The SMRK Hamiltonian for General L-Functions

Spectral Arithmetic Dynamics Beyond the Riemann Zeta Function

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Contents

Chapter 1

From the Riemann Zeta Function to General L-Functions

1.1 Context and Motivation

The spectral construction developed in the first SMRK whitepaper establishes a self-adjoint arithmetic Hamiltonian whose trace calculus naturally reproduces explicit formulae associated with the Riemann zeta function. While the zeta function provides the most elementary and universal example, it represents only the first member of a much larger family of arithmetic objects: general L-functions.

The purpose of the present work is to extend the SMRK spectral program from the Riemann zeta function to general L-functions in a systematic and structurally stable manner. The emphasis is not on reproducing individual L-functions by ad hoc modifications, but on identifying a universal operator-theoretic framework capable of encoding their common features.

1.2 Why General L-Functions

General L-functions arise throughout modern number theory, including Dirichlet L-functions, automorphic L-functions, and Artin L-functions. Despite their diversity, these objects share a small set of structural properties:

- an Euler product encoding local prime data,
- analytic continuation and functional equation,
- explicit formulae linking prime powers to spectral data.

Any spectral approach that aims to be more than a curiosity must be compatible with this broader landscape. Restricting attention to $\zeta(s)$ alone risks obscuring which aspects of the construction are universal and which are accidental.

1.3 Structural Lessons from the Zeta Case

The zeta-function case already reveals several principles that guide the generalization:

- Arithmetic dynamics is generated locally at primes.
- Explicit formulae arise from trace identities, not from point spectra.
- Renormalization is universal and independent of fine arithmetic data.

- Functional equations correspond to symmetry, not coincidence.

Crucially, none of these principles depend on special properties of $\zeta(s)$. They depend only on the existence of Euler factors and logarithmic prime weights.

1.4 Euler Factors as Local Dynamical Data

A general L-function admits an Euler product of the form

$$L(s,\pi) = \prod_{p\in\mathbb{P}} \prod_{j=1}^{d} \left(1 - \alpha_{p,j}(\pi)\, p^{-s}\right)^{-1}, \tag{1.1}$$

where π denotes arithmetic or automorphic data, and $\alpha_{p,j}(\pi)$ are local parameters.

From the SMRK perspective, this suggests the following interpretation:

Each prime p carries internal dynamical data encoded in the collection $\{\alpha_{p,j}\}$.

The role of the Hamiltonian is to promote this local data to global spectral dynamics.

1.5 Universality of Renormalization

One of the key outcomes of the zeta-based construction was the identification of a universal ultraviolet divergence arising from large primes. This divergence was shown to be scalar and removable by a cutoff-dependent constant.

In the present work, we show that:

- the same divergence appears for all L-functions,
- its structure is independent of π,
- renormalization does not depend on representation-theoretic details.

This universality is essential: it ensures that the SMRK framework is robust under extension to families of L-functions.

1.6 What This Work Aims to Establish

The goals of this whitepaper are the following:

- to define a generalized SMRK Hamiltonian incorporating Euler factors,
- to prove self-adjointness and spectral confinement,
- to construct trace objects admitting generalized von Mangoldt weights,
- to derive explicit formulae for general L-functions,
- to clarify the role of functional equations as gauge symmetries.

As in the zeta case, no claim is made regarding the Generalized Riemann Hypothesis. The focus is on structural compatibility and falsifiability.

1.7 Relation to the Langlands Program

Although this work is not formulated in the language of the Langlands program, its perspective is compatible with Langlands philosophy: local data at primes assemble into global spectral objects.

The SMRK Hamiltonian provides an operator-theoretic realization of this assembly process. Whether deeper connections exist is left as an open question, but the framework developed here provides a concrete setting in which such connections can be explored.

1.8 Organization of the Whitepaper

Chapters 2–4 introduce local Euler data, internal degrees of freedom, and generalized prime-shift operators.

Chapters 5–6 define the generalized SMRK Hamiltonian and establish its self-adjointness via renormalized quadratic forms.

Chapters 7–9 develop the spectral and trace calculus and derive explicit formulae for general L-functions.

Chapters 10–12 analyze functional equations, gauge symmetries, and concrete examples, including Dirichlet and automorphic L-functions.

Several appendices collect technical material and numerical protocols.

1.9 Perspective

The extension from $\zeta(s)$ to general L-functions marks a transition from a single spectral construction to a genuine arithmetic spectral program.

The central question is no longer whether one operator can encode one function, but whether a unified operator framework can accommodate entire families of arithmetic objects.

This work is intended as a step in that direction.

Chapter 2

Euler Products and Local Data

2.1 General Form of Euler Products

Let $L(s, \pi)$ be an L-function associated with arithmetic or automorphic data π. We assume that $L(s, \pi)$ admits an Euler product of the form

$$L(s, \pi) = \prod_{p \in \mathbb{P}} L_p(s, \pi), \qquad L_p(s, \pi) = \prod_{j=1}^{d} \left(1 - \alpha_{p,j}(\pi)\, p^{-s}\right)^{-1}, \tag{2.1}$$

where d is the degree of the L-function and $\alpha_{p,j}(\pi) \in \mathbb{C}$ are the local Euler parameters.

Throughout this work, π is treated as a label for local data; no representation-theoretic assumptions beyond (2.1) are required.

2.2 Unramified and Ramified Primes

For all but finitely many primes p, the local factor $L_p(s, \pi)$ is unramified. At unramified primes, the parameters $\{\alpha_{p,j}(\pi)\}_{j=1}^{d}$ are well-defined and satisfy bounds of the form

$$|\alpha_{p,j}(\pi)| \leq p^{\theta}, \tag{2.2}$$

for some $\theta \geq 0$ (e.g. $\theta = 0$ in the Ramanujan–Petersson conjecture).

Ramified primes are finite in number and contribute local factors that may differ from (2.1). Since the SMRK Hamiltonian involves infinite prime sums, ramified primes play no role in ultraviolet behavior and may be treated as finite-rank perturbations.

2.3 Local Logarithmic Derivatives

The logarithmic derivative of $L(s, \pi)$ admits the expansion

$$-\frac{L'}{L}(s, \pi) = \sum_{p \in \mathbb{P}} \sum_{k=1}^{\infty} \left(\sum_{j=1}^{d} \alpha_{p,j}(\pi)^k \right) \frac{\log p}{p^{ks}}, \qquad \Re s > 1. \tag{2.3}$$

This motivates the definition of a generalized von Mangoldt function.

2.4 Generalized von Mangoldt Function

Define $\Lambda_\pi : \mathbb{N} \to \mathbb{C}$ by

Chapter 3

Arithmetic Hilbert Spaces with Internal Degrees of Freedom

3.1 Motivation

In Chapter 2 we identified the local Euler parameters $\{\alpha_{p,j}(\pi)\}_{j=1}^{d}$ as finite-dimensional internal data associated with each prime. To incorporate this structure into the spectral framework, we introduce arithmetic Hilbert spaces with internal degrees of freedom.

The guiding principle is that arithmetic dynamics acts on integers, while local representation-theoretic data acts internally and linearly.

3.2 Underlying Arithmetic Hilbert Space

Let

$$\mathcal{H}_{\mathrm{arith}} := \ell^2(\mathbb{N}, 1/n) = \left\{ \psi : \mathbb{N} \to \mathbb{C} \;\middle|\; \sum_{n \geq 1} \frac{|\psi(n)|^2}{n} < \infty \right\}.$$

This space was used in the zeta-based SMRK construction and remains the natural arithmetic configuration space. The weight $1/n$ ensures logarithmic confinement and compatibility with multiplicative shifts.

3.3 Internal Hilbert Space

Let

$$\mathcal{H}_{\mathrm{int}} := \mathbb{C}^d$$

equipped with its standard Hermitian inner product. The dimension d equals the degree of the L-function.

Elements of $\mathcal{H}_{\mathrm{int}}$ encode the local Euler data associated with π.

3.4 Total Arithmetic Hilbert Space

We define the full arithmetic Hilbert space by the Hilbert tensor product

$$\mathcal{H}_\pi := \mathcal{H}_{\mathrm{arith}} \otimes \mathcal{H}_{\mathrm{int}} \simeq \ell^2(\mathbb{N}, 1/n; \mathbb{C}^d). \tag{3.1}$$

An element $\Psi \in \mathcal{H}_\pi$ is a vector-valued arithmetic function

$$\Psi(n) \in \mathbb{C}^d, \qquad \sum_{n \geq 1} \frac{\|\Psi(n)\|^2}{n} < \infty.$$

The inner product on $\mathcal{H}_\pi$ is given by

$$\langle \Psi, \Phi \rangle = \sum_{n \geq 1} \frac{1}{n} \langle \Psi(n), \Phi(n) \rangle_{\mathbb{C}^d}. \tag{3.2}$$

3.5 Canonical Basis and Finite Support Core

Let $\{e_j\}_{j=1}^d$ be the standard basis of $\mathbb{C}^d$. For each $n \in \mathbb{N}$ and $1 \leq j \leq d$, define the basis vectors

$$\delta_{n,j}(m) = \begin{cases} e_j, & m = n, \\ 0, & m \neq n. \end{cases}$$

The linear span of $\{\delta_{n,j}\}$ with finite n-support defines a dense subspace

$$\mathcal{D}_0 \subset \mathcal{H}_\pi,$$

which will serve as the common core domain for all operators introduced later.

3.6 Diagonal Multiplication Operators

Any arithmetic weight $V : \mathbb{N} \to \mathbb{C}$ acts diagonally on $\mathcal{H}_\pi$ by

$$(V\Psi)(n) := V(n)\Psi(n). \tag{3.3}$$

In particular, the generalized von Mangoldt potential $\Lambda_\pi(n)$ introduced in Chapter 2 defines a diagonal operator on $\mathcal{H}_\pi$.

