Numerical Results for the SMRK Hamiltonian

Spectral Matching, Explicit Formula Residuals, and Stability Analysis

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December 20, 2025

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Abstract

We present numerical results for the SMRK Hamiltonian, an arithmetic transfer operator proposed as a candidate within the Hilbert–Pólya program. Building on a previously established experimental framework, we evaluate the operator against certified zeta zero data, explicit formula diagnostics, and stability under systematic refinement.

The results reported here include spectral comparisons with nontrivial zeros of the Riemann zeta function, level spacing statistics, explicit formula residual analyses, and convergence behavior under increasing resolution. Both positive and negative findings are documented.

The purpose of this work is not to claim correctness of the SMRK Hamiltonian, but to demonstrate how concrete arithmetic operators can be quantitatively evaluated within a falsification-oriented experimental framework.

1 Experimental Setup

1.1 Purpose and scope

This section specifies the precise numerical setup used to generate the results reported in this work. All methodological principles, including certified zero computation and explicit formula diagnostics, are inherited from the experimental framework defined in Whitepaper IV and are not repeated here.

The present section focuses exclusively on fixing the concrete realization of the SMRK Hamiltonian, its numerical truncation, and the procedures used to extract spectral and trace data. All results in subsequent sections should be interpreted relative to this setup.

1.2 Hilbert space and basis

The SMRK Hamiltonian is realized as an operator acting on the Hilbert space

$$\mathcal{H} = \ell^2(\mathbb{N}),$$

with the canonical basis $\{|n\rangle\}_{n\geq 1}$.

For numerical purposes, we introduce a finite-dimensional truncation

$$\mathcal{H}_N = \mathrm{span}\{|1\rangle, |2\rangle, \ldots, |N\rangle\},$$

where N denotes the truncation dimension. All reported spectra are obtained from operators restricted to $\mathcal{H}_N$, with N varied systematically to assess stability under refinement.

1.3 Definition of the SMRK Hamiltonian

In the experiments reported here, the SMRK Hamiltonian is defined as an arithmetic transfer operator of the form

$$(H_{\mathrm{SMRK}}\psi)(n) = \sum_{p \le P_{\max}} \frac{1}{p}\Big(\psi(pn) + \mathbf{1}_{p|n}\,\psi(n/p)\Big) + \big(\alpha\Lambda(n) + \beta \log n\big)\,\psi(n),$$

where:

- the sum runs over prime numbers p,
- $P_{\max}$ is a prime cutoff parameter,
- $\Lambda(n)$ denotes the von Mangoldt function,
- $\mathbf{1}_{p|n}$ is the indicator function of divisibility,
- $\alpha, \beta \in \mathbb{R}$ are scalar parameters.

This form represents a symmetric prime-indexed transfer Hamiltonian with a diagonal arithmetic potential. The choice of weights $1/p$ and the inclusion of the logarithmic term are motivated by compatibility with explicit formula structures, but are not assumed to be optimal or unique.

1.4 Matrix representation and truncation

On the truncated space $\mathcal{H}_N$, the operator H_{SMRK} is represented as an $N \times N$ real symmetric matrix. Entries corresponding to transitions outside the truncated index range are discarded.

Explicitly, matrix elements are given by

$$\langle m|H_{\mathrm{SMRK}}|n\rangle = \sum_{p \le P_{\max}} \frac{1}{p}\left(\delta_{m,pn} + \delta_{m,n/p}\,\mathbf{1}_{p|n}\right) + \delta_{m,n}\big(\alpha\Lambda(n) + \beta \log n\big),$$

for $1 \le m, n \le N$.

The truncation introduces boundary effects, particularly for large n near N. These effects are monitored by varying both N and $P_{\max}$ and by excluding edge regions from sensitive diagnostics where appropriate.

1.5 Numerical diagonalization

Eigenvalues of the truncated Hamiltonian are computed using standard dense or sparse symmetric eigensolvers, depending on the truncation size.

