Self-Adjoint Prime-Shift Operators

Quadratic Forms, Regularization, and Arithmetic Dynamics

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Contents

Abstract

We construct and analyze a class of prime-shift operators acting on $\ell^2(\mathbb{N})$ that encode multiplicative arithmetic dynamics through prime-induced transitions. The operators are defined as weighted adjacency operators on an arithmetic graph with edges $n \leftrightarrow pn$, equipped with explicit regularizations to control divergence of the prime sum.

The main objective of this work is to establish a rigorous self-adjoint framework for such operators. To this end, we introduce associated quadratic forms, study their boundedness properties under suitable weight choices, and prove the existence of closed, densely defined forms. This allows the construction of canonical self-adjoint realizations via Friedrichs extension.

We show that the resulting operators admit well-defined resolvents for $\operatorname{Im} z \neq 0$ and possess stable spectral properties under regularization limits. While no explicit identification with zeta or L-functions is attempted here, the analysis provides a mathematically sound operator-theoretic foundation for subsequent trace and spectral investigations.

The framework developed in this paper isolates the analytic core of prime-shift dynamics and prepares the ground for trace formulas and spectral identification results pursued in subsequent work.

1 Introduction and Strategy

The purpose of this paper is to place prime-shift operators into a rigorous self-adjoint framework suitable for spectral analysis. While operators encoding multiplicative arithmetic dynamics have appeared in various formal or heuristic contexts, their analytic foundations are often left implicit. In particular, formal symmetry alone is insufficient for spectral interpretation.

Our objective is therefore precise: to construct a well-defined self-adjoint operator associated with prime-shift dynamics, obtained as a canonical extension of a densely defined symmetric operator on $\ell^2(\mathbb{N})$.

1.1 Why Self-Adjointness Matters

Spectral statements about arithmetic operators require self-adjointness for their validity. Without it, eigenvalues need not be real, spectral measures may fail to exist, and trace-based constructions become ill-defined.

In the context of arithmetic dynamics, this issue is particularly acute. The operators of interest involve infinite sums over primes, with weights that are typically not absolutely summable. As a result, naive operator definitions may fail to converge or admit multiple inequivalent self-adjoint extensions.

Any attempt to connect arithmetic operators to spectral invariants must therefore begin with a careful analysis of domains, closures, and extensions.

1.2 Formal Symmetry Is Not Enough

Prime-shift operators defined by pairing forward and backward transitions $n \leftrightarrow pn$ are formally symmetric on natural dense domains such as $c_{00}(\mathbb{N})$. However, formal symmetry does not imply self-adjointness, nor does it guarantee uniqueness of extension.

In infinite-dimensional settings, symmetric operators may admit infinitely many self-adjoint extensions, each corresponding to different boundary conditions at infinity. Distinguishing a canonical choice requires additional analytic input.

1.3 Strategy of the Paper

The strategy adopted in this work is based on quadratic form methods. Rather than attempting to define the operator directly, we associate to it a quadratic form encoding prime-shift interactions.

This approach has several advantages:

- it avoids pointwise convergence issues,
- it allows direct control of boundedness and lower semi-boundedness,
- it provides access to canonical self-adjoint extensions via standard functional-analytic theorems.

Once a closed, densely defined quadratic form is established, the existence of a unique self-adjoint operator associated with it follows from the Friedrichs extension theorem.

1.4 Regularization and Limits

Throughout the paper, all constructions are carried out within explicitly regularized settings. Prime sums are truncated or damped, and operators are defined initially on finite or weighted subspaces.

The role of regularization is not to approximate an infinite operator naively, but to provide a controlled family of operators whose analytic properties may be studied uniformly. Limits, when taken, are understood in appropriate operator or form senses.

1.5 Scope and Limitations

This paper deliberately restricts its scope. We do not attempt to derive trace formulas, to identify spectral data with zeta or L-functions, or to make any claims regarding the Riemann Hypothesis.

Instead, we isolate the analytic core of prime-shift dynamics and establish a solid operator-theoretic foundation upon which such investigations may later be built.

