4D to 3D reduction of Seiberg duality for SU(N) susy gauge theories with adjoint matter: a partition function approach

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1 Physics

—— INTRODUCTION OUTLINE ——

~ More symmetry = more tools for studying t\heories

~ Milder divergences

~ Renormalization constraints

~ Non renormalization theorems (perturbative)

~ Holomorphicity, couplings as background fields

~ Exact results (superpotential, exact beta function)

~ Moduli space

1.1 Introduction

Supersymmetric quantum field theories enjoy an enlarged group of symmetries compared to other field theories. Since the symmetry group is a non trivial combination of internal and spacetime symmetries, they have many unexpected features and new techniques were found to study them. Almost all of the new tools found are available only for supersymmetric field theories, making them the theater for many advances in physics.

A more technical introduction on supersymmetry and its representation on fields can be found in appendix A.

In this section we will analyse more advanced features of supersymmetric field theories that has been used intensively in the discovery and in the analysis of electric magnetic duality and its generalisations.

1.1.1 General renormalization properties

A remarkable feature of supersymmetry is the constraint that the additional symmetry imposes on the renormalization properties of the theories.

One of the first aspects that brought attention to supersymmetry was that divergences of loop diagrams were milder because of the cancellation between diagrams with bosons and fermions running in the loops.

Nowadays we know powerful theorems that restrict the behaviour of supersymmetric field theories during renormalization. In order to preserve supersymmetry, the renormalization process has to preserve the Hilbert space structure. For example the wave function renormalization of different *particles* inside a multiplet must be the same, otherwise the renormalized lagrangian is not supersymmetric invariant anymore.

Moreover, in the supersymmetry algebra P^2 is still a Casimir operator i.e. it commutes with every operator in the algebra: particles in the same multiplet must have the same mass. Renormalization cannot break this condition, otherwise it would break supersymmetry.

For a *Super Yang Mills* theory with $\mathcal{N}=1$ we have the additional requirement that gV, where g is the coupling and V is the vector superfield, cannot be renormalized by symmetry considerations.

Adding more supersymmetry the wave function renormalization of the various field are even more constrained by symmetry. For example, for $\mathcal{N}=4$ *SYM* the fields and the coupling are not renormalized at all.

Beta function for SYM and SQCD

Another nice feature of supersymmetric field theories is that some quantities can be calculated exactly. The first object of this kind that we encounter is the β function of four dimensional $\mathcal{N}=1$ *Super Yang Mills* and *Super QCD* theories.

It is given by the *NSVZ β function*

$$\beta(\alpha) = -\frac{\alpha^2}{2\pi}\left[3\,T(Adj) - \sum_i T(R_i)(1-\gamma_i)\right]\left(1-\frac{\alpha\,T(Adj)}{2\pi}\right)^{-1} \qquad \alpha = \frac{g^2}{4\pi} \tag{1.1}$$

where γ_i are the anomalous dimensions of the matter fields and $T(R_i)$ are the dynkin indices of their representation.
The anomalous dimensions are defined as

$$\gamma_i = -\frac{d\,\log(Z_i)}{d\,\log(\mu)} \tag{1.2}$$

where Z_i is the wave function renormalization coefficient. For example for gauge group $SU(N)$ we have

$$T(N) = \frac{1}{2} \qquad T(Adj) = N$$

The *NSVZ β function* was first calculated using instanton methods in [1]. Over the years it has been calculated in other ways using the fact that the action is holomorphic in the complexified coupling

$$\frac{1}{g_h^2} = \frac{4\pi i}{g_c^2} + \frac{\theta_{YM}}{2\pi} \tag{1.3}$$

Using the holomorphic coupling the action for the vector field is written as

$$\mathcal{L}_h(V_h) = \frac{1}{32\pi i}\int d^2\theta \frac{1}{g_h^2} W^a(V_h)W^a(V_h) + h.c. \tag{1.4}$$

whereas with the canonical normalization for the vector field is

$$\mathcal{L}_c(V_c) = \frac{1}{32\pi i}\int d^2\theta \left(\frac{4\pi i}{g_c^2} + \frac{\theta_{YM}}{2\pi}\right) W^a(g_cV_c)W^a(g_cV_c) + h.c. \tag{1.5}$$

Using the canonical normalization g_cV_c is a real superfield, imposing that g_c is real. For this reason with the canonical normalization the lagrangian is not holomorphic in the combination $\frac{4\pi i}{g^2} + \frac{\theta}{2\pi}$. Thanks to holomorphy, the holomorphic coupling is only renormalized at one-loop and the β function can be computed but its expression is different from *NSVZ β function.* In fact, the *NSVZ β function* is defined using the canonical (or physical) coupling constant and receives contribution from all orders in perturbation theory.

At first sight, one should expect that the expressions should match since the first two orders in α of the β function are scheme independent. The reason why the two expressions differ is that the Jacobian of the transformation between canonical and holomorphic normalization is anomalous. Once the anomaly is taken into account the two expressions for the β function agree. An explicit calculation relating the comparison between the two different approach can be found in [2].