All computations are performed with explicitly recorded numerical precision. For large N, only the lowest portion of the spectrum is computed, as higher eigenvalues are more sensitive to truncation artifacts.

Eigenvalues are ordered increasingly and denoted by

$$\lambda_1^{(N)} \leq \lambda_2^{(N)} \leq \cdots .$$

1.6 Reference zeta zero data

Spectral comparisons are performed against certified nontrivial zeros of the Riemann zeta function on the critical line.

Let $\{\gamma_n\}$ denote the ordered set of imaginary parts of zeros used as reference data. All zero sets employed in this work satisfy the certification criteria defined in Whitepaper IV, including checkpoint validation.

The number of reference zeros used in each comparison is matched to the number of reliably computed eigenvalues to avoid extrapolation bias.

1.7 Parameter ranges and refinement strategy

The following parameters are varied to assess stability:

- truncation dimension N,
- prime cutoff $P_{\max}$,
- potential coefficients α and β,
- numerical precision and solver tolerances.

Parameter sweeps are performed systematically, with each configuration recorded as a separate reproducible run. Only trends that persist across multiple refinement levels are interpreted as meaningful.

1.8 Summary

This section fixes the concrete numerical realization of the SMRK Hamiltonian used throughout this paper. By explicitly specifying the operator form, truncation scheme, and reference data, it ensures that all reported results are reproducible and interpretable within a well-defined context.

2 Spectral Matching Results

2.1 Objective of spectral comparison

The primary numerical test for the SMRK Hamiltonian is the comparison of its spectrum with the imaginary parts of the nontrivial zeros of the Riemann zeta function.

This section reports direct spectral matching results between the eigenvalues of truncated SMRK Hamiltonians and certified zeta zero data. The purpose is to quantify the degree of agreement, its dependence on truncation parameters, and its stability under refinement.

2.2 Eigenvalue–zero alignment

Let

$$\lambda_1^{(N)} \leq \lambda_2^{(N)} \leq \cdots$$

denote the ordered eigenvalues of the truncated SMRK Hamiltonian acting on $\mathcal{H}_N$, and let

$$\gamma_1 \leq \gamma_2 \leq \cdots$$

denote the ordered imaginary parts of certified nontrivial zeros of the Riemann zeta function.

Spectral alignment is assessed by comparing $\lambda_n^{(N)}$ with γ_n for indices n within a stable spectral window. The comparison is restricted to ranges where boundary effects and numerical instabilities are negligible.

2.3 Normalized spectral deviation

To quantify spectral agreement, we define the normalized deviation

$$\Delta_n^{(N)} = \frac{|\lambda_n^{(N)} - \gamma_n|}{\gamma_n}.$$

This quantity allows comparison across different spectral heights and truncation dimensions. Aggregate measures such as

$$\overline{\Delta}^{(N)} = \frac{1}{M} \sum_{n=1}^{M} \Delta_n^{(N)}$$

are reported for fixed M and varied N, where M denotes the number of reliably resolved eigenvalues.

2.4 Dependence on truncation dimension

Spectral deviation is examined as a function of the truncation dimension N. As N increases, one of the following behaviors is observed:

- monotone decrease of $\overline{\Delta}^{(N)}$,
- saturation at a finite error level,
- growth or oscillation indicating instability.

Only the first behavior is interpreted as evidence of asymptotic spectral correspondence. The other behaviors signal truncation artifacts or structural incompatibility.

2.5 Low-lying versus higher eigenvalues

Low-lying eigenvalues often exhibit larger relative deviations, due to stronger sensitivity to boundary conditions and diagonal potential terms.

For this reason, spectral agreement is evaluated separately for:

- low-index eigenvalues,
- intermediate spectral ranges,
- the highest reliably computed eigenvalues.

This stratification helps distinguish localized discrepancies from global spectral trends.

2.6 Robustness under parameter variation

The spectral comparison is repeated for multiple values of the parameters α, β, $P_{\max}$, and N. Parameter regimes yielding qualitatively similar spectral alignment are identified.

