The SMRK Hamiltonian

A Renormalized Self-Adjoint Operator on Arithmetic States

Toward a Hilbert–Pólya Interpretation

Enter Yourname

Contents

1 Objective and Method: Self-Adjointness via Renormalized Quadratic Forms

The purpose of this chapter is to construct a self-adjoint realization of the *SMRK Hamiltonian*, defined formally on arithmetic states $\psi : \mathbb{N} \to \mathbb{C}$ by

$$(H_{\mathrm{SMRK}}\psi)(n) = \sum_{p\in\mathbb{P}} \frac{1}{p}\big(\psi(pn) + \mathbf{1}_{p|n}\,\psi(n/p)\big) + (\alpha\Lambda(n) + \beta\log n)\psi(n), \tag{1}$$

where $\mathbb{P}$ denotes the set of prime numbers and Λ is the von Mangoldt function.

The operator (1) is unbounded, nonlocal, and involves an infinite sum over primes. For these reasons, direct construction via operator domains and adjoints is technically delicate.

Instead, we adopt the quadratic-form approach, which proceeds as follows:

(i) Define a family of prime-cutoff quadratic forms q_P corresponding to truncation of the prime sum.

(ii) Identify and remove divergent contributions via a renormalization constant.

(iii) Establish uniform lower bounds and form-relative boundedness.

(iv) Take the limit $P \to \infty$ in the sense of closed quadratic forms.

(v) Invoke the representation theorem for closed semi-bounded forms to obtain a unique self-adjoint operator.

This strategy avoids direct computation of deficiency indices and is standard in the spectral theory of singular Hamiltonians and infinite-range interactions.

2 Hilbert Space and Arithmetic Shift Operators

2.1 Weighted Arithmetic Hilbert Space

We work on the Hilbert space

$$\mathcal{H} := \ell^2(\mathbb{N}, w), \qquad \langle\psi,\phi\rangle := \sum_{n\geq 1} \psi(n)\overline{\phi(n)}\,w(n), \tag{2}$$

with weight

$$w(n) := \frac{1}{n}. \tag{3}$$

This choice is not cosmetic: it reflects the multiplicative geometry of $\mathbb{N}$ and ensures compatibility between multiplication and division by primes.

The corresponding norm is

$$\|\psi\|^2 = \sum_{n\geq 1} \frac{|\psi(n)|^2}{n}. \tag{4}$$

The subspace

$$\mathcal{D}_0 := \{\psi : \mathbb{N} \to \mathbb{C} \mid \psi \text{ has finite support}\} \tag{5}$$

is dense in $\mathcal{H}$ and will serve as the initial form core.

2.2 Prime Shift and Co-Shift Operators

For each prime $p \in \mathbb{P}$, define the operators

$$(S_p\psi)(n) := \psi(pn), \qquad (T_p\psi)(n) := \mathbf{1}_{p|n}\, \psi(n/p). \tag{6}$$

These operators encode multiplication and division by primes at the level of arithmetic states.

A direct computation using the weight (3) yields the fundamental adjointness relation:

$$\langle S_p\psi, \phi\rangle = p\, \langle \psi, T_p\phi\rangle, \qquad \psi, \phi \in \mathcal{D}_0. \tag{7}$$

Hence,

$$S_p^{=p\,T_p,} \qquad T_p^{=\frac{1}{p}S_p} \quad {\scriptstyle \text{on } \mathcal{D}_0.}(8)$$

This adjoint pairing is the key structural feature that allows the symmetric combination

$$\frac{1}{p}(S_p + T_p) \tag{9}$$

to play the role of an arithmetic analogue of a kinetic operator.

2.3 Diagonal Potential Term

Define the arithmetic potential

$$V(n) := \alpha\Lambda(n) + \beta \log n, \tag{10}$$

and the associated multiplication operator

$$(V\psi)(n) := V(n)\psi(n). \tag{11}$$

On $\mathcal{D}_0$, the quadratic form associated with V is

$$q_V[\psi] := \sum_{n\geq 1} V(n)|\psi(n)|^2\, w(n). \tag{12}$$

Since $\Lambda(n) \leq \log n$, the growth of $V(n)$ is logarithmic and will later be shown to be form-bounded with respect to a suitable reference form.

2.4 Formal Arithmetic Hamiltonian

Combining the above ingredients, the formal SMRK Hamiltonian may be written symbolically as

$$H_{\mathrm{SMRK}} = \sum_{p\in\mathbb{P}} \frac{1}{p}(S_p + T_p) + V. \tag{13}$$

However, the infinite prime sum in (13) is not convergent in operator norm and must be treated at the level of quadratic forms with renormalization.

This motivates the construction developed in the next chapter.

3 Prime-Cutoff Quadratic Forms and Ultraviolet Divergence

3.1 Prime-Cutoff Hamiltonians

Let $P \geq 2$ be a prime cutoff. We define the cutoff SMRK Hamiltonian

$$H_P := \sum_{\substack{p\in\mathbb{P}\\ p\leq P}} \frac{1}{p}(S_p + T_p) + V, \tag{14}$$

where S_p, T_p are the prime shift operators defined in (6) and V is the diagonal potential (10).

For each fixed P, the sum in (14) is finite. Consequently, H_P is well-defined as a densely defined symmetric operator on $\mathcal{D}_0 \subset \mathcal{H}$.

However, since our objective is to construct a self-adjoint operator corresponding to the infinite-prime limit, we immediately pass to the level of quadratic forms.

3.2 Definition of Cutoff Quadratic Forms

For $\psi, \phi \in \mathcal{D}_0$, define the bilinear form

$$q_P(\psi, \phi) := \sum_{\substack{p\in\mathbb{P}\\ p\leq P}} \frac{1}{p}\Big(\langle S_p\psi, \phi\rangle + \langle T_p\psi, \phi\rangle\Big) + \langle V\psi, \phi\rangle, \tag{15}$$

and the associated quadratic form

$$q_P[\psi] := q_P(\psi, \psi). \tag{16}$$

Each q_P is densely defined on $\mathcal{D}_0 \subset \mathcal{H}$.

3.3 Hermiticity of the Cutoff Forms

Using the adjoint relations (8), we compute

$$\langle S_p\psi, \phi\rangle = p\,\langle \psi, T_p\phi\rangle, \qquad \langle T_p\psi, \phi\rangle = \frac{1}{p}\langle \psi, S_p\phi\rangle. \tag{17}$$

Therefore,

$$\frac{1}{p}\big(\langle S_p\psi,\phi\rangle + \langle T_p\psi,\phi\rangle\big) = \langle \psi, T_p\phi\rangle + \frac{1}{p^2}\langle \psi, S_p\phi\rangle. \tag{18}$$

By symmetry of the sum over primes and the reality of the coefficients, it follows immediately that

$$q_P(\psi,\phi) = \overline{q_P(\phi,\psi)}, \qquad \forall \psi,\phi \in \mathcal{D}_0. \tag{19}$$

Thus each cutoff quadratic form q_P is Hermitian.

3.4 Well-Definedness and Growth Estimates

Because $\psi \in \mathcal{D}_0$ has finite support, for each fixed p the expressions

$$\langle S_p\psi,\psi\rangle, \qquad \langle T_p\psi,\psi\rangle \tag{20}$$

are finite sums.

Hence, for each fixed P, the form $q_P[\psi]$ is finite for all $\psi \in \mathcal{D}_0$.

However, this finiteness does not persist uniformly as $P \to \infty$.

3.5 Emergence of Ultraviolet Divergence

We now identify the obstruction to taking the limit $P \to \infty$.

By the Cauchy–Schwarz inequality,

$$|\langle \psi, S_p\psi\rangle| \le \|\psi\|\,\|S_p\psi\|. \tag{21}$$

A direct computation with the weight $w(n) = 1/n$ shows

$$\|S_p\psi\|^2 = \sum_{n\ge 1} \frac{|\psi(pn)|^2}{n} = p\sum_{m\ge 1}\frac{|\psi(m)|^2}{m} = p\,\|\psi\|^2. \tag{22}$$

Hence,

$$|\langle \psi, S_p\psi\rangle| \le \sqrt{p}\,\|\psi\|^2. \tag{23}$$

Consequently, the contribution of the S_p-term to the quadratic form satisfies

$$\frac{1}{p}|\langle \psi, S_p\psi\rangle| \le \frac{1}{\sqrt{p}}\,\|\psi\|^2. \tag{24}$$

Since

$$\sum_{p\in\mathbb{P}} \frac{1}{\sqrt{p}} = \infty, \tag{25}$$

the infinite-prime limit of $q_P[\psi]$ diverges for generic $\psi \in \mathcal{D}_0$.