1.6 Organization of the Paper

The paper is organized as follows. In Section 2 we define the arithmetic Hilbert space and discuss suitable weighting schemes. Section 3 introduces regularized prime-shift operators. Section 4 develops the associated quadratic forms and establishes their basic properties. Section 5 constructs self-adjoint extensions via the Friedrichs method. Section 6 discusses resolvent existence and basic spectral properties. The final section summarizes the results and outlines directions for further work.

2 Arithmetic Hilbert Space

In this section we fix the Hilbert space setting in which prime-shift operators will be analyzed. The choice of space and inner product is not merely technical; it determines the admissible domains, boundedness properties, and the viability of self-adjoint realizations.

2.1 The Canonical Arithmetic Hilbert Space

We begin with the standard Hilbert space

$$\ell^2(\mathbb{N}) = \left\{ \psi : \mathbb{N} \to \mathbb{C} \;\middle|\; \sum_{n=1}^{\infty} |\psi(n)|^2 < \infty \right\},$$

equipped with the inner product

$$\langle \psi, \phi \rangle = \sum_{n=1}^{\infty} \overline{\psi(n)}\, \phi(n).$$

This space provides the simplest setting in which arithmetic states may be represented as square-summable amplitudes indexed by natural numbers. The dense subspace $c_{00}(\mathbb{N})$ of finitely supported functions serves as a natural initial domain for all operators considered in this work.

2.2 Weighted Hilbert Spaces

For certain choices of prime weights and regularizations, it is advantageous to consider weighted Hilbert spaces of the form

$$\ell^2(\mathbb{N}, \mu) = \left\{ \psi : \mathbb{N} \to \mathbb{C} \;\middle|\; \sum_{n=1}^{\infty} |\psi(n)|^2 \mu(n) < \infty \right\},$$

where $\mu : \mathbb{N} \to (0, \infty)$ is a weight function.

The corresponding inner product is

$$\langle \psi, \phi \rangle_\mu = \sum_{n=1}^{\infty} \overline{\psi(n)}\, \phi(n)\, \mu(n).$$

Weighted spaces allow partial compensation for growth induced by prime multiplication. Typical choices include polynomial weights $\mu(n) = n^{-\alpha}$ or logarithmic weights $\mu(n) = (\log n)^{-\beta}$, with parameters chosen to balance forward and backward prime shifts.

2.3 Unitary Equivalence of Weighted Spaces

For strictly positive weights μ, the space $\ell^2(\mathbb{N}, \mu)$ is unitarily equivalent to $\ell^2(\mathbb{N})$ via the transformation

$$(U_\mu \psi)(n) = \mu(n)^{1/2}\, \psi(n).$$

Under this transformation, operators acting on weighted spaces may be conjugated to operators on $\ell^2(\mathbb{N})$ with modified coefficients. This observation allows us to work primarily in the unweighted space while retaining the analytic flexibility provided by weights.

2.4 Domain Considerations

All operators in this paper are initially defined on $c_{00}(\mathbb{N})$, which is dense in both $\ell^2(\mathbb{N})$ and $\ell^2(\mathbb{N}, \mu)$. This choice ensures that algebraic manipulations are unambiguous and that quadratic forms are well defined.

Closures and extensions of operators are taken with respect to the chosen Hilbert space norm. When weighted spaces are employed, this is stated explicitly.

2.5 Growth Under Prime Shifts

A critical issue is the behavior of norms under prime multiplication. For the forward shift $(T_p^+\psi)(n) = \psi(pn)$, one has formally

$$\|T_p^+\psi\|^2 = \sum_{n=1}^{\infty} |\psi(pn)|^2 = \sum_{m\equiv 0 \pmod p} |\psi(m)|^2,$$

which is bounded by $\|\psi\|^2$ but does not decay with p.

Backward shifts $(T_p^-\psi)(n) = \mathbf{1}_{p|n}\psi(n/p)$ exhibit analogous behavior. This lack of decay necessitates either damping of prime weights or the use of weighted spaces in order to obtain closed quadratic forms.

2.6 Choice Adopted in This Work

In what follows, we formulate all constructions initially in $\ell^2(\mathbb{N})$ with explicit regularization of prime weights. Weighted spaces are introduced only when analytically convenient and are always reduced to the unweighted case by unitary equivalence.

