Spectral Program for the SMRK Hamiltonian

Projective Weight Geometry on $\mathbb{CP}^1$ and a Computation-First Route

Enter Yourname

Abstract

We develop a computation-first spectral program for the SMRK Hamiltonian acting on $\ell^2(\mathbb{N})$, starting from self-adjointness and moving toward numerical and analytic control of the spectrum. A central feature is a projective weight formalism: prime-transfer weights are treated as points on the Riemann sphere $\mathbb{CP}^1$, with the Möbius involution $z \mapsto -1/z$ encoding an expansion–contraction duality. The distinguished pair 12 and $-1/12$ appears as a canonical antipodal calibration in the projective weight space. We provide (i) a precise operator definition with $\mathbb{CP}^1$-valued weights, (ii) self-adjointness and domain remarks, (iii) a truncation-to-sparse-matrix spectral algorithm, and (iv) a list of falsifiable spectral signatures relevant to Hilbert–Pólya-style aspirations.

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1 Overview: from self-adjointness to spectrum

1.1 Goal

We assume that a self-adjoint realization of the SMRK Hamiltonian has been established on a dense domain $D(H) \subset \ell^2(\mathbb{N})$. The next indispensable task is to understand its spectrum $\sigma(H)$:

$$\sigma(H) \subset \mathbb{R}, \qquad \sigma(H) = \sigma_{\mathrm{pp}}(H) \cup \sigma_{\mathrm{ac}}(H) \cup \sigma_{\mathrm{sc}}(H),$$

where σ_{pp} denotes point spectrum (eigenvalues), σ_{ac} absolutely continuous spectrum, and σ_{sc} singular continuous spectrum.

In the present program we prioritize *computability*: we define a canonical finite-volume truncation H_N on $\mathrm{span}\{\delta_1, \ldots, \delta_N\}$ and study the stability of numerical spectral features as $N \to \infty$.

1.2 The two pillars

The spectral program rests on two pillars.

(P1) Transfer structure on $\ell^2(\mathbb{N})$. The SMRK Hamiltonian is built from multiplicative transfers $n \mapsto pn$ and $n \mapsto n/p$. This makes it a "multiplicative adjacency + potential" operator on the directed graph of $\mathbb{N}$.

(P2) Projective weight geometry on $\mathbb{CP}^1$. Instead of fixed scalar weights (e.g. $1/p$), we treat weights as *projective pairs*: each prime p carries a class $W_p = [A_p : B_p] \in \mathbb{CP}^1$ controlling forward/backward amplitudes. The Möbius involution

$$J : \ z \mapsto -\frac{1}{z}$$

is singled out as the expansion–contraction duality. The calibration pair

$$12 \longleftrightarrow -\frac{1}{12}$$

is interpreted as a canonical antipodal dual pair in the Riemann-sphere compactification of the weight-line.

1.3 Why spectrum is the real checkpoint

Self-adjointness guarantees real spectrum, but *not* its shape. To be meaningful for any Hilbert–Pólya-type ambition, we need at minimum:

- a controlled spectral measure and stable numerical spectrum under truncation,
- evidence for (or against) discrete vs. continuous spectral components,
- falsifiable signatures (level spacing, scaling laws, trace/determinant surrogates).

This whitepaper therefore treats "compute the spectrum" as the first truly discriminative test: it can validate the model, falsify it, or force refinement of weights/potentials/domains.

2 The $\mathbb{CP}^1$-weighted SMRK Hamiltonian on $\ell^2(\mathbb{N})$

2.1 Hilbert space and basic transfers

Let $\ell^2(\mathbb{N})$ be the Hilbert space of square-summable complex sequences $\psi : \mathbb{N} \to \mathbb{C}$ with inner product $\langle \phi, \psi \rangle = \sum_{n \geq 1} \overline{\phi(n)} \psi(n)$.

For each prime $p \in \mathbb{P}$ define the forward transfer operator T_p and its formal backward counterpart:

$$(T_p \psi)(n) := \psi(pn), \tag{1}$$

$$(T_p^\star \psi)(n) := \mathbb{1}_{p|n} \, \psi(n/p). \tag{2}$$

On suitable dense domains, these encode the elementary expansion/contraction along the p-multiplicative direction.

2.2 Projective weights as points on the Riemann sphere

For each prime p assign a projective weight

$$W_p = [A_p : B_p] \in \mathbb{CP}^1, \qquad (A_p, B_p) \in \mathbb{C}^2 \setminus \{(0,0)\}.$$

The interpretation is: A_p weights the forward transfer $n \mapsto pn$ and B_p weights the backward transfer $n \mapsto n/p$. Only the ratio matters, consistent with a projective "gauge" freedom.

We single out the Möbius involution

$$J(z) = -\frac{1}{z},$$

which in homogeneous coordinates acts as

$$J : \ [A_p : B_p] \mapsto [-B_p : A_p].$$

This is the fundamental exchange of forward/backward directions with an orientation reversal.

2.3 Operator definition

We define the $\mathbb{CP}^1$-weighted SMRK operator in its formal action as

$$(H\psi)(n) = \sum_{p\in\mathbb{P}} \Big(A_p\, \psi(pn) + B_p\, \mathbb{1}_{p|n}\, \psi(n/p) \Big) \;+\; V(n)\psi(n), \tag{3}$$

where $V : \mathbb{N} \to \mathbb{R}$ is a real-valued potential (diagonal term), e.g.

$$V(n) = \alpha\, \Lambda(n) + \beta \log n,$$

with Λ the von Mangoldt function and $\alpha, \beta \in \mathbb{R}$.

Remark 1 (Scalar SMRK as a gauge choice)**.** The usual scalar-weight version with weights $w_p \in \mathbb{R}$ corresponds to the gauge slice $A_p = B_p = w_p$ (or a closely related normalization depending on conventions). The projective formalism strictly contains this case while enabling Möbius-covariant deformations.

2.4 The calibration pair $12 \leftrightarrow -1/12$

To encode the "two infinities compactified to a sphere" intuition, we view weight ratios

$$z_p := \frac{A_p}{B_p} \in \mathbb{C} \cup \{\infty\} \cong \mathbb{CP}^1.$$

Then the involution J acts as $z_p \mapsto -1/z_p$. The distinguished dual pair

$$12 \longleftrightarrow -\frac{1}{12}$$

is interpreted as an antipodal calibration in $\mathbb{CP}^1$ under J. Operationally, this means that if a regime (a prime family, a dyadic shell proxy, or an effective flow) is tuned to $z \approx 12$, its J-dual regime is automatically tuned to $z \approx -1/12$.

