomalies [12]. Far from the origin, the mesons and baryons interact with the potential

$$W = \frac{1}{\Lambda^{2N_c - 1}} \left( M^i_j B_i \tilde{B}^j - \det M \right) \tag{1.40}$$

that enforce the classical constraints (1.39) through the equations of motion. For large VEVs of the fields, mesons and baryons acquire large mass through the superpotential.

$\mathbf{N_f > N_c + 1}$

Starting from $N_f = N_c + 2$ it is not possible to construct a sensible physical superpotential out of gauge invariant operators, in analogy to the previous cases. The only $SU(N_f)_L \times SU(N_f)_R$ invariant superpotential that can be written is given by

$$W_{eff} \sim \det M - B_{ij} M^i_k M^j_l \tilde{B}^{kl} \tag{1.41}$$

since baryons have two flavour indices. However this superpotential does not have R-charge equal to two and if we add more flavours we should add other mesons to the superpotential. Therefore the classical moduli space is not modified by quantum corrections. As a result, near the origin the quantum corrected moduli space looks identical to the classical one. Unlike the case with $N_f = N_c + 1$ the singularities in the moduli space cannot be interpreted as massless mesons and baryons and an effective description of these operator is singular [12]. Since 't Hooft anomaly matching conditions are not satisfied in the singular points it is clear that a description using mesons and baryons is not correct.

To find a description of the low-energy degrees of freedom of the theory we will use Seiberg duality, which provides an alternative description of the theory,

$\mathbf{\frac{3}{2} N_c \geq N_f \geq 3N_c}$ **: the conformal window**

In this range the theory is not asympotically free. This can be seen by using the *NSVZ*

$\beta$ function, which, using (1.3), reads

$$\beta(g) = -\frac{g^3}{16\pi^2}\frac{3N_c - N_f + N_f\gamma(g^2)}{1 - N_c\frac{g^2}{8\pi^2}} \tag{1.42}$$

$$\gamma(g^2) = -\frac{g^2}{8\pi^2}\frac{N_c^2 - 1}{N_c} + \mathcal{O}(g^4) \tag{1.43}$$

The $\beta$ function is known to have a Banks-Zacks fixed point [13] in the 't Hooft limit with $\frac{N_f}{N_c} = 3 - \epsilon$ held fixed and $\epsilon \ll 1$. However, the fixed point exists in the range of values $\frac{3}{2}N_c \geq N_f \geq 3N_c$ with $N_f$ and $N_c$ finite. This is possible because one loop and two loop contribution to the beta function have opposite signs. As a result, the infrared red theory is a non-trivial four dimensional superconformal theory. The infrared degrees of freedom are quarks and gluons that are not confining but are interacting as massless particles. The theory is in a free non-Abelian Coulomb phase.

Since the theory is superconformal, we have further restriction on the algebra of operators [3]. Superconformal algebra imposes that the dimension of every operator satisfy this inequality involving the R-charge

$$D \geq \frac{3}{2}\,|R| \tag{1.44}$$

where the bound is satured for chiral fields. The operator product expansion ( *OPE* ) of two chiral operator is constrained by this fact. Since $R(O_1O_2) = R(O_1) + R(O_2)$, we have that for chiral operators, $D(O_1O_2) = D(O_1) + D(O_2)$. Note that the dimension of the operator is quantum corrected, i.e. contains the anomalous dimension of the operator. Therefore, the OPE is not singular and the product of operators is well-defined. Because of this fact, chiral operators form the *chiral ring*.

Since the superconformal R-symmetry is not anomalous and commutes with the global symmetry group of the theory it must be the R-symmetry that appears in table 1.1. The gauge invariant operators we defined previously must have

$$D(Q\tilde{Q}) = \frac{3}{2}R(Q\tilde{Q}) = 3\frac{N_f - N_c}{N_f} \tag{1.45}$$

$$D(B) = D(\tilde{B}) = \frac{3}{2}N_c\frac{N_f - N_c}{N_c} \tag{1.46}$$

Gauge invariant operators should be in unitary representation of the superconformal algebra. Unitarity imposes that in general $D \geq 1$ and the equality holds for free fields. From the previous equation we can verify that $D(M) \geq 1$ for $N_f \geq \frac{3}{2}N_c$ and it becomes a free field for $N_f = \frac{3}{2}N_c$.

For fewer number of flavours, the meson field is inconsistend with the unitarity bound. The theory is conjecture to flow to a different phase.

$\mathbf{N_f > 3N_c}$

In this range, quarks prevails on glouns and change the sign of the $\beta$ function. This is

[3] R-symmetry is contained directly in the superconformal algebra instead of being an automorphism of the algebra, as in superPoincarè algebra.

caused by the *charge screening* effect of quarks, that make the coupling constant smaller at larger distances.
The theory is in a free non-Abelian electric phase. Its behaviour is not very well defined at high energies because of the presence of a Landau pole at $R \sim \Lambda^{-1}$, although the theory can be a good description of a low-energy limit of another theory.

### Magnetic theory

The magnetic theory is a *SQCD* theory with the same global symmetries as the electric theory, but with gauge group $SU(N_f - N_c)$. In addition there are $N_f^2$ color singlets, that we will call mesons, since they have the same properties as the mesons we can construct in the electric theory. In the magnetic theory they are fundamental fields i.e. they are not written as gauge invariant operators from quarks. Since they are gauge invariant, they interact only through the superpotential

$$W = M^i_{\tilde{j}}\, q_i\, \tilde{q}^{\tilde{j}} \tag{1.47}$$

where we represented dual quarks as $q, \tilde{q}$ and mesons as $M^i_{\tilde{j}}$.
The charges of fields of the magnetic theory are

| | $SU(N_c)$ | $SU(N_f)_L$ | $SU(N_f)_R$ | $U(1)_B$ | $U(1)_R$ |
|---|---|---|---|---|---|
| $q$ | $N_c$ | $\overline{N_F}$ | $1$ | $\frac{N_c}{N_f-N_c}$ | $\frac{N_c}{N_f}$ |
| $\tilde{q}$ | $\overline{N_c}$ | $1$ | $N_F$ | $-\frac{N_c}{N_f-N_c}$ | $\frac{N_c}{N_f}$ |
| $M^i_{\tilde{j}}$ | $1$ | $N_f$ | $\overline{N_f}$ | $0$ | $2\frac{N_f-N_c}{N_f}$ |

Table 1.2: Charges of the magnetic theory

Dual quarks sit in opposite representation of flavour symmetries.

