Armstrong Topology Solutions

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Contents

0 Definitions

0.1 Functions

(1) Let $f : A \to B$ be a function. Then,

- f is onto if, and only if, there exists a function $g : B \to A$ such that $fg = 1_B$.
- f is one-to-one if, and only if, there exists a function $g : B \to A$ such that $gf = 1_A$ (provided A is non-empty).
- f Is in 1-1 correspondence if there exists a function $g : B \to A$ such that $fg = 1_B$ and $gf = 1_A$. In this case, g is unique and is called the inverse function of f, typically denoted f^{-1}.

1 Extra Proofs and Lemmas

1.1 Group Theory & Abstract Algebra

(1) Split Exact Sequence: Given an exact sequence of abeilean groups and homomorphisms,

$$0 \to G \xrightarrow{\theta} H \xrightarrow{\phi} K \to 0,$$

and a homomorphism $\psi : K \to H$, such that $\phi\psi \equiv 1_K$, we have $H \cong G \oplus K$.

Proof. Define $\alpha : G \oplus K \to H$ by $\alpha(g \oplus k) = \theta(g) + \psi(k)$: it is easy to see that α is a homomorphism. Also, α is (1-1), for if $\alpha(g \oplus k) = 0$, we have

$$0 = \phi(\theta(g) + \psi(k)) = \phi\psi(k) = k;$$

but, then $\theta(g) = 0$, so that $g = 0$ since θ is one-to-one.

Moreover, α is onto, since given $h \in H$, we have

$$\phi(h - \psi\phi(h)) = \phi(h) - \phi\psi\phi(h) = 0$$

Thus, there exists $g \in G$ such that $h - \psi\phi(k) = \theta(g)$, that is,

$$h = \theta(g) + \psi\phi(h) = \alpha(g \oplus \psi(h))$$

□

1.2 Limit Points, Closure & Density

(1) Let (X, τ) be a topological space, and $A \subset X$. We show that $\overline{A} = A \cup L_A$, where L_A is the set of all accumulation points of A:

Proof. If $\overline{A} = A$, then by Theorem 2.2, pg. 29, $L_A \subset A$ and we are done. So, suppose that A is not closed. Then, A does not contain all of its limit points; consequently, $L_A \neq \varnothing$. Further, the set $B = A \cup L_A$ is closed by Theorem 2.2, pg. 29.

To conclude the proof, we show that any closed set C containing A, contains B. Indeed; Let $C \subset X$ be closed, such that $A \subset C$. Now, for $a \in L_A$ and open set O_a containing a, we have that $A \cap (O_a - \{a\}) \neq \varnothing$. But, as $A \subset C$, we have $C \cap (O_a - \{a\}) \neq \varnothing$. Therefore, $a \subset C$. So, $L_A \subset C$, implying $A \cup L_A \subset C$. By Theorem 2.3, pg. 30, $\overline{A} = B = A \cup L_A$. □

(2) Let (X, τ) be a topological space, and $A \subset X$. We prove that L_A, defined above, contains all limit points of sequences contained in A:[1]

Proof. Let $\{a_n\}_{n=1}^{\infty} \subset A$, such that $a_n \to a$. Then, $a \in L_{\{a_n : n \in \mathbb{N}\}}$ and every neighbourhood of a contains a point of $\{a_n : n \in \mathbb{N}\} - \{a\} = B$. As $\{a_n : n \in \mathbb{N}\} \subset A$, every neighbourhood of a has a point in $A - \{a\}$; $a \in L_A$. □

(3) Let (X, τ) be a topological space, and $A \subset X$ such that $\overline{A} = X$. We prove that $A \cap O \neq \varnothing$, for all $O \neq \varnothing$, $O \in \tau$:

Proof. Suppose that for some $O \in \tau$, $O \neq \varnothing$, $A \cap O = \varnothing$. By a previous lemma, we have $\overline{A} = (L_A - A) \cup O^c$, implying $O^c = A$. But then we have $X = O^c = \overline{O^c} = \overline{A}$, as O is open. But, contrarily, this implies that $(O^c)^c = O = X^c = \varnothing$. □

(4) We prove that the intersection of a closed set and a compact set is always compact:

Proof. Let (X, τ) be a topological space. Let $H, K \subset X$, such that H is closed and K is compact. Consider $H \cap K$. Now, if $\{O_\alpha\}_\alpha$ is an open cover of $H \cap K$, then $K \subset \bigcup_{\alpha \in N} O_\alpha \cup (X - H)$. But, since H is closed, $X - H$ is open. In conclusion, as

$$H \cap K \subset \bigcup_{\alpha \in N} O_\alpha \cup (X - H)$$

such a finite subcover of $H \cap K$ exists. □

(5) We prove that if (X, d) is a metric space with the induced topology, then $C \subset X$ is closed if, and only if, whenever $\{a_n\}_{n=1}^{\infty}$ is a sequence in C, with $\{a_n\} \to L$, we have $L \in C$:

[1]This assumes the general definition of limit points of a set

Proof.