1.1.2 Superpotential: holomorphy and non-renormalization

Other than renormalization constraints, supersymmetry provides non-renormalization theorems for certain objects, such as the superpotential. In [3] it has been demonstrated that the superpotential is tree-level exact, i.e. it does not receive correction in perturbation theory. However it usually receive contributions from non perturbative dynamics.

Perturbative calculations can be done using supergraphs, i.e. Feynman diagrams with superfields. The advantage of this approach is that supersymmetry is explicit and many semplification occur naturally. The demonstration is based on the fact that for general supersymmetric field theories, supergraph loops diagrams yield a term that can be written in the form

$$\int d^4x_1 \dots d^4x_n d^2\theta d^2\bar{\theta}\; G(x_1, \dots, x_n)\, F_1(x_1, \theta, \bar{\theta}) \dots F_n(x_n, \theta, \bar{\theta}) \tag{1.6}$$

where $G(x_1, \dots, x_n)$ is translationally invariant function.
The importance of this result is that all contribution from Feynman diagrams are given by a single integral over full superspace ($d^2\theta d^2\bar{\theta}$) whereas the superpotential must be written as an integral in half-superspace($d^2\theta$ only) of chiral fields. Exploiting the fact that a product of chiral fields is a chiral field, the most general form of a superpotential is

$$W(\lambda, \Phi) = \sum_{n=1}^{\infty} \left(\int d^2\theta\, \lambda_n \Phi^n \;+\; \int d^2\bar{\theta}\, \lambda_n^{\dagger} \bar{\Phi}^n \right) \tag{1.7}$$

The second term of the superpotential is added in order to give a real lagrangian after the integration in superspace. From the definition, we can see that the superpotential is holomorphic in the fields and in the coupling constants.

Fifteen years later, Seiberg [4] provided a proof of this theorem using a different approach. He noted that the coupling constants λ_n can be treated as background fields, i.e. chiral superfields with no dynamics.

Using this observation we can assign transformation laws to the coupling constants, making the lagrangian invariant under a larger symmetry. Fields and coupling constants are charged under this symmetry and only certain combinations of them can appear in the superpotential. In addition, in a suitable weak coupling limit the effective superpotential must be identical with the tree-level one. These conditions, taken together, fix the expansion of the superpotential to the expression of the tree-level potential. A more detailed discussion can be found in [5] and [6].

1.1.3 Moduli space

Supersymmetric field theories have a larger set of vacua compared to ordinary field theories because of the presence of many scalar fields in the supermultiplets.

Lorentz invariance of the vacuum forbids fields with spin different from zero to acquire a vacuum expectation value. With the same reasoning, derivatives of scalar fields must be set to zero because of translational invariance of the vacuum. The scalar potential is the only term in the Lagrangian and in the Hamiltonian that can differ from zero. As a result, the minimums of the scalar potential are in one-to-one correspondence with the states of minimal energy of the theory.

For $4D$ $\mathcal{N}=1$ gauge theories with matter, the scalar potential (involving only squarks in this case) is given by

$$V(\phi_i, \bar{\phi}_j) = F\bar{F} + \frac{1}{2}D^2 \overset{on-shell}{=} \frac{\partial W}{\partial \phi_i} F^i \frac{\partial \bar{W}}{\partial \bar{\phi}_i} \bar{F}^i + \frac{g^2}{2} \sum_a |\bar{\phi}_j (T^a)^j_i \phi^j + \xi^a|^2 \geq 0 \tag{1.8}$$

ξ^a is the Fayet-Iliopulos coefficient and differs from zero only for abelian factors of the gauge group. The last equality is valid since D and F are auxiliary scalar field of vector and chiral fields respectively. Their value is set by their equations of motion

$$\bar{F}_i = \frac{\partial \bar{W}}{\partial \bar{\phi}_i} \qquad D^a = -g\bar{\phi}T^a\phi - g\xi^a \tag{1.9}$$

Supersymmetric vacua are described by the sets of values of the scalar *VEVs* that give a zero scalar potential. This requirement is equivalent to two different sets of equations, called $D-term$ and $F-term$ equations

$$\bar{F}^i(\phi) = 0 \qquad D^a(\phi, \bar{\phi}) = 0 \tag{1.10}$$

F-term equations are present only if there is a superpotential.
Some theories do not have supersymmetric vacua and in these cases supersymmetry is spontaneously broken.
The *classical moduli space* is the set of solution of these equations for scalar *VEVs* and represents the classical supersymmetric vacua of the theory. Gauge transformation should be taken into account in order to avoid redundance in the description. The moduli space describe physically inequivalent vacua, since the mass spectrum of the theory depends on the *VEVs* of the scalar fields, that differ in every point of the moduli space.

Thanks to supersymmetry radiative corrections do not lift the energy of the ground state and the vacuum remains supersymmetric. As a result, only superpotentials generated from non perturbative dynamics can lift the moduli space. We will see examples of this phenomenon in the analysis of SQCD.