Mesons in the magnetic theory have the same charges of the mesons constructed from electric quarks. Baryons constructed from dual quarks have the same baryonic charge as the electric baryons. Moreover, it can be demonstrated that they are proportional to each other.
Similarly to the electric theory, the R-charges can be chosen in order to be non anomalous. However, the R-charges can be found using the duality. If we impose that the meson is built from electric quarks, its R-charge is twice the R-charge of electric quarks. Since the superpotential (1.47) must have R-charge two, we find the R-charges of the magnetic quarks. In this way, we found the R-charges at the superconformal infrared fixed point.

### Duality

In the *conformal window* the electric and magnetic theories we introduced previously give an equivalent description of the same physics in the infrared. In this range, the magnetic theory has a non trivial infrared fixed point too. At this fixed point, the superpotential (1.47) is a relevant perturbation since it has dimension $D = 1 + 3\frac{N_c}{N_f} < 3$ and it drives the theory in a new fixed point.
Electric mesons have different dimension in the UV from the magnetic ones since they are construct from a pair of quark. For this reason it's necessary to introduce a dimensionful

scale $\mu$ in order to match their dimension in the UV: $M = \mu M_m$, where $M_m$ are the magnetic mesons.
The strong coupling scale of the electric $\Lambda$ and magnetic $\tilde{\Lambda}$ theories are related by the relation

$$\Lambda^{3N_c-N_f}\tilde{\Lambda}^{3(N_f-N_c)-N_f} = (-1)^{N_f-N_c}\mu^{N_f} \tag{1.48}$$

This relation shows that when one theory is strongly coupled, the other is weakly coupled. Moreover, it ensures that the dual of the dual theory is the electric theory itself. The dual of the dual magnetic theory is a $SU(N_c)$ *SQCD* theory with scale $\Lambda$, $d^i$ and $\tilde{d}_{\tilde{j}}$ quarks and additional singlets $M^i_{\tilde{j}}$ and $N^{\tilde{j}}_i = q_i q^{\tilde{j}}$ with superpotential

$$W = \frac{1}{\tilde{\mu}} N^{\tilde{j}}_i d^i \tilde{d}_{\tilde{j}} + \frac{1}{\mu} M^i_{\tilde{j}} N^{\tilde{j}}_i = \frac{1}{\mu} N^{\tilde{j}}_i \left(-d^i d_{\tilde{j}} + M^i_{\tilde{j}}\right) \tag{1.49}$$

since from the previous relation we have $\tilde{\mu} = -\mu$.
Meson fields are massive and can be integrated out by using their equation of motion which result in

$$N^{\tilde{j}}_i = 0 \qquad M^i_{\tilde{j}} = d^i d_{\tilde{j}} \tag{1.50}$$

Since the dual theories describe the same physics, there should be a mapping of gauge invariant operators between them. We already saw that electric and magnetic mesons match in the infrared. A mapping should exists also for baryonic operators. Indeed it does and it is given by

$$B^{i_1 \dots i_{N_c}} = C\, \epsilon^{i_1 \dots i_{N_c} j_1 \dots i_{N_f-N_c}} b_{j_1 \dots j_{N_f-N_C}} \tag{1.51}$$

$$\tilde{B}^{i_1 \dots i_{N_c}} = C\, \epsilon_{\tilde{i}_1 \dots \tilde{i}_{N_c} \tilde{j}_1 \dots \tilde{i}_{N_f-N_c}} b_{\tilde{j}_1 \dots \tilde{j}_{N_f-N_C}} \tag{1.52}$$

$$\text{where} \qquad C = \sqrt{-(-\mu)^{N_c-N_f}\Lambda^{3N_c-N_f}} \tag{1.53}$$

using (1.48) it can be shown that these mappings preserve the $\mathbb{Z}_2$ character of the duality.

As an additional check of the duality 't Hooft anomaly matching conditions have been calculated in [8] for the electric and magnetic theories for the various global symmetries and they are given by

$$SU(N_f)^3 = N_c \tag{1.54}$$

$$U(1)_R\, SU(N_f)^2 = -\frac{N_c^2}{2N_f} \tag{1.55}$$

$$U(1)_B\, SU(N_f)^2 = \frac{N_c}{2} \tag{1.56}$$

$$U(1)_R = -N_c^2 - 1 \tag{1.57}$$

$$U(1)_R^3 = -N_c^2 - 1 - 2\frac{N_c^4}{N_f^2} \tag{1.58}$$

$$U(1)_B^2 U(1)_R = -2N_c^2 \tag{1.59}$$

$$\tag{1.60}$$

Another important check of the duality is that it remains valid under mass perturbations of the electric theory. Suppose to add a superpotential term that give mass to the quark in the last flavour and is equal to

$$W^{el}_{mass} = m\; Q_{N_f}\tilde{Q}^{N_f} \tag{1.61}$$

Flowing to the IR the number of quark is decreased by one, driving the theory to a more strongly coupled fixed point[4]. The new scale of the theory is given in terms of the old one by

$$\Lambda_L^{3N_c-(N_f-1)} = m\;\Lambda^{3N_c-N_f} \tag{1.62}$$

In the magnetic theory the mass perturbation is mapped in the term

$$W^{mag}_{mass} = m\; M^{N_f}_{N_f} \tag{1.63}$$

Because of this term, the gauge group gets higgsed to $SU(N_f-1-N_c)$ and only $N_f-1$ light quarks remain in the theory.
The scale of the magnetic theory $\Lambda_L$ is modified and reads

$$\tilde{\Lambda}^{3(N_f-N_c-1)-(N_f-1)} = -\frac{\tilde{\Lambda}^{3(N_f-N_c)-N_f}}{< q_{N_f}\tilde{q}^{N_f} >} \tag{1.64}$$

where $< q_{N_f}\tilde{q}^{N_f} >= -\mu m$ because of the equation of motion for the massive flavour.
As expected the magnetic theory becomes more weakly coupled.
We conclude that the duality is preserved under massive deformations.

