A Self-Adjoint Operator Framework Toward a Hilbert–Pólya Program

Enter Yourname

Contents

1 Mathematical Setting and Definitions

This section establishes the precise mathematical framework in which the operator studied in this work is defined. No spectral, arithmetic, or conjectural claims are made here. The sole purpose of this section is to specify the underlying Hilbert space, notation, and basic objects with complete rigor.

1.1 Underlying Hilbert Space

Let $\mathbb{N} = \{1, 2, 3, \dots\}$ denote the set of positive integers. We consider the Hilbert space

$$\mathcal{H} := \ell^2(\mathbb{N}),$$

consisting of all complex-valued sequences

$$\psi : \mathbb{N} \to \mathbb{C}$$

such that

$$\|\psi\|_{\mathcal{H}}^2 = \sum_{n=1}^{\infty} |\psi(n)|^2 < \infty.$$

The inner product on $\mathcal{H}$ is defined by

$$\langle \psi, \phi \rangle = \sum_{n=1}^{\infty} \overline{\psi(n)}\, \phi(n), \qquad \psi, \phi \in \mathcal{H}.$$

This Hilbert space is separable and admits the standard orthonormal basis

$$\{e_n\}_{n \in \mathbb{N}}, \qquad e_n(m) = \delta_{n,m}.$$

1.2 Arithmetic Notation

We use the following standard arithmetic functions throughout:

- $\mathcal{P}$ denotes the set of prime numbers.
- $\Lambda(n)$ denotes the von Mangoldt function:

$$\Lambda(n) = \begin{cases} \log p, & \text{if } n = p^k \text{ for some prime } p \text{ and } k \geq 1, \\ 0, & \text{otherwise.} \end{cases}$$

All sums over primes are taken over $\mathcal{P}$ and are understood in the sense specified explicitly when introduced.

1.3 Dense Subspace

Let $\mathcal{D} \subset \mathcal{H}$ denote the vector space of finitely supported sequences:

$$\mathcal{D} = \{\psi \in \mathcal{H} \mid \exists N \in \mathbb{N} \text{ such that } \psi(n) = 0 \text{ for all } n > N\}.$$

The subspace $\mathcal{D}$ is dense in $\mathcal{H}$.

Every element of $\ell^2(\mathbb{N})$ can be approximated in norm by its finite truncations.

All operators introduced in this work will be defined initially on $\mathcal{D}$.

1.4 Multiplicative Shift Operators

For each prime $p \in \mathcal{P}$, we define the forward and backward multiplicative shift operators

$$(T_p^+\psi)(n) := \psi(pn), \qquad (T_p^-\psi)(n) := \begin{cases} \psi(n/p), & p \mid n, \\ 0, & p \nmid n. \end{cases}$$

These operators map $\mathcal{D}$ into itself.

For each prime p, the operators T_p^+ and T_p^- are well-defined linear operators on $\mathcal{D}$.

If ψ has finite support, then both pn and n/p range over finite sets. Hence the resulting sequences are also finitely supported.

1.5 Diagonal Arithmetic Operators

Let $V : \mathbb{N} \to \mathbb{R}$ be a real-valued function. We define the associated multiplication operator

$$(M_V\psi)(n) := V(n)\,\psi(n), \qquad \psi \in \mathcal{D}.$$

In particular, we will consider diagonal operators generated by combinations of $\Lambda(n)$ and $\log n$. No assumptions on boundedness are made at this stage.

1.6 Operator Framework

The operator studied in subsequent sections will be constructed as a linear combination of:

- multiplicative shift operators $T_p^\pm$,
- diagonal multiplication operators M_V,

defined on the common dense domain $\mathcal{D}$.

At this stage, no claims are made regarding:

- symmetry,
- closability,
- self-adjointness,

- spectral properties.

These questions are addressed systematically in the following sections.

1.7 Scope of This Section

This section is purely definitional. All statements made here are elementary and verifiable directly from the definitions. No conjectural or heuristic assumptions are used.

2 Domain and Symmetry of the Operator

In this section we define the operator studied in this work on a common dense domain and establish its symmetry. No claims regarding self-adjointness or spectral properties are made here. The goal is to verify that the operator is a legitimate symmetric operator on a Hilbert space, thereby qualifying as a candidate for a Hilbert–Pólya type construction.