An alternative description of the space of classical *D-flat* directions is given by the space of all holomorphic gauge invariant polynomials of scalar fields modulo classical relations between them [7]. As a result, gauge invariant polynomials of operators parametrize the classical moduli space of the theory. Using this description it's easier to find the moduli space of the theory in consideration. If a superpotential is present, *F-term* equations should be imposed on the gauge invariant polynomial used to describe *D-flat* direction. We will use this convenient description in the next chapters.

1.2 Electric-magnetic duality in three and four dimensions

——— Roba da dire nel capitolo ———
~Phases of gauge theories?(asympt. free, non abelian coulomb/magnetic..)
~Region where duality is correct because of unitary?
~Use of superconformal algebra
~Anomaly free global symmetries
~Different gauge group: scale invariant theory
~Construct the dual from gauge invariant operators: they make sense and a
~Superpotential allowed by charges
~'t'Hooft anomalies
~Different group con be used

1.3 Four dimensional dualities

We will start our analysis on eletric-magnetic duality studying the first pair of theories that were discovered to be dual in [8]. We are gonna analyse the properties of these theory in order to better understand the features of the duality.

Seiberg Duality — $SUN(N)$ SQCD with N_f flavors

The electric theory is a $SU(N_c)$ supersymmetric gauge theory with N_f flavours. Its non anomalous global symmetry group is $SU(N_f)_L \times SU(N_f)_R \times U(1)_B \times U(1)_R$.

The classical lagrangian doesn't have a superpotential and in terms of superfield is written as

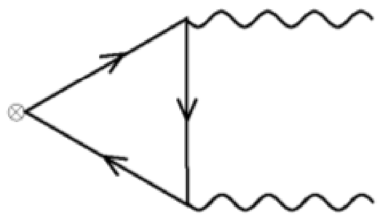
$$\mathcal{L} = \int d^2\theta\, Tr(W_\alpha W^\alpha) + \int d^2\theta\, d^2\bar{\theta}\, Q^\dagger e^V Q + \int d^2\theta\, d^2\bar{\theta}\, \tilde{Q}^\dagger e^{-V} \tilde{Q} \tag{1.11}$$

Q and $\tilde{Q}$ represent left and right quark superfield respectively.
The charges of the fields are summarized in the table below. Mixing the R-symmetry

	$SU(N_c)$	$SU(N_f)_L$	$SU(N_f)_R$	$U(1)_B$	$U(1)_R$
Q	N_c	N_F	1	1	$\frac{N_f-N_c}{N_f}$
$\tilde{Q}$	$\overline{N_c}$	1	$\overline{N_F}$	-1	$\frac{N_f-N_c}{N_f}$

Table 1.1: Charges of the electric theory

with the baryon symmetry we can set the quark R-charges to be equal. Moreover, the value of the R-charge is fixed by the triangle anomaly $SU(N_c)^2\, U(1)_R$, given by diagrams with two exiting gluons and R-symmetry current inserted in the cross



Every fermion in the theory contributes to the anomaly which, as a result, is proportional to the R-charge of the fermion running in the loop and the Dynkin index of its representation

$$R_{gaugino}T(\mathrm{Ad}) + \sum_f (R_f - 1)T(r) = 0$$
$$N_c + \frac{1}{2}2N_f(R_Q - 1) = 0 \quad \rightarrow \quad R_Q = \frac{N_f - N_c}{N_f}$$

where we set the gaugino R-charge to 1 in order to have gluons without charge.

Classical moduli space

Since there is no superpotential, the classical moduli space of the theory is given by *D-terms* only. They can be read from the on-shell lagrangian and are given by

$$D^a = g\left(Q^{*i}T^a Q_i - \tilde{Q}^{*i}T^a\tilde{Q}_i\right) = 0 \tag{1.12}$$

where T^a are the gauge group generators in fundamental or antifundamental representation, color indices are suppressed and i is a flavour index.

After considering gauge and global symmetries, the squark *VEVs*, represented as $N_f \times N_c$ matrices, that satisfy the D-term equation are

$$Q = \tilde{Q} = \begin{pmatrix} a_1 & & & & \vdots \\ & a_2 & & & \vdots \\ & & \ddots & & \vdots \\ & & & a_{N_f} & \vdots \end{pmatrix} \tag{1.13}$$

for $N_f \leq N_c$ and a_i generic and

$$Q = \begin{pmatrix} a_1 & & & \\ & a_2 & & \\ & & \ddots & \\ & & & a_{N_c} \\ \dots & \dots & \dots & \dots \end{pmatrix} \quad \tilde{Q} = \begin{pmatrix} \tilde{a}_1 & & & \\ & \tilde{a}_2 & & \\ & & \ddots & \\ & & & \tilde{a}_{N_c} \\ \dots & \dots & \dots & \dots \end{pmatrix} \tag{1.14}$$

for $N_f \geq N_c$ and $|a_i|^2 - |\tilde{a}_i|^2 = a$ indepdent of i.

We can try to understand why for different number of flavors the theory has qualitatively different moduli spaces. As we said in the previous section, we can solve the D-term equation by finding holomorphic gauge invariant polynomial in the operators and modding out classical relations between them. For $N_f \leq N_c$ we can only construct *mesons* out of squarks

$$M^i_j = Q^i \tilde{Q}_j \tag{1.15}$$

where color indices are summed. Mesons have maximal rank since $N_f \leq N_c$ and there are no classical to impose on them. They can be diagonalized in the same way we put the squarks *VEVs* in diagonal form.

