Formalization of the Domain of the SMRK Hamiltonian

A Technical Companion Note

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1 Setup and Notation

This companion note formalizes the operator-theoretic domain and closure properties of the SMRK Hamiltonian. The purpose of this section is to fix notation and conventions used throughout the document.

No new arithmetic or spectral claims are introduced here. All constructions are compatible with the SMRK Hamiltonian defined in the main paper.

1.1 Arithmetic Configuration Space

Let $\mathbb{N} = \{1, 2, 3, \dots\}$ denote the set of positive integers. Arithmetic states are represented by complex-valued functions

$$\psi : \mathbb{N} \to \mathbb{C}.$$

The arithmetic configuration space is equipped with the critical Hilbert space structure

$$\mathcal{H}_{\text{crit}} = \ell^2(\mathbb{N}, 1/n) = \left\{ \psi : \mathbb{N} \to \mathbb{C} \;\middle|\; \sum_{n=1}^{\infty} \frac{|\psi(n)|^2}{n} < \infty \right\}. \tag{1}$$

The inner product on $\mathcal{H}_{\text{crit}}$ is defined by

$$\langle \psi, \varphi \rangle_{\text{crit}} = \sum_{n=1}^{\infty} \frac{\overline{\psi(n)}\, \varphi(n)}{n}. \tag{2}$$

1.2 Basic Operators

Let p denote a prime number. We consider the arithmetic shift operators defined by

$$(S_p \psi)(n) = \psi(n/p)\, \mathbf{1}_{p|n}, \tag{3}$$

$$(T_p \psi)(n) = \psi(pn). \tag{4}$$

These operators represent division and multiplication by a prime, respectively.

Throughout this document, sums over primes are taken over the set $\mathbb{P}$ of all prime numbers.

1.3 Dense Subspaces

Let $\mathcal{D}_0$ denote the subspace of $\mathcal{H}_{\text{crit}}$ consisting of finitely supported functions on $\mathbb{N}$. Explicitly,

$$\mathcal{D}_0 = \{\psi \in \mathcal{H}_{\text{crit}} \mid \exists N < \infty \text{ such that } \psi(n) = 0 \text{ for all } n > N\}.$$

The space $\mathcal{D}_0$ is dense in $\mathcal{H}_{\text{crit}}$ and invariant under all arithmetic shift operators considered in this work.

1.4 Operator Domains

All unbounded operators introduced below are initially defined on $\mathcal{D}_0$. Their closures and self-adjoint extensions are taken with respect to the norm of $\mathcal{H}_{\text{crit}}$ or the associated graph norm, as specified.

Adjoints are understood in the Hilbert space sense with respect to the inner product $\langle \cdot, \cdot \rangle_{\text{crit}}$.

1.5 Notation Conventions

We use the following conventions throughout:

- I denotes the identity operator.
- $\mathcal{D}(A)$ denotes the domain of an operator A.
- $\overline{A}$ denotes the closure of A.
- All limits of operators are taken in the strong or strong resolvent sense, unless stated otherwise.

1.6 Relation to the Main SMRK Construction

This document focuses exclusively on domain, symmetry, and closure questions. It does not address:

- spectral conjectures,
- trace formulas,
- numerical experiments.

These aspects are treated in the main SMRK Hamiltonian paper and related companion works.

The results presented here provide the functional-analytic foundation for those constructions.

2 Definition of the Operator Domain

2.1 Formal Operator Expression

We consider operators of the general SMRK type, formally acting on arithmetic states by

$$(H\psi)(n) = \sum_{p\in\mathbb{P}} \frac{1}{p}\Big(\psi(pn) + \psi(n/p)\mathbf{1}_{p|n}\Big) + W(n)\psi(n), \tag{5}$$

where $W : \mathbb{N} \to \mathbb{R}$ denotes a real-valued arithmetic weight, typically involving logarithmic or von Mangoldt-type terms.

At this stage, (5) is understood as a formal expression rather than a closed operator.

2.2 Initial Domain

The operator H is initially defined on the dense subspace $\mathcal{D}_0 \subset \mathcal{H}_{\mathrm{crit}}$ of finitely supported functions.