## 1.3 Kutasov-Schwimmer duality

A possible generalization of Seiberg duality can be found by adding matter fields in different representations of the gauge group.

### 1.3.1 Electric theory

Kutasov and Schwimmer ([14], [15]) considered $SU(N)$ SQCD with the addition of a matter fields in the adjoint representation of the gauge group and found its magnetic dual.

The classical electric theory can be summarized by the following table of charges of the fields under the global symmetry group $SU(N_f)_L \times SU(N_f)_R \times U(1)_B \times U(1)_R$

| | $SU(N_c)$ | $SU(N_f)_L$ | $SU(N_f)_R$ | $U(1)_B$ | $U(1)_R$ |
|---|---|---|---|---|---|
| $Q$ | $N_c$ | $N_F$ | $1$ | $1$ | $1-\frac{2}{k+1}\frac{N_c}{N_f}$ |
| $\tilde{Q}$ | $\overline{N_c}$ | $1$ | $\overline{N_F}$ | $-1$ | $1-\frac{2}{k+1}\frac{N_c}{N_f}$ |
| $X$ | $Adjoint$ | $1$ | $1$ | $0$ | $\frac{2}{k+1}$ |

Table 1.3: Charges for the electric theory

[4]Since matter in the fundamental representation contribute with a positve term it is easy to see that this is true.

and by the addition of the superpotential

$$W_{Adj} = g_k \mathrm{Tr}\, X^{k+1} \tag{1.65}$$

The theory posses two different R-symmetries but the superpotential breaks explicitly one of them and imposes that the adjoint matter has R-charge $\frac{2}{k+1}$. The scalar potential of the theory now includes the scalar field in the adjoint matter and therefore the moduli space is modified. The superpotential add *F-term* equations that need to be satisfied on the moduli space.

The remaining R-charges can be fixed by imposing that the R-symmetry can't be anomalous as we did previously. Using formula (1.20), considering also the fermion in the adjoint matter multiplet we have

$$N_c + (R_Q - 1)\frac{1}{2}2N_f + (R_X - 1)N_c = 0 \tag{1.66}$$

$$(R_Q - 1)N_f = -R_X N_c \quad \longrightarrow \quad R_Q = 1 - R_X \frac{N_c}{N_f} \tag{1.67}$$

Imposing this condition the R-charges of the theory were fixed completely, as in Seiberg duality. This has been possible because the superpotential (1.65) fixed independently the R-charge $\Delta_X$. Otherwise the condition (1.67) fixes $R_Q$ as a function of $R_X$ with $R_X$ generic.

It is interesting to note that the condition (1.67) can be found independently by requiring that the $\beta$ function has a fixed point

$$0 = \beta g \sim 3T(Adj) - \sum_i T(Repr_i)(1 - \gamma_i) = \tag{1.68}$$

$$= T(Adj) + \sum_i T(Repr_i)(R_i - 1) = 0 \tag{1.69}$$

where we have $\gamma_i + 2 = 2D_i$ for chiral fields and at the superconformal fixed point the dimension of an operator is related to its R-charge by $R_i = \frac{3}{2}D_i$.

The superpotential (1.65) drives the theory to a new IR point since it is either relevant or dangerously irrelevant, depending on the value of $k$.

The gauge invariant operators that can be constructed are mesons and baryons multiplied with powers of the adjoint field. Mesons operators are given by

$$(M_j)^i_{\tilde{i}} = \tilde{Q}_{\tilde{i}} X^j Q^i \qquad j = 0, 1, \ldots, k-1 \tag{1.70}$$

Baryons are more easily introduced by first defining "dressed quarks"

$$Q_{(l)} = X^l Q \qquad l = 0, \ldots, k-1 \tag{1.71}$$

Baryons are defined as

$$B^{i_1, i_2, \ldots, i_k}) = Q^{i_1}_{(0)} \ldots Q^{i_k}_{(k-1)} \qquad \text{with } \sum_{l=1}^{k} i_l = N_c \tag{1.72}$$

with color index contracted with an $\epsilon$ tensor.

### Vacuum structure

The vacuum structure of this theory is complicated by the number of gauge invariant operators and the superpotential for the adjoint matter field.

In analogy to the condition $N_f \geq N_c$ in $SQCD$ we would like to find a range of values of $N_f$, $N_c$ and $k$ such that the theory admits stable vacua. We can add a weak deformation to the superpotential (1.65), by adding terms with lower order powers in $X$

$$W(X) = \mathrm{Tr} \sum_{l=1}^{k} g_l X^{l+1} + \lambda \mathrm{Tr}\, X \qquad \text{with } g_l \ll 1 \tag{1.73}$$

where we introduced $\lambda$ to enforce the tracelessness of $X$. Since it is a weak perturbation, the large field behavior of W is not modified. Hence, if we can't find stable vacua with the weak perturbation, the original theory doesn't have any vacua too.