When $N_f \geq N_c$ the mesons cannot have maximal rank anymore, it can be at most N_c. There are additional gauge invariant operators that can be constructed: *baryons*, that are defined as

$$B_{i_1,\ldots,i_{N_f-N_c}} = \epsilon_{i_1,\ldots,i_{N_f-N_c},j_1,\ldots,j_{N_c}} \epsilon^{a_1,\ldots,a_{N_c}} Q^{j_1}_{a_1} \ldots Q^{j_{N_c}}_{a_{N_c}} \tag{1.16}$$

$$\tilde{B}_{i_1,\ldots,i_{N_f-N_c}} = \epsilon^{i_1,\ldots,i_{N_f-N_c},j_1,\ldots,j_{N_c}} \epsilon_{a_1,\ldots,a_{N_c}} \tilde{Q}^{a_1}_{j_1} \ldots \tilde{Q}^{a_{N_c}}_{j_{N_c}} \tag{1.17}$$

Mesons and baryons can be written down explicitly (ignoring null components for baryons)

$$M = \begin{pmatrix} a_1\tilde{a}_1 & & & & \\ & a_2\tilde{a}_2 & & & \\ & & \ddots & & \\ & & & a_{N_c}\tilde{a_{N_c}} & \\ & & & & \end{pmatrix} \tag{1.18}$$

$$B_{12\ldots N_c} = a_1 a_2 \ldots a_{N_c} \tag{1.19}$$

$$\tilde{B}_{12\ldots N_c} = \tilde{a}_1 \tilde{a}_2 \ldots \tilde{a_{N_c}} \tag{1.20}$$

We can see that if the meson has rank less than N_c, then B or $\tilde{B}$ (or both) has to vanish and the other has rank one. If the meson rank is N_c both B and $\tilde{B}$ have rank one.

Moreover that are classical constraints between mesons and baryons, but depend on the number of flavor. For example for $N_f = N_c$ we have $det\,(M) - B\tilde{B} = 0$

Kutasov-Schwimmer duality

1.4 3D dualities

1.4.1 Supersymmetry in 3 dimensions

Spinors in three dimensions have different properties than their four dimension counterpart.
The dimension of the representation in an arbitrary dimension D is given by $2^{\frac{D}{2}}$ for D even, while $2^{\frac{D-1}{2}}$ for D odd. Hence, in three dimension we have a two dimensional representation.

In odd dimensions representations are irreducible and Weyl spinors do not exist: in odd dimensions the chirality operator (γ_{D+1} or γ^*) is proportional to the identity. This is related to the fact that representations in odd dimensions are constructed by taking the representation in one dimension less, which is even, and adding the chirality operator as the D-th matrix in the Clifford algebra.
Gamma matrices can be chosen to be real and we can impose Majorana condition, lowering the degress of freedom of the representation from four (two complex numbers) to two.
Because minimal spinors in three dimensions have half the degrees of freedom of their four dimensional counterpart, the $4d\,\mathcal{N}=1$ superspace formalism can be easily extended to $3d\,\mathcal{N}=2$ theories.
The supersymmetry algebra can be found by dimensional reduction from the $d=4\,\mathcal{N}=1$ supersymmetry algebra.

$$\{Q_\alpha, Q_\beta\} = \{\bar{Q}_\alpha, \bar{Q}_\beta\} = 0 \qquad \{Q_\alpha, \bar{Q}_\beta\} = 2\sigma^\mu_{\alpha\beta} P_\mu + 2i\epsilon_{\alpha\beta} Z \tag{1.21}$$

Since minimal spinors in four dimension are complex these relations include twice the number of minimal supercharges for three dimensional theories. The central charge Z is the component P_3 of the momentum, along the reduced direction. Because of the presence of the central charge in the algebra, now states must satisfy a BPS bound of the form $M \geq Z$, which imply that for massless representations have null central charge.
The automorphism of this algebra is $U(1)_R \simeq SO(2)_R$, as in four dimensions.

Superspace formalism is similar to what we introduced previously. Covariant derivatives are defined in the same way as in $4D$, with proper changes (e.g. gamma matrices). Chiral and real superfield can be defined in three dimensions in the same way we did in four dimensions.

(Anti)Chiral superfields contains *on-shell* one complex scalar and a complex spinor. Vector superfield contains an additional real scalar field with respect from the four dimensional superfield. The scalar field is just the last component of the vector field of the superfield, after dimensional reduction. Its variation under a supersymmetry transformation match those of the vector field because of this reason.

1.4.2 Aharony duality

1.4.3 Kutasov-Schwimmer duality

Appendices

A | Supersymmetry and superfields

A.1 Supersymmetry algebra

The supersymmetry algebra is an extension of the Poincarè group involving anticommutators together with commutators. Since it is not a ordinary Lie algebra, Coleman-Mandula theorem does not apply for theories that are invariant under it.

