4D to 3D reduction of Seiberg duality for $SU(N)$ susy gauge theories with adjoint matter: a partition function approach

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1 Four dimensional dualities

—— INTRODUCTION OUTLINE ——
~ More symmetry = more tools for studying t\heories
~ Milder divergences
~ Renormalization constraints
~ Non renormalization theorems (perturbative)
~ Holomorphicity, couplings as background fields
~ Exact results (superpotential, exact beta function)
~ Moduli space

1.1 Introduction

Supersymmetric quantum field theories enjoy an enlarged group of symmetries compared to other field theories. Since the symmetry group is a non trivial combination of internal and spacetime symmetries, they have many unexpected features and new techniques were found to study them. Almost all of the new tools found are available only for supersymmetric field theories, making them the theater for many advances in physics.

A more technical introduction on supersymmetry and its representation on fields can be found in appendix A.

In this section we will analyse more advanced features of supersymmetric field theories that has been used intensively in the discovery and in the analysis of electric magnetic duality and its generalisations.

1.1.1 General renormalization properties

A remarkable feature of supersymmetry is the constraint that the additional symmetry imposes on the renormalization properties of the theories.

One of the first aspects that brought attention to supersymmetry was that divergences of loop diagrams were milder because of the cancellation between diagrams with bosons and fermions running in the loops.

Nowadays we know powerful theorems that restrict the behaviour of supersymmetric field theories during renormalization. In order to preserve supersymmetry, the renormalization process has to preserve the Hilbert space structure. For example the wave function renormalization of different *particles* inside a multiplet must be the same, otherwise the renormalized lagrangian is not supersymmetric invariant anymore.

Moreover, in the supersymmetry algebra P^2 is still a Casimir operator i.e. it commutes with every operator in the algebra: particles in the same multiplet must have the same mass. Renormalization cannot break this condition, otherwise it would break supersymmetry.

For a *Super Yang Mills* theory with $\mathcal{N}=1$ we have the additional requirement that gV, where g is the coupling and V is the vector superfield, cannot be renormalized by symmetry considerations.

Adding more supersymmetry the wave function renormalization of the various field are even more constrained by symmetry. For example, for $\mathcal{N}=4$ *SYM* the fields and the coupling are not renormalized at all.

Beta function for SYM and SQCD

Another nice feature of supersymmetric field theories is that some quantities can be calculated exactly. The first object of this kind that we encounter is the β function of four dimensional $\mathcal{N}=1$ *Super Yang Mills* and *Super QCD* theories.

It is given by the *NSVZ β function*

$$\beta(g) = -\frac{g^3}{16\pi^2}\left[3\,T(Adj) - \sum_i T(R_i)(1-\gamma_i)\right]\left(1-\frac{g^2\,T(Adj)}{8\pi^2}\right)^{-1} \qquad \alpha = \frac{g^2}{4\pi} \tag{1.1}$$

where γ_i are the anomalous dimensions of the matter fields and $T(R_i)$ are the Dynkin indices of their representation.
The anomalous dimensions are defined as

$$\gamma_i = -\frac{d\,\log(Z_i)}{d\,\log(\mu)} \tag{1.2}$$

where Z_i is the wave function renormalization coefficient. In general, anomalous dimensions are not know exactly.
The Dynkin index of the gauge group $SU(N)$ are

$$T(N) = \frac{1}{2} \qquad T(Adj) = N \tag{1.3}$$

The *NSVZ β function* was first calculated using instanton methods in [1]. Over the years it has been calculated in other ways using the fact that the action is holomorphic in the complexified coupling

$$\tau = \frac{4\pi i}{g_c^2} + \frac{\theta_{YM}}{2\pi} \tag{1.4}$$

Using the holomorphic coupling the action for the vector field is written as

$$\mathcal{L}_h(V_h) = \frac{1}{16\pi i}\int d^2\theta\;\tau\;W^a(V_h)W^a(V_h) + h.c. \tag{1.5}$$

whereas with the canonical normalization for the vector field is

$$\mathcal{L}_c(V_c) = \frac{1}{16\pi i}\int d^2\theta\left(\frac{4\pi i}{g_c^2} + \frac{\theta_{YM}}{2\pi}\right)W^a(g_cV_c)W^a(g_cV_c) + h.c. \tag{1.6}$$

Using the canonical normalization g_cV_c is a real superfield, imposing that g_c is real. For this reason with the canonical normalization the lagrangian is not holomorphic in τ.

Thanks to holomorphicity, the holomorphic coupling is only renormalized at one-loop and the β function can be computed exactly at one loop but its expression is different from *NSVZ β function*. The cause of this mismatch is that the *NSVZ β function* is defined using the canonical (or physical) coupling constant and receives contribution from all orders in perturbation theory.

At first sight, one should expect that the expressions should match since the first two orders in α of the β function are scheme independent. The reason why the two expressions differ is that the Jacobian of the transformation between canonical and holomorphic normalization is anomalous. Once the anomaly is taken into account the two expressions for the β function agree.
An explicit calculation relating the comparison between the two different approach can be found in [2].

1.1.2 Superpotential: holomorphy and non-renormalization

Other than renormalization constraints, supersymmetry provides non-renormalization theorems for certain objects, such as the superpotential. In [3] it has been demonstrated that the superpotential is tree-level exact, i.e. it does not receive correction in perturbation theory. However it usually receive contributions from non perturbative dynamics.

Perturbative calculations can be done using supergraphs, i.e. Feynman diagrams with superfields. The advantage of this approach is that supersymmetry is explicit and many semplification occur naturally. The demonstration is based on the fact that for general supersymmetric field theories, supergraph loops diagrams with n external leg yield a term that can be written in the form

$$\int d^4x_1 \dots d^4x_n d^2\theta d^2\bar{\theta}\; G(x_1, \dots, x_n)\, F_1(x_1, \theta, \bar{\theta}) \dots F_n(x_n, \theta, \bar{\theta}) \tag{1.7}$$

where $G(x_1, \dots, x_n)$ is translationally invariant function.
The importance of this result is that all contribution from Feynman diagrams are given by a single integral over full superspace ($d^2\theta d^2\bar{\theta}$) whereas the superpotential must be written as an integral in half-superspace($d^2\theta$ only) of chiral fields. Exploiting the fact that a product of chiral fields is a chiral field, the most general form of a superpotential is

$$W(\lambda, \Phi) = \sum_{n=1}^{\infty} \left(\int d^2\theta\, \lambda_n \Phi^n \;+\; \int d^2\bar{\theta}\, \lambda_n^\dagger \bar{\Phi}^n \right) \tag{1.8}$$

The second term of the superpotential is added in order to give a real lagrangian after the integration in superspace. From the definition, we can see that the superpotential is holomorphic in the fields and in the coupling constants.