3.7 Internal Matrix Actions

For any fixed matrix $M \in \mathrm{Mat}_{d\times d}(\mathbb{C})$, define the internal action

$$(M\Psi)(n) := M\,\Psi(n). \tag{3.4}$$

Such operators are bounded on $\mathcal{H}_\pi$, with operator norm equal to the matrix norm of M.

This allows local Euler matrices $A_p(\pi)$ to act internally without affecting arithmetic summability.

3.8 Adjointness and Compatibility

The adjoint of an operator of the form $V \otimes M$ is given by

$$(V \otimes M)^* = \overline{V} \otimes M^*.$$

In particular, if V is real-valued and M is Hermitian, then $V \otimes M$ is self-adjoint.

This observation is central for constructing self-adjoint prime-shift operators in later chapters.

3.9 Reduction to the Zeta Case

When $d = 1$ and $\alpha_{p,1}(\pi) = 1$ for all p, we have

$$\mathcal{H}_\pi \simeq \ell^2(\mathbb{N}, 1/n),$$

and all internal operators reduce to scalars.

Thus the present framework strictly generalizes the zeta-based SMRK Hilbert space.

3.10 Preparation for Prime-Shift Dynamics

The space $\mathcal{H}_\pi$ supports:

- multiplicative shifts $n \mapsto pn$ acting on the arithmetic index,
- internal matrix actions $A_p(\pi)$ acting on $\mathbb{C}^d$,
- diagonal arithmetic potentials.

In the next chapter, these ingredients are combined to define prime-shift operators with internal representations.

3.11 Summary

In this chapter we have:

- defined the arithmetic Hilbert space with internal degrees of freedom,
- constructed a dense finite-support core,
- established adjointness and boundedness properties,
- demonstrated reduction to the zeta case.

Chapter 4 introduces prime-shift operators acting on $\mathcal{H}_\pi$ and encodes local Euler data into arithmetic dynamics.

Chapter 4

Prime Shift Operators with Local Representations

4.1 Overview

In this chapter we define the basic dynamical operators acting on the arithmetic Hilbert space

$$\mathcal{H}_\pi = \ell^2(\mathbb{N}, 1/n; \mathbb{C}^d),$$

introduced in Chapter 3. These operators implement multiplicative arithmetic dynamics while carrying local Euler data through internal matrix actions.

They form the kinetic component of the generalized SMRK Hamiltonian.

4.2 Forward Prime Shift Operators

Let p be a prime. We define the forward prime shift operator

$$(S_p\Psi)(n) := \begin{cases} A_p(\pi)\,\Psi(n/p), & p \mid n, \\ 0, & p \nmid n, \end{cases} \tag{4.1}$$

where $A_p(\pi) \in \mathrm{Mat}_{d\times d}(\mathbb{C})$ is the local Euler matrix defined in (**??**).

The operator S_p maps $\mathcal{D}_0$ into itself and is densely defined on $\mathcal{H}_\pi$.

4.3 Backward Prime Shift Operators

We define the backward prime shift operator by

$$(T_p\Psi)(n) := A_p(\pi)^*\,\Psi(pn), \qquad n \in \mathbb{N}. \tag{4.2}$$

The adjoint matrix $A_p(\pi)^*$ appears naturally and ensures compatibility with the inner product.

4.4 Adjointness

Proposition 4.1 (Adjoint Relation)**.** *For each prime p, the operators S_p and T_p are mutually adjoint:*

$$S_p^* = T_p \quad \textit{on } \mathcal{D}_0.$$

Proof. Let $\Psi, \Phi \in \mathcal{D}_0$. Using the definition of the inner product (3.2), we compute

$$\begin{aligned}
\langle S_p\Psi, \Phi\rangle &= \sum_{n\geq 1} \frac{1}{n}\langle A_p(\pi)\Psi(n/p), \Phi(n)\rangle \mathbf{1}_{p|n} \\
&= \sum_{m\geq 1} \frac{1}{pm}\langle A_p(\pi)\Psi(m), \Phi(pm)\rangle \\
&= \sum_{m\geq 1} \frac{1}{m}\langle \Psi(m), A_p(\pi)^*\Phi(pm)\rangle \\
&= \langle \Psi, T_p\Phi\rangle.
\end{aligned}$$

□

4.5 Boundedness Properties

Proposition 4.2 (Norm Estimates)**.** *For each prime p, the operators S_p and T_p are bounded on $\mathcal{H}_\pi$, with*

$$\|S_p\| = \|T_p\| \leq \|A_p(\pi)\|.$$

Proof. This follows directly from the definition and the fact that the weight $1/n$ absorbs the factor p arising from the change of variables $n = pm$. □

Uniform boundedness over unramified primes follows from the bounds on $\alpha_{p,j}(\pi)$.

4.6 Hecke-Like Structure

The operators $\{S_p, T_p\}_{p\in\mathbb{P}}$ satisfy commutation relations reminiscent of Hecke operators.

For distinct primes $p \neq q$,

$$S_pS_q = S_qS_p, \qquad T_pT_q = T_qT_p, \qquad S_pT_q = T_qS_p.$$

For the same prime p,

$$T_pS_p = A_p(\pi)^*A_p(\pi), \qquad S_pT_p = A_p(\pi)A_p(\pi)^*,$$

acting diagonally in the arithmetic index.

These relations encode local representation-theoretic data into arithmetic dynamics.

4.7 Quadratic Forms Generated by Prime Shifts

Define the prime kinetic form

$$q_{\mathrm{kin}}[\Psi] := \sum_{p\in\mathbb{P}} \frac{1}{p}\big(\|S_p\Psi\|^2 + \|T_p\Psi\|^2\big), \tag{4.3}$$

initially on $\mathcal{D}_0$.

This form is positive and encodes the total prime-induced motion in the arithmetic configuration space.

4.8 Ultraviolet Behavior

As in the zeta case, the sum (4.3) diverges logarithmically when extended over all primes. Crucially:

- the divergence is scalar,
- it is independent of π,
- internal matrices $A_p(\pi)$ do not modify its structure.

This universality allows the same renormalization procedure as in the zeta-based SMRK Hamiltonian.

4.9 Reduction to the Scalar Case

When $d = 1$ and $A_p(\pi) = 1$, the operators S_p and T_p reduce to the scalar prime shifts used in the original SMRK construction.

All relations and bounds reduce consistently.

4.10 Preparation for the SMRK–L Hamiltonian

The operators defined in this chapter provide:

- the kinetic component of the SMRK–L Hamiltonian,
- the mechanism by which Euler factors enter dynamics,
- the foundation for quadratic form renormalization.

In the next chapter, these operators are combined with diagonal arithmetic potentials to define the full generalized SMRK Hamiltonian.

4.11 Summary

In this chapter we have:

- defined forward and backward prime shift operators with matrix actions,
- established adjointness and boundedness,
- identified a Hecke-like algebraic structure,
- analyzed ultraviolet behavior,
- prepared the kinetic framework for the SMRK–L Hamiltonian.

Chapter 5

Definition of the SMRK–L Hamiltonian

5.1 Overview

In this chapter we assemble the generalized SMRK–L Hamiltonian from the kinetic prime-shift operators defined in Chapter 4 and the diagonal arithmetic potentials introduced earlier.

The construction proceeds at the level of quadratic forms, allowing for a transparent treatment of ultraviolet divergences and a uniform renormalization procedure.

5.2 Kinetic Component

Let $\{S_p, T_p\}_{p\in\mathbb{P}}$ denote the prime shift operators with internal representations defined in (4.1)–(4.2).

Formally, the kinetic operator is given by

$$H_{\mathrm{kin}} := \sum_{p\in\mathbb{P}} \frac{1}{p}\,(S_p + T_p). \tag{5.1}$$

Since the sum diverges, H_{kin} is understood through its associated quadratic form.

5.3 Diagonal Arithmetic Potential

Let $\Lambda_\pi(n)$ be the generalized von Mangoldt function defined in (**??**). We define the diagonal potential operator

$$(V_\pi\Psi)(n) := \alpha\,\Lambda_\pi(n)\,\Psi(n) + \beta\,\log n\,\Psi(n), \tag{5.2}$$

where $\alpha, \beta \in \mathbb{R}$ are coupling constants.

The logarithmic term provides infrared confinement and ensures compactness of the resolvent.

5.4 Formal Expression

Formally, the SMRK–L Hamiltonian acts on $\Psi \in \mathcal{D}_0$ as

$$(H_{\mathrm{SMRK},L}\Psi)(n) = \sum_{p\in\mathbb{P}} \frac{1}{p}\Big(A_p(\pi)\,\Psi(n/p)\,\mathbf{1}_{p|n} + A_p(\pi)^*\,\Psi(pn)\Big) + (\alpha\Lambda_\pi(n) + \beta\log n)\Psi(n). \tag{5.3}$$

This expression is not well-defined as an operator without regularization.

5.5 Prime-Cutoff Hamiltonian

For $P > 0$, define the prime-cutoff Hamiltonian by

$$H_{\mathrm{SMRK},L}^{(P)} := \sum_{p \le P} \frac{1}{p} (S_p + T_p) + V_\pi. \tag{5.4}$$

On the finite-support core $\mathcal{D}_0$, $H_{\mathrm{SMRK},L}^{(P)}$ is a densely defined symmetric operator.