The supersymmetry algebra is divided into two subalgebras, the bosonic and fermionic part. The bosonic part contains Poincarè Lie algebra $(M_{\mu\nu}, P_\mu)$ while fermionic subalgebra is generated by the *supercharges* $(Q^I_\alpha, \bar{Q}^I_{\dot{\alpha}})$ with $I = 1, \ldots, \mathcal{N}$. When more than one pair of supercharges is present we refer to extended supersymmetry.

The supercharges sit in spinorial representations of the Lorentz group, respectively $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$.

We will not repeat the bosonic subalgebra, since is given by the Poincarè Lie algebra. The fermionic generators satisfy anticommutation rules between themselves and commutation rules with bosonic generators. For this reason, the supersymmetry algebra is defined in mathematical literature as a graded Lie algebra with grade one.
The (anti)commutation rules in four dimensions are

$$[P_\mu, Q^I_\alpha] = 0 \tag{A.1}$$

$$[P_\mu, \bar{Q}^I_{\dot{\alpha}}] = 0 \tag{A.2}$$

$$[M_{\mu\nu}, Q^I_\alpha] = i(\sigma_{\mu\nu})^\beta_\alpha Q^I_\beta \tag{A.3}$$

$$[M_{\mu\nu}, \bar{Q}^I_{\dot{\alpha}}] = i(\bar{\sigma}_{\mu\nu})^{\dot{\alpha}}_{\dot{\beta}} \bar{Q}^I_{\dot{\beta}} \tag{A.4}$$

$$\{Q^I_\alpha, \bar{Q}^J_{\dot{\beta}}\} = 2\sigma^\mu_{\alpha\dot{\beta}} P_\mu \delta^{IJ} \tag{A.5}$$

$$\{Q^I_\alpha, Q^J_\beta\} = \epsilon_{\alpha\beta} Z^{IJ} \qquad Z^{IJ} = -Z^{JI} \tag{A.6}$$

$$\{Q^I_{\dot{\alpha}}, Q^J_{\dot{\beta}}\} = \epsilon_{\dot{\alpha}\dot{\beta}} \left(Z^{IJ}\right)^* \tag{A.7}$$

This set of commutation rules can be found using symmetry arguments and enforcing the consistency of the algebra using the graded Jacobi identity.
It is important to stress the fact that Z^{IJ} are operators that span an invariant subalgebra: they are *central charges*. They play an important role especially in massive representations.

There is an additional symmetry that is not present in the previous commutation rules: R-symmetry. It is an automorphism of the algebra that act on the supercharges. For generic $\mathcal{N}$ the R-Symmetry group is $U(\mathcal{N})$.

A.2 Representations

Since the supercharges do not commute with Lorentz generators, their action on a state will result in a state with different spin: they generate a symmetry between bosons and fermions.
Representations of supersymmetry contain particle with different spin but same mass and they are organized in supermultiplets. The mass of particles in the same multiplet must be the same because P^2 is still a Casimir operator of the supersymmetry algebra, while the Pauli-Lubanski operator W^2 isn't anymore.

Moreover, the supersymmetry algebra imposes that every state must have positive energy and that every supermultiplet must contain the same number of bosonic and fermionic degrees of freedom *on-shell* .

Various supermultiplets exist and their properties depend on the number of supercharges of the theory and on what they represent e.g matter, glue or gravity.

Massless supermultiplet are typically shorter than massive multiplet because in the massless case half of the supercharges are represented trivially. Massive representation of extended supersymmetry can be shortened in case some of the central charges of the algebra are equal to twice the mass of the multiplet. These states are usually called (ultra)short multiplet or BPS states.

We will introduce the multiplets that can be defined for $4d\,\mathcal{N}=1$ theories and only later we will explain the differences with $3d\,\mathcal{N}=2$ theories. Representations are similar because in both cases we have the same number of supercharges.

For four dimensional theories, we can define two different multiplet that are invariant under supersymmetry transformations. The matter or chiral multiplet contains a complex scalar (*squark*) and a Weyl fermion (*quark*). It identifies the matter content of the theory. The vector or gauge multiplet contains a Weyl fermion (*gaugino*) and a vector (*gluon or photon*). Particles in the same multiplet transform in the same representation of global or gauge symmetries. For this reason the gaugino cannot represent matter.

A representation of these multiplets on fields can be easily found using the *superspace* formalism that we will introduce in the next section. In this formalism it is possible to represent fields that are *off-shell*, in contrast with multiplets that we introduced previously that are *on-shell* since they represent states in Hilbert space.

A.2.1 Superfields and superspace in four dimensions

Supersymmetry representations on fields can be found more sistematically using the formulation of *superspace* instead of acting directly with supercharges and verifying that the algebra closes.

A simple formulation of superspace exist for theories with 4 supercharges while for theories with a bigger number of supercharges its definition is much more complex. We will be interested only in theories with 4 supercharges such as theories in 4D with $\mathcal{N}=1$ or 3D with $\mathcal{N}=2$.

