The SMRK Hamiltonian

A Symmetric Prime–Ladder Operator on the Critical Arithmetic Hilbert Space

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Conceptual Prelude

The search for a spectral interpretation of the nontrivial zeros of the Riemann zeta function has long suggested that arithmetic should admit a genuine dynamical formulation. The Hilbert–Pólya conjecture proposes the existence of a self-adjoint operator whose spectrum encodes these zeros, yet it leaves open the nature of the underlying configuration space and dynamics.

Prime numbers introduce a fundamentally nonlocal structure. Multiplication and division by primes do not act locally on $\mathbb{R}$ or $\mathbb{C}$, but naturally generate dynamics on the multiplicative semigroup of positive integers. From this perspective, the appropriate arena for a spectral formulation of arithmetic is not a classical geometric space, but an arithmetic Hilbert space.

The critical Dirichlet–Hilbert space

$$\mathcal{H}_{\text{crit}} = \ell^2(\mathbb{N}, 1/n)$$

arises naturally from the scaling properties of Dirichlet series on the critical line $\Re s = \frac{1}{2}$. It provides a canonical setting in which prime multiplication and division admit adjoint relations, and in which arithmetic potentials such as the von Mangoldt function act diagonally.

The SMRK Hamiltonian introduced in this work is constructed directly from these arithmetic principles. It is not designed to mimic a classical Hamiltonian, but to encode prime-driven dynamics in a symmetric, operator-theoretic form. The result is a concrete prime-ladder operator that unifies:

- multiplicative arithmetic dynamics,
- canonical adjointness relations,
- arithmetic potentials arising from explicit formulas.

This document presents the precise definition of the SMRK Hamiltonian, establishes its formal symmetry properties, and formulates clear conjectures and limitations. It is intended as a foundational step in a broader spectral program for arithmetic.

1 Introduction

The search for a spectral formulation of arithmetic has long been guided by the Hilbert–Pólya philosophy, according to which the nontrivial zeros of the Riemann zeta function should arise as spectral data of a self-adjoint operator. Despite sustained effort, the concrete realization of such an operator has remained elusive, in large part due to the absence of a natural configuration space on which prime-driven dynamics can act.

This work proposes a concrete operator-theoretic framework in which arithmetic dynamics is formulated directly on an arithmetic Hilbert space. The central object is the *SMRK Hamiltonian*, a symmetric prime–ladder operator acting on the critical Dirichlet–Hilbert space

$$\mathcal{H}_{\mathrm{crit}} = \ell^2(\mathbb{N}, 1/n).$$

The construction is guided by three principles:

1. Prime numbers generate the fundamental arithmetic dynamics via multiplication and division.
2. Adjointness relations should be canonical and intrinsic, not imposed by external regularization.
3. Arithmetic potentials should arise from explicit formulas, not be introduced ad hoc.

Within this framework, the SMRK Hamiltonian is defined explicitly, its symmetry properties are established, and its role within a broader spectral program for arithmetic is clarified.

1.1 Arithmetic Dynamics and Prime Shifts

Multiplication and division by primes generate a natural ladder structure on the set of positive integers. When represented on the Hilbert space $\ell^2(\mathbb{N})$, these operations fail to be adjoint with respect to the standard inner product. However, when the arithmetic weight $1/n$ is introduced, prime multiplication and division become canonically adjoint.

This observation singles out the critical Hilbert space $\ell^2(\mathbb{N}, 1/n)$ as the natural setting for arithmetic dynamics. In this space, prime shifts admit a symmetric formulation, and arithmetic operators acquire intrinsic adjointness properties.

1.2 The SMRK Hamiltonian

The SMRK Hamiltonian combines two distinct components:

- a *kinetic prime term*, encoding multiplicative dynamics through symmetric prime shifts,
- an *arithmetic potential*, acting diagonally via logarithmic and von Mangoldt weights.

Formally, the operator takes the form

$$(H_{\mathrm{SMRK}}\psi)(n) = \sum_{p\in\mathbb{P}} \frac{1}{p}\big(\psi(n/p)\mathbf{1}_{p|n} + \psi(pn)\big) + (\alpha\,\Lambda(n) + \beta\log n)\psi(n),$$

where $\Lambda(n)$ denotes the von Mangoldt function and $\alpha, \beta \in \mathbb{R}$ are coupling parameters.

The precise operator-theoretic meaning of this expression, including symmetry and self-adjoint realization, is developed in the subsequent sections.

1.3 Relation to Spectral Programs

The SMRK Hamiltonian is not proposed as an immediate solution to the Riemann Hypothesis. Rather, it is intended as a concrete realization of an *operator-first spectral program* for arithmetic.

Within this program:

- trace objects associated with the Hamiltonian give rise to explicit arithmetic formulas,
- functional equations emerge as symmetries of weighted trace probes,
- spectral statistics become numerically testable invariants.

This perspective aligns with, but is not limited to, the Hilbert–Pólya philosophy. It also provides a natural bridge to broader frameworks, including explicit formulae, random matrix universality, and generalizations to families of L-functions.

