① Density independent example of pop. dynamics
Exponential Growth

Lecture 2

Discrete Time model solution

$N_{t+1} = \lambda N_t$

$N_T = N_0 \lambda^T$

Continuous Time model solution

$\frac{dN}{dt} = rN(t)$

$N(T) = N(0)e^{rT}$

How relate two?

$$\lambda^T = e^{rT}$$

$$\boxed{\lambda = e^r}$$

discrete per capita growth rate

intrinsic per capita rate of growth

$$\ln \lambda = \ln(e^r) = r \ln e = r$$

$$\boxed{\ln \lambda = r}$$

We can use Taylor series expansion to relate r and λ

$$e^r \approx 1 + r + \frac{r^2}{2} + \ldots$$

For small r

$$\boxed{\lambda \approx 1 + r}$$

why don't we see $\lambda \approx r$?

$\lambda = 1$ means constant population size

$r = 0$ in continuous time means constant population size

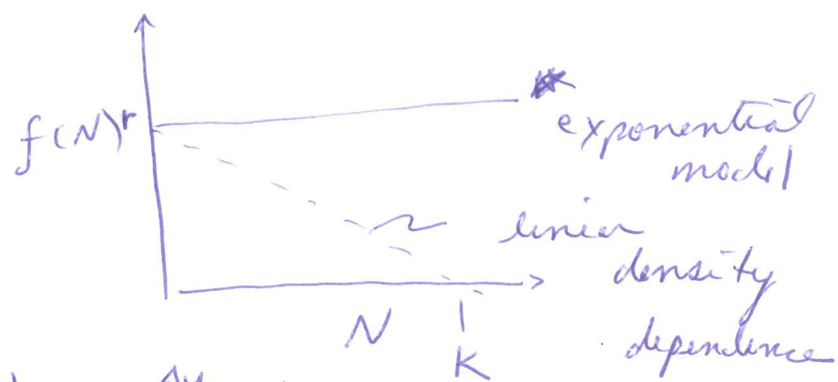
Go through examples + photos

Density dependence in continuous time

Before $\frac{dN}{dt} = r\,N(t)$

Now consider $f(N)$ = density dependent per capita growth rate

$$\frac{dN}{dt} = f(N)\,N$$



Logistic model

$$\frac{dN}{dt} = r\left(1-\frac{N}{K}\right)N$$

$$f(N) = r - \frac{\Delta y}{\Delta x}N$$
$$= r - \frac{r}{K}N$$
$$= r\left(1-\frac{N}{K}\right)$$

carrying capacity = value population density where $f(N)=0$

Solve for $N(t)$

1. Separate variable

$$\frac{dN}{N\left(1-\frac{N}{K}\right)} = r\,dt$$

2. Integrate both sides

$$\int \frac{dN}{N\left(1-\frac{N}{K}\right)} = \int r\,dt$$

Need to know

) $\frac{1}{N(1-\frac{N}{K})} = \frac{1}{N} - \frac{1}{N-K}$ } Partial fractions trick

$\frac{1}{N(1-\frac{N}{K})} = \frac{A}{N} + \frac{B}{1-\frac{N}{K}}$

$1 = A(1-\frac{N}{K}) + BN$

$= A - \frac{AN}{K} + BN$

$1 = N(B-\frac{A}{K}) + A$

$B = \frac{A}{K}$

b) $\int \frac{dN}{N-N_0} = \ln(N-N_0)$

$$\int_{t=0}^{t=T} \frac{dN}{N} - \frac{dN}{N-K} = \int_{t=0}^{t=T} r\,dt$$

$$\Big[\ln N(t) - \ln(N(t)-K)\Big]_{t=0}^{t=T} = rT$$

$$\ln\left(\frac{N(t)}{N(t)-K}\right)\Bigg|_{t=0}^{t=T} = rT$$

$$\ln\left(\frac{N(T)}{N(T)-K}\right) - \ln\left(\frac{N(0)}{N(0)-K}\right) = rT$$

