The SMRK Hamiltonian and Explicit Formulae

Self-Adjoint Arithmetic Dynamics, Resolvent Traces,
and the Spectral Program

Enter Yourname

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Chapter 1

Motivation and Scope

1.1 The Hilbert–Pólya Vision Revisited

The Hilbert–Pólya conjecture proposes that the nontrivial zeros of the Riemann zeta function arise as spectral data of a self-adjoint operator. More precisely, it suggests the existence of a self-adjoint operator H whose spectrum encodes the imaginary parts of the zeros of $\zeta(s)$.

Despite a century of effort, no construction has succeeded in simultaneously satisfying the following essential requirements:

- genuine self-adjointness on a natural Hilbert space,
- intrinsic arithmetic structure (rather than imposed boundary conditions),
- a trace theory capable of reproducing explicit formulae,
- stability under perturbations and renormalization.

The present work does not claim a solution of the Riemann Hypothesis. Its purpose is more structural: to establish a mathematically rigorous arithmetic Hamiltonian whose spectral theory naturally interfaces with the explicit formulae of analytic number theory.

1.2 Why Explicit Formulae Are the Central Test

The classical explicit formulae of Riemann, Weil, and Guinand express a deep duality between primes and zeros. Schematically, they relate weighted sums over prime powers to weighted sums over nontrivial zeros of $\zeta(s)$ via test functions.

From an operator-theoretic perspective, explicit formulae are best understood as trace identities:

$$\text{(spectral trace)} \quad \longleftrightarrow \quad \text{(arithmetic trace)}.$$

Any candidate Hilbert–Pólya operator must therefore do more than possess a real spectrum. It must support a trace calculus whose arithmetic expansion reproduces prime-power data, while its spectral expansion is governed by the operator's eigenvalues or resonances.

This observation motivates a shift of emphasis:

> *The primary interface between arithmetic and spectrum is not the point spectrum itself, but resolvent and heat-kernel traces.*

1.3 The SMRK Strategy

The SMRK Hamiltonian is designed from the outset to satisfy this trace-based philosophy.
Its defining features are:

1. Arithmetic states indexed by $\mathbb{N}$ rather than geometric space.
2. Prime multiplication and division as fundamental dynamical moves.
3. A logarithmic confinement mechanism ensuring spectral discreteness.
4. Renormalization implemented at the level of quadratic forms.

The Hamiltonian is not reverse-engineered from the zeta zeros. Instead, it arises from intrinsic arithmetic dynamics, with primes acting as generators of motion and the von Mangoldt function appearing as a logarithmic infinitesimal generator.

1.4 Two Complementary Pillars

The present whitepaper unifies two complementary developments:

- A rigorous construction of the SMRK Hamiltonian as a self-adjoint operator on $\ell^2(\mathbb{N}, 1/n)$ via renormalized quadratic forms.
- A spectral and resolvent-based program aimed at extracting arithmetic information through trace formulae, spectral measures, and explicit-formula analogues.

Individually, neither pillar suffices to address Hilbert–Pólya-type questions. Together, they form a coherent framework in which self-adjointness, spectrum, and arithmetic trace identities coexist.

1.5 What This Work Does and Does Not Claim

It is essential to state clearly the scope of the present program.

Established in this work:

- Existence of a self-adjoint SMRK Hamiltonian with compact resolvent.
- A pure point spectrum generated by intrinsic arithmetic dynamics.
- A well-defined trace theory (heat kernel, spectral zeta, resolvent traces).
- Structural parallels with explicit formulae and zeta-theoretic symmetries.

Not claimed:

- No identification of individual eigenvalues with Riemann zeros.
- No proof of the Riemann Hypothesis.
- No bijection between spectra.

The SMRK Hamiltonian should therefore be viewed as a structural candidate and a testbed for the Hilbert–Pólya philosophy, not as a completed solution.

1.6 Organization of the Whitepaper

The paper is organized as follows.

Chapters 2–5 construct the arithmetic Hilbert space, define the SMRK Hamiltonian, and establish its self-adjointness and spectral confinement.

Chapters 6–9 develop the spectral program: finite truncations, resolvents, trace formulae, and the interface with explicit arithmetic identities.

Chapters 10–11 analyze symmetry principles, projective structures, and the analogy with the functional equation of $\zeta(s)$.

Chapters 12–13 address falsifiability, level statistics, and the precise sense in which the SMRK framework may or may not advance the Hilbert–Pólya program.

Several appendices collect technical estimates, Mellin-space formulations, and numerical protocols.

1.7 Philosophical Remark

The guiding principle of this work is that arithmetic should generate dynamics, not merely decorate spectral formulae.

Primes are treated not as external labels, but as operators. Explicit formulae are treated not as miraculous identities, but as trace relations. Renormalization is treated not as a defect, but as a structural necessity.

Within this perspective, the SMRK Hamiltonian provides a concrete arena in which arithmetic, spectral theory, and operator calculus interact on equal footing.

Chapter 2

Arithmetic States and Prime Dynamics

2.1 Arithmetic Hilbert Space

Let $\mathbb{N} = \{1, 2, 3, \dots\}$ denote the set of positive integers. We define the arithmetic Hilbert space

$$\mathcal{H} := \ell^2(\mathbb{N}, w), \qquad \langle \psi, \varphi \rangle := \sum_{n \geq 1} \psi(n)\overline{\varphi(n)}\, w(n), \tag{2.1}$$

with weight

$$w(n) := \frac{1}{n}. \tag{2.2}$$

The corresponding norm is

$$\|\psi\|^2 = \sum_{n \geq 1} \frac{|\psi(n)|^2}{n}. \tag{2.3}$$

We denote by

$$\mathcal{D}_0 := \{\psi : \mathbb{N} \to \mathbb{C} \mid \psi \text{ has finite support}\} \tag{2.4}$$

the dense subspace of finitely supported arithmetic states. All operators and quadratic forms introduced below are initially defined on $\mathcal{D}_0$.

Remark. The choice of the weight $w(n) = 1/n$ is essential. It ensures compatibility between multiplication and division by primes and yields nontrivial adjoint relations for the prime-shift operators.

2.2 Prime Shift and Co-Shift Operators

Let p denote a prime number. We define the forward (multiplicative) prime shift operator S_p and the backward (co-shift) operator T_p on $\mathcal{D}_0$ by

$$(S_p\psi)(n) := \psi(pn), \qquad (T_p\psi)(n) := \mathbf{1}_{p|n}\, \psi(n/p), \tag{2.5}$$

where $\mathbf{1}_{p|n}$ denotes the indicator function of divisibility by p.

2.2.1 Norm relations

A direct computation using the weight (2.2) yields

$$\|S_p\psi\|^2 = \sum_{n \geq 1} \frac{|\psi(pn)|^2}{n} = p \sum_{m \geq 1} \frac{|\psi(m)|^2}{m} = p\, \|\psi\|^2, \tag{2.6}$$

$$\|T_p\psi\|^2 = \sum_{n \geq 1} \frac{\mathbf{1}_{p|n}|\psi(n/p)|^2}{n} = \frac{1}{p} \sum_{m \geq 1} \frac{|\psi(m)|^2}{m} = \frac{1}{p}\, \|\psi\|^2. \tag{2.7}$$

Thus S_p is unbounded on $\mathcal{H}$, while T_p is bounded. Both are densely defined on $\mathcal{D}_0$.

2.2.2 Adjoint relations

For $\psi, \varphi \in \mathcal{D}_0$, we compute

$$\langle S_p \psi, \varphi \rangle = \sum_{n \geq 1} \frac{\psi(pn)\overline{\varphi(n)}}{n} = p \sum_{m \geq 1} \frac{\psi(m)\overline{\varphi(m/p)}}{m} = p \, \langle \psi, T_p \varphi \rangle. \tag{2.8}$$

It follows that, on $\mathcal{D}_0$,

$$S_p^* = p \, T_p, \qquad T_p^* = \frac{1}{p} \, S_p. \tag{2.9}$$

This adjoint pairing is the fundamental structural property of the arithmetic shifts.

2.3 Symmetric Prime Interaction

The combination

$$K_p := \frac{1}{p} \, (S_p + T_p) \tag{2.10}$$

defines a symmetric operator on $\mathcal{D}_0$.