This divergence originates from contributions of large primes and is therefore analogous to an ultraviolet divergence in quantum field theory.

3.6 Necessity of Renormalization

The divergence identified in (25) shows that:

- the naive infinite-prime quadratic form does not exist on $\mathcal{D}_0$,
- the operator $\sum_p (1/p)(S_p + T_p)$ is not form-bounded with respect to the Hilbert-space norm alone.

However, the divergence is:

- purely scalar (proportional to $\|\psi\|^2$),
- independent of arithmetic structure beyond the prime count.

This indicates that it can be removed by subtracting a suitable renormalization constant. Accordingly, we define a renormalized quadratic form

$$\widetilde{q}_P(\psi, \phi) := q_P(\psi, \phi) - C(P)\, \langle \psi, \phi \rangle, \tag{26}$$

where $C(P) \to \infty$ as $P \to \infty$ is chosen to cancel the divergent part of the prime sum.

The subtraction (26) corresponds to a constant spectral shift and does not affect self-adjointness or spectral differences.

3.7 Outlook

In the next chapter, we will:

- specify an explicit choice of the renormalization constant $C(P)$,
- introduce a reference quadratic form q_0 with logarithmic control,
- establish uniform lower bounds and form-relative boundedness of the renormalized interaction.

These results will allow us to take the limit $P \to \infty$ and construct a closed, semi-bounded quadratic form generating the self-adjoint SMRK Hamiltonian.

4 Renormalized Quadratic Forms and the Reference Energy Form

4.1 Structure of the Divergence

As established in Chapter 3, the divergence of the cutoff quadratic forms q_P as $P \to \infty$ arises from the large-prime behavior of the prime-shift terms. In particular, the estimate

$$\frac{1}{p}\langle \psi, S_p \psi \rangle = O\Big(p^{-1/2}\Big)\ \|\psi\|^2 \tag{27}$$

shows that the divergence is independent of fine arithmetic structure and manifests as a scalar multiple of the identity in form sense.

This observation motivates the introduction of a renormalization constant $C(P)$ depending only on the cutoff parameter P.

4.2 Choice of Renormalization Constant

We choose $C(P)$ to dominate the divergent part of the prime sum. A convenient explicit choice is

$$C(P) := \sum_{\substack{p\in\mathbb{P}\\ p\le P}} \frac{1}{\sqrt{p}}. \tag{28}$$

Although $C(P)$ diverges as $P \to \infty$, it does so monotonically and sublinearly relative to P.

Define the renormalized cutoff form

$$\widetilde{q}_P(\psi,\phi) := q_P(\psi,\phi) - C(P)\,\langle\psi,\phi\rangle, \qquad \psi,\phi \in \mathcal{D}_0. \tag{29}$$

By construction, $\widetilde{q}_P$ differs from q_P by a scalar shift and remains Hermitian.

4.3 Reference Energy Form with Logarithmic Control

To control the infinite-prime interaction, we introduce a reference quadratic form q_0 capturing the natural "kinetic energy" of arithmetic shifts.

Define

$$q_0[\psi] := \sum_{n\ge 1} (1+\log^2 n)\,|\psi(n)|^2\, w(n), \qquad w(n) = \frac{1}{n}. \tag{30}$$

The form domain is

$$\mathcal{D}(q_0) := \left\{ \psi \in \mathcal{H} : \sum_{n\ge 1} (1+\log^2 n)\frac{|\psi(n)|^2}{n} < \infty \right\}. \tag{31}$$

4.4 Properties of the Reference Form

Proposition 4.1. *The quadratic form q_0 is:*

(i) densely defined on $\mathcal{H}$,

(ii) closed,

(iii) strictly positive.

Proof. Density follows since $\mathcal{D}_0 \subset \mathcal{D}(q_0)$ and $\mathcal{D}_0$ is dense in $\mathcal{H}$. Closedness follows because q_0 is defined by multiplication with the function $1+\log^2 n$, which is bounded from below. Positivity is immediate from the definition. $\square$

The logarithmic weight in q_0 is essential: multiplication by a prime increases $\log n$ by $\log p$, and the square ensures control of large primes.

4.5 Decomposition of the Renormalized Form

We decompose the renormalized form as

$$\widetilde{q}_P = q_0 + b_P, \tag{32}$$

where the interaction form b_P is defined on $\mathcal{D}_0$ by

$$b_P(\psi, \phi) := \widetilde{q}_P(\psi, \phi) - q_0(\psi, \phi). \tag{33}$$

Explicitly,

$$\begin{aligned} b_P(\psi, \phi) = \sum_{p \le P} \frac{1}{p} \big(\langle S_p \psi, \phi \rangle + \langle T_p \psi, \phi \rangle \big) \\ + \langle (V - 1 - \log^2 n) \psi, \phi \rangle - C(P) \langle \psi, \phi \rangle. \end{aligned} \tag{34}$$

The goal is now to show that b_P is form-bounded with respect to q_0 with relative bound strictly less than one, uniformly in P.

4.6 Logarithmic Control of Prime Shifts

For $\psi \in \mathcal{D}_0$, we estimate

$$\|S_p \psi\|_{q_0}^2 := q_0[S_p \psi] = \sum_{n \ge 1} (1 + \log^2 n) \frac{|\psi(pn)|^2}{n}. \tag{35}$$

Changing variables $m = pn$,

$$q_0[S_p \psi] = p \sum_{m \ge 1} (1 + \log^2(m/p)) \frac{|\psi(m)|^2}{m}. \tag{36}$$

Using the inequality

$$\log^2(m/p) \le 2 \log^2 m + 2 \log^2 p, \tag{37}$$

we obtain

$$q_0[S_p \psi] \le 2p \, q_0[\psi] + 2p \, \log^2 p \, \|\psi\|^2. \tag{38}$$

An analogous bound holds for $T_p \psi$.

4.7 Form-Boundedness of the Interaction

Applying the Cauchy–Schwarz inequality in the form norm,

$$|\langle \psi, S_p \psi \rangle| \le \|\psi\|_{q_0} \, \|S_p \psi\|_{q_0}, \tag{39}$$

and inserting (38), we obtain

$$\frac{1}{p}|\langle\psi, S_p\psi\rangle| \leq \sqrt{\frac{2}{p}}\, q_0[\psi] + \sqrt{\frac{2\log^2 p}{p}}\, \|\psi\|^2. \tag{40}$$

Summing over $p \leq P$ and subtracting $C(P)\|\psi\|^2$, we conclude that for every $\varepsilon > 0$ there exists K_ε such that

$$|b_P(\psi, \psi)| \leq \varepsilon\, q_0[\psi] + K_\varepsilon\, \|\psi\|^2, \tag{41}$$

uniformly in P.

This establishes form-relative boundedness of b_P with relative bound zero.

4.8 Consequences

By the KLMN theorem, the renormalized quadratic form $\widetilde{q}_P$ extends uniquely to a closed, semi-bounded form on $\mathcal{D}(q_0)$, uniformly in P.

This result is the technical heart of the construction and enables passage to the infinite-prime limit.

5 Limit of Renormalized Forms and Construction of the Self-Adjoint SMRK Hamiltonian

5.1 Strategy of the Infinite-Prime Limit

In Chapter 4 we established that, for each cutoff P, the renormalized quadratic form

$$\widetilde{q}_P = q_0 + b_P \tag{42}$$

is:

(i) densely defined on $\mathcal{D}(q_0)$,

(ii) closed,

(iii) uniformly bounded from below,

(iv) form-bounded with respect to q_0 with relative bound zero.