Superspace can be seen as the extension of Minkowsky space with *fermionic coordinates* i.e. *Grassman numbers* θ^α , $\bar{\theta}^{\dot{\alpha}}$. They anticommute between themselves and commute with everything else.

$$\{\theta^\alpha, \theta^\beta\} = 0 \quad \{\bar\theta^{\dot\alpha}, \bar\theta^{\dot\beta}\} = 0 \quad \{\theta^\alpha, \bar\theta^{\dot\beta}\} = 0 \qquad \alpha, \dot\alpha = 1, 2 \tag{A.8}$$

Derivation and integration in Grassmann variables are summarized by these rules

$$\partial_\alpha = \frac{\partial}{\partial\theta_\alpha} \quad \partial^\alpha = -\epsilon^{\alpha\beta}\partial_\beta \quad \bar\partial_{\dot\alpha} = \frac{\partial}{\partial\bar\theta^{\dot\alpha}} \quad \bar\partial^{\dot\alpha} = -\epsilon^{\dot\alpha\dot\beta}\partial_{\dot\beta} \quad \partial_\alpha\theta^\beta = \delta_\alpha^\beta \quad \partial_\alpha\bar\theta^{\dot\alpha} = 0 \tag{A.9}$$

$$\int d\theta = 0 \quad \int d\theta\,\theta = 1 \quad d^2\theta = \frac{1}{2}d\theta^1 d\theta^2 \quad \int d^2\theta = \frac{1}{4}\epsilon^{\alpha\beta}\partial_\alpha\partial\beta \tag{A.10}$$

For a more detailed introduction on Grassmann numbers and their properties see [**?**].

Using Grassmann numbers we can write the anticommutators in the supersymmetry algebra as commutators defining $\theta Q = \theta^\alpha Q_\alpha$ and $\bar\theta\bar Q = \bar\theta^{\dot\alpha}\bar Q_{\dot\alpha}$

$$[\theta Q, \bar\theta\bar Q] = 2\theta\sigma^\mu\bar\theta P_\mu \quad , \quad [\theta Q, \theta Q] = [\bar\theta\bar Q, \bar\theta\bar Q] = 0 \tag{A.11}$$

Using this trick we are able to represent the supersymmetry algebra as a Lie algebra. An element of the superPoincarè group can be found exponentiating the generators

$$G(x, \theta, \bar\theta, \omega) = \exp\left(ixP + i\theta Q + i\bar\theta\bar Q + \frac{1}{2}i\omega M\right) \tag{A.12}$$

The superspace is defined as the 4+4 dimensions group coset

$$M_{4|1} = \frac{\text{SuperPoincarè}}{\text{Lorentz}} \tag{A.13}$$

in analogy to Minkowsky space that can be defined as the coset between Poincarè and Lorentz groups.

A generic point in superspace is parametrized by $(x^\mu, \theta^\alpha, \bar\theta^{\dot\alpha})$. A superfields is a field in superspace i.e. function of the superspace coordinates. Since θ coordinates anticommute, the expansion of a superfield in fermionic coordinates stops at some point. The most general superfield $Y = Y(x, \theta, \bar\theta)$ is given by

$$\begin{aligned} Y(x, \theta, \bar\theta) = f(x) + \theta\psi_1(x) + \bar\theta\bar\psi_2(x) + \theta\theta g_1(x) + \bar\theta\bar\theta g_2(x) \\ + \theta\sigma^\mu\theta v_\mu(x) + \theta\theta\bar\theta\bar\lambda(x) + \bar\theta\bar\theta\theta\rho(x) + \theta\theta\bar\theta\bar\theta s(x) \end{aligned} \tag{A.14}$$

fields with uncontracted θ such as $\psi_1, \psi_2, \lambda, \rho$ are spinors while v_μ is a vector.

Supercharges can be represented as differential operators that act on superfield. Their expression is

$$\begin{cases} Q_\alpha &= -i\partial_\alpha - \sigma^\mu_{\alpha\dot\beta}\bar\theta^{\dot\beta}\partial_\mu \\ \bar Q_{\dot\alpha} &= +i\dot\partial_{\dot\alpha} + \theta^\beta\sigma^\mu_{\beta\dot\alpha}\bar\theta^{\dot\beta}\partial_\mu \end{cases} \tag{A.15}$$

An infinitesimal supersymmetry transformation on a superfield is defined by

$$\delta_{\epsilon,\bar\epsilon}Y = \left(i\epsilon Q + i\bar\epsilon\bar Q\right)Y \tag{A.16}$$

The powerfulness of the superfield formalism is due to the fact that an integral in full superspace coordinates of a superfield is supersymmetric invariant.

$$\delta_{\epsilon,\bar{\epsilon}} \int d^4x \, d^2\theta \, d^2\bar{\theta} \; Y = \int d^4x \, d^2\theta \, d^2\bar{\theta} \; \delta_{\epsilon,\bar{\epsilon}} Y = 0 \tag{A.17}$$

The first equality holds because the Grassmann measure is inviariant under translation while the second is true because we can see that the variation of the superfield is either killed by the integration in the θ variables or is proportional to a spacetime derivative that does not contribute after integration in space.

