4D to 3D reduction of Seiberg duality for SU(N) susy gauge theories with adjoint matter: a partition function approach

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1 Physics

——— INTRODUCTION OUTLINE———

~ More symmetry = more tools for studying t\heories

~ State structure: multiplet & superspace

~ Milder divergences

~ Renormalization constraints

~ Non renormalization theorems (perturbative)

~ Holomorphicity, couplings as background fields

(important smoothness of weak coupling limits, e.g. classic imit g in well defined.

~ Exact results (superpotential, exact beta function)

~Moduli space

1.1 Introduction

Supersymmetric quantum field theories enjoy an enlarged group of symmetries compared to other field theories. Since the symmetry group is a non trivial combination of internal and spacetime symmetries, they have many unexpected features and new techniques were found to study them. Almost all of the new tools found are available only for supersymmetric field theories, making them the theater for many advances in physics.

An more technical introduction on supersymmetry and its representation on fields can be found in appendix A.

In this section we will analyse the features of supersymmetric field theories that are crucial to the discovery of electric magnetic duality and its generalisations.

1.1.1 General renormalization properties

A remarkable feature of supersymmetry is the constraint that the additional symmetry imposes on the renormalization properties of the theories.

One of the first aspects that brought attention to supersymmetry was that divergences of loop diagrams were milder because of the cancellation between diagrams with bosons and fermions running in the loops.

Nowadays we know powerful theorems that restrict the behaviour of supersymmetric field theories during renormalization. In order to preserve supersymmetry, the renormalization process has to preserve the Hilbert space structure. For example the wave function renormalization of different *particles* inside a multiplet must be the same, otherwise the renormalized lagrangian is not supersymmetric invariant anymore.

Moreover, in the supersymmetry algebra P^2 is still a Casimir operator i.e. it commutes with every operator in the algebra: particles in the same multiplet must have the same mass. Renormalization cannot break this condition, otherwise it would break supersymmetry.

For a *Super Yang Mills* theory with $\mathcal{N} = 1$ we have the additional requirement that gV, where g is the coupling and V is the vector superfield, cannot be renormalized by symmetry considerations.

Adding more supersymmetry the wave function renormalization of the various field are even more constrained by symmetry. For example, for $\mathcal{N} = 4$ *SYM* the fields and the coupling are not renormalized at all.

Beta function for SYM and SQCD

Another nice feature of supersymmetric field theories is that some quantities can be calculated exactly. The first object of this kind that we encounter is the β function of four dimensional $\mathcal{N} = 1$ *Super Yang Mills* and *Super QCD* theories.

It is given by the *NSVZ β function*

$$\beta(\alpha) = -\frac{\alpha^2}{2\pi}\left[3\,T(Adj) - \sum_i T(R_i)(1-\gamma_i)\right]\left(1 - \frac{\alpha\,T(Adj)}{2\pi}\right)^{-1} \qquad \alpha = \frac{g^2}{4\pi} \tag{1.1}$$

where γ_i are the anomalous dimensions of the matter fields and $T(R_i)$ are the dynkin indices of their representation.
The anomalous dimensions are defined as

$$\gamma_i = -\frac{d\,\log(Z_i)}{d\,\log(\mu)} \tag{1.2}$$

where Z_i is the wave function renormalization coefficient. For example for gauge group $SU(N)$ we have

$$T(N) = \frac{1}{2} \qquad T(Adj) = N$$

The *NSVZ β function* was first calculated using instanton methods in [1]. Over the years it has been calculated in other ways using the fact that the action is holomorphic in the complexified coupling

$$\frac{1}{g_h^2} = \frac{1}{g^2} + i\frac{\theta}{8\pi^2} \tag{1.3}$$

Using the holomorphic coupling the action for the vector field is written as

$$\mathcal{L}_h(V_h) = \frac{1}{16}\int d^2\theta \frac{1}{g_h^2} W^a(V_h)W^a(V_h) \tag{1.4}$$

whereas with the canonical normalization for the vector field is

$$\mathcal{L}_c(V_c) = \frac{1}{16}\int d^2\theta \left(\frac{1}{g_c^2} + i\frac{\theta}{8\pi^2}\right) W^a(g_cV_c)W^a(g_cV_c) \tag{1.5}$$