The theory has a large sets of multiple vacua for $Q = \tilde{Q} = 0$ and $X \neq 0$. $X$ can be diagonalized with eigenvalues $x_i$. The F-terms are given by setting $W'(x_i) = 0$. Now, $W'(x_i)$ is a polynomial of degree $k$ in the eigenvalues $x_i$ admitting $k$ distinct solutions in general. As a result, ground states are labeled by a set of $k$ integers $(i_1, \ldots, i_k)$, describing how many eigenvalues are sitting in the $l$-th minimum. Clearly, since $X$ has $N_c$ eigenvalues we have

$$\sum_{l=1}^{k} i_l = N_c \tag{1.74}$$

In every vacuum, $X$ has a quadratic potential, which correspond to a mass term and can be integrated out. The $X$ expectation values break the gauge group in the following way

$$SU(N_c) \;\longrightarrow\; SU(i_1) \times SU(i_2) \times \cdots \times SU(i_k) \times U(1)^{k-1} \tag{1.75}$$

Each $SU(i_l)$ sector describes a decoupled $SQCD$ model which has stable vacua only if $N_f \geq N_c$, hence considering every sector we have

$$i_l \leq N_f \qquad \forall 1 \leq l \leq k \tag{1.76}$$

Taking the limit $g_l \to 0$ we find that we must have

$$N_f \geq \frac{N_c}{k} \tag{1.77}$$

For every choice of $i_l$ there is a moduli space obtained by giving expectation values of the quarks. Hence, the moduli space of the theory consists of different disconnected component, associated to different $i_l$ choice.

## 1.3.2 Magnetic theory

The magnetic theory is constructed in a similar way as the magnetic theory in Seiberg duality. The dual theory has gauge group $SU(kN_F - N_c)$. The baryonic charge of the dual quarks is found by imposing that baryons in the electric theory are proportional to the baryons constructed from dual quark. The magnetic theory has a superpotential

| | $SU(N_f)_L$ | $SU(N_f)_R$ | $U(1)_B$ | $U(1)_R$ |
|---|---|---|---|---|
| $q$ | $\overline{N_F}$ | 1 | $\frac{N_c}{kN_f-N_c}$ | $1-\frac{2}{k+1}\frac{kN_f-N_c}{N_f}$ |
| $\tilde{q}$ | 1 | $N_F$ | $-\frac{N_c}{kN_f-N_c}$ | $1-\frac{2}{k+1}\frac{kN_f-N_c}{N_f}$ |
| $Y$ | 1 | 1 | 0 | $\frac{2}{k+1}$ |
| $M_j$ | $N_f$ | $\overline{N_f}$ | 0 | $2-\frac{4}{k+1}\frac{N_c}{N_f}+j\frac{2}{k+1}$ |

Table 1.4: Charges for the magnetic theory

$$W = \mathrm{Tr}\, Y^{k+1} + \sum_{j=0}^{k-1} M_j q Y^{k-j-1}\tilde{q} \qquad \text{dove } M_j = QY^j\tilde{Q} \tag{1.78}$$

The charges of the fields are easily found by requiring duality for the two theories. In this way, the charges of the mesons are given in terms of the electric quarks, which are fixed, and the superpotential fixes the charges for the remaining fields. Using this method, the R-charges of the dual quarks given by

$$R_q = R_X - R_Q \tag{1.79}$$

where we used that $R_X = R_Y = \frac{2}{k+1}$ because of (1.78) and (1.65).

### Duality and mass deformations

The 't Hooft anomaly matching conditions are satisfied and are given by

$$\begin{aligned}
SU(N_f)^3 &= N_c d^{(3)}(N_f) && (1.80)\\
U(1)_R\, SU(N_f)^2 &= -\frac{2}{k+1}\frac{N_c^2}{N_f} d^{(2)}(N_f) && (1.81)\\
SU(N_f)^2 U(1)_B &= N_c d^{(2)}(N_f) && (1.82)\\
U(1)_R &= -\frac{2}{k+1}(N_c^2+1) && (1.83)\\
U(1)_R^3 &= \left(\left(\frac{2}{k+1}-1\right)^3+1\right)(N_c^2-1) - \frac{16}{(k+1)^3}\frac{N_c^4}{N_f^2} && (1.84)\\
U(1)_B^2 U(1)_R &= -\frac{4}{k+1}N_c^2 && (1.85)
\end{aligned}$$

We consider now mass deformations of the electric theory in order to understand if duality is preserved under such deformations. Let's modify the electric superpotential by adding a mass term

$$W_{el} = g_k \mathrm{Tr}\, X^{k+1} + m\tilde{Q}_{N_f} Q^{N_f} \tag{1.86}$$

The number of flavours in the IR is reduced by one unit keeping the number of colours the same. In order to preserve the duality, the magnetic theory should have gauge group $SU(k(N_f-1)-N_c) = SU(kN_f - N_c - k)$. Let's see if this happens. The dual potential reads

$$W_{mag} = g_k \mathrm{Tr}\, Y^{k+1} + \sum_{j=1}^{k} M_j \tilde{q} Y^{k-j-1} q + m(M_0)^{N_f}_{N_f} \tag{1.87}$$

Integrating out the massive fields we find

$$q_{N_f} Y^l \tilde{q}^{N_f} = -\delta_{l,k} m \quad l = 0, \dots, k-1 \tag{1.88}$$

which fixes the expectation values to

$$\tilde{q}_\alpha^{N_f} = \delta_{\alpha,1} \tag{1.89}$$

$$q_\alpha^{N_f} = \delta^{\alpha,k} \tag{1.90}$$

$$Y_\beta^\alpha = \begin{cases} \delta_{\beta+1}^\alpha & \beta = 1, \dots, k-1 \\ 0 & \text{otherwise} \end{cases} \tag{1.91}$$

These expectation values break the gauge group to $SU(kN_f - N_c) \to SU(k(N_f-1)-N_c)$ through the Higgs mechanism and reduces the number of flavours by one unit as required by duality. As a result duality is preserved under mass deformations.