Fifteen years later, Seiberg [4] provided a proof of this theorem using a different approach. He noted that the coupling constants λ_n can be treated as background fields, i.e. chiral superfields with no dynamics.

Using this observation we can assign transformation laws to the coupling constants, making the lagrangian invariant under a larger symmetry. Fields and coupling constants are charged under this symmetry and only certain combinations of them can appear in the superpotential. In addition, in a suitable weak coupling limit the effective superpotential must be identical with the tree-level one. These conditions, taken together, fix the expansion of the superpotential to the expression of the tree-level potential. A more detailed discussion can be found in [5] and [6].

1.1.3 Moduli space

Supersymmetric field theories have a larger set of vacua compared to ordinary field theories because of the presence of many scalar fields in the supermultiplets.

Lorentz invariance of the vacuum forbids fields with spin different from zero to acquire a vacuum expectation value. With the same reasoning, derivatives of scalar fields must be set to zero because of translational invariance of the vacuum. The scalar potential is the only term in the Lagrangian and in the Hamiltonian that can differ from zero. As a result, the minimums of the scalar potential are in one-to-one correspondence with the states of minimal energy of the theory.

For $4D$ $\mathcal{N}=1$ gauge theories with matter, the scalar potential for the squarks reads

$$V(\phi_i,\bar{\phi}_j) = F\bar{F} + \frac{1}{2}D^2 \overset{on-shell}{=} \frac{\partial W}{\partial \phi_i}F^i \frac{\partial \bar{W}}{\partial \bar{\phi}_i}\bar{F}^i + \frac{g^2}{2}\sum_a |\bar{\phi}_j (T^a)^j_i \phi^j + \xi^a|^2 \geq 0 \tag{1.9}$$

ξ^a is the Fayet-Iliopulos coefficient and differs from zero only for abelian factors of the gauge group. The last equality is valid since D and F are auxiliary fields with no dynamics. Their value is set by their equations of motion

$$\bar{F}_i = \frac{\partial \bar{W}}{\partial \bar{\phi}_i} \qquad D^a = -g\bar{\phi}T^a\phi - g\xi^a \tag{1.10}$$

Supersymmetric vacua are described by the sets of values of the scalar *VEVs* that give a zero scalar potential. This requirement is equivalent to two different sets of equations, called $D - term$ and $F - term$ equations

$$\bar{F}^i(\phi) = 0 \qquad D^a(\phi,\bar{\phi}) = 0 \tag{1.11}$$

F-term equations are present only if there is a superpotential while the *D-term* equations are always present.
If the minimum of the scalar potential is different from zero the vacuum is not supersymmetric. In this case supersymmetry is spontaneously broken. Another possible situation is that the scalar potential has no minimum at all: the theory does not have any stable vacua.

The *classical moduli space* is the set of solution of these equations for scalar *VEVs* and represents the classical supersymmetric vacua of the theory. Gauge transformation should be taken into account in order to avoid redundance in the description. The moduli space describe physically inequivalent vacua, since the mass spectrum of the theory depends on the *VEVs* of the scalar fields, that differ in every point of the moduli space.

Because of supersymmetry radiative corrections do not lift the energy of the ground state and the vacuum remains supersymmetric. As a result, only superpotentials generated from non perturbative dynamics can lift the moduli space. We will see examples of this phenomenon in the analysis of SQCD.

An alternative description of the space of classical *D-flat* directions is given by the space of all holomorphic gauge invariant polynomials of scalar fields modulo classical relations between them [7]. As a result, gauge invariant polynomials of operators parametrize the classical moduli space of the theory. Using this description it's easier to find the moduli space of the theory in consideration. If a superpotential is present, *F-term* equations should be imposed on the gauge invariant polynomial used to describe *D-flat* direction. We will use this convenient description in the next chapters.

1.1.4 Phases of gauge theories

The dynamics of gauge theories can be classified according to the low-energy effective potential $V(R)$ between two test charges separated by a large distance R. The possible forms of the potential, up to addictive constant, are

$$\text{Coulomb} \qquad V(R) \sim \frac{1}{R} \tag{1.12}$$

$$\text{free electric} \qquad V(R) \sim \frac{1}{R\,\log(R\Lambda)} \tag{1.13}$$

$$\text{free magnetic} \qquad V(R) \sim \frac{\log(R\Lambda)}{R} \tag{1.14}$$

$$\text{Higgs} \qquad V(R) \sim \text{constant} \tag{1.15}$$

$$\text{confining} \qquad V(R) \sim \sigma\, R \tag{1.16}$$

The first three phases feature massless gauge fields and their potential is $V(R) \sim g^2(R)/R$ and they differ because of the renormalization of the charge in the IR. In the Coulomb phase, $g^2_{IR} = \text{constant}$, while in the free abelian/non-Abelian phase the coupling constant goes to zero as $g^2(R) \sim 1/\log(R\,\Lambda)$. The free electric phases is possible for abelian or non-Abelian theories.In the latter case for asymptotically free theories it's necessary that the renormalization group has a non trivial infrared fixed point. The free magnetic phases is generated by magnetic monopoles acting as source of the field. Since magnetic and electric charges are related by Dirac quantization condition, the running of the coupling constant for magnetic monopoles is the inverse of electric charges.
The situation is completely different in the last two cases. In the Higgs phase gauge fields are massive and the potential is given by a Yukawa potential, exponential suppressed at long distances that results in a constant value. The confining phase can be described by tube of confined gauge flux between the charges which, at large distances, acts as a string with costant tension, yielding a linear potential.

1.2 $SU(N)$ SQCD with N_f flavours

We will start our analysis on eletric-magnetic duality studying the first pair of theories that were discovered to be dual in [8]. We are gonna analyse the properties of these theory in order to better understand the features of the duality.

The electric theory is a $SU(N_c)$ supersymmetric gauge theory with N_f flavours. Its non anomalous global symmetry group is

$$SU(N_f)_L \times SU(N_f)_R \times U(1)_B \times U(1)_R$$

. The classical lagrangian doesn't have a superpotential and in terms of superfield is written as

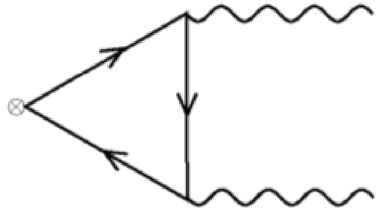
$$\mathcal{L} = \tau \int d^2\theta\, Tr(W_\alpha W^\alpha) + \int d^2\theta\, d^2\bar{\theta}\, Q^\dagger e^V Q + \int d^2\theta\, d^2\bar{\theta}\, \tilde{Q}^\dagger e^{-V} \tilde{Q} \tag{1.17}$$

Q and $\tilde{Q}$ represent left and right quark superfield respectively.
The charges of the fields are summarized in the table below.