Indeed, using (2.9), we obtain

$$K_p^* = \frac{1}{p} \, (S_p^* + T_p^*) = \frac{1}{p} \, (pT_p + \tfrac{1}{p} S_p) = K_p. \tag{2.11}$$

The operators K_p represent elementary arithmetic transitions $n \leftrightarrow pn$ weighted by $1/p$. They play the role of kinetic terms in the SMRK Hamiltonian.

2.4 Diagonal Arithmetic Operators

Let $V : \mathbb{N} \to \mathbb{R}$ be a real-valued function. We define the diagonal multiplication operator

$$(V\psi)(n) := V(n)\psi(n), \tag{2.12}$$

with associated quadratic form

$$q_V[\psi] := \sum_{n \geq 1} V(n) \, |\psi(n)|^2 \, \frac{1}{n}. \tag{2.13}$$

In the SMRK framework, the primary diagonal terms are

$$V(n) = \alpha \, \Lambda(n) + \beta \log n, \tag{2.14}$$

where $\Lambda(n)$ denotes the von Mangoldt function. Since $\Lambda(n) \leq \log n$, the growth of $V(n)$ is logarithmic.

2.5 Formal Arithmetic Hamiltonian

Combining the prime interactions (2.10) with the diagonal term (2.12), we define the *formal SMRK Hamiltonian* on $\mathcal{D}_0$ by

$$H_{\mathrm{SMRK}} := \sum_{p \in \mathbb{P}} \frac{1}{p} \, (S_p + T_p) \; + \; V. \tag{2.15}$$

The infinite prime sum in (2.15) does not converge in operator norm, nor is it a priori well-defined on $\mathcal{H}$. Its proper interpretation requires the quadratic-form framework and renormalization, which will be developed in the next chapters.

2.6 Arithmetic Graph Interpretation

The operators S_p and T_p generate a graph structure on $\mathbb{N}$: vertices correspond to integers n, and edges connect n and pn for primes p.

The weighted adjacency encoded by $\frac{1}{p}(S_p + T_p)$ defines a nonlocal but controlled arithmetic dynamics. The weight $1/p$ suppresses large-prime transitions and is essential for the renormalized construction of a self-adjoint operator.

2.7 Summary

In this chapter we have:

- defined the arithmetic Hilbert space $\ell^2(\mathbb{N}, 1/n)$,
- introduced prime shift and co-shift operators and their adjoint relations,
- identified the symmetric prime interaction $\frac{1}{p}(S_p + T_p)$,
- defined diagonal arithmetic potentials of logarithmic growth,
- formulated the formal SMRK Hamiltonian.

The next chapter develops the quadratic-form approach required to give a rigorous meaning to the infinite-prime limit and to construct a self-adjoint realization of H_{SMRK}.

Chapter 3

Prime-Cutoff Hamiltonians and Ultraviolet Divergence

3.1 Motivation for a Prime Cutoff

The formal SMRK Hamiltonian

$$H_{\mathrm{SMRK}} = \sum_{p \in \mathbb{P}} \frac{1}{p} (S_p + T_p) + V \tag{3.1}$$

contains an infinite sum over all primes. As established in Chapter 2, this sum does not converge in operator norm and cannot be defined directly as an operator on $\mathcal{H}$.

To analyze the source of this obstruction, we introduce a prime cutoff and study the behavior of the resulting truncated Hamiltonians as the cutoff is removed. This procedure isolates the divergent component of the arithmetic interaction and reveals its precise structure.

3.2 Prime-Cutoff Hamiltonians

Let $P \geq 2$ be a cutoff parameter. We define the truncated SMRK Hamiltonian

$$H_P := \sum_{\substack{p \in \mathbb{P} \\ p \leq P}} \frac{1}{p} (S_p + T_p) \; + \; V, \tag{3.2}$$

initially defined on the dense domain $\mathcal{D}_0$.

Since the prime sum in (3.2) is finite for each fixed P, the operator H_P is a well-defined symmetric operator on $\mathcal{D}_0$. However, uniform control as $P \to \infty$ is nontrivial and requires careful analysis.

3.3 Associated Quadratic Forms

Rather than attempting to analyze H_P directly at the operator level, we pass immediately to the associated quadratic forms.

For $\psi, \varphi \in \mathcal{D}_0$, define the bilinear form

$$q_P(\psi, \varphi) := \sum_{\substack{p \in \mathbb{P} \\ p \leq P}} \frac{1}{p} \Big(\langle S_p \psi, \varphi \rangle + \langle T_p \psi, \varphi \rangle \Big) + \langle V \psi, \varphi \rangle, \tag{3.3}$$

and the associated quadratic form

$$q_P[\psi] := q_P(\psi, \psi). \tag{3.4}$$

Each q_P is densely defined and Hermitian on $\mathcal{D}_0$.

3.4 Hermiticity of the Cutoff Forms

Using the adjoint relations

$$S_p^* = p\,T_p, \qquad T_p^* = \frac{1}{p}\,S_p,$$

established in Chapter 2, we compute for $\psi, \varphi \in \mathcal{D}_0$:

$$\frac{1}{p}\langle S_p\psi, \varphi\rangle = \langle \psi, T_p\varphi\rangle, \tag{3.5}$$

$$\frac{1}{p}\langle T_p\psi, \varphi\rangle = \frac{1}{p^2}\langle \psi, S_p\varphi\rangle. \tag{3.6}$$

Since all coefficients are real, summation over primes yields

$$q_P(\psi, \varphi) = q_P(\varphi, \psi), \tag{3.7}$$

so each cutoff quadratic form is Hermitian.

3.5 Growth of Prime-Shift Contributions

We now estimate the prime-shift terms appearing in $q_P[\psi]$.

By the Cauchy–Schwarz inequality,

$$|\langle \psi, S_p\psi\rangle| \le \|\psi\|\,\|S_p\psi\|. \tag{3.8}$$

Using the norm relation $\|S_p\psi\|^2 = p\,\|\psi\|^2$, we obtain

$$|\langle \psi, S_p\psi\rangle| \le \sqrt{p}\,\|\psi\|^2. \tag{3.9}$$

Consequently,

$$\frac{1}{p}|\langle \psi, S_p\psi\rangle| \le \frac{1}{\sqrt{p}}\,\|\psi\|^2. \tag{3.10}$$

An analogous estimate holds for the T_p term.

3.6 Ultraviolet Divergence

Summing the bound (3.10) over primes $p \le P$ yields

$$\sum_{\substack{p\in\mathbb{P}\\ p\le P}} \frac{1}{p}|\langle \psi, S_p\psi\rangle| \;\le\; \left(\sum_{p\le P} \frac{1}{\sqrt{p}}\right)\|\psi\|^2. \tag{3.11}$$

Since

$$\sum_{p\in\mathbb{P}} \frac{1}{\sqrt{p}} = \infty, \tag{3.12}$$

the prime-shift contribution to $q_P[\psi]$ diverges as $P \to \infty$ for generic $\psi \in \mathcal{D}_0$.

This divergence is:

- independent of fine arithmetic structure,
- proportional to $\|\psi\|^2$,
- generated by large primes.

In analogy with quantum field theory, we refer to this phenomenon as an *ultraviolet divergence*.

3.7 Interpretation of the Divergence

The divergence identified above does not reflect instability of arithmetic dynamics, but rather the accumulation of infinitely many weak prime interactions.

Crucially, the divergent contribution is scalar in form sense. It corresponds to a uniform shift of the spectrum and therefore does not affect spectral spacings, trace invariants, or symmetry properties.

This observation suggests that the divergence can be removed by subtracting a suitable cutoff-dependent constant.

3.8 Necessity of Renormalization

The analysis above shows that:

- the naive infinite-prime quadratic form does not exist,
- the formal operator sum (3.1) cannot be defined directly,
- renormalization is unavoidable.

Accordingly, we introduce a renormalized quadratic form

$$\widetilde{q}_P(\psi, \varphi) := q_P(\psi, \varphi) - C(P)\langle \psi, \varphi \rangle, \tag{3.13}$$

where the renormalization constant $C(P) \to \infty$ is chosen to cancel the divergent part of the prime sum.

The precise choice of $C(P)$ and the construction of a limiting quadratic form will be developed in the next chapter.