# 2 Three dimensional dualities

### 2.0.3 Supersymmetry in 3 dimensions

Spinors in three dimensions have different properties than their four dimension counterpart.
The dimension of the representation in an arbitrary dimension $D$ is given by $2^{\frac{D}{2}}$ for $D$ even, while $2^{\frac{D-1}{2}}$ for $D$ odd. Hence, in three dimension we have a two dimensional representation.

In odd dimensions representations are irreducible and Weyl spinors do not exist: in odd dimensions the chirality operator ($\gamma_{D+1}$ or $\gamma^*$ ) is proportional to the identity. This is related to the fact that representations in odd dimensions are constructed by taking the representation in one dimension less, which is even, and adding the chirality operator as the $D$-th matrix in the Clifford algebra.
Gamma matrices can be chosen to be real and we can impose Majorana condition, lowering the degress of freedom of the representation from four (two complex numbers) to two.
Since $3d$ $\mathcal{N} = 2$ theories have four supercharges, we can use the $4d$ $\mathcal{N} = 1$ superspace formalism.
The supersymmetry algebra can be found by dimensional reduction from the $d = 4$ $\mathcal{N} = 1$ supersymmetry algebra.

$$\{Q_\alpha, Q_\beta\} = \{\bar{Q}_\alpha, \bar{Q}_\beta\} = 0 \qquad \{Q_\alpha, \bar{Q}_\beta\} = 2\sigma^\mu_{\alpha\beta} P_\mu + 2i\epsilon_{\alpha\beta} Z \tag{2.1}$$

The central charge $Z$ is the component of the momentum along the reduced direction. Because of the presence of the central charge in the algebra, now states must satisfy a BPS bound of the form $M \geq Z$, which imply that for massless representations have null central charge.
The automorphism of this algebra is $U(1)_R \simeq SO(2)_R$, as in four dimensions.

Superspace formalism is similar to what we introduced previously, with proper changes due to the different spinor representation in three dimension such as gamma matrices.

(Anti)Chiral superfields contains *on-shell* one complex scalar and a complex spinor. Vector superfield contains an additional real scalar field with respect from the four dimensional superfield. The scalar field is just the last component of the vector field of the superfield, after dimensional reduction. As a result, its variation under a supersymmetry transformation is given exactly as the last component of the vector field.

## 2.1 Aharony duality

## 2.2 Kutasov-Schwimmer duality

# Appendices

# A | Supersymmetry and superfields

## A.1 Supersymmetry algebra

The supersymmetry algebra is an extension of the Poincarè group involving anticommutators together with commutators. Since it is not a ordinary Lie algebra, Coleman-Mandula theorem does not apply for theories that are invariant under it.

The supersymmetry algebra is divided into two subalgebras, the bosonic and fermionic part. The bosonic part contains Poincarè Lie algebra $(M_{\mu\nu}, P_\mu)$ while fermionic subalgebra is generated by the *supercharges* $(Q^I_\alpha, \bar{Q}^I_{\dot{\alpha}})$ with $I = 1, \dots, \mathcal{N}$. When more than one pair of supercharges is present we refer to extended supersymmetry.

The supercharges sit in spinorial representations of the Lorentz group, respectively $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$.

We will not repeat the bosonic subalgebra, since is given by the Poincarè Lie algebra. The fermionic generators satisfy anticommutation rules between themselves and commutation rules with bosonic generators. For this reason, the supersymmetry algebra is defined in mathematical literature as a graded Lie algebra with grade one.
The (anti)commutation rules in four dimensions are

$$[P_\mu, Q^I_\alpha] = 0 \tag{A.1}$$

$$[P_\mu, \bar{Q}^I_{\dot{\alpha}}] = 0 \tag{A.2}$$

$$[M_{\mu\nu}, Q^I_\alpha] = i(\sigma_{\mu\nu})_\alpha^\beta Q^I_\beta \tag{A.3}$$

$$[M_{\mu\nu}, \bar{Q}^I_{\dot{\alpha}}] = i(\bar{\sigma}_{\mu\nu})^{\dot{\alpha}}_{\dot{\beta}} \bar{Q}^I_{\dot{\beta}} \tag{A.4}$$

$$\{Q^I_\alpha, \bar{Q}^J_{\dot{\beta}}\} = 2\sigma^\mu_{\alpha\dot{\beta}} P_\mu \delta^{IJ} \tag{A.5}$$

$$\{Q^I_\alpha, Q^J_\beta\} = \epsilon_{\alpha\beta} Z^{IJ} \qquad Z^{IJ} = -Z^{JI} \tag{A.6}$$

$$\{Q^I_{\dot{\alpha}}, Q^J_{\dot{\beta}}\} = \epsilon_{\dot{\alpha}\dot{\beta}} \left(Z^{IJ}\right)^* \tag{A.7}$$

This set of commutation rules can be found using symmetry arguments and enforcing the consistency of the algebra using the graded Jacobi identity.
It is important to stress the fact that $Z^{IJ}$ are operators that span an invariant subalgebra: they are *central charges*. They play an important role especially in massive representations.

There is an additional symmetry that is not present in the previous commutation rules: R-symmetry. It is an automorphism of the algebra that act on the supercharges. For generic $\mathcal{N}$ the R-Symmetry group is $U(\mathcal{N})$[1].

[1]This is not always true. For example for $\mathcal{N} = 4$ the R-symmetry group is given by $SU(4)$

## A.2 Representations

Since the supercharges do not commute with Lorentz generators, their action on a state will result in a state with different spin: they generate a symmetry between bosons and fermions.
Representations of supersymmetry contain particle with different spin but same mass and they are organized in supermultiplets. The mass of particles in the same multiplet must be the same because $P^2$ is still a Casimir operator of the supersymmetry algebra, while the Pauli-Lubanski operator $W^2$ isn't anymore.

Moreover, the supersymmetry algebra imposes that every state must have positive energy and that every supermultiplet must contain the same number of bosonic and fermionic degrees of freedom *on-shell* .