	$SU(N_c)$	$SU(N_f)_L$	$SU(N_f)_R$	$U(1)_B$	$U(1)_R$
Q	N_c	N_F	1	1	$\frac{N_f-N_c}{N_f}$
$\tilde{Q}$	$\overline{N_c}$	1	$\overline{N_F}$	-1	$\frac{N_f-N_c}{N_f}$

Table 1.1: Charges of the electric theory

Mixing the R-symmetry with the baryon symmetry we can set the quark R-charges to be equal. Since there's no superpotential their R-charges are not fixed relative to each other.
The axial symmetry $U(1)_A$ is anomalous and cannot be present. However it can mix with the *natural* $U(1)_{R_0}$ R-symmetry of the theory, which is anomalous by itself, in order to give a non anomalous $U(1)_R$.
In fact, the value of the R-charge is fixed by the triangle anomaly $SU(N_c)^2\, U(1)_R$, given by diagrams with two exiting gluons and R-symmetry current inserted in the cross



Every fermion in the theory contributes to the anomaly which, as a result, is proportional to the R-charge of the fermion running in the loop and the Dynkin index of its representation

$$R_{gaugino} T(\text{Ad}) + \sum_f (R_f - 1) T(r) = 0$$

$$N_c + \frac{1}{2} 2N_f(R_Q - 1) = 0 \quad \rightarrow \quad R_Q = \frac{N_f - N_c}{N_f}$$

where we set the gaugino R-charge to 1 in order to have gluons without charge.

Classical moduli space

Since there is no superpotential, the classical moduli space of the theory is given by *D-terms* only. They can be read from the on-shell lagrangian and are given by

$$D^a = g\left(Q^{*i} T^a Q_i - \tilde{Q}^{*i} T^a \tilde{Q}_i\right) = 0 \tag{1.18}$$

where T^a are the gauge group generators in fundamental or antifundamental representation and i is a flavour index.

After considering gauge and global symmetries, the squark *VEVs*, represented as $N_f \times N_c$ matrices, that satisfy the D-term equation are for $N_f \leq N_c$ and a_i generic

$$Q = \tilde{Q} = \begin{pmatrix} a_1 & & & & \vdots \\ & a_2 & & & \vdots \\ & & \ddots & & \vdots \\ & & & a_{N_f} & \vdots \end{pmatrix} \tag{1.19}$$

for $N_f \geq N_c$ and $|a_i|^2 - |\tilde{a}_i|^2 = a$ independent of i.

$$Q = \begin{pmatrix} a_1 & & & \\ & a_2 & & \\ & & \ddots & \\ & & & a_{N_c} \\ \dots & \dots & \dots & \dots \end{pmatrix} \quad \tilde{Q} = \begin{pmatrix} \tilde{a}_1 & & & \\ & \tilde{a}_2 & & \\ & & \ddots & \\ & & & \tilde{a}_{N_c} \\ \dots & \dots & \dots & \dots \end{pmatrix} \tag{1.20}$$

For $N_f \leq N_c$ in a generic point of the moduli space the gauge group is broken to $SU(N_c - N_f)$ while for $N_f \geq N_c$ is broken completely. The gauge group breaks through the super Higgs mechanism, where every broken generator gets absorbed by the (originally) massless vector superfield in order to make a massive vector superfield [1] The mass of the gauge superfield is given by the VEVs of the squarks.

As we said in the previous section, we can study the classical moduli space by finding holomorphic gauge invariant polynomial in the operators and modding out classical relations between them. For $N_f \leq N_c$ we can only construct *mesons* out of squarks

$$M^i_j = Q^i \tilde{Q}_j \tag{1.21}$$

where color indices are summed. Mesons have maximal rank since $N_f \leq N_c$ and there are no classical to impose on them. They can be diagonalized in the same way we put the squarks *VEVs* in diagonal form.
When $N_f \geq N_c$ the mesons cannot have maximal rank anymore, it can be at most N_c. There are additional gauge invariant operators that can be constructed: *baryons*, that are defined as

$$B_{i_1,\dots,i_{N_f-N_c}} = \epsilon_{i_1,\dots,i_{N_f-N_c},j_1,\dots,j_{N_c}} \epsilon^{a_1,\dots,a_{N_c}} Q^{j_1}_{a_1} \dots Q^{j_{N_c}}_{a_{N_c}} \tag{1.22}$$

$$\tilde{B}_{i_1,\dots,i_{N_f-N_c}} = \epsilon^{i_1,\dots,i_{N_f-N_c},j_1,\dots,j_{N_c}} \epsilon_{a_1,\dots,a_{N_c}} \tilde{Q}^{a_1}_{j_1} \dots \tilde{Q}^{a_{N_c}}_{j_{N_c}} \tag{1.23}$$

Mesons and baryons can be written down using the *VEVs* we found solving the *D-term*

[1]Remember that massive representation of supersymmetry have twice the degrees of freedom of massless ones, because in the latter half of the supercharges are represented trivially.

equations (ignoring null components for baryons)

$$M = \begin{pmatrix} a_1\tilde{a}_1 & & & \\ & a_2\tilde{a}_2 & & \\ & & \ddots & \\ & & & a_{N_c}\tilde{a_{N_c}} \\ & & & \end{pmatrix} \tag{1.24}$$

$$B \simeq a_1 a_2 \dots a_{N_c} \tag{1.25}$$

$$\tilde{B} \simeq \tilde{a}_1 \tilde{a}_2 \dots \tilde{a_{N_c}} \tag{1.26}$$

We can see that if the mesons have rank less than N_c, then B or $\tilde{B}$ (or both) has to vanish and the other has rank one. If the mesons' rank is N_c both B and $\tilde{B}$ have rank one.

That are classical constraints that should be imposed between mesons and baryons, but depend on the number of flavourssis . For example for $N_f = N_c$ the classical constraint is $det\,(M) - B\tilde{B} = 0$.