Using the canonical normalization g_cV_c is a real superfield, imposing that g_c is real. For this reason with the canonical normalization the lagrangian is not holomorphic in the combination $\frac{1}{g^2} + i\frac{\theta}{8\pi^2}$. Thanks to holomorphy, the holomorphic coupling is only renormalized at one-loop and the β function can be computed but its expression is different from *NSVZ β function.* In fact, the *NSVZ β function* is defined using the canonical (or physical) coupling constant and receives contribution from all orders in perturbation theory.

At first sight, one should expect that the expressions should match since the first two orders in α of the β function are scheme independent. The reason why the two expressions differ is that the Jacobian of the transformation between canonical and holomorphic normalization is anomalous. Once the anomaly is taken into account the two expressions for the β function agree. An explicit calculation can be found in [2].

1.1.2 Superpotential: holomorphy and non-renormalization

Other than renormalization constraints, supersymmetry provides non-renormalization theorems for certain objects, such as the superpotential. In [3] it has been demonstrated that the superpotential is tree-level exact, i.e. it does not receive correction in perturbation theory. However it usually receive contributions from non perturbative dynamics.

Perturbative calculations can be done using supergraphs, i.e. Feynman diagrams with superfields. The advantage of this approach is that supersymmetry is explicit and many semplification occur naturally. The demonstration is based on the fact that for general supersymmetric field theories, supergraph loops diagrams yield a term that can be written in the form

$$\int d^4x_1 \dots d^4x_n d^2\theta d^2\bar{\theta}\, G(x_1, \dots, x_n)\, F_1(x_1, \theta, \bar{\theta}) \dots F_n(x_n, \theta, \bar{\theta}) \tag{1.6}$$

where $G(x_1, \dots, x_n)$ is translationally invariant function.
The importance of this result is that all contribution from Feynman diagrams are given by a single integral over full superspace ($d^2\theta d^2\bar{\theta}$) whereas the superpotential must be written as an integral in half-superspace($d^2\theta$ only) of chiral fields. Exploiting the fact that a product of chiral fields is a chiral field, the most general form of a superpotential is

$$W(\lambda, \Phi) = \sum_{n=1}^{\infty} \left(\int d^2\theta\, \lambda_n \Phi^n \;+\; \int d^2\bar{\theta}\, \lambda_n^\dagger \bar{\Phi}^n \right) \tag{1.7}$$

The second term of the superpotential is added in order to give a real lagrangian after the integration in superspace. From the definition, we can see that the superpotential is holomorphic in the fields and in the coupling constants.

Fifteen years later, Seiberg [4] provided a proof of this theorem using a different approach. He noted that the coupling constants λ_n can be treated as background fields, i.e. chiral superfields with no dynamics.

Using this observation we can assign transformation laws to the coupling constants, making the lagrangian invariant under a larger symmetry. Fields and coupling constants are charged under this symmetry and only certain combinations of them can appear in the superpotential. In addition, in a suitable weak coupling limit the effective superpotential must be identical with the tree-level one. These conditions, taken together, fix the expansion of the superpotential to the expression of the tree-level potential. A more detailed discussion can be found in [5] and [6].

1.1.3 Moduli space

Supersymmetric field theories have a larger set of vacua compared to ordinary field theories. This is related to the fact that chiral fields, which represent matter, contain a scalar field.

Lorentz invariance of the vacuum forbids fields with spin different from zero to acquire a vacuum expectation value. With the same reasoning, derivatives of scalar fields must be set to zero because of translational invariance of the vacuum. The scalar potential is the only term in the lagrangian that can differ from zero and in fact it is the only object that can be different from zero in the Hamiltonian. As a result, the minimal of the scalar potential are in one-to-one correspondence with the states of minimal energy of the theory.

