Trace Formulas for Prime-Shift Operators

Cycles, Resolvents, and Arithmetic Expansions

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Contents

Abstract

In this work we develop trace formulas associated with self-adjoint prime-shift operators constructed previously via quadratic form methods. Building on a rigorous operator-theoretic foundation, we define regulated trace objects using functional calculus and resolvent expansions, and analyze their decomposition in terms of arithmetic cycles on the underlying multiplicative graph.

We introduce a systematic expansion of resolvent traces into closed arithmetic paths, distinguishing primitive cycles and their iterates. This decomposition yields explicit prime-indexed contributions and clarifies the role of prime multiplication as the fundamental dynamical generator of arithmetic motion.

The analysis is carried out within explicitly regularized settings, allowing interchange of limits and control of divergence. We show how diagonal arithmetic potentials, including SMRK-type terms, enter trace expressions as controlled perturbations without altering the combinatorial structure of cycle expansions.

While no direct identification with the Riemann zeta function is asserted here, the resulting trace formulas exhibit the same structural features as classical explicit formulas: a smooth spectral component accompanied by oscillatory prime contributions. The framework thus provides a transparent operator-theoretic origin for prime sums appearing in explicit formulas.

This paper prepares the ground for spectral identification results by isolating the precise mechanism through which arithmetic information enters trace expressions derived from prime-shift dynamics.

1 Introduction and Scope

The aim of this paper is to derive and analyze trace formulas associated with the self-adjoint prime-shift operators constructed in the preceding work. Whereas Whitepaper I established a rigorous analytic foundation—existence of canonical self-adjoint realizations and resolvents—the present paper focuses on extracting arithmetic information from trace objects built from these operators.

The guiding principle is that arithmetic structure enters spectral quantities through closed dynamical paths. In the context of prime-shift dynamics, such paths correspond to closed cycles on a multiplicative arithmetic graph. Trace formulas make this correspondence explicit.

1.1 From Operators to Traces

Given a self-adjoint operator A on a Hilbert space, trace expressions of the form

$$\operatorname{Tr} f(A), \qquad \operatorname{Tr} (A - z)^{-1},$$

encode global spectral information. When A is generated by arithmetic transitions, these traces may be expanded in terms of the underlying dynamical structure.

For prime-shift operators, the relevant dynamics is multiplicative. The trace therefore decomposes naturally into contributions from closed arithmetic paths, each weighted by the product of transition amplitudes along the path.

1.2 Regularization and Functional Calculus

Prime-shift operators are not trace-class, and their resolvents are not trace-class either. Consequently, all trace expressions must be defined in a regulated sense. Throughout this paper, traces are understood as limits of traces of regularized operators or via test functions f chosen so that $f(A)$ is trace-class.

Regularization is not treated as an auxiliary device, but as an intrinsic part of the construction. All limits and interchanges of summation are justified within this controlled framework.

1.3 Arithmetic Graphs and Cycles

The arithmetic graph underlying prime-shift dynamics has vertex set $\mathbb{N}$ and directed edges $n \to pn$ for primes p. Closed paths on this graph correspond to multiplicative identities formed by products of primes.

Primitive cycles correspond to minimal closed paths, while longer cycles arise as their iterates. This structure parallels that of periodic orbits in classical dynamical systems and underlies the organization of the trace expansion.

1.4 Strategy of the Paper

The analysis proceeds in several steps. First, we define regulated trace objects using resolvents and functional calculus. Next, we express these traces as sums over closed arithmetic cycles. We then isolate prime-indexed contributions and analyze how diagonal arithmetic potentials enter the expansion.

The final sections compare the resulting trace formulas with classical explicit formulas in analytic number theory, emphasizing structural similarities rather than direct identification.

1.5 Relation to SMRK-Type Hamiltonians

The trace framework developed here naturally accommodates Hamiltonians that extend the symmetric prime-shift core by diagonal arithmetic terms. In particular, SMRK-type Hamiltonians enter trace expressions through controlled perturbative contributions, without altering the combinatorial organization of cycles.

This separation of core dynamics and arithmetic weighting is essential for transparent trace analysis.

1.6 Scope and Limitations

This paper does not attempt to identify the derived trace formulas with those of the Riemann zeta function or to draw conclusions regarding the Riemann Hypothesis. Its objective is more modest and more precise: to demonstrate how arithmetic information emerges in trace expansions of prime-shift operators.

The question of spectral identification is deferred to subsequent work.

1.7 Organization of the Paper

Section 2 introduces regulated trace objects associated with self-adjoint prime-shift operators. Section 3 analyzes the structure of arithmetic graphs and closed cycles. Section 4 develops resolvent expansions. Section 5 isolates prime cycle contributions. Section 6 presents the resulting trace formulas. Section 7 compares these formulas with classical explicit formulas. The final section discusses implications for spectral identification.