Singularities of the moduli space can be investigated using the gauge invariant description we just introduced. The part of the lagrangian that describes flat directions can be written in terms of mesons and baryons. The lagrangian involving mesons features a non trivial Kahler potential that reads

$$K = 2\mathrm{Tr}\sqrt{M^\dagger M} \tag{1.27}$$

that generate a singular metric whenever the meson matrix is not invertible. This happens when some of the VEVs are zero, i.e. in points of the moduli space of enhanced gauge symmetry. The appearance of this singularities is related to the fact that some (or all) gluons are now massless and should be included in the low-energy description.

Quantum moduli space

Quantum dynamics modifies the structure of the moduli space of the theory in a different way depending on the number of flavours.

$\mathbf{N_f = 0}$

For pure *Super Yang Mills*, i.e. no quarks, the theory exihibit a discrete set of N_c vacua. Without quarks a non anomalous R-symmetry cannot be found, and the R-symmetry is broken down to the discreet symmetry $\mathbb{Z}_{2N_c}$. Using holomorphy and symmetry arguments, the form of the non perturbative potential can be found and it can be shown that induces the gaugino to condensate, meaning that

$$\langle \lambda\lambda \rangle = -\frac{32\pi^2}{N_c} a\,\Lambda^3 \tag{1.28}$$

where Λ is the dynamically generated scale of the theory defined as

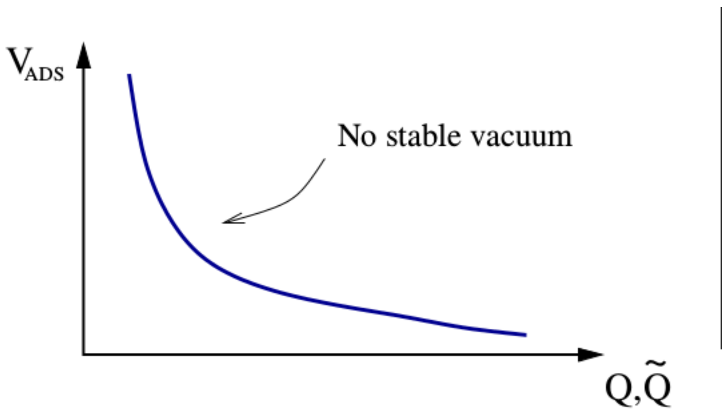
$$\Lambda = \mu e^{-\frac{2\pi i \tau}{b_0}} \qquad \tau = \frac{4\pi i}{g^2(\mu)} + \frac{\theta_{YM}}{2\pi} \qquad b_0 = 3N_c - N_f \tag{1.29}$$

where τ is the complexified gauge coupling. $|\Lambda|$ is defined as the scale at which the coupling constant blows up.
The gaugino condensation breaks R-symmetry to $\mathbb{Z}_2$ and in fact there are N_c physically different vacua labelled by different phases of the gaugino condensate.

$\mathbf{N_f < N_c}$
The quantum corrections for *SQCD* with $N_f < N_c$ flavours completely lift the moduli space through the Affleck-Dine-Seiberg (ADS) superpotential ([9][10]) which reads

$$W_{eff} = (N_c - N_f)\left(\frac{\Lambda^{3N_c-N_f}}{\det M}\right)^{\frac{1}{N_c-N_f}} \tag{1.30}$$

It is the only superpotential that is compatible with the symmetries of the theory and with the other properties of the superpotential we introduced in section 1.1.2. We can see that the ADS superpotential do not exist for $N_F \geq N_c$ since the exponent diverges for $N_f = N_c$ or the determinant vanishes for $N_f \geq N_c$ since the mesons do not have maximal rank. Note that this superpotential is non perturbative and thus it is not in contrast with the renormalization theorem of section 1.1.2. The superpotential is generated either by instantons for $N_f = N_C - 1$ or by gaugino condensation for other number or flavours. The effect of this superpotential is that the theory does not have ground state. The slope of the potential goes to zero only for $\det M \to \infty$.



This situation is the perfect example when, unlike the classical moduli space, quantum corrections lift completely the moduli space and the theory does not posses a vacuum anymore.

$\mathbf{N_f = N_c}$
When the number of flavours is equal the number of colours of the theory, the classical moduli space was subject to the constraint

$$\det M - B\tilde{B} = 0 \tag{1.31}$$

in the quantum corrected moduli space mesons and baryons satisfy [11]

$$\det M - B\tilde{B} = \Lambda^{2N_c} \tag{1.32}$$

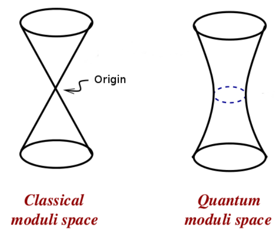


Figure 1.1: Schematical representation of the quantum and classical moduli space near the origin

which flows to the classical constraint in the classical limit ($\Lambda \to 0$).

The effect of this relation is that the origin does not belong to the moduli space anymore and the moduli space is now smooth. For large expectation values of M, B and $\tilde{B}$ the classical and the quantum moduli space look similar, while in the origin quantum correction modify drastically the structure of the moduli space. Moreover, the subspace with B or $\tilde{B}$ is zero, is not singular anymore while classical the meson matrix was constrained to have zero determinant.

Since the origin is not in the quantum moduli space the global symmetry group 1.2 is necessarily broken in some way, depending on the position of the moduli space

$$M^i_j = \Lambda^2 \delta^i_j \quad B = \tilde{B} = 0 \quad \to \quad SU(N_f)_V \times U(1)_B \times U(1)_R \tag{1.33}$$

$$M^i_j = 0 \quad B = -\tilde{B} = \Lambda^{N_c} \quad \to \quad SU(N_f)_L \times SU(N_f)_R \times U(1)_R \tag{1.34}$$

$\mathbf{N_f = N_c + 1}$

In the case $N_f = N_c + 1$ the classical moduli space is constrained by

$$\det M \left(\frac{1}{M}\right)^j_i - B_i \tilde{B}^j = 0 \qquad M^i_j B_i = M^i_j \tilde{B}^j = 0 \tag{1.35}$$

and quantum corrections do not modify it. In the previous section we noted that the singularities in the classical moduli space are associated to the appearance of massless gluons. In the quantum picture, the interpretation of the singularities is different: they are associated with additional massless mesons and baryons. At the origin of the moduli space the theory is strongly coupled and the global symmetry 1.2 is unbroken and it can be checked with 't Hooft anomalies that mesons and baryons physical and contribute to the anomalies.

Far from the origin, mesons and baryons interact with an effective potential

$$W_{eff} = \frac{1}{\Lambda^{2N_c - 1}} \left(M^i_j B_i \tilde{B}^j - \det M \right) \tag{1.36}$$

that enforce the classical constraints 1.35 through the equations of motion. Moreover, the superpotential give large mass to mesons and baryons.