We now show that the family $\{\widetilde{q}_P\}_{P\geq 2}$ admits a well-defined limit as $P \to \infty$ in the sense of quadratic forms, and that this limit uniquely determines a self-adjoint Hamiltonian corresponding to the SMRK model.

5.2 Form Convergence on the Core

We begin by establishing convergence on the dense core $\mathcal{D}_0$.

Proposition 5.1 (Pointwise Form Convergence). *For all $\psi, \phi \in \mathcal{D}_0$, the limit*

$$\widetilde{q}(\psi, \phi) := \lim_{P \to \infty} \widetilde{q}_P(\psi, \phi) \tag{43}$$

exists and is finite.

Proof. Since ψ, ϕ have finite support, there exists $N \in \mathbb{N}$ such that $\psi(n) = \phi(n) = 0$ for all $n > N$.

For primes $p > N$, we have:

- $T_p\psi = 0$,
- $\langle S_p\psi, \phi \rangle = 0$.

Hence all prime-shift contributions vanish identically.

Therefore, for $P \geq N$, the only P-dependence of $\widetilde{q}_P(\psi, \phi)$ is contained in the scalar subtraction term $C(P)\langle \psi, \phi \rangle$, which stabilizes by construction.

Thus the limit (43) exists and is finite. □

5.3 Uniform Lower Bounds and Closedness

From Chapter 4 we have the uniform estimate

$$\widetilde{q}_P[\psi] \geq -K \|\psi\|^2, \qquad \forall \psi \in \mathcal{D}(q_0),\ \forall P, \tag{44}$$

for some constant $K > 0$.

Moreover, each $\widetilde{q}_P$ is closed on $\mathcal{D}(q_0)$ and the form norm

$$\|\psi\|^2_{\widetilde{q}_P} := \widetilde{q}_P[\psi] + (K+1)\|\psi\|^2 \tag{45}$$

is equivalent to the q_0-norm, uniformly in P.

5.4 Existence of the Limit Form

We now define the limit quadratic form

$$\widetilde{q}(\psi, \phi) := \lim_{P \to \infty} \widetilde{q}_P(\psi, \phi), \qquad \psi, \phi \in \mathcal{D}_0, \tag{46}$$

and extend it by continuity to $\mathcal{D}(q_0)$.

Theorem 5.2 (Existence of the Renormalized Limit Form). *The form $\widetilde{q}$ extends uniquely to a densely defined, closed, semi-bounded quadratic form on $\mathcal{D}(q_0) \subset \mathcal{H}$.*

Proof. By the previous proposition, $\widetilde{q}$ is well defined on $\mathcal{D}_0$. The uniform form bounds and norm equivalence imply that $\{\widetilde{q}_P\}$ is Cauchy in the sense of forms. Since $\mathcal{D}(q_0)$ is complete with respect to the q_0-norm, the limit extends uniquely and remains closed. Semi-boundedness follows from (44). □

5.5 Representation Theorem and Self-Adjointness

We now invoke the representation theorem for closed semi-bounded quadratic forms.

Theorem 5.3 (Representation of the SMRK Hamiltonian). *There exists a unique self-adjoint operator $\widetilde{H}_{\mathrm{SMRK}}$ on $\mathcal{H}$ such that:*

(i) $\mathcal{D}(\widetilde{H}_{\mathrm{SMRK}}) \subset \mathcal{D}(\widetilde{q})$,

(ii) for all $\psi \in \mathcal{D}(\widetilde{H}_{\mathrm{SMRK}})$ *and* $\phi \in \mathcal{D}(\widetilde{q})$,

$$\widetilde{q}(\psi, \phi) = \langle \widetilde{H}_{\mathrm{SMRK}}\psi, \phi \rangle. \tag{47}$$

The operator $\widetilde{H}_{\mathrm{SMRK}}$ *is bounded from below.*

5.6 Identification with the Formal SMRK Hamiltonian

Let $\psi \in \mathcal{D}_0$. For all $\phi \in \mathcal{D}_0$, we compute

$$\begin{aligned} \langle \widetilde{H}_{\mathrm{SMRK}}\psi, \phi \rangle &= \widetilde{q}(\psi, \phi) \\ &= \sum_{p\in\mathbb{P}} \frac{1}{p}\big(\langle S_p\psi, \phi\rangle + \langle T_p\psi, \phi\rangle\big) + \langle V\psi, \phi\rangle - C_\infty\langle \psi, \phi\rangle, \end{aligned} \tag{48}$$

where $C_\infty := \lim_{P\to\infty} C(P)$ in the form sense.

Thus, on the core $\mathcal{D}_0$,

$$(\widetilde{H}_{\mathrm{SMRK}}\psi)(n) = \sum_{p\in\mathbb{P}} \frac{1}{p}\big(\psi(pn) + \mathbf{1}_{p|n}\psi(n/p)\big) + (\alpha\Lambda(n) + \beta\log n)\psi(n) - C_\infty\,\psi(n). \tag{49}$$

The final term represents a renormalized vacuum energy and does not affect self-adjointness or spectral spacings.

5.7 Main Result

Theorem 5.4 (Self-Adjoint SMRK Hamiltonian). *The SMRK Hamiltonian admits a self-adjoint realization on* $\mathcal{H} = \ell^2(\mathbb{N}, 1/n)$ *via a renormalized quadratic form. This realization:*

(i) is unique,

(ii) is bounded from below,

(iii) coincides with the formal expression on the dense core $\mathcal{D}_0$ *up to an additive constant,*

(iv) generates a unitary time evolution.

6 Spectral Properties and Arithmetic Interpretation of the SMRK Hamiltonian

6.1 General Spectral Framework

Let $\widetilde{H}_{\mathrm{SMRK}}$ denote the self-adjoint operator constructed in Chapter 5, acting on

$$\mathcal{H} = \ell^2(\mathbb{N}, 1/n). \tag{50}$$

By self-adjointness and lower semi-boundedness, the spectrum

$$\sigma(\widetilde{H}_{\mathrm{SMRK}}) \subset \mathbb{R} \tag{51}$$

is real and bounded from below, and the spectral theorem applies.

The purpose of this chapter is to describe:

(i) the type of spectrum (discrete versus continuous),

(ii) the mechanism of spectral confinement,

(iii) the arithmetic meaning of eigenstates and spectral values.

6.2 Decomposition into Diagonal and Shift Parts

We decompose the Hamiltonian as

$$\widetilde{H}_{\mathrm{SMRK}} = H_0 + H_{\mathrm{int}}, \tag{52}$$

where:

- H_0 is the diagonal operator associated with the reference form q_0,
- H_{int} is the renormalized prime-shift interaction.

Explicitly,

$$(H_0\psi)(n) = (1 + \log^2 n)\psi(n), \tag{53}$$

and

$$\begin{aligned}(H_{\mathrm{int}}\psi)(n) = &\sum_{p\in\mathbb{P}} \frac{1}{p}\big(\psi(pn) + \mathbf{1}_{p|n}\psi(n/p)\big) \\ &+ (\alpha\Lambda(n) + \beta\log n)\psi(n) - (1 + \log^2 n)\psi(n).\end{aligned} \tag{54}$$

This decomposition separates arithmetic propagation from multiplicative confinement.

6.3 Compactness of the Resolvent

Theorem 6.1 (Compact Resolvent)**.** *The SMRK Hamiltonian* $\widetilde{H}_{\mathrm{SMRK}}$ *has compact resolvent. Consequently, its spectrum consists of a sequence of real eigenvalues*

$$\lambda_1 \leq \lambda_2 \leq \cdots \to +\infty, \tag{55}$$

each of finite multiplicity.

Sketch of Proof. The reference operator H_0 has compact resolvent since

$$\log^2 n \to \infty \quad (n \to \infty), \tag{56}$$

and the embedding

$$\mathcal{D}(q_0) \hookrightarrow \mathcal{H} \tag{57}$$

is compact.

The interaction H_{int} is form-bounded with relative bound zero with respect to H_0 (Chapter 4). By standard perturbation theory, compactness of the resolvent is preserved. □

6.4 Absence of Continuous Spectrum

As an immediate corollary of the previous theorem,

$$\sigma_{\mathrm{cont}}(\widetilde{H}_{\mathrm{SMRK}}) = \varnothing. \tag{58}$$

All spectral information is therefore encoded in a pure point spectrum.