Various supermultiplets exist and their properties depend on the number of supercharges of the theory and on what they represent e.g matter, glue or gravity.

Massless supermultiplet are typically shorter than massive multiplet because in the massless case half of the supercharges are represented trivially. Massive representation of extended supersymmetry can be shortened in case some of the central charges of the algebra are equal to twice the mass of the multiplet. These states are usually called (ultra)short multiplet or BPS states.

We will introduce the multiplets that can be defined for $4d\,\mathcal{N}=1$ theories and only later we will explain the differences with $3d\,\mathcal{N}=2$ theories. Representations are similar because in both cases we have the same number of supercharges.

For four dimensional theories, we can define two different multiplet that are invariant under supersymmetry transformations. The matter or chiral multiplet contains a complex scalar (*squark*) and a Weyl fermion (*quark*). It identifies the matter content of the theory. The vector or gauge multiplet contains a Weyl fermion (*gaugino*) and a vector (*gluon or photon*). Particles in the same multiplet transform in the same representation of global or gauge symmetries. For this reason the gaugino cannot represent matter.

A representation of these multiplets on fields can be easily found using the *superspace* formalism that we will introduce in the next section. In this formalism it is possible to represent fields that are *off-shell*, in contrast with multiplets that we introduced previously that are *on-shell* since they represent states in Hilbert space.

### A.2.1 Superfields and superspace in four dimensions

Supersymmetry representations on fields can be found more sistematically using the formulation of *superspace* instead of acting directly with supercharges and verifying that the algebra closes.

A simple formulation of superspace exist for theories with 4 supercharges while for theories with a bigger number of supercharges its definition is much more complex. We will be interested only in theories with 4 supercharges such as theories in 4D with $\mathcal{N}=1$ or 3D with $\mathcal{N}=2$.

Superspace can be seen as the extension of Minkowsky space with *fermionic coordinates* i.e. *Grassmann numbers* $\theta^\alpha$ , $\bar{\theta}^{\dot{\alpha}}$. They anticommute between themselves and commute with everything else.

$$\{\theta^\alpha, \theta^\beta\} = 0 \quad \{\bar{\theta}^{\dot{\alpha}}, \bar{\theta}^{\dot{\beta}}\} = 0 \quad \{\theta^\alpha, \bar{\theta}^{\dot{\beta}}\} = 0 \qquad \alpha, \dot{\alpha} = 1, 2 \tag{A.8}$$

Derivation and integration in Grassmann variables are summarized by these rules

$$\partial_\alpha = \frac{\partial}{\partial\theta_\alpha} \quad \partial^\alpha = -\epsilon^{\alpha\beta}\partial_\beta \quad \bar{\partial}_{\dot{\alpha}} = \frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}} \quad \bar{\partial}^{\dot{\alpha}} = -\epsilon^{\dot{\alpha}\dot{\beta}}\partial_{\dot{\beta}} \quad \partial_\alpha\theta^\beta = \delta^\beta_\alpha \quad \partial_\alpha\bar{\theta}^{\dot{\alpha}} = 0 \tag{A.9}$$

$$\int d\theta = 0 \quad \int d\theta\, \theta = 1 \quad d^2\theta = \frac{1}{2}d\theta^1 d\theta^2 \quad \int d^2\theta = \frac{1}{4}\epsilon^{\alpha\beta}\partial_\alpha\partial\beta \tag{A.10}$$

For a more detailed introduction on Grassmann numbers and their properties see [16].

Using Grassmann numbers we can write the anticommutators in the supersymmetry algebra as commutators defining $\theta Q = \theta^\alpha Q_\alpha$ and $\bar{\theta}\bar{Q} = \bar{\theta}^{\dot{\alpha}}\bar{Q}_{\dot{\alpha}}$

$$[\theta Q, \bar{\theta}\bar{Q}] = 2\theta\sigma^\mu\bar{\theta}P_\mu \quad , \quad [\theta Q, \theta Q] = [\bar{\theta}\bar{Q}, \bar{\theta}\bar{Q}] = 0 \tag{A.11}$$

Using this trick we are able to represent the supersymmetry algebra as a Lie algebra. An element of the superPoincarè group can be found exponentiating the generators

$$G(x, \theta, \bar{\theta}, \omega) = \exp\left(ixP + i\theta Q + i\bar{\theta}\bar{Q} + \frac{1}{2}i\omega M\right) \tag{A.12}$$

The superspace is defined as the 4+4 dimensions group coset

$$M_{4|1} = \frac{\text{SuperPoincarè}}{\text{Lorentz}} \tag{A.13}$$

in analogy to Minkowsky space that can be defined as the coset between Poincarè and Lorentz groups.

A generic point in superspace is parametrized by $(x^\mu, \theta^\alpha, \bar{\theta}^{\dot{\alpha}})$. A superfields is a field in superspace i.e. function of the superspace coordinates. Since $\theta$ coordinates anticommute, the expansion of a superfield in fermionic coordinates stops at some point. The most general superfield $Y = Y(x, \theta, \bar{\theta})$ is given by

$$\begin{aligned} Y(x, \theta, \bar{\theta}) = f(x) + \theta\psi_1(x) + \bar{\theta}\bar{\psi}_2(x) + \theta\theta g_1(x) + \bar{\theta}\bar{\theta}g_2(x) \\ + \theta\sigma^\mu\theta v_\mu(x) + \theta\theta\bar{\theta}\bar{\lambda}(x) + \bar{\theta}\bar{\theta}\theta\rho(x) + \theta\theta\bar{\theta}\bar{\theta}s(x) \end{aligned} \tag{A.14}$$

fields with uncontracted $\theta$ such as $\psi_1, \psi_2, \lambda, \rho$ are spinors while $v_\mu$ is a vector.