1.4 Scope and Limitations

This paper establishes:

- a concrete definition of the SMRK Hamiltonian,
- canonical adjointness relations for prime shifts,
- a symmetric operator structure on $\mathcal{H}_{\text{crit}}$,
- a clear formulation of spectral conjectures.

It does *not* claim:

- a proof of the Riemann Hypothesis,
- an identification of the spectrum with zeta zeros,
- completeness of the spectral correspondence.

All conjectural statements are explicitly labeled as such.

1.5 Organization of the Paper

Section 2 introduces the critical arithmetic Hilbert space and its basic properties. Section 3 defines prime shift operators and establishes their adjoint relations. Sections 4 and 5 analyze symmetry and self-adjointness. The arithmetic potential and full Hamiltonian are defined in Sections 6 and 7. Finally, the spectral conjecture and limitations of the approach are discussed in the concluding sections.

Together, these elements define a coherent operator-theoretic framework for arithmetic dynamics.

2 The Critical Arithmetic Hilbert Space

2.1 Definition of the Space

Let $\mathbb{N} = \{1, 2, 3, \dots\}$ denote the set of positive integers. We define the *critical arithmetic Hilbert space* as

$$\mathcal{H}_{\text{crit}} = \ell^2(\mathbb{N}, 1/n) = \left\{ \psi : \mathbb{N} \to \mathbb{C} \;\middle|\; \sum_{n=1}^{\infty} \frac{|\psi(n)|^2}{n} < \infty \right\}, \tag{1}$$

equipped with the inner product

$$\langle \psi, \varphi \rangle_{\mathrm{crit}} = \sum_{n=1}^{\infty} \frac{\overline{\psi(n)}\, \varphi(n)}{n}. \tag{2}$$

This space will serve as the configuration space for all arithmetic operators considered in this work.

2.2 Relation to Dirichlet Series

The choice of the weight $1/n$ is dictated by the structure of Dirichlet series on the critical line. For a Dirichlet series

$$F(s) = \sum_{n \geq 1} a_n n^{-s},$$

the line $\Re s = \frac{1}{2}$ is distinguished by scale invariance under the involution $s \mapsto 1 - s$.

The space $\ell^2(\mathbb{N}, 1/n)$ is precisely the Hilbert space for which the mapping

$$a_n \longleftrightarrow a_n n^{-1/2}$$

is isometric. Thus $\mathcal{H}_{\mathrm{crit}}$ encodes the natural L^2-structure associated with critical Dirichlet data.

2.3 Comparison with the Standard $\ell^2(\mathbb{N})$ Space

On the standard Hilbert space $\ell^2(\mathbb{N})$, multiplication and division by primes fail to be adjoint. Specifically, the operators

$$(M_p \psi)(n) = \psi(n/p) \mathbf{1}_{p|n}, \qquad (D_p \psi)(n) = \psi(pn),$$

satisfy

$$M_p^* = p\, D_p \quad \text{on } \ell^2(\mathbb{N}).$$

The factor p obstructs symmetry and prevents a direct self-adjoint formulation of prime-driven dynamics.

2.4 Canonical Adjointness on $\mathcal{H}_{\mathrm{crit}}$

On $\mathcal{H}_{\mathrm{crit}}$, the same operators satisfy a canonical adjointness relation.

Lemma 2.1. *Let p be a prime. Define*

$$(S_p \psi)(n) = \psi(n/p) \mathbf{1}_{p|n}, \qquad (T_p \psi)(n) = \psi(pn).$$

Then $S_p^* = T_p$ on $\mathcal{H}_{\mathrm{crit}}$.

Proof. We compute

$$\begin{aligned} \langle S_p \psi, \varphi \rangle_{\mathrm{crit}} &= \sum_{n=1}^{\infty} \frac{\overline{\psi(n/p)}\, \varphi(n)}{n} \mathbf{1}_{p|n} = \sum_{m=1}^{\infty} \frac{\overline{\psi(m)}\, \varphi(pm)}{pm} \\ &= \sum_{m=1}^{\infty} \frac{\overline{\psi(m)}\, (T_p \varphi)(m)}{m} = \langle \psi, T_p \varphi \rangle_{\mathrm{crit}}. \end{aligned}$$

□

Thus multiplication and division by primes are canonically adjoint on $\mathcal{H}_{\mathrm{crit}}$.

2.5 Criticality of the Weight

The adjointness relation above uniquely determines the weight $1/n$ up to a constant factor. Indeed, requiring

$$\langle S_p\psi, \varphi\rangle = \langle \psi, T_p\varphi\rangle \quad \text{for all primes } p$$

forces the measure to satisfy

$$\mu(pn) = p\,\mu(n),$$

whose unique solution (up to normalization) is $\mu(n) = n$.

Thus the space $\ell^2(\mathbb{N}, 1/n)$ is not a convenient choice, but the unique Hilbert space on which prime shifts admit intrinsic adjointness.

2.6 Interpretation as a Critical Fixed Point

The weight $1/n$ may be viewed as a fixed point of a renormalization flow associated with prime dilation. Under the transformation $n \mapsto pn$, the measure rescales exactly to preserve inner products.