2.1 Definition of the Operator

Let $\alpha, \beta \in \mathbb{R}$ be fixed real parameters. We define a linear operator

$$H : \mathcal{D} \to \mathcal{H}$$

by the action

$$(H\psi)(n) = \sum_{p \in \mathcal{P}} \frac{1}{p}\Big(\psi(pn) + \mathbf{1}_{p|n}\, \psi(n/p)\Big) + \big(\alpha\, \Lambda(n) + \beta\, \log n\big)\psi(n), \tag{1}$$

for all $\psi \in \mathcal{D}$ and $n \in \mathbb{N}$. Here $\mathbf{1}_{p|n}$ denotes the indicator function of the divisibility relation.

For every $\psi \in \mathcal{D}$, the sum in (1) is finite for each n, and $H\psi \in \mathcal{D}$.

Since ψ has finite support, $\psi(pn) \neq 0$ for only finitely many primes p. Likewise, for fixed n, the condition $p \mid n$ holds for only finitely many primes. Thus the sum over p is finite. Moreover, the support of $H\psi$ is contained in a finite set determined by the support of ψ.

2.2 Domain Considerations

Throughout this work, the operator H is initially defined on the dense subspace $\mathcal{D} \subset \mathcal{H}$. All algebraic manipulations in this section take place on $\mathcal{D}$.

No extension beyond $\mathcal{D}$ is assumed at this stage. Questions of closability and self-adjointness are deferred to the next section.

2.3 Adjoint Relations of the Shift Operators

Let $p \in \mathcal{P}$ be fixed. For $\psi, \phi \in \mathcal{D}$, we compute

$$\langle T_p^+ \psi, \phi \rangle = \sum_{n=1}^{\infty} \overline{\psi(pn)}\, \phi(n) = \sum_{m=1}^{\infty} \overline{\psi(m)}\, \mathbf{1}_{p|m}\, \phi(m/p) = \langle \psi, T_p^- \phi \rangle.$$

On the domain $\mathcal{D}$, the operators T_p^+ and T_p^- are formal adjoints:

$$(T_p^+)^* = T_p^-, \qquad (T_p^-)^* = T_p^+.$$

2.4 Symmetry of the Diagonal Part

Let $V(n) = \alpha\, \Lambda(n) + \beta \log n$. Since $V(n) \in \mathbb{R}$ for all n, the associated multiplication operator M_V satisfies

$$\langle M_V \psi, \phi \rangle = \langle \psi, M_V \phi \rangle \qquad \forall \psi, \phi \in \mathcal{D}.$$

2.5 Symmetry of the Full Operator

We now establish the main result of this section.

The operator H defined in (1) is symmetric on $\mathcal{D}$, i.e.

$$\langle H\psi, \phi \rangle = \langle \psi, H\phi \rangle \qquad \forall \psi, \phi \in \mathcal{D}.$$

By linearity, it suffices to verify symmetry separately for each component.

The prime-shift part satisfies

$$\left\langle \sum_{p \in \mathcal{P}} \frac{1}{p} (T_p^+ + T_p^-)\psi, \phi \right\rangle = \left\langle \psi, \sum_{p \in \mathcal{P}} \frac{1}{p} (T_p^+ + T_p^-)\phi \right\rangle,$$

by the adjoint relations established above.

The diagonal part is symmetric by construction. Combining these observations yields the claim.

2.6 Remarks

- Symmetry alone does not imply self-adjointness.
- At this stage, the operator H may admit multiple self-adjoint extensions, or none.
- The purpose of this section is solely to establish H as a well-defined symmetric operator on a dense domain.

The question of essential self-adjointness is addressed in the next section.

3 Essential Self-Adjointness

This section addresses the central structural question of the operator H: whether the symmetric operator defined on the dense domain $\mathcal{D}$ admits a unique self-adjoint extension. This property is a necessary condition for any Hilbert–Pólya type interpretation.

3.1 Preliminaries

Let H denote the symmetric operator introduced in Section 2, with domain $\mathcal{D} \subset \mathcal{H} = \ell^2(\mathbb{N})$.

Recall that a symmetric operator H is called *essentially self-adjoint* if its closure $\overline{H}$ is self-adjoint, or equivalently, if H admits a unique self-adjoint extension.

We emphasize that symmetry alone does not guarantee this property. The analysis below is therefore indispensable.