Supercharges can be represented as differential operators that act on superfield. Their expression is

$$\begin{cases} Q_\alpha &= -i\partial_\alpha - \sigma^\mu_{\alpha\dot{\beta}}\bar{\theta}^{\dot{\beta}}\partial_\mu \\ \bar{Q}_{\dot{\alpha}} &= +i\partial_{\dot{\alpha}} + \theta^\beta\sigma^\mu_{\beta\dot{\alpha}}\bar{\theta}^{\dot{\beta}}\partial_\mu \end{cases} \tag{A.15}$$

An infinitesimal supersymmetry transformation on a superfield is defined by

$$\delta_{\epsilon,\bar{\epsilon}}Y = \left(i\epsilon Q + i\bar{\epsilon}\bar{Q}\right)Y \tag{A.16}$$

The powerfulness of the superfield formalism is due to the fact that an integral in full superspace coordinates of a superfield is supersymmetric invariant.

$$\delta_{\epsilon,\bar{\epsilon}} \int d^4x \, d^2\theta \, d^2\bar{\theta} \; Y = \int d^4x \, d^2\theta \, d^2\bar{\theta} \; \delta_{\epsilon,\bar{\epsilon}} Y = 0 \tag{A.17}$$

The first equality holds because the Grassmann measure is inviariant under translation while the second is true because we can see that the variation of the superfield is either killed by the integration in the $\theta$ variables or is proportional to a spacetime derivative that does not contribute after integration in space.

Using this fact we can construct supersymmetric invariant lagrangians by integrating superfields in superspace. Clearly, in order to find a physically significant lagrangian we should choose the superfield we wish to integrate wisely. More importantly, we want to use irreducible representation of supersymmetry i.e. the supermultiplets we introduced before. We need to find conditions that can be imposed on a general superfield that are invariant under a supersymmetry transformation.

### Chiral superfield

One way to achieve this goal is to find an operator that commute with the supercharges and annihilate the superfield An example of such operator is the *covariant derivative*

$$\begin{cases} D_\alpha = & \partial_\alpha + i\sigma^\mu_{\alpha\dot{\beta}}\bar{\theta}^{\dot{\beta}}\partial_\mu \\ \bar{D}_{\dot{\alpha}} = & \dot{\partial}_{\dot{\alpha}} + i\theta^\beta\sigma^\mu_{\beta\dot{\alpha}}\bar{\theta}^{\dot{\beta}}\partial_\mu \end{cases} \tag{A.18}$$

We can define a (anti)chiral superfield $\Phi$

$$\bar{D}_{\dot{\alpha}}\Phi = 0 \quad \text{chiral} \qquad\qquad D_\alpha\Psi = 0 \quad \text{anti-chiral} \tag{A.19}$$

This condition reduces the number of components of the superfield. It can be easily demonstrated that if $\Psi$ is chiral, then $\bar{\Psi}$ is anti-chiral. As a result a chiral field cannot be real.

The expansion of a chiral fields in components is given by

$$\Phi(x,\theta,\bar{\theta}) = \phi(x) + \sqrt{2}\theta\psi(x) + i\theta\sigma^\mu\bar{\theta}\partial\mu\phi(x) - \theta\theta F(x) - \frac{i}{\sqrt{2}}\theta\theta\partial\mu\psi(x)\sigma^\mu\bar{\theta} - \frac{1}{4}\theta\theta\bar{\theta}\bar{\theta}(x) \tag{A.20}$$

We can see that a chiral field is composed by three fields: two complex scalars ($\phi$ and $F$) and a spinor ($\psi$).

The chiral superfield identifies the matter multiplet we introduced previously. It contains an additional bosonic field ($F(x)$) that is present because superfields provide an *off-shell* representation of supersymmetry and it is needed in order to close the algebra. It is called *auxiliary field* because it will not have kinetic terms in every Lagrangian that can be constructed.

### Real or Vector Field

We can impose that the superfield is real. In this way we find the *real* or *vector* multiplet. Its general expression in component is messy and a simplification can be made noting that $\Phi + \bar{\Phi}$ is a vector superfield if $\Phi$ is chiral. Choosing an appropriate chiral field, the real superfield can be put in what is called Wess-Zumino gauge. We stress the fact that the Wess-Zumino gauge is not supersymmetric invariant: after a supersymmetry transformation the vector superfield acquire its general expression involving many other field components. In this gauge the vector superfield can be written as

$$V_{WZ}(x,\theta,\bar{\theta}) = \theta\sigma^{\mu}\bar{\theta}v_{\mu}(x) + i\theta\theta\bar{\theta}\bar{\lambda}(x) - i\bar{\theta}\bar{\theta}\theta\lambda(x) + \frac{1}{2}\theta\theta\bar{\theta}\bar{\theta}D(x) \tag{A.21}$$

The vector superfields represents the vector multiplet (which contains radiation) and similarly to the chiral superfields contains an auxiliary field ($D(x)$).

### A.2.2 R-symmetry

R-symmetry was first introduced with the supersymmetry algebra. For the theories we will consider in superspace it is given by a global $U(1)_R$. It is defined by as a transformation of the Grassmann coordinates

$$\theta \;\rightarrow\; e^{i\alpha}\theta \qquad \bar{\theta} \;\rightarrow\; e^{-i\alpha}\bar{\theta} \tag{A.22}$$

$\alpha$ parametrizes the transformation. As a result supercharges transform under the transformation

$$Q \rightarrow\; e^{-i\alpha}Q \qquad \bar{Q} \;\rightarrow\; e^{+i\alpha}\bar{Q} \tag{A.23}$$

From this we find the commutator relations between supercharges and R-symmetry generator $R$

$$[R,Q] = -Q \qquad [R,\bar{Q}] = \bar{Q} \tag{A.24}$$

The R-charge of a superfield is defined by

$$Y(x,\theta,\bar{\theta}) \;\rightarrow\; e^{iR_Y\alpha}Y(x,\theta,\bar{\theta}) \tag{A.25}$$

Different component field in the superfield have different R-charge and are related because of A.24. For a chiral field we have

$$R[\phi] = R\,[\Phi] \qquad R[\psi] = R\,[\Phi] - 1 \qquad R[F] = R\,[\Phi] - 2 \tag{A.26}$$

The corresponding antichiral field carry opposite charges.