2.5 The spectral problem

Assuming H admits a self-adjoint realization on a domain $D(H) \subset \ell^2(\mathbb{N})$, we seek:

- qualitative spectral type (point vs. continuous components),
- quantitative spectral invariants (spectral density, scaling, level statistics),
- computational scheme for stable approximations via truncations H_N.

The next chapter will introduce the projective/Möbius and $SU(1,1)$ structures as a controllable "weight gauge", and then we move to the sparse truncation algorithm and numerical spectrum extraction.

3 The $\mathbb{CP}^1$-weighted SMRK Hamiltonian on $\ell^2(\mathbb{N})$

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4 Self-Adjointness, Domains, and Gauge Constraints

4.1 Why self-adjointness is the non-negotiable checkpoint

For any operator intended to play a Hilbert–Pólya-type role, self-adjointness is not a technical luxury but a structural necessity. Only self-adjoint operators admit:

- a real spectrum,
- a complete spectral resolution,
- a well-defined functional calculus (resolvents, determinants, traces),
- physically meaningful unitary dynamics e^{-itH}.

In the present setting, self-adjointness must be reconciled with three competing features:

1. unbounded multiplicative transfers ($n \mapsto pn$),
2. infinitely many prime channels,
3. projective (rather than scalar) weights.

This chapter clarifies which parts are already under control, and which are enforced by suitable domain and gauge choices.

4.2 Formal symmetry of the $\mathbb{CP}^1$-weighted SMRK operator

Recall the formal definition

$$(H\psi)(n) = \sum_{p\in\mathbb{P}} \Big(A_p\, \psi(pn) + B_p\, \mathbb{1}_{p|n}\, \psi(n/p) \Big) + V(n)\psi(n),$$

with real-valued potential $V(n)$.

On the algebraic core

$$\mathcal{D}_0 := \operatorname{span}\{\delta_n : n \in \mathbb{N}\},$$

the operator is densely defined. A direct computation shows:

$$\langle \phi, H\psi \rangle = \sum_{p,n} \overline{\phi(n)} A_p \psi(pn) + \sum_{p,n} \overline{\phi(n)} B_p \mathbb{1}_{p|n} \psi(n/p) + \sum_{n} \overline{\phi(n)} V(n)\psi(n).$$

Changing summation variables $m = pn$ in the first sum yields

$$\sum_{p,m} \overline{\phi(m/p)} A_p \mathbb{1}_{p|m} \psi(m),$$

which matches the adjoint of the backward term provided

$$B_p = \overline{A_p} \quad \text{for all } p \in \mathbb{P}. \tag{7}$$

Proposition 1 (Formal symmetry)**.** *If $V(n) \in \mathbb{R}$ and the weight pairs satisfy $B_p = \overline{A_p}$, then H is formally symmetric on $\mathcal{D}_0$:*

$$\langle \phi, H\psi \rangle = \langle H\phi, \psi \rangle \quad \textit{for all } \phi, \psi \in \mathcal{D}_0.$$

4.3 Interpretation as a gauge condition

Condition (7) has a clean geometric meaning. In projective terms,

$$W_p = [A_p : B_p] \in \mathbb{CP}^1 \quad \text{with} \quad B_p = \overline{A_p}$$

means that weights lie on a distinguished real slice of $\mathbb{CP}^1$. After normalization, one may write

$$W_p = [e^{i\theta_p} : e^{-i\theta_p}],$$

so that the affine ratio becomes

$$z_p = \frac{A_p}{B_p} = e^{2i\theta_p} \in S^1.$$

Remark 3 (Unit-circle gauge)**.** This shows that formal self-adjointness restricts weights to the unit circle $S^1 \subset \mathbb{CP}^1$, which is precisely the boundary preserved by $SU(1,1)$. Thus the hyperbolic/Möbius gauge introduced in Chapter 3 is not decorative: it is the natural symmetry group compatible with adjointness.

4.4 Domain issues and essential self-adjointness

Formal symmetry alone does not guarantee self-adjointness. One must address whether the closure of H is self-adjoint, i.e. whether H is *essentially self-adjoint* on $\mathcal{D}_0$.

Two features are decisive:

(i) Growth of the potential. If $V(n) \to +\infty$ sufficiently fast (e.g. logarithmic or stronger growth), then standard results for discrete Schrödinger-type operators imply essential self-adjointness. In particular, diagonal terms of the form

$$V(n) = \beta \log n, \qquad \beta > 0,$$

act as a confining potential in logarithmic scale.

(ii) Relative boundedness of transfer terms. The prime-transfer part must be relatively bounded with respect to the diagonal operator. Under mild summability assumptions on $|A_p|$ (e.g. $\sum_p |A_p| < \infty$ or suitable weighted bounds), the Kato–Rellich theorem applies.

Theorem 1 (Essential self-adjointness, informal statement)**.** *Assume:*

- $B_p = \overline{A_p}$ *for all* p*,*
- $V(n) \in \mathbb{R}$ *with* $V(n) \to +\infty$ *as* $n \to \infty$*,*
- *the transfer part is relatively bounded with respect to* V*.*

Then H *is essentially self-adjoint on* $\mathcal{D}_0$*.*

A fully rigorous proof would require explicit norm estimates; however, for the purposes of this program, the conditions above are sufficient to justify both analytic and numerical work.

4.5 Self-adjointness and the involution J

Recall the Möbius involution

$$J : [A : B] \mapsto [-B : A].$$

Under the self-adjoint gauge $B = \overline{A}$, J acts as

$$A \mapsto -\overline{A},$$

which preserves $|A|$ and hence preserves the unit-circle condition. Thus J is compatible with self-adjointness and acts as a *symmetry* of the admissible weight space.

Remark 4 (Compatibility with $12 \leftrightarrow -1/12$)**.** When working in a real-affine chart $z \in \mathbb{R} \cup \{\infty\}$, self-adjointness forces a restriction to $|z| = 1$ after normalization. The pair $12 \leftrightarrow -1/12$ should therefore be understood as a *projective calibration* before fixing gauge. After gauge fixing, it is represented not by absolute values but by antipodal points on the unit circle, preserving the same involutive structure.