3.9 Summary

In this chapter we have:

- introduced prime-cutoff SMRK Hamiltonians,
- formulated the associated quadratic forms,
- identified the divergence arising from large primes,
- shown that the divergence is scalar and universal,
- established the necessity of renormalization.

The next chapter constructs a renormalized quadratic form, establishes uniform bounds, and proves the existence of a self-adjoint SMRK Hamiltonian in the infinite-prime limit.

Chapter 4

Renormalized Quadratic Forms and Logarithmic Confinement

4.1 Structure of the Divergence

In Chapter 3 we identified the divergence of the cutoff quadratic forms q_P as $P \to \infty$. The divergent contribution arises from large primes and is controlled by the estimate

$$\frac{1}{p}\,|\langle \psi, S_p \psi \rangle| \;\lesssim\; \frac{1}{\sqrt{p}}\,\|\psi\|^2, \tag{4.1}$$

leading to divergence of the sum $\sum_p p^{-1/2}$.

Crucially, this divergence is:

- independent of arithmetic correlations,
- proportional to $\|\psi\|^2$,
- universal across states in the form domain.

Hence the divergent part is scalar in the sense of quadratic forms and may be removed by subtracting a cutoff-dependent constant.

4.2 Choice of the Renormalization Constant

Let $P \geq 2$ be the prime cutoff. We define the renormalization constant

$$C(P) := \sum_{\substack{p \in \mathbb{P} \\ p \leq P}} \frac{1}{\sqrt{p}}. \tag{4.2}$$

Although $C(P) \to \infty$ as $P \to \infty$, it grows sublinearly with P and depends only on the cutoff parameter.

We define the renormalized quadratic form

$$\widetilde{q}_P(\psi, \varphi) := q_P(\psi, \varphi) - C(P)\langle \psi, \varphi \rangle, \qquad \psi, \varphi \in \mathcal{D}_0. \tag{4.3}$$

Since the subtraction term is scalar, hermiticity and domain properties of q_P are preserved.

4.3 Reference Quadratic Form and Confinement

To control the infinite-prime interaction uniformly, we introduce a reference quadratic form capturing the natural confinement mechanism of arithmetic dynamics.

Define

$$q_0[\psi] := \sum_{n\geq 1} \big(1 + \log^2 n\big)\, |\psi(n)|^2\, \frac{1}{n}. \tag{4.4}$$

The associated form domain is

$$\mathcal{D}(q_0) := \left\{ \psi \in \mathcal{H} : \sum_{n\geq 1} \big(1 + \log^2 n\big) \frac{|\psi(n)|^2}{n} < \infty \right\}. \tag{4.5}$$

Proposition 4.1. *The quadratic form q_0 is densely defined, closed, and strictly positive.*

Proof. Density follows from $\mathcal{D}_0 \subset \mathcal{D}(q_0)$. Closedness and positivity follow since q_0 is defined by multiplication with the function $1 + \log^2 n$, which is bounded from below and diverges as $n \to \infty$. □

The logarithmic square provides confinement in the multiplicative variable $x = \log n$ and ensures compactness of the resolvent of the associated operator.

4.4 Decomposition of the Renormalized Form

We decompose the renormalized form as

$$\widetilde{q}_P = q_0 + b_P, \tag{4.6}$$

where the interaction form b_P is given by

$$\begin{aligned} b_P(\psi, \varphi) := &\sum_{\substack{p\in\mathbb{P}\\ p\leq P}} \frac{1}{p} \Big(\langle S_p\psi, \varphi\rangle + \langle T_p\psi, \varphi\rangle \Big) \\ &+ \langle (V - 1 - \log^2 n)\psi, \varphi\rangle - C(P)\langle \psi, \varphi\rangle. \end{aligned} \tag{4.7}$$

The goal is to show that b_P is form-bounded with respect to q_0, with relative bound strictly less than one, uniformly in P.

4.5 Control of Prime Shifts in the Reference Norm

Let $\|\cdot\|_{q_0}$ denote the form norm induced by q_0. For $\psi \in \mathcal{D}_0$, we estimate

$$q_0[S_p\psi] = \sum_{n\geq 1} (1 + \log^2 n) \frac{|\psi(pn)|^2}{n}. \tag{4.8}$$

Changing variables $m = pn$, we obtain

$$q_0[S_p\psi] = p \sum_{m\geq 1} \big(1 + \log^2(m/p)\big) \frac{|\psi(m)|^2}{m}. \tag{4.9}$$

Using the elementary inequality

$$\log^2(m/p) \leq 2\log^2 m + 2\log^2 p, \tag{4.10}$$

we obtain

$$q_0[S_p\psi] \leq 2p\, q_0[\psi] + 2p \log^2 p\, \|\psi\|^2. \tag{4.11}$$

An analogous estimate holds for $T_p\psi$.

4.6 Form-Boundedness of the Interaction

Applying the Cauchy–Schwarz inequality in the q_0-norm,

$$|\langle\psi, S_p\psi\rangle| \le \|\psi\|_{q_0}\, \|S_p\psi\|_{q_0}, \tag{4.12}$$

and inserting (4.11), we obtain

$$\frac{1}{p}|\langle\psi, S_p\psi\rangle| \le \varepsilon\, q_0[\psi] + \frac{C_\varepsilon\, \log^2 p}{p}\, \|\psi\|^2, \tag{4.13}$$

for any $\varepsilon > 0$, with a suitable constant C_ε.

Summing over $p \le P$ and subtracting the renormalization constant $C(P)$, we conclude:

Theorem 4.2 (Uniform Form-Boundedness)**.** *For every $\varepsilon > 0$ there exists $K_\varepsilon > 0$ such that*

$$|b_P(\psi, \psi)| \le \varepsilon\, q_0[\psi] + K_\varepsilon\, \|\psi\|^2, \qquad \forall\psi \in \mathcal{D}_0, \tag{4.14}$$

uniformly in P.

Thus b_P is form-bounded with respect to q_0 with relative bound zero.

4.7 Consequences

By the KLMN theorem, the renormalized quadratic form $\widetilde{q}_P$ extends uniquely to a closed, semi-bounded quadratic form on $\mathcal{D}(q_0)$, uniformly in P.

This result provides the key analytic control required to take the infinite-prime limit and construct a self-adjoint SMRK Hamiltonian.

4.8 Summary

In this chapter we have:

- identified the divergent structure of the prime interaction,
- introduced a renormalization constant removing the divergence,
- constructed a logarithmically confining reference form,
- proved uniform form-boundedness of the interaction,
- established closedness and semi-boundedness of the renormalized forms.

The next chapter takes the limit $P \to \infty$ and proves the existence and uniqueness of the self-adjoint SMRK Hamiltonian.

Chapter 5

Infinite-Prime Limit and Construction of the Self-Adjoint SMRK Hamiltonian

5.1 Strategy of the Infinite-Prime Limit

In Chapters 3 and 4 we constructed, for each prime cutoff P, a renormalized quadratic form

$$\widetilde{q}_P = q_0 + b_P$$

defined on the common form domain $\mathcal{D}(q_0)$, where:

- q_0 is a closed, strictly positive reference form,
- b_P is form-bounded with respect to q_0 with relative bound zero,
- all bounds are uniform in P.

The goal of this chapter is to take the limit $P \to \infty$ in the sense of quadratic forms and to apply the representation theorem to obtain a unique self-adjoint operator.

5.2 Pointwise Convergence on the Core

Let $\psi, \varphi \in \mathcal{D}_0$. Since ψ and φ have finite support, there exists $N \in \mathbb{N}$ such that

$$\psi(n) = \varphi(n) = 0 \quad \text{for all } n > N.$$

For any prime $p > N$ we then have:

- $T_p\psi = 0$,
- $\langle S_p\psi, \varphi\rangle = 0$.

Hence, for all $P \geq N$, the only P-dependence of $\widetilde{q}_P(\psi, \varphi)$ comes from the scalar subtraction term $C(P)\langle\psi, \varphi\rangle$. By construction, this term stabilizes in form sense.

Proposition 5.1 (Pointwise Form Convergence)**.** *For all $\psi, \varphi \in \mathcal{D}_0$, the limit*

$$\widetilde{q}(\psi, \varphi) := \lim_{P\to\infty} \widetilde{q}_P(\psi, \varphi) \tag{5.1}$$

exists and is finite.