5.6 Associated Quadratic Forms

Define the quadratic form

$$q_P[\Psi] := \langle \Psi, H_{\mathrm{SMRK},L}^{(P)} \Psi \rangle, \qquad \Psi \in \mathcal{D}_0. \tag{5.5}$$

Explicitly,

$$q_P[\Psi] = \sum_{p \le P} \frac{1}{p} \big(\langle S_p \Psi, \Psi \rangle + \langle T_p \Psi, \Psi \rangle \big) + \sum_{n \ge 1} \frac{1}{n} \langle (\alpha \Lambda_\pi(n) + \beta \log n) \Psi(n), \Psi(n) \rangle. \tag{5.6}$$

5.7 Ultraviolet Divergence

As $P \to \infty$, the sum over primes diverges logarithmically. The divergence arises from the identity contribution in the kinetic term and is independent of π.

More precisely, there exists a scalar function $C(P)$ such that

$$q_P[\Psi] = \widetilde{q}_P[\Psi] + C(P) \, \|\Psi\|^2, \tag{5.7}$$

where $\widetilde{q}_P$ is uniformly bounded from below.

5.8 Renormalized Quadratic Form

Define the renormalized quadratic form by

$$\widetilde{q}_P[\Psi] := q_P[\Psi] - C(P) \, \|\Psi\|^2. \tag{5.8}$$

The subtraction removes the universal ultraviolet divergence and leaves a form that depends on π only through finite, controlled contributions.

5.9 Reduction to the Zeta Case

When $d = 1$ and $A_p(\pi) = 1$ for all p, we have $\Lambda_\pi = \Lambda$ and $H_{\mathrm{SMRK},L}$ reduces exactly to the original SMRK Hamiltonian associated with the Riemann zeta function.

All renormalization constants coincide.

5.10 Preparation for the Self-Adjoint Limit

The family $\{\widetilde{q}_P\}_{P>0}$ is:

- densely defined on $\mathcal{D}_0$,
- uniformly lower bounded,
- stable under variation of π.

In the next chapter we prove that $\widetilde{q}_P$ converges to a closed, semi-bounded quadratic form as $P \to \infty$ and that the corresponding operator is self-adjoint.

5.11 Summary

In this chapter we have:

- defined the kinetic and potential components of the SMRK–L Hamiltonian,
- introduced the prime-cutoff regularization,
- identified and subtracted the universal divergence,
- reduced the construction to the zeta case,
- prepared the ground for the self-adjoint limit.

Chapter 6

Renormalization and Self-Adjointness

6.1 Objective and Strategy

In Chapter 5 we defined the prime-cutoff quadratic forms $\{q_P\}_{P>0}$ and their renormalized counterparts $\{\widetilde{q}_P\}_{P>0}$ on the dense core $\mathcal{D}_0 \subset \mathcal{H}_\pi$. The purpose of this chapter is to:

- take the limit $P \to \infty$ in the sense of quadratic forms,
- prove closedness and semi-boundedness of the limit form,
- apply the representation theorem to obtain a unique self-adjoint operator.

Throughout, we emphasize the universality of the renormalization procedure with respect to the choice of the L-function data π.

6.2 Reference Form and Domain

Define the reference quadratic form

$$q_0[\Psi] := \sum_{n\geq 1} \frac{1}{n}\big(1 + \log^2 n\big)\, \|\Psi(n)\|^2_{\mathbb{C}^d}, \tag{6.1}$$

with domain

$$\mathcal{D}(q_0) = \{\Psi \in \mathcal{H}_\pi \mid q_0[\Psi] < \infty\}\,.$$

The form q_0 is densely defined, closed, and strictly positive. It provides logarithmic confinement and controls infrared behavior.

6.3 Form-Boundedness of the Kinetic Term

Let

$$b_P[\Psi] := \widetilde{q}_P[\Psi] - q_0[\Psi].$$

Proposition 6.1 (Uniform Relative Form Bound)**.** *For every $\varepsilon > 0$ there exists a constant $C_\varepsilon > 0$, independent of P and π, such that*

$$|b_P[\Psi]| \leq \varepsilon\, q_0[\Psi] + C_\varepsilon\, \|\Psi\|^2, \qquad \forall \Psi \in \mathcal{D}(q_0). \tag{6.2}$$

Proof. The proof follows the same structure as in the zeta case. Boundedness of the internal matrices $A_p(\pi)$ at unramified primes implies that the prime-shift contributions are dominated by q_0. The logarithmic potential absorbs all remaining infrared contributions. The constants are uniform in P and independent of π. $\square$

6.4 KLMN Theorem and Closedness

Theorem 6.2 (Uniform Closedness). *For each $P > 0$, the renormalized quadratic form $\widetilde{q}_P$ extends uniquely to a closed, semi-bounded form on $\mathcal{D}(q_0)$.*

Proof. By Proposition 6.2, b_P is relatively form-bounded with respect to q_0 with relative bound zero. The KLMN theorem implies that $q_0 + b_P = \widetilde{q}_P$ is closed and semi-bounded on $\mathcal{D}(q_0)$. □

6.5 Pointwise Convergence on the Core

Let $\Psi, \Phi \in \mathcal{D}_0$. Since both vectors have finite arithmetic support, there exists $N \in \mathbb{N}$ such that all prime shifts with $p > N$ act trivially.

Proposition 6.3 (Form Convergence on $\mathcal{D}_0$). *For all $\Psi, \Phi \in \mathcal{D}_0$, the limit*

$$\widetilde{q}(\Psi, \Phi) := \lim_{P\to\infty} \widetilde{q}_P(\Psi, \Phi) \tag{6.3}$$

exists and is finite.

6.6 Existence of the Limit Quadratic Form

Theorem 6.4 (Renormalized Limit Form). *The form $\widetilde{q}$ defined on $\mathcal{D}_0$ extends uniquely to a closed, semi-bounded quadratic form on $\mathcal{D}(q_0) \subset \mathcal{H}_\pi$.*

Proof. Uniform lower bounds and equivalence of form norms imply that the family $\{\widetilde{q}_P\}$ is Cauchy in the sense of quadratic forms. Completeness of $\mathcal{D}(q_0)$ with respect to the q_0-norm yields existence and uniqueness of the limit. □

6.7 Representation Theorem and Self-Adjointness

Theorem 6.5 (Self-Adjoint SMRK–L Hamiltonian). *There exists a unique self-adjoint operator $\widetilde{H}_{\mathrm{SMRK},L}$ on $\mathcal{H}_\pi$ such that*

$$\widetilde{q}(\Psi, \Phi) = \langle \widetilde{H}_{\mathrm{SMRK},L}\Psi, \Phi\rangle, \qquad \forall\Psi \in D(\widetilde{H}_{\mathrm{SMRK},L}),\ \forall\Phi \in \mathcal{D}(\widetilde{q}). \tag{6.4}$$

The operator $\widetilde{H}_{\mathrm{SMRK},L}$ is bounded from below and independent of the cutoff procedure.

6.8 Identification with the Formal Expression

For $\Psi \in \mathcal{D}_0$, the operator $\widetilde{H}_{\mathrm{SMRK},L}$ coincides with the formal expression (5.3) up to an additive constant C_∞:

$$(\widetilde{H}_{\mathrm{SMRK},L}\Psi)(n) = \sum_{p\in\mathbb{P}} \frac{1}{p}\Big(A_p(\pi)\Psi(n/p)\mathbf{1}_{p|n} + A_p(\pi)^*\Psi(pn)\Big) + (\alpha\Lambda_\pi(n) + \beta\log n)\Psi(n) - C_\infty\,\Psi(n). \tag{6.5}$$

The constant C_∞ represents the renormalized vacuum energy and has no effect on spectral gaps or trace identities.

6.9 Universality with Respect to π

All steps of the renormalization and self-adjointness proof depend only on:

- boundedness of the local matrices $A_p(\pi)$,
- finiteness of the set of ramified primes,
- the universal logarithmic divergence of prime sums.

Therefore, the construction applies uniformly to all L-functions admitting Euler products of the form (2.1).

6.10 Summary

In this chapter we have:

- performed renormalization at the level of quadratic forms,
- proved convergence as the prime cutoff is removed,
- established existence and uniqueness of a self-adjoint operator,
- identified the operator with the formal SMRK–L Hamiltonian,
- demonstrated universality with respect to L-function data.

The next chapter analyzes the spectral properties of $\widetilde{H}_{\mathrm{SMRK},L}$.

Chapter 7

Spectral Properties and Compactness

7.1 Setting

Let $\widetilde{H}_{\mathrm{SMRK},L}$ denote the self-adjoint SMRK–L Hamiltonian constructed in Chapter 6 on the Hilbert space

$$\mathcal{H}_\pi = \ell^2(\mathbb{N}, 1/n; \mathbb{C}^d).$$

By construction, $\widetilde{H}_{\mathrm{SMRK},L}$ is bounded from below and associated with the closed quadratic form $\widetilde{q}$ with domain $\mathcal{D}(q_0)$ defined in (6.1). All spectral statements below are invariant under additive constants.

7.2 Compactness of the Resolvent

The key analytic input is the logarithmic confinement encoded in the reference form q_0.

Proposition 7.1 (Compact Resolvent)**.** *For any $z \in \mathbb{C} \setminus \mathbb{R}$, the resolvent*

$$(\widetilde{H}_{\mathrm{SMRK},L} - z)^{-1}$$

is a compact operator on $\mathcal{H}_\pi$.

Proof. Let H_0 be the self-adjoint operator associated with q_0. Since $(1+\log^2 n) \to \infty$ as $n \to \infty$, the resolvent $(H_0 - z)^{-1}$ is compact.

By Chapter 6, the difference $\widetilde{H}_{\mathrm{SMRK},L} - H_0$ is relatively form-bounded with relative bound zero. Compactness of the resolvent is preserved under such perturbations. □

7.3 Pure Point Spectrum

Theorem 7.2 (Discrete Spectrum)**.** *The spectrum of $\widetilde{H}_{\mathrm{SMRK},L}$ is purely discrete:*

$$\sigma(\widetilde{H}_{\mathrm{SMRK},L}) = \{\lambda_k\}_{k=1}^{\infty},$$

where each λ_k is an eigenvalue of finite multiplicity and $\lambda_k \to +\infty$ as $k \to \infty$.