$\mathbf{N_f > N_c + 1}$

Starting from $N_f = N_c + 2$ it is not possible to construct a sensible physical superpotential out of gauge invariant operators, in analogy to the previous cases. The only $SU(N_f)_L \times SU(N_f)_R$ invariant superpotential that can be written is given by

$$W_{eff} \sim \det M - B_{ij} M^i_k M^j_l \tilde{B}^{kl} \tag{1.37}$$

since baryons have two flavour indices. However this superpotential does not have R-charge equal to two and if we add more flavours we should add other mesons to the superpotential.
The classical constraints on mesons and baryons are satisfied quantum mechanically. Unlike the case with $N_f = N_c+1$ the singularities in the moduli space cannot be interpreted as massless mesons and baryons and an effective description of these operator is singular [11]. Since 't Hoof anomaly matching conditions are not satisfied in the singular points it is clear that a description using mesons and baryons is not correct.

To find a description of the low-energy degrees of freedom of the theory we will use Seiberg duality, which provides an alternative description of the theory,

$\mathbf{\frac{3}{2}N_c \geq N_f \geq 3N_c}$
In this range the theory is not asympotically free. This can be seen by using the *NSVZ* β function,using 1.3, reads

$$\beta(g) = -\frac{g^3}{16\pi^2}\frac{3N_c - N_f + N_f\gamma(g^2)}{1 - N_c\frac{g^2}{8\pi^2}} \tag{1.38}$$

$$\gamma(g^2) = -\frac{g^2}{8\pi^2}\frac{N_c^2-1}{N_c} + \mathcal{O}(g^4) \tag{1.39}$$

Even though the β function is exact, the anomalous dimension is only known perturbatively.
In [8] was shown that exist a non trivial fixed point in the renormalization group flow for $\frac{3}{2}N_c \geq N_f \geq 3N_c$. This is possible because one loop and two loop contribution to the beta function have opposite signs. As a result, the infrared red theory is a non-trivial four dimensional superconformal theory. The infrared degrees of freedom are quarks and gluons that are not confining but are interacting as massless particles. The theory is in a free non-Abelian Coulomb phase.

Since the theory is superconformal, we have further restriction on the algebra of operators [2] Superconformal algebra imposes that the dimension of every operator satisfy this inequality involving the R-charge

$$D \geq \frac{3}{2}|R| \tag{1.40}$$

where the bound is satured for chiral fields. The operator product expansion (*OPE*) of two chiral operator is constrained by this fact. Since $R(O_1O_2) = R(O_1) + R(O_2)$, we have that for chiral operators, $D(O_1O_2) = D(O_1) + D(O_2)$. Therefore, the OPE is not singular and the product of operators is well-defined. Because of this fact, chiral operators form the *chiral ring*.

Since the superconformal R-symmetry is not anomalous and commutes with the global symmetry group of the theory it must be the R-symmetry that appears in 1.1. The gauge

[2] R-symmetry is contained directly in the superconformal algebra instead of being an automorphism of the algebra, as in superPoincarè algebra.

invariant operators we defined previously must have

$$D(Q\tilde{Q}) = \frac{3}{2}R(Q\tilde{Q}) = 3\frac{N_f - N_c}{N_f} \tag{1.41}$$

$$D(B) = D(\tilde{B}) = \frac{3}{2}N_c\frac{N_f - N_c}{N_c} \tag{1.42}$$

Gauge invariant operators should be in unitary representation of the superconformal algebra. Unitarity imposes that in general $D \geq 1$ and the equality holds for free fields. From the previous equation we can verify that $D(M) \geq 1$ for $N_f \geq \frac{3}{2}N_c$ and it becomes a free field for $N_f = \frac{3}{2}N_c$.

For fewer number of flavours, the meson field is inconsistend with the unitarity bound. The theory is conjecture to flow to a different phase.

$\mathbf{N_f > N_c}$

In this range, quarks prevails on glouns and change the sign of the β function. This is caused by the *charge screening* effect of quarks, that make the coupling constant smaller at larger distances.

The theory is in a free non-Abelian electric phase. Its behaviour is not very well defined at high energies because of the presence of a Landau pole at $R \sim \Lambda^{-1}$, although the theory can be a very good description of a low energy limit of another theory.

1.3 Seiberg duality

Electric magnetic duality relates the dynamics of two different gauge theories in their infrared fixed point. In the case of Seiberg duality, the electric theory is given by the theory we analysed previously.

The magnetic theory is a theory with the same global symmetries as the electric theory, but with gauge group $SU(N_f - N_c)$. In addition there are N_f^2 chargeless fields, that we will call mesons, since they have the same properties as the mesons we can construct in the electric theory. In the magnetic theory they are fundamental fields i.e. they are not written as gauge invariant operators from quarks. Since they are gauge invariant, they interact only through the superpotential

$$W = M^i_{\tilde{j}} q_i \tilde{q}^{\tilde{j}} \tag{1.43}$$

where we represented dual quarks as $q, \tilde{q}$ and mesons as $M^i_{\tilde{j}}$.
The charges for the magnetic theory are

	$SU(N_c)$	$SU(N_f)_L$	$SU(N_f)_R$	$U(1)_B$	$U(1)_R$
q	N_c	N_F	1	$\frac{N_c}{N-f-N_c}$	$\frac{N_c}{N_f}$
$\tilde{q}$	$\overline{N_c}$	1	$\overline{N_F}$	$-\frac{N_c}{N-f-N_c}$	$\frac{N_c}{N_f}$
$M^i_{\tilde{j}}$	1	N_f	$\overline{N_f}$	0	$2\frac{N_f-N_c}{N_f}$

Table 1.2: Charge of matter content of the magnetic theory

Appendices

A Supersymmetry and superfields

A.1 Supersymmetry algebra

The supersymmetry algebra is an extension of the Poincarè group involving anticommutators together with commutators. Since it is not a ordinary Lie algebra, Coleman-Mandula theorem does not apply for theories that are invariant under it.

The supersymmetry algebra is divided into two subalgebras, the bosonic and fermionic part. The bosonic part contains Poincarè Lie algebra $(M_{\mu\nu}, P_\mu)$ while fermionic subalgebra is generated by the *supercharges* $(Q^I_\alpha, \bar{Q}^I_{\dot{\alpha}})$ with $I = 1, \dots, \mathcal{N}$. When more than one pair of supercharges is present we refer to extended supersymmetry.

The supercharges sit in spinorial representations of the Lorentz group, respectively $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$.