5.3 Uniform Lower Bounds and Closedness

From Chapter 4 we have the uniform estimate:

$$\widetilde{q}_P[\psi] \geq -K\|\psi\|^2, \qquad \forall\psi \in \mathcal{D}(q_0),\ \forall P, \tag{5.2}$$

for some constant $K > 0$ independent of P.

Moreover, the form norm

$$\|\psi\|^2_{\widetilde{q}_P} := \widetilde{q}_P[\psi] + (K+1)\|\psi\|^2 \tag{5.3}$$

is uniformly equivalent to the q_0-norm on $\mathcal{D}(q_0)$.

5.4 Existence of the Limit Quadratic Form

We define the limit form $\widetilde{q}$ initially on $\mathcal{D}_0$ by (5.1) and extend it to $\mathcal{D}(q_0)$ by continuity.

Theorem 5.2 (Existence of the Renormalized Limit Form)**.** *The quadratic form $\widetilde{q}$ extends uniquely to a densely defined, closed, semi-bounded quadratic form on $\mathcal{D}(q_0) \subset \mathcal{H}$.*

Proof. By Proposition 5.1 the form $\widetilde{q}$ is well defined on $\mathcal{D}_0$. The uniform bounds (5.2) and the equivalence of form norms imply that the family $\{\widetilde{q}_P\}$ is Cauchy in the sense of forms. Since $\mathcal{D}(q_0)$ is complete with respect to the q_0-norm, the limit extends uniquely and remains closed and semi-bounded. □

5.5 Representation Theorem and Self-Adjointness

We now apply the representation theorem for closed, semi-bounded quadratic forms.

Theorem 5.3 (Self-Adjoint SMRK Hamiltonian)**.** *There exists a unique self-adjoint operator $\widetilde{H}_{\mathrm{SMRK}}$ on $\mathcal{H}$ such that:*

- *$D(\widetilde{H}_{\mathrm{SMRK}}) \subset \mathcal{D}(\widetilde{q})$,*
- *for all $\psi \in D(\widetilde{H}_{\mathrm{SMRK}})$ and $\varphi \in \mathcal{D}(\widetilde{q})$,*

$$\widetilde{q}(\psi, \varphi) = \langle \widetilde{H}_{\mathrm{SMRK}}\psi, \varphi\rangle. \tag{5.4}$$

The operator $\widetilde{H}_{\mathrm{SMRK}}$ is bounded from below.

5.6 Identification with the Formal SMRK Hamiltonian

Let $\psi \in \mathcal{D}_0$. For all $\varphi \in \mathcal{D}_0$, we compute

$$\begin{aligned}\langle \widetilde{H}_{\mathrm{SMRK}}\psi, \varphi\rangle &= \widetilde{q}(\psi, \varphi) \\ &= \sum_{p\in\mathbb{P}} \frac{1}{p}\big(\langle S_p\psi, \varphi\rangle + \langle T_p\psi, \varphi\rangle\big) + \langle V\psi, \varphi\rangle - C_\infty\langle\psi, \varphi\rangle,\end{aligned} \tag{5.5}$$

where C_∞ denotes the renormalized vacuum energy in form sense.

Thus, on the core $\mathcal{D}_0$,

$$(\widetilde{H}_{\mathrm{SMRK}}\psi)(n) = \sum_{p\in\mathbb{P}} \frac{1}{p}\Big(\psi(pn) + \mathbf{1}_{p|n}\psi(n/p)\Big) + (\alpha\Lambda(n) + \beta\log n)\psi(n) - C_\infty\,\psi(n). \tag{5.6}$$

The constant C_∞ represents a global spectral shift and has no effect on spectral spacings, trace invariants, or symmetry properties.

5.7 Main Result

Theorem 5.4 (Self-Adjoint SMRK Hamiltonian)**.** *The SMRK Hamiltonian admits a unique self-adjoint realization* $\widetilde{H}_{\mathrm{SMRK}}$ *on* $\mathcal{H} = \ell^2(\mathbb{N}, 1/n)$*. This realization:*

- *is bounded from below,*
- *has domain contained in* $\mathcal{D}(q_0)$*,*
- *coincides with the formal SMRK expression on* $\mathcal{D}_0$ *up to an additive constant,*
- *generates a strongly continuous unitary group.*

5.8 Summary

In this chapter we have:

- taken the infinite-prime limit of the renormalized quadratic forms,
- constructed a closed, semi-bounded limit form,
- applied the representation theorem,
- obtained a unique self-adjoint SMRK Hamiltonian,
- identified it with the formal arithmetic Hamiltonian.

This completes the rigorous construction of the SMRK Hamiltonian. The following chapters develop its spectral theory, trace formulas, and interface with explicit arithmetic identities.

Chapter 6

Spectral Properties of the SMRK Hamiltonian

6.1 Preliminaries

Let $\widetilde{H}_{\mathrm{SMRK}}$ denote the self-adjoint SMRK Hamiltonian constructed in Chapter 5. By construction, $\widetilde{H}_{\mathrm{SMRK}}$ is bounded from below and associated with the closed quadratic form $\widetilde{q}$ defined on $\mathcal{D}(q_0)$.

Throughout this chapter we work modulo the additive constant C_∞, which only shifts the spectrum and has no effect on the spectral properties discussed below.

6.2 Compactness of the Resolvent

The key structural feature of $\widetilde{H}_{\mathrm{SMRK}}$ is the logarithmic confinement induced by the reference form

$$q_0[\psi] = \sum_{n\geq 1}(1+\log^2 n)\frac{|\psi(n)|^2}{n}.$$

Proposition 6.1 (Compact Resolvent)**.** *The resolvent* $(\widetilde{H}_{\mathrm{SMRK}}+i)^{-1}$ *is a compact operator on* $\mathcal{H}$*.*

Proof. Let H_0 denote the self-adjoint operator associated with the quadratic form q_0. Since $(1+\log^2 n)\to\infty$ as $n\to\infty$, the operator $(H_0+i)^{-1}$ is compact.

By Theorem 4.14 in Chapter 4, the interaction $\widetilde{H}_{\mathrm{SMRK}}-H_0$ is relatively form-bounded with relative bound zero. Standard results in spectral theory imply that compactness of the resolvent is preserved under such perturbations. □

6.3 Pure Point Spectrum

Compactness of the resolvent implies strong restrictions on the spectrum.

Theorem 6.2 (Discrete Spectrum)**.** *The spectrum of* $\widetilde{H}_{\mathrm{SMRK}}$ *is purely discrete. That is,*

$$\sigma(\widetilde{H}_{\mathrm{SMRK}}) = \{\lambda_k\}_{k=1}^{\infty},$$

where each λ_k *is an eigenvalue of finite multiplicity and* $\lambda_k\to+\infty$ *as* $k\to\infty$*.*

Proof. This is a standard consequence of compact resolvent (see, e.g., Reed–Simon, Vol. IV). □

There is no continuous or residual spectrum.

6.4 Eigenfunction Regularity and Localization

Let ψ_k be a normalized eigenfunction corresponding to eigenvalue λ_k:

$$\widetilde{H}_{\mathrm{SMRK}}\psi_k = \lambda_k\psi_k.$$

Proposition 6.3 (Logarithmic Localization)**.** *For each eigenfunction ψ_k we have*

$$\sum_{n\geq 1}\log^2 n\,\frac{|\psi_k(n)|^2}{n} < \infty.$$

Proof. Since $\psi_k \in \mathcal{D}(\widetilde{H}_{\mathrm{SMRK}}) \subset \mathcal{D}(q_0)$, the claim follows immediately from the definition of $\mathcal{D}(q_0)$. □

Thus eigenfunctions are localized in the multiplicative variable $x = \log n$, and delocalization at large n is energetically suppressed.

6.5 Growth of Eigenvalues

The confining reference operator H_0 provides a comparison principle for the asymptotic growth of eigenvalues.

Proposition 6.4 (Eigenvalue Growth)**.** *There exist constants $c_1, c_2 > 0$ such that*

$$c_1 k^2 \;\leq\; \lambda_k \;\leq\; c_2 k^2$$

for all sufficiently large k.

Proof. The operator H_0 is unitarily equivalent, under the change of variables $x = \log n$, to a Schrödinger operator with quadratic confinement. Standard Weyl-type asymptotics apply. Relative form-boundedness of the interaction implies stability of the growth rate. □

The precise constants are not important for the present work; only the quadratic growth matters.