4.6 Consequences for spectral analysis

At this stage we have:

- a densely defined, essentially self-adjoint operator H,
- real spectrum $\sigma(H) \subset \mathbb{R}$,
- a compact, well-controlled weight space (a gauge slice of $\mathbb{CP}^1$),
- a built-in involutive symmetry compatible with adjointness.

This is precisely the minimal analytic infrastructure required before attempting any serious spectral computation.

4.7 What remains unresolved

Self-adjointness does *not* determine:

- whether the spectrum is discrete or continuous,
- the density of states,
- level statistics,
- any connection to zeta zeros.

These questions are empirical and analytic at once. The next chapter therefore abandons abstract considerations and turns to a concrete, falsifiable procedure: finite-volume truncation and sparse-matrix spectral computation.

Self-adjointness makes the spectral problem meaningful. Computation decides whether it is interesting.

5 Finite-Volume Truncation and Numerical Spectrum

5.1 Why truncation is unavoidable (and legitimate)

The SMRK Hamiltonian acts on the infinite-dimensional space $\ell^2(\mathbb{N})$ and involves unbounded multiplicative transfers. Analytic spectral characterization in full generality is currently out of reach. However, spectral theory provides a standard and reliable bridge between infinite operators and computation: *finite-volume truncation*.

The guiding principle is simple:

> If a spectral feature (eigenvalue cluster, gap, density profile) is stable under increasing truncation size, it is a strong candidate for a genuine feature of the infinite operator.

This chapter defines a canonical truncation H_N, shows that it yields sparse symmetric matrices, and explains which numerical observables are meaningful and which are not.

5.2 Canonical truncation of the SMRK Hamiltonian

Let

$$\mathcal{H}_N := \operatorname{span}\{\delta_1, \dots, \delta_N\} \subset \ell^2(\mathbb{N})$$

and define the truncated operator

$$H_N := P_N H P_N,$$

where P_N is the orthogonal projection onto $\mathcal{H}_N$.

Explicitly, for $1 \le n \le N$,

$$(H_N\psi)(n) = \sum_{p\in\mathbb{P}:\, pn\le N} A_p\, \psi(pn) + \sum_{p\in\mathbb{P}:\, p\mid n} B_p\, \psi(n/p) + V(n)\psi(n). \tag{8}$$

Remark 5. The truncation is *not* a naive cutoff in prime space but a cutoff in the multiplicative graph of $\mathbb{N}$. This preserves the local transfer structure while discarding only transitions leaving $\{1, \dots, N\}$.

5.3 Matrix structure and sparsity

In the standard basis $\{\delta_n\}$, the matrix $(H_N)_{mn}$ satisfies:

$$(H_N)_{mn} \neq 0 \quad \Longleftrightarrow \quad \begin{cases} m = pn \le N, \\ n = pm \le N, \\ m = n. \end{cases}$$

Key properties:

- Each row contains $O(\log n)$ nonzero off-diagonal entries (one per prime divisor or multiple).
- The total number of nonzero entries scales as $O(N \log\log N)$.
- Under the self-adjoint gauge $B_p = \overline{A_p}$, the matrix is real symmetric (or Hermitian).

This makes H_N a *sparse symmetric matrix*, ideal for Lanczos/Arnoldi-type algorithms.

5.4 Numerical diagonalization strategy

Direct diagonalization scales as $O(N^3)$ and is infeasible for large N. Instead, we use iterative solvers to extract parts of the spectrum.

Standard choices:

- `Lanczos` for extremal eigenvalues,
- `ARPACK / eigsh` for selected windows,
- shift–invert to probe interior spectral regions.

Typical workflow:

1. Fix N (e.g. 10^4–10^5 depending on resources).
2. Assemble H_N as a sparse CSR/CSC matrix.
3. Compute $k \ll N$ eigenvalues $\lambda_1^{(N)}, \dots, \lambda_k^{(N)}$.
4. Increase N and track convergence.

5.5 Spectral observables that *do* converge

Not all numerical outputs are meaningful. The following quantities are robust:

(i) Low-lying eigenvalues. If $V(n) \to +\infty$, the bottom of the spectrum is discrete. Eigenvalues that stabilize as N increases are genuine candidates for point spectrum of H.

(ii) Local density of states. By binning eigenvalues in energy windows and normalizing by N, one can estimate the density of states $\rho(E)$ and its scaling behavior.

(iii) Level spacing statistics. After unfolding, nearest-neighbor spacings

$$s_j = \lambda_{j+1} - \lambda_j$$

can be compared against Poisson, GOE, or GUE statistics. This is a diagnostic, not a proof.

5.6 What does *not* converge (and should be ignored)

- Individual high-energy eigenvalues (they drift with N).
- Raw traces $\mathrm{Tr}(H_N^k)$ without normalization.
- Determinants $\det(H_N)$ (diverge exponentially with N).

Only normalized or renormalized quantities should be compared across truncations.

5.7 Truncation as a probe of spectral type

Empirically:

- **Discrete spectrum:** isolated eigenvalues stabilize rapidly.
- **Absolutely continuous spectrum:** eigenvalues fill intervals densely as N grows.
- **Singular continuous spectrum:** irregular clustering without stabilization.

Thus truncation provides not just numbers, but a qualitative fingerprint of the spectral type.

5.8 Relation to the projective weight geometry

The truncation procedure is sensitive to the choice of weights $W_p \in \mathbb{CP}^1$:

- weights near the equator $|z_p| = 1$ typically yield balanced forward/backward transport,
- strongly polar weights $|z_p| \gg 1$ or $|z_p| \ll 1$ push spectral mass toward edges,
- Möbius-related weight configurations produce numerically identical spectra up to unitary equivalence, serving as a nontrivial consistency check.

In particular, calibrating a regime near the antipodal pair $12 \leftrightarrow -1/12$ allows one to test the stability of spectral features under projective duality.

5.9 Interpretational checkpoint

At this stage, the project has crossed an important threshold:

- the operator is self-adjoint,
- its finite-volume approximants are computable,
- spectral claims are falsifiable by explicit numerics.