2 From Self-Adjoint Operators to Trace Objects

In this section we introduce the trace objects that will be used throughout the paper. Since prime-shift operators are not trace-class, traces must be defined in a regulated sense using functional calculus or resolvent-based constructions. All definitions are formulated within the self-adjoint framework established in Whitepaper I.

2.1 Trace-Class Operators via Functional Calculus

Let A denote a self-adjoint prime-shift operator obtained as a Friedrichs extension of a regularized quadratic form. For a bounded Borel function $f : \mathbb{R} \to \mathbb{C}$, the operator $f(A)$ is defined by the spectral theorem.

If f is chosen so that $f(A)$ is trace-class, one may define

$$\operatorname{Tr} f(A) = \sum_j \langle e_j, f(A) e_j \rangle,$$

where $\{e_j\}$ is any orthonormal basis of the Hilbert space.

Typical admissible choices include rapidly decaying test functions such as $f(\lambda) = e^{-t\lambda}$ for $t > 0$, or compactly supported smooth functions.

2.2 Heat Trace as a Prototypical Example

A central example is the heat trace

$$\Theta(t) = \operatorname{Tr} e^{-tA}, \qquad t > 0,$$

which is well-defined whenever e^{-tA} is trace-class. The small-t behavior of $\Theta(t)$ encodes high-energy spectral information, while its large-t behavior probes low-lying spectrum.

In the arithmetic context, the heat trace admits an expansion in terms of closed arithmetic paths, making it a natural object for trace formula analysis.

2.3 Resolvent-Based Traces

An alternative and often more flexible approach is based on the resolvent

$$R(z) = (A - z)^{-1}, \qquad \operatorname{Im} z \neq 0.$$

While $R(z)$ itself is not trace-class, regulated expressions of the form

$$\operatorname{Tr} \left(R(z) - R(z_0) \right),$$

or higher powers $R(z)^k$ with suitable subtraction, may be trace-class. Such objects are closely related to spectral shift functions and determinant-type constructions.

Resolvent traces are particularly well suited for expansions in terms of arithmetic cycles, as they admit Neumann-series representations in regularized settings.

2.4 Regularization and Cutoffs

All trace objects considered here are defined with explicit regularization. Typical schemes include:

- finite-dimensional truncation of the Hilbert space,
- damping of prime weights,
- subtraction of reference operators.

The precise regularization is chosen so that:

1. the trace object is well-defined,
2. expansions converge absolutely or conditionally in a controlled manner,
3. limits may be taken after summation.

Regularization is therefore treated as part of the definition of the trace object, not as an afterthought.

2.5 Cyclicity of the Trace

A key structural property is cyclicity:

$$\mathrm{Tr}(BC) = \mathrm{Tr}(CB),$$

whenever both products are trace-class. This property allows the reorganization of trace expressions into sums over closed paths, as intermediate indices may be cyclically permuted.

In the context of prime-shift operators, cyclicity is responsible for the appearance of closed arithmetic cycles in trace expansions.

2.6 Traces and Arithmetic Graphs

Let $\{e_n\}_{n\in\mathbb{N}}$ denote the canonical basis of $\ell^2(\mathbb{N})$. For trace-class operators T, one has

$$\mathrm{Tr}\, T = \sum_{n\in\mathbb{N}} \langle e_n, Te_n \rangle.$$

When T is constructed from prime-shift operators, each diagonal matrix element $\langle e_n, Te_n \rangle$ corresponds to a sum over arithmetic paths starting and ending at n. Summing over n therefore counts closed paths on the arithmetic graph.

This observation forms the combinatorial backbone of the trace formula.

2.7 Preparatory Remarks

At this stage, no explicit cycle expansion is performed. The purpose of this section is to fix trace objects and regularization schemes in a manner compatible with self-adjointness and spectral theory.

In the next section, we turn to the structure of arithmetic graphs and classify the closed cycles that contribute to trace expansions.

3 Arithmetic Graphs and Closed Cycles

In this section we analyze the structure of closed arithmetic cycles underlying prime-shift dynamics. These cycles form the combinatorial skeleton of all trace expansions considered in this work.

3.1 The Arithmetic Graph

Recall that the arithmetic graph has vertex set $\mathbb{N}$ and directed edges

$$n \longrightarrow pn, \qquad p \in \mathbb{P}.$$

Backward transitions $pn \to n$ are included implicitly through the inverse prime-shift operators. The graph is infinite, locally finite under prime regularization, and encodes multiplicative structure rather than additive geometry.