This approach isolates the essential analytic features of prime-shift dynamics without committing prematurely to a specific weighting scheme.

2.7 Summary

The arithmetic Hilbert space provides the foundational setting for prime-shift operators. Careful attention to domains, weights, and norm behavior under prime shifts is essential for the construction of well-defined quadratic forms and self-adjoint operators.

In the next section we introduce explicit regularizations of prime-shift operators and define the operator families to which quadratic form methods will be applied.

3 Prime-Shift Operators and Regularization

In this section we introduce explicit regularizations of prime-shift operators. The purpose of regularization is twofold: to ensure that all expressions are well-defined on natural dense domains, and to provide operator families whose analytic properties can be studied uniformly as regularization parameters vary.

3.1 Formal Prime-Shift Operator

Formally, prime-shift dynamics is generated by the expression

$$(H\psi)(n) = \sum_{p\in\mathbb{P}} w(p)\big(\psi(pn) + \mathbf{1}_{p|n}\psi(n/p)\big),$$

where $w(p)$ is a real-valued weight function. For typical arithmetic choices, such as $w(p) = p^{-1}$, the sum diverges and the expression does not define a bounded or densely defined operator on $\ell^2(\mathbb{N})$.

This necessitates the introduction of explicit regularization.

3.2 Prime Cutoff Regularization

The simplest regularization is a hard cutoff on the prime sum. For $P < \infty$, we define

$$(H^{(P)}\psi)(n) = \sum_{\substack{p\in\mathbb{P}\\ p\le P}} w(p)\big(\psi(pn) + \mathbf{1}_{p|n}\psi(n/p)\big).$$

For fixed P, the operator $H^{(P)}$ is well-defined on $c_{00}(\mathbb{N})$ and extends to a densely defined symmetric operator on $\ell^2(\mathbb{N})$. The dependence on P provides a controlled family of operators whose behavior may be analyzed as $P \to \infty$.

3.3 Soft Cutoff and Damped Weights

To obtain smoother control over large primes, we also consider damped weight functions of the form

$$w_{P,\alpha}(p) = \frac{e^{-p/P}}{p^{\alpha}}, \qquad \alpha > 0.$$

The corresponding operator is defined by

$$(H^{(P,\alpha)}\psi)(n) = \sum_{p\in\mathbb{P}} w_{P,\alpha}(p)\big(\psi(pn) + \mathbf{1}_{p|n}\psi(n/p)\big).$$

For each fixed (P,α), the sum converges absolutely for $\psi \in c_{00}(\mathbb{N})$. Moreover, for $\alpha > 1$ the prime sum converges uniformly in P, yielding improved boundedness properties.

3.4 Regularized Operator Families

In what follows, we work with a family of operators

$$\{H^{(P,\alpha)}\}_{P>0,\ \alpha>\alpha_0},$$

where $\alpha_0 \geq 0$ is chosen to ensure the desired analytic properties of the associated quadratic forms.

This family interpolates between strongly regularized operators and the formally unregularized prime-shift dynamics. Analytic statements are made uniformly with respect to (P, α) wherever possible.

3.5 Symmetry on the Core Domain

For each choice of regularization parameters, the operator $H^{(P,\alpha)}$ is symmetric on $c_{00}(\mathbb{N})$. Indeed, for $\psi, \phi \in c_{00}(\mathbb{N})$, one has

$$\langle \psi, H^{(P,\alpha)}\phi \rangle = \langle H^{(P,\alpha)}\psi, \phi \rangle,$$

as each term corresponding to a transition $n \leftrightarrow pn$ appears in conjugate pairs.

This formal symmetry provides the starting point for the quadratic form analysis developed in the next section.

3.6 Inclusion of Local Potentials

More general operator families may include an additional diagonal term

$$(V\psi)(n) = V(n)\psi(n),$$

with $V : \mathbb{N} \to \mathbb{R}$. Such terms may be used to stabilize the spectrum or to incorporate slowly varying arithmetic contributions.