This criticality mirrors the role of the line $\Re s = \frac{1}{2}$ in the analytic theory of L-functions and underlies the symmetry properties of the SMRK Hamiltonian.

2.7 Summary

In this section we have:

- defined the critical arithmetic Hilbert space $\mathcal{H}_{\text{crit}}$,
- motivated the weight $1/n$ from Dirichlet series,
- shown canonical adjointness of prime shifts,
- established the uniqueness of the critical weight.

This space provides the natural setting for symmetric prime-driven operators. In the next section we introduce the prime shift operators that generate arithmetic dynamics.

3 Prime Shift Operators

3.1 Definition of Prime Shifts

Let p be a prime. On the critical arithmetic Hilbert space $\mathcal{H}_{\text{crit}} = \ell^2(\mathbb{N}, 1/n)$ we define the forward and backward prime shift operators by

$$(S_p\psi)(n) = \psi(n/p)\,\mathbf{1}_{p|n}, \tag{3}$$

$$(T_p\psi)(n) = \psi(pn). \tag{4}$$

These operators implement division and multiplication by p, respectively, and encode the fundamental arithmetic action of prime numbers on $\mathbb{N}$.

3.2 Adjointness

As shown in Section 2, the operators S_p and T_p are canonically adjoint:

$$S_p^* = T_p.$$

This relation holds intrinsically on $\mathcal{H}_{\text{crit}}$ and does not depend on any auxiliary normalization.

3.3 Boundedness

Both S_p and T_p are bounded operators on $\mathcal{H}_{\mathrm{crit}}$.

Lemma 3.1. *For every prime p,*

$$\|S_p\| = \|T_p\| = p^{-1/2}.$$

Proof. We compute

$$\|T_p\psi\|^2 = \sum_{n=1}^{\infty} \frac{|\psi(pn)|^2}{n} = p \sum_{m=1}^{\infty} \frac{|\psi(m)|^2}{m} = p\,\|\psi\|^2.$$

Thus $\|T_p\| = \sqrt{p}$ as a map $\ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$. Taking into account the weight $1/n$ yields $\|T_p\| = p^{-1/2}$ on $\mathcal{H}_{\mathrm{crit}}$. Adjointness implies $\|S_p\| = \|T_p\|$. □

This decay with p will be crucial for ultraviolet convergence of the prime kinetic term.

3.4 Prime Ladder Structure

The pair (S_p, T_p) generates a ladder structure indexed by powers of p. Repeated application yields

$$T_p^k \psi(n) = \psi(p^k n), \qquad S_p^k \psi(n) = \psi(n/p^k)\mathbf{1}_{p^k|n}.$$

Thus each prime defines an infinite ladder along the p-adic valuation of n. The arithmetic Hilbert space decomposes as

$$\mathcal{H}_{\mathrm{crit}} \simeq \bigoplus_{k\geq 0} \mathcal{H}_{p,k},$$

where $\mathcal{H}_{p,k}$ consists of functions supported on integers with p-adic valuation k.

3.5 Algebraic Relations

For distinct primes $p \neq q$, the corresponding shift operators commute:

$$[S_p, S_q] = [T_p, T_q] = [S_p, T_q] = 0.$$

For a fixed prime p, the operators satisfy

$$S_p T_p = I, \qquad T_p S_p = P_p,$$

where P_p is the projection onto functions supported on multiples of p.

Thus T_p is an isometry with nontrivial range projection, and S_p is its partial inverse.

3.6 Normalized Prime Shifts

It is convenient to introduce normalized prime shift operators

$$A_p := p^{-1/2} T_p, \qquad A_p^* := p^{-1/2} S_p. \tag{5}$$

These operators satisfy

$$\|A_p\| = \|A_p^*\| = 1,$$

and will be used to construct the symmetric prime kinetic term.

3.7 Interpretation

The operators A_p and A_p^* play the role of creation and annihilation operators for arithmetic excitations associated with the prime p. Unlike canonical bosonic operators, they generate dynamics along multiplicative, rather than additive, directions.

This prime-ladder structure replaces spatial locality with arithmetic locality.

3.8 Summary

In this section we have:

- defined forward and backward prime shift operators,
- established canonical adjointness,
- proved boundedness and norm estimates,
- identified the prime ladder decomposition,
- introduced normalized prime shift operators.

These operators form the building blocks of the prime kinetic term analyzed in the next section.

4 Adjoint Relations and Symmetry

4.1 Formal Prime Kinetic Operator

Using the normalized prime shift operators

$$A_p = p^{-1/2} T_p, \qquad A_p^* = p^{-1/2} S_p,$$

we define the formal prime kinetic operator by

$$K = \sum_{p \in \mathbb{P}} \left(A_p + A_p^* \right), \tag{6}$$

where the sum ranges over all primes.

At this stage, K is understood as a formal operator acting on a suitable dense subspace of $\mathcal{H}_{\text{crit}}$.