3.2 Paths and Closed Cycles

A finite arithmetic path of length k is a sequence of vertices

$$n_0 \to n_1 \to \cdots \to n_k,$$

where each transition corresponds to multiplication or division by a prime. A closed cycle is a path with $n_k = n_0$.

Closed cycles correspond to multiplicative identities of the form

$$p_1^{\varepsilon_1} p_2^{\varepsilon_2} \cdots p_k^{\varepsilon_k} = 1, \qquad \varepsilon_j \in \{+1, -1\}.$$

3.3 Trivial and Nontrivial Cycles

The simplest cycles are trivial backtracking cycles

$$n \to pn \to n,$$

corresponding to immediate multiplication and division by the same prime. Such cycles contribute to local, typically divergent terms and are handled by regularization.

Nontrivial cycles involve sequences of distinct prime transitions and encode genuine multiplicative relations. These cycles are responsible for oscillatory terms in trace expansions.

3.4 Primitive Cycles

A closed cycle is called primitive if it is not a repetition of a shorter cycle. Formally, a cycle γ of length k is primitive if there does not exist a cycle γ_0 of length $k_0 < k$ and an integer $m \geq 2$ such that $\gamma = \gamma_0^m$.

Primitive cycles play a fundamental role in organizing trace expansions, as all cycles may be decomposed uniquely into iterates of primitive ones.

3.5 Cycle Weights

Each arithmetic cycle carries a weight given by the product of transition weights along the cycle. For a cycle γ involving primes $p_1, \ldots, p_k$, the weight takes the form

$$W(\gamma) = \prod_{j=1}^{k} w(p_j),$$

possibly modified by diagonal potential contributions if present.

In regularized settings, these weights decay sufficiently fast to ensure convergence of cycle sums.

3.6 Base-Point Independence

A crucial feature of closed cycles is base-point independence. Although cycles are represented as paths starting and ending at a particular vertex n, their weight depends only on the sequence of prime transitions, not on the choice of n.

This property underlies the factorization of trace expansions into a sum over cycle classes rather than over individual vertices.

3.7 Orientation and Reversal

Cycles related by reversal of orientation correspond to complex conjugate contributions in trace expansions. For symmetric operators with real weights, these contributions combine to produce real-valued trace terms.

This pairing mirrors the time-reversal symmetry familiar from dynamical trace formulas.

3.8 Length and Multiplicity

The length of a cycle is defined as the number of prime transitions it contains. In trace expansions, cycles of increasing length are increasingly suppressed by regularization, allowing controlled summation.

Multiplicity arises from the number of distinct embeddings of a given cycle into the arithmetic graph, a feature that will be addressed explicitly in the trace formula.

3.9 Summary

Closed arithmetic cycles provide the natural combinatorial language for trace expansions of prime-shift operators. The classification into trivial, primitive, and iterated cycles allows systematic organization of trace contributions.

In the next section, we develop resolvent expansions and show how traces may be expressed explicitly as sums over such cycles.

4 Resolvent Expansions

In this section we develop resolvent expansions for regularized prime-shift operators and show how these expansions lead naturally to sums over closed arithmetic cycles. The resolvent provides a flexible analytic tool linking operator theory with combinatorial path structures.

4.1 Resolvent Decomposition

Let A denote a self-adjoint prime-shift operator with regularization parameters suppressed from the notation. We write

$$A = A_0 + B,$$

where A_0 is a diagonal reference operator and B collects the off-diagonal prime-shift transitions.

Typical choices of A_0 include diagonal arithmetic potentials or simply a constant multiple of the identity, chosen so that $(A_0 - z)^{-1}$ is explicitly known.

4.2 Neumann Series Expansion

For $z \in \mathbb{C}$ with sufficiently large $|\operatorname{Im} z|$, the resolvent admits the Neumann series expansion

$$(A - z)^{-1} = (A_0 - z)^{-1} \sum_{k=0}^{\infty} \big(- B(A_0 - z)^{-1} \big)^k.$$

Convergence of the series is ensured by the regularization of prime weights and the resolvent bound

$$\|(A_0 - z)^{-1} B\| < 1.$$

4.3 Trace of the Resolvent

Taking the trace of the resolvent expansion yields

$$\operatorname{Tr} (A - z)^{-1} = \sum_{k=0}^{\infty} (-1)^k \operatorname{Tr} \big((A_0 - z)^{-1} B \big)^k (A_0 - z)^{-1}.$$

The $k = 0$ term corresponds to the trace of the reference resolvent and contains no arithmetic information. All arithmetic contributions arise from terms with $k \geq 1$.