Throughout this paper, the inclusion of $V(n)$ is optional and does not affect the core regularization arguments.

3.7 Summary

Explicit regularization is essential for placing prime-shift dynamics on firm analytic footing. The families $H^{(P)}$ and $H^{(P,\alpha)}$ provide concrete, well-defined operators suitable for quadratic form methods.

In the next section we associate quadratic forms to these operators and analyze their boundedness and closability properties, paving the way toward canonical self-adjoint realizations.

4 Quadratic Forms

In this section we associate quadratic forms to the regularized prime-shift operators introduced previously. The quadratic form approach provides a robust framework for controlling unbounded operators and for constructing canonical self-adjoint realizations.

4.1 Definition of the Quadratic Form

Let $H^{(P,\alpha)}$ denote one of the regularized prime-shift operators defined in Section 3. We define the associated quadratic form

$$Q_{P,\alpha}(\psi) = \langle \psi, H^{(P,\alpha)}\psi \rangle, \qquad \psi \in c_{00}(\mathbb{N}).$$

Explicitly,

$$Q_{P,\alpha}(\psi) = \sum_{p\in\mathbb{P}} w_{P,\alpha}(p) \sum_{n\in\mathbb{N}} \overline{\psi(n)}\big(\psi(pn) + \mathbf{1}_{p|n}\psi(n/p)\big).$$

By symmetry of the prime-shift terms, $Q_{P,\alpha}(\psi)$ is real-valued for all $\psi \in c_{00}(\mathbb{N})$.

4.2 Symmetrized Representation

Using the change of variables $m = pn$ in the backward term, the quadratic form may be written in the symmetrized form

$$Q_{P,\alpha}(\psi) = 2\sum_{p\in\mathbb{P}} w_{P,\alpha}(p) \sum_{n\in\mathbb{N}} \operatorname{Re}\big(\overline{\psi(n)}\,\psi(pn)\big).$$

This representation makes explicit the pairing of forward and backward prime transitions and will be used to establish boundedness estimates.

4.3 Form Boundedness

For $\psi \in c_{00}(\mathbb{N})$, the Cauchy–Schwarz inequality yields

$$|\overline{\psi(n)}\,\psi(pn)| \le \frac{1}{2}\big(|\psi(n)|^2 + |\psi(pn)|^2\big).$$

Substituting into the symmetrized form gives

$$|Q_{P,\alpha}(\psi)| \le \sum_{p\in\mathbb{P}} w_{P,\alpha}(p) \sum_{n\in\mathbb{N}} \big(|\psi(n)|^2 + |\psi(pn)|^2\big).$$

Interchanging sums and using the fact that $\sum_n |\psi(pn)|^2 \le \sum_n |\psi(n)|^2$, we obtain

$$|Q_{P,\alpha}(\psi)| \le 2\Big(\sum_{p\in\mathbb{P}} w_{P,\alpha}(p)\Big)\|\psi\|^2.$$

Thus, whenever $\sum_p w_{P,\alpha}(p) < \infty$, the quadratic form is bounded with respect to the $\ell^2(\mathbb{N})$ norm.

4.4 Lower Semiboundedness

If the weights $w_{P,\alpha}(p)$ are nonnegative, the quadratic form satisfies

$$Q_{P,\alpha}(\psi) \geq -C_{P,\alpha}\|\psi\|^2,$$

for some finite constant $C_{P,\alpha}$. In particular, for exponentially damped weights or for $\alpha > 1$, the form is bounded below.

Lower semiboundedness is the key hypothesis required for the application of Friedrichs' extension theorem.

4.5 Closability of the Form

Since $Q_{P,\alpha}$ is densely defined and bounded with respect to the Hilbert space norm, it is closable. We denote its closure by $\overline{Q}_{P,\alpha}$, with form domain $\mathcal{D}(\overline{Q}_{P,\alpha})$.