3.2 Quadratic Form Associated to H

For $\psi \in \mathcal{D}$, define the quadratic form

$$Q[\psi] := \langle \psi, H\psi \rangle. \tag{2}$$

By symmetry of H, the form Q is real-valued. Explicitly,

$$\begin{aligned} Q[\psi] = \sum_{p \in \mathcal{P}} \frac{1}{p} \left(\sum_{n=1}^{\infty} \overline{\psi(n)}\, \psi(pn) + \sum_{n=1}^{\infty} \overline{\psi(pn)}\, \psi(n) \right) \\ + \sum_{n=1}^{\infty} (\alpha\, \Lambda(n) + \beta\, \log n) |\psi(n)|^2. \end{aligned} \tag{3}$$

All sums are finite for $\psi \in \mathcal{D}$.

3.3 Lower Semiboundedness

There exists a constant $C \in \mathbb{R}$ such that

$$Q[\psi] \geq -C\, \|\psi\|_{\mathcal{H}}^2 \qquad \forall \psi \in \mathcal{D}.$$

The prime-shift contribution satisfies the estimate

$$\left| \sum_{p \in \mathcal{P}} \frac{1}{p} \sum_{n=1}^{\infty} \overline{\psi(n)}\, \psi(pn) \right| \leq \left(\sum_{p \in \mathcal{P}} \frac{1}{p^2} \right)^{1/2} \|\psi\|_{\mathcal{H}}^2,$$

by Cauchy–Schwarz and the square-summability of $(1/p)$.

The diagonal terms involving $\Lambda(n)$ and $\log n$ are real-valued. Since $\Lambda(n) \geq 0$ and $\log n \geq 0$ for $n \geq 2$, any possible negative contribution can be absorbed into a global constant depending on α and β.

Combining these estimates yields the claim.

3.4 Closability and Friedrichs Extension

By Lemma 3.3, the quadratic form Q is lower semibounded. It follows that Q is closable and admits a unique closed extension $\overline{Q}$.

The operator H admits a self-adjoint Friedrichs extension H_F associated with the closed form $\overline{Q}$.

This is a standard consequence of the representation theorem for closed, densely defined, lower semibounded quadratic forms.

3.5 Essential Self-Adjointness Criterion

The existence of a Friedrichs extension alone does not imply essential self-adjointness. To establish uniqueness, one must show that no other self-adjoint extensions exist.

We therefore consider the deficiency spaces

$$\mathcal{N}_{\pm} := \ker(H^* \mp i).$$

[Gatekeeper Theorem] If the deficiency indices of H satisfy

$$\dim \mathcal{N}_{+} = \dim \mathcal{N}_{-} = 0,$$

then H is essentially self-adjoint on $\mathcal{D}$.

This theorem is standard; see, e.g., Reed and Simon, Vol. II.

3.6 Reduction of the Problem

At this point, the question of essential self-adjointness is reduced to the absence of nontrivial $\ell^2(\mathbb{N})$ solutions of

$$H^*\psi = \pm i\psi.$$

Equivalently, one must exclude square-summable solutions of the difference equation

$$\sum_{p\in\mathcal{P}} \frac{1}{p}\Big(\psi(pn) + \mathbf{1}_{p|n}\psi(n/p)\Big) + \big(\alpha\,\Lambda(n) + \beta\,\log n\big)\psi(n) = \pm i\,\psi(n). \tag{4}$$

3.7 Status of the Result

- The operator H is symmetric and lower semibounded.
- A canonical self-adjoint (Friedrichs) extension exists.
- Essential self-adjointness is reduced to the nonexistence of ℓ^2 solutions of (4).

A proof of the vanishing of the deficiency spaces would establish essential self-adjointness and thereby legitimize the operator as a Hilbert–Pólya candidate.

This analysis is deferred to future work.

4 Spectral Framework

This section outlines the spectral-theoretic setting relevant to the operator H, independently of any arithmetic interpretation. No claims are made here regarding the location of eigenvalues or their relation to zeros of zeta or L-functions.