$(\implies)$: Suppose, to the contrary, that $\overline{C} = C$, but $L \notin C$. Thus, there exists some $\epsilon > 0$, such that $B_\epsilon(L) \cap C = \varnothing$, as L is not a limit point of C. But then, $a_n \notin B_\epsilon(L)$, for all $n \geqslant N$, $N \in \mathbb{N}$ is sufficiently large; A contradiction.

$(\impliedby)$: Suppose, to the contrary, that $\overline{C} \neq C$. Then, by extra lemma[2], there is some $l \in L_C$, such that $L \notin C$. Thus, for each $n \in \mathbb{N}$, we pick $a_n \in B_{1/n}(L) \cap C \neq \varnothing$. Consequently, $\{a_n\}_{n=1}^{\infty}$ is a sequence in C such that $a_n \to L \in C$, by hypothesis; A contradiction.

□

1.3 Separation

(1) We show that a compact T_2 space T_3. Consequently, we show that it is T_4:

Proof. Let X be the compact T_2 space. We first show that X is T_3:

Let $A \subset X$, such that A is closed. Then A is compact. Further, let $b \in X$ such that $b \notin A$. Now, for each $a \in A$, there exists an open set O_a, and some open set O_b^a, such that $O_a \cap O_b^a = \varnothing$. It follows that $A \subset \bigcup_{a \in N} O_a \subset \bigcup_a O_a$ and that $\bigcup_{a \in N} O_a = O_A$ is open. Further, $b \in \bigcap_{a \in N} O_b^a = O_b$, is open. And by construction, we have $O_A \cap O_b = \varnothing$. So, X is T_3.

To show that X is T_4, let $A, B \subset X$, be disjoint and closed. Then, arguing as above, we have two disjoint open subsets $O_B \in \tau$, $O_A \in \tau$ with $O_A \cap O_B = \varnothing$. □

1.4 Compactness

(1) We show that a homeomorphism between locally compact T_2 spaces, X, Y, extends to a homeomorphism between the Alexandroff compactifications; in other-words, locally compact homeomorphic T_2 spaces have homeomorphic one point compactifications.

Proof. Suppose that $f : X \to Y$ is the homeomorphism. Define $g : X \cup \{\infty_1\} \to Y \cup \{\infty_2\}$ as follows:

$$g(x) = \begin{cases} f(x) & x \neq \infty_1 \\ \infty_2 & x = \infty_1 \end{cases}$$

It is clear that as f is a bijection, so is g. We show that g is a homeomorphism:

[2] reference this

Suppose that U is open in Y. Then, $g^{-1}[U] = f^{-1}[U]$, which is open in $X \cup \{\infty_1\}$. Now, if $\mathcal{O}$ is open in $Y \cup \{\infty_2\}$, we have $\mathcal{O} = K \cup \{\infty_2\}$, where $K \subset Y$ is compact. Then,

$$g^{-1}[\mathcal{O}] = g^{-1}[K \cup \{\infty_2\}] = g^{-1}[K] \cup g^{-1}[\{\infty_2\}] = g^{-1}[K] \cup \{\infty_1\}$$

which is open in $X \cup \{\infty_1\}$.

Consequently, as g is one-to-one and onto, since f is, g is a homeomorphism by Theorem 3.7, pg. 48. □

(2) We show that if (X, τ) is a topological space, and $S \subset X$ is compact, then if W is a neighbourhood of S, there exists another neighbourhood G, such that

$$S \subset G \subset W$$

Proof. By exercise 21, section 3 pg. 55, and the fact that $X \cup \{y\} \cong X$, the result follows. □

1.5 Connectedness

(1) We show that if X is locally connected, then every connected component of X is open in X; hence X is the disjoint union of its connected components:

Proof. Let $x \in Y$, where Y is a connected component of X. By definition, x is contained in some open connected subset U of X. Since Y is a maximal connected set containing x, we have $x \in U \subset Y$. This shows that Y is open in X. □

1.6 Quotient Maps & Quotient Topology

(1) We show that if $q : X \to Y$ is a quotient map[3], then the topology of Y is the largest which makes q continuous:

Proof. Suppose that τ was some other topology on Y, such that q was continuous. We conclude by showing that $\tau \subset \tau_Y$:

Indeed; if $\mathcal{O} \in \tau$, then $q^{-1}[\mathcal{O}]$ is open in X. But then, $q[q^{-1}[\mathcal{O}]] = \mathcal{O} \in \tau_Y$. Thus, $\tau \subset \tau_Y$. □

(2) Suppose that (X, τ) is a topological space, and A a non-empty set. We show that if $f : X \to A$, then the quotient topology on A, τ_A is indeed a topology:

[3] Armstrong does not do a good job describing what the topology on Y is. A simple exercise shows that by letting $\mathcal{O}$ be open in Y whenever $q^{-1}[\mathcal{O}]$ is open in X, we have a topology on Y; call this τ_Y. Further, Armstrong does not do an adequate job describing what a quotient map is: $q : X \to Y$ is a quotient map if it is onto, continuous with respect to τ_Y, and such that $g^{-1}[\mathcal{O}]$ is open in X implies $\mathcal{O}$ open in Y. We can summarize by saying that q is onto, and such that $\mathcal{O}$ is open in Y iff $g^{-1}[\mathcal{O}]$ is open in X.

Proof. By definition, as noted above, $\mathcal{O}$ is open in A, if $f^{-1}[\mathcal{O}]$ is open in X:

(a) As $f^{-1}[\varnothing] = \varnothing$, and $f^{-1}[A] = X$, $\varnothing, A \in \tau_A$.

(b) As
$$f^{-1}[\bigcup_{j \in J} \mathcal{O}_j] = \bigcup_{j \in J} f^{-1}[\mathcal{O}_j],$$
we see that $\bigcup_{j \in J} \mathcal{O}_j \in \tau_A$.

(c) Lastly,
$$f^{-1}[\bigcap_{j=1}^{\infty} \mathcal{O}_j] = \bigcap_{j=1}^{\infty} f^{-1}[\mathcal{O}_j]$$
So, $\bigcap_{j=1}^{\infty} \mathcal{O}_j \in \tau_A$.

□

(3) We show that projective real n-space, $\mathbb{P}^n$, is Hausdorff[4]:

Proof. Let $q : S^n \to \mathbb{P}^n$ be the quotient map, and suppose that $u, v \in \mathbb{P}^n$, $u \neq v$; there are such $x, y \in S^n$, such that $q^{-1}[\{u\}] = \{x, -x\}$, and $q^{-1}[\{v\}] = \{y, -y\}$. We exhibit an $\epsilon \in \mathbb{R}$ such that the open neighbourhoods of radius ϵ, centered at u, v are disjoint:

Let
$$\epsilon = \frac{1}{2} \min\{||x - y||, ||x + y||\},$$
and set $U = B(x, \epsilon)$, and $V = B(y, \epsilon)$. It follows form the construction of U, V, that $U, V, -U, -V$ are pairwise disjoint, and open neighbourhoods of $x, y, -x, -y \in S^n$. Moreover, as noted above, $q^{-1}(q[U]) = U \cup -U$, and $q^{-1}(q[V]) = V \cup -V$. To conclude, we show that $q[U] \cap q[V]$ are disjoint neighbourhoods for $u, v \in \mathbb{P}^n$.

Indeed;
$$\begin{aligned} q[U] \cap q[V] &= q[B(x, \epsilon) \cap S^n] \cap q[B(y, \epsilon) \cap S^n] \\ &= q[B(x, \epsilon)] \cap q[B(y, \epsilon)] \cap \mathbb{P}^n \\ &= B(u, q(\epsilon)) \cap B(v, q(\epsilon)) \cap \mathbb{P}^n \\ &= B_1 \cap B_2 \cap \mathbb{P}^n \end{aligned}$$

Now, if $B_1 \cap B_2 \neq \varnothing$, then we would have $q^{-1}[B_1 \cap B_2] = U \cap V \neq \varnothing$, contary to assumption. Thus,
$$q[U] \cap q[V] = \varnothing \cap \mathbb{P}^n = \varnothing$$

□

[4]Credit for the initial idea of the proof goes to Brain M. Scott. In addtion, we use the notation $B(z, z')$ for the open ball of radius z', centered at z.

1.7 Topological Groups

(1) Suppose that $(G, \tau, *)$ is a topological group. Fixing $x \in G$, we show that $x\mathcal{O} \in \tau$ if, and only if, $\mathcal{O} \in \tau$:

Proof.

($\implies$): Suppose that $x\mathcal{O} \in \tau$. It follows from pg. 75, that $L_{x^{-1}} : G \to G$ is a homeomorphism and an open map: thus, $L_{x^{-1}}[\mathcal{O}] \in \tau$.