The closure may be described explicitly as the completion of $c_{00}(\mathbb{N})$ with respect to the form norm

$$\|\psi\|_Q^2 = \|\psi\|^2 + \overline{Q}_{P,\alpha}(\psi).$$

4.6 Inclusion of Diagonal Potentials

If a real-valued diagonal potential $V(n)$ is included, the quadratic form becomes

$$Q_{P,\alpha}^V(\psi) = Q_{P,\alpha}(\psi) + \sum_{n\in\mathbb{N}} V(n)\,|\psi(n)|^2.$$

Provided $V(n)$ is bounded below or form-bounded with respect to $Q_{P,\alpha}$, all conclusions above remain valid. This allows incorporation of logarithmic or von Mangoldt–type terms as relatively bounded perturbations.

4.7 Relation to the SMRK Hamiltonian

The SMRK Hamiltonian may be written schematically as

$$H_{\mathrm{SMRK}} = H_{\text{prime-shift}} + V_{\mathrm{SMRK}}(n),$$

where $H_{\text{prime-shift}}$ denotes the symmetric core analyzed here and $V_{\mathrm{SMRK}}(n)$ is a real-valued diagonal arithmetic potential.

The present quadratic form analysis establishes a rigorous foundation for the prime-shift component of H_{SMRK}. Diagonal arithmetic terms may be treated as relatively form-bounded perturbations and incorporated at later stages without affecting self-adjointness of the core operator.

4.8 Summary

The regularized prime-shift operators admit densely defined, lower semibounded, closable quadratic forms. These forms provide the natural analytic starting point for constructing canonical self-adjoint realizations.

In the next section we use Friedrichs' extension theorem to associate a unique self-adjoint operator to the closed quadratic form $\overline{Q}_{P,\alpha}$.

5 Self-Adjoint Extensions

In this section we construct self-adjoint realizations of the regularized prime-shift operators using quadratic form methods. The key result is the existence of a canonical self-adjoint operator associated with the closed, lower semibounded quadratic forms introduced in the previous section.

5.1 From Quadratic Forms to Operators

Let $\overline{Q}_{P,\alpha}$ denote the closure of the quadratic form $Q_{P,\alpha}$ defined on $c_{00}(\mathbb{N})$. By construction, $\overline{Q}_{P,\alpha}$ is densely defined, closed, and lower semibounded on $\ell^2(\mathbb{N})$.

By the representation theorem for closed quadratic forms, there exists a unique self-adjoint operator $A_{P,\alpha}$ such that

$$\overline{Q}_{P,\alpha}(\psi,\phi) = \langle \psi, A_{P,\alpha}\phi\rangle, \qquad \phi \in \mathcal{D}(A_{P,\alpha}), \quad \psi \in \mathcal{D}(\overline{Q}_{P,\alpha}).$$

The operator $A_{P,\alpha}$ is called the operator associated with the quadratic form $\overline{Q}_{P,\alpha}$.

5.2 Friedrichs Extension

The operator $A_{P,\alpha}$ may be viewed as the Friedrichs extension of the symmetric operator $H^{(P,\alpha)}$ initially defined on $c_{00}(\mathbb{N})$. Among all self-adjoint extensions of $H^{(P,\alpha)}$, the Friedrichs extension is distinguished by minimality of the quadratic form and preservation of lower semiboundedness.

This construction provides a canonical choice of self-adjoint realization, independent of auxiliary boundary conditions at infinity.

5.3 Domain of the Self-Adjoint Operator

The domain $\mathcal{D}(A_{P,\alpha})$ consists of those $\phi \in \mathcal{D}(\overline{Q}_{P,\alpha})$ for which there exists $\chi \in \ell^2(\mathbb{N})$ such that

$$\overline{Q}_{P,\alpha}(\psi,\phi) = \langle \psi, \chi\rangle \quad \text{for all } \psi \in \mathcal{D}(\overline{Q}_{P,\alpha}).$$

In this case, one sets $A_{P,\alpha}\phi = \chi$. While an explicit pointwise characterization of $\mathcal{D}(A_{P,\alpha})$ is generally difficult, this weak formulation suffices for spectral and resolvent analysis.

5.4 Inclusion of Diagonal Potentials

If a diagonal potential $V(n)$ is included, and if V is relatively form-bounded with respect to $\overline{Q}_{P,\alpha}$ with relative bound strictly less than one, then the perturbed quadratic form

$$\overline{Q}^{V}_{P,\alpha} = \overline{Q}_{P,\alpha} + \sum_n V(n)|\psi(n)|^2$$

remains closed and lower semibounded.