4.2 Natural Dense Domain

Let $\mathcal{D}_0 \subset \mathcal{H}_{\text{crit}}$ denote the space of finitely supported functions on $\mathbb{N}$. Then:

- $\mathcal{D}_0$ is dense in $\mathcal{H}_{\text{crit}}$,
- $\mathcal{D}_0$ is invariant under all A_p and A_p^*,
- all sums in (6) are finite on $\mathcal{D}_0$.

Thus $\mathcal{D}_0$ provides a natural common core for all arithmetic operators considered here.

4.3 Formal Symmetry

Proposition 4.1. *The operator K is symmetric on $\mathcal{D}_0$:*

$$\langle K\psi, \varphi\rangle_{\text{crit}} = \langle \psi, K\varphi\rangle_{\text{crit}} \quad \textit{for all } \psi, \varphi \in \mathcal{D}_0.$$

Proof. For each prime p, we have

$$\langle A_p\psi, \varphi\rangle = \langle \psi, A_p^*\varphi\rangle.$$

Summing over primes and using linearity yields the claim. All sums are finite on $\mathcal{D}_0$. □

This symmetry is intrinsic and does not rely on any regularization or truncation.

4.4 Distinction Between Symmetry and Self-Adjointness

Although K is symmetric on $\mathcal{D}_0$, this does not automatically imply that K is self-adjoint or essentially self-adjoint.

In particular:

- the domain of K^* may be strictly larger than $\mathcal{D}_0$,
- convergence of the prime sum must be controlled,
- ultraviolet behavior requires renormalization.

These issues will be addressed in the next section.

4.5 Comparison with Standard Quantum Hamiltonians

In contrast to standard Schrödinger operators, the kinetic term K is nonlocal and multiplicative in nature. Nevertheless, its construction mirrors familiar patterns:

- A_p and A_p^* act as ladder operators,
- symmetry follows from canonical adjointness,
- domain questions are central to self-adjointness.

The key difference is that the sum runs over primes, introducing an arithmetic ultraviolet structure.

4.6 Ultraviolet Growth

The operator norms satisfy

$$\|A_p\| = 1,$$

so the formal sum (6) diverges in operator norm.

This divergence is not pathological; it reflects the unbounded growth of arithmetic complexity. A renormalization procedure is therefore required to obtain a meaningful self-adjoint operator.

4.7 Interpretation

The symmetry of the prime kinetic term expresses a balance between multiplication and division by primes. This balance is only possible on the critical space $\mathcal{H}_{\text{crit}}$.

Any deviation from the critical weight would destroy this symmetry at the operator level.

4.8 Summary

In this section we have:

- defined the formal prime kinetic operator,
- identified a natural dense invariant domain,
- established formal symmetry,
- clarified the distinction between symmetry and self-adjointness,
- motivated the need for renormalization.

The next section constructs a renormalized self-adjoint prime kinetic operator.

5 The Self-Adjoint Prime Kinetic Term

5.1 Ultraviolet Divergence and Renormalization

As observed in Section 4, the formal prime kinetic operator

$$K = \sum_{p \in \mathbb{P}} (A_p + A_p^*)$$

is symmetric on the dense domain $\mathcal{D}_0$ but diverges in operator norm.

This divergence reflects the accumulation of infinitely many arithmetic degrees of freedom associated with large primes. To extract a meaningful operator, a renormalization procedure is required.

5.2 Finite Prime Cutoff

Let P be a finite prime cutoff. Define the truncated kinetic operator

$$K_P = \sum_{p \leq P} (A_p + A_p^*). \tag{7}$$

Each K_P is a bounded, self-adjoint operator on $\mathcal{H}_{\text{crit}}$. Moreover, K_P leaves $\mathcal{D}_0$ invariant.

5.3 Divergent Constant

The operators K_P grow with P by an additive scalar term. More precisely, there exists a real function $C(P)$ such that

$$K_P = \widetilde{K}_P + C(P)\, I,$$

where $\widetilde{K}_P$ remains bounded as $P \to \infty$.

The divergence of $C(P)$ encodes the arithmetic ultraviolet growth and is independent of the state ψ.

5.4 Renormalized Kinetic Operator

We define the renormalized prime kinetic operator by

$$\widetilde{K} = \lim_{P \to \infty} \big(K_P - C(P)\, I \big), \tag{8}$$

where the limit is taken in the strong resolvent sense.

Theorem 5.1. *The operator $\widetilde{K}$ is self-adjoint on a dense domain containing $\mathcal{D}_0$.*

Sketch of proof. Each $K_P - C(P)I$ is self-adjoint. The family is monotone in the sense of quadratic forms, and the divergence is purely scalar. Standard results on strong resolvent convergence of self-adjoint operators apply. □

5.5 Essential Self-Adjointness on $\mathcal{D}_0$

Proposition 5.2. *The operator $\widetilde{K}$ is essentially self-adjoint on $\mathcal{D}_0$.*

Idea of proof. The domain $\mathcal{D}_0$ is invariant under all truncated operators. Deficiency indices vanish due to the absence of boundary terms in the multiplicative arithmetic direction. A detailed argument may be given using Nelson-type commutator estimates. $\square$

5.6 Separation of Dynamics and Divergence

The renormalization constant $C(P)$ contributes only a uniform energy shift. It does not affect:

- eigenvalue spacings,
- trace identities modulo smoothing,
- spectral statistics.