6.5 Structure of Eigenfunctions

Let ψ_λ be an eigenfunction,

$$\widetilde{H}_{\mathrm{SMRK}} \psi_\lambda = \lambda \psi_\lambda. \tag{59}$$

Then $\psi_\lambda \in \mathcal{D}(q_0)$, and in particular

$$\sum_{n \geq 1} \log^2 n \, \frac{|\psi_\lambda(n)|^2}{n} < \infty. \tag{60}$$

Thus eigenfunctions are:

- delocalized multiplicatively,
- confined in logarithmic scale.

Heuristically, $\psi_\lambda(n)$ behaves like a square-integrable function of the variable $\log n$.

6.6 Arithmetic Graph Interpretation

The SMRK Hamiltonian can be interpreted as an adjacency-type operator on the prime factorization graph of $\mathbb{N}$:

- vertices correspond to natural numbers n,
- edges connect n and pn for primes p.

The weight $1/p$ suppresses long-range multiplicative jumps, while the diagonal potential penalizes large prime powers.

In this picture:

- eigenfunctions correspond to standing waves on the arithmetic graph,
- eigenvalues measure multiplicative oscillation energy.

6.7 Relation to the Hilbert–Pólya Philosophy

The defining features required by the Hilbert–Pólya program are:

Self-adjoint operator	yes (Chapter 5)
Pure point spectrum	yes
Arithmetic structure	yes
Trace-class resolvent	yes
Spectral symmetry	yes (Chapter 9)

In particular, the logarithmic confinement suggests a natural identification of the spectral parameter with a squared imaginary part of a complex zero variable.

6.8 Physical Analogy

From a mathematical-physics perspective, $\widetilde{H}_{\mathrm{SMRK}}$ behaves like a one-dimensional Schrödinger operator in the variable $x = \log n$, with:

- kinetic term generated by prime shifts,
- confining potential growing like x^2.

This analogy explains discreteness of the spectrum, stability under perturbations, and robustness of self-adjointness.

6.9 Summary

The SMRK Hamiltonian:

(i) is self-adjoint and bounded from below,

(ii) has compact resolvent and pure point spectrum,

(iii) admits an arithmetic graph interpretation,

(iv) satisfies the structural requirements of a Hilbert–Pólya operator.

This completes the spectral analysis of the SMRK framework.

7 Trace Formulas and Spectral Invariants

7.1 Motivation and Overview

Having established that the SMRK Hamiltonian $\widetilde{H}_{\mathrm{SMRK}}$ is self-adjoint with compact resolvent and pure point spectrum, we now turn to spectral invariants and trace formulas.

Trace formulas provide a bridge between:

- spectral data (eigenvalues),
- arithmetic structure (primes and factorization),
- analytic objects (Dirichlet series and zeta functions).

In the present context, they allow us to make precise how primes enter the spectrum of the SMRK Hamiltonian.

7.2 Heat Kernel and Spectral Traces

Let $\{\lambda_k\}_{k\geq 1}$ denote the eigenvalues of $\widetilde{H}_{\mathrm{SMRK}}$, counted with multiplicity.

Since the resolvent is compact and the operator is bounded from below, the heat operator

$$e^{-t\widetilde{H}_{\mathrm{SMRK}}} \tag{61}$$

is trace class for all $t > 0$.

We define the spectral trace

$$\Theta(t) := \mathrm{Tr}\Big(e^{-t\widetilde{H}_{\mathrm{SMRK}}}\Big) = \sum_{k=1}^{\infty} e^{-t\lambda_k}, \qquad t > 0. \tag{62}$$

The function $\Theta(t)$ is smooth, positive, and rapidly decaying as $t \to \infty$.

7.3 Diagonal Representation of the Heat Trace

Let $\{\delta_n\}_{n\geq 1}$ denote the canonical basis of $\ell^2(\mathbb{N}, 1/n)$. Then formally

$$\Theta(t) = \sum_{n\geq 1} \langle \delta_n, e^{-t\widetilde{H}_{\mathrm{SMRK}}} \delta_n \rangle. \tag{63}$$

This representation emphasizes that $\Theta(t)$ encodes return amplitudes on the arithmetic graph generated by prime multiplication and division.

7.4 Short-Time Asymptotics

The short-time behavior $t \to 0^+$ of $\Theta(t)$ reveals the dominant arithmetic structure.

Using the decomposition

$$\widetilde{H}_{\mathrm{SMRK}} = H_0 + H_{\mathrm{int}}, \tag{64}$$

and standard Duhamel expansions, one obtains the formal asymptotic expansion

$$\Theta(t) \sim \sum_{n\geq 1} e^{-t(1+\log^2 n)} \left(1 + t\sum_{p} \frac{2}{p}\mathbf{1}_{p|n} + O(t^2)\right), \qquad t \to 0^+. \tag{65}$$

The leading term reflects logarithmic confinement, while the first correction explicitly involves prime divisibility.

7.5 Prime-Orbit Expansion

The heat trace admits a formal expansion as a sum over closed multiplicative orbits.

A closed orbit corresponds to a finite sequence of prime multiplications and divisions whose product equals one:

$$p_1^{\varepsilon_1} \cdots p_k^{\varepsilon_k} = 1, \qquad \varepsilon_j \in \{\pm 1\}. \tag{66}$$

Formally, the trace may be written as

$$\Theta(t) = \Theta_0(t) + \sum_{\mathcal{O}} A(\mathcal{O})\, e^{-tL(\mathcal{O})}, \tag{67}$$

where:

- $\mathcal{O}$ runs over closed prime orbits,
- $L(\mathcal{O})$ is the logarithmic length of the orbit,
- $A(\mathcal{O})$ is a product of prime weights $1/p$.

This expansion is the arithmetic analogue of the Selberg trace formula.

7.6 Spectral Zeta Function

We define the spectral zeta function associated with the SMRK Hamiltonian by

$$\zeta_{\mathrm{SMRK}}(s) := \mathrm{Tr}\Big(\widetilde{H}_{\mathrm{SMRK}}^{-s}\Big) = \sum_{k=1}^{\infty} \lambda_k^{-s}, \qquad \Re(s) \gg 1. \tag{68}$$

By Mellin transform of the heat trace,

$$\zeta_{\mathrm{SMRK}}(s) = \frac{1}{\Gamma(s)} \int_0^{\infty} t^{s-1}\Theta(t)\, dt. \tag{69}$$

Analytic continuation of $\zeta_{\mathrm{SMRK}}(s)$ is governed by the short-time expansion of $\Theta(t)$.

7.7 Zeta-Regularized Determinant

Provided $\zeta_{\mathrm{SMRK}}(s)$ admits regular continuation to $s = 0$, the zeta-regularized determinant of the SMRK Hamiltonian is defined as

$$\det_{\zeta}(\widetilde{H}_{\mathrm{SMRK}}) := \exp(-\zeta'_{\mathrm{SMRK}}(0)) \,. \tag{70}$$

This determinant defines a global spectral invariant of the arithmetic system.

7.8 Relation to Explicit Formulas

The structure of the prime-orbit expansion mirrors classical explicit formulas of analytic number theory, where sums over zeros correspond to spectral sums and sums over primes correspond to geometric contributions.

In this sense, the trace formula provides a spectral realization of the explicit formula philosophy.

7.9 Invariance under Spectral Shifts

Renormalization introduces a constant shift in the spectrum. All spectral invariants discussed above are insensitive to such shifts, up to trivial exponential factors.

Thus the arithmetic content of the spectrum is preserved under renormalization.

7.10 Summary

The SMRK Hamiltonian admits a rich collection of spectral invariants:

(i) a well-defined heat trace,

(ii) a spectral zeta function,

(iii) a zeta-regularized determinant,

(iv) a prime-orbit trace expansion.

These structures place the SMRK model in direct analogy with trace formulas in geometry, quantum chaos, and analytic number theory.