Appendices

A | Supersymmetry and superfields

A.1 Supersymmetry algebra

The supersymmetry algebra is an extension of the Poincarè group involving anticommutators together with commutators. Since it is not a ordinary Lie algebra, Coleman-Mandula theorem does not apply for theories that are invariant under it.

The supersymmetry algebra is divided into two subalgebras, the bosonic and fermionic part. The bosonic part contains Poincarè Lie algebra $(M_{\mu\nu}, P_\mu)$ while fermionic subalgebra is generated by the *supercharges* $(Q^I_\alpha, \bar{Q}^I_{\dot{\alpha}})$ with $I = 1, \ldots, \mathcal{N}$. When more than one pair of supercharges is present we refer to extended supersymmetry.

The supercharges sit in spinorial representations of the Lorentz group, respectively $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$.

We will not repeat the bosonic subalgebra, since is given by the Poincarè Lie algebra. The fermionic generators satisfy anticommutation rules between themselves and commutation rules with bosonic generators. For this reason, the supersymmetry algebra is defined in mathematical literature as a graded Lie algebra with grade one.
The (anti)commutation rules in four dimensions are

$$[P_\mu, Q^I_\alpha] = 0 \tag{A.1}$$

$$[P_\mu, \bar{Q}^I_{\dot{\alpha}}] = 0 \tag{A.2}$$

$$[M_{\mu\nu}, Q^I_\alpha] = i(\sigma_{\mu\nu})^\beta_\alpha Q^I_\beta \tag{A.3}$$

$$[M_{\mu\nu}, \bar{Q}^I_{\dot{\alpha}}] = i(\bar{\sigma}_{\mu\nu})^{\dot{\alpha}}_{\dot{\beta}} \bar{Q}^I_{\dot{\beta}} \tag{A.4}$$

$$Q^I_\alpha, \bar{Q}^J_{\dot{\beta}} = 2\sigma^\mu_{\ \alpha\dot{\beta}} P_\mu \delta^{IJ} \tag{A.5}$$

$$Q^I_\alpha, Q^J_\beta = \epsilon_{\alpha\beta} Z^{IJ} \qquad Z^{IJ} = -Z^{JI} \tag{A.6}$$

$$Q^I_{\dot{\alpha}}, Q^J_{\dot{\beta}} = \epsilon_{\dot{\alpha}\dot{\beta}} \left(Z^{IJ}\right)^* \tag{A.7}$$

This set of commutation rules can be found using symmetry arguments and enforcing the consistency of the algebra using the graded Jacobi identity.
It is important to stress the fact that Z^{IJ} are operators that span an invariant subalgebra: they are *central charges*. They play an important role especially in massive representations.

There is an additional symmetry that is not present in the previous commutation rules: R-symmetry. It is an automorphism of the algebra that act on the supercharges. For generic $\mathcal{N}$ the R-Symmetry group is $U(\mathcal{N})$.

A.1.1 Representations

Since the supercharges do not commute with Lorentz generators, their action on a state will result in a state with different spin: they generate a symmetry between bosons and fermions.
Representations of supersymmetry contain particle with different spin but same mass and they are organized in supermultiplets.

Various supermultiplets exist and their properties depend on the number of supercharges of the theory and on what they represent e.g matter, glue or gravity.

We will introduce the multiplets that can be defined for $4d$ $\mathcal{N} = 1$ theories and only later we will explain the differences with $3d$ $\mathcal{N} = 2$ theories. For four dimensional theories, we can define two different multiplet that are invariant under supersymmetry transformations. The matter or chiral multiplet contains a complex scalar (*squark*) and a Weyl fermion (*quark*). It identifies the matter content of the theory. The vector or gauge multiplet contains a Weyl fermion (*gaugino*) and a vector (*gluon*. Particles in the same multiplet transform in the same representation of global or gauge symmetries. For this reason the gaugino cannot represent matter.

A representation of these multiplets on fields can be easily found using the *superspace* formalism that we will introduce later. In this formalism it is possible to represent fields that are *off-shell*, in contrast with multiplets that we introduced previously that are *on-shell* since they represent states in Hilbert space.
Matter and gauge multiplets are represented by (anti)chiral and real superfields respectively.

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