($\impliedby$): Suppose that $\mathcal{O} \in \tau$. Then, as noted on pg. 75, $L_x : G \to G$ is a homeomorphism and an open map: thus, $L_x[\mathcal{O}] = x\mathcal{O} \in \tau$.

□

(2) Suppose that $(G, \tau, *)$ is a topological group. We show that $\mathcal{O}$ is a neighbourhood of $x \in G$, if, and only if, $x^{-1}\mathcal{O}$ is a neighbourhood of e:

Proof. Suppose that $x \in G$, and without loss of generality, $\mathcal{O}$ is an open set containing x. Then, by a previous lemma[5], $L_{x^{-1}}[\mathcal{O}] = x^{-1}\mathcal{O} \in \tau$, and contains $x^{-1}x = e$.

By considering L_x, similar logic shows the reverse implication. □

(3) We show that $\mathbb{R}$, with the Euclidean topology and addition is a topological group:

Proof. The fact that $(\mathbb{R}, +)$ is a group is clear. To conclude, we show that $+ \equiv m : \mathbb{R} \to \mathbb{R}$, and $-1 \equiv i : \mathbb{R} \to \mathbb{R}$ are continuous:

Now, $i(x) = -x$, which is polynomial, and so continuous. And, as $m(x, y) = x + y = P_1(x, y) + P_2(x, y)$, where P_i is the projection mapping from $\mathbb{R}^2$, m is continuous; it is the sum of two continuous functions.

A similar argument show that $(\mathbb{R}^n, \tau, +)$ is a topological group. □

(4) Let (G, τ, m) be a topological group. We show that the topological automorphisms of (G, τ, m), for a subgroup of $(Aut)(G)$. We denote the set of all topological automorphism of G by $\mathrm{Aut}_\tau(G)$.

Proof. It is well known that $(\mathrm{Aut}(G), \circ)$ is a group. We show that $(\mathrm{Aut}_\tau(G), \circ)$ is a group:

As the composition of automorphism/homeomorphisms is another automorphism/homeomorphism, the fact that $(\mathrm{Aut}_\tau(G), \circ)$ is closed is clear. To conclude, as $f \in \mathrm{Aut}_\tau(G)$ is a homeomorphism, $f^{-1} \in \mathrm{Aut}_\tau(G)$. Furthermore, $e : (G, \tau) \to (G, \tau)$ given by $e(x) = x$ is a homeomorphism. So, $e \in \mathrm{Aut}_\tau(G)$.

[5] reference this

By the subgroup test,

$$\operatorname*{Aut}_{\tau}(G) \leqslant \operatorname{Aut}(G)$$

□

(5) Suppose that (G, τ, m) is a T_2, finite, topological group. We claim $\tau = \mathcal{P}(G)$:

Proof. As G is T_2, $\{x\}$ is open for each $x \in G$. Thus, as the union of open sets is open, it follows that $\tau = \mathcal{P}(G)$. □

(6) We claim that all topological groups of order 2 are topologically isomorphic.

Proof. Let (G, τ_G, m) be the topological group of order 2. Consider the topological group $(Z_2, \tau_{Z_2}, +)$ and the map $\phi : G \to Z_2$, given by $\phi(e_G) = 0$, and $\phi(g) = 1$. The fact that ϕ is a group isomorphism is well-known. Further, ϕ is 1-1 and onto, and ϕ^{-1} is continuous, as $\tau_G = \mathcal{P}(G)$ and $\tau_{Z_2} = \mathcal{P}(Z_2)$. This concludes the proof. □

1.8 Homotopy Type

(1) We prove that if A is a subspace of a topological space X and $G : X \times I \to X$ is deformation retract relative to A, then X and A have the same homotopy type[6]:

Proof. Indeed: Let $f : A \to X$ be the inclusion map and $g : X \to A$ be defined by $g(x) = G(x, 1)$. We have previously shown that f and g are continuous. It is only left to show that $f \circ g \simeq 1_X$ and $g \circ f \simeq 1_A$:

Direct computation shows that

$$(g \circ f)(a) = g(f(a)) = g(a) = G(a, 1) = a,$$

while $f \circ g \simeq_G 1_X$. This completes the proof. □

(2) We show that any non-empty convex subset, X, of a euclidean space is homotopy equivalent to a point:

Proof. Without loss of generality, we assume that $X \subset \mathbb{E}^n$. We first show the existence of a deformation retract, $G : X \times I \hookrightarrow \{x\}$, where $x \in X$ is fixed:

Let G be defined as $G(y, t) = (1 - t)y + tx$, for all $y \in X$. Then, clearly $G(y, 0) = y$, and $G(y, 1) = x \in \{x\}$ for all $y \in X$. So, G is deformation retract. By the above, X and A are of the same homotopy type: null-homotopic. □

[6] Armstrong does not do a good job of defining a deformation retract. We use the following definition. "Strong" deformation retract: A is a 'strong' deformation retract of X iff there exists a map $D : X \times I \to X$ such that $D(a, t) = a$ for every $a \in A$, $D(x, 0) = x$ and $D(x, 1) \in A$ for all $x \in X$. See this post on SE.