Conversely, parameter choices that produce spurious agreement only within narrow tuning windows are flagged as unstable. Such regimes are documented but not treated as evidence of structural validity.

2.7 Summary of spectral findings

The results of this section demonstrate that the SMRK Hamiltonian can exhibit partial spectral alignment with the nontrivial zeros of the Riemann zeta function over finite ranges and resolutions.

However, spectral matching alone is insufficient to establish relevance to the Riemann Hypothesis. Accordingly, these results are interpreted as preliminary consistency checks, to be complemented by statistical, arithmetic, and stability diagnostics in subsequent sections.

3 Level Spacing Statistics

3.1 Motivation for spacing analysis

While pointwise spectral matching tests whether individual eigenvalues approximate the imaginary parts of zeta zeros, it does not capture the local statistical structure of the spectrum.

Level spacing statistics probe short-range correlations between neighboring eigenvalues. For the nontrivial zeros of the Riemann zeta function, extensive numerical evidence indicates agreement with the Gaussian Unitary Ensemble (GUE) of random matrix theory.

In this section, we examine whether the spectrum of the SMRK Hamiltonian exhibits compatible local statistics, independently of absolute alignment.

3.2 Nearest-neighbor spacings

Let

$$\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_M$$

denote a sequence of reliably computed eigenvalues of the truncated SMRK Hamiltonian.

The nearest-neighbor spacings are defined as

$$s_n = \lambda_{n+1} - \lambda_n, \qquad n = 1, \ldots, M-1.$$

Raw spacings depend on the local density of eigenvalues. To allow comparison across spectral regions, the spectrum must be unfolded.

3.3 Spectral unfolding

Unfolding transforms the eigenvalues to a new sequence with approximately unit mean spacing. Let $N(\lambda)$ denote the integrated spectral density, estimated either empirically or via a smooth fit.

The unfolded eigenvalues are given by

$$\tilde{\lambda}_n = N(\lambda_n),$$

and the unfolded spacings by

$$\tilde{s}_n = \tilde{\lambda}_{n+1} - \tilde{\lambda}_n.$$

Care is taken to ensure that the unfolding procedure does not artificially impose random-matrix-like behavior. Multiple unfolding methods are tested to assess robustness.

3.4 Comparison with random matrix ensembles

The empirical distribution of unfolded spacings $\tilde{s}_n$ is compared with standard random matrix ensembles. The primary reference is the Gaussian Unitary Ensemble (GUE), whose spacing distribution is

$$P_{\mathrm{GUE}}(s) = \frac{32}{\pi^2} s^2 e^{-4s^2/\pi}.$$

Alternative ensembles, such as the Gaussian Orthogonal Ensemble (GOE) and Poisson statistics, are used as controls.

Deviation between the empirical distribution and reference distributions is quantified using the Kolmogorov–Smirnov distance and related measures.

3.5 Finite-size and boundary effects

Spacing statistics are sensitive to finite-size effects, particularly for low-lying eigenvalues and near truncation boundaries.

To mitigate these effects, statistics are computed only on interior spectral windows, excluding regions strongly affected by truncation or diagonal dominance.

Results are reported for multiple window sizes to assess consistency.

3.6 Results and observed trends

Across tested parameter regimes, the spacing statistics of the SMRK Hamiltonian exhibit mixed behavior.

In certain configurations, the unfolded spacings show partial agreement with GUE predictions over limited spectral ranges. In other regimes, significant deviations are observed, including excess level clustering or near-Poisson behavior.

These observations indicate that local spectral correlations are highly sensitive to parameter choices and truncation details.

3.7 Interpretation and limitations

Agreement with GUE-like statistics is interpreted as supportive evidence of compatibility with known zeta zero phenomenology, but not as confirmation of arithmetic relevance.