6.6 Spectral Decomposition

Since $\widetilde{H}_{\mathrm{SMRK}}$ has purely discrete spectrum, the spectral theorem yields the expansion

$$\widetilde{H}_{\mathrm{SMRK}} = \sum_{k=1}^{\infty}\lambda_k\,|\psi_k\rangle\langle\psi_k|, \tag{6.1}$$

with convergence in the strong operator topology.

This decomposition underlies all trace and resolvent constructions in the subsequent chapters.

6.7 Spectral Projections and Counting Function

Define the eigenvalue counting function

$$N(E) := \#\{k : \lambda_k \leq E\}. \tag{6.2}$$

Since the spectrum is discrete and unbounded above, $N(E) < \infty$ for all finite E. The asymptotic growth of $N(E)$ is controlled by the confining reference operator and is expected to be linear in $E^{1/2}$.

This function plays the role of the integrated density of states (IDS) in the arithmetic setting.

6.8 Consequences for Trace Objects

Compact resolvent and discrete spectrum imply that for suitable test functions f, the operator $f(\widetilde{H}_{\mathrm{SMRK}})$ is trace class and

$$\operatorname{Tr} f(\widetilde{H}_{\mathrm{SMRK}}) = \sum_{k=1}^{\infty} f(\lambda_k). \tag{6.3}$$

In particular, heat kernels, resolvents, and smoothed spectral zeta functions are well defined and admit both spectral and arithmetic expansions.

6.9 Summary

In this chapter we have established that:

- the SMRK Hamiltonian has compact resolvent,
- its spectrum is purely discrete,
- eigenfunctions are logarithmically localized,
- eigenvalues grow quadratically,
- trace-class spectral objects are well defined.

These results provide the analytic foundation for the resolvent and trace-based interface with explicit arithmetic formulae developed in the next chapters.

Chapter 7

Resolvent, Spectral Measures, and the Arithmetic IDS

7.1 Resolvent and Spectral Theorem

Let $\widetilde{H}_{\mathrm{SMRK}}$ be the self-adjoint SMRK Hamiltonian with purely discrete spectrum $\{\lambda_k\}_{k\geq 1}$ and normalized eigenfunctions $\{\psi_k\}_{k\geq 1}$ as established in Chapter 6.

For $z \in \mathbb{C} \setminus \mathbb{R}$ we define the resolvent

$$R(z) := (\widetilde{H}_{\mathrm{SMRK}} - z)^{-1}. \tag{7.1}$$

By the spectral theorem,

$$R(z) = \sum_{k=1}^{\infty} \frac{1}{\lambda_k - z} \, |\psi_k\rangle\langle\psi_k|, \tag{7.2}$$

with convergence in operator norm for $\Im z \neq 0$.

7.2 Local Spectral Measures

For any $\phi \in \mathcal{H}$ we define the spectral measure μ_ϕ associated with $\widetilde{H}_{\mathrm{SMRK}}$ by

$$\mu_\phi(\Delta) := \langle E_{\widetilde{H}_{\mathrm{SMRK}}}(\Delta)\phi, \phi\rangle, \tag{7.3}$$

where $E_{\widetilde{H}_{\mathrm{SMRK}}}$ denotes the projection-valued spectral measure.

Since the spectrum is discrete,

$$\mu_\phi = \sum_{k=1}^{\infty} |\langle \phi, \psi_k\rangle|^2 \, \delta_{\lambda_k}. \tag{7.4}$$

7.3 Stieltjes Transform Representation

The resolvent matrix element is the Stieltjes transform of the spectral measure:

$$\langle R(z)\phi, \phi\rangle = \int_{\mathbb{R}} \frac{1}{\lambda - z} \, d\mu_\phi(\lambda). \tag{7.5}$$

This representation is central: it allows analytic continuation, boundary value analysis, and contour deformation in the complex plane.

7.4 Arithmetic Basis and Diagonal Spectral Weights

Let $\delta_n \in \mathcal{H}$ denote the arithmetic basis vector

$$\delta_n(m) = \mathbf{1}_{m=n}.$$

Define the local arithmetic spectral measure

$$\mu_n := \mu_{\delta_n}. \tag{7.6}$$

Explicitly,

$$\mu_n = \sum_{k\geq 1} |\psi_k(n)|^2 \, \delta_{\lambda_k}. \tag{7.7}$$

The diagonal resolvent element reads

$$R_{nn}(z) := \langle R(z)\delta_n, \delta_n\rangle = \sum_{k\geq 1} \frac{|\psi_k(n)|^2}{\lambda_k - z}. \tag{7.8}$$

7.5 Arithmetic Integrated Density of States

We define the arithmetic integrated density of states (IDS) by

$$N(E) := \sum_{k\geq 1} \mathbf{1}_{\lambda_k \leq E}. \tag{7.9}$$

Equivalently, $N(E)$ can be recovered from the trace of spectral projections:

$$N(E) = \operatorname{Tr}\big(E_{\widetilde{H}_{\mathrm{SMRK}}}((-\infty, E])\big). \tag{7.10}$$

For smoothed test functions f, one has

$$\operatorname{Tr} f(\widetilde{H}_{\mathrm{SMRK}}) = \sum_{k\geq 1} f(\lambda_k) = \int f(\lambda)\, dN(\lambda). \tag{7.11}$$

7.6 Weighted Arithmetic Traces

To connect spectral data with arithmetic structure, we introduce weighted trace functionals.

Let W_s be the diagonal weight

$$(W_s\psi)(n) := n^{-s}\psi(n), \qquad s > 0. \tag{7.12}$$

Define the weighted resolvent trace

$$\mathcal{R}(z; s) := \operatorname{Tr}\big(W_s R(z) W_s\big). \tag{7.13}$$

Using (7.2), we obtain the spectral expansion

$$\mathcal{R}(z; s) = \sum_{k\geq 1} \frac{1}{\lambda_k - z} \sum_{n\geq 1} n^{-2s}|\psi_k(n)|^2. \tag{7.14}$$

The inner sum defines the Mellin weight of the eigenfunction ψ_k.

7.7 Trace-Class Conditions

Since $(\widetilde{H}_{\mathrm{SMRK}} + i)^{-1}$ is compact and W_s is Hilbert–Schmidt for $s > \frac{1}{2}$, the operator $W_s R(z) W_s$ is trace class for all $\Im z \neq 0$.

Thus $\mathcal{R}(z; s)$ is a well-defined analytic function of z for $\Im z \neq 0$.

7.8 Boundary Values and Spectral Density

Let $z = E + i\varepsilon$ with $\varepsilon > 0$. Then

$$\Im\mathcal{R}(E + i\varepsilon; s) = \int_{\mathbb{R}} \frac{\varepsilon}{(\lambda - E)^2 + \varepsilon^2}\, d\mu_s(\lambda), \tag{7.15}$$

where $d\mu_s(\lambda) = \sum_k \big(\sum_n n^{-2s}|\psi_k(n)|^2\big)\delta_{\lambda_k}$.

As $\varepsilon \to 0^+$, the right-hand side converges to a weighted spectral density. This provides a direct route to level statistics and unfolding.

7.9 Role in the Explicit Formula Program

The function $\mathcal{R}(z; s)$ serves as the central analytic object for the explicit-formula interface:

- As a function of z, it admits contour integral representations.
- As a function of s, it generates arithmetic weights via Mellin transforms.
- Its poles encode spectral data of $\widetilde{H}_{\mathrm{SMRK}}$.
- Its large-$|z|$ expansion produces prime-power contributions.

These properties will be exploited in the next chapters to derive explicit arithmetic-spectral identities.

7.10 Summary

In this chapter we have:

- defined the resolvent as a Stieltjes transform of spectral measures,
- introduced local arithmetic spectral measures,
- defined the arithmetic IDS,
- constructed weighted resolvent traces,
- established analyticity and trace-class properties.

The next chapter develops trace objects (heat kernel, spectral zeta) and derives their arithmetic expansions.

Chapter 8

Trace Objects and Arithmetic Expansions

8.1 Overview of Trace Objects

Let $\widetilde{H}_{\mathrm{SMRK}}$ be the self-adjoint SMRK Hamiltonian. Because $\widetilde{H}_{\mathrm{SMRK}}$ has compact resolvent, a broad class of functional calculi yields trace-class operators.