(3) We show that (X, τ) is contractible[7] if, and only if, every map $f : X \to Y$, (Y, τ_Y) a topological space, is null-homotopic:

Proof.

($\implies$): Suppose that (X, τ) is contractible to $x_0 \in X$. Let F be the contraction, and $f : X \to Y$ any map. The claim is that $f \circ F : X \times I \to Y$ is a homotopy between f, and $f(x_0)$.

Indeed; We note that the composition of two continuous functions is continuous and

$$(f \circ F)(x, 0) = f(F(x, 0)) = f(x)$$

As well as,

$$(f \circ F)(x, 1) = f(F(x, 1)) = f(x_0)$$

($\impliedby$): Suppose that every map $f : X \to Y$ is null-homotopic. Then, in particular, $i : X \to X$ is null-homotopic. Thus, there exists a map $F : X \times I \to X$, such that $F(x, 0) = x = i(x)$, and $F(x, 1) = x_0 = i(x_0)$.

$\square$

(4) Let (X, τ) be a topological space. We show that the cone on X, CX is contractible. We consider the "cone tip" as $X \times \{0\}$:

Proof. Consider the function $\overline{H} : (X \times I) \times I \to X \times I$, given by $\overline{H}((x, v), s) = (x, v(1 - s))$. By the product topology, it follows that $\overline{H}$ is continuous. [8] Let $\pi : X \times I \to CX = {}^{X \times I}/_{X \times \{0\}}$ be the canonical map. The claim is that $\pi \circ \overline{H} : (X \times I) \times I \to CX$ is the desired homotopy.

Direct computation shows that, for all $x \in X$, $v \in V$,

$$\pi(\overline{H}((x, v), 0)) = \pi((x, v)) = (x, v),$$

and

$$\pi(\overline{H}((x, v), 1)) = \pi((x, 0)) = (x, 0)$$

Thus, $H \equiv \pi \circ \overline{H}$ is the desired homotopy. $\square$

[7] A space (X, τ) is contractible (to a point $x_0 \in X$) provided that there exists a map $F : X \times I \to X$ such that $F(x, 0) = x$ and $F(x, 1) = x_0$.

[8] For every $t \in I$, we can associate $\pi(\overline{H}((x, v), t))$ with $(c, v(1 - t), t)$, showing that the product topology of $CX \times I$ agrees with that of $(X \times I) \times I$.

2 Introduction

2.1 Topological Invariants

1. We prove that $v(T) - e(T) = 1$ for any tree T:

   Proof. We proceed by induction on the number of edges in T:

   - Basis: If $e(T) = 1$, then $v(T) = 2$, and so, $v(T) - e(T) = 1$.
   - Hypothesis: Suppose that $v(T) - e(T) = 1$ for any tree T, such that $e(T) = n$, for some $n \in \mathbb{N}$.
   - Step: Suppose that T is a tree, such that $e(T) = n + 1$. Now, removing any edge will disconnect the tree, since by definition, T contains no loops. Say that we remove and edge $e_0 \in T$, breaking up T into T_1, T_2. Now, clearly, $v(T) = v(T_1) + v(T_1)$, and $e(T) = n + 1 = e(T_1) + e(T_2) + 1$. Thus, by the inductive hypothesis,

   $$\begin{aligned} v(T) - e(T) &= v(T_1) + v(T_2) - e(T_1) - e(T_2) - 1 \\ &= 1 + 1 - 1 \\ &= 1 \end{aligned}$$

   □

2. We show that inside any graph we can always find a tree which contains all the vertexes:

   Proof. Let G be a finite graph. If G is already a tree, we are done. So, suppose that G is not a tree. Then, by the comments on pg. 3, G contains a (without loss of generality, minimal) loop, L. Now, we remove some edge $e_L \in L$. Now, as L is a loop, e_L does not disconnect L, and so $L - e_L$ is a tree. We continue in this way, as long as the new graph formed by removing an edge is not a tree.