Conversely, failure to reproduce GUE statistics does not necessarily invalidate the SMRK Hamiltonian. Finite-size effects, incomplete unfolding, or unresolved truncation artifacts may obscure universal behavior.

Accordingly, level spacing statistics are treated as a diagnostic tool, to be considered alongside spectral matching, explicit formula residuals, and stability tests.

3.8 Summary

Level spacing analysis provides an important statistical perspective on the SMRK Hamiltonian spectrum. While partial agreement with random matrix predictions can be observed in selected regimes, the results emphasize the necessity of complementary diagnostics and careful interpretation.

4 Explicit Formula Residual Results

4.1 Purpose of the explicit formula test

Spectral agreement and level spacing statistics do not by themselves guarantee arithmetic relevance. The explicit formula provides a direct bridge between the distribution of primes and the spectral data.

In this section, we evaluate the SMRK Hamiltonian using explicit formula diagnostics, comparing prime-side quantities with zero-side reconstructions derived from the operator spectrum. The resulting residuals serve as a stringent test of arithmetic compatibility.

4.2 Prime-side reference quantities

Let $\Lambda(n)$ denote the von Mangoldt function. We consider smoothed prime-weighted sums of the form

$$\Psi_f(X) = \sum_{n=1}^{\infty} \Lambda(n)\, f(\log n - \log X),$$

where f is a smooth test function with compact support or rapid decay.

In numerical implementations, the sum is truncated according to the support of f and recorded with controlled error bounds. These quantities serve as independent arithmetic reference data.

4.3 Zero-side reconstruction from the SMRK spectrum

Given a finite set of eigenvalues $\{\lambda_n\}$ of the truncated SMRK Hamiltonian, we construct the zero-side approximation

$$\Psi_{f,H}(X) = X\widehat{f}(0) - \sum_{n} X^{\rho_n}\, \widehat{f}(\rho_n), \qquad \rho_n = \tfrac{1}{2} + i\lambda_n,$$

where $\widehat{f}$ denotes the Mellin transform of f.

This construction mirrors the truncated explicit formula used for certified zeta zeros, with operator eigenvalues playing the role of imaginary parts of nontrivial zeros.

4.4 Definition of residuals

The explicit formula residual associated with the SMRK Hamiltonian is defined as

$$R_H(X) = \Psi_f(X) - \Psi_{f,H}(X).$$

Under ideal circumstances, $R_H(X)$ should consist primarily of truncation effects, smoothing artifacts, and controlled lower-order terms. Persistent structured deviations indicate incompatibility between the operator-induced spectrum and the arithmetic data.

4.5 Residual magnitude and scaling

Residuals are evaluated over a logarithmic grid of X values. Their magnitude is quantified using norms such as

$$\|R_H\|^2 = \int_{X_{\min}}^{X_{\max}} |R_H(X)|^2\, dX,$$

as well as pointwise statistics.

The dependence of residual norms on truncation dimension, prime cutoff, and smoothing parameters is systematically recorded.

4.6 Stability under refinement

A central diagnostic is the behavior of residuals under refinement. Specifically, we examine whether:

- residual magnitudes decrease as truncation dimension increases,
- residual structure remains stable under changes in the test function f,
- oscillatory features persist across refinement levels.

Residual patterns that remain stable under refinement are interpreted as structural. Those that diminish or change qualitatively are attributed to truncation artifacts.

4.7 Observed behavior

Numerical experiments reveal that explicit formula residuals for the SMRK Hamiltonian exhibit nontrivial structure.

In certain parameter regimes, residual magnitudes decrease modestly with increasing resolution, suggesting partial arithmetic compatibility. In other regimes, persistent oscillatory residuals remain, indicating a mismatch between the operator spectrum and prime-weighted sums.

These findings demonstrate that explicit formula diagnostics are more discriminating than spectral or spacing tests alone.

4.8 Interpretation and implications

The residual analysis confirms that arithmetic compatibility is a strong and restrictive constraint. Even operators that perform well in spectral matching may fail explicit formula tests.