We will not repeat the bosonic subalgebra, since is given by the Poincarè Lie algebra. The fermionic generators satisfy anticommutation rules between themselves and commutation rules with bosonic generators. For this reason, the supersymmetry algebra is defined in mathematical literature as a graded Lie algebra with grade one.
The (anti)commutation rules in four dimensions are

$$[P_\mu, Q^I_\alpha] = 0 \tag{A.1}$$
$$[P_\mu, \bar{Q}^I_{\dot{\alpha}}] = 0 \tag{A.2}$$
$$[M_{\mu\nu}, Q^I_\alpha] = i(\sigma_{\mu\nu})^\beta_\alpha Q^I_\beta \tag{A.3}$$
$$[M_{\mu\nu}, \bar{Q}^I_{\dot{\alpha}}] = i(\bar{\sigma}_{\mu\nu})^{\dot{\alpha}}_{\dot{\beta}} \bar{Q}^I_{\dot{\beta}} \tag{A.4}$$
$$\{Q^I_\alpha, \bar{Q}^J_{\dot{\beta}}\} = 2\sigma^\mu_{\alpha\dot{\beta}} P_\mu \delta^{IJ} \tag{A.5}$$
$$\{Q^I_\alpha, Q^J_\beta\} = \epsilon_{\alpha\beta} Z^{IJ} \qquad Z^{IJ} = -Z^{JI} \tag{A.6}$$
$$\{Q^I_{\dot{\alpha}}, Q^J_{\dot{\beta}}\} = \epsilon_{\dot{\alpha}\dot{\beta}} \left(Z^{IJ}\right)^* \tag{A.7}$$

This set of commutation rules can be found using symmetry arguments and enforcing the consistency of the algebra using the graded Jacobi identity.
It is important to stress the fact that Z^{IJ} are operators that span an invariant subalgebra: they are *central charges*. They play an important role especially in massive representations.

There is an additional symmetry that is not present in the previous commutation rules: R-symmetry. It is an automorphism of the algebra that act on the supercharges. For generic $\mathcal{N}$ the R-Symmetry group is $U(\mathcal{N})$.

A.2 Representations

Since the supercharges do not commute with Lorentz generators, their action on a state will result in a state with different spin: they generate a symmetry between bosons and fermions.
Representations of supersymmetry contain particle with different spin but same mass and they are organized in supermultiplets. The mass of particles in the same multiplet must be the same because P^2 is still a Casimir operator of the supersymmetry algebra, while the Pauli-Lubanski operator W^2 isn't anymore.

Moreover, the supersymmetry algebra imposes that every state must have positive energy and that every supermultiplet must contain the same number of bosonic and fermionic degrees of freedom *on-shell* .

Various supermultiplets exist and their properties depend on the number of supercharges of the theory and on what they represent e.g matter, glue or gravity.

Massless supermultiplet are typically shorter than massive multiplet because in the massless case half of the supercharges are represented trivially. Massive representation of extended supersymmetry can be shortened in case some of the central charges of the algebra are equal to twice the mass of the multiplet. These states are usually called (ultra)short multiplet or BPS states.

We will introduce the multiplets that can be defined for $4d$ $\mathcal{N} = 1$ theories and only later we will explain the differences with $3d$ $\mathcal{N} = 2$ theories. Representations are similar because in both cases we have the same number of supercharges.

For four dimensional theories, we can define two different multiplet that are invariant under supersymmetry transformations. The matter or chiral multiplet contains a complex scalar (*squark*) and a Weyl fermion (*quark*). It identifies the matter content of the theory. The vector or gauge multiplet contains a Weyl fermion (*gaugino*) and a vector (*gluon or photon*). Particles in the same multiplet transform in the same representation of global or gauge symmetries. For this reason the gaugino cannot represent matter.

A representation of these multiplets on fields can be easily found using the *superspace* formalism that we will introduce in the next section. In this formalism it is possible to represent fields that are *off-shell*, in contrast with multiplets that we introduced previously that are *on-shell* since they represent states in Hilbert space.

A.2.1 Superfields and superspace in four dimensions

Supersymmetry representations on fields can be found more sistematically using the formulation of *superspace* instead of acting directly with supercharges and verifying that the algebra closes.

A simple formulation of superspace exist for theories with 4 supercharges while for theories with a bigger number of supercharges its definition is much more complex. We will be interested only in theories with 4 supercharges such as theories in 4D with $\mathcal{N} = 1$ or 3D with $\mathcal{N} = 2$.

Superspace can be seen as the extension of Minkowsky space with *fermionic coordinates* i.e. *Grassman numbers* θ^α , $\bar{\theta}^{\dot{\alpha}}$. They anticommute between themselves and commute with everything else.

$$\{\theta^\alpha,\theta^\beta\}=0 \quad \{\bar\theta^{\dot\alpha},\bar\theta^{\dot\beta}\}=0 \quad \{\theta^\alpha,\bar\theta^{\dot\beta}\}=0 \qquad \alpha,\dot\alpha=1,2 \tag{A.8}$$

Derivation and integration in Grassmann variables are summarized by these rules

$$\partial_\alpha=\frac{\partial}{\partial\theta_\alpha} \quad \partial^\alpha=-\epsilon^{\alpha\beta}\partial_\beta \quad \bar\partial_{\dot\alpha}=\frac{\partial}{\partial\bar\theta^{\dot\alpha}} \quad \bar\partial^{\dot\alpha}=-\epsilon^{\dot\alpha\dot\beta}\partial_{\dot\beta} \quad \partial_\alpha\theta^\beta=\delta_\alpha^\beta \quad \partial_\alpha\bar\theta^{\dot\alpha}=0 \tag{A.9}$$

$$\int d\theta=0 \quad \int d\theta\,\theta=1 \quad d^2\theta=\frac{1}{2}d\theta^1d\theta^2 \quad \int d^2\theta=\frac{1}{4}\epsilon^{\alpha\beta}\partial_\alpha\partial\beta \tag{A.10}$$

For a more detailed introduction on Grassmann numbers and their properties see [12].