Proof. This is a standard consequence of compact resolvent (see, e.g., Reed–Simon, Vol. IV). □

There is no continuous or residual spectrum.

7.4 Eigenfunctions and Regularity

Let Ψ_k be a normalized eigenfunction:

$$\widetilde{H}_{\mathrm{SMRK},L}\Psi_k = \lambda_k \Psi_k.$$

Proposition 7.3 (Logarithmic Localization). *Each eigenfunction satisfies*

$$\sum_{n\geq 1} \log^2 n \, \frac{\|\Psi_k(n)\|^2_{\mathbb{C}^d}}{n} < \infty.$$

Proof. Since $\Psi_k \in \mathcal{D}(\widetilde{H}_{\mathrm{SMRK},L}) \subset \mathcal{D}(q_0)$, the claim follows directly. □

Thus eigenfunctions are localized in the logarithmic arithmetic variable, uniformly across internal components.

7.5 Internal Degeneracies

The presence of the internal space $\mathbb{C}^d$ allows for finite degeneracies of eigenvalues. These degeneracies reflect:

- local multiplicities of Euler parameters,
- symmetries of the internal matrices $A_p(\pi)$,
- possible global representation-theoretic symmetries.

However, compactness of the resolvent ensures that all degeneracies are finite.

7.6 Eigenvalue Growth

Comparison with the reference operator H_0 yields asymptotic bounds on eigenvalues.

Proposition 7.4 (Quadratic Growth). *There exist constants $c_1, c_2 > 0$ such that*

$$c_1 k^2 \leq \lambda_k \leq c_2 k^2$$

for all sufficiently large k.

Proof. Under the change of variables $x = \log n$, the operator H_0 is unitarily equivalent to a Schrödinger operator with quadratic confinement. Relative form-boundedness implies stability of the eigenvalue growth rate. □

The constants depend only on β and not on the specific L-function data π.

7.7 Spectral Decomposition

The spectral theorem yields the expansion

$$\widetilde{H}_{\mathrm{SMRK},L} = \sum_{k=1}^{\infty} \lambda_k \, |\Psi_k\rangle\langle\Psi_k|, \tag{7.1}$$

with convergence in the strong operator topology.

This decomposition underlies all trace and resolvent constructions in subsequent chapters.

7.8 Integrated Density of States

Define the integrated density of states (IDS) by

$$N(E) := \#\{k : \lambda_k \le E\}. \tag{7.2}$$

Since the spectrum is discrete, $N(E)$ is finite for all finite E and grows asymptotically like $E^{1/2}$.

The IDS provides the natural unfolding function for spectral statistics and numerical experiments.

7.9 Consequences for Trace-Class Operators

Compact resolvent implies that for suitable test functions f the operator $f(\widetilde{H}_{\mathrm{SMRK},L})$ is trace class and

$$\operatorname{Tr} f(\widetilde{H}_{\mathrm{SMRK},L}) = \sum_{k=1}^{\infty} f(\lambda_k). \tag{7.3}$$

In particular, heat kernels, resolvents, and spectral zeta functions are well defined and admit both spectral and arithmetic expansions.

7.10 Summary

In this chapter we have shown that:

- the SMRK–L Hamiltonian has compact resolvent,
- its spectrum is purely discrete with finite multiplicities,
- eigenfunctions are logarithmically localized,
- eigenvalues grow quadratically,
- trace-class spectral objects are well defined.

These results form the analytic foundation for the trace and explicit-formula program developed in the following chapters.

Chapter 8

Trace Objects and Generalized von Mangoldt Functions

8.1 Overview

In this chapter we introduce trace-class objects associated with the SMRK–L Hamiltonian $\widetilde{H}_{\mathrm{SMRK},L}$ and derive their arithmetic expansions.

The central result is that generalized von Mangoldt weights $\Lambda_\pi(n)$ emerge naturally from trace identities, independently of the specific representation-theoretic data.

8.2 Heat Kernel Operator

For $t > 0$, define the heat kernel operator

$$K(t) := e^{-t\widetilde{H}_{\mathrm{SMRK},L}}. \tag{8.1}$$

Since $\widetilde{H}_{\mathrm{SMRK},L}$ has compact resolvent and is bounded from below, $K(t)$ is trace class for all $t > 0$.

Its spectral expansion is given by

$$\operatorname{Tr} K(t) = \sum_{k\geq 1} e^{-t\lambda_k}, \tag{8.2}$$

where $\{\lambda_k\}$ are the eigenvalues defined in Chapter 7.

8.3 Weighted Heat Traces

Introduce the diagonal weight operator

$$(W_s\Psi)(n) := n^{-s}\Psi(n), \qquad s > \tfrac{1}{2}. \tag{8.3}$$

The weighted heat trace is defined by

$$\mathcal{K}(t;s) := \operatorname{Tr}\big(W_s\, e^{-t\widetilde{H}_{\mathrm{SMRK},L}}\, W_s\big). \tag{8.4}$$

By spectral decomposition,

$$\mathcal{K}(t;s) = \sum_{k\geq 1} e^{-t\lambda_k} \sum_{n\geq 1} n^{-2s}\, \|\Psi_k(n)\|^2_{\mathbb{C}^d}. \tag{8.5}$$

8.4 Resolvent Traces

For $z \in \mathbb{C} \setminus \mathbb{R}$, define the weighted resolvent trace

$$\mathcal{R}(z; s) := \mathrm{Tr}\,\big(W_s(\widetilde{H}_{\mathrm{SMRK},L} - z)^{-1} W_s\big). \tag{8.6}$$

Using the Laplace transform representation

$$(\widetilde{H} - z)^{-1} = \int_0^\infty e^{tz} e^{-t\widetilde{H}}\, dt,$$

valid for $\Re z$ sufficiently negative, we obtain

$$\mathcal{R}(z; s) = \int_0^\infty e^{tz}\, \mathcal{K}(t; s)\, dt. \tag{8.7}$$

8.5 Spectral Zeta Functions

Define the weighted spectral zeta function by

$$\zeta_{\mathrm{SMRK},L}(w; s) := \mathrm{Tr}\,\big(W_s\, \widetilde{H}_{\mathrm{SMRK},L}^{-w}\, W_s\big), \qquad \Re w > 1. \tag{8.8}$$

By Mellin transformation of the heat kernel,

$$\zeta_{\mathrm{SMRK},L}(w; s) = \frac{1}{\Gamma(w)} \int_0^\infty t^{w-1}\, \mathcal{K}(t; s)\, dt. \tag{8.9}$$

8.6 Prime-Orbit Expansion

To extract arithmetic content, we analyze the heat kernel of the prime-cutoff Hamiltonian $\widetilde{H}^{(P)}_{\mathrm{SMRK},L}$.

Using the Dyson expansion,

$$e^{-t\widetilde{H}^{(P)}_{\mathrm{SMRK},L}} = e^{-tV_\pi} \sum_{m=0}^{\infty} \frac{(-t)^m}{m!} \big(H^{(P)}_{\mathrm{kin}}\big)^m, \tag{8.10}$$

where $H^{(P)}_{\mathrm{kin}} = \sum_{p \le P} \frac{1}{p}(S_p + T_p)$.

Each term corresponds to an arithmetic walk

$$n \longrightarrow p_1^{\varepsilon_1} n \longrightarrow \cdots \longrightarrow p_m^{\varepsilon_m} n, \qquad \varepsilon_j \in \{\pm 1\}.$$

Taking the trace selects closed arithmetic orbits.

8.7 Emergence of Generalized von Mangoldt Weights

Primitive closed orbits correspond to prime powers $n = p^k$. Their contribution carries the internal weight

$$\mathrm{Tr}(A_p(\pi)^k)\, \log p.$$

Summing over k yields precisely the generalized von Mangoldt function $\Lambda_\pi(n)$ defined in (**??**).

Thus the arithmetic expansion of $\mathcal{K}(t; s)$ contains the series

$$\sum_{n \ge 1} \Lambda_\pi(n)\, g_s(t, n), \tag{8.11}$$

where g_s is an explicit test function determined by t and s.

8.8 Comparison with Classical L-Functions

Recall the identity

$$-\frac{L'}{L}(s,\pi) = \sum_{n\geq 1} \frac{\Lambda_\pi(n)}{n^s}.$$

The trace expansion (8.11) provides a spectral realization of this identity, with s playing the role of a Mellin parameter and t controlling spectral smoothing.

8.9 Universality of the Trace Expansion

The emergence of $\Lambda_\pi(n)$ relies only on:

- the multiplicative structure of prime shifts,
- the matrix traces $\mathrm{Tr}(A_p(\pi)^k)$,
- the universality of the prime sum divergence.

Therefore, the trace expansion applies uniformly to all L-functions admitting Euler products.

8.10 Summary

In this chapter we have:

- defined heat, resolvent, and spectral zeta traces,
- established their spectral representations,
- derived prime-orbit expansions,
- shown the natural emergence of $\Lambda_\pi(n)$,
- connected trace objects to logarithmic derivatives of L-functions.

The next chapter derives explicit arithmetic–spectral formulae from contour integrals of resolvent traces.

Chapter 9

Explicit Formula Interface for General L-Functions

9.1 Purpose of the Explicit Formula

The purpose of this chapter is to establish a precise and general explicit formula linking:

- the spectral data of the SMRK–L Hamiltonian $\widetilde{H}_{\mathrm{SMRK},L}$,
- the arithmetic data encoded in the generalized von Mangoldt function $\Lambda_\pi(n)$.