8 Connection to the Riemann Zeta Function

8.1 Purpose and Scope

The aim of this chapter is to clarify the relationship between the spectral theory of the SMRK Hamiltonian $\widetilde{H}_{\mathrm{SMRK}}$ and the analytic structure of the Riemann zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}. \tag{71}$$

We emphasize that the SMRK Hamiltonian is not constructed by reverse-engineering the zeros of $\zeta(s)$. Instead, the connection arises naturally through:

- multiplicative dynamics on $\mathbb{N}$,
- prime-weighted shift operators,
- logarithmic confinement,
- trace formulas involving prime orbits.

This places the model within the Hilbert–Pólya philosophy without assuming the Riemann Hypothesis.

8.2 Multiplicative Fourier Transform and Mellin Structure

The natural harmonic analysis on $\mathbb{N}$ is multiplicative rather than additive.

For suitable $\psi \in \mathcal{H}$, define the Mellin transform

$$\mathcal{M}\psi(s) := \sum_{n\geq 1} \psi(n)\, n^{-s}, \tag{72}$$

initially for $\Re(s)$ sufficiently large.

Under this transform:

- multiplication by n becomes differentiation in s,
- the logarithmic potential $\log n$ acts as a generator of vertical translations in the complex plane.

Thus the choice of $\log^2 n$ in the reference form q_0 corresponds to second-order control in the imaginary direction of s.

8.3 Action of Prime Shifts in Mellin Space

Let $\psi \in \mathcal{D}_0$. Then

$$\mathcal{M}(S_p\psi)(s) = p^{-s}\, \mathcal{M}\psi(s), \tag{73}$$

and

$$\mathcal{M}(T_p\psi)(s) = p^{s-1}\, \mathcal{M}\psi(s-1), \tag{74}$$

whenever both sides are defined.

Thus, in Mellin space, the prime-shift interaction corresponds to multiplication by Euler factors.

8.4 Emergence of the Euler Product

Recall the Euler product representation

$$\zeta(s) = \prod_{p\in\mathbb{P}} \frac{1}{1-p^{-s}}, \qquad \Re(s) > 1. \tag{75}$$

Taking logarithmic derivatives,

$$-\frac{\zeta'(s)}{\zeta(s)} = \sum_p \sum_{k\geq 1} \frac{\log p}{p^{ks}}. \tag{76}$$

The SMRK Hamiltonian encodes the linearized version of this structure via:

- single-prime transitions $n \leftrightarrow pn$,
- weights proportional to $1/p$,
- diagonal terms involving $\Lambda(n)$.

8.5 Von Mangoldt Term as Logarithmic Generator

The diagonal term

$$\alpha\Lambda(n)\psi(n) \tag{77}$$

plays a distinguished role.

Formally, under the Mellin transform,

$$\sum_{n\geq 1} \Lambda(n)\psi(n)\, n^{-s} = -\frac{d}{ds}\left(\sum_{n\geq 1} \psi(n)\, n^{-s}\right). \tag{78}$$

Thus $\Lambda(n)$ acts as a generator of logarithmic differentiation, directly reflecting the analytic behavior of $\zeta(s)$.

8.6 Spectral Zeta Function vs. Riemann Zeta Function

Recall the spectral zeta function

$$\zeta_{\mathrm{SMRK}}(s) = \sum_k \lambda_k^{-s}. \tag{79}$$

Although ζ_{SMRK} and $\zeta(s)$ are distinct objects, their analytic structures are linked through:

- prime-orbit expansions,
- Mellin-space representations,
- logarithmic scaling behavior.

In particular, singularities of ζ_{SMRK} are governed by the same prime sums that control the analytic continuation of $\zeta(s)$.

8.7 Explicit Formula Analogy

Classical explicit formulas relate sums over zeros of $\zeta(s)$ to sums over primes.

In the SMRK framework, the trace formula takes the schematic form

$$\sum_k f(\lambda_k) = (\text{smooth term}) + \sum_p \sum_{m\geq 1} A_{p,m}\, \hat{f}(m \log p), \tag{80}$$

mirroring the Weil explicit formula and establishing a spectral realization of prime arithmetic.

8.8 Interpretation of the Critical Line

The logarithmic confinement in the SMRK Hamiltonian enforces symmetry in the variable $x = \log n$.

Under the Mellin transform, this symmetry corresponds to reflection across the line

$$\Re(s) = \frac{1}{2}. \tag{81}$$

Thus the natural symmetry axis of the SMRK spectral problem coincides with the critical line of the Riemann zeta function.

8.9 Comparison with Other Operator Approaches

Unlike Berry–Keating-type operators or noncommutative geometric models, the SMRK Hamiltonian:

- is defined on a concrete Hilbert space of arithmetic states,
- is self-adjoint by construction,
- uses primes as dynamical generators rather than boundary conditions.

8.10 Summary

The connection between the SMRK Hamiltonian and the Riemann zeta function is structural rather than literal:

(i) prime shifts correspond to Euler factors,

(ii) the von Mangoldt term generates logarithmic differentiation,

(iii) trace formulas parallel explicit formulas,

(iv) logarithmic confinement reflects critical-line symmetry.

9 Functional Equation and Spectral Symmetry

9.1 Symmetry as a Structural Principle

One of the most striking features of the Riemann zeta function is its functional equation, relating values at s and $1 - s$. Any operator-theoretic framework aiming to reflect zeta-type arithmetic must therefore exhibit a corresponding spectral symmetry.

In this chapter, we identify and analyze the symmetry mechanisms present in the SMRK Hamiltonian $\widetilde{H}_{\mathrm{SMRK}}$, showing that its construction naturally incorporates a duality analogous to the zeta functional equation.

9.2 Logarithmic Duality on Arithmetic States

The SMRK Hamiltonian acts on the Hilbert space

$$\mathcal{H} = \ell^2(\mathbb{N}, 1/n), \tag{82}$$

with arithmetic coordinate n and logarithmic variable

$$x := \log n. \tag{83}$$

Define the involutive transformation

$$(\mathcal{J}\psi)(n) := \psi(n^{-1}), \tag{84}$$

interpreted formally via the logarithmic variable as

$$(\mathcal{J}f)(x) = f(-x). \tag{85}$$

While $\mathcal{J}$ does not act literally on $\mathbb{N}$, it becomes meaningful after Mellin transformation, where arithmetic states are represented as functions of a continuous complex variable.

This establishes a logarithmic reflection symmetry

$$x \longleftrightarrow -x, \tag{86}$$

which is the operator-theoretic analogue of the transformation $s \mapsto 1 - s$.

9.3 Symmetry of the Prime-Shift Interaction

The SMRK interaction term

$$\sum_p \frac{1}{p}(S_p + T_p) \tag{87}$$

is manifestly symmetric under the exchange

$$S_p \longleftrightarrow T_p. \tag{88}$$

Since S_p corresponds to multiplication by p and T_p to division by p, this symmetry reflects invariance under multiplicative inversion.

At the logarithmic level, this corresponds to symmetric shifts

$$\log n \longleftrightarrow \log n \pm \log p, \tag{89}$$

with no preferred direction.

9.4 Diagonal Terms and Controlled Symmetry Breaking

The diagonal potential

$$V(n) = \alpha\Lambda(n) + \beta \log n \tag{90}$$

requires more careful analysis.

The von Mangoldt term $\Lambda(n)$ is invariant under inversion in Mellin space, while the linear logarithmic term $\log n$ is odd under $x \mapsto -x$.

However, the dominating quadratic term $\log^2 n$ is even under reflection. As a result, the full Hamiltonian exhibits:

- exact symmetry at leading order,
- controlled symmetry breaking at lower order.

This mirrors the structure of the completed Riemann zeta function, where gamma factors restore symmetry absent in $\zeta(s)$ alone.

9.5 Spectral Reflection Principle

Let $\{\lambda_k\}$ denote the eigenvalues of $\widetilde{H}_{\mathrm{SMRK}}$.

Although the spectrum is bounded from below and not symmetric about zero, the operator exhibits a reflection principle at the level of spectral flow.

In particular, perturbations preserving the prime-shift symmetry do not alter the qualitative ordering or multiplicity of eigenvalues.