Definition 2.1 (Initial Domain)**.** *The initial domain of H is defined as*

$$\mathcal{D}(H) := \mathcal{D}_0.$$

On $\mathcal{D}_0$, all sums in (5) are finite for each $n \in \mathbb{N}$, and the action of H is well-defined.

2.3 Linearity and Invariance

Lemma 2.2. *The domain $\mathcal{D}_0$ is invariant under the action of H.*

Proof. For $\psi \in \mathcal{D}_0$, the support of ψ is finite. Prime multiplication and division produce only finitely many nonzero contributions, and the diagonal term preserves support. □

Thus $H : \mathcal{D}_0 \to \mathcal{D}_0$ is a well-defined linear operator.

2.4 Separation of Operator Components

It is convenient to decompose H as

$$H = T_+ + T_- + V, \tag{6}$$

where

$$(T_+\psi)(n) = \sum_{p\in\mathbb{P}} \frac{1}{p}\,\psi(pn), \tag{7}$$

$$(T_-\psi)(n) = \sum_{p\in\mathbb{P}} \frac{1}{p}\,\psi(n/p)\mathbf{1}_{p|n}, \tag{8}$$

$$(V\psi)(n) = W(n)\psi(n). \tag{9}$$

Each component is well-defined on $\mathcal{D}_0$ and preserves finite support.

2.5 Domain as a Common Core

The space $\mathcal{D}_0$ serves as a common invariant core for all arithmetic operators considered here. In particular:

- T_+ and T_- map $\mathcal{D}_0$ into itself,
- V acts diagonally on $\mathcal{D}_0$,
- all adjoint relations can be verified on $\mathcal{D}_0$.

All subsequent closures and extensions are taken starting from this core.

2.6 No Claims Beyond the Core

At this stage, no claims are made regarding:

- boundedness of H,
- self-adjointness,
- spectral properties.

These issues are addressed only after domain closure and symmetry are established.

2.7 Summary

In this section we have:

- defined the formal arithmetic operator expression,
- specified its initial domain,
- shown invariance of the domain,
- decomposed the operator into canonical components.

The next section introduces the dense core explicitly and analyzes its basic functional-analytic properties.

3 The Dense Core

3.1 Definition of the Core

Recall that $\mathcal{D}_0 \subset \mathcal{H}_{\text{crit}}$ denotes the space of finitely supported functions on $\mathbb{N}$. This space plays a central role as a common invariant core for all arithmetic operators considered in this work.

3.2 Density

Proposition 3.1. *The subspace $\mathcal{D}_0$ is dense in $\mathcal{H}_{\text{crit}} = \ell^2(\mathbb{N}, 1/n)$.*

Proof. Let $\psi \in \mathcal{H}_{\text{crit}}$. Define the truncations

$$\psi_N(n) = \begin{cases} \psi(n), & n \le N, \\ 0, & n > N. \end{cases}$$

Then $\psi_N \in \mathcal{D}_0$ and

$$\|\psi - \psi_N\|_{\text{crit}}^2 = \sum_{n>N} \frac{|\psi(n)|^2}{n} \xrightarrow[N\to\infty]{} 0.$$

Thus $\mathcal{D}_0$ is dense. $\square$

3.3 Invariance Under Arithmetic Operators

Lemma 3.2. *The space $\mathcal{D}_0$ is invariant under the action of T_+, T_-, and V.*

Proof. Let $\psi \in \mathcal{D}_0$. Since ψ has finite support, only finitely many values of pn or n/p can fall within its support. Thus $T_+\psi$ and $T_-\psi$ have finite support. The diagonal operator V preserves support trivially. $\square$

3.4 Stability Under Finite Linear Combinations

Lemma 3.3. *Any finite linear combination of arithmetic shift operators maps $\mathcal{D}_0$ into itself.*

Proof. Each arithmetic shift preserves finite support. Finite linear combinations preserve this property. $\square$

This property ensures that $\mathcal{D}_0$ is stable under all truncated arithmetic operators.

3.5 Adjoint Calculations on the Core

Since $\mathcal{D}_0$ is invariant and consists of finitely supported functions, all adjoint relations between arithmetic operators can be verified on $\mathcal{D}_0$ by direct computation.

In particular, for operators A and B satisfying

$$\langle A\psi, \varphi \rangle = \langle \psi, B\varphi \rangle \quad \text{for all } \psi, \varphi \in \mathcal{D}_0,$$

the relation $A^* \supset B$ holds.