The SMRK Hamiltonian does not uniformly pass the explicit formula diagnostics across all tested regimes. This outcome does not invalidate the operator-based approach, but highlights the sensitivity of arithmetic structure to operator details.

4.9 Summary

Explicit formula residuals provide the most stringent numerical test applied in this study. They reveal both promising behavior in selected regimes and clear failure modes in others.

These results underscore the necessity of combining spectral, statistical, and arithmetic diagnostics when evaluating candidate operators for relevance to the Riemann Hypothesis.

5 Stability Under Refinement

5.1 Motivation

Numerical agreement observed at a fixed resolution may arise from truncation artifacts, parameter tuning, or accidental cancellations. To distinguish such effects from structurally

meaningful behavior, it is essential to study stability under systematic refinement.

In this section, we examine how the numerical properties of the SMRK Hamiltonian evolve as resolution parameters are varied. Only features that persist across multiple refinement levels are treated as significant.

5.2 Refinement dimensions

Refinement is performed along several independent axes:

- truncation dimension N of the Hilbert space,
- prime cutoff parameter $P_{\max}$,
- numerical precision and solver tolerances,
- spectral window size used for diagnostics,
- smoothing parameters in explicit formula tests.

Each refinement dimension is varied independently and in selected coupled sweeps, allowing interactions between parameters to be identified.

5.3 Spectral stability

Spectral stability is assessed by monitoring the behavior of eigenvalues $\lambda_n^{(N)}$ as N increases.

For fixed index n, a stable spectral feature is characterized by convergence

$$\lambda_n^{(N)} \longrightarrow \lambda_n \qquad \text{as } N \to \infty,$$

up to controlled numerical error.

In practice, convergence is evaluated by tracking relative changes

$$\frac{|\lambda_n^{(N_2)} - \lambda_n^{(N_1)}|}{|\lambda_n^{(N_1)}|}$$

for increasing truncation levels $N_1 < N_2$.

Eigenvalues exhibiting erratic drift, splitting, or reordering under refinement are classified as unstable and excluded from sensitive diagnostics.

5.4 Stability of spectral matching

The normalized spectral deviation defined in Section 2 is examined across refinement levels. A stable matching regime is indicated by:

- decreasing or bounded deviation as N and $P_{\max}$ increase,
- consistent ordering of matched eigenvalues,

- absence of abrupt error growth near truncation boundaries.

Failure of these conditions signals that apparent agreement at low resolution does not extrapolate.

5.5 Stability of level spacing statistics

Level spacing statistics are recomputed for increasing truncation dimensions and for different interior spectral windows.

Stability is indicated by:

- similar unfolded spacing distributions across refinement levels,
- comparable goodness-of-fit metrics with respect to reference ensembles,
- robustness to moderate changes in unfolding procedures.

Strong dependence of spacing statistics on truncation size or window selection is interpreted as evidence of finite-size dominance.

5.6 Stability of explicit formula residuals

Explicit formula residuals provide the most sensitive refinement test. Residual functions $R_H(X)$ are examined for:

- reduction in overall magnitude with increasing resolution,
- persistence or decay of structured oscillatory components,
- qualitative similarity across smoothing parameter choices.

Residual features that remain stable under simultaneous refinement of truncation dimension, prime cutoff, and smoothing scale are interpreted as intrinsic to the operator structure.

5.7 Cross-parameter interactions

Refinement along one axis may amplify or suppress instability along another. For example, increasing $P_{\max}$ without a corresponding increase in N can introduce boundary artifacts, while increased N with insufficient prime coverage may underrepresent arithmetic structure.

Such interactions are explicitly monitored, and only parameter regimes exhibiting coherent behavior across multiple axes are considered reliable.

5.8 Summary of refinement behavior

The refinement analysis reveals that the SMRK Hamiltonian exhibits a mixture of stable and unstable features.

Certain spectral and statistical properties persist under refinement, suggesting genuine structural behavior. Other apparent agreements degrade as resolution increases, indicating truncation-induced artifacts.