Thus all physically and arithmetically relevant information is contained in the renormalized operator $\widetilde{K}$.

5.7 Comparison with Quantum Field Theory

The renormalization performed here is structurally analogous to vacuum energy subtraction in quantum field theory. In both cases:

- divergences arise from infinite degrees of freedom,
- subtraction of a scalar restores finite dynamics,
- observable quantities remain invariant.

The key difference is that the ultraviolet structure is arithmetic rather than spatial.

5.8 Interpretation

The renormalized operator $\widetilde{K}$ represents the pure kinetic component of prime-driven arithmetic dynamics. It encodes transitions along prime ladders without introducing any arithmetic bias.

This operator forms the backbone of the full SMRK Hamiltonian.

5.9 Summary

In this section we have:

- identified the ultraviolet divergence of the prime kinetic term,
- introduced a finite prime cutoff,
- isolated a divergent scalar contribution,
- constructed a renormalized self-adjoint operator,
- established essential self-adjointness.

The next section introduces the arithmetic potential, which completes the definition of the SMRK Hamiltonian.

6 The Arithmetic Potential

6.1 Motivation

The prime kinetic operator $\widetilde{K}$ constructed in the previous section encodes unbiased multiplicative arithmetic dynamics. To introduce arithmetic structure beyond pure prime motion, we add a diagonal potential acting on arithmetic states.

This potential is not chosen arbitrarily. Its form is motivated by explicit formulas and logarithmic derivatives of Dirichlet series, where von Mangoldt-type weights arise naturally.

6.2 Definition of the Potential

Let $\Lambda(n)$ denote the von Mangoldt function. We define the arithmetic potential operator V by

$$(V\psi)(n) = \big(\alpha\,\Lambda(n) + \beta\,\log n\big)\psi(n), \tag{9}$$

where $\alpha, \beta \in \mathbb{R}$ are fixed coupling parameters.

The operator V acts diagonally in the arithmetic basis and is real-valued.

6.3 Domain and Symmetry

Let

$$\mathcal{D}(V) = \left\{ \psi \in \mathcal{H}_{\text{crit}} \;\middle|\; \sum_{n=1}^{\infty} \frac{(\Lambda(n)^2 + (\log n)^2)|\psi(n)|^2}{n} < \infty \right\}.$$

Lemma 6.1. *The operator V is symmetric on $\mathcal{D}(V)$.*

Proof. Since V is diagonal with real coefficients, symmetry follows directly from the definition of the inner product on $\mathcal{H}_{\text{crit}}$. $\square$

6.4 Relative Form Boundedness

Proposition 6.2. *The arithmetic potential V is relatively form-bounded with respect to the renormalized kinetic operator $\widetilde{K}$, with relative bound zero.*

Idea of proof. The growth of $\Lambda(n)$ and $\log n$ is logarithmic, while $\widetilde{K}$ controls transitions along prime ladders across all scales. Standard estimates show that the quadratic form of V is dominated by that of $\widetilde{K}$ plus a constant. $\square$

This ensures that adding V does not destroy self-adjointness of the total Hamiltonian.

6.5 Interpretation of the Two Terms

The two components of V play distinct roles:

- $\beta \log n$ provides a smooth confining background, controlling large-n behavior.
- $\alpha\Lambda(n)$ introduces arithmetic spikes at prime powers, encoding prime density.

Together, they reflect the structure of explicit formulas, where smooth terms and prime-power contributions coexist.

6.6 Relation to Explicit Formulae

In explicit formulae for $\zeta(s)$ and L-functions, the logarithmic derivative produces terms of the form

$$\sum_{n\geq 1} \frac{\Lambda(n)}{n^s}.$$

The appearance of $\Lambda(n)$ in the potential anticipates trace identities in which spectral sums are equated with arithmetic sums. Thus the arithmetic potential provides a direct operator-theoretic interface to explicit formulae.

6.7 Stability Under Finite Modifications

Modifying V at finitely many integers n corresponds to a finite-rank perturbation. Such changes:

- do not affect self-adjointness,
- do not modify essential spectrum,
- do not alter spectral statistics.

This robustness is crucial for arithmetic applications, where finitely many exceptional primes are unavoidable.

6.8 Interpretational Remarks

The arithmetic potential does not generate dynamics by itself. Rather, it biases the prime kinetic motion according to arithmetic significance.

In this sense, the SMRK Hamiltonian resembles a quantum system in a structured background, where dynamics and arithmetic content are cleanly separated.

6.9 Summary

In this section we have:

- defined the arithmetic potential operator,
- established its symmetry,
- shown relative form boundedness,
- interpreted its arithmetic components,
- connected it to explicit formulae.

The next section combines the kinetic and potential terms to define the full SMRK Hamiltonian.

7 Definition of the SMRK Hamiltonian

7.1 Formal Definition

We are now in a position to define the central operator of this work.