Using Grassmann numbers we can write the anticommutators in the supersymmetry algebra as commutators defining $\theta Q=\theta^\alpha Q_\alpha$ and $\bar\theta\bar Q=\bar\theta^{\dot\alpha}\bar Q_{\dot\alpha}$

$$[\theta Q,\bar\theta\bar Q]=2\theta\sigma^\mu\bar\theta P_\mu \quad , \quad [\theta Q,\theta Q]=[\bar\theta\bar Q,\bar\theta\bar Q]=0 \tag{A.11}$$

Using this trick we are able to represent the supersymmetry algebra as a Lie algebra. An element of the superPoincarè group can be found exponentiating the generators

$$G(x,\theta,\bar\theta,\omega)=\exp\left(ixP+i\theta Q+i\bar\theta\bar Q+\frac{1}{2}i\omega M\right) \tag{A.12}$$

The superspace is defined as the 4+4 dimensions group coset

$$M_{4|1}=\frac{\text{SuperPoincarè}}{\text{Lorentz}} \tag{A.13}$$

in analogy to Minkowsky space that can be defined as the coset between Poincarè and Lorentz groups.

A generic point in superspace is parametrized by $(x^\mu,\theta^\alpha,\bar\theta^{\dot\alpha})$. A superfields is a field in superspace i.e. function of the superspace coordinates. Since θ coordinates anticommute, the expansion of a superfield in fermionic coordinates stops at some point. The most general superfield $Y=Y(x,\theta,\bar\theta)$ is given by

$$\begin{aligned}Y(x,\theta,\bar\theta)=f(x)+\theta\psi_1(x)+\bar\theta\bar\psi_2(x)+\theta\theta g_1(x)+\bar\theta\bar\theta g_2(x)\\ +\theta\sigma^\mu\theta v_\mu(x)+\theta\theta\bar\theta\bar\lambda(x)+\bar\theta\bar\theta\theta\rho(x)+\theta\theta\bar\theta\bar\theta s(x)\end{aligned} \tag{A.14}$$

fields with uncontracted θ such as $\psi_1,\psi_2,\lambda,\rho$ are spinors while v_μ is a vector.

Supercharges can be represented as differential operators that act on superfield. Their expression is

$$\begin{cases}Q_\alpha &= -i\partial_\alpha-\sigma^\mu_{\alpha\dot\beta}\bar\theta^{\dot\beta}\partial_\mu\\ \bar Q_{\dot\alpha} &= +i\dot\partial_{\dot\alpha}+\theta^\beta\sigma^\mu_{\beta\dot\alpha}\bar\theta^{\dot\beta}\partial_\mu\end{cases} \tag{A.15}$$

An infinitesimal supersymmetry transformation on a superfield is defined by

$$\delta_{\epsilon,\bar\epsilon}Y=\left(i\epsilon Q+i\bar\epsilon\bar Q\right)Y \tag{A.16}$$

The powerfulness of the superfield formalism is due to the fact that an integral in full superspace coordinates of a superfield is supersymmetric invariant.

$$\delta_{\epsilon,\bar{\epsilon}} \int d^4x\, d^2\theta\, d^2\bar{\theta}\; Y = \int d^4x\, d^2\theta\, d^2\bar{\theta}\; \delta_{\epsilon,\bar{\epsilon}} Y = 0 \tag{A.17}$$

The first equality holds because the Grassmann measure is inviariant under translation while the second is true because we can see that the variation of the superfield is either killed by the integration in the θ variables or is proportional to a spacetime derivative that does not contribute after integration in space.

Using this fact we can construct supersymmetric invariant lagrangians by integrating superfields in superspace. Clearly, in order to find a physically significant lagrangian we should choose the superfield we wish to integrate wisely. More importantly, we want to use irreducible representation of supersymmetry i.e. the supermultiplets we introduced before. We need to find conditions that can be imposed on a general superfield that are invariant under a supersymmetry transformation.

Chiral superfield

One way to achieve this goal is to find an operator that commute with the supercharges and annihilate the superfield An example of such operator is the *covariant derivative*

$$\begin{cases} D_\alpha = & \partial_\alpha + i\sigma^\mu_{\alpha\dot{\beta}}\bar{\theta}^{\dot{\beta}}\partial_\mu \\ \bar{D}_{\dot{\alpha}} = & \dot{\partial}_{\dot{\alpha}} + i\theta^\beta\sigma^\mu_{\beta\dot{\alpha}}\bar{\theta}^{\dot{\beta}}\partial_\mu \end{cases} \tag{A.18}$$

We can define a (anti)chiral superfield Φ

$$\bar{D}_{\dot{\alpha}}\Phi = 0 \quad \text{chiral} \qquad\qquad D_\alpha\Psi = 0 \quad \text{anti-chiral} \tag{A.19}$$

This condition reduces the number of components of the superfield. It can be easily demonstrated that if Ψ is chiral, then $\bar{\Psi}$ is anti-chiral. As a result a chiral field cannot be real.

The expansion of a chiral fields in components is give by

$$\Phi(x,\theta,\bar{\theta}) = \phi(x) + \sqrt{2}\theta\psi(x) + i\theta\sigma^\mu\bar{\theta}\partial\mu\phi(x) - \theta\theta F(x) - \frac{i}{\sqrt{2}}\theta\theta\partial\mu\psi(x)\sigma^\mu\bar{\theta} - \frac{1}{4}\theta\theta\bar{\theta}\bar{\theta}(x) \tag{A.20}$$

We can see that a chiral field is composed by three fields: two complex scalars (ϕ and F) and a spinor (ψ).

The chiral superfield identifies the matter multiplet we introduced previously. It contains an additional bosonic field ($F(x)$) that is present because superfields provide an *off-shell* representation of supersymmetry and it is needed in order to close the algebra. It is called *auxiliary field* because it will not have kinetic terms in every Lagrangian that can be constructed.

Real or Vector Field

We can impose that the superfield is real. In this way we find the *real* or *vector* multiplet. Its general expression in component is messy and a simplification can be made noting that $\Phi + \bar{\Phi}$ is a vector superfield if Φ is chiral. Choosing an appropriate chiral field, the real superfield can be put in what is called Wess-Zumino gauge. We stress the fact that the Wess-Zumino gauge is not supersymmetric invariant: after a supersymmetry transformation the vector superfield acquire its general expression involving many other field components. In this gauge the vector superfield can be written as

$$V_{WZ}(x,\theta,\bar{\theta}) = \theta\sigma^\mu\bar{\theta}v_\mu(x) + i\theta\theta\bar{\theta}\bar{\lambda}(x) - i\bar{\theta}\bar{\theta}\theta\lambda(x) + \frac{1}{2}\theta\theta\bar{\theta}\bar{\theta}D(x) \tag{A.21}$$

The vector superfields represents the vector multiplet (which contains radiation) and similarly to the chiral superfields contains an auxiliary field ($D(x)$).