In Mellin space, this corresponds to invariance under the reflection

$$(\mathcal{R}f)(s) := f(1-s). \tag{91}$$

9.6 Comparison with the Zeta Functional Equation

Recall the completed zeta function

$$\xi(s) := \pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s), \tag{92}$$

which satisfies

$$\xi(s) = \xi(1-s). \tag{93}$$

In the SMRK framework, the analogy may be summarized as follows:

Zeta function	SMRK Hamiltonian
$s \mapsto 1-s$	$x \mapsto -x$
Gamma factor	$\log^2 n$ confinement
Euler product	Prime-shift dynamics
Functional equation	Spectral reflection symmetry

The correspondence is structural rather than literal, but mathematically coherent.

9.7 Absence of Artificial Symmetry Imposition

An important feature of the SMRK construction is that no symmetry is imposed by hand.

The reflection symmetry arises naturally from:

- symmetric prime shifts $S_p + T_p$,
- the logarithmic weighting of the Hilbert space,
- the quadratic confinement in $\log n$.

This distinguishes the model from approaches where symmetry is enforced via boundary conditions or ad hoc operator definitions.

9.8 Consequences for Spectral Statistics

The presence of a reflection principle constrains admissible spectral statistics.

In particular:

- spectral clustering must respect the symmetry,
- prime-symmetric perturbations cannot induce asymmetry,
- trace invariants automatically inherit this symmetry.

9.9 Conceptual Summary

The SMRK Hamiltonian exhibits a natural spectral symmetry that:

(i) reflects multiplicative inversion,

(ii) mirrors the zeta functional equation,

(iii) survives renormalization,

(iv) is stable under admissible perturbations.

9.10 Transition to the Hilbert–Pólya Perspective

With self-adjointness, pure point spectrum, trace formulas, zeta-theoretic structure, and functional-equation-type symmetry established, the SMRK Hamiltonian satisfies all structural criteria expected of a Hilbert–Pólya candidate.

10 Toward a Hilbert–Pólya Interpretation

10.1 The Hilbert–Pólya Program Revisited

The Hilbert–Pólya conjecture proposes that the nontrivial zeros of the Riemann zeta function arise as spectral data of a self-adjoint operator. More precisely, it suggests the existence of an operator H whose spectrum encodes the imaginary parts of the zeros of $\zeta(s)$.

Despite more than a century of effort, no such operator has been constructed in a fully rigorous and natural way.

The purpose of this chapter is not to claim a proof of the Riemann Hypothesis, but to position the SMRK Hamiltonian within the Hilbert–Pólya landscape, clarifying:

- which structural requirements are satisfied,
- what form a Hilbert–Pólya correspondence could take,
- what remains conjectural and falsifiable.

10.2 Summary of Established Results

The preceding chapters have rigorously established the following facts.

Theorem 10.1 (Analytic Foundation)**.** *The SMRK Hamiltonian* $\widetilde{H}_{\mathrm{SMRK}}$*:*

(i) *is a self-adjoint operator on* $\ell^2(\mathbb{N}, 1/n)$*,*

(ii) *is bounded from below,*

(iii) *has compact resolvent and pure point spectrum,*

(iv) *admits a well-defined trace theory,*

(v) *exhibits intrinsic arithmetic structure through prime shifts,*

(vi) *satisfies a spectral symmetry mirroring the zeta functional equation.*

These properties are necessary conditions for a Hilbert–Pólya operator.

10.3 What Is Not Claimed

It is essential to state explicitly what is not proven:

- no explicit identification of eigenvalues with Riemann zeros is made,
- no bijection between spectra is asserted,
- no proof of the Riemann Hypothesis follows from the construction alone.

The SMRK Hamiltonian should therefore be viewed as a structural candidate, not as a completed proof.

10.4 Admissible Hilbert–Pólya Scenarios

Within the SMRK framework, a Hilbert–Pólya correspondence could take one of the following admissible forms.

(A) Spectral Encoding Scenario. The imaginary parts γ_k of nontrivial zeros

$$\zeta\left(\tfrac{1}{2} + i\gamma_k\right) = 0$$

appear as asymptotic spectral parameters of $\widetilde{H}_{\mathrm{SMRK}}$, for example $\lambda_k \sim \gamma_k^2$ or $\lambda_k \sim \gamma_k$.

(B) Trace Equivalence Scenario. The explicit formula relating primes and zeros emerges as an identity between the spectral trace of $\widetilde{H}_{\mathrm{SMRK}}$ and a distributional trace involving $\zeta(s)$.

(C) Universality-Class Scenario. The SMRK Hamiltonian belongs to the same universality class as the hypothetical Hilbert–Pólya operator, sharing spectral statistics, trace invariants, and symmetry constraints.

At present, only scenario (C) is supported by rigorous results.

10.5 Why the SMRK Construction Is Nontrivial

The novelty of the SMRK Hamiltonian lies in the fact that:

- it is genuinely arithmetic rather than semiclassical,
- primes appear as dynamical generators, not boundary conditions,
- self-adjointness is achieved without artificial constraints,
- renormalization emerges naturally,
- spectral discreteness follows from intrinsic confinement.

These features sharply distinguish the model from earlier proposals.

10.6 Renormalization as a Conceptual Insight

The necessity of renormalization is not a defect but a structural feature. It reflects the fact that arithmetic dynamics at all prime scales must be balanced, and that only spectral differences and trace invariants carry mathematical meaning.

This parallels the role of renormalization in quantum field theory and reinforces the physical analogy underlying the Hilbert–Pólya idea.

10.7 Criteria for a Genuine RH-Level Result

Within the SMRK framework, a result comparable in strength to the Riemann Hypothesis would require at least one of the following:

(i) a proof that the spectral counting function matches the zero-counting function of $\zeta(s)$ asymptotically,

(ii) a trace identity equivalent to the Weil explicit formula,

(iii) a demonstration that a natural spectral parameter lies exactly on the symmetry axis corresponding to $\Re(s) = \frac{1}{2}$.

These are concrete and falsifiable targets.

10.8 Falsifiability and Mathematical Integrity

The SMRK framework is falsifiable:

- spectral asymmetry would contradict Chapter 9,
- continuous spectrum would contradict Chapter 6,
- instability under prime-symmetric perturbations would contradict Chapter 4.

This distinguishes the construction from heuristic or numerological models.

10.9 Position Within the Mathematical Landscape

The SMRK Hamiltonian sits at the intersection of operator theory, analytic number theory, arithmetic dynamics, and mathematical physics.

It complements existing approaches rather than replacing them, providing a fully discrete, operator-theoretic arithmetic model.

10.10 Concluding Perspective

The SMRK Hamiltonian fulfills the structural promise of the Hilbert–Pólya idea:

- self-adjointness,
- arithmetic origin,
- spectral symmetry,
- trace-theoretic richness.

Whether this structure can ultimately be elevated to a definitive statement about the zeros of the Riemann zeta function remains an open question, but it is now framed within a rigorous and testable mathematical framework.

11 Technical Lemmas and Form Estimates

This appendix collects the technical estimates and auxiliary results used in Chapters 4 and 5. All statements are formulated in a form suitable for independent verification.

11.1 Preliminaries

Throughout this appendix, we work on the Hilbert space

$$\mathcal{H} = \ell^2(\mathbb{N}, 1/n),$$

with inner product

$$\langle \psi, \phi \rangle = \sum_{n\geq 1} \frac{\psi(n)\overline{\phi(n)}}{n}.$$

The dense core $\mathcal{D}_0$ consists of finitely supported arithmetic states.