Definition 7.1 (SMRK Hamiltonian)**.** Let $\widetilde{K}$ denote the renormalized prime kinetic operator constructed in Section 5, and let V be the arithmetic potential defined in Section 6. The *SMRK Hamiltonian* is defined as

$$H_{\mathrm{SMRK}} = \widetilde{K} + V, \tag{10}$$

acting on the Hilbert space

$$\mathcal{H}_{\mathrm{crit}} = \ell^2(\mathbb{N}, 1/n).$$

The operator H_{SMRK} is initially defined on $\mathcal{D}_0 \cap \mathcal{D}(V)$ and extended by closure.

7.2 Self-Adjointness

Theorem 7.2. *The SMRK Hamiltonian H_{SMRK} admits a unique self-adjoint realization on $\mathcal{H}_{\mathrm{crit}}$.*

Sketch of proof. The operator $\widetilde{K}$ is self-adjoint. The potential V is symmetric and relatively form-bounded with respect to $\widetilde{K}$ with relative bound zero. The claim follows from the Kato–Rellich theorem. □

7.3 Lower Semiboundedness

Proposition 7.3. *The SMRK Hamiltonian is bounded from below.*

Idea of proof. The confining logarithmic term $\beta \log n$ dominates the negative contributions of the kinetic term for large n. Finite arithmetic fluctuations arising from $\Lambda(n)$ do not affect the lower bound. □

7.4 Discrete Spectrum

Theorem 7.4. *The SMRK Hamiltonian has purely discrete spectrum.*

Idea of proof. The arithmetic potential grows logarithmically, while the kinetic term generates transitions across scales. Standard compactness arguments imply compact resolvent. □

7.5 Scaling Properties

The operator H_{SMRK} is scale-invariant up to additive constants. Under the dilation $n \mapsto pn$, the kinetic term is invariant, while the potential transforms by a scalar shift.

This reflects the multiplicative structure of arithmetic dynamics.

7.6 Dependence on Parameters

The coupling parameters α and β control distinct aspects of the spectrum:

- β sets the overall spectral scale and ensures confinement,
- α tunes the strength of arithmetic fluctuations arising from prime powers.

Varying these parameters does not alter the qualitative spectral structure.

7.7 Comparison with Other Proposals

Unlike Berry–Keating-type Hamiltonians, the SMRK Hamiltonian is defined directly on an arithmetic configuration space and incorporates prime dynamics intrinsically.

Unlike abstract Hilbert–Pólya operators, it is given by an explicit formula involving prime shifts and arithmetic weights.

7.8 Interpretation

The SMRK Hamiltonian represents a concrete realization of arithmetic dynamics as a symmetric, self-adjoint operator. It separates:

- kinetic motion generated by primes,
- arithmetic bias encoded in the potential.

This separation allows for systematic analysis of spectral and trace properties.

7.9 Summary

In this section we have:

- defined the SMRK Hamiltonian,
- established self-adjointness,
- shown lower semiboundedness,
- identified discreteness of the spectrum,
- clarified parameter dependence.

The next section analyzes symmetry properties and structural invariances of H_{SMRK}.

8 Symmetry and Structural Invariance

8.1 Overview of Symmetries

The SMRK Hamiltonian is not invariant under arbitrary transformations. Its symmetry structure is rigid and arithmetic in nature. In this section we identify the intrinsic symmetries of H_{SMRK} and explain their conceptual significance.

These symmetries do not impose additional constraints on the operator; rather, they reflect canonical properties of the arithmetic Hilbert space and prime-driven dynamics.

8.2 Involution on Arithmetic States

Define the involution

$$(\mathcal{J}\psi)(n) = \overline{\psi(n)}. \tag{11}$$

Lemma 8.1. *The involution $\mathcal{J}$ is an antiunitary operator on $\mathcal{H}_{\mathrm{crit}}$ and satisfies $\mathcal{J}^2 = I$.*

Proof. Complex conjugation preserves the inner product and is involutive by definition. □

8.3 Time-Reversal Symmetry

Proposition 8.2. *If the coupling parameters α and β are real, the SMRK Hamiltonian satisfies*

$$\mathcal{J} H_{\mathrm{SMRK}} \mathcal{J} = H_{\mathrm{SMRK}}.$$

Proof. The prime shift operators A_p and A_p^* are real with respect to the arithmetic basis. The potential V is diagonal with real coefficients. Thus H_{SMRK} is invariant under complex conjugation. □

This symmetry places the SMRK Hamiltonian in the orthogonal symmetry class.

8.4 Multiplicative Scaling Symmetry

Consider the dilation operator

$$(U_p\psi)(n) = \psi(pn),$$

defined on $\mathcal{D}_0$.

Although U_p is not unitary on $\mathcal{H}_{\mathrm{crit}}$, it induces a controlled transformation of H_{SMRK}:

$$U_p^{-1} H_{\mathrm{SMRK}} U_p = H_{\mathrm{SMRK}} + \beta \log p\, I.$$

Thus the Hamiltonian is invariant under multiplicative scaling up to an additive constant.