No appeal to conjecture or analogy is needed here. If the spectrum behaves pathologically, the model fails. If it stabilizes and exhibits structured statistics, deeper analytic questions become justified.

From here on, every claim about the SMRK Hamiltonian can be tested against data.

6 Resolvent, Spectral Measures, and Trace Surrogates

6.1 Why resolvent-based objects are central

Once self-adjointness and computable truncations are in place, the most robust analytic objects are not eigenvalues themselves, but quantities derived from the *resolvent*. For a self-adjoint operator H and $z \in \mathbb{C} \setminus \mathbb{R}$, define

$$R(z) := (H - zI)^{-1}.$$

The resolvent encodes the entire spectral data of H and is the correct bridge between:

- infinite-dimensional spectral theory,
- finite-volume numerical approximations,
- trace- and determinant-type constructions relevant for zeta-like objects.

Unlike raw traces or determinants, resolvent-based quantities admit controlled limits.

6.2 Spectral theorem and spectral measures

By the spectral theorem for self-adjoint operators, there exists a projection-valued measure $E_H(\lambda)$ such that

$$H = \int_{\mathbb{R}} \lambda \, dE_H(\lambda).$$

For any vector $\psi \in \ell^2(\mathbb{N})$, this induces a finite Borel measure

$$\mu_\psi(B) := \langle \psi, E_H(B)\psi \rangle, \qquad B \subset \mathbb{R} \text{ Borel},$$

called the *spectral measure associated to* ψ.

Key facts:

- μ_ψ completely determines the spectral decomposition of ψ.
- H has purely point spectrum iff all μ_ψ are pure point measures.
- Absolutely continuous components correspond to densities $d\mu_\psi/d\lambda$.

Thus the spectral problem reduces to understanding these measures.

6.3 Resolvent as Stieltjes transform

For $\operatorname{Im} z \neq 0$, the resolvent matrix element satisfies

$$\langle \psi, R(z)\psi \rangle = \int_{\mathbb{R}} \frac{1}{\lambda - z} \, d\mu_\psi(\lambda). \tag{9}$$

This is the Stieltjes transform of the spectral measure. Taking the boundary value $z = E + i\varepsilon$ with $\varepsilon \downarrow 0$ yields:

$$\lim_{\varepsilon \downarrow 0} \frac{1}{\pi} \operatorname{Im} \langle \psi, R(E + i\varepsilon)\psi \rangle = \frac{d\mu_\psi}{dE}(E) \quad \text{(a.e., for the a.c. part)}.$$

Remark 6. This formula is the practical key: spectral densities can be extracted from resolvent matrix elements without diagonalizing the operator.

6.4 Finite-volume resolvent and convergence

For truncations H_N, define

$$R_N(z) := (H_N - zI)^{-1}.$$

Then for fixed ψ with finite support,

$$\langle \psi, R_N(z)\psi \rangle \longrightarrow \langle \psi, R(z)\psi \rangle \quad \text{as } N \to \infty,$$

for all $z \in \mathbb{C} \setminus \mathbb{R}$.

This convergence is:

- stable under numerical error,
- insensitive to high-energy spectral drift,
- compatible with sparse linear solvers.

Thus the resolvent is the preferred numerical observable.

6.5 Diagonal spectral measures and local probes

A particularly natural choice is $\psi = \delta_n$. Then

$$\mu_n := \mu_{\delta_n}$$

measures how spectral weight is distributed at arithmetic site n.

Interpretation:

- localization: μ_n concentrates on discrete eigenvalues,
- delocalization: μ_n has a smooth density,
- arithmetic structure: variation of μ_n with n probes number-theoretic geometry.

In practice, one computes

$$G_n(z) := \langle \delta_n, R_N(z)\delta_n \rangle$$

and extracts $\operatorname{Im} G_n(E + i\varepsilon)$ for small ε.

6.6 Trace per unit volume

The full trace $\operatorname{Tr}(R(z))$ diverges. Instead, define a *trace per unit volume*:

$$\tau_N(z) := \frac{1}{N}\operatorname{Tr}(R_N(z)) = \frac{1}{N}\sum_{j=1}^{N} \frac{1}{\lambda_j^{(N)} - z}.$$

As $N \to \infty$, $\tau_N(z)$ often converges to a deterministic limit

$$\tau(z) = \int_{\mathbb{R}} \frac{1}{\lambda - z}\, d\nu(\lambda),$$

where ν is the *integrated density of states* (IDS).

Definition 1 (Integrated density of states)**.** If the limit

$$\nu(E) := \lim_{N\to\infty} \frac{1}{N} \#\{\lambda_j^{(N)} \le E\}$$

exists, it is called the integrated density of states of H.

The IDS is invariant under finite-rank perturbations and hence robust under truncation.

6.7 Trace identities and prime-transfer structure

Formally, one may expand the resolvent as

$$R(z) = -\frac{1}{z}\sum_{k\ge 0} \left(\frac{H}{z}\right)^k,$$

suggesting traces of powers H^k. In the SMRK case, H^k corresponds to closed multiplicative walks:

$$n \to p_1 n \to p_2 p_1 n \to \cdots \to n.$$

Although $\operatorname{Tr}(H^k)$ diverges, normalized traces encode:

- combinatorics of prime factorizations,
- interference between forward/backward transfers,
- cancellations enforced by projective duality.

This is the precise sense in which the model is compatible with trace-formula heuristics, without yet asserting any zeta identity.

6.8 What can and cannot be claimed at this stage

At the level of resolvent and spectral measures, one can legitimately claim:

- existence of well-defined spectral measures,
- computable density-of-states limits,
- meaningful trace-per-volume quantities.

One *cannot* yet claim:

- identification of $\xi(s)$ with a determinant,
- direct correspondence between zeros and eigenvalues,
- RH-level conclusions.

These require additional analytic input and will be addressed separately.

6.9 Why this matters for the RH interface

The resolvent framework is the exact analytic layer where Hilbert–Pólya-type ideas must eventually live: zeros of $\xi(s)$ would correspond not to raw eigenvalues, but to singularities or structured features of a spectral transform.

By establishing resolvent convergence and spectral measures now, we ensure that any later RH-facing claim rests on a solid operator-theoretic foundation, rather than on unstable finite-dimensional artifacts.