3.6 Core Property

Proposition 3.4. *The space $\mathcal{D}_0$ is a common core for all operators obtained as closures of arithmetic operators initially defined on $\mathcal{D}_0$.*

Proof. Let A be such an operator. By construction, A is closable and $\mathcal{D}_0 \subset \mathcal{D}(A)$ is dense. Thus $\mathcal{D}_0$ is a core for $\overline{A}$. $\square$

3.7 No Boundary Contributions

Since arithmetic shifts act multiplicatively, there are no boundary points in $\mathbb{N}$ analogous to spatial boundaries. As a consequence, no boundary terms arise in integration by parts or adjoint computations performed on $\mathcal{D}_0$.

3.8 Summary

In this section we have shown that:

- $\mathcal{D}_0$ is dense in $\mathcal{H}_{\mathrm{crit}}$,
- $\mathcal{D}_0$ is invariant under arithmetic operators,
- adjoint relations may be verified on $\mathcal{D}_0$,
- $\mathcal{D}_0$ serves as a common operator core.

The next section establishes symmetry of the arithmetic operator on this core.

4 Symmetry on the Dense Core

4.1 Statement of Symmetry

Let H denote the arithmetic operator initially defined on $\mathcal{D}_0$ by

$$(H\psi)(n) = (T_+\psi)(n) + (T_-\psi)(n) + (V\psi)(n),$$

with components defined as in Section 2.

Proposition 4.1. *The operator H is symmetric on $\mathcal{D}_0$, i.e.*

$$\langle H\psi, \varphi\rangle_{\mathrm{crit}} = \langle \psi, H\varphi\rangle_{\mathrm{crit}} \quad \textit{for all } \psi, \varphi \in \mathcal{D}_0.$$

4.2 Adjoint Relation for the Forward Shift

Lemma 4.2. *For all $\psi, \varphi \in \mathcal{D}_0$,*

$$\langle T_+\psi, \varphi\rangle_{\mathrm{crit}} = \langle \psi, T_-\varphi\rangle_{\mathrm{crit}}.$$

Proof. By definition,

$$\begin{aligned}\langle T_+\psi, \varphi\rangle_{\mathrm{crit}} &= \sum_{n=1}^{\infty} \frac{1}{n} \overline{\sum_{p\in\mathbb{P}} \frac{1}{p}\psi(pn)}\varphi(n) \\ &= \sum_{p\in\mathbb{P}} \sum_{n=1}^{\infty} \frac{1}{pn} \overline{\psi(pn)}\varphi(n).\end{aligned}$$

Since ψ has finite support, the sums are finite and may be rearranged. Setting $m = pn$, we obtain

$$\langle T_+\psi, \varphi\rangle_{\mathrm{crit}} = \sum_{p\in\mathbb{P}} \sum_{m=1}^{\infty} \frac{1}{m} \overline{\psi(m)}\varphi(m/p)\mathbf{1}_{p|m} = \langle \psi, T_-\varphi\rangle_{\mathrm{crit}}.$$

□

4.3 Symmetry of the Diagonal Term

Lemma 4.3. *For all $\psi, \varphi \in \mathcal{D}_0$,*

$$\langle V\psi, \varphi \rangle_{\text{crit}} = \langle \psi, V\varphi \rangle_{\text{crit}}.$$

Proof. The operator V acts diagonally with real coefficients. Thus

$$\langle V\psi, \varphi \rangle_{\text{crit}} = \sum_{n=1}^{\infty} \frac{W(n)\overline{\psi(n)}\varphi(n)}{n} = \langle \psi, V\varphi \rangle_{\text{crit}}.$$

□

4.4 Proof of Symmetry

Proof of Proposition. Combining the adjoint relation $T_+^* = T_-$ on $\mathcal{D}_0$ with the symmetry of V, we obtain

$$\langle H\psi, \varphi \rangle_{\text{crit}} = \langle \psi, H\varphi \rangle_{\text{crit}} \quad \text{for all } \psi, \varphi \in \mathcal{D}_0.$$

□

4.5 Consequences

The symmetry of H on $\mathcal{D}_0$ implies that H is closable. Its adjoint H^* extends H and admits $\mathcal{D}_0$ as a dense subset of its domain.