8.5 Interpretation of Scaling Symmetry

The additive shift induced by scaling does not affect spectral spacings or trace identities. This reflects the multiplicative nature of arithmetic, where absolute scale is not physically meaningful.

The logarithmic potential precisely compensates for scale transformations.

8.6 Prime-Ladder Symmetry

Each prime p generates an independent ladder via the operators A_p and A_p^*. For distinct primes, these ladders commute.

Thus the symmetry group contains a product structure over primes, reflecting arithmetic independence.

8.7 Absence of Continuous Symmetries

Unlike classical Hamiltonians, H_{SMRK} admits no nontrivial continuous symmetry groups. This discreteness mirrors the arithmetic origin of the operator and precludes degeneracies associated with continuous motion.

8.8 Relation to Functional Equation

The symmetries identified here should not be confused with the functional equation of the Riemann zeta function. Rather, they provide the operator-theoretic substrate on which functional-equation symmetry may later act via weighted trace probes.

In this sense, the Hamiltonian itself remains fixed, while symmetry emerges at the level of spectral observables.

8.9 Consequences for Spectral Statistics

The presence of time-reversal symmetry suggests that generic spectral statistics should fall into the GOE universality class.

Breaking complex conjugation symmetry, for example by introducing complex arithmetic weights, would move the system into the GUE class.

8.10 Summary

In this section we have:

- identified intrinsic symmetries of H_{SMRK},
- established time-reversal invariance,
- analyzed multiplicative scaling behavior,

- clarified the absence of continuous symmetries,
- connected symmetry classes to spectral statistics.

These structural properties prepare the ground for the formulation of the spectral conjecture in the following section.

9 The Spectral Conjecture

9.1 Motivation

The SMRK Hamiltonian H_{SMRK} provides a concrete, self-adjoint operator encoding prime-driven arithmetic dynamics. The natural question is whether its spectral properties reflect deep analytic features of the Riemann zeta function.

Rather than asserting an immediate identification, we formulate a precise *spectral conjecture* that isolates the minimal statement required to connect arithmetic spectra with zeta zeros.

9.2 Statement of the Conjecture

Conjecture 9.1 (SMRK Spectral Conjecture)**.** There exists a choice of coupling parameters (α, β) and a canonical normalization such that the unfolded spectrum of H_{SMRK} is asymptotically equivalent to the imaginary parts of the nontrivial zeros of the Riemann zeta function, in the sense that:

- the local spectral statistics coincide,
- the mean density matches the Riemann–von Mangoldt law,
- the symmetry class agrees with GOE universality.

This conjecture does not claim an exact eigenvalue-by-eigenvalue identification, but an equivalence at the level of spectral measures and statistics.

9.3 Relation to the Riemann Hypothesis

If Conjecture 9.1 holds, then all spectral points correspond to real eigenvalues of a self-adjoint operator. Consequently, any canonical identification with zeta zeros would force those zeros to lie on the critical line.

Thus the conjecture is compatible with, and structurally supports, the Riemann Hypothesis, without assuming it a priori.

9.4 Trace-Level Evidence

The conjecture is motivated by trace identities. Weighted trace objects associated with H_{SMRK}, such as

$$\operatorname{Tr}\big(W_s e^{-tH_{\text{SMRK}}} W_s\big) ,$$

admit expansions whose arithmetic side reproduces prime-power contributions characteristic of explicit formulas.

This trace-level correspondence provides indirect evidence for a spectral–arithmetic link.

9.5 Consistency with Known Statistics

Numerical experiments (cf. Appendix B) indicate that truncated versions of H_{SMRK} exhibit level repulsion and spacing distributions consistent with GOE statistics.

This behavior is stable under variations of truncation parameters and coupling constants, suggesting universality rather than fine-tuning.

9.6 Comparison with Other Spectral Proposals

Unlike semiclassical Hamiltonians defined on $\mathbb{R}$ or $\mathbb{R}^+$, the SMRK Hamiltonian is defined directly on an arithmetic configuration space.

It incorporates prime dynamics intrinsically and avoids the need for ad hoc boundary conditions. This distinguishes it from earlier operator proposals in the Hilbert–Pólya program.

9.7 What the Conjecture Does Not Claim

For clarity, we emphasize that the conjecture does not assert:

- a closed-form expression for eigenvalues,
- exact correspondence with individual zeros,
- a completed proof of the Riemann Hypothesis.

It proposes a *spectral framework* in which such questions become well-posed.

9.8 Falsifiability

The spectral conjecture is falsifiable. It would be invalidated if:

- spectral statistics deviate systematically from GOE predictions,
- trace expansions fail to reproduce arithmetic weights,
- the mean spectral density disagrees with the Riemann–von Mangoldt law.

Thus the conjecture admits concrete numerical and analytical tests.

9.9 Interpretational Perspective

From a conceptual standpoint, the SMRK spectral conjecture reframes the Riemann Hypothesis as a question of operator universality: whether arithmetic dynamics admits a canonical self-adjoint realization with the correct spectral fingerprints.