Eigenvalues are fragile. Resolvents and spectral measures are structural.

7 $SU(1,1)$ Möbius Gauge and Spectral Stability

7.1 Why a gauge principle is needed

Up to this point, the SMRK Hamiltonian has been defined with projective weights

$$W_p \in \mathbb{CP}^1,$$

and self-adjointness has been enforced by restricting to a suitable real slice. However, without further structure, the space of admissible weights is still too large: different choices of W_p may lead to operators that are formally self-adjoint but spectrally unrelated.

To meaningfully compare spectra, truncations, and resolvent limits, we must identify a *gauge symmetry* on the weight space such that:

- gauge-related weight configurations yield unitarily equivalent (or spectrally equivalent) operators,
- physically relevant spectral data are gauge-invariant,
- the remaining degrees of freedom correspond to genuine deformations of the model.

The correct symmetry group accomplishing this is $SU(1,1)$ acting by Möbius transformations on the projective weight space $\mathbb{CP}^1$.

7.2 The group $SU(1,1)$ and its Möbius action

Recall that

$$SU(1,1) = \left\{ \begin{pmatrix} \alpha & \beta \\ \overline{\beta} & \overline{\alpha} \end{pmatrix} \in SL(2,\mathbb{C}) \;\middle|\; |\alpha|^2 - |\beta|^2 = 1 \right\}.$$

An element $g \in SU(1,1)$ acts on $\mathbb{CP}^1$ by the Möbius transformation

$$z \longmapsto g \cdot z := \frac{\alpha z + \beta}{\overline{\beta} z + \overline{\alpha}}.$$

Key invariants of this action:

- the unit disk $\mathbb{D} = \{z : |z| < 1\}$ is preserved,
- the unit circle $S^1 = \partial\mathbb{D}$ is preserved,
- antipodal involutions such as $z \mapsto -1/z$ are contained in the normalizer.

Thus $SU(1,1)$ is the natural symmetry group of the self-adjoint weight slice.

7.3 Gauge action on SMRK weights

Let $\{W_p = [A_p : B_p]\}_{p\in\mathbb{P}}$ be a collection of projective weights satisfying the self-adjointness condition $B_p = \overline{A_p}$.

Define a global gauge transformation by fixing $g \in SU(1,1)$ and setting

$$W_p \longmapsto g \cdot W_p \quad \text{for all } p.$$

Definition 2 ($SU(1,1)$ gauge equivalence)**.** Two weight configurations $\{W_p\}$ and $\{W'_p\}$ are called gauge-equivalent if there exists $g \in SU(1,1)$ such that $W'_p = g \cdot W_p$ for all p.

This transformation is *prime-independent*: it acts uniformly on all prime channels, reflecting a global change of representation rather than a dynamical deformation.

7.4 Gauge covariance of the operator

Under an $SU(1,1)$ gauge transformation, the forward/backward coefficients (A_p, B_p) transform linearly:

$$\begin{pmatrix} A_p \\ B_p \end{pmatrix} \longmapsto \begin{pmatrix} \alpha & \beta \\ \overline{\beta} & \overline{\alpha} \end{pmatrix} \begin{pmatrix} A_p \\ B_p \end{pmatrix}.$$

Because this transformation preserves the Hermitian form $|A_p|^2 - |B_p|^2$, it preserves the adjointness relation between forward and backward transfer terms.

Proposition 2 (Gauge covariance)**.** *If H is the $\mathbb{CP}^1$-weighted SMRK Hamiltonian associated to weights $\{W_p\}$, and H' is defined using $\{g \cdot W_p\}$ for some $g \in SU(1,1)$, then H and H' are unitarily equivalent up to a bounded diagonal conjugation. In particular, they have identical spectra.*

Remark 7. At the level of finite truncations H_N, this equivalence holds modulo boundary effects, which vanish in the $N \to \infty$ limit.

7.5 Fixing a canonical gauge

Gauge freedom allows us to impose convenient normalizations. Common choices include:

Unit-circle gauge. Choose W_p such that $|z_p| = 1$ for all p, i.e.

$$z_p = e^{i\theta_p}.$$

This emphasizes balanced forward/backward transport.

Real-axis gauge. Use a Möbius transformation to place selected weights on $\mathbb{R} \cup \{\infty\}$, useful for visualizing antipodal pairs such as

$$12 \leftrightarrow -\frac{1}{12}.$$

Calibration gauge. Fix a reference shell or prime family so that its effective weight ratio equals 12. All other weights are then measured relative to this calibration.

Remark 8. Different gauges correspond to different coordinate charts on the same geometric object. No spectral statement should depend on the choice of gauge.

7.6 The involution J inside $SU(1,1)$

The involution

$$J(z) = -\frac{1}{z}$$

is represented (up to phase) by an element of $SU(1,1)$. It exchanges the interior and exterior of the unit disk and preserves S^1.

In gauge language:

- J is a discrete gauge symmetry,
- it pairs expansion-dominated and contraction-dominated regimes,
- it enforces spectral pairing at the level of transfer amplitudes.

The calibration pair $12 \leftrightarrow -1/12$ is precisely a J-orbit.

7.7 Gauge invariants and genuine parameters

Once the $SU(1,1)$ gauge freedom is factored out, the remaining parameters of the model fall into two categories:

- **Gauge invariants:** cross-ratios of weight quadruples, relative phase differences, and quantities entering resolvent limits and IDS.
- **Physical deformations:** prime-dependent variations not removable by a global Möbius transformation, diagonal potential parameters (α, β).

Only the latter can change spectral statistics in a meaningful way.

7.8 Consequences for numerical experiments

For finite-volume computations, the gauge principle provides a critical consistency check:

- gauge-related weight choices must yield identical unfolded level statistics,
- resolvent-based observables should agree up to numerical error,
- discrepancies signal truncation artifacts rather than genuine spectral effects.

Thus $SU(1,1)$ gauge covariance is not merely aesthetic; it is a practical tool for validating numerical results.

7.9 Position in the overall program

At this point, the structure is as follows:

- weights live on a compact geometric space $\mathbb{CP}^1$,
- self-adjointness selects an $SU(1,1)$-invariant slice,
- gauge equivalence removes non-physical degrees of freedom,
- remaining spectral data are meaningful and testable.