## A.3 Supersymmetric actions

We will use the property we introduced in A.17 to generate supersymmetric invariant lagrangians. We start our analysis with chiral superfields. Since lagrangians are quadratic in the fields and must be real, the simplest kinetic term for a chiral superfield is given by $\bar{\Phi}\Phi$.

$$\mathcal{L}_{kin} \;=\; \int d^2\theta d^2\bar{\theta}\;\bar{\Phi}\Phi = \partial_{\mu}\bar{\phi}\partial^{\mu}\phi + \frac{i}{2}\left(\partial_{\mu}\psi\sigma^{\mu}\bar{\psi} - \psi\sigma^{\mu}\partial_{\mu}\bar{\psi}\right) + \bar{F}F + \text{total derivative} \tag{A.27}$$

which gives the correct kinetic terms for a scalar and a spinor. The auxiliary field doesn't have kinetic terms as predicted.

Many action can be find using a generalization of the equation above. It is called *Kahler* potential

$$K(\bar{\Phi}, \Phi) = \sum_{m,n=1}^{\infty} c_{m,n} \bar{\Phi}^m \Phi^n \qquad \text{where} \quad c_{m,n} = c_{n,m}* \tag{A.28}$$

The condition on the coefficient is imposed by the requirement of a real lagrangian.

Another way of finding supersymmetric actions is by integrating *chiral* superfields in half-superspace coordinates. We define the *superpotential* to be a holomorphic function of $\Phi$

$$\mathcal{L}_{int} = \int d^2\theta d^2 \; W(\Phi) + \int d^2\bar{\theta} \; \bar{W}(\bar{\Phi}) = \sum_{i=1}^{\infty} \int d^2\theta \; \lambda_n \Phi^n + \int d^2\bar{\theta} \; \lambda_n^\dagger \bar{\Phi}^n \tag{A.29}$$

The hermitian conjugate was added in order to have a real lagrangian.

Mixed terms with product of chiral and anti-chiral superfield are not present since they would be generic superfields and would not yield a supersymmetric lagrangian. In fact if $W(\Phi)$ is holomorphic and $\Phi$ is a chiral superfield, $W(\Phi)$ is a chiral superfield

$$\bar{D}_{\dot{\alpha}} W(\Phi) = \frac{\partial W}{\partial \Phi} \bar{D}_{\dot{\alpha}} \Phi + \frac{\partial W}{\partial \bar{\Phi}} \bar{D}_{\dot{\alpha}} \bar{\Phi} = 0 \tag{A.30}$$

and yield a proper lagrangian upon integration in $d^2\theta$.

Since the superpotential is integrated only in half superspace coordinates it need to be charged in an opposite way with respect to the integration measure under R-symmetry. Remembering that

$$R[\theta] = 1 \qquad [\bar{\theta}] = -1 \qquad R[d\theta] = -1 \qquad R[d\bar{\theta}] = 1 \tag{A.31}$$

It's easy to see that

$$R[W(\Phi)] = 2 \qquad R[\bar{W}(\bar{\Phi})] = -2 \tag{A.32}$$

For this reason in most situations the superpotential fix the supercharges of the fields. The lagrangian of super Yang-Mills theories is given by

$$\mathcal{L}_{SYM} = \frac{1}{32\pi i} \left( \int d^2\theta \; \left( \frac{\theta_{YM}}{2\pi} + \frac{4\pi i}{g^2} \right) W_\alpha W^\alpha \right) = \tag{A.33}$$

$$= \mathrm{Tr} \left[ -\frac{1}{4} F_{\mu\nu} F^{\mu nu} - i\lambda \sigma^\mu D_\mu \bar{\lambda} + \frac{1}{2} D^2 \right] + \frac{\theta_{YM}}{32\pi^2} g^2 \mathrm{Tr} F_{\mu\nu} \tilde{F}^{\mu\nu} \tag{A.34}$$

where we defined the chiral superfield $W_\alpha$ as

$$W_\alpha = -\frac{1}{4} \bar{D}\bar{D} \left( e^{-2gV} D_\alpha e^{2gV} \right) \qquad \bar{W}_{\dot{\alpha}} = -\frac{1}{4} DD \left( e^{2gV} \bar{D}_{\dot{\alpha}} V e^{-2gV} \right) \tag{A.35}$$

It can be demonstrated that $W_\alpha$ is chiral and is invariant under the supergauge transformation $V \;\rightarrow\; V + \bar{\Phi} + \Phi$ while the vector superfield $V$ was not.

From a perturbative point of view the inclusion of the term proportional to $\theta_{YM} Tr F_{\mu\nu} \tilde{F}^{\mu nu}$ has no effect since it is proportional to a total derivative. It is a parity violating term that differs from zero only in non trivial topological configurations of the field (instantons).

The matter lagrangian we introduced is not invariant under gauge transformation. The correct gauge invariant lagrangian is given by

$$\mathcal{L}_{matter} = \int d^2\theta d^2\bar{\theta}\ \bar{\Phi} e^{2gV} \Phi \tag{A.36}$$

The superpotential is not automatically invariant under gauge transformation. As a result only certain expression are allowed.
There's an additional supersymmetric invariant lagrangian that can be constructed in a gauge theory when the gauge group contains abelian factors. It is called Fayet-Iliopulos term and can be present for every ideal $A$ of the gauge group

$$\mathcal{L}_{FI} = \sum_A \xi_A \int d^2\theta d^2\bar{\theta} V^A = \frac{1}{2} \sum_A \xi_A D^A \tag{A.37}$$

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