These findings reinforce the necessity of refinement-based validation and motivate the careful interpretation of numerical agreement as provisional rather than definitive.

6 Negative Results and Failure Cases

6.1 Role of negative results

Negative results play a central role in the experimental framework adopted in this work. Because the objective is falsification rather than confirmation, documenting failure modes is as important as reporting partial successes.

This section records parameter regimes, numerical configurations, and operator variants for which the SMRK Hamiltonian fails to meet one or more diagnostic criteria. These results delimit the space of plausible operator constructions and help prevent overinterpretation of isolated positive findings.

6.2 Spectral overfitting at low truncation

In several parameter configurations, good spectral alignment with zeta zeros was observed only at low truncation dimensions. As the truncation dimension N increased, spectral deviations grew or eigenvalue ordering became unstable.

Such behavior is interpreted as overfitting to finite spectral data. Operators exhibiting this pattern were excluded from further analysis, even if initial spectral comparisons appeared favorable.

6.3 Instability under prime cutoff variation

The prime cutoff parameter $P_{\max}$ plays a critical role in shaping the arithmetic structure of the SMRK Hamiltonian.

In some regimes, increasing $P_{\max}$ without proportional increase in N led to pronounced boundary effects, manifesting as spectral distortion and increased explicit formula residuals.

Conversely, overly restrictive prime cutoffs failed to capture sufficient arithmetic structure, resulting in near-Poisson spectral statistics. These regimes were classified as inadequate.

6.4 Failure of explicit formula compatibility

Several operator variants exhibited acceptable spectral matching and level spacing statistics, yet failed explicit formula diagnostics.

In these cases, explicit formula residuals remained large or structurally unstable under refinement, indicating a fundamental mismatch between the operator-induced spectrum and prime-weighted sums.

These failures demonstrate that explicit formula tests are strictly stronger than purely spectral diagnostics.

6.5 Parameter sensitivity and fine tuning

Some configurations produced apparent agreement with diagnostics only within narrow parameter windows. Small perturbations of parameters α, β, or $P_{\max}$ resulted in rapid degradation of spectral or arithmetic scores.

Such fine-tuned regimes are not considered structurally meaningful. The framework treats robustness to moderate parameter variation as a necessary condition for relevance.

6.6 Numerical pathologies

Additional failure modes arose from numerical issues, including:

- loss of orthogonality in eigenvectors at large N,
- sensitivity to solver tolerances,
- accumulation of round-off errors in explicit formula sums.

While these issues can sometimes be mitigated through increased precision or alternative algorithms, they nonetheless impose practical limits on achievable resolution.

6.7 Implications for operator design

The documented failures highlight several lessons for future operator constructions:

- spectral matching alone is insufficient,
- arithmetic structure must be encoded robustly,
- stability under refinement is a non-negotiable requirement,
- fine tuning should be viewed with skepticism.

These observations inform the design of modified or alternative operators and motivate the exploration of broader operator families.

6.8 Summary

The negative results reported here demonstrate the discriminating power of the experimental framework. By explicitly recording failure cases, the study narrows the landscape of plausible operator models and strengthens the credibility of any surviving candidates.

7 Discussion and Interpretation

7.1 Summary of numerical findings

The numerical experiments reported in this work provide a multifaceted assessment of the SMRK Hamiltonian as a candidate operator within the Hilbert–Pólya program.

Across spectral matching, level spacing statistics, explicit formula diagnostics, and refinement analysis, the results reveal a nuanced picture. The SMRK Hamiltonian exhibits partial compatibility with known properties of the nontrivial zeros of the Riemann zeta function, but also displays clear limitations and failure modes.

These findings underscore the importance of evaluating operator proposals along multiple independent dimensions, rather than relying on a single diagnostic.

7.2 Interpretation of spectral agreement

The observed spectral alignment between SMRK eigenvalues and zeta zeros over finite ranges suggests that arithmetic transfer operators can encode nontrivial spectral structure.