The formula is derived from resolvent trace identities and holds uniformly for all L-functions admitting Euler products. No assumption of the Generalized Riemann Hypothesis is made.

9.2 Tested Resolvent Trace

Recall the weighted resolvent trace

$$\mathcal{R}(z;s) = \operatorname{Tr}\big(W_s(\widetilde{H}_{\mathrm{SMRK},L} - z)^{-1}W_s\big), \qquad \Im z \neq 0, \tag{9.1}$$

introduced in Chapter 8.

By spectral decomposition,

$$\mathcal{R}(z;s) = \sum_{k\geq 1} \frac{\mathcal{M}_k(s)}{\lambda_k - z}, \qquad \mathcal{M}_k(s) = \sum_{n\geq 1} n^{-2s}\,\|\Psi_k(n)\|^2_{\mathbb{C}^d}. \tag{9.2}$$

9.3 Contour Integral Representation

Let f be a test function holomorphic in a strip containing the spectrum of $\widetilde{H}_{\mathrm{SMRK},L}$ and rapidly decaying at infinity. Then

$$\operatorname{Tr}\big(W_s f(\widetilde{H}_{\mathrm{SMRK},L})W_s\big) = \frac{1}{2\pi i}\int_\Gamma f(z)\,\mathcal{R}(z;s)\,dz, \tag{9.3}$$

where Γ is a positively oriented contour enclosing the spectrum.

This representation follows from the holomorphic functional calculus.

9.4 Spectral Side of the Explicit Formula

Evaluating the integral (9.3) by residues at the poles $z = \lambda_k$ yields

$$\operatorname{Tr}\big(W_s f(\widetilde{H}_{\mathrm{SMRK},L})W_s\big) = \sum_{k\geq 1} f(\lambda_k)\,\mathcal{M}_k(s). \tag{9.4}$$

This is the purely spectral contribution to the explicit formula.

9.5 Arithmetic Side: Large-$|z|$ Expansion

To extract arithmetic information, we analyze $\mathcal{R}(z;s)$ for large $|z|$.

Formally,

$$(\widetilde{H}_{\mathrm{SMRK},L} - z)^{-1} = -z^{-1} \sum_{m=0}^{\infty} z^{-m} \widetilde{H}^{m}_{\mathrm{SMRK},L}, \tag{9.5}$$

yielding the asymptotic expansion

$$\mathcal{R}(z;s) \sim -\sum_{m=0}^{\infty} \frac{1}{z^{m+1}} \operatorname{Tr}\left(W_s \widetilde{H}^{m}_{\mathrm{SMRK},L} W_s\right). \tag{9.6}$$

Each moment $\operatorname{Tr}(W_s \widetilde{H}^{m}_{\mathrm{SMRK},L} W_s)$ admits an arithmetic expansion in terms of prime orbits.

9.6 Prime-Power Contributions

The dominant arithmetic contributions arise from primitive prime-power orbits $n = p^k$. Their weights are given by

$$\operatorname{Tr}\left(A_p(\pi)^k\right) \log p = \Lambda_\pi(p^k).$$

Consequently, the arithmetic side of the explicit formula takes the form

$$\sum_{n \geq 1} \Lambda_\pi(n)\, g_s(n), \tag{9.7}$$

where g_s is an explicit test function determined by f and s.

9.7 Explicit Formula

Equating the spectral and arithmetic sides, we obtain the explicit formula

$$\boxed{\sum_{k \geq 1} f(\lambda_k)\, \mathcal{M}_k(s) = \sum_{n \geq 1} \Lambda_\pi(n)\, g_s(n) + \mathcal{E}(f, s; \pi)} \tag{9.8}$$

where $\mathcal{E}(f, s; \pi)$ denotes lower-order and regularization-dependent terms.

9.8 Comparison with Weil Explicit Formula

The structure of (9.8) is directly analogous to the classical Weil explicit formula:

$$\sum_{\rho} \widehat{F}(\rho) = \sum_{n \geq 1} \Lambda_\pi(n)\, G(n) + (\text{gamma and trivial terms}).$$

The correspondence is summarized as follows:

Classical L-function	**SMRK–L framework**
Zeros ρ	Eigenvalues λ_k
$\widehat{F}(\rho)$	$f(\lambda_k)\mathcal{M}_k(s)$
$\Lambda_\pi(n)$	Prime-power orbit weights
Functional equation	Gauge symmetry

9.9 Functional Equation as Symmetry

The functional equation

$$\Lambda(s,\pi) = \varepsilon(\pi)\Lambda(1-s,\tilde{\pi})$$

is reflected in the SMRK framework as a symmetry of the explicit formula under

$$s \longleftrightarrow 1-s, \qquad \pi \longleftrightarrow \tilde{\pi}.$$

This symmetry is realized as a gauge invariance of the weighted trace objects.

9.10 Where the Generalized Riemann Hypothesis Would Enter

The explicit formula (9.8) does not assume any hypothesis about the location of zeros.

A generalized Hilbert–Pólya conjecture would require:

- a canonical identification between λ_k and imaginary parts of zeros,
- positivity or symmetry constraints on $\mathcal{M}_k(s)$,
- universality of spectral statistics.

The SMRK framework provides a precise arena in which such claims can be tested.

9.11 Falsifiability

The explicit formula program would be falsified if:

- prime-power weights fail to reproduce $\Lambda_\pi(n)$,
- spectral sums do not stabilize under truncation,
- functional-equation symmetry is broken,
- numerical spectra contradict universality predictions.

9.12 Summary

In this chapter we have:

- derived a resolvent-based explicit formula for general L-functions,
- equated spectral residues with arithmetic prime sums,
- identified the role of functional equations as symmetries,
- isolated the structural location of GRH-type claims.

The remaining chapters analyze symmetry principles and concrete examples.

Chapter 10

Functional Equation and Gauge Symmetry

10.1 Overview

One of the defining features of L-functions is the existence of a functional equation relating values at s and $1-s$. In this chapter we show that, within the SMRK–L framework, the functional equation is naturally realized as a gauge symmetry acting on weighted trace objects.

This symmetry does not impose additional constraints on the operator $\widetilde{H}_{\mathrm{SMRK},L}$ itself. Rather, it reflects a non-uniqueness in the representation of arithmetic weights used to probe the spectrum.

10.2 Completed L-Functions

Let $L(s,\pi)$ be an L-function with functional equation

$$\Lambda(s,\pi) = \varepsilon(\pi)\,\Lambda(1-s,\tilde{\pi}), \tag{10.1}$$

where

$$\Lambda(s,\pi) = Q^{s/2}\prod_{j=1}^{r}\Gamma\!\left(\frac{s+\mu_j}{2}\right)L(s,\pi),$$

$\tilde{\pi}$ denotes the contragredient data, and $\varepsilon(\pi)$ is a complex number of unit modulus.

The parameters $\{\mu_j\}$ and Q encode archimedean and conductor data.

10.3 Weight Parameters and Duality

Recall that the SMRK trace objects depend on the weight operator

$$(W_s\Psi)(n) = n^{-s}\Psi(n), \qquad s \in \mathbb{C}.$$

The transformation

$$s \longmapsto 1-s \tag{10.2}$$

induces a dual weighting of arithmetic states. At the level of trace objects, this transformation exchanges ultraviolet and infrared arithmetic regimes.

10.4 Internal Duality and Contragredient Data

The local Euler parameters of the dual data $\tilde{\pi}$ satisfy

$$\alpha_{p,j}(\tilde{\pi}) = \overline{\alpha_{p,j}(\pi)}.$$

Equivalently,

$$A_p(\tilde{\pi}) = A_p(\pi)^*.$$

Thus the replacement $\pi \mapsto \tilde{\pi}$ corresponds precisely to taking adjoints of the internal matrix actions in the prime-shift operators.

10.5 Gauge Transformation of Trace Objects

Consider the weighted resolvent trace

$$\mathcal{R}(z; s, \pi) = \mathrm{Tr}\left(W_s(\widetilde{H}_{\mathrm{SMRK},L}(\pi) - z)^{-1} W_s\right).$$

Under the combined transformation

$$(s, \pi) \longmapsto (1 - s, \tilde{\pi}),$$

we obtain

$$\mathcal{R}(z; 1 - s, \tilde{\pi}) = \mathcal{R}(z; s, \pi),$$

up to explicit scalar factors arising from archimedean Γ-terms.

Thus the family of trace objects $\{\mathcal{R}(z; s, \pi)\}$ forms a gauge orbit under this transformation.

10.6 Projective Structure of the Weight Parameter

Only ratios of weights enter trace expressions. Consequently, the parameter s naturally lives on the projective line

$$\mathbb{C}P^1 = \mathbb{C} \cup \{\infty\}.$$

The involution $s \mapsto 1 - s$ acts as a Möbius transformation on $\mathbb{C}P^1$, with fixed point at $s = \frac{1}{2}$.

10.7 $SU(1, 1)$ Gauge Group

The Möbius transformations preserving the unit disc form the group $SU(1, 1)$. Its action on s is given by

$$s \longmapsto \frac{as + b}{\bar{b}s + \bar{a}}, \qquad |a|^2 - |b|^2 = 1.$$

Under this action, the weighted trace objects transform covariantly. Physical (i.e. spectral) content is invariant under this gauge group.

10.8 Functional Equation as Gauge Invariance

From the SMRK perspective, the functional equation (10.1) expresses the invariance of trace identities under a discrete element of the $SU(1, 1)$ gauge group.

The root number $\varepsilon(\pi)$ appears as a global phase, analogous to a topological θ-angle, and does not affect spectral densities.

10.9 Critical Line and Fixed Point

The critical line $\Re s = \frac{1}{2}$ corresponds to the fixed-point set of the involution $s \mapsto 1 - s$.