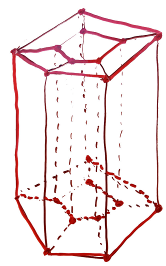
   To conclude, we note that this process must stop at some loop, because $e(G) < \infty$; further, we have not removed any vertexes. □

3. We prove that $v(\Gamma) - e(\Gamma) \leqslant 1$ for any graph Γ, with equality, precisely when Γ is a tree:

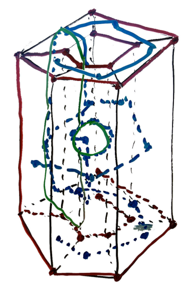
   Proof. We have previously shown the equality condition. Using the above proof, we select a subtree, with the same vertexes as Γ. Call this tree Γ'. Then, $v(\Gamma') - e(\Gamma') = 1$. As Γ' was created by removing edges, and not vertexes, we have $v(\Gamma) = v(\Gamma')$, and $e(\Gamma) \leqslant e(\Gamma')$. So,

   $$e(\Gamma) - e(\Gamma) \leqslant v(\Gamma') - e(\Gamma') = 1$$

   □



(a) The tree that contains every vertex.



(b) Blue: Dual graph. Green: loops.

4. We find a tree in the polyhedron of Fig. 1.3 which contains all the vertexes and construct the dual graph Γ and show that Γ contains loops:

   Proof. Please refer to figures 1a and 1b □

5. Having done Problem 4, thicken both T and Γ in the polyhedron. T is a tree, so thickening it gives a disc. We investigate what happens when you thicken Γ?

   Proof. Γ is basically two loops connected at a point, with some other edges connected that do not make any more loops. So thickening should produce something homeomorphic to what is shown in Problem 11 (b). □

6. Let P be a regular polyhedron in which each face has p edges and for which q faces meet at each vertex. We use Euler's formula to prove that,

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{2} + \frac{1}{e}$$

   Proof. We count the number of faces of P as follows: Since each face of P has p edges, $pf = 2e$. We count the number of vertexes as follows: Since each vertex has q faces which meet on it, $qv = 2e$. In all, $f = {}^{2e}/_{p}$, and $v = {}^{2e}/_{q}$. Assuming that Euler's formula holds for P, we have

$$v - e + f = \frac{2e}{q} - e + \frac{2e}{p} = 2$$

   And the result follows. □

7. We deduce that there are only 5 regular (convex) polyhedra.

   Proof. If P is a polyhedra, it must satisfy the above criterion.Further, we assume that $p \geqslant 3$, since if not, we cannot construct a polygon. We use

Shlüt notation. By checking routinely, for values up to $p, q \leqslant 5$ we have the following set of polyhedra:

$$\{\{3,3\},\{3,5\},\{5,3\},\{3,4\},\{4,3\}\}$$

Now, if $p \geqslant 6$, then we must have

$$\frac{1}{6} + \frac{1}{q} = \frac{1}{2} + \frac{1}{e}$$

Further, as $0 \leqslant {}^1\!/_q \leqslant {}^1\!/_5$, we have

$$\begin{aligned} \frac{1}{6} + \frac{1}{q} &= \frac{q+6}{6q} \\ &< \frac{12}{6q} = \frac{2}{q} \\ &< \frac{1}{2} + \frac{1}{e} \end{aligned}$$

A contradiction.

Please see figures 2a, 2b, 2c, 2d, and 2e. □

8. We check that $v - e + f = 0$ for the polyhedron shown in Fig. 1.3 and find a polyhedron which can be deformed into a pretzel and calculate its Euler number:

   Proof. For the polyhedron in Fig. 1.3, $v = 20$, $e = 40$, $f = 20$. Therefore $v - e + f = 0$. Figure 3 is basically a donut with two holes, i.e. a "pretzel".

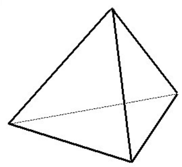
   Figure 3 has 38 faces (10 on the top, 10 on the bottom, 10 inside the hole, and 8 around the outside sides). It has 76 edges (29 on the top, 29 on the bottom, 10 vertical ones inside the holes and 8 vertical ones around the outside sides), and it has 36 vertexes (18 on the top, 18 on the bottom); Therefore $v - e + f = -2$. □

9. This is left to the reader as an exercise.

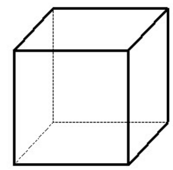
10. We find a homeomorphism from the real line to the open interval $(0, 1)$, and show that any two open intervals are homeomorphic:

    Proof. We show that the real line, $\mathbb{R}$ is homeomorphic to $(-\pi/2, \pi/2)$.This amount to showing that $arctan : \mathbb{R} \to (-\pi/2, \pi/2)$ is continuous, 1-1, onto and that $tan \equiv arctan^{-1}$ is continuous. Thus, $\mathbb{R} \simeq (-\pi/2, \pi/2)$.