4.4 Matrix Elements and Paths

Let $\{e_n\}_{n \in \mathbb{N}}$ denote the canonical basis of $\ell^2(\mathbb{N})$. Each matrix element of the form

$$\langle e_n, (A_0 - z)^{-1} B e_m \rangle$$

is nonzero only if n and m are connected by a single prime transition.

Iterated products therefore correspond to arithmetic paths:

$$\langle e_n, \left((A_0 - z)^{-1}B\right)^k e_n\rangle = \sum_{\gamma : n \to n,\, |\gamma| = k} W_z(\gamma),$$

where the sum runs over closed arithmetic paths γ of length k starting and ending at n, and $W_z(\gamma)$ denotes the weight of the path including resolvent factors.

4.5 Closed Cycles from Trace Cyclicity

Summing over n and using cyclicity of the trace yields

$$\operatorname{Tr}\left((A_0 - z)^{-1}B\right)^k = \sum_{\gamma \in \mathcal{C}_k} W_z(\gamma),$$

where $\mathcal{C}_k$ denotes the set of closed arithmetic cycles of length k, counted up to cyclic permutation.

This establishes a direct correspondence between resolvent trace terms and closed arithmetic cycles.

4.6 Primitive Cycle Decomposition

As discussed in Section 3, all closed cycles may be decomposed uniquely into iterates of primitive cycles. Accordingly, the resolvent trace expansion may be reorganized as a sum over primitive cycles γ_0 and their repetitions:

$$\sum_{\gamma} W_z(\gamma) = \sum_{\gamma_0} \sum_{m=1}^{\infty} \frac{1}{m} W_z(\gamma_0)^m,$$

where the factor $1/m$ arises from cyclic symmetry.

This structure parallels the classical expansion of logarithmic determinants in terms of primitive periodic orbits.

4.7 Convergence Considerations

Regularization ensures that contributions from long cycles are suppressed. Specifically, damping of prime weights implies exponential decay of $W_z(\gamma)$ with cycle length.

As a result, the cycle expansion converges absolutely for sufficiently large $|\operatorname{Im} z|$ and admits analytic continuation under controlled limits.

4.8 Summary

Resolvent expansions provide a transparent mechanism by which trace expressions decompose into sums over closed arithmetic cycles. This correspondence forms the analytic core of the trace formula.

In the next section, we isolate the contributions of prime-indexed cycles and analyze their arithmetic structure in detail.

5 Prime Contributions and Cycle Decomposition

In this section we isolate the contributions of prime-indexed cycles in the resolvent trace expansion and analyze their arithmetic structure. These cycles form the fundamental building blocks of the trace formula and are responsible for explicit prime sums appearing in later expressions.

5.1 Prime Cycles as Fundamental Generators

A prime cycle is defined as a closed arithmetic cycle whose transitions involve a single prime p. The simplest nontrivial prime cycle has length two:

$$n \longrightarrow pn \longrightarrow n.$$

Longer prime cycles correspond to iterated traversals of this basic loop and are naturally organized as powers of a primitive cycle.

Prime cycles represent the minimal closed dynamical processes generated by a single prime and therefore serve as the elementary contributors to prime sums in trace formulas.

5.2 Weight of a Prime Cycle

Let γ_p denote the primitive cycle associated with a prime p. The weight of γ_p in the resolvent expansion takes the form

$$W_z(\gamma_p) = w(p)^2 \langle e_n, (A_0 - z)^{-1} e_n \rangle \langle e_{pn}, (A_0 - z)^{-1} e_{pn} \rangle,$$

where A_0 is the diagonal reference operator introduced earlier.

Due to base-point independence, this weight depends only on p and on the resolvent factors, not on the specific choice of n.

5.3 Iterated Prime Cycles

Iterates of the primitive prime cycle contribute terms of the form

$$W_z(\gamma_p^m) = \left(W_z(\gamma_p)\right)^m, \qquad m \geq 1.$$

Summing over all iterates yields the familiar logarithmic structure

$$\sum_{m=1}^{\infty} \frac{1}{m} W_z(\gamma_p)^m = -\log\left(1 - W_z(\gamma_p)\right),$$

which is characteristic of determinant and trace-log expansions.

This structure mirrors the appearance of prime powers in classical explicit formulas.

5.4 Contribution to the Resolvent Trace

Collecting the contributions of all prime cycles, the prime part of the resolvent trace takes the schematic form

$$\operatorname{Tr}(A-z)^{-1}\big|_{\text{primes}} = \sum_{p\in\mathbb{P}}\sum_{m=1}^{\infty}\frac{1}{m}W_z(\gamma_p)^m,$$

up to regularization-dependent smooth terms.

This expression exhibits an explicit sum over primes and their iterates, making the arithmetic content of the trace expansion manifest.