This completes the geometric and analytic preparation. The next step is to ask a genuinely discriminating question: *what do the level statistics look like, and how do they compare with random-matrix universality classes?*

That question, and its implications for a Hilbert–Pólya interpretation, will be addressed in the subsequent chapter.

8 RH Interface: Level Statistics, Unfolding, and Falsifiable Predictions

8.1 Why level statistics matter (and what they do *not* prove)

The purpose of this chapter is diagnostic rather than deductive. Level statistics do not prove the Riemann Hypothesis (RH); they test whether a given operator is a *credible* Hilbert–Pólya candidate.

The logical hierarchy is:

1. existence of a self-adjoint operator (Ch. 4),
2. computable and stable spectral data (Ch. 5–6),
3. universality of spectral fluctuations (this chapter),
4. only then: trace/determinant identities linking to $\xi(s)$ (future work).

Thus, level statistics provide a necessary but not sufficient checkpoint. Failure here falsifies the model; success only justifies further analysis.

8.2 Finite-volume spectrum and ordering

For a truncation H_N, let

$$\lambda_1^{(N)} \le \lambda_2^{(N)} \le \cdots \le \lambda_N^{(N)}$$

denote the ordered eigenvalues (counted with multiplicity). Raw spacings $\lambda_{j+1}^{(N)} - \lambda_j^{(N)}$ are *not* universal: they depend on the local density of states and therefore on energy.

Universality emerges only after removing the global density trend.

8.3 Integrated density of states and unfolding

Let $\nu(E)$ be the integrated density of states (IDS) introduced in Ch. 6. In practice, ν is approximated numerically by

$$\nu_N(E) := \frac{1}{N}\#\{\lambda_j^{(N)} \le E\},$$

smoothed over a suitable energy window.

Definition 3 (Unfolding)**.** The unfolded eigenvalues are defined by

$$\tilde{\lambda}_j := \nu_N\left(\lambda_j^{(N)}\right).$$

Their mean spacing is asymptotically unity.

All statistical tests in this chapter are performed on the unfolded sequence $\{\tilde{\lambda}_j\}$.

Remark 9. Without unfolding, apparent "random-matrix behavior" can be entirely spurious. Unfolding is therefore not optional but mandatory.

8.4 Universality classes: Poisson, GOE, and GUE

Let $s_j = \tilde{\lambda}_{j+1} - \tilde{\lambda}_j$ be nearest-neighbor spacings.

Three reference distributions play a central role:

Poisson (integrable systems).

$$P_{\text{Pois}}(s) = e^{-s}.$$

No level repulsion; $P(0) = 1$.

GOE (time-reversal invariant chaos).

$$P_{\text{GOE}}(s) = \frac{\pi}{2} s\, e^{-\pi s^2/4}.$$

Linear level repulsion: $P(s) \sim s$ as $s \downarrow 0$.

GUE (broken time-reversal symmetry).

$$P_{\text{GUE}}(s) = \frac{32}{\pi^2} s^2 e^{-4s^2/\pi}.$$

Quadratic level repulsion: $P(s) \sim s^2$.

Montgomery's pair-correlation conjecture and extensive numerical work by Odlyzko suggest that the nontrivial zeros of the Riemann zeta function exhibit GUE statistics.

8.5 Expected behavior for a Hilbert–Pólya candidate

If the SMRK Hamiltonian is to be a serious Hilbert–Pólya candidate, the following behavior is expected *in an appropriate energy window*:

- unfolded spacing statistics close to GUE,
- clear level repulsion ($P(0) = 0$),
- stability under increasing truncation size N,
- invariance under $SU(1,1)$ gauge transformations of weights,
- robustness under the involution $J : z \mapsto -1/z$.

None of these properties is guaranteed by self-adjointness alone; they are emergent, testable features.

8.6 Gauge invariance as a consistency check

As shown in Ch. 7, global $SU(1,1)$ transformations act as a gauge symmetry on weights. Therefore:

Proposition 3 (Gauge invariance of level statistics)**.** *If two weight configurations are related by a global $SU(1,1)$ transformation, their unfolded level statistics must agree up to statistical error.*

Violation of this invariance indicates that observed statistics are dominated by truncation artifacts or boundary effects rather than intrinsic spectral behavior.

8.7 Positive signals and negative signals

The level-statistics interface provides a clean falsifiability criterion.

Positive signals (model survives).

- GUE-like spacing distribution over a growing energy window,
- convergence of statistics as $N \to \infty$,
- independence from gauge choice and calibration,
- consistency across different spectral probes (spacing, number variance).

Negative signals (model fails).

- Poisson statistics or absence of level repulsion,
- statistics drifting with N without stabilization,
- strong sensitivity to gauge or calibration choice,
- dominant localization visible in spectral measures.

Negative outcomes do not disprove RH; they disqualify the operator.

8.8 Relation to the projective duality

The involution

$$J(z) = -\frac{1}{z}$$

encodes expansion–contraction duality at the level of weights. A necessary consistency check is that unfolded level statistics are invariant under the induced transformation on the operator.

This mirrors, at a purely spectral level, the functional symmetry $\xi(s) = \xi(1-s)$ of the completed zeta function, without asserting any direct identity.

8.9 What would count as genuine progress toward RH

At this stage, *progress* means narrowing the search space:

- ruling out large classes of operators by negative statistical tests,
- identifying a small class with stable GUE behavior and strong symmetry control,
- justifying deeper analytic work (trace formulae, determinants) only for survivors.

A positive outcome here does *not* imply RH. It implies that the operator deserves further analytic investment.

8.10 Summary

This chapter establishes a strict interface between the SMRK Hamiltonian and the statistical phenomenology associated with zeta zeros:

- level statistics are computed only after unfolding,
- universality classes provide a falsifiable benchmark,
- $SU(1,1)$ gauge invariance is a non-negotiable consistency condition,
- GUE-like behavior is a necessary (but not sufficient) signal.

A viable Hilbert–Pólya candidate must first survive statistics.

A Appendix A: Determinant Program and the ξ-Interface

A.1 Purpose and status of this appendix

This appendix is explicitly *programmatic*. It does not claim a proof of any identity involving the Riemann zeta function. Its purpose is to precisely formulate what a determinant-based route *would have to look like* if the SMRK Hamiltonian were to support a Hilbert–Pólya interpretation.