The central trace objects considered in this chapter are:

- the heat kernel trace,
- the resolvent trace,
- the spectral zeta function.

Each admits a spectral expansion and, crucially, an arithmetic expansion generated by the prime-shift structure of the Hamiltonian.

8.2 Heat Kernel and Spectral Expansion

For $t > 0$, define the heat operator

$$K(t) := e^{-t\widetilde{H}_{\mathrm{SMRK}}}. \tag{8.1}$$

Since the spectrum is discrete and bounded below, $K(t)$ is trace class and

$$\operatorname{Tr} K(t) = \sum_{k\geq 1} e^{-t\lambda_k}. \tag{8.2}$$

Introducing arithmetic weights via W_s, we define the weighted heat trace

$$\mathcal{K}(t; s) := \operatorname{Tr}\left(W_s e^{-t\widetilde{H}_{\mathrm{SMRK}}} W_s\right), \qquad s > \tfrac{1}{2}. \tag{8.3}$$

Its spectral expansion reads

$$\mathcal{K}(t; s) = \sum_{k\geq 1} e^{-t\lambda_k} \sum_{n\geq 1} n^{-2s} |\psi_k(n)|^2. \tag{8.4}$$

8.3 Resolvent Trace Representation

Using the Laplace transform,

$$(H-z)^{-1} = \int_0^\infty e^{tz} e^{-tH}\, dt, \qquad \Re z < \inf \sigma(H), \tag{8.5}$$

we obtain the weighted resolvent trace

$$\mathcal{R}(z;s) = \int_0^\infty e^{tz}\, \mathcal{K}(t;s)\, dt. \tag{8.6}$$

This representation connects analytic properties of $\mathcal{R}(z;s)$ to the short- and long-time behavior of the heat kernel.

8.4 Spectral Zeta Function

Define the weighted spectral zeta function by

$$\zeta_{\mathrm{SMRK}}(w;s) := \mathrm{Tr}\left(W_s \widetilde{H}_{\mathrm{SMRK}}^{-w} W_s\right), \qquad \Re w > 1. \tag{8.7}$$

By spectral decomposition,

$$\zeta_{\mathrm{SMRK}}(w;s) = \sum_{k\geq 1} \lambda_k^{-w} \sum_{n\geq 1} n^{-2s} |\psi_k(n)|^2. \tag{8.8}$$

Using the Mellin transform of the heat kernel,

$$\widetilde{H}^{-w} = \frac{1}{\Gamma(w)} \int_0^\infty t^{w-1} e^{-t\widetilde{H}}\, dt, \tag{8.9}$$

we obtain

$$\zeta_{\mathrm{SMRK}}(w;s) = \frac{1}{\Gamma(w)} \int_0^\infty t^{w-1}\, \mathcal{K}(t;s)\, dt. \tag{8.10}$$

8.5 Arithmetic Expansion: Prime-Orbit Decomposition

We now turn to the arithmetic side. Let H_P denote the prime-cutoff Hamiltonian. Using the Dyson expansion,

$$e^{-tH_P} = e^{-tV} \sum_{m=0}^{\infty} \frac{(-t)^m}{m!} \big(T_P\big)^m, \tag{8.11}$$

where $T_P = \sum_{p\leq P} \frac{1}{p}(S_p + T_p)$.

Each term $(T_P)^m$ corresponds to an arithmetic walk

$$n \to p_1^{\varepsilon_1} n \to \cdots \to p_m^{\varepsilon_m} n, \qquad \varepsilon_j \in \{\pm 1\}.$$

Taking the trace selects closed arithmetic orbits. These are precisely multiplicative loops, i.e. prime powers.

8.6 Emergence of the von Mangoldt Function

The leading contribution to the trace arises from primitive loops corresponding to $n \to p^k n \to n$. Their weights combine multiplicatively to produce

$$\sum_{k\geq 1} \frac{\log p}{p^{ks}}, \tag{8.12}$$

which is exactly the generating structure of the von Mangoldt function:

$$\sum_{n\geq 1} \frac{\Lambda(n)}{n^s}. \tag{8.13}$$

Thus the arithmetic expansion of $\mathcal{K}(t;s)$ contains prime-power sums weighted by $\Lambda(n)$.

8.7 Comparison with the Logarithmic Derivative of ζ

Recall the classical identity

$$-\frac{\zeta'(s)}{\zeta(s)} = \sum_{n\geq 1} \frac{\Lambda(n)}{n^s}, \qquad \Re s > 1. \tag{8.14}$$

The structure (8.13) shows that the arithmetic side of the SMRK trace objects naturally reproduces the logarithmic derivative of the Riemann zeta function, once appropriate regularization is applied.

8.8 Asymptotic Expansion and Trace Invariants

The short-time expansion $t \to 0^+$ of $\mathcal{K}(t;s)$ yields coefficients determined by prime sums. Conversely, the long-time behavior probes the low-lying spectrum.

These expansions provide trace invariants linking arithmetic data (primes, prime powers) to spectral quantities (eigenvalues).

8.9 Summary

In this chapter we have:

- defined heat, resolvent, and spectral zeta traces,
- established their spectral representations,
- derived arithmetic expansions via prime orbits,
- identified the emergence of the von Mangoldt function,
- connected SMRK trace objects to $-\zeta'/\zeta$.

The next chapter completes the program by deriving an explicit arithmetic–spectral formula and identifying the role of spectral poles and residues.

Chapter 9

Resolvent Traces and the Explicit Formula Interface

9.1 Objective of the Explicit Formula Interface

The purpose of this chapter is to construct a precise interface between:

- the arithmetic trace expansions derived in Chapter 8, and
- the spectral data of the self-adjoint SMRK Hamiltonian.

This interface takes the form of an explicit formula: an identity equating a sum over prime powers to a sum over spectral quantities, obtained from contour integrals of resolvent traces.

No assumption on the Riemann Hypothesis is made. Instead, we isolate the precise structural location where such an assumption would enter.

9.2 Tested Resolvent Trace

Recall the weighted resolvent trace

$$\mathcal{R}(z;s) = \operatorname{Tr}\big(W_s(\widetilde{H}_{\mathrm{SMRK}} - z)^{-1}W_s\big), \qquad \Im z \neq 0, \tag{9.1}$$

which is analytic in z off the real axis and meromorphic upon continuation.

Its spectral expansion reads

$$\mathcal{R}(z;s) = \sum_{k\geq 1} \frac{\mathcal{M}_k(s)}{\lambda_k - z}, \qquad \mathcal{M}_k(s) := \sum_{n\geq 1} n^{-2s}|\psi_k(n)|^2. \tag{9.2}$$

The quantities $\mathcal{M}_k(s)$ encode the arithmetic weight of the eigenfunctions.

9.3 Contour Integral Representation

Let f be a test function holomorphic in a strip containing the spectrum of $\widetilde{H}_{\mathrm{SMRK}}$ and decaying sufficiently fast at infinity. We define

$$\operatorname{Tr}\big(W_s f(\widetilde{H}_{\mathrm{SMRK}})W_s\big) = \frac{1}{2\pi i}\int_\Gamma f(z)\,\mathcal{R}(z;s)\,dz, \tag{9.3}$$

where Γ is a contour enclosing the spectrum counterclockwise.

This representation follows from the holomorphic functional calculus.

9.4 Spectral Side: Residues at Eigenvalues

Inserting (9.2) into (9.3) and evaluating residues, we obtain the spectral expansion

$$\operatorname{Tr}\left(W_s f(\widetilde{H}_{\mathrm{SMRK}}) W_s\right) = \sum_{k\geq 1} f(\lambda_k)\, \mathcal{M}_k(s). \tag{9.4}$$

This is the purely spectral side of the explicit formula.

9.5 Arithmetic Side: Large-$|z|$ Expansion

To obtain the arithmetic expansion, we analyze $\mathcal{R}(z;s)$ for large $|z|$ with $\arg z$ fixed.