4.1 Self-Adjoint Extensions and Spectra

Let H be the symmetric operator defined on $\mathcal{D} \subset \mathcal{H}$. Whenever a self-adjoint extension $\widetilde{H}$ of H exists, the spectral theorem applies and yields a projection-valued measure $E_{\widetilde{H}}(\lambda)$ such that

$$\widetilde{H} = \int_{\mathbb{R}} \lambda \, dE_{\widetilde{H}}(\lambda).$$

The spectrum $\sigma(\widetilde{H})$ decomposes into:

$$\sigma(\widetilde{H}) = \sigma_{\mathrm{pp}} \, \cup \, \sigma_{\mathrm{ac}} \, \cup \, \sigma_{\mathrm{sc}},$$

corresponding to pure point, absolutely continuous, and singular continuous parts.

At this stage, no assumption is made regarding which components are present.

4.2 Resolvent and Spectral Measures

For $z \in \mathbb{C} \setminus \mathbb{R}$, the resolvent operator

$$R(z) := (\widetilde{H} - z)^{-1}$$

is bounded and analytic. Spectral information is encoded in matrix elements of the resolvent,

$$\langle \psi, R(z)\phi \rangle = \int_{\mathbb{R}} \frac{1}{\lambda - z} \, d\mu_{\psi,\phi}(\lambda),$$

where $\mu_{\psi,\phi}$ is the complex spectral measure associated with $\psi, \phi \in \mathcal{H}$.

4.3 Trace and Determinant Structures

If suitable regularization conditions are satisfied, one may formally consider trace expressions of the form

$$\operatorname{Tr} f(\widetilde{H}) = \int_{\mathbb{R}} f(\lambda) \, dN(\lambda),$$

where $N(\lambda)$ denotes the spectral counting function.

In particular, zeta-regularized determinants may be introduced via

$$\det\Big(\widetilde{H} - z\Big) := \exp\Big(-\frac{d}{ds}\Big|_{s=0} \operatorname{Tr}\,(\widetilde{H} - z)^{-s}\Big),$$

whenever the trace is well-defined.

No claim is made here that such determinants exist or converge for H; this section merely establishes the formal framework.

4.4 Conditional Spectral Interpretation

Should H admit a unique self-adjoint extension, any identification between spectral data of $\widetilde{H}$ and arithmetic objects must proceed through explicit trace or resolvent identities.

The logical direction emphasized throughout this work is:

$$\text{operator-theoretic structure} \Rightarrow \text{spectral identities} \Rightarrow \text{arithmetic consequences},$$

never the reverse.

4.5 Scope

This section is intentionally abstract. It introduces no arithmetic assumptions and does not rely on conjectures. Its role is to fix the spectral language required for subsequent analysis.

5 Numerical Experiments

This section reports numerical experiments designed to probe structural features of the operator H. These results are not presented as evidence for the Riemann Hypothesis, but as consistency checks of the operator framework.

5.1 Finite Truncations

The operator H is approximated by finite-dimensional truncations acting on $\ell^2(\{1, \ldots, N\})$. Prime sums are truncated to $p \leq P(N)$.

Such truncations break exact symmetry and self-adjointness, but provide insight into qualitative behavior.

5.2 Spectral Observations

Numerical spectra of truncated operators exhibit:

- predominantly real eigenvalues,
- increasing spectral density with N,
- sensitivity to parameter choices (α, β).

These observations are consistent with, but do not imply, the existence of a self-adjoint infinite-dimensional limit.

5.3 Stability Tests

Eigenvalue distributions were tested against:

- changes in truncation size,
- variations in prime cutoffs,
- perturbations of diagonal terms.

Qualitative features persist under moderate perturbations.

5.4 Limitations

Finite truncations cannot resolve:

- essential self-adjointness,
- deficiency indices,
- true spectral type.

Numerical experiments serve only as supporting evidence and motivation for further analysis.

5.5 Reproducibility

All numerical experiments are reproducible using deterministic algorithms and fixed parameter choices. Implementation details are documented separately.

6 Riemann Hypothesis as a Corollary

This section clarifies the precise logical relationship between the operator-theoretic framework developed in this work and the Riemann Hypothesis. No new analytical results are introduced. The purpose is solely to state conditional implications in a transparent, non-circular manner.

6.1 Statement of the Riemann Hypothesis

The Riemann Hypothesis asserts that all nontrivial zeros ρ of the Riemann zeta function $\zeta(s)$ satisfy

$$\mathrm{Re}(\rho) = \tfrac{1}{2}.$$

Equivalently, all nontrivial zeros may be written as

$$\rho = \tfrac{1}{2} + i\gamma, \qquad \gamma \in \mathbb{R}.$$

6.2 Conditional Hilbert–Pólya Principle

The Hilbert–Pólya principle may be formulated abstractly as follows:

> If there exists a self-adjoint operator whose spectral data encode the nontrivial zeros of $\zeta(s)$, then the Riemann Hypothesis holds.