Everything here is conditional and falsifiable. Nothing in this appendix is used elsewhere in the paper.

A.2 Why determinants appear naturally

For a self-adjoint operator H, eigenvalues encode spectral data additively. Zeta functions and L-functions, by contrast, encode arithmetic data multiplicatively. The bridge between these two languages is the *spectral determinant*.

Formally, if $\{\lambda_j\}$ are eigenvalues of H, one would like to define

$$D(z) \overset{\text{formal}}{=} \prod_j (\lambda_j - z).$$

This product diverges and must be regularized.

The key question is not whether a determinant exists, but whether a *canonical regularization* exists that is stable, gauge-invariant, and compatible with arithmetic structure.

A.3 Zeta-regularized determinants

A standard approach is zeta regularization. Given a self-adjoint operator H bounded below, define

$$\zeta_H(s) := \sum_j \lambda_j^{-s},$$

for $\mathrm{Re}(s)$ sufficiently large, and extend meromorphically.

If $\zeta_H(s)$ is regular at $s = 0$, one defines

$$\det{}_\zeta(H) := \exp\big(-\zeta_H'(0)\big).$$

Remark 10. For infinite-dimensional operators, existence of $\zeta_H'(0)$ is highly nontrivial. This appendix treats it as a *target property*, not an assumption.

A.4 Resolvent-based determinant surrogates

An alternative formulation avoids explicit eigenvalue products. For $z \in \mathbb{C} \setminus \mathbb{R}$, define the resolvent trace per unit volume

$$\tau(z) := \lim_{N\to\infty} \frac{1}{N} \mathrm{Tr}\,(H_N - zI)^{-1}.$$

Formally, if a determinant density exists, it should satisfy

$$\frac{d}{dz} \log D(z) = -\tau(z).$$

Integrating,

$$\log D(z) = -\int^z \tau(w)\, dw + \text{const.}$$

This formulation:

- is compatible with the resolvent framework of Chapter 6,
- bypasses direct eigenvalue products,
- is stable under finite-rank perturbations.

A.5 Transfer operators and determinant factorization

The SMRK Hamiltonian decomposes naturally into transfer components:

$$H = \sum_{p\in\mathbb{P}} H_p + V,$$

where H_p encodes the p-multiplicative forward/backward transfers.

In favorable situations, determinant-like objects factorize:

$$D(z) \sim \prod_p D_p(z),$$

where D_p captures the contribution of the p-channel.

Remark 11. Such factorization is formal and must be justified. Its arithmetic appeal lies in the resemblance to Euler products, but resemblance alone is not evidence.

A.6 The role of projective weights and Möbius symmetry

Because weights live on $\mathbb{CP}^1$ and are acted upon by $SU(1,1)$, any determinant candidate must satisfy:

- invariance under global $SU(1,1)$ gauge transformations,
- invariance under the involution $J : z \mapsto -1/z$,
- independence from the choice of affine chart.

These conditions sharply restrict admissible regularizations. In particular, any determinant depending on absolute values of weights is automatically excluded.

A.7 Hypothetical ξ-correspondence

A Hilbert–Pólya-type goal would be an identity of the schematic form

$$D\left(\tfrac{1}{2}+it\right) \;\propto\; \xi\left(\tfrac{1}{2}+it\right),$$

where ξ is the completed Riemann xi-function.

This would imply:

- zeros of $\xi(s)$ correspond to spectral singularities of H,
- the critical line $\mathrm{Re}(s)=\frac{1}{2}$ arises from self-adjointness,
- functional symmetry $\xi(s)=\xi(1-s)$ reflects projective duality.

Remark 12. This appendix does *not* claim such an identity. It merely states what form such an identity would have to take to be compatible with the framework developed in this paper.

A.8 Falsifiability criteria for the determinant program

The determinant route is rejected if any of the following occurs:

- no gauge-invariant regularization can be defined,
- determinant surrogates depend sensitively on truncation,
- resolvent integrals fail to converge,
- projective duality is broken at the determinant level.

Conversely, if a stable, gauge-invariant determinant surrogate emerges from resolvent data, the model merits deeper analytic investigation.

A.9 Summary of Appendix A

This appendix outlines a *conditional determinant program*:

- determinants are introduced only via resolvent limits,
- projective and gauge symmetries are treated as constraints, not decoration,
- no claim is made beyond falsifiable structural compatibility.

Determinants are not assumed. They must be earned from resolvents.

B Appendix B: Numerical Protocol and Reproducibility

B.1 Purpose of this appendix

This appendix specifies a minimal numerical protocol for all spectral claims made in Chapters 5–8. Its role is purely methodological: to ensure that every reported spectral feature is reproducible, parameter-controlled, and falsifiable.

No theoretical claims are introduced here.

B.2 Finite-volume construction

All numerical experiments are based on the truncated operators

$$H_N := P_N H P_N, \quad \mathcal{H}_N = \mathrm{span}\{\delta_1, \ldots, \delta_N\},$$

with N chosen large enough to ensure stability of observables under further increases of truncation size.

Recommended values. Typical experiments use

$$N \in [10^4, 10^5],$$

depending on available memory and solver performance.

B.3 Sparse matrix assembly

The matrix elements of H_N are constructed as follows:

- diagonal entries: $(H_N)_{nn} = V(n)$,
- forward transfers: $(H_N)_{n,pn} = A_p$ if $pn \le N$,
- backward transfers: $(H_N)_{n,n/p} = \overline{A_p}$ if $p \mid n$.

Only primes $p \le N$ need to be considered. The resulting matrix is sparse, symmetric (or Hermitian), with $O(N \log\log N)$ nonzero entries.

B.4 Eigenvalue computation

Eigenvalues are computed using iterative solvers for sparse Hermitian matrices:

- Lanczos algorithm for extremal eigenvalues,
- ARPACK (`eigsh`) for interior spectral windows,
- shift–invert mode for focused energy regions.

Full diagonalization is neither required nor recommended.

B.5 Energy windows

Level statistics are computed only in energy windows where:

- the local density of states is approximately constant,
- boundary effects from truncation are negligible,
- convergence under increasing N is observed.

Edge regions of the spectrum are excluded.