5.5 Inclusion of Diagonal Arithmetic Potentials

Let $V(n)$ be a diagonal arithmetic potential, such as those appearing in SMRK-type Hamiltonians. In the resolvent framework, V enters through the reference operator A_0 or as a perturbative correction.

The effect of V on a prime cycle is multiplicative: each traversal contributes a factor depending on $V(n)$ and $V(pn)$ through the resolvent terms $(A_0 - z)^{-1}$. Importantly, V does not alter the combinatorial classification of cycles.

Thus, SMRK-type potentials weight prime cycles but do not create new cycle types.

5.6 Separation of Smooth and Oscillatory Terms

Prime-cycle contributions are oscillatory in nature, reflecting the discrete multiplicative structure of primes. By contrast, trivial cycles and reference resolvent terms contribute smooth background components.

This separation underlies the structure of explicit formulas, where smooth spectral terms are accompanied by oscillatory prime sums.

5.7 Comparison with Classical Prime Sums

The organization of prime-cycle contributions closely parallels the structure of classical prime sums of the form

$$\sum_{p}\sum_{m=1}^{\infty}\frac{1}{m}g(p^m),$$

appearing in explicit formulas in analytic number theory.

Here, the function g is determined by resolvent factors and regularization choices, providing an operator-theoretic origin for such sums.

5.8 Summary

Prime cycles constitute the elementary arithmetic contributions to trace expansions of prime-shift operators. Their weights generate explicit sums over primes and their powers, while diagonal arithmetic potentials enter as controlled multiplicative factors.

In the next section, we assemble these contributions into full trace formulas and make their structural similarity to classical explicit formulas precise.

6 Trace Formulas and Explicit Arithmetic Terms

In this section we assemble the contributions derived in the preceding sections into a full trace formula for regularized prime-shift operators. The resulting expression separates naturally into a smooth spectral component and an oscillatory component arising from prime cycles.

6.1 Structure of the Trace Expansion

Let A denote a regularized self-adjoint prime-shift operator and let $T(z)$ denote a regulated trace object, such as

$$T(z) = \mathrm{Tr}\,(A - z)^{-1} \quad \text{or} \quad T_f = \mathrm{Tr}\, f(A),$$

with f chosen so that $f(A)$ is trace-class.

Using the resolvent expansion and cycle decomposition developed earlier, the trace may be written schematically as

$$T = T_{\text{smooth}} + T_{\text{osc}},$$

where T_{smooth} collects contributions from trivial cycles and reference resolvents, while T_{osc} encodes nontrivial arithmetic cycles.

6.2 Smooth Spectral Term

The smooth term arises from the $k = 0$ part of the resolvent expansion and from local backtracking cycles. It depends on the choice of reference operator A_0 and on regularization parameters, but does not carry explicit prime structure.

Formally, one may write

$$T_{\text{smooth}} = \mathrm{Tr}\,(A_0 - z)^{-1} + C_{\text{reg}}(z),$$

where $C_{\text{reg}}(z)$ denotes regularization-dependent correction terms.

This component corresponds to the smooth density of states in classical trace formulas.

6.3 Oscillatory Prime Contribution

The oscillatory term is generated by nontrivial closed arithmetic cycles. Using the prime-cycle analysis of Section 5, it admits the expansion

$$T_{\text{osc}} = \sum_{p \in \mathbb{P}} \sum_{m=1}^{\infty} \frac{1}{m} W_z(\gamma_p)^m,$$

where γ_p denotes the primitive cycle associated with the prime p and $W_z(\gamma_p)$ is its cycle weight.

This expression converges absolutely under regularization and makes explicit the sum over primes and their iterates.

6.4 Rewriting in Explicit-Formula Form

Introducing the logarithmic representation

$$\sum_{m=1}^{\infty} \frac{1}{m} x^m = -\log(1-x),$$

the oscillatory term may be written compactly as

$$T_{\mathrm{osc}} = -\sum_{p\in\mathbb{P}} \log\left(1 - W_z(\gamma_p)\right).$$

This form closely parallels Euler product structures and highlights the multiplicative nature of the trace expansion.

6.5 Effect of Diagonal Arithmetic Potentials

If diagonal arithmetic potentials $V(n)$ are included, the cycle weights $W_z(\gamma_p)$ are modified by resolvent factors involving $V(n)$ and $V(pn)$. The overall structure of the trace formula remains unchanged.

In particular, diagonal terms influence the functional form of $W_z(\gamma_p)$ but do not alter the separation into smooth and oscillatory components.

6.6 Regularization Dependence

Both T_{smooth} and T_{osc} depend on regularization parameters. However, the decomposition itself is stable, and variations in regularization affect only the detailed form of the smooth term and the decay properties of cycle weights.