However, the lack of uniform convergence under refinement indicates that spectral agreement alone does not establish relevance to the Riemann Hypothesis. Instead, such agreement should be viewed as an initial filter: necessary, but far from sufficient.

This interpretation aligns with the broader understanding that many constructions can reproduce finite spectral data without capturing the underlying arithmetic dynamics.

7.3 Statistical structure versus arithmetic content

Level spacing statistics provide insight into local spectral correlations. Partial agreement with GUE-like behavior in selected regimes is consistent with known phenomenology of zeta zeros.

At the same time, the sensitivity of spacing statistics to truncation and parameter choices highlights their limited discriminating power. Statistical compatibility does not uniquely characterize arithmetic origin and must be interpreted in conjunction with explicit arithmetic tests.

7.4 Explicit formula diagnostics as a decisive filter

Among all diagnostics employed, explicit formula residuals emerge as the most restrictive. They directly test whether the operator-induced spectrum can reproduce prime-weighted sums, rather than merely mimic spectral patterns.

The persistence of structured residuals in many parameter regimes demonstrates that encoding arithmetic content is substantially more demanding than achieving spectral resemblance. This observation reinforces the view that arithmetic compatibility is the defining constraint for any operator relevant to the Riemann Hypothesis.

7.5 Stability and structural significance

Stability under refinement serves as a practical proxy for structural validity. Features that degrade as resolution increases are unlikely to reflect intrinsic properties of the underlying operator.

The mixed stability behavior observed for the SMRK Hamiltonian suggests that certain aspects of its construction capture meaningful structure, while others remain sensitive to truncation artifacts or parameter tuning.

This distinction provides guidance for future refinements and modifications of arithmetic transfer operators.

7.6 Implications for the Hilbert–Pólya program

The results presented here do not support the conclusion that the SMRK Hamiltonian realizes the Hilbert–Pólya conjecture. They do, however, demonstrate that the conjecture can be approached as a falsifiable experimental program.

By embedding candidate operators within a disciplined numerical framework, it becomes possible to systematically exclude incompatible models and focus analytical effort on constructions that survive stringent tests.

This shift from speculative construction to constrained evaluation represents a methodological advance in operator-based research on RH.

7.7 Lessons for future operator constructions

Several general lessons emerge:

- arithmetic structure must be encoded robustly,
- spectral agreement must persist under refinement,
- explicit formula compatibility is essential,
- fine-tuned parameter regimes are suspect.

These lessons apply not only to the SMRK Hamiltonian, but to any future operator proposal aimed at explaining zeta zero statistics.

7.8 Concluding perspective

The numerical study presented in this work illustrates both the promise and the limitations of arithmetic Hamiltonian approaches to the Riemann Hypothesis.

While no definitive conclusions can be drawn regarding RH itself, the framework succeeds in transforming operator proposals into testable scientific objects. In doing so, it helps clarify which directions merit deeper analytical investigation and which are unlikely to succeed.

The value of this work lies not in resolving the Riemann Hypothesis, but in advancing a structured pathway toward understanding the interplay between arithmetic, operators, and spectral phenomena.

A Numerical Parameters and Configurations

This appendix records the numerical parameters used in the experiments reported in this work. All results in Sections 2–6 should be interpreted relative to these configurations.

Unless stated otherwise, each parameter set corresponds to an independent reproducible run, as defined by the reproducibility protocol in Whitepaper IV.

A.1 Truncation parameters

The SMRK Hamiltonian is truncated to finite-dimensional subspaces $\mathcal{H}_N \subset \ell^2(\mathbb{N})$. The truncation dimension N is varied to assess stability.

Typical values used in the experiments include:

```
N  {200, 400, 800, 1200, 2000}
```

Only interior spectral regions, sufficiently separated from truncation boundaries, are used for sensitive diagnostics.

A.2 Prime cutoff

The prime cutoff parameter $P_{\max}$ limits the range of primes included in the transfer terms.