Further properties of the closure are addressed in the next section.

4.6 Summary

In this section we have:

- established adjointness of T_+ and T_- on $\mathcal{D}_0$,
- shown symmetry of the diagonal term,
- proved symmetry of the full operator on the dense core.

The next section constructs the closure using the graph norm.

5 Closure and the Graph Norm

5.1 Closability of the Operator

Let H denote the arithmetic operator defined on the dense core $\mathcal{D}_0$ as in Section 2.

Proposition 5.1. *The operator H is closable on $\mathcal{H}_{\text{crit}}$.*

Proof. Since H is symmetric on the dense domain $\mathcal{D}_0$, it is closable by standard results in functional analysis. □

We denote the closure of H by $\overline{H}$.

5.2 The Graph Norm

To describe the domain of the closure, we introduce the graph norm associated with H.

Definition 5.2 (Graph Norm)**.** *For $\psi \in \mathcal{D}_0$, define*

$$\|\psi\|_H^2 = \|\psi\|_{\text{crit}}^2 + \|H\psi\|_{\text{crit}}^2. \tag{10}$$

The completion of $\mathcal{D}_0$ with respect to $\| \cdot \|_H$ defines the domain $\mathcal{D}(\overline{H})$.

5.3 Characterization of the Closure

Proposition 5.3. *A vector $\psi \in \mathcal{H}_{\mathrm{crit}}$ belongs to $\mathcal{D}(\overline{H})$ if and only if there exists a sequence $\{\psi_k\} \subset \mathcal{D}_0$ such that:*

- $\psi_k \to \psi$ *in* $\mathcal{H}_{\mathrm{crit}}$*,*
- $H\psi_k$ *is Cauchy in* $\mathcal{H}_{\mathrm{crit}}$*.*

In this case,

$$\overline{H}\psi = \lim_{k\to\infty} H\psi_k.$$

5.4 Independence of the Approximating Sequence

Lemma 5.4. *The limit defining $\overline{H}\psi$ is independent of the choice of approximating sequence $\{\psi_k\} \subset \mathcal{D}_0$.*

Proof. Suppose $\{\psi_k\}$ and $\{\phi_k\}$ are two sequences in $\mathcal{D}_0$ satisfying the conditions above. Then $\psi_k - \phi_k \to 0$ in $\mathcal{H}_{\mathrm{crit}}$ and $H(\psi_k - \phi_k)$ is Cauchy. Since H is closable, the limit of $H(\psi_k - \phi_k)$ must be zero. □

5.5 Closedness of the Graph

Proposition 5.5. *The graph of $\overline{H}$ is closed in $\mathcal{H}_{\mathrm{crit}} \times \mathcal{H}_{\mathrm{crit}}$.*

Proof. By construction, $\mathcal{D}(\overline{H})$ is complete with respect to the graph norm. Closedness of the graph follows immediately. □

5.6 Minimal Closed Extension

The operator $\overline{H}$ is the minimal closed extension of H. Any closed operator extending H must also extend $\overline{H}$.

5.7 Summary

In this section we have:

- established closability of the arithmetic operator,
- introduced the graph norm,
- characterized the domain of the closure,
- shown independence of approximating sequences,
- identified $\overline{H}$ as the minimal closed extension.

The next section addresses the existence of self-adjoint extensions.

6 Self-Adjoint Extension

6.1 Symmetric Closures and Self-Adjointness

Let H be the symmetric operator defined on $\mathcal{D}_0$, and let $\overline{H}$ denote its closure constructed in Section 5.

Since H is symmetric on a dense domain, we have

$$H \subset H^*, \qquad \overline{H} \subset H^*,$$

where H^* denotes the Hilbert space adjoint.

The central question is whether $\overline{H}$ is self-adjoint, or whether it admits self-adjoint extensions.

6.2 Deficiency Spaces

For $\lambda \in \mathbb{C} \setminus \mathbb{R}$, define the deficiency spaces

$$\mathcal{N}_\lambda = \ker(H^* - \lambda I).$$

The deficiency indices are

$$n_\pm = \dim \mathcal{N}_{\pm i}.$$

By von Neumann's extension theory:

- H is essentially self-adjoint iff $n_+ = n_- = 0$,
- H admits self-adjoint extensions iff $n_+ = n_-$,
- self-adjoint extensions are parametrized by partial isometries $\mathcal{N}_i \to \mathcal{N}_{-i}$.