The validity of this principle rests entirely on two independent requirements:

1. the existence of a canonical self-adjoint operator,
2. a rigorous identification between its spectral data and the zeros of $\zeta(s)$.

6.3 Logical Structure of the Present Framework

Within the framework developed in this work, the following conditional chain is identified:

(A) The symmetric operator H defined on $\mathcal{D}$ is essentially self-adjoint.

(B) The spectrum (or suitably regularized spectral determinant) of the unique self-adjoint extension $\widetilde{H}$ coincides with the set of imaginary parts of nontrivial zeros of $\zeta(s)$.

[Conditional Corollary] If conditions **(A)** and **(B)** hold, then the Riemann Hypothesis is true.

Under condition **(A)**, the operator $\widetilde{H}$ is self-adjoint, hence its spectrum is real. Under condition **(B)**, this spectrum coincides with the set $\{\gamma\}$ appearing in the nontrivial zeros $\rho = \frac{1}{2} + i\gamma$ of $\zeta(s)$. Therefore all such γ are real, which is exactly the Riemann Hypothesis.

6.4 Non-Circularity

It is essential to emphasize that neither condition **(A)** nor **(B)** assumes the Riemann Hypothesis. In particular:

- essential self-adjointness is a purely operator-theoretic property,
- spectral identification must be derived independently of any a priori assumptions on the location of zeros.

Thus the implication established above is strictly one-directional.

6.5 Status and Limitations

At present, neither condition **(A)** nor condition **(B)** has been established in full generality. The present work isolates these conditions explicitly and demonstrates that no additional hidden assumptions are required.

Consequently, the Riemann Hypothesis appears here not as a premise, but as a conditional corollary of a well-defined analytical program.

6.6 Interpretational Remark

Whether or not the program ultimately succeeds, the reformulation of the Riemann Hypothesis as a question of operator-theoretic uniqueness and spectral correspondence provides a concrete and falsifiable pathway for further investigation.

7 Numerical Experiments

This section reports numerical experiments designed to probe structural features of the operator H. These results are not presented as evidence for the Riemann Hypothesis, but as consistency checks of the operator framework.

7.1 Finite Truncations

The operator H is approximated by finite-dimensional truncations acting on $\ell^2(\{1, \ldots, N\})$. Prime sums are truncated to $p \leq P(N)$.

Such truncations break exact symmetry and self-adjointness, but provide insight into qualitative behavior.

7.2 Spectral Observations

Numerical spectra of truncated operators exhibit:

- predominantly real eigenvalues,
- increasing spectral density with N,
- sensitivity to parameter choices (α, β).

These observations are consistent with, but do not imply, the existence of a self-adjoint infinite-dimensional limit.

7.3 Stability Tests

Eigenvalue distributions were tested against:

- changes in truncation size,
- variations in prime cutoffs,
- perturbations of diagonal terms.

Qualitative features persist under moderate perturbations.

7.4 Limitations

Finite truncations cannot resolve:

- essential self-adjointness,
- deficiency indices,
- true spectral type.

Numerical experiments serve only as supporting evidence and motivation for further analysis.

7.5 Reproducibility

All numerical experiments are reproducible using deterministic algorithms and fixed parameter choices. Implementation details are documented separately.

8 Failure Modes and Limitations

This section summarizes the principal ways in which the program developed in this work could fail. The purpose is to make all assumptions explicit and to delimit the scope of the conclusions.

8.1 Failure of Essential Self-Adjointness

If the deficiency equation admits nontrivial ℓ^2 solutions, the operator H would admit multiple self-adjoint extensions. In that case:

- no canonical spectrum exists,
- any arithmetic interpretation becomes extension-dependent,
- the Hilbert–Pólya framework collapses at the structural level.

8.2 Spectral Mismatch

Even if H is essentially self-adjoint, its spectrum may fail to correspond to arithmetic data. Possible issues include:

- presence of continuous spectrum,
- absence of discrete spectral points,
- lack of trace-class properties needed for explicit formulas.

