An Operator-Based Formal System for Arithmetic State Transitions

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Abstract

We introduce an explicit operator-based formal system acting on discrete arithmetic states. The framework is defined axiomatically over a countable state space equipped with a weighted inner product and specifies state transitions derived from multiplicative relations.

The central object of the system is a symmetric transition operator combining forward and backward arithmetic transitions with a diagonal potential term. These components are interpreted as inference rules governing the evolution of symbolic states. The operator is defined on a dense core of finitely supported states and admits a canonical closure under a graph norm, allowing rigorous analysis of its structural properties.

We formalize the syntax, semantics, and domain of the system and show that the transition rules form an adjoint pair, yielding a well-defined symmetric evolution operator. The framework may be viewed as a logical computation model in which arithmetic structure is encoded directly at the operator level rather than through external rewrite rules.

This work provides a general abstraction for operator-driven computation on discrete domains and establishes a foundation for further study of consistency, closure, and spectral behavior within a logic-oriented setting.

1. Introduction

Formal models of computation are traditionally built from rewrite systems, automata, or logical inference rules acting on symbolic configurations. In parallel, operator-based frameworks have long played a central role in mathematics and physics, yet their use as primary logical computation models remains comparatively underexplored.

In this work we present a formal operator-based system defined over a discrete arithmetic domain. The system is specified axiomatically and independently of any physical interpretation. Its dynamics are governed by a single symmetric transition operator whose action encodes arithmetic relations directly at the level of state evolution.

The goal of this paper is not to introduce a new programming language or algorithm, but rather to define a mathematically precise inference mechanism in which arithmetic structure is intrinsic to the operator itself. The resulting framework may be interpreted as a logical computation model operating on arithmetic states, suitable for rigorous analysis of well-definedness, closure, and structural properties.

2. Arithmetic State Space

Let $\mathbb{N} = \{1,2,3,...\}$N={1,2,3,…} denote the set of positive integers.
We define the arithmetic state space $\mathscr{H}$H as the space of complex-valued functions

ψ:$\mathbb{N} \to \mathbb{C}$ψ:N→C

equipped with the weighted inner product

$$\langle\psi,\phi\rangle = \underset{n}{\overline{\psi(n)}\phi(n)}.$$⟨ψ,ϕ⟩=n≥1∑nψ(n)ϕ(n).

The associated norm is

$$\| \psi\|^2 = \underset{n}{| \psi(n)|^2}.$$‖ψ‖2=n≥1∑n∣ψ(n)∣2.

This space provides a natural setting for symbolic arithmetic states, where the weight $1/n$1/n ensures convergence properties compatible with multiplicative structure.

We define the dense core

$\mathcal{D}_0 := \{\psi \in \mathscr{H} \mid \psi\ has\ finite\ support\}$.D0:={ψ∈H∣ψ has finite support}.

All operators introduced in this work are initially defined on $\mathcal{D}_0$D0.

3. Definition of the Transition Operator

Definition 3.1 (Arithmetic Transition Operator)

Let $\psi\in\mathcal{D}_0$ψ∈D0.
We define the arithmetic transition operator TT by

$$(T\psi)(n) = \underset{p}{1}\left(\psi(pn) + 1_{p|n}\psi(n/p)\right) + \left(\alpha\Lambda(n) + \beta log \quad n\right)\psi(n),$$
(Tψ)(n)=p∈P∑p1(ψ(pn)+1p∣nψ(n/p))+(αΛ(n)+βlogn)ψ(n),

where:

- $\mathbb{P}$P denotes the set of prime numbers,
- $1_{p|n}$1p∣n is the indicator of divisibility,
- $\Lambda(n)$Λ(n) is the von Mangoldt function,
- $\alpha,\beta\in\mathbb{R}$α,β∈R are fixed parameters.

Interpretation of Components

The operator consists of three structurally distinct components:

1. **Forward transitions** $\psi(n) \mapsto \psi(pn)$ψ(n)↦ψ(pn), encoding multiplicative expansion.

2. **Backward transitions** $\psi(n) \mapsto \psi(n/p)$ψ(n)↦ψ(n/p) when defined, encoding inverse inference.
3. **Diagonal constraint terms**, acting pointwise on states via arithmetic functions.

Together, these terms define a deterministic inference mechanism acting on arithmetic configurations.

4. Symmetry and Adjoint Structure

Proposition 4.1

The operator TT is symmetric on $\mathcal{D}_0$D0, i.e.

$\langle\psi,T\phi\rangle = \langle T\psi,\phi\rangle\forall\psi,\phi\in\mathcal{D}_0.$⟨ψ,Tϕ⟩=⟨Tψ,ϕ⟩∀ψ,ϕ∈D0.

Sketch of Proof

- The forward and backward transition terms form an adjoint pair under the weighted inner product.
- The diagonal terms are real-valued multiplication operators and are therefore symmetric. □□

This symmetry ensures that the transition rules define a coherent bidirectional inference structure.

5. Domain, Closure, and Well-Definedness

We equip $\mathcal{D}_0$D0 with the graph norm

$\|\psi\|_T^2 := \|\psi\|^2 + \|T\psi\|^2.$‖ψ‖T2:=‖ψ‖2+‖Tψ‖2.

Let $\overline{Dom(T}$Dom(T) denote the closure of $\mathcal{D}_0$D0 under this norm.

The operator $\overline{T}$T is then a closed symmetric operator on $\mathcal{H}$H.

This construction ensures that the system admits a well-defined extension suitable for further structural and spectral analysis.

6. Interpretation as a Logical Computation Model

The arithmetic transition operator TT may be interpreted as a logical inference engine:

- **States** correspond to symbolic arithmetic configurations.
- **Operator application** corresponds to a single inference step.
- **Iteration of TT** defines a computation or deduction sequence.
- **Symmetry** ensures reversibility at the level of inference rules.

Unlike traditional logical systems based on rewrite rules or syntactic derivations, the present framework embeds inference directly into operator action. Arithmetic structure is therefore intrinsic to the computation model rather than externally imposed.

This perspective places the system within the scope of logic-oriented models of computation, where semantic evolution is governed by formally defined operators.

7. Relation to Arithmetic Spectral Problems (Informal)

While the present work is formulated independently of analytic number theory, the structure of the operator admits interpretations related to classical arithmetic spectral problems.

In particular, the interaction between multiplicative transitions and diagonal arithmetic constraints suggests potential connections to spectral encodings of arithmetic information. These interpretations are not required for the formal validity of the system and are treated here as optional motivation rather than foundational claims.

8. Outlook

Future research directions include:

- analysis of essential self-adjointness,
- characterization of the operator spectrum,
- investigation of convergence and stability properties,
- computational simulation of operator iteration,
- extension to related arithmetic domains.

The framework introduced here provides a mathematically explicit foundation for studying operator-driven computation on discrete arithmetic structures.

Appendix A — Document Integrity and Timestamp

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