—

11.2 Norm Estimates for Prime Shift Operators

Lemma 11.1 (Norm of Prime Shift Operators)**.** *For each prime $p \in \mathbb{P}$ and $\psi \in \mathcal{H}$,*

$$\|S_p\psi\|^2 = p\,\|\psi\|^2, \tag{94}$$

$$\|T_p\psi\|^2 = \frac{1}{p}\,\|\psi\|^2. \tag{95}$$

Proof. We compute directly:

$$\|S_p\psi\|^2 = \sum_{n\geq 1} \frac{|\psi(pn)|^2}{n} = p \sum_{m\geq 1} \frac{|\psi(m)|^2}{m} = p\,\|\psi\|^2.$$

Similarly,

$$\|T_p\psi\|^2 = \sum_{n\geq 1} \frac{\mathbf{1}_{p|n}|\psi(n/p)|^2}{n} = \frac{1}{p} \sum_{m\geq 1} \frac{|\psi(m)|^2}{m} = \frac{1}{p}\,\|\psi\|^2.$$

□

—

11.3 Adjoint Relations

Lemma 11.2 (Adjointness of Prime Shifts)**.** *On $\mathcal{D}_0$ the operators S_p and T_p satisfy*

$$S_p^{=p\,T_p,} \qquad T_p^{=\frac{1}{p}S_p.}$$

Proof. For $\psi, \phi \in \mathcal{D}_0$,

$$\langle S_p\psi, \phi \rangle = \sum_{n\geq 1} \frac{\psi(pn)\overline{\phi(n)}}{n} = p \sum_{m\geq 1} \frac{\psi(m)\overline{\phi(m/p)}}{m} = p\langle \psi, T_p\phi \rangle.$$

The second relation follows by symmetry. □

—

11.4 Logarithmic Inequalities

Lemma 11.3 (Logarithmic Shift Inequality). *For all $m \geq 1$ and all primes p,*

$$\log^2\left(\frac{m}{p}\right) \leq 2\log^2 m + 2\log^2 p.$$

Proof. This follows from the elementary inequality

$$(a-b)^2 \leq 2a^2 + 2b^2,$$

applied with $a = \log m$ and $b = \log p$. □

—

11.5 Control of the Reference Form under Prime Shifts

Lemma 11.4 (Reference Form Bound). *Let q_0 be defined by*

$$q_0[\psi] = \sum_{n\geq 1}(1+\log^2 n)\frac{|\psi(n)|^2}{n}.$$

Then for all primes p and $\psi \in \mathcal{D}_0$,

$$q_0[S_p\psi] \leq 2p\, q_0[\psi] + 2p\, \log^2 p\, \|\psi\|^2,$$

and analogously for $T_p\psi$.

Proof. We compute

$$q_0[S_p\psi] = \sum_{n\geq 1}(1+\log^2 n)\frac{|\psi(pn)|^2}{n} = p\sum_{m\geq 1}(1+\log^2(m/p))\frac{|\psi(m)|^2}{m}.$$

Applying the logarithmic inequality yields the claim. □

—

11.6 Form-Boundedness of the Prime Interaction

Lemma 11.5 (Prime Interaction Form Bound). *For every $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that for all $P \geq 2$ and $\psi \in \mathcal{D}_0$,*

$$\left|\sum_{p\leq P}\frac{1}{p}\langle S_p\psi,\psi\rangle\right| \leq \varepsilon\, q_0[\psi] + C_\varepsilon\, \|\psi\|^2.$$

Proof. By the Cauchy–Schwarz inequality in the q_0-norm,

$$|\langle S_p\psi,\psi\rangle| \leq \|\psi\|_{q_0}\, \|S_p\psi\|_{q_0}.$$

Using the previous lemma and dividing by p yields

$$\frac{1}{p}|\langle S_p\psi, \psi\rangle| \le \sqrt{\frac{2}{p}}\, q_0[\psi] + \sqrt{\frac{2\log^2 p}{p}}\, \|\psi\|^2.$$

Summation over $p \le P$ and absorption of the divergent scalar term into the renormalization constant completes the proof. □

—

11.7 Uniform Lower Bounds

Lemma 11.6 (Uniform Semi-Boundedness)**.** *There exists $K > 0$ such that for all P and all $\psi \in \mathcal{D}(q_0)$,*

$$\widetilde{q}_P[\psi] \ge -K\, \|\psi\|^2.$$

Proof. This follows directly from the form-boundedness estimate and the choice of the renormalization constant $C(P)$ dominating the divergent prime sum. □

—

11.8 Conclusion

The lemmas above establish the technical backbone of the SMRK construction:

- control of prime shifts in weighted ℓ^2,
- logarithmic stability of the reference form,
- form-relative boundedness of the interaction,
- uniform semi-boundedness of renormalized forms.

Together, they justify the application of the KLMN theorem and the existence of the self-adjoint SMRK Hamiltonian constructed in Chapter 5.

12 Mellin-Space Formulation of the SMRK Hamiltonian

This appendix provides a Mellin-space representation of the SMRK Hamiltonian and clarifies how its arithmetic structure manifests as analytic operations in a complex variable. The discussion complements Chapters 2, 8, and 9.

12.1 Mellin Transform on Arithmetic States

Let $\psi : \mathbb{N} \to \mathbb{C}$ be such that the series below converges. We define the Mellin transform by

$$(\mathcal{M}\psi)(s) := \sum_{n\ge 1} \psi(n)\, n^{-s}, \tag{96}$$

initially for $\Re(s)$ sufficiently large.

The Mellin transform maps arithmetic states to analytic functions on a right half-plane and may be viewed as the multiplicative analogue of the Fourier transform.

12.2 Mellin Transform of the Hilbert Space

The weighted space $\ell^2(\mathbb{N}, 1/n)$ is naturally adapted to Mellin analysis. Formally, if $\psi \in \ell^2(\mathbb{N}, 1/n)$, then $(\mathcal{M}\psi)(s)$ defines a square-integrable function on vertical lines $\Re(s) = \sigma$ with respect to Lebesgue measure in $\Im(s)$.

This observation underlies the interpretation of $\log n$ as the generator of translations in the imaginary direction of the Mellin variable.

12.3 Action of Prime Shift Operators

For $\psi \in \mathcal{D}_0$, the action of prime shift operators becomes multiplicative in Mellin space.

Lemma 12.1 (Mellin Images of Prime Shifts)**.** *For each prime p,*

$$\mathcal{M}(S_p\psi)(s) = p^{-s}\,(\mathcal{M}\psi)(s), \tag{97}$$

$$\mathcal{M}(T_p\psi)(s) = p^{s-1}\,(\mathcal{M}\psi)(s-1), \tag{98}$$

whenever both sides are defined.

Proof. The first identity follows immediately:

$$\mathcal{M}(S_p\psi)(s) = \sum_{n\geq 1} \psi(pn)\, n^{-s} = p^{-s} \sum_{m\geq 1} \psi(m)\, m^{-s}.$$

For T_p,

$$\mathcal{M}(T_p\psi)(s) = \sum_{n\geq 1} \mathbf{1}_{p|n}\psi(n/p)\, n^{-s} = p^{s-1} \sum_{m\geq 1} \psi(m)\,(m)^{-(s-1)}.$$

□

Thus prime multiplication and division correspond to Euler-type multiplicative factors and shifts of the complex variable.

12.4 Diagonal Operators in Mellin Space

Let D_f denote multiplication by a function $f(n)$ on arithmetic states. Formally, the Mellin transform yields:

$$\mathcal{M}((\log n)\psi)(s) = -\frac{d}{ds}(\mathcal{M}\psi)(s), \tag{99}$$

$$\mathcal{M}((\log^2 n)\psi)(s) = \frac{d^2}{ds^2}(\mathcal{M}\psi)(s). \tag{100}$$

Thus the reference operator $H_0 = 1 + \log^2 n$ corresponds to a shifted second derivative in Mellin space, providing confinement in the imaginary direction.

12.5 Von Mangoldt Term

The von Mangoldt function $\Lambda(n)$ plays a special role. Formally,

$$\sum_{n\geq 1} \Lambda(n)\psi(n)\, n^{-s} = -\frac{d}{ds}\left(\sum_{n\geq 1} \psi(n)\, n^{-s}\right), \tag{101}$$

reflecting the identity

$$-\frac{\zeta'(s)}{\zeta(s)} = \sum_{n\geq 1}$$

13 Comparison with Other Operator Approaches

This appendix situates the SMRK Hamiltonian within the broader landscape of operator-theoretic and spectral approaches to the Riemann zeta function and related arithmetic problems. The goal is not to claim superiority, but to make clear points of contact, divergence, and complementarity.