In this case, the associated operator $A_{P,\alpha}^{V}$ is self-adjoint and represents a canonical perturbation of the prime-shift operator. This framework covers a wide class of arithmetic diagonal terms, including those appearing in SMRK-type Hamiltonians.

5.5 Uniqueness and Canonical Nature

The Friedrichs extension yields a unique self-adjoint operator associated with the quadratic form $\overline{Q}_{P,\alpha}$. No further choices are required, and the construction is stable under perturbations that preserve form boundedness.

This canonical nature is essential for later spectral and trace-based investigations, as it eliminates ambiguities arising from nonunique extensions.

5.6 Summary

The regularized prime-shift operators admit canonical self-adjoint realizations obtained via Friedrichs extension of their associated quadratic forms. This establishes a rigorous operator-theoretic foundation for arithmetic prime-shift dynamics.

In the next section we analyze the resolvent of the self-adjoint operator $A_{P,\alpha}$ and discuss basic spectral properties relevant for subsequent trace and identification results.

6 Resolvent and Spectral Properties

In this section we analyze the resolvent of the self-adjoint operators constructed in the previous section and record basic spectral properties relevant for later trace and identification arguments. No explicit spectral decomposition is attempted; the focus is on existence, stability, and analytic control.

6.1 Existence of the Resolvent

Let $A_{P,\alpha}$ denote the self-adjoint operator obtained as the Friedrichs extension associated with the closed quadratic form $\overline{Q}_{P,\alpha}$. By self-adjointness, the resolvent

$$R_{P,\alpha}(z) = (A_{P,\alpha} - z)^{-1}$$

exists as a bounded operator on $\ell^2(\mathbb{N})$ for all $z \in \mathbb{C}$ with $\operatorname{Im} z \neq 0$.

Moreover, the resolvent satisfies the standard estimate

$$\|R_{P,\alpha}(z)\| \leq \frac{1}{|\operatorname{Im} z|},$$

which is uniform in the regularization parameters (P, α) as long as the lower bound of the quadratic form is controlled uniformly.

6.2 Lower Spectral Bound

Since the quadratic form $\overline{Q}_{P,\alpha}$ is lower semibounded, the spectrum of $A_{P,\alpha}$ is bounded from below. That is, there exists a constant $C_{P,\alpha} \in \mathbb{R}$ such that

$$\sigma(A_{P,\alpha}) \subset [C_{P,\alpha}, \infty).$$

In particular, after a constant energy shift if necessary, one may assume that $A_{P,\alpha}$ is nonnegative. This normalization is often convenient for trace and determinant constructions.

6.3 Stability Under Regularization

The family of operators $\{A_{P,\alpha}\}$ depends continuously on the regularization parameters in the strong resolvent sense. Specifically, if $(P_n, \alpha_n) \to (P, \alpha)$, then

$$(A_{P_n,\alpha_n} - z)^{-1} \xrightarrow{\mathrm{s}} (A_{P,\alpha} - z)^{-1}$$

for all z with $\operatorname{Im} z \neq 0$.

This stability follows from the corresponding convergence of the quadratic forms $\overline{Q}_{P_n,\alpha_n}$ to $\overline{Q}_{P,\alpha}$ and standard results on convergence of self-adjoint operators generated by forms.

6.4 Perturbations by Diagonal Potentials

If a diagonal potential $V(n)$ is included and is relatively form-bounded with respect to $\overline{Q}_{P,\alpha}$ with relative bound strictly less than one, then the perturbed operator $A^V_{P,\alpha}$ remains self-adjoint and lower semibounded.

In this case, the resolvent identity

$$(A^V_{P,\alpha} - z)^{-1} = (A_{P,\alpha} - z)^{-1} - (A_{P,\alpha} - z)^{-1} V (A^V_{P,\alpha} - z)^{-1}$$

holds in the sense of bounded operators. This relation provides the starting point for resolvent expansions used in trace formula derivations.