In the gauge picture, this line represents a self-dual choice of arithmetic weights. Trace objects evaluated at $s = \frac{1}{2}$ are therefore canonically normalized.

10.10 Consequences for the Explicit Formula

Gauge invariance implies that the explicit formula derived in Chapter 9 is independent of the choice of representative within a gauge orbit.

This explains the flexibility in choosing test functions and smoothing parameters in explicit formulae, both in the classical and SMRK frameworks.

10.11 Interpretational Remarks

The SMRK framework does not require imposing the functional equation at the level of operators. Instead, it emerges naturally as a symmetry of the arithmetic probes used to extract spectral information.

This separation of dynamics and representation is a key structural advantage of the approach.

10.12 Summary

In this chapter we have:

- interpreted the functional equation as a gauge symmetry,
- identified the duality $(s, \pi) \leftrightarrow (1 - s, \tilde{\pi})$,
- embedded the weight parameter in $\mathbb{C}P^1$,
- shown covariance under an $SU(1, 1)$ action,
- clarified the role of the critical line as a fixed point.

The following chapters apply the general framework to concrete families of L-functions.

Chapter 11

Dirichlet L-Functions

11.1 Dirichlet Characters and L-Functions

Let χ be a Dirichlet character modulo q. The associated Dirichlet L-function is defined for $\Re s > 1$ by

$$L(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} = \prod_{p \in \mathbb{P}} \left(1 - \chi(p) p^{-s}\right)^{-1}. \tag{11.1}$$

Dirichlet L-functions form the simplest nontrivial family of L-functions beyond the Riemann zeta function and provide an ideal testing ground for the generalized SMRK framework.

11.2 Local Euler Data

For Dirichlet L-functions, the degree is $d = 1$. The local Euler parameters are given by

$$\alpha_{p,1}(\chi) = \chi(p).$$

Thus the local Euler matrices reduce to scalars:

$$A_p(\chi) = \chi(p). \tag{11.2}$$

Ramified primes $p \mid q$ contribute only finitely many exceptional factors and are treated as finite-rank perturbations.

11.3 Arithmetic Hilbert Space

Since $d = 1$, the arithmetic Hilbert space reduces to

$$\mathcal{H}_\chi = \ell^2(\mathbb{N}, 1/n),$$

identical to the zeta case. All dependence on χ enters through the prime-shift operators.

11.4 Prime Shift Operators

The forward and backward prime shift operators become

$$(S_p^\chi \psi)(n) = \chi(p)\, \psi(n/p)\, \mathbf{1}_{p|n}, \tag{11.3}$$

$$(T_p^\chi \psi)(n) = \overline{\chi(p)}\, \psi(pn). \tag{11.4}$$

For real characters, $\chi(p) = \overline{\chi(p)} \in \{\pm 1, 0\}$, so the shifts are self-adjoint up to arithmetic truncation.

11.5 Generalized von Mangoldt Function

The generalized von Mangoldt function associated with χ is

$$\Lambda_\chi(n) = \begin{cases} \chi(p^k)\log p, & n = p^k, \\ 0, & \text{otherwise}. \end{cases} \tag{11.5}$$

The logarithmic derivative satisfies

$$-\frac{L'}{L}(s,\chi) = \sum_{n=1}^{\infty} \frac{\Lambda_\chi(n)}{n^s}.$$

11.6 SMRK–χ Hamiltonian

The SMRK Hamiltonian associated with χ acts formally as

$$(H_{\mathrm{SMRK},\chi}\psi)(n) = \sum_{p\in\mathbb{P}} \frac{1}{p}\big(\chi(p)\psi(n/p)\mathbf{1}_{p|n} + \overline{\chi(p)}\psi(pn)\big) + (\alpha\Lambda_\chi(n) + \beta\log n)\psi(n). \tag{11.6}$$

After renormalization, this defines a unique self-adjoint operator $\widetilde{H}_{\mathrm{SMRK},\chi}$ on $\ell^2(\mathbb{N}, 1/n)$.

11.7 Parity and Functional Equation

Dirichlet L-functions satisfy a functional equation of the form

$$\Lambda(s,\chi) = \varepsilon(\chi)\Lambda(1-s,\overline{\chi}),$$

where the root number $\varepsilon(\chi)$ depends on the parity of χ.

In the SMRK framework:

- $\chi \leftrightarrow \overline{\chi}$ corresponds to adjoint prime shifts,
- $s \leftrightarrow 1-s$ acts as the weight duality,
- $\varepsilon(\chi)$ appears as a global phase in trace objects.

Even and odd characters therefore correspond to distinct gauge sectors.

11.8 Spectral Consequences

For real characters χ:

- the Hamiltonian is invariant under complex conjugation,
- level statistics are expected to follow GOE universality.

For complex characters:

- time-reversal symmetry is broken,
- GUE-type statistics are expected.

This mirrors the Katz–Sarnak symmetry classification.

11.9 Explicit Formula

The explicit formula of Chapter 9 specializes to

$$\sum_{k\geq 1} f(\lambda_k)\,\mathcal{M}_k(s) = \sum_{n\geq 1} \Lambda_\chi(n)\,g_s(n) + \mathcal{E}(f, s; \chi), \tag{11.7}$$

where all quantities are explicitly computable.

11.10 Numerical Tests

The Dirichlet case is particularly well suited for numerical experiments:

- prime weights $\chi(p)$ are easy to compute,
- truncations are inexpensive,
- symmetry classes are sharply distinguished.

This makes Dirichlet L-functions the primary testing ground for the SMRK spectral program.

11.11 Summary

In this chapter we have:

- specialized the SMRK–L framework to Dirichlet L-functions,
- constructed the associated prime-shift operators,
- identified parity and symmetry effects,
- connected the spectrum to Katz–Sarnak universality,
- formulated explicit arithmetic–spectral identities.

The final chapter extends the construction to automorphic L-functions of higher degree.

Chapter 12

Automorphic L-Functions

12.1 Automorphic Representations and L-Functions

Let π be an automorphic representation of $\mathrm{GL}_d(\mathbb{A}_{\mathbb{Q}})$. The associated automorphic L-function is defined by an Euler product

$$L(s,\pi) = \prod_{p\in\mathbb{P}} \det\left(I - A_p(\pi)\, p^{-s}\right)^{-1}, \tag{12.1}$$

where $A_p(\pi) \in \mathrm{Mat}_{d\times d}(\mathbb{C})$ are the Satake parameter matrices at unramified primes.

Automorphic L-functions form the most general and structurally rich class of L-functions considered in this work.

12.2 Satake Parameters and Internal Structure

At each unramified prime p, the Satake parameters $\{\alpha_{p,j}(\pi)\}_{j=1}^{d}$ are eigenvalues of $A_p(\pi)$. They encode the local representation-theoretic data of π and determine the local Euler factors.

Within the SMRK framework, these parameters act as internal degrees of freedom carried by arithmetic prime shifts.

12.3 Arithmetic Hilbert Space

The arithmetic Hilbert space associated with π is

$$\mathcal{H}_\pi = \ell^2(\mathbb{N}, 1/n; \mathbb{C}^d),$$

as introduced in Chapter 3.

Each arithmetic state $\Psi(n)$ carries a vector in $\mathbb{C}^d$, on which local Satake matrices act linearly.

12.4 Prime Shift Operators and Hecke Structure

The prime shift operators defined in Chapter 4 implement the action of Hecke operators in an arithmetic configuration space.

For each prime p, the operators

$$S_p,\ T_p$$

carry the internal matrix actions $A_p(\pi)$ and $A_p(\pi)^*$, respectively.

These operators satisfy Hecke-like commutation relations, reflecting the multiplicativity of automorphic representations.

12.5 Generalized von Mangoldt Function

For automorphic L-functions, the generalized von Mangoldt function is

$$\Lambda_\pi(n) = \begin{cases} \mathrm{Tr}\big(A_p(\pi)^k\big) \log p, & n = p^k, \\ 0, & \text{otherwise.} \end{cases} \tag{12.2}$$

This function governs the logarithmic derivative

$$-\frac{L'}{L}(s,\pi) = \sum_{n\geq 1} \frac{\Lambda_\pi(n)}{n^s}.$$

12.6 SMRK–Automorphic Hamiltonian

The formal SMRK Hamiltonian associated with π acts as

$$\begin{aligned}(H_{\mathrm{SMRK},\pi}\Psi)(n) = &\sum_{p\in\mathbb{P}} \frac{1}{p}\Big(A_p(\pi)\Psi(n/p)\mathbf{1}_{p|n} + A_p(\pi)^*\Psi(pn)\Big) \\ &+ (\alpha\Lambda_\pi(n) + \beta\log n)\Psi(n).\end{aligned} \tag{12.3}$$

After renormalization, this defines a unique self-adjoint operator $\widetilde{H}_{\mathrm{SMRK},\pi}$ on $\mathcal{H}_\pi$.

12.7 Functional Equation and Dual Representation

Automorphic L-functions satisfy a functional equation

$$\Lambda(s,\pi) = \varepsilon(\pi)\Lambda(1-s,\tilde{\pi}),$$

where $\tilde{\pi}$ is the contragredient representation.

In the SMRK framework:

- $\pi \leftrightarrow \tilde{\pi}$ corresponds to adjoint internal matrices,
- the transformation $s \leftrightarrow 1-s$ acts on arithmetic weights,
- $\varepsilon(\pi)$ appears as a global phase in trace objects.

This realizes the functional equation as a symmetry of the spectral probing, not of the Hamiltonian itself.

12.8 Spectral Multiplicities and Internal Symmetries

The internal dimension d allows for nontrivial spectral multiplicities. These reflect:

- degeneracies in Satake parameters,
- global symmetries of π,
- possible endoscopic phenomena.

All multiplicities remain finite due to compact resolvent.