Using this fact we can construct supersymmetric invariant lagrangians by integrating superfields in superspace. Clearly, in order to find a physically significant lagrangian we should choose the superfield we wish to integrate wisely. More importantly, we want to use irreducible representation of supersymmetry i.e. the supermultiplets we introduced before. We need to find conditions that can be imposed on a general superfield that are invariant under a supersymmetry transformation.

Chiral superfield

One way to achieve this goal is to find an operator that commute with the supercharges and annihilate the superfield An example of such operator is the *covariant derivative*

$$\begin{cases} D_\alpha = & \partial_\alpha + i\sigma^\mu_{\alpha\dot{\beta}}\bar{\theta}^{\dot{\beta}}\partial_\mu \\ \bar{D}_{\dot{\alpha}} = & \dot{\partial}_{\dot{\alpha}} + i\theta^\beta\sigma^\mu_{\beta\dot{\alpha}}\bar{\theta}^{\dot{\beta}}\partial_\mu \end{cases} \tag{A.18}$$

We can define a (anti)chiral superfield Φ

$$\bar{D}_{\dot{\alpha}}\Phi = 0 \quad \text{chiral} \qquad\qquad D_\alpha \Psi = 0 \quad \text{anti-chiral} \tag{A.19}$$

This condition reduces the number of components of the superfield. It can be easily demonstrated that if Ψ is chiral, then $\bar{\Psi}$ is anti-chiral. As a result a chiral field cannot be real.

The expansion of a chiral fields in components is give by

$$\Phi(x,\theta,\bar{\theta}) = \phi(x) + \sqrt{2}\theta\psi(x) + i\theta\sigma^\mu\bar{\theta}\partial\mu\phi(x) - \theta\theta F(x) - \frac{i}{\sqrt{2}}\theta\theta\partial\mu\psi(x)\sigma^\mu\bar{\theta} - \frac{1}{4}\theta\theta\bar{\theta}\bar{\theta}(x) \tag{A.20}$$

We can see that a chiral field is composed by three fields: two complex scalars (ϕ and F) and a spinor (ψ).

The chiral superfield identifies the matter multiplet we introduced previously. It contains an additional bosonic field ($F(x)$) that is present because superfields provide an *off-shell* representation of supersymmetry and it is needed in order to close the algebra. It is called *auxiliary field* because it will not have kinetic terms in every Lagrangian that can be constructed.

Real or Vector Field

We can impose that the superfield is real. In this way we find the *real* or *vector* multiplet. Its general expression in component is messy and a simplification can be made noting that $\Phi + \bar{\Phi}$ is a vector superfield if Φ is chiral. Choosing an appropriate chiral field, the real superfield can be put in what is called Wess-Zumino gauge. We stress the fact that the Wess-Zumino gauge is not supersymmetric invariant: after a supersymmetry transformation the vector superfield acquire its general expression involving many other field components. In this gauge the vector superfield can be written as

$$V_{WZ}(x, \theta, \bar{\theta}) = \theta\sigma^{\mu}\bar{\theta}v_{\mu}(x) + i\theta\theta\bar{\theta}\bar{\lambda}(x) - i\bar{\theta}\bar{\theta}\theta\lambda(x) + \frac{1}{2}\theta\theta\bar{\theta}\bar{\theta}D(x) \tag{A.21}$$

The vector superfields represents the vector multiplet (which contains radiation) and similarly to the chiral superfields contains an auxiliary field ($D(x)$).

A.2.2 R-symmetry

R-symmetry was first introduced with the supersymmetry algebra. For the theories we will consider in superspace it is given by a global $U(1)_R$. It is defined by as a transformation of the Grassmann coordinates

$$\theta \ \to \ e^{i\alpha}\theta \qquad \bar{\theta} \ \to \ e^{-i\alpha}\bar{\theta} \tag{A.22}$$

α parametrized the transformation. As a result supercharges transform under the transformation

$$Q \to \ e^{-i\alpha}Q \qquad \bar{Q} \ \to \ e^{+i\alpha}\bar{Q} \tag{A.23}$$

From this we find the commutator relations between supercharges and R-symmetry generator R

$$[R, Q] = -Q \qquad [R, \bar{Q}] = \bar{Q} \tag{A.24}$$

The R-charge of a superfield is defined by

$$Y(x, \theta, \bar{\theta}) \ \to \ e^{iR_Y\alpha}Y(x, \theta, \bar{\theta}) \tag{A.25}$$

Different component field in the superfield have different R-charge and are related because of A.24. For a chiral field we have

$$R[\phi] = R\,[\Phi] \qquad R[\psi] = R\,[\Phi] - 1 \qquad R[F] = R\,[\Phi] - 2 \tag{A.26}$$

The corresponding antichiral field carry opposite charges.