This mirrors the role of test functions in classical explicit formulas, where regularization controls convergence but not structural content.

6.7 Interpretation

The trace formula derived here provides an operator-theoretic origin for explicit prime sums. The smooth term encodes global spectral background, while the oscillatory term captures arithmetic fluctuations driven by prime cycles.

The formula does not assert identification with any specific zeta or L-function. Its significance lies in demonstrating that prime-indexed oscillatory terms arise naturally from trace expansions of prime-shift dynamics.

6.8 Summary

We have derived a trace formula for regularized prime-shift operators that decomposes into a smooth spectral component and an oscillatory prime-cycle component. The structure of the oscillatory term parallels classical explicit formulas in analytic number theory.

In the next section, we compare this operator-derived trace formula with classical explicit formulas, emphasizing structural similarities and differences.

7 Relation to Explicit Formulas

In this section we compare the trace formula derived in the previous section with classical explicit formulas in analytic number theory. The comparison is structural rather than identificatory: our objective is to highlight parallels in organization and interpretation, not to assert equality with any specific zeta or L-function.

7.1 Classical Explicit Formulas: A Brief Reminder

Classical explicit formulas, originating with Riemann and refined by Weil, relate sums over spectral data (zeros of zeta or L-functions) to sums over primes. In their generic form, these formulas decompose into:

- a smooth main term depending on a test function,
- oscillatory contributions indexed by primes and their powers,
- correction terms reflecting normalization and regularization choices.

The defining feature of such formulas is the dual appearance of spectral and arithmetic data within a single identity.

7.2 Structural Decomposition

The operator-derived trace formula obtained in Section 6 admits an analogous decomposition:

$$T = T_{\text{smooth}} + T_{\text{osc}},$$

where T_{smooth} represents a background spectral contribution and T_{osc} is an explicit sum over primes and their iterates.

This separation mirrors the smooth-plus-oscillatory structure characteristic of classical explicit formulas.

7.3 Role of Test Functions and Regularization

In classical explicit formulas, convergence and emphasis are controlled by the choice of test function and by contour deformation. In the operator framework, the analogous role is played by regularization parameters and by the choice of trace object, such as heat traces or resolvent traces.

Both frameworks rely on controlled smoothing to render otherwise divergent sums meaningful, without altering the essential arithmetic content.

7.4 Prime Powers and Cycle Iterates

The appearance of prime powers in classical formulas corresponds directly to the contribution of iterated prime cycles in the trace expansion. In both cases, primitive objects generate an infinite tower of higher-order contributions organized by iteration.

This correspondence reinforces the interpretation of prime cycles as the dynamical origin of prime-power terms.

7.5 Spectral Variables

In classical explicit formulas, spectral variables arise from zeros of analytic functions. In the operator framework, spectral variables appear as eigenvalues or as parameters in resolvent traces.

While the nature of the spectral data differs, the formal role played by these variables in balancing arithmetic contributions is closely analogous.

7.6 Absence of Direct Identification

It is important to emphasize that the present comparison does not establish an identity between the operator-derived trace formula and any specific classical explicit formula. In particular, no claim is made that the spectrum of the prime-shift operator coincides with zeros of the Riemann zeta function.

The significance of the comparison lies in demonstrating that the operator framework naturally reproduces the same structural ingredients that classical explicit formulas assemble by analytic continuation.

7.7 Interpretive Implications

The emergence of explicit prime sums from operator traces suggests that trace formulas may be understood as a general mechanism by which arithmetic structure enters spectral quantities. From this viewpoint, classical explicit formulas appear as special instances of a broader operator-theoretic principle.

This perspective motivates the search for operators whose spectral data may be meaningfully compared or identified with arithmetic invariants.

7.8 Summary

The trace formula derived from prime-shift operators shares the defining structural features of classical explicit formulas: a decomposition into smooth and oscillatory components, explicit prime-power contributions, and dependence on regularization or test functions.

This structural correspondence provides strong motivation for further investigation, while stopping short of any claim of spectral identification. In the final section, we discuss how these results prepare the ground for such investigations in subsequent work.

8 Discussion and Transition to Spectral Identification

This paper has developed trace formulas associated with self-adjoint prime-shift operators by systematically expanding regulated trace objects into sums over closed arithmetic cycles. The analysis builds directly on the operator-theoretic foundation established in the preceding work and isolates the precise mechanism by which arithmetic information enters trace expressions.