6.5 Spectral Measures

By the spectral theorem, the self-adjoint operator $A_{P,\alpha}$ admits a projection-valued spectral measure $E_{P,\alpha}(\lambda)$ such that

$$A_{P,\alpha} = \int_{\mathbb{R}} \lambda \, dE_{P,\alpha}(\lambda).$$

For any $\psi \in \ell^2(\mathbb{N})$, the scalar measure

$$\mu_\psi(\lambda) = \langle \psi, E_{P,\alpha}(\lambda)\psi \rangle$$

encodes the spectral distribution of $A_{P,\alpha}$ with respect to ψ.

While explicit expressions for these measures are not available at this stage, their existence is sufficient for defining trace-like objects via functional calculus in subsequent work.

6.6 Absence of Pathologies

The constructions above exclude several potential pathologies. In particular, the operators $A_{P,\alpha}$:

- have real spectrum,
- admit bounded resolvents off the real axis,
- depend stably on regularization parameters,
- allow controlled diagonal perturbations.

These properties are essential prerequisites for any attempt to derive trace formulas or to identify spectral data with arithmetic invariants.

6.7 Summary

The self-adjoint prime-shift operators constructed via quadratic form methods admit well-defined resolvents and stable spectral measures. Their analytic behavior is robust under regularization and under admissible perturbations.

With these results, the analytic foundations of prime-shift dynamics are firmly established. The subsequent development of trace formulas and spectral identification may therefore proceed on a rigorous operator-theoretic basis.

7 Discussion and Outlook

This paper has established a rigorous self-adjoint framework for a class of regularized prime-shift operators encoding multiplicative arithmetic dynamics. The construction proceeds through quadratic form methods and yields canonical self-adjoint realizations via Friedrichs extension.

The main objective has been analytic rather than arithmetic: to isolate and stabilize the operator-theoretic core of prime-shift dynamics in a setting where spectral and resolvent-based tools apply without ambiguity.

7.1 Summary of Results

The principal results obtained in this work may be summarized as follows:

- explicit regularized families of prime-shift operators were defined on $\ell^2(\mathbb{N})$,
- associated quadratic forms were shown to be densely defined, lower semibounded, and closable,
- canonical self-adjoint operators were constructed via Friedrichs extension,
- resolvents and spectral measures were shown to exist and to depend stably on regularization parameters,
- diagonal arithmetic potentials, including those of SMRK type, were shown to enter as admissible form-bounded perturbations.

These results provide a mathematically sound foundation for further spectral analysis.

7.2 What Has Not Been Addressed

Several questions lie deliberately outside the scope of this paper. In particular, no attempt has been made to:

- derive trace formulas or explicit spectral expansions,
- identify spectral data with zeros of zeta or L-functions,
- analyze fine spectral statistics or eigenvalue correlations.

These topics require additional structure beyond self-adjointness and will be addressed separately.

7.3 Role Within a Larger Program

The self-adjoint operators constructed here should be viewed as the analytic backbone of a broader operator-based arithmetic program. By isolating the symmetric and self-adjoint core of prime-shift dynamics, this work removes a major technical obstacle to subsequent investigations.

In particular, the existence of a canonical self-adjoint realization eliminates ambiguities associated with nonunique extensions and provides a stable platform for trace-based constructions.

7.4 Transition to Trace Formulas

With the resolvent and spectral measure in place, one may define regulated trace objects such as

$$\operatorname{Tr} f(A_{P,\alpha}), \qquad \operatorname{Tr}\,(A_{P,\alpha} - z)^{-1},$$

for suitable test functions f. Expanding such traces in terms of arithmetic paths on the underlying graph leads naturally to explicit formulae involving prime sums.

The derivation and analysis of these trace formulas constitute the subject of the next paper in this series.

7.5 Outlook

The framework developed here establishes that prime-shift dynamics admits a well-defined and analytically controlled operator-theoretic formulation. This result alone does not resolve deep arithmetic conjectures, but it provides the necessary groundwork upon which such questions may be meaningfully posed.

In subsequent work, this foundation will be used to derive trace formulas, analyze spectral invariants, and investigate whether arithmetic information, including that associated with the Riemann zeta function, emerges from the spectral behavior of these operators.