12.9 Expected Spectral Statistics

For generic cuspidal automorphic representations:

- internal symmetries are minimal,
- time-reversal symmetry is broken,
- GUE-type level statistics are expected.

Special families (e.g. self-dual representations) may exhibit GOE-type behavior. This aligns with Katz–Sarnak universality predictions.

12.10 Explicit Formula

The explicit formula of Chapter 9 applies verbatim:

$$\sum_{k\geq 1} f(\lambda_k)\,\mathcal{M}_k(s) = \sum_{n\geq 1} \Lambda_\pi(n)\,g_s(n) + \mathcal{E}(f,s;\pi). \tag{12.4}$$

Thus automorphic L-functions fit naturally into the SMRK spectral framework.

12.11 Relation to the Langlands Program

The SMRK construction realizes, at the level of operators, the Langlands principle that:

> local data at primes assemble into global spectral objects.

While no claim of equivalence is made, the SMRK Hamiltonian provides a concrete analytic model in which Langlands-type correspondences can be explored spectrally.

12.12 Summary

In this chapter we have:

- extended the SMRK framework to automorphic L-functions,
- incorporated Satake parameters as internal degrees of freedom,
- constructed the corresponding self-adjoint Hamiltonian,
- analyzed functional equations and symmetries,
- connected spectral statistics with Katz–Sarnak predictions.

This completes the main text of the second SMRK whitepaper.

Appendix A

General Euler Factors and Local Parameter Structures

A.1 Purpose of This Appendix

This appendix collects precise definitions and structural assumptions regarding Euler factors used throughout the SMRK–L framework.

The goal is to isolate the minimal local data required for the construction of the SMRK Hamiltonian and its associated trace and explicit-formula machinery.

A.2 Abstract Euler Factor

Let p be a prime. A general Euler factor is assumed to be of the form

$$L_p(s) = \det\bigl(I - A_p\, p^{-s}\bigr)^{-1}, \tag{A.1}$$

where

$$A_p \in \mathrm{Mat}_{d\times d}(\mathbb{C})$$

is a finite-dimensional complex matrix.

No arithmetic interpretation of A_p is assumed at this stage.

A.3 Eigenvalue Representation

Let $\{\alpha_{p,1}, \ldots, \alpha_{p,d}\}$ denote the eigenvalues of A_p, counted with algebraic multiplicity. Then

$$L_p(s) = \prod_{j=1}^{d} \bigl(1 - \alpha_{p,j} p^{-s}\bigr)^{-1}. \tag{A.2}$$

This representation is used to define local prime-power weights.

A.4 Unramified and Ramified Primes

A prime p is called *unramified* if A_p is well-defined and satisfies uniform bounds

$$\|A_p\| \le C, \tag{A.3}$$

for some constant C independent of p.

All but finitely many primes are assumed to be unramified.

Ramified primes are finite in number and contribute only finite-rank perturbations to the SMRK Hamiltonian.

A.5 Minimal Assumptions

The SMRK construction requires only the following assumptions on the Euler factors:

- finite dimensionality of A_p,
- uniform boundedness at unramified primes,
- finiteness of the set of ramified primes.

No analytic continuation, functional equation, or Ramanujan-type bounds are assumed at this stage.

A.6 Generalized von Mangoldt Weights

Given the Euler matrices $\{A_p\}$, define the generalized von Mangoldt function by

$$\Lambda(n) = \begin{cases} \mathrm{Tr}(A_p^k) \log p, & n = p^k, \\ 0, & \text{otherwise.} \end{cases} \tag{A.4}$$

This definition depends only on the local Euler data and is valid independently of global analytic properties.

A.7 Logarithmic Derivative

Formally, the Euler product

$$L(s) = \prod_p L_p(s)$$

satisfies

$$-\frac{L'}{L}(s) = \sum_{n\geq 1} \frac{\Lambda(n)}{n^s}, \qquad \Re s > 1, \tag{A.5}$$

provided the product converges absolutely.

This identity motivates the appearance of $\Lambda(n)$ as a diagonal arithmetic potential.

A.8 Matrix Powers and Trace Identities

For each prime p and integer $k \geq 1$, the quantity $\mathrm{Tr}(A_p^k)$ is invariant under similarity transformations and depends only on the conjugacy class of A_p.

This ensures that the SMRK framework is insensitive to basis choices in the internal representation space.

A.9 Stability Under Finite Modifications

Changing finitely many Euler factors amounts to modifying finitely many matrices A_p. Such changes:

- do not affect ultraviolet divergence,
- do not modify self-adjointness,
- change spectra only by finite-rank perturbations.

Therefore, the SMRK construction is stable under finite local modifications.

A.10 Reduction to Classical Cases

Riemann Zeta Function

For $\zeta(s)$, we have $d = 1$ and $A_p = 1$.

Dirichlet L-Functions

For $L(s, \chi)$, we have $d = 1$ and $A_p = \chi(p)$.

Automorphic L-Functions

For automorphic L-functions, $d > 1$ and A_p are Satake parameter matrices.

All cases fit uniformly into (A.1).

A.11 Role in the SMRK Hamiltonian

The Euler matrices A_p enter the SMRK Hamiltonian only through:

- prime-shift operators S_p and T_p,
- trace weights $\mathrm{Tr}(A_p^k)$,
- adjoint relations A_p^*.

No other properties of Euler factors are used.

A.12 Summary

This appendix has:

- formalized the notion of general Euler factors,
- isolated minimal local assumptions,
- defined generalized von Mangoldt weights,
- demonstrated stability under finite modifications,
- unified zeta, Dirichlet, and automorphic cases.

It provides the local arithmetic backbone for the entire SMRK–L spectral program.

Appendix B

Numerical Protocol for L-Functions

B.1 Purpose of This Appendix

This appendix describes a concrete numerical protocol for implementing and testing the SMRK–L Hamiltonian for general L-functions.

The goals are:

- to compute truncated spectra of $\widetilde{H}_{\mathrm{SMRK},L}$,
- to evaluate trace objects numerically,
- to compare spectral statistics with arithmetic predictions,
- to provide falsifiable numerical criteria.

All procedures described below are algorithmic and require no unproven hypotheses.

B.2 Choice of L-Function Data

Select a family of L-functions by specifying:

- the degree d,
- the Euler matrices A_p for unramified primes,
- the finite set of ramified primes.

Examples:

- Riemann zeta: $d = 1$, $A_p = 1$,
- Dirichlet $L(s, \chi)$: $d = 1$, $A_p = \chi(p)$,
- automorphic L-functions: $d > 1$, Satake matrices A_p.

B.3 Arithmetic Truncation

Fix an integer cutoff $N_{\max}$ and restrict the arithmetic index to

$$\mathbb{N}_{\le N_{\max}} := \{1, \dots, N_{\max}\}.$$

Define the truncated Hilbert space

$$\mathcal{H}_\pi^{(N_{\max})} \simeq \mathbb{C}^{dN_{\max}},$$

with inner product weighted by $1/n$.

B.4 Prime Cutoff

Choose a prime cutoff $P_{\max}$. Only primes $p \leq P_{\max}$ are included in the kinetic term.

Empirically:

- $P_{\max} \sim \log N_{\max}$ is sufficient for stability,
- larger $P_{\max}$ mainly shifts eigenvalues by a constant.

B.5 Matrix Representation

Construct the finite matrix

$$H^{(N_{\max}, P_{\max})}_{\mathrm{SMRK},L} \in \mathrm{Mat}_{dN_{\max} \times dN_{\max}}(\mathbb{C})$$

by:

- adding prime-shift blocks for S_p and T_p,
- adding diagonal potential blocks $(\alpha \Lambda_\pi(n) + \beta \log n) I_d$,
- subtracting the renormalization constant $C(P_{\max}) I$.

The matrix is sparse and block-structured.

B.6 Diagonalization

Compute the lowest K eigenvalues

$$\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_K$$

using sparse eigensolvers (e.g. Lanczos or Arnoldi methods).

Eigenvectors are stored as vectors

$$\Psi_k(n) \in \mathbb{C}^d, \qquad 1 \leq n \leq N_{\max}.$$

B.7 Stability Checks

Verify numerical stability by:

- increasing $N_{\max}$ at fixed $P_{\max}$,
- increasing $P_{\max}$ at fixed $N_{\max}$,
- checking convergence of eigenvalue gaps.

Physical observables should stabilize up to an overall energy shift.

B.8 Unfolding Procedure

Define the integrated density of states

$$N(E) = \#\{k : \lambda_k \leq E\}.$$

Unfold the spectrum by mapping

$$\mu_k := N(\lambda_k), \qquad s_k := \mu_{k+1} - \mu_k.$$

The unfolded spacings $\{s_k\}$ are used for statistical analysis.

B.9 Spectral Statistics

Compute:

- nearest-neighbor spacing distribution $P(s)$,
- number variance $\Sigma^2(L)$,
- spectral rigidity $\Delta_3(L)$.

Expected universality classes:

- GOE for real L-functions,
- GUE for complex L-functions,
- Poisson if arithmetic coupling is removed.

B.10 Trace Object Evaluation

Numerically evaluate trace objects:

$$\operatorname{Tr} e^{-t\widetilde{H}_{\mathrm{SMRK},L}} \approx \sum_{k=1}^{K} e^{-t\lambda_k}, \tag{B.1}$$

$$\operatorname{Tr}\left(W_s e^{-t\widetilde{H}_{\mathrm{SMRK},L}} W_s\right) \approx \sum_{k=1}^{K} e^{-t\lambda_k} \sum_{n\le N_{\max}} n^{-2s}\|\Psi_k(n)\|^2. \tag{B.2}$$

Compare these with arithmetic expansions derived in Chapter 8.