8.3 Regularization Ambiguities

Any use of spectral traces or determinants requires regularization. Different regularization schemes may yield inequivalent results. Failure to control this dependence would undermine arithmetic conclusions.

8.4 Parameter Dependence

The parameters α and β play a structural role. Certain choices may destroy lower semiboundedness, invalidate closability, or introduce pathological spectral behavior.

8.5 Logical Separation from the Riemann Hypothesis

No part of the present framework assumes the Riemann Hypothesis. Conversely, the failure of this program would not constitute evidence against the truth of the Riemann Hypothesis itself.

8.6 Summary

The framework succeeds only if:

- essential self-adjointness is established,
- spectral quantities are well-defined and stable,
- arithmetic identification is derived, not imposed.

All other outcomes are explicitly acknowledged and documented.

9 Analysis of the Deficiency Equation

This section provides a preliminary analytical investigation of the deficiency equation introduced in Section **??**. The analysis is partial and exploratory, intended to clarify structure rather than to deliver a complete proof.

9.1 Restatement of the Equation

We consider square-summable solutions $\psi \in \ell^2(\mathbb{N})$ of

$$\sum_{p \in \mathcal{P}} \frac{1}{p}\Big(\psi(pn) + \mathbf{1}_{p|n}\psi(n/p)\Big) + \big(\alpha\,\Lambda(n) + \beta\,\log n\big)\psi(n) = \pm i\,\psi(n). \tag{5}$$

9.2 Growth Considerations

The diagonal term $\beta \log n$ grows unboundedly with n. Formally, this suggests that any solution $\psi(n)$ must decay sufficiently rapidly to compensate for logarithmic growth.

In particular, polynomially decaying solutions are excluded for $\beta \neq 0$.

9.3 Prime-Scale Coupling

The equation couples values of ψ across multiplicative scales. This destroys any simple local recurrence and prevents reduction to standard one-dimensional difference equations.

Such non-locality is expected to suppress square-summable solutions, though a rigorous proof remains open.

9.4 Heuristic Absence of ℓ^2 Solutions

Assuming generic oscillatory behavior induced by the imaginary term $\pm i$, and taking into account logarithmic confinement, it is heuristically unlikely that nontrivial ℓ^2 solutions exist.

This heuristic supports, but does not prove, the conjecture that the deficiency indices vanish.

9.5 Open Problem

Prove or disprove that equation (5) *admits no nontrivial solutions in* $\ell^2(\mathbb{N})$*.*

This problem represents the central analytical challenge of the program.

10 Conclusion and Research Program

This work has developed a fully explicit operator-theoretic framework motivated by the Hilbert–Pólya program. The construction is grounded in a rigorously defined symmetric operator acting on a concrete Hilbert space, with all structural assumptions stated openly and without reliance on unproven arithmetic hypotheses.

10.1 Summary of Results

The main achievements of this work may be summarized as follows:

- A concrete Hilbert space $\ell^2(\mathbb{N})$ and dense domain $\mathcal{D}$ were specified explicitly.
- A symmetric operator H built from prime-indexed shift operators and arithmetic diagonal terms was defined and shown to be well-posed.
- Essential self-adjointness was reduced to the absence of $\ell^2(\mathbb{N})$ solutions of a single, explicit deficiency equation.
- A general spectral framework was established independently of arithmetic conjectures.
- Structural interfaces with classical explicit formulas were identified, while carefully avoiding circular reasoning.
- Numerical experiments were presented strictly as supporting evidence, not as proofs.

10.2 Legitimacy of the Hilbert–Pólya Program

The framework presented here satisfies the minimal structural requirements of a legitimate Hilbert–Pólya program:

- the operator is defined independently of spectral assumptions,
- symmetry is rigorously established,
- the obstruction to self-adjointness is precisely localized,
- arithmetic consequences are formulated conditionally.

No claim of having proven the Riemann Hypothesis is made. Instead, the work isolates a concrete and falsifiable analytical problem whose resolution would determine the viability of the approach.

10.3 Future Directions

The natural continuation of this research consists of three parallel tracks:

1. **Analytical resolution of the deficiency equation.** Proving the absence of square-summable solutions would establish essential self-adjointness and complete the operator-theoretic foundation.
2. **Rigorous spectral-arithmetic identities.** Deriving trace or resolvent formulas linking spectral data of H to arithmetic expressions remains a central objective.
3. **Refined numerical and experimental studies.** Large-scale computations may guide conjectures, but must remain subordinate to analytical arguments.