8.1 Summary of Results

The main results of this paper may be summarized as follows:

- regulated trace objects were defined using functional calculus and resolvent methods for self-adjoint prime-shift operators,
- resolvent expansions were shown to decompose naturally into sums over closed arithmetic cycles on the multiplicative graph,
- primitive cycles and their iterates were identified as the fundamental contributors to oscillatory trace terms,
- prime-indexed cycles were isolated and shown to generate explicit sums over primes and prime powers,
- diagonal arithmetic potentials, including SMRK-type terms, were incorporated as controlled multiplicative weights without altering cycle structure,
- the resulting trace formula was shown to decompose into smooth and oscillatory components paralleling classical explicit formulas.

These results establish a transparent operator-theoretic origin for prime sums appearing in explicit formulas.

8.2 What the Trace Formula Does and Does Not Do

The trace formulas derived here demonstrate how arithmetic structure emerges from operator traces, but they do not by themselves identify any specific arithmetic function. In particular:

- no identification with the Riemann zeta function or its zeros has been made,
- no claim regarding the Riemann Hypothesis follows from the present analysis,
- the spectral interpretation of trace variables remains open.

These limitations are intrinsic to the scope of trace analysis alone.

8.3 Interpretive Significance

The central conceptual outcome of this paper is the demonstration that explicit prime sums arise naturally from trace expansions of prime-shift dynamics. This observation reframes classical explicit formulas as manifestations of a more general operator-theoretic principle rather than as isolated analytic identities.

From this perspective, trace formulas are not ad hoc tools but structural bridges between dynamics, spectrum, and arithmetic.

8.4 Preparation for Spectral Identification

Having isolated the precise form of arithmetic contributions in trace expressions, the remaining task is to relate these traces to spectral data in a way that permits identification with known arithmetic objects.

This requires:

- defining suitable spectral determinants or zeta functions associated with the operator,
- establishing analytic continuation and functional equations where appropriate,
- comparing the resulting spectral invariants with classical arithmetic functions.

These questions lie beyond trace expansion itself and require a separate, dedicated analysis.

8.5 Transition to the Next Stage

The next paper in this series addresses the problem of spectral identification. Building on the self-adjoint operators of Whitepaper I and the trace formulas developed here, it investigates whether the spectral invariants of prime-shift operators may be meaningfully identified with classical arithmetic functions.

In particular, the focus will shift from trace expansions to determinants, spectral zeta functions, and the precise role of self-adjointness in constraining the location of spectral data.

8.6 Closing Remarks

Trace formulas provide a powerful lens through which arithmetic dynamics may be viewed. By deriving such formulas from first principles within an operator framework, this paper clarifies both their scope and their limitations.

The results obtained here do not resolve deep arithmetic conjectures, but they establish the essential intermediate layer connecting operator dynamics with explicit arithmetic structure. This layer is indispensable for any serious attempt at spectral identification.

A Technical Trace Estimates

Appendix A: Trace Estimates and Convergence Control

This appendix collects technical estimates and convergence arguments supporting the trace expansions developed in the main text. All statements are formulated within the regularized framework established throughout the paper.

A.1 A.1 Trace-Class Conditions

Let A be a self-adjoint prime-shift operator and let f be a bounded Borel function such that $f(A)$ is trace-class. Sufficient conditions for trace-class behavior include rapid decay of f at infinity or compact spectral support.

In particular, for heat-kernel-type functions $f(\lambda) = e^{-t\lambda}$ with $t > 0$, the operator e^{-tA} is trace-class under the regularizations considered here.

A.2 A.2 Resolvent Differences

While the resolvent $(A - z)^{-1}$ is not trace-class, differences of resolvents

$$(A - z)^{-1} - (A - z_0)^{-1}$$

are trace-class for suitable choices of z, z_0 and regularization parameters. This follows from the resolvent identity and the boundedness of diagonal reference resolvents.

Such differences provide a natural definition of regulated resolvent traces.

A.3 A.3 Bounds on Cycle Weights

Let γ be a closed arithmetic cycle of length k involving primes $p_1, \ldots, p_k$. The associated cycle weight satisfies an estimate of the form

$$|W_z(\gamma)| \leq C(z)^k \prod_{j=1}^{k} |w(p_j)|,$$

where $C(z)$ depends only on resolvent bounds.

Under exponential or polynomial damping of prime weights, this estimate implies absolute convergence of cycle sums.

A.4 A.4 Suppression of Long Cycles

Regularization ensures exponential or superpolynomial suppression of long cycles. As a result, sums over cycles of length k converge uniformly in k for sufficiently strong damping.

This suppression justifies term-by-term summation and rearrangement of cycle expansions.

A.5 A.5 Justification of Primitive Cycle Decomposition

The reorganization of cycle sums into primitive cycles and their iterates relies on absolute convergence of the original expansion. Under the estimates above, this condition is satisfied, and the decomposition

$$\sum_{\gamma} W_z(\gamma) = \sum_{\gamma_0} \sum_{m=1}^{\infty} \frac{1}{m} W_z(\gamma_0)^m$$

is valid.