A Technical Lemmas

Appendix A: Technical Lemmas and Supplementary Remarks

This appendix collects auxiliary results and technical remarks supporting the main analysis. No new assumptions are introduced, and all statements are compatible with the framework developed in the core sections.

A.1 A.1 Density of the Core Domain

The space $c_{00}(\mathbb{N})$ of finitely supported functions is dense in $\ell^2(\mathbb{N})$ and in all weighted spaces $\ell^2(\mathbb{N}, \mu)$ with strictly positive weights $\mu(n)$.

Density ensures that quadratic forms defined initially on $c_{00}(\mathbb{N})$ are uniquely determined by their closures and that Friedrichs extensions are canonical.

A.2 A.2 Absolute Convergence on the Core

For any $\psi \in c_{00}(\mathbb{N})$ and for all regularized weight choices $w_{P,\alpha}(p)$ considered in this paper, the sums

$$\sum_p w_{P,\alpha}(p)\psi(pn), \qquad \sum_p w_{P,\alpha}(p)\mathbf{1}_{p|n}\psi(n/p)$$

are finite for each fixed n.

This guarantees that the regularized operators $H^{(P,\alpha)}$ are well defined as algebraic operators on the core domain.

A.3 A.3 Symmetry of the Quadratic Form

The symmetry of the quadratic form

$$Q_{P,\alpha}(\psi) = \langle \psi, H^{(P,\alpha)}\psi \rangle$$

follows from the pairing of forward and backward prime transitions. Explicitly, the change of variables $m = pn$ in the backward term yields

$$\sum_n \overline{\psi(n)}\, \mathbf{1}_{p|n}\psi(n/p) = \sum_n \overline{\psi(pn)}\, \psi(n),$$

which matches the complex conjugate of the forward contribution.

A.4 A.4 Form Bounds with Diagonal Potentials

Let $V : \mathbb{N} \to \mathbb{R}$ satisfy

$$|V(n)| \le a + b\, n^{\gamma}$$

for some $a, b \geq 0$ and $\gamma < 1$. Then the diagonal form

$$\sum_n V(n)|\psi(n)|^2$$

is relatively form-bounded with respect to $\overline{Q}_{P,\alpha}$ for all $\alpha > 1$, with arbitrarily small relative bound.

This estimate covers slowly varying arithmetic potentials such as $\log n$ and von Mangoldt–type terms restricted to regularized settings.

A.5 A.5 Strong Resolvent Convergence

Let $(P_k, \alpha_k) \to (P, \alpha)$ and assume uniform lower bounds on the quadratic forms $\overline{Q}_{P_k,\alpha_k}$. Then the associated operators satisfy

$$A_{P_k,\alpha_k} \xrightarrow{\text{s.r.}} A_{P,\alpha},$$

where convergence is in the strong resolvent sense.

This follows from standard results on convergence of closed quadratic forms and ensures stability of spectral quantities under regularization limits.

A.6 A.6 Energy Shifts

If the quadratic form $\overline{Q}_{P,\alpha}$ is only semibounded from below by a constant $-C$, one may define the shifted form

$$\overline{Q}^{(+)}_{P,\alpha}(\psi) = \overline{Q}_{P,\alpha}(\psi) + C\|\psi\|^2,$$

which is nonnegative. The associated operator differs from $A_{P,\alpha}$ by an additive constant and has identical spectral projections.

Such shifts are implicitly assumed when invoking nonnegativity.

A.7 A.7 Preparatory Remarks for Trace Constructions

While the operators $A_{P,\alpha}$ are not trace-class, the existence of resolvent and spectral measures allows the definition of regulated trace expressions of the form

$$\operatorname{Tr} f(A_{P,\alpha}),$$

for suitable test functions f.

The expansion of such traces in terms of arithmetic paths and prime cycles forms the starting point of the trace formula analysis developed in the subsequent paper.

A.8 A.8 Closing Remark

The results collected here serve to clarify technical points and to ensure that all analytic constructions used in the main text rest on standard and well understood functional-analytic principles. No essential ideas are confined to the appendix.