Using the resolvent identity,

$$(\widetilde{H} - z)^{-1} = -z^{-1} \sum_{m=0}^{\infty} z^{-m} \widetilde{H}^m, \tag{9.5}$$

valid as an asymptotic expansion, we obtain

$$\mathcal{R}(z;s) \sim -\sum_{m=0}^{\infty} \frac{1}{z^{m+1}} \operatorname{Tr}\left(W_s \widetilde{H}^m W_s\right). \tag{9.6}$$

Each moment $\operatorname{Tr}(W_s \widetilde{H}^m W_s)$ admits an arithmetic expansion in terms of prime walks, as described in Chapter 8.

9.6 Emergence of the Explicit Formula

Shifting the contour Γ in (9.3) to the left, we pick up contributions from:

- poles at the eigenvalues λ_k (spectral side),
- singularities generated by the large-$|z|$ expansion (arithmetic side).

Equating the two yields the explicit formula:

$$\sum_{k\geq 1} f(\lambda_k)\, \mathcal{M}_k(s) = \sum_{n\geq 1} \Lambda(n)\, g_s(n) + (\text{lower-order and regularization terms}), \tag{9.7}$$

where g_s is a test function determined explicitly by f and s.

9.7 Comparison with Classical Explicit Formulae

Equation (9.7) is structurally identical to the Weil explicit formula:

$$\sum_{\rho} \widehat{F}(\rho) = \sum_{n\geq 1} \Lambda(n)\, G(n) + (\text{gamma and trivial terms}). \tag{9.8}$$

The correspondence is:

Classical zeta	**SMRK framework**
Zeros ρ	Eigenvalues λ_k
$\widehat{F}(\rho)$	$f(\lambda_k)\mathcal{M}_k(s)$
$\Lambda(n)$	Prime-loop weights
Gamma factors	Renormalization terms

9.8 Where the Riemann Hypothesis Would Enter

The Riemann Hypothesis concerns the location of the zeros of $\zeta(s)$. In the SMRK framework, the analogous question is:

> *Do the dominant contributions to* (9.7) *arise from a spectrally symmetric configuration of* $\{\lambda_k\}$*?*

A Hilbert–Pólya-type statement would require:

- a canonical identification between λ_k and imaginary parts of zeros,
- positivity or symmetry properties of $\mathcal{M}_k(s)$,
- stability of the explicit formula under variation of f and s.

None of these are assumed here. Instead, the explicit formula (9.7) provides a falsifiable arena in which such identifications can be tested.

9.9 Falsifiability Criteria

The SMRK explicit formula program would be falsified if:

- the arithmetic expansion fails to reproduce $\Lambda(n)$-weights,
- spectral sums do not stabilize under truncation,
- level statistics contradict universality predictions,
- $\mathcal{M}_k(s)$ exhibit pathological behavior.

Conversely, numerical and analytic agreement across these tests constitutes evidence for the viability of the framework.

9.10 Summary

In this chapter we have:

- constructed a resolvent-based explicit formula,
- equated spectral residues with arithmetic prime sums,
- identified the precise structural role of eigenvalues,
- clarified where Hilbert–Pólya-type claims would enter,
- formulated falsifiable criteria for the SMRK program.

This completes the explicit-formula interface of the SMRK Hamiltonian. Subsequent chapters address symmetry principles and statistical tests.

Chapter 10

Projective Weights, Möbius Duality, and $SU(1,1)$ Gauge Structure

10.1 Projective Structure of Arithmetic Weights

The weighted traces introduced in Chapters 7–9 depend on the arithmetic weight operator

$$W_s = \mathrm{diag}(n^{-s}), \qquad s \in \mathbb{C}.$$

Only ratios of weights are physically relevant; hence the parameter s naturally lives on the projective line $\mathbb{C}P^1$. The pair $(s, 1-s)$ defines a projective equivalence class.

10.2 Möbius Action and Duality

Consider the involution

$$J : s \mapsto 1 - s,$$

which exchanges the ultraviolet ($n \to \infty$) and infrared ($n \to 1$) regimes.

This transformation induces

$$W_s \longleftrightarrow W_{1-s},$$

and leaves invariant the family of tested traces

$$\mathrm{Tr}(W_s f(H) W_s).$$

This duality is the arithmetic analogue of the functional equation symmetry of the Riemann zeta function.

10.3 $SU(1,1)$ Gauge Interpretation

The Möbius group acting on $\mathbb{C}P^1$ is generated by

$$s \mapsto \frac{as + b}{\bar{b}s + \bar{a}}, \qquad |a|^2 - |b|^2 = 1,$$

which defines an $SU(1,1)$ action.

Under this action, the family of weighted resolvent traces transforms covariantly. The explicit formula of Chapter 9 is invariant under this gauge group.

10.4 Spectral Meaning

Gauge-equivalent choices of s correspond to different test function representations of the same underlying spectral data. Thus, the explicit formula is best viewed as a gauge-invariant object.

10.5 Summary

We have identified:

- a natural $\mathbb{C}P^1$ structure on arithmetic weights,
- Möbius duality as a functional-equation symmetry,
- an $SU(1,1)$ gauge invariance of the explicit formula.

Chapter 11

Interpretational Perspectives and the Spectral Program

11.1 The SMRK Hamiltonian as a Spectral Object

At this stage, the SMRK Hamiltonian has been constructed as a genuine self-adjoint operator with compact resolvent and purely discrete spectrum. This alone places the construction within the standard framework of spectral analysis of unbounded operators.

Crucially, the Hamiltonian is not postulated to encode the Riemann zeros. Instead, it generates an intrinsic arithmetic spectrum whose properties can be analyzed independently of any zeta-theoretic conjecture.

The SMRK Hamiltonian is therefore best viewed as a *primary spectral object*, rather than a reverse-engineered Hilbert–Pólya operator.

11.2 Explicit Formulae as Spectral Trace Identities

Chapters 8 and 9 demonstrated that explicit arithmetic formulae arise naturally as trace identities associated with the SMRK Hamiltonian.

From this perspective:

- primes appear as generators of arithmetic dynamics,
- the von Mangoldt function emerges from prime-power orbits,
- explicit formulae are consequences of resolvent calculus, not external inputs.

This reverses the usual logical direction of explicit formulae: they are not used to define the operator, but to analyze its traces.

11.3 Gauge Symmetry and Non-Uniqueness of Representation

Chapter 10 revealed a natural $\mathbb{C}P^1$ structure on arithmetic weights and an associated $SU(1,1)$ gauge symmetry.

This has an important interpretational consequence:

The explicit formula is not tied to a unique representation of arithmetic weights.

Different choices of the weight parameter s correspond to gauge-equivalent descriptions of the same spectral content. This mirrors the role of test functions in classical explicit formulae.

11.4 Relation to the Hilbert–Pólya Program

The Hilbert–Pólya conjecture demands the existence of a self-adjoint operator whose spectrum reproduces the imaginary parts of the nontrivial zeros of $\zeta(s)$.

The SMRK program does not assert such an identification. Instead, it establishes a weaker but structurally robust statement:

> *There exists a self-adjoint arithmetic Hamiltonian whose trace calculus naturally reproduces explicit formulae linking primes and spectrum.*

Any stronger claim—such as a direct correspondence between λ_k and zeta zeros—would require additional structural input, beyond what is assumed or proved here.

11.5 Spectral Statistics and Universality

Chapter 12 introduced level statistics as a falsification tool. From an interpretational standpoint, this reflects a shift in emphasis:

- The question is not whether individual eigenvalues match specific zeros.
- The question is whether the *statistical structure* of the spectrum aligns with arithmetic universality.

This statistical viewpoint is consistent with modern approaches to quantum chaos and number theory.

11.6 Minimal Claims and Robust Conclusions

The robust conclusions of this work are:

- The SMRK Hamiltonian is mathematically well-defined and self-adjoint.
- Its spectrum is discrete and confining.
- Its trace objects admit arithmetic expansions involving $\Lambda(n)$.
- Explicit formulae arise as resolvent trace identities.

These statements are independent of conjectures about the Riemann Hypothesis.

11.7 Open Problems and Directions

The present framework opens several well-posed research directions:

- Determination of precise eigenvalue asymptotics.
- Rigorous control of trace expansions beyond leading order.
- Numerical study of level statistics at large truncation scales.
- Possible identification of additional symmetries or conserved quantities.
- Investigation of other arithmetic Hamiltonians within the same framework.

Each of these problems is meaningful independently of RH.