$$e^{\ln\left[\frac{N(T)/N(T)-K}{N(0)/N(0)-K}\right]} = e^{rT}$$

$$N(T)/N(T)-K = \frac{N(0)}{N(0)-K} e^{rT} = \alpha(T)$$

$$N(T) = N(T)\alpha(T) - K\alpha(T)$$

$$N(T)(1-\alpha) = -K\alpha$$

$$N(T) = \frac{-K\alpha}{1-\alpha} = -K\frac{N(0)}{(N(0)-K)}e^{rT} \Big/ 1 - \frac{N(0)}{(N(0)-K)}e^{rT}$$

$$) = \frac{-K N(0) e^{rT}}{N(0) - K - N(0) e^{rT}} = \frac{-K N(0) e^{rT}}{-K + N(0)[1 - e^{rT}]}$$

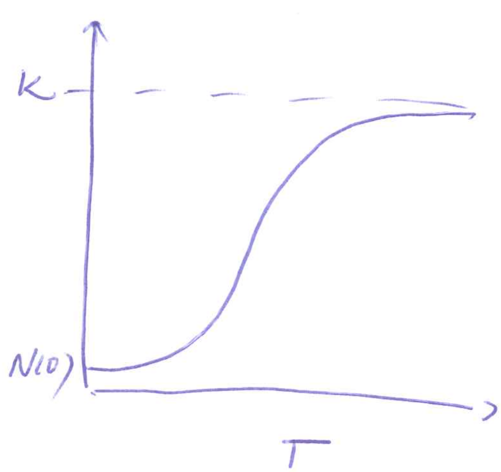
$$= \frac{N(0) e^{rT}}{1 - \frac{N(0)}{K}[1 - e^{rT}]}$$

$$N(T) = \frac{N(0) e^{rT}}{1 + N(0)\left(\frac{e^{rT} - 1}{K}\right)}$$

What happens for T and $N(0)$ small? $N(T) = N(0)$

$T \to \infty$? $N(T) \to K$

have $$\frac{K N(0) e^{rT}}{K + N(0)[e^{rT} - 1]}$$



$$\underset{T \to \infty}{N(T)} = \frac{K N(0) e^{rT}}{K + N(0)(e^{rT} - 1)} = \frac{K N(0) e^{rT}}{N(0) e^{rT}} = K$$

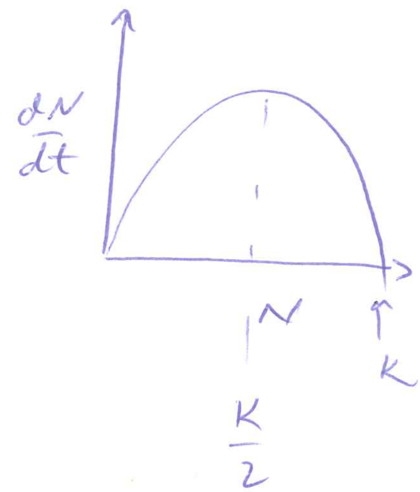
$$\underset{T \to 0}{N(T)} = \frac{K N(0)}{K} = N(0)$$

When is the growth rate maximum?

$$\frac{dN}{dt} = rN - \frac{rN^2}{K}$$

$\underbrace{\qquad\qquad}$

equation of parabola



$$\frac{d}{dN}\left(\frac{dN}{dt}\right) = 0$$

$$r - \frac{2Nr}{K} = 0$$

$$\frac{2N\cancel{r}}{K} = \cancel{r}$$

$$N = \frac{K}{2} \checkmark$$

$\boxed{\text{SLIDES}}$

Next moving onto equilibrium analysis

① what is the equilibrium population size, meaning population size that is not changing with time?

② Is the population stable, meaning it returns to equilibrium state following perturbations from that state?