A.3 Supersymmetric actions

We will use the property we introduced in A.17 to generate supersymmetric invariant lagrangians. We start our analysis with chiral superfields. Since lagrangians are quadratic in the fields and must be real, our best shot at finding the correct superfield to integrate is given by $\bar{\Phi}\Phi$.

$$\mathcal{L}_{kin} \ = \int d^2\theta d^2\bar{\theta}\ \bar{\Phi}\Phi = \partial_{\mu}\bar{\phi}\partial^{\mu}\phi + \frac{i}{2}\left(\partial_{\mu}\psi\sigma^{\mu}\bar{\psi} - \psi\sigma^{\mu}\partial_{\mu}\bar{\psi}\right) + \bar{F}F + \text{total derivative} \tag{A.27}$$

We found the correct kinetic terms for the scalar and spinor field. The auxiliary field doesn't have kinetic terms as predicted. Many action can be find using a generalization of the equation above. It is called *Kahler* potential

$$K(\bar{\Phi}, \Phi) = \sum_{m,n=1}^{\infty} c_{m,n} \bar{\Phi}^m \Phi^n \qquad \text{where} \quad c_{m,n} = c_{n,m}* \tag{A.28}$$

The condition on the coefficient is imposed by the requirement of a real action.

Actually we can produce supersymmetric actions also integrating *chiral* superfields in half-superspace coordinates. We define the *superpotential* to be a holomorphic function of Φ

$$\mathcal{L}_{int} = \int d^2\theta d^2 \; W(\Phi) + \int d^2\bar{\theta} \; \bar{W}(\bar{\Phi}) = \sum_{i=1}^{\infty} \int d^2\theta \; \lambda_n \Phi^n + \int d^2\bar{\theta} \; \lambda_n^{\dagger} \bar{\Phi}^n \tag{A.29}$$

In order to have a real lagrangian we added the hermitian conjugate.

Mixed terms with product of chiral and anti-chiral superfield are not present since they would be generic superfields and would not yield a supersymmetric lagrangian. In fact if $W(\Phi)$ is holomorphic and Φ is a chiral superfield, $W(\Phi)$ is a chiral superfield

$$\bar{D}_{\dot{\alpha}} W(\Phi) = \frac{\partial W}{\partial \Phi} \bar{D}_{\dot{\alpha}} \Phi + \frac{\partial W}{\partial \bar{\Phi}} \bar{D}_{\dot{\alpha}} \bar{\Phi} = 0 \tag{A.30}$$

and yield a proper lagrangian upon integration in $d^2\theta$.

Since the superpotential is integrated only in half superspace coordinates it need to be charged in an opposite way in order to provide a lagrangian invariant under R-symmetry. Remembering that

$$R[\theta] = 1 \qquad [\bar{\theta}] = -1 \qquad R[d\theta] = -1 \qquad R[d\bar{\theta}] = 1 \tag{A.31}$$

It's easy to see that

$$R[W(\Phi)] = 2 \qquad R[\bar{W}(\bar{\Phi})] = -2 \tag{A.32}$$

For this reason in most situations the superpotential fix the supercharges of the fields.

The lagrangian of super Yang-Mills theories is given by

$$\mathcal{L}_{SYM} = \frac{1}{32\pi i} \left(\int d^2\theta \left(\frac{\theta_{YM}}{2\pi} + \frac{4\pi i}{g^2} \right) W_\alpha W^\alpha \right) = \tag{A.33}$$

$$= \mathrm{Tr} \left[-\frac{1}{4} F_{\mu\nu} F^{\mu nu} - i\lambda\sigma^\mu D_\mu \bar{\lambda} + \frac{1}{2} D^2 \right] + \frac{\theta_{YM}}{32\pi^2} g^2 \mathrm{Tr} F_{\mu\nu} \tilde{F}^{\mu nu} \tag{A.34}$$

where we the chiral superfield as

$$W_\alpha = -\frac{1}{4} \bar{D}\bar{D} \left(e^{-2gV} D_\alpha e^{2gV} \right) \qquad \bar{W}_{\dot{\alpha}} = -\frac{1}{4} DD \left(e^{2gV} \bar{D}_{\dot{\alpha}} V e^{-2gV} \right) \tag{A.35}$$

It can be demonstrated that W_α is chiral and is invariant under the supergauge transformation $V \rightarrow V + \bar{\Phi} + \Phi$ while the vector superfield V was not.

From a perturbative point of view the inclusion of the term proportional to $\theta_{YM} Tr F_{\mu\nu} \tilde{F}^{\mu nu}$ has no effect since it is proportional to a total derivative. It is a parity violating term

that differs from zero in non trivial topological configurations of the field (instantons).
The matter lagrangian we introduced is not invariant under gauge transformation. The correct invariant lagrangian is given by

$$\mathcal{L}_{matter} = \int d^2\theta d^2\bar{\theta}\ \bar{\Phi} e^{2gV} \Phi \tag{A.36}$$

The superpotential is not automatically invariant under gauge transformation. As a result only certain expression are allowed.
There's an additional supersymmetric invariant lagrangian that can be constructed in a gauge theory when the gauge group contains abelian factors. It is called Fayet-Iliopulos term and can be present for every ideal A of the gauge group

$$\mathcal{L}_{FI} = \sum_A \xi_A \int d^2\theta d^2\bar{\theta} V^A = \frac{1}{2} \sum_A \xi_A D^A \tag{A.37}$$

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