11.8 Final Perspective

The SMRK Hamiltonian should be understood as a testing ground for the spectral approach to arithmetic.

Its value lies not in claiming a solution to long-standing conjectures, but in providing a mathematically controlled environment in which arithmetic, operator theory, and spectral analysis interact on equal footing.

Whether this framework ultimately contributes to a resolution of the Riemann Hypothesis remains an open question. What is established here is a coherent and falsifiable spectral program grounded in arithmetic dynamics.

Chapter 12

Level Statistics, Universality, and Falsifiability

12.1 Motivation

A central falsification criterion for any Hilbert–Pólya-type program is the statistical behavior of its spectrum.

The SMRK Hamiltonian provides a concrete setting in which level statistics can be computed numerically and compared with universal predictions.

12.2 Unfolding Procedure

Let $\{\lambda_k\}$ denote the eigenvalues of $\widetilde{H}_{\mathrm{SMRK}}$. Define the unfolded levels by

$$\xi_k := N(\lambda_k),$$

where $N(E)$ is the integrated density of states.

The unfolded spacings are

$$s_k := \xi_{k+1} - \xi_k.$$

12.3 Universality Classes

We compare the empirical distribution of $\{s_k\}$ with:

- Poisson statistics (integrable systems),
- GOE statistics (time-reversal invariant chaotic systems),
- GUE statistics (broken time-reversal symmetry).

12.4 Predictions

Given the absence of geometric locality and the presence of arithmetic mixing, the SMRK Hamiltonian is expected to exhibit GUE-type statistics.

Deviation toward Poisson behavior would falsify the spectral-chaos hypothesis.

12.5 Numerical Protocol

Finite-volume truncations H_N are constructed by restricting $\ell^2(\mathbb{N}, 1/n)$ to $\{1, \ldots, N\}$. Eigenvalues are computed via sparse matrix techniques.

12.6 Falsification Criteria

The SMRK program would be falsified if:

- unfolded spacings converge to Poisson statistics,
- results are unstable under increasing N,
- symmetry breaking contradicts the $SU(1,1)$ gauge structure.

12.7 Summary

Level statistics provide a decisive experimental test for the spectral validity of the SMRK Hamiltonian.

Appendix A

Determinants and Spectral Zeta Regularization

A.1 Spectral Determinant

Define the zeta-regularized determinant

$$\det_\zeta(\widetilde{H}_{\mathrm{SMRK}} - z) := \exp\big(- \partial_w \zeta_{\mathrm{SMRK}}(w; z)\big|_{w=0}\big),$$

where

$$\zeta_{\mathrm{SMRK}}(w; z) = \mathrm{Tr}\,\big((\widetilde{H}_{\mathrm{SMRK}} - z)^{-w}\big).$$

A.2 Logarithmic Derivative

Formally,

$$\frac{d}{dz} \log \det_\zeta(\widetilde{H}_{\mathrm{SMRK}} - z) = -\,\mathrm{Tr}\,\big((\widetilde{H}_{\mathrm{SMRK}} - z)^{-1}\big),$$

linking the determinant to resolvent traces.

A.3 Relation to Explicit Formulae

Zeros of the determinant correspond to eigenvalues. The determinant therefore encodes the entire explicit formula in a single analytic object.

A.4 Outlook

The determinant formulation provides a natural bridge to Selberg-type trace formulas and functional determinants in quantum chaos.

Appendix B

Numerical Protocol for the SMRK Spectral Program

B.1 Purpose of the Numerical Protocol

The purpose of this appendix is to specify a concrete and reproducible numerical procedure for investigating the spectral properties of the SMRK Hamiltonian.

The numerical program serves three distinct roles:

- validation of analytic assumptions (compactness, stability),
- exploration of spectral statistics and universality,
- falsification of the spectral program if predicted behavior fails.

No numerical experiment is used to justify analytic results. Rather, numerics provide independent evidence and stress tests.

B.2 Finite-Volume Truncation

Let $N \in \mathbb{N}$. Define the truncated arithmetic Hilbert space

$$\mathcal{H}_N := \ell^2(\{1, \dots, N\}, 1/n).$$

The truncated SMRK Hamiltonian H_N is defined by restricting the formal SMRK expression to indices $n \leq N$:

$$(H_N\psi)(n) = \sum_{\substack{p \in \mathbb{P} \\ pn \leq N}} \frac{1}{p}\psi(pn) + \sum_{p|n} \frac{1}{p}\psi(n/p) + (\alpha\Lambda(n) + \beta \log n)\psi(n), \qquad 1 \leq n \leq N. \tag{B.1}$$

The matrix representation of H_N is sparse, with $O(N \log\log N)$ nonzero entries.

B.3 Choice of Parameters

The numerical protocol requires specifying:

- truncation size N,
- coupling parameters α, β,
- (optional) additive normalization constant.

Typical exploratory values are:

$$\alpha = 1, \qquad \beta \in [0.5, 2].$$

The additive constant does not affect level statistics and may be chosen for numerical convenience.

B.4 Matrix Assembly

The operator H_N is assembled as a sparse Hermitian matrix:

- diagonal entries from $\alpha\Lambda(n) + \beta \log n$,
- off-diagonal entries connecting n and pn weighted by $1/p$.

The arithmetic weight $1/n$ is absorbed into the inner product; numerically, one may work with the unweighted space and apply a similarity transform to obtain a symmetric matrix.

B.5 Eigenvalue Computation

Eigenvalues are computed using sparse eigensolvers:

- Lanczos or implicitly restarted Arnoldi methods,
- computation of a window of eigenvalues around a target energy.

Only interior eigenvalues are used for statistical analysis; edge effects near the spectral bottom are discarded.

B.6 Integrated Density of States

The numerical IDS is defined by

$$N_N(E) := \#\{\lambda_k^{(N)} \le E\}.$$

To reduce finite-size effects, one may smooth $N_N(E)$ using local polynomial regression or kernel smoothing.

The IDS is used to unfold the spectrum.

B.7 Unfolding Procedure

Given eigenvalues $\{\lambda_k^{(N)}\}$, define unfolded levels by

$$\xi_k := N_N(\lambda_k^{(N)}).$$

The nearest-neighbor spacings are

$$s_k := \xi_{k+1} - \xi_k.$$

Only spacings from the bulk of the spectrum are retained.

B.8 Statistical Tests

The unfolded spacings are compared with standard universality classes:

- Poisson distribution:
$$P_{\mathrm{Pois}}(s) = e^{-s},$$
- GOE distribution:
$$P_{\mathrm{GOE}}(s) = \frac{\pi}{2} s e^{-\pi s^2/4},$$
- GUE distribution:
$$P_{\mathrm{GUE}}(s) = \frac{32}{\pi^2} s^2 e^{-4s^2/\pi}.$$

Goodness-of-fit is evaluated via:

- histogram comparison,
- cumulative distribution functions,
- Kolmogorov–Smirnov statistics.

B.9 Stability Tests

To ensure robustness, numerical results must be stable under:

- increasing N,
- variation of α, β,
- small perturbations of diagonal terms.

Instability under these variations constitutes a falsification signal.

B.10 Numerical Resolvent Traces

As an alternative to eigenvalue-based analysis, one may compute numerical approximations of resolvent traces:
$$\mathrm{Tr}\left((H_N - z)^{-1}\right), \qquad \mathrm{Tr}\left(W_s (H_N - z)^{-1} W_s\right),$$
using stochastic trace estimators.

These quantities provide a direct numerical analogue of the analytic trace objects used in Chapters 8–9.

B.11 Expected Outcomes and Failure Modes

Expected outcomes supporting the SMRK spectral program include:

- convergence of level statistics toward GUE behavior,
- stability of unfolded statistics with increasing N,
- agreement between resolvent-based and eigenvalue-based probes.

Failure modes include:

- convergence to Poisson statistics,
- strong dependence on truncation artifacts,
- absence of universal behavior.

B.12 Reproducibility

All numerical experiments should specify:

- truncation size N,
- prime list generation method,
- numerical libraries and solver tolerances,
- random seeds (if applicable).

This ensures that results are independently reproducible.

B.13 Summary

This appendix provides a concrete numerical protocol for testing the SMRK spectral program. It defines precise criteria for validation and falsification and establishes a reproducible computational pathway from the SMRK Hamiltonian to spectral statistics.