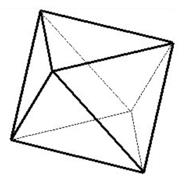
    To conclude the proof, we show that any open subset, $(a, b) \subset \mathbb{R}$ is homeomorphic to $(-\pi/2, \pi/2)$, $a \neq b$:



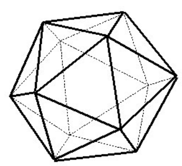
(a) Tetrahedron



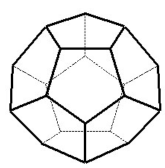
(b) Cube



(c) Octahedron



(d) Dodecahedron



(e) Icosahedron

Figure 2: All five platonic solids.

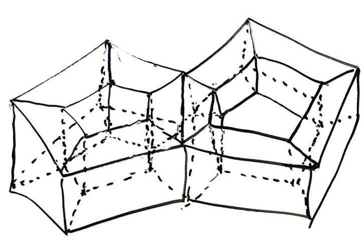


Figure 3: "pretzel"

Indeed, under the mapping $f(x) = (b-a)(\frac{x}{\pi} + \frac{1}{2}) + a$, we have

$$(-\pi/2, \pi/2) \to (a, b)$$

As f is a degree 1 polynomial in $\mathbb{R}$, it follows that it is 1-1, onto and continuous with a continuous inverse. $\square$

11. This is left to the reader as an exercise.

12. We find a homeomorphism from $S^2 - N = S^2 - \{(0,0,1)\}$ to $\mathbb{E}^2$:

Proof. As pointed out in Armstrong, we find a formula for stereographic projection. From the fact that $S^2 = \{x \in \mathbb{R}^3 : ||x||_2 = 1\}$, and rules of calculus, any line passing through $v = (x, y, z) \in S^2 - N$ and N is of the following form:

$$p(t) = (0,0,1) + t(x,y,z) \equiv < tx, ty, 1 - t(1-x) > \tag{1}$$

For $t \in [0, \infty)$. As the z component of p is zero exactly when $t_0 = 1/_{1-z}$, we substitute t_0 back in (1) to obtain our formula for stereographic projection:

$$\Pi(x,y,z) = \big(\frac{x}{1-z}, \frac{y}{1-z}\big)$$

It is left to show that Π is 1-1, onto, continuous, with a continuous inverse.

1-1: By component wise comparison, the result follows.

Onto: Given $(x_0, y_0) \in \mathbb{R}^2$, we have, after some simple algebra,

$$\Pi\Big(\frac{x_0}{r}, \frac{y_0}{r}, \frac{x_0^2 + y_0^2}{r}\Big) = (x_0, y_0),$$

where $r = 1 + x_0^2 + y_0^2$.

Continuity: As Π is a linear transformation in $\mathbb{R}^2$, we know that from rules of calculus/real analysis that it is continuous.

Cont. Inverse: Π^{-1} is as outline above in the derivation of onto. For the same reasons the function is continuous.

$\square$

13. Let $x, y \in S^2$. We find a homeomorphism from S^2 to S^2 which takes x to y. We do the same for the plane, and torus:

Proof. We assume that results about matrix groups are applicable.

- S^2: Consider S^2 and $SO(3)$. Let $x \in S^2$. Then, by the Gram-Schmidt Orthogonalization process, there exists $X \in SO(3)$ for which x is the first column of X. Likewise for y, its corresponding matrix Y. Then, $YX^{-1} \in SO(3)$ and maps x to y. As $SO(3)$ is a group, $(YX^{-1})^{-1} \in SO(3)$. Thus, YX^{-1} is a homeomorphism which takes x to y.

- $\mathbb{E}^2$: Fix $(x, y), (z, w) \in \mathbb{R}^2$. Then, the function, f that sends $(a, b) \in \mathbb{R}^2$ to
$$((x + z) - a, (y + w) - b)$$
sends (x, y) to (z, w). As f is linear, it is continuous. Likewise, its inverse is continuous.
- Torus: Consider the torus as $S^1 \times S^1$. Then component wise examination shows that the example above proves the result.