B.6 Integrated density of states (IDS)

The IDS is estimated by

$$\nu_N(E) = \frac{1}{N} \#\{\lambda_j^{(N)} \le E\},$$

optionally smoothed by local averaging. All unfolding procedures are based on ν_N.

B.7 Unfolding procedure

Given ordered eigenvalues $\{\lambda_j^{(N)}\}$, the unfolded sequence is

$$\tilde{\lambda}_j = \nu_N(\lambda_j^{(N)}).$$

Only unfolded eigenvalues are used for statistical analysis. Raw spacings are never interpreted.

B.8 Level statistics

The following observables are computed:

- nearest-neighbor spacing distribution $P(s)$,
- cumulative spacing distribution,
- number variance (optional).

Results are compared against Poisson, GOE, and GUE reference curves.

B.9 Gauge consistency checks

All numerical experiments are repeated under:

- different $SU(1,1)$ gauge choices,
- involution $J : z \mapsto -1/z$ applied to weights,
- mild variations of calibration parameters.

Only gauge-invariant spectral features are considered meaningful.

B.10 Resolvent-based observables

As a complementary check, diagonal resolvent elements

$$G_n(z) = \langle \delta_n, (H_N - zI)^{-1} \delta_n \rangle$$

are computed for $z = E + i\varepsilon$ with fixed $\varepsilon > 0$. Spectral densities are extracted from $\operatorname{Im} G_n(z)$.

B.11 Convergence criteria

A numerical claim is accepted only if:

- it is stable under $N \to N'$ with $N'/N \geq 1.5$,
- it persists across gauge transformations,
- it is insensitive to reasonable solver tolerances.

Failure of any criterion invalidates the claim.

B.12 Reproducibility statement

All numerical results presented in this work are, in principle, reproducible from:

- the operator definition in Chapter 2,
- the gauge constraints in Chapters 4 and 7,
- the protocol specified in this appendix.

Implementation details (language, libraries, hardware) do not affect the qualitative conclusions when the protocol is followed.

Numerics do not replace theory. They decide whether theory deserves to continue.

C Minimal Assumptions and Failure Modes

C.1 Purpose of this appendix

This appendix serves two closely related goals:

- to make *explicit* the minimal assumptions under which the SMRK spectral program is meaningful,
- to clearly identify failure modes that would invalidate the operator as a Hilbert–Pólya candidate.

This appendix is intentionally conservative. Any violation listed here is to be interpreted as a *stop condition*, not as a technical inconvenience.

C.2 Minimal structural assumptions

The entire framework developed in this paper relies only on the following assumptions.

(A1) Well-defined Hilbert space. The operator acts on $\ell^2(\mathbb{N})$ with its standard inner product. No completion, weighting, or renormalization of the Hilbert space is assumed.

(A2) Densely defined operator. The SMRK Hamiltonian is defined at least on the algebraic core

$$\mathcal{D}_0 = \mathrm{span}\{\delta_n : n \in \mathbb{N}\},$$

which is dense in $\ell^2(\mathbb{N})$.

(A3) Self-adjoint realization. There exists a self-adjoint (or essentially self-adjoint) realization of H compatible with the projective weight constraints and diagonal potential.

This assumption is structural and non-negotiable.

(A4) Gauge compatibility. All physically meaningful quantities are invariant under global $SU(1,1)$ Möbius gauge transformations of weights.

(A5) Existence of thermodynamic limits. Key observables (resolvent matrix elements, IDS, unfolded statistics) admit limits as $N \to \infty$ independent of truncation details.

No assumptions beyond (A1)–(A5) are used anywhere in the paper.

C.3 Assumptions explicitly *not* made

Equally important is what is *not* assumed:

- no a priori relation to the Riemann zeta function,
- no trace formula,
- no determinant identity,
- no random-matrix universality conjecture,
- no semiclassical limit.

All such structures, if present, must *emerge* from analysis or numerics.

C.4 Failure modes: operator-level

The following conditions invalidate the program at the operator level:

(F1) Non-self-adjointness. If no self-adjoint extension exists, the spectral interpretation collapses. The model must be abandoned or fundamentally altered.

(F2) Domain instability. If different reasonable domain choices lead to inequivalent spectra, the operator is ill-defined for Hilbert–Pólya purposes.

(F3) Gauge breaking. If global $SU(1,1)$ transformations alter spectral observables, the projective weight interpretation fails.

C.5 Failure modes: spectral and numerical

The following are decisive numerical failures:

(F4) Lack of convergence. If resolvent limits, IDS, or unfolded statistics do not stabilize as $N \to \infty$, no infinite-volume operator interpretation is justified.

(F5) Poisson statistics. Persistent Poissonian level statistics in bulk energy windows indicate integrable or localized behavior incompatible with RH phenomenology.

(F6) Strong localization. If spectral measures μ_{δ_n} remain sharply localized across scales, the operator fails to generate global spectral mixing.

(F7) Truncation dominance. If observed features are sensitive to boundary conditions, solver tolerances, or prime cutoffs, they are numerical artifacts.

C.6 Failure modes: determinant program

The determinant route (Appendix A) is invalidated if:

- no gauge-invariant regularization exists,
- resolvent integrals diverge or oscillate uncontrollably,
- determinant surrogates depend on affine chart choice,

- projective duality fails at determinant level.

Failure here does not invalidate the operator itself, but definitively blocks the determinant approach.

C.7 Positive survival criteria

Conversely, the operator *survives* this appendix if:

- self-adjointness is robust and domain-independent,
- spectral measures exist and are non-pathological,
- IDS and unfolded statistics converge,
- gauge invariance is numerically exact,
- GUE-like statistics persist in growing windows.

Survival does not imply RH. It implies that the model remains a legitimate object of further study.

C.8 Interpretational discipline

This appendix enforces a strict interpretational rule:

> Negative results falsify the model. Positive results justify deeper analysis, nothing more.

Any claim exceeding this rule is outside the scope of this work.

C.9 Summary

This appendix delineates the exact logical boundaries of the SMRK program:

- minimal assumptions are few and explicit,
- failure modes are clear and decisive,
- no ambiguity remains about what would invalidate the approach.

A serious RH program must define how it can fail.

References

[1] M. Reed, B. Simon, *Methods of Modern Mathematical Physics I: Functional Analysis.*

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