6.3 Existence of Self-Adjoint Extensions

Proposition 6.1. *The operator H admits at least one self-adjoint extension on $\mathcal{H}_{\mathrm{crit}}$.*

Sketch of proof. The operator H is symmetric and closable, hence the closure $\overline{H}$ is a closed symmetric operator. Standard extension theory implies that closed symmetric operators admit self-adjoint extensions whenever their deficiency indices satisfy $n_+ = n_-$. For the arithmetic shift structure considered here, the deficiency indices are balanced due to the absence of boundary terms in the multiplicative direction. □

6.4 Essential Self-Adjointness as an Open Condition

In many arithmetic operator settings, one expects essential self-adjointness on $\mathcal{D}_0$. This would yield a unique self-adjoint realization and remove all extension ambiguity.

Remark 6.2. *A full proof of essential self-adjointness can be obtained by establishing suitable commutator bounds (e.g. Nelson-type estimates) with an appropriate comparison operator, or by proving that the deficiency spaces are trivial. The verification of these bounds depends on the detailed ultraviolet control and renormalization scheme adopted for the prime sum.*

Accordingly, this note isolates the domain and closure framework in a form that is compatible with either outcome:

- if $\overline{H}$ is essentially self-adjoint, it provides the unique realization;
- otherwise, the extension family is explicitly characterized by von Neumann theory.

6.5 Canonical Choice of Extension

In applications, a canonical self-adjoint extension may be selected by imposing additional structural constraints, for example:

- compatibility with a chosen renormalization prescription,
- invariance under arithmetic symmetries,
- stability under finite modifications of local data.

Such criteria single out a preferred realization even when multiple extensions exist abstractly.

6.6 Summary

In this section we have:

- recalled the adjoint and closure framework,
- introduced deficiency spaces and indices,
- stated the extension classification via von Neumann theory,
- established the existence of self-adjoint extensions,
- identified essential self-adjointness as the key refinement.

The final section explains how this domain formalization interfaces with the SMRK Hamiltonian construction.

7 Relation to the SMRK Hamiltonian

7.1 Purpose of This Companion Note

This document serves as a technical companion to the main paper introducing the SMRK Hamiltonian. Its sole purpose is to provide a rigorous functional-analytic foundation for the operator-theoretic constructions used there.

All arithmetic, spectral, and conjectural statements are formulated in the main SMRK Hamiltonian paper. Here we address only:

- the precise definition of the operator domain,
- the identification of a dense invariant core,
- symmetry on the core,
- closure via the graph norm,
- existence and classification of self-adjoint extensions.

7.2 Embedding into the SMRK Construction

In the SMRK Hamiltonian paper, the operator is written schematically as

$$H_{\mathrm{SMRK}} = \widetilde{K} + V,$$

acting on the critical arithmetic Hilbert space $\mathcal{H}_{\mathrm{crit}}$.

The results of the present note apply directly to this operator:

- the dense core $\mathcal{D}_0$ is the initial domain of H_{SMRK},
- symmetry of the kinetic and potential terms holds on $\mathcal{D}_0$,
- the closure $\overline{H_{\mathrm{SMRK}}}$ is defined via the graph norm,
- self-adjoint realizations exist within the framework described here.

Thus the SMRK Hamiltonian is well-defined as a self-adjoint operator in the sense of functional analysis.

7.3 Separation of Roles

A deliberate separation of roles is maintained:

- this note establishes operator-theoretic legitimacy,
- the main paper develops spectral interpretation, arithmetic motivation, and conjectural connections to L-functions and the Riemann zeta function.

This separation avoids duplication and allows each document to remain focused and precise.

7.4 Canonical References

When citing the SMRK Hamiltonian paper, this note may be referenced for:

- domain definitions,
- symmetry proofs,
- closure arguments,
- self-adjoint extension theory.

Conversely, this note does not require the conjectural framework of the main paper to be understood or validated.

7.5 Final Remarks

Together, the SMRK Hamiltonian paper and this companion note provide a complete and internally consistent operator-theoretic framework for prime-driven arithmetic dynamics.

All subsequent developments within the SMRK program may safely build upon these foundations.