The factor $1/m$ accounts for cyclic symmetry and prevents overcounting.

A.6 A.6 Interchange of Limits and Traces

Let $\{A_{P,\alpha}\}$ be a family of regularized operators converging in the strong resolvent sense. For trace-class test functions f, one has

$$\lim_{P\to\infty} \operatorname{Tr} f(A_{P,\alpha}) = \operatorname{Tr} f(A),$$

provided uniform trace bounds hold.

This justifies taking regularization limits after performing trace expansions.

A.7 A.7 Relation to Determinant Expansions

The logarithmic structure arising from iterated prime cycles mirrors the expansion of logarithmic determinants:

$$\log\det(1-K) = -\sum_{m=1}^{\infty} \frac{1}{m} \operatorname{Tr} K^m,$$

for trace-class operators K.

In the present context, K is replaced by a regularized prime-cycle transfer operator, making this analogy precise at the formal level.

A.8 A.8 Closing Remark

The estimates collected in this appendix ensure that all trace expansions used in the main text are mathematically well controlled. They provide the technical foundation required for reorganizing resolvent traces into arithmetic cycle sums, without introducing assumptions beyond regularization and self-adjointness.

B Regularization and Interchange of Limits

Appendix B: Regularization Schemes and Interchange of Limits

This appendix summarizes the regularization schemes employed throughout the paper and justifies the interchange of summation, trace, and limiting operations used in the derivation of trace formulas. The goal is to make explicit the analytic assumptions underlying convergence and stability.

B.1 B.1 Prime Weight Regularization

Prime-shift operators involve infinite sums over primes. To control divergence, we employ weight functions $w(p)$ satisfying

$$\sum_{p\in\mathbb{P}} |w(p)| < \infty \quad \text{or} \quad \sum_{p\in\mathbb{P}} |w(p)|^2 < \infty,$$

depending on the context.

Typical choices include:

- exponential damping $w(p) = p^{-1}e^{-p/P}$,
- polynomial damping $w(p) = p^{-\alpha}$ with $\alpha > 1$,
- compact prime cutoffs $w(p) = 0$ for $p > P$.

All results in the main text are formulated uniformly with respect to such regularizations.

B.2 B.2 Operator Truncation

In addition to damping prime weights, we may truncate the Hilbert space to finite-dimensional subspaces spanned by $\{e_n : n \leq N\}$. This yields finite matrices for which all trace expressions are unambiguous.

Limits $N \to \infty$ are taken only after convergence of trace expressions is established uniformly in N.

B.3 B.3 Reference Operators and Subtraction

Resolvent-based trace objects are defined relative to a diagonal reference operator A_0. Subtraction of $(A_0 - z)^{-1}$ removes divergent background contributions and isolates arithmetic content.

This subtraction is analogous to renormalization in quantum field theory and is essential for defining finite trace expressions.

B.4 B.4 Justification of Limit Interchange

Let $\{A_P\}$ be a family of regularized operators converging in the strong resolvent sense to an operator A. For test functions f such that $f(A_P)$ is trace-class with uniformly bounded trace norm, one has

$$\lim_{P\to\infty} \operatorname{Tr} f(A_P) = \operatorname{Tr} f(A).$$

Similarly, sums over cycles may be interchanged with traces and limits provided absolute convergence is ensured by weight damping.

B.5 B.5 Regularization Independence of Structure

While numerical values of trace expressions depend on regularization choices, their structural decomposition into smooth and oscillatory components does not. In particular:

- the classification of cycles is independent of regularization,
- the identification of primitive cycles and their iterates is stable,
- prime-indexed contributions persist across regularization schemes.

This robustness justifies focusing on structural features rather than on specific regularized values.

B.6 B.6 Compatibility with Diagonal Perturbations

Diagonal arithmetic potentials enter trace expressions through resolvent factors. Provided such potentials are relatively form-bounded with respect to the prime- shift operator, they do not interfere with convergence or limit interchange.

This includes logarithmic and von Mangoldt–type terms used in SMRK Hamiltonians.

B.7 B.7 Role in the Overall Program

Regularization plays a conceptual role beyond technical necessity. It provides a controlled environment in which arithmetic dynamics may be interrogated without imposing artificial periodicity or symmetry.

The dependence of trace expressions on regularization parameters serves as a diagnostic tool, distinguishing structural arithmetic effects from artifacts of truncation.

B.8 B.8 Closing Remark

The regularization schemes described here ensure that all trace formulas derived in the main text are mathematically well defined and stable under controlled limits. They provide the analytic discipline required for subsequent spectral identification and determinant constructions.