Representative values include:

```
P_max  {11, 23, 47, 97, 197}
```

For each truncation dimension N, $P_{\max}$ is chosen such that prime-induced transitions do not dominate boundary effects.

A.3 Diagonal potential coefficients

The diagonal arithmetic potential is parameterized by real coefficients α and β:

$$U(n) = \alpha\Lambda(n) + \beta \log n.$$

Explored parameter ranges include:

```
[0.0, 1.0]
[-1.0, 1.0]
```

Parameter sweeps are performed with step sizes chosen to detect sensitivity without fine tuning.

A.4 Numerical precision and solvers

Eigenvalue computations are performed using symmetric eigensolvers, with numerical precision adjusted according to truncation size.

Typical settings include:

```
- double precision for N  800
- extended precision for N > 800
- solver tolerance  10^{-10}
```

Precision settings are logged for each run and included in the reproducibility manifest.

A.5 Reference zero data

Certified nontrivial zeros of the Riemann zeta function are used as reference data.

For each truncation level, only as many zeros are used as can be reliably matched to stable eigenvalues. Zero sets satisfy the certification criteria described in Whitepaper IV, including checkpoint validation.

A.6 Explicit formula parameters

Explicit formula diagnostics use smoothed test functions f with compact support or rapid decay.

Representative choices include:

```
- Gaussian kernels with   {0.1, 0.2, 0.3}
- compact bump functions with fixed width
```

Residuals are evaluated on logarithmic grids spanning multiple scales in X.

A.7 Run identification

Each numerical run is assigned a deterministic run identifier, computed as a hash of:

- code revision identifier,
- serialized configuration parameters,
- environment summary.

This identifier is referenced by all output artifacts and manifests associated with the run.

A.8 Notes on extensibility

The parameter ranges reported here are not intended to be exhaustive. They represent a baseline configuration sufficient to reveal both partial agreement and failure modes.

Future studies may extend these ranges or introduce additional parameters, provided that reproducibility and stability analysis are maintained.

B Data Availability and Provenance

This appendix documents the availability and provenance of numerical data generated for the experiments reported in this work.

The intent is to ensure that all reported results can be independently verified, reproduced, or extended.

B.1 Generated datasets

Primary datasets produced by the numerical experiments include:

- lists of SMRK Hamiltonian eigenvalues,
- certified zeta zero reference lists,
- spectral deviation tables,
- level spacing data,
- explicit formula residual arrays,
- checkpoint and refinement logs.

Each dataset is treated as an immutable artifact once generated.

B.2 File formats

To ensure long-term accessibility, data are stored in stable, open formats:

- CSV or TSV for tabular data,
- plain text manifests for metadata,
- optional binary formats (e.g. NPY) accompanied by schema documentation.

All files include headers describing column meanings and units.

B.3 Manifests and checksums

Each experimental run produces a manifest file recording:

```
- run identifier
- creation timestamp (UTC)
- code revision hash
- configuration hash
- environment summary
- list of artifacts with SHA-256 checksums
```

Manifests serve as the primary integrity anchor for the entire run.

B.4 Access and distribution

At the time of writing, numerical datasets are maintained by the author and can be made available upon reasonable request.

Future releases may include:

- public repository snapshots,
- archival storage with persistent identifiers,
- cryptographic timestamping of manifests and data bundles.

Any public release will preserve the original hashes to maintain continuity with the results reported here.

B.5 Reproducibility statement

All results in this paper are reproducible in principle from the recorded code revisions, configuration parameters, and environment descriptions.

While numerical precision and platform differences may introduce minor variation, the qualitative behaviors reported (spectral trends, residual structures, and stability properties) are robust under recomputation.

B.6 Citation and reuse

Researchers wishing to reuse the data or methodology are encouraged to cite both this work and Whitepaper IV, which defines the underlying experimental framework.

Reuse of data should preserve the associated manifest files to maintain provenance and reproducibility.