10.4 Perspective

The approach presented here does not assume the truth of the Riemann Hypothesis, nor does it depend on numerical verification of its zeros. Instead, it seeks to reformulate the problem as a question of operator-theoretic uniqueness and spectral structure.

Whether this program ultimately succeeds or fails, the explicit nature of the constructions ensures that any obstruction will be visible, concrete, and mathematically meaningful.

10.5 Closing Remark

The value of the present framework lies not in its claims, but in the clarity with which it exposes both its strengths and its limitations. In this sense, it aims to contribute not a conclusion, but a well-defined path forward.

A Technical Lemmas and Estimates

This appendix collects auxiliary results used implicitly throughout the main text. All statements are elementary and included for completeness.

A.1 Boundedness of Prime-Weighted Sums

The sequence $(1/p)_{p\in\mathcal{P}}$ belongs to $\ell^2(\mathcal{P})$, i.e.

$$\sum_{p\in\mathcal{P}} \frac{1}{p^2} < \infty.$$

This follows from comparison with the convergent series $\sum_{n=2}^{\infty} 1/n^2$ and the fact that $\mathcal{P} \subset \mathbb{N}$.

A.2 Norm Estimate for Shift Operators

For each prime p, the operators T_p^+ and T_p^- defined on $\mathcal{D}$ satisfy

$$\|T_p^{\pm}\psi\|_{\mathcal{H}} \leq \|\psi\|_{\mathcal{H}}, \qquad \forall \psi \in \mathcal{D}.$$

For T_p^+,

$$\|T_p^+\psi\|^2 = \sum_{n=1}^{\infty} |\psi(pn)|^2 \leq \sum_{m=1}^{\infty} |\psi(m)|^2.$$

The argument for T_p^- is analogous.

A.3 Diagonal Growth Control

For any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$\log n \leq C_\varepsilon\, n^\varepsilon \qquad \forall n \geq 1.$$

This is a standard consequence of sub-polynomial growth of the logarithm.

A.4 Form-Boundedness of Diagonal Terms

The multiplication operator $M_{\log}$ defined by $(M_{\log}\psi)(n) = \log n\, \psi(n)$ is relatively form-bounded with respect to the identity operator.

By Lemma A.3, $\log n$ grows slower than any power of n. Since $\psi \in \ell^2(\mathbb{N})$, the claim follows by standard estimates.

A.5 Purpose of This Appendix

The results collected here justify the boundedness and convergence claims used in Sections 3–9. No originality is claimed for these lemmas.

B Numerical Protocols and Reproducibility

This appendix documents the numerical procedures used in Section 7. The purpose is transparency and reproducibility, not numerical proof.

B.1 Finite-Dimensional Truncation

Numerical experiments approximate the operator H by truncation to the finite-dimensional space

$$\mathcal{H}_N := \ell^2(\{1, \dots, N\}).$$

Prime sums are truncated to primes $p \leq P(N)$, with $P(N)$ chosen such that $pn \leq N$ for non-negligible contributions.

B.2 Matrix Representation

The truncated operator H_N is represented as an $N \times N$ matrix with entries

$$(H_N)_{m,n} = \sum_{p \in \mathcal{P}_N} \frac{1}{p}(\delta_{m,pn} + \delta_{m,n/p}) + \delta_{m,n}(\alpha\,\Lambda(n) + \beta\,\log n),$$

where $\mathcal{P}_N$ denotes the truncated prime set.

B.3 Spectral Computation

Eigenvalues of H_N are computed using standard dense linear algebra routines. No attempt is made to extrapolate finite-dimensional spectra to the infinite-dimensional limit.

B.4 Stability Checks

Computations were repeated under:

- increasing truncation sizes N,
- different prime cutoffs $P(N)$,
- small perturbations of parameters (α, β).

Only qualitative, stable features are reported.

B.5 Limitations

Finite truncations:

- do not preserve exact self-adjointness,
- cannot detect deficiency indices,
- may introduce artificial boundary effects.

Accordingly, numerical results are interpreted only as consistency checks for the operator framework.

B.6 Reproducibility

All experiments are deterministic and reproducible. Exact implementation details, parameter values, and scripts are available upon request or in the accompanying repository.