A.2.2 R-symmetry

R-symmetry was first introduced with the supersymmetry algebra. For the theories we will consider in superspace it is given by a global $U(1)_R$. It is defined by as a transformation of the Grassmann coordinates

$$\theta \;\rightarrow\; e^{i\alpha}\theta \qquad \bar{\theta} \;\rightarrow\; e^{-i\alpha}\bar{\theta} \tag{A.22}$$

α parametrizes the transformation. As a result supercharges transform under the transformation

$$Q \rightarrow\; e^{-i\alpha}Q \qquad \bar{Q} \;\rightarrow\; e^{+i\alpha}\bar{Q} \tag{A.23}$$

From this we find the commutator relations between supercharges and R-symmetry generator R

$$[R,Q] = -Q \qquad [R,\bar{Q}] = \bar{Q} \tag{A.24}$$

The R-charge of a superfield is defined by

$$Y(x,\theta,\bar{\theta}) \;\rightarrow\; e^{iR_Y\alpha}Y(x,\theta,\bar{\theta}) \tag{A.25}$$

Different component field in the superfield have different R-charge and are related because of A.24. For a chiral field we have

$$R[\phi] = R\,[\Phi] \qquad R[\psi] = R\,[\Phi] - 1 \qquad R[F] = R\,[\Phi] - 2 \tag{A.26}$$

The corresponding antichiral field carry opposite charges.

A.3 Supersymmetric actions

We will use the property we introduced in A.17 to generate supersymmetric invariant lagrangians. We start our analysis with chiral superfields. Since lagrangians are quadratic in the fields and must be real, the simplest kinetic term for a chiral superfield is given by $\bar{\Phi}\Phi$.

$$\mathcal{L}_{kin} \;=\; \int d^2\theta d^2\bar{\theta}\;\bar{\Phi}\Phi = \partial_\mu\bar{\phi}\partial^\mu\phi + \frac{i}{2}\left(\partial_\mu\psi\sigma^\mu\bar{\psi} - \psi\sigma^\mu\partial_\mu\bar{\psi}\right) + \bar{F}F + \text{total derivative} \tag{A.27}$$

which gives the correct kinetic terms for a scalar and a spinor. The auxiliary field doesn't have kinetic terms as predicted.

Many action can be find using a generalization of the equation above. It is called *Kahler* potential

$$K(\bar{\Phi},\Phi)=\sum_{m,n=1}^{\infty}c_{m,n}\bar{\Phi}^m\Phi^n \qquad \text{where} \quad c_{m,n}=c_{n,m}* \tag{A.28}$$

The condition on the coefficient is imposed by the requirement of a real lagrangian.

Another way of finding supersymmetric actions is by integrating *chiral* superfields in half-superspace coordinates. We define the *superpotential* to be a holomorphic function of Φ

$$\mathcal{L}_{int}=\int d^2\theta d^2\ W(\Phi)+\int d^2\bar{\theta}\ \bar{W}(\bar{\Phi})=\sum_{i=1}^{\infty}\int d^2\theta\ \lambda_n\Phi^n+\int d^2\bar{\theta}\ \lambda_n^\dagger\bar{\Phi}^n \tag{A.29}$$

The hermitian conjugate was added in order to have a real lagrangian.

Mixed terms with product of chiral and anti-chiral superfield are not present since they would be generic superfields and would not yield a supersymmetric lagrangian. In fact if $W(\Phi)$ is holomorphic and Φ is a chiral superfield, $W(\Phi)$ is a chiral superfield

$$\bar{D}_{\dot{\alpha}}W(\Phi)=\frac{\partial W}{\partial\Phi}\bar{D}_{\dot{\alpha}}\Phi+\frac{\partial W}{\partial\bar{\Phi}}\bar{D}_{\dot{\alpha}}\bar{\Phi}=0 \tag{A.30}$$

and yield a proper lagrangian upon integration in $d^2\theta$.

Since the superpotential is integrated only in half superspace coordinates it need to be charged in an opposite way with respect to the integration measure under R-symmetry. Remembering that

$$R[\theta]=1 \qquad [\bar{\theta}]=-1 \qquad R[d\theta]=-1 \qquad R[d\bar{\theta}]=1 \tag{A.31}$$

It's easy to see that

$$R[W(\Phi)]=2 \qquad R[\bar{W}(\bar{\Phi})]=-2 \tag{A.32}$$

For this reason in most situations the superpotential fix the supercharges of the fields.

The lagrangian of super Yang-Mills theories is given by

$$\mathcal{L}_{SYM}\ =\frac{1}{32\pi i}\left(\int d^2\theta\ \left(\frac{\theta_{YM}}{2\pi}+\frac{4\pi i}{g^2}\right)W_\alpha W^\alpha\right)= \tag{A.33}$$

$$=\mathrm{Tr}\left[-\frac{1}{4}F_{\mu\nu}F^{\mu nu}-i\lambda\sigma^\mu D_\mu\bar{\lambda}+\frac{1}{2}D^2\right]+\frac{\theta_{YM}}{32\pi^2}g^2\mathrm{Tr}F_{\mu\nu}\tilde{F}^{\mu nu} \tag{A.34}$$

where we defined the chiral superfield W_α as

$$W_\alpha=-\frac{1}{4}\bar{D}\bar{D}\left(e^{-2gV}D_\alpha e^{2gV}\right) \qquad \bar{W}_{\dot{\alpha}}=-\frac{1}{4}DD\left(e^{2gV}\bar{D}_{\dot{\alpha}}Ve^{-2gV}\right) \tag{A.35}$$

It can be demonstrated that W_α is chiral and is invariant under the supergauge transformation $V\ \rightarrow\ V+\bar{\Phi}+\Phi$ while the vector superfield V was not.

From a perturbative point of view the inclusion of the term proportional to $\theta_{YM}TrF_{\mu\nu}\tilde{F}^{\mu nu}$ has no effect since it is proportional to a total derivative. It is a parity violating term that differs from zero only in non trivial topological configurations of the field (instantons).

The matter lagrangian we introduced is not invariant under gauge transformation. The correct gauge invariant lagrangian is given by

$$\mathcal{L}_{matter} = \int d^2\theta d^2\bar{\theta}\; \bar{\Phi} e^{2gV} \Phi \tag{A.36}$$

The superpotential is not automatically invariant under gauge transformation. As a result only certain expression are allowed.
There's an additional supersymmetric invariant lagrangian that can be constructed in a gauge theory when the gauge group contains abelian factors. It is called Fayet-Iliopulos term and can be present for every ideal A of the gauge group

$$\mathcal{L}_{FI} = \sum_A \xi_A \int d^2\theta d^2\bar{\theta} V^A = \frac{1}{2} \sum_A \xi_A D^A \tag{A.37}$$

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