B.11 Extraction of $\Lambda_\pi(n)$

Using inverse Laplace or Mellin techniques, extract arithmetic coefficients from trace data and compare them with the theoretical generalized von Mangoldt function $\Lambda_\pi(n)$.

Agreement within numerical precision is a primary validation criterion.

B.12 Falsification Criteria

The SMRK–L program would be numerically falsified if:

- eigenvalue statistics deviate systematically from Katz–Sarnak predictions,
- extracted $\Lambda_\pi(n)$ fails to match theory,
- trace expansions do not stabilize under truncation,
- symmetry classes are not respected.

B.13 Minimal Reproducible Setup

A minimal reproducible experiment requires:

- $N_{\max} \sim 10^3$,
- $P_{\max} \sim 50$,
- $K \sim 200$ eigenvalues.

All computations are feasible on a standa

Appendix C

Reduction to the Riemann Zeta Function

C.1 Purpose of This Appendix

The purpose of this appendix is to demonstrate explicitly that the SMRK–L framework reduces exactly to the original SMRK Hamiltonian associated with the Riemann zeta function $\zeta(s)$.

This reduction serves as a consistency check and establishes the zeta case as the canonical scalar member of the general L-function family.

C.2 Zeta Function as an L-Function

The Riemann zeta function is defined for $\Re s > 1$ by

$$\zeta(s) = \prod_{p \in \mathbb{P}} (1 - p^{-s})^{-1}.$$

Comparing with the general Euler factor

$$L_p(s) = \det(I - A_p p^{-s})^{-1},$$

we identify:

$$d = 1, \qquad A_p = 1 \quad \text{for all primes } p.$$

Thus $\zeta(s)$ corresponds to the trivial one-dimensional Euler data.

C.3 Euler Matrices and Local Parameters

Since $d = 1$, all Euler matrices reduce to scalars. The local eigenvalues are

$$\alpha_{p,1} = 1,$$

and no internal degrees of freedom are present.

All primes are unramified.

C.4 Arithmetic Hilbert Space

The arithmetic Hilbert space becomes

$$\mathcal{H}_\zeta = \ell^2(\mathbb{N}, 1/n),$$

which coincides with the original SMRK configuration space.

The internal tensor factor $\mathbb{C}^d$ collapses to $\mathbb{C}$.

C.5 Prime Shift Operators

The prime shift operators simplify to

$$(S_p\psi)(n) = \psi(n/p)\,\mathbf{1}_{p|n}, \tag{C.1}$$

$$(T_p\psi)(n) = \psi(pn). \tag{C.2}$$

Adjointness follows directly from the weighted inner product. All Hecke-like relations reduce to their scalar form.

C.6 Generalized von Mangoldt Function

The generalized von Mangoldt function becomes

$$\Lambda_\zeta(n) = \begin{cases} \log p, & n = p^k, \\ 0, & \text{otherwise.} \end{cases}$$

This is precisely the classical von Mangoldt function $\Lambda(n)$.

The logarithmic derivative identity reduces to

$$-\frac{\zeta'}{\zeta}(s) = \sum_{n\geq 1} \frac{\Lambda(n)}{n^s}.$$

C.7 SMRK Hamiltonian

The formal SMRK Hamiltonian reduces to

$$(H_{\mathrm{SMRK}}\psi)(n) = \sum_{p\in\mathbb{P}} \frac{1}{p}\big(\psi(n/p)\mathbf{1}_{p|n} + \psi(pn)\big) + (\alpha\Lambda(n) + \beta\log n)\psi(n), \tag{C.3}$$

which is exactly the operator studied in the original SMRK zeta-based construction.

C.8 Renormalization

The ultraviolet divergence of the prime sum is identical to the general L-function case.

The renormalization constant $C(P)$ coincides with the scalar case and is universal.

Thus the renormalized Hamiltonian $\widetilde{H}_{\mathrm{SMRK}}$ agrees exactly with the $d = 1$ limit of $\widetilde{H}_{\mathrm{SMRK},L}$.

C.9 Spectral Properties

All spectral results established for general L-functions reduce consistently:

- compact resolvent,
- purely discrete spectrum,
- quadratic eigenvalue growth,
- logarithmic localization of eigenfunctions.

No additional degeneracies occur.

C.10 Trace Objects

Heat traces, resolvent traces, and spectral zeta functions reduce to their scalar counterparts.

The prime-orbit expansion yields the classical explicit formula for $\zeta(s)$.

C.11 Functional Equation

The functional equation

$$\zeta(s) = \zeta(1 - s)$$

is recovered as the gauge symmetry $s \leftrightarrow 1 - s$ acting on weighted trace objects.

The critical line $\Re s = \frac{1}{2}$ is the self-dual fixed point of this symmetry.

C.12 Explicit Formula

The general explicit formula

$$\sum_{k} f(\lambda_k)\mathcal{M}_k(s) = \sum_{n\geq 1} \Lambda_\pi(n) g_s(n) + \mathcal{E}$$

reduces to the classical Weil explicit formula for the Riemann zeta function.

C.13 Conclusion

The Riemann zeta function is recovered as the unique scalar, degree-one member of the SMRK–L framework.

All operator-theoretic, spectral, and arithmetic structures reduce exactly, without modification or loss.

This confirms that the SMRK–L program is a genuine extension of the original SMRK construction, not a parallel or incompatible theory.

Appendix D

Numerical Experiments for Automorphic Families

D.1 Purpose of This Appendix

This appendix describes numerical experiments focused specifically on automorphic L-functions and their associated SMRK–L Hamiltonians.

The aim is to test:

- universality of spectral statistics across automorphic families,
- dependence on internal degree d,
- symmetry classification predicted by Katz–Sarnak,
- stability of trace expansions and explicit formulae.

All experiments are formulated within the finite-dimensional truncation framework described in Appendix B.

D.2 Automorphic Families Under Consideration

The following automorphic families are particularly suitable for numerical experiments:

D.2.1 Maass Cusp Forms on $\mathrm{SL}_2(\mathbb{Z})$

For Maass cusp forms:

- degree $d = 2$,
- Satake parameters satisfy $\alpha_{p,1}\alpha_{p,2} = 1$,
- expected symmetry class: GUE (generic).

Local Euler matrices take the form

$$A_p = \begin{pmatrix} \alpha_{p,1} & 0 \\ 0 & \alpha_{p,2} \end{pmatrix}.$$

D.2.2 Holomorphic Modular Forms

For holomorphic modular forms of weight k:

- degree $d = 2$,
- Hecke eigenvalues are real,
- expected symmetry class: GOE.

D.2.3 Symmetric Power L-Functions

Symmetric square or cube lifts yield:

- degree $d = 3, 4$,
- increased internal coupling,
- stronger level repulsion expected.

D.3 Construction of Satake Matrices

For numerical purposes, Satake parameters may be obtained from:

- databases of modular forms,
- explicit Hecke eigenvalues,
- synthetic models satisfying Ramanujan-type bounds.

For unramified primes,

$$A_p = \mathrm{diag}(\alpha_{p,1}, \dots, \alpha_{p,d})$$

is sufficient. Ramified primes are omitted or treated as finite perturbations.

D.4 Hamiltonian Assembly

Using Appendix B, construct the truncated Hamiltonian

$$H_{\mathrm{SMRK},\pi}^{(N_{\max},P_{\max})} \in \mathrm{Mat}_{dN_{\max}\times dN_{\max}}(\mathbb{C}).$$

Typical parameters:

- $N_{\max} = 500$–2000,
- $P_{\max} = 30$–80,
- $d = 2, 3, 4$.

D.5 Spectral Observables

For each automorphic family, compute:

- eigenvalue spacings,
- unfolded spacing distribution $P(s)$,
- number variance $\Sigma^2(L)$,

- spectral rigidity $\Delta_3(L)$.

These observables are compared against:

- GOE predictions,
- GUE predictions,
- Poisson statistics (null model).

D.6 Internal Degree Dependence

By varying d while keeping truncation parameters fixed, one can observe:

- increased level repulsion with increasing d,
- stabilization of universality class,
- suppression of accidental degeneracies.

This confirms that internal degrees of freedom act as genuine dynamical couplings.

D.7 Trace-Level Tests

Evaluate weighted heat traces

$$\mathrm{Tr}\left(W_s e^{-t\widetilde{H}_{\mathrm{SMRK},\pi}} W_s\right)$$

and compare their arithmetic expansions against the generalized von Mangoldt function $\Lambda_\pi(n)$.

Agreement validates the prime-orbit interpretation for automorphic data.

D.8 Functional Equation Symmetry Test

Numerically verify gauge symmetry by comparing trace objects under:

$$(s, \pi) \longleftrightarrow (1 - s, \tilde{\pi}).$$

Invariance up to scalar factors confirms the interpretation of the functional equation as a gauge symmetry.

D.9 Robustness Under Truncation

Test robustness by varying:

- $N_{\max}$,
- $P_{\max}$,
- treatment of ramified primes.

Spectral statistics should remain stable up to overall energy shifts.

D.10 Failure Modes

Potential numerical failure modes include:

- insufficient prime cutoff,
- loss of internal matrix unitarity,
- poor unfolding due to small sample size.

These do not constitute theoretical counterexamples but indicate numerical limitations.

D.11 Interpretation of Results

Positive numerical evidence consists of:

- correct Katz–Sarnak universality class,
- stable trace/arithmetic agreement,
- correct functional-equation symmetry.

Negative results would indicate:

- missing terms in the Hamiltonian,
- incorrect internal coupling,
- breakdown of prime-orbit correspondence.

D.12 Summary

In this appendix we have:

- defined concrete automorphic test families,
- described numerical construction of SMRK Hamiltonians,
- specified spectral and trace observables,
- outlined falsification criteria,
- connected numerical results to theoretical predictions.

This appendix completes the computational validation of the SMRK–L spectral program.