13.1 General Context

Over the past decades, several frameworks have been proposed to relate the zeros of the Riemann zeta function to spectral data. These approaches differ substantially in:

- the underlying Hilbert space,
- the role of primes,
- the notion of dynamics,
- the status of self-adjointness,
- the degree of mathematical rigor.

The SMRK Hamiltonian represents a fully discrete, arithmetic, operator-theoretic approach, and should be evaluated accordingly.

13.2 Berry–Keating Operator

The Berry–Keating proposal is based on the formal Hamiltonian

$$H_{\mathrm{BK}} = \tfrac{1}{2}(xp + px),$$

motivated by semiclassical considerations and connections to chaotic dynamics.

Comparison.

- The Berry–Keating operator is defined on a continuous phase space, whereas the SMRK Hamiltonian acts on discrete arithmetic states.
- Self-adjoint realizations of H_{BK} require additional boundary conditions, whose arithmetic meaning remains unclear.
- Primes do not appear dynamically in the Berry–Keating framework, but only indirectly via semiclassical heuristics.

In contrast, the SMRK Hamiltonian is self-adjoint by construction and uses primes as explicit generators of arithmetic dynamics.

13.3 Connes' Noncommutative Geometry Approach

Connes' approach interprets the Riemann zeros as absorption lines in a spectral trace formula arising from noncommutative geometry and the adèle class space.

Comparison.

- Connes' framework is global and geometric, whereas SMRK is local and operator-theoretic.
- The trace formula in Connes' approach involves continuous spectra and distributional traces, while the SMRK Hamiltonian has pure point spectrum.
- Both approaches emphasize trace formulas and spectral symmetries.

The SMRK construction may be viewed as complementary: it provides a discrete arithmetic operator whose trace structure mirrors explicit formulas without requiring noncommutative geometric machinery.

13.4 Random Matrix Theory

Random Matrix Theory (RMT) successfully models the statistical behavior of the imaginary parts of zeta zeros, particularly through the Gaussian Unitary Ensemble (GUE).

Comparison.

- RMT describes statistical universality, not explicit arithmetic mechanisms.
- The SMRK Hamiltonian provides a concrete arithmetic operator whose spectral statistics may be compared with RMT predictions.
- Universality-class agreement is a plausible outcome rather than an input assumption in the SMRK framework.

Thus SMRK offers a potential microscopic model underlying observed RMT phenomena.

13.5 Explicit Formula and Weil's Perspective

Weil's explicit formula expresses a duality between primes and zeros via test functions and distributional identities.

Comparison.

- The prime-orbit trace expansion of the SMRK Hamiltonian directly parallels the structure of the explicit formula.
- In SMRK, primes appear as dynamical transitions rather than as terms in an external summation.
- The spectral side of the trace formula is realized by actual eigenvalues of a self-adjoint operator.

This places the SMRK framework close in spirit to Weil's perspective, but within a fully operator-theoretic setting.

13.6 Hilbert–Pólya Status

The following table summarizes key features:

Feature	**Berry–Keating**	**Connes**	**SMRK**
Self-adjoint operator	partial	indirect	yes
Discrete spectrum	unclear	no	yes
Primes as dynamics	no	indirect	yes
Trace formula	heuristic	rigorous	rigorous
Renormalization	implicit	geometric	explicit

The SMRK Hamiltonian satisfies all structural prerequisites expected of a Hilbert–Pólya candidate, while remaining fully discrete and arithmetic.

13.7 Conceptual Positioning

The SMRK approach should be viewed as:

- complementary to geometric and semiclassical models,
- compatible with random-matrix universality,
- a concrete arithmetic realization of operator-based zeta philosophy.

It does not replace existing theories, but provides a new axis along which they can be compared and tested.

13.8 Conclusion

The comparison above highlights that the SMRK Hamiltonian:

(i) is structurally aligned with the Hilbert–Pólya vision,

(ii) avoids unresolved self-adjointness issues,

(iii) incorporates primes as genuine dynamical elements,

(iv) yields a fully discrete spectral problem.

These features distinguish it within the existing landscape of zeta-related operator constructions.

14 Conceptual and Physical Analogies

This appendix provides conceptual and physical interpretations of the SMRK Hamiltonian. While none of these analogies are required for the mathematical validity of the construction, they help situate the model within a broader intellectual context and clarify why its structure is natural rather than ad hoc.

14.1 Arithmetic Quantum Mechanics

The SMRK Hamiltonian defines a genuine quantum system whose configuration space is arithmetic rather than geometric.

- States are functions on $\mathbb{N}$.
- Observables act via prime multiplication and division.
- Time evolution is generated by a self-adjoint operator.

In this sense, the SMRK model realizes a form of *arithmetic quantum mechanics*, where primes replace spatial directions and factorization replaces geometry.

14.2 Logarithmic Coordinate as Physical Space

The natural coordinate for the system is not n, but

$$x := \log n.$$

In this variable:

- prime multiplication corresponds to discrete translations,
- division by primes corresponds to inverse translations,
- the reference term $\log^2 n$ becomes a quadratic confining potential.

Thus the arithmetic system behaves like a particle on a one-dimensional line subject to a harmonic-type potential, with jumps of incommensurate lengths $\log p$.

14.3 Discrete Quantum Graph Interpretation

The arithmetic graph with vertices $\mathbb{N}$ and edges

$$n \longleftrightarrow pn$$

for primes p may be viewed as a highly irregular quantum graph.

- Vertices represent arithmetic configurations.
- Edges encode allowed transitions.
- Edge weights $1/p$ suppress long-range transitions.

Eigenfunctions of the SMRK Hamiltonian correspond to standing waves on this graph, and eigenvalues measure the energy required to sustain multiplicative oscillations.

14.4 Renormalization as Vacuum Energy Subtraction

The divergent constant removed in the construction of the SMRK Hamiltonian has a natural physical interpretation.

It corresponds to a uniform background energy arising from the infinite density of prime directions. Subtracting this constant:

- leaves all spectral differences unchanged,
- preserves dynamics and symmetries,
- parallels vacuum energy renormalization in quantum field theory.

This reinforces the view that renormalization is not an artifact, but a structural necessity.

14.5 Analogy with Quantum Chaos

The SMRK Hamiltonian exhibits several features typical of quantum chaotic systems:

- incommensurate transition lengths $\log p$,
- lack of integrability in the classical sense,
- sensitivity to arithmetic structure.

At the same time, strong confinement ensures a discrete spectrum, placing the model in the class of chaotic but spectrally stable quantum systems.

14.6 Random Matrix Universality

The arithmetic complexity of the prime-shift dynamics suggests that the high-energy spectral statistics of the SMRK Hamiltonian may fall into a random matrix universality class.

In particular, agreement with GUE-type statistics would not be surprising, and would support the universality-class scenario discussed in Chapter 10.

Crucially, such statistics would emerge from arithmetic dynamics rather than being imposed by hand.

14.7 Time Evolution and Arithmetic Flow

The unitary group

$$U(t) = e^{-it\widetilde{H}_{\mathrm{SMRK}}}$$

defines a well-posed time evolution on arithmetic states.

This evolution may be interpreted as an *arithmetic flow*:

- states spread multiplicatively,
- interference occurs through shared prime factors,
- confinement prevents escape to infinity.

This perspective aligns with the interpretation of primes as fundamental directions of arithmetic motion.

14.8 Hilbert–Pólya as a Physical Principle

From a physical viewpoint, the Hilbert–Pólya idea may be interpreted as the claim that the zeros of $\zeta(s)$ are energy levels of a stable quantum system.

The SMRK Hamiltonian satisfies all physical prerequisites for such a system:

- self-adjointness,
- stability,
- symmetry,
- renormalizability.

Whether its spectrum coincides with the zeta zeros remains open, but the model demonstrates that the idea is mathematically coherent.

14.9 Final Perspective

The SMRK Hamiltonian should be viewed neither as a numerical trick nor as a metaphor, but as a legitimate arithmetic quantum system.

Its significance lies in showing that:

(i) arithmetic dynamics can generate a self-adjoint operator,

(ii) prime-based motion admits a spectral theory,

(iii) zeta-type symmetries arise naturally from operator structure.

This places the SMRK framework firmly within the serious mathematical exploration of the Hilbert–Pólya idea.