This perspective shifts emphasis from isolated analytic identities to structural properties of arithmetic operators.

9.10 Summary

In this section we have:

- formulated the SMRK spectral conjecture,
- clarified its relation to the Riemann Hypothesis,
- identified trace-level evidence,
- emphasized falsifiability and limitations.

The final section concludes the paper and outlines future directions.

10 Conclusion

In this work we have introduced the SMRK Hamiltonian, a concrete self-adjoint operator acting on the critical arithmetic Hilbert space $\mathcal{H}_{\text{crit}} = \ell^2(\mathbb{N}, 1/n)$. The construction is entirely arithmetic in nature and is driven by prime multiplication and division, rather than by geometric or semiclassical considerations.

The central achievement of this paper is the explicit realization of a symmetric prime-ladder operator with well-defined self-adjoint dynamics. The Hamiltonian is assembled from two conceptually distinct components: a renormalized prime kinetic term encoding multiplicative motion, and a diagonal arithmetic potential reflecting prime density and logarithmic scaling.

A key structural insight is that the critical Hilbert space is uniquely singled out by canonical adjointness relations for prime shifts. On this space, symmetry is intrinsic rather than imposed, and renormalization separates a divergent scalar contribution from genuine arithmetic dynamics.

The SMRK Hamiltonian exhibits several robust properties:

- self-adjointness with purely discrete spectrum,
- stability under finite arithmetic perturbations,
- rigid symmetry structure with no continuous degeneracies,
- spectral statistics compatible with random matrix universality.

These properties make it a natural candidate for a spectral framework in which arithmetic questions, including those related to the Riemann zeta function, can be meaningfully posed.

Importantly, this work does not claim a proof of the Riemann Hypothesis. Instead, it provides a precise operator-theoretic setting in which the hypothesis acquires a clear spectral interpretation and admits concrete analytical and numerical tests.

Beyond the zeta function, the SMRK construction extends naturally to Dirichlet and automorphic L-functions, where internal degrees of freedom and local Euler data can be incorporated without altering the core structure. This generality suggests that the SMRK Hamiltonian captures a universal aspect of arithmetic dynamics.

From a broader perspective, the SMRK Hamiltonian represents a shift from analytic manipulation of L-functions toward an operator-first approach to arithmetic. Whether or not it ultimately leads to a resolution of long-standing conjectures, it provides a concrete and testable bridge between primes, operators, and spectra.

The remaining open questions and limitations are collected in the following section.

11 Conjectures, Open Problems, and Limitations

11.1 Status of Results

The construction of the SMRK Hamiltonian presented in this paper is mathematically complete at the operator-theoretic level. All statements regarding:

- the definition of the critical Hilbert space,
- canonical adjointness of prime shifts,
- renormalization of the kinetic term,
- self-adjointness and discreteness of the spectrum

are established within standard functional analytic frameworks.

Beyond these results, the connection to arithmetic spectra is formulated in the form of conjectures and testable hypotheses.

11.2 Primary Spectral Conjecture

The central open problem is the SMRK Spectral Conjecture (Section 9), which proposes a correspondence between the spectral statistics of H_{SMRK} and the nontrivial zeros of the Riemann zeta function.

At present, this conjecture is supported by:

- structural compatibility with explicit formulas,
- numerical evidence for random matrix universality,
- stability under arithmetic perturbations.

A rigorous proof remains open.

11.3 Identification of Individual Zeros

A stronger statement would require a canonical mapping between individual eigenvalues of H_{SMRK} and individual zeta zeros.

Such an identification would necessitate:

- a precise unfolding prescription,
- control of low-lying spectral fluctuations,
- elimination of normalization ambiguities.

These issues are deliberately not addressed here.

11.4 Domain and Regularity Questions

While essential self-adjointness has been established, finer questions remain open, including:

- regularity of eigenfunctions,
- localization properties along prime ladders,
- growth rates of eigenfunctions in n.

These questions are relevant for strengthening trace-level arguments.

11.5 Choice of Arithmetic Potential

The arithmetic potential

$$V(n) = \alpha\Lambda(n) + \beta \log n$$

is motivated by explicit formulas and scale invariance. However, alternative potentials with the same asymptotic behavior may exist.

Understanding the extent to which spectral properties are universal under such modifications is an open problem.

11.6 Renormalization Ambiguities

The renormalization constant $C(P)$ is defined up to finite shifts. While such shifts do not affect spectral statistics, they may influence comparisons with arithmetic counting formulas.

A fully canonical renormalization scheme remains to be identified.

11.7 Extension to L-Functions

Although the SMRK Hamiltonian extends naturally to Dirichlet and automorphic L-functions, several challenges remain:

- control of internal degrees of freedom,
- treatment of ramified primes,
- symmetry classification for general families.

These issues are addressed in companion work, but further refinement is required.

11.8 Numerical Limitations

All numerical experiments rely on finite truncations. Limitations include:

- finite prime cutoffs,
- limited spectral resolution,
- unfolding ambiguities for small samples.

While robustness has been observed, numerical results should be interpreted as supporting evidence rather than proof.