□

14. This is left to the reader as an exercise.
15. This is left to the reader as an exercise.
16. This is left to the reader as an exercise.
17. Define $f : [0, 1) \to C$, by $f(x) = e^{2\pi i x}$. We prove that f is a continuous bijection. In addition, we find a point $x \in [0, 1)$ and a neighbourhood N, of x in $[0, 1)$, such that $f[N]$ is not a neighbourhood of $f(x)$ in C; consequently, this means f is not a homeomorphism.

    Proof. We first show that f is a continuous bijection:

    Via Euler's formula,
$$e^{2\pi i x} = \cos(2\pi x) + i \sin(2\pi x), \quad \forall x \in \mathbb{R}$$
    Using this fact, we show that f is 1-1 and onto.

    - 1-1: Suppose that $\cos(2\pi x) + i \sin(2\pi x) = \cos(2\pi y) + i \sin(2\pi y)$ for some $x, y \in [0, 1)$. Then, by component-wise comparison of complex numbers we have $\cos(2\pi y) = \cos(2\pi x)$, iff, $2\pi y = 2\pi x$, iff $x = y$, as $x, y \in [0, 1)$; likewise for sin. Thus, f is 1-1.
    - Onto: Let $y \in C$. Then, $y = \cos(2\pi y_0) + i \sin(2\pi y_1)$ for some $(y_0, y_1) \in \mathbb{R}^2$. But as $f : [0, 1) \to C$, we have $0 \leqslant 2\pi y_0, 2\pi y_1 \leqslant 2\pi$. Consequently, f is onto.
    - Continuity: By the definition of continuity in $\mathbb{C}$, and the fact that cos, sin are continuous, we have that f is continuous.

    Next, consider $[0, 1/2) \subset [0, 1)$. This is a open neighbourhood in $[0, 1)$, by definition. But, $f[[0, 1/2)]$ is C intersected with the upper half of the plane minus $z = -1$. Which is not a neighbourhood of $z = 1 \in C$, because any open ball centered around $z = 1$ must contain the lower plane. Therefore, f is not a homeomorphism. □

18. This is left to the reader as an exercise.
19. This is left to the reader as an exercise.

20. We prove that the radial projection of the tetrahedron to the sphere is a homeomorphism:

    Proof. A solid proof of this relies on the fact that a circumscribed polygon with center of mass 0 is homeomorphic to S^n via radial projection. The result of which are shown in chapter 5. $\square$

21. Let C denote the unit circle in the complex plane and D the disc which it bounds. Given two points $x, y \in D - C$, we find a homeomorphism from D to D which interchanges x and y and leaves all points of C fixed:

    Proof. This is more or less intuitively obvious. But writing down an explicit function is not so easy. First note that for any $a \in \mathbb{C}$, the function $f(z) = \frac{z-a}{1-\bar{A}z}$ takes S^1 to itself. To see this suppose $|z| = 1$. then

$$\begin{aligned} \left|\frac{z-a}{1-\overline{a}z}\right| &= \left|\frac{z-a}{\overline{z}z-\overline{a}z}\right| \\ &= \left|\frac{z-a}{(\overline{z}-\overline{a})z}\right| \\ &= \left|\frac{z-a}{\overline{z-a}}\right| \frac{1}{|z|} \\ &= 1 \end{aligned}$$

    If $|a| < 1$ then the denominator never vanishes so this is a continuous function on D. Also, $f(0) = -a$, so if $|a| < 1$ then since f takes C to itself and 0 maps to $-a \in (D - C)$, f must take all of D to itself. The inverse f^{-1} is therefore also a continuous function from D to D that takes C to C. Now suppose $x, y \in \mathbb{C}$ with $|x| < 1$ and $|y| < 1$, let

$$f_1(z) = \frac{z-x}{1-\bar{x}z}$$

    and

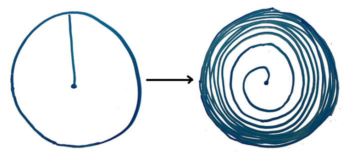
$$f_2(z) = \frac{z-ty_1}{1-t\bar{y_1}z}$$

    where $y_1 = f_1(y)$ and $t = \frac{1-\sqrt{1-|y_1|^2}}{|y_1|^2}$. As shown above, both f_1 and f_2 take C to itself. Finally let

$$g(z) = ze^{i(1-|z|)\pi/(1-|x_2|)}$$

    where $x_2 = f_2(0)$. Then $g(x_2) = -x_2$ and $g(-x_2) = x_2$ and $g(z) = z \ \forall\ z \in C$. Since $z \mapsto |z|$ is continuous, g is built up from sums, products and compositions of continuous functions and therefore g is continuous.

    The function $f_1^{-1} f_2^{-1} g f_2 f_1$ is therefore a continuous function from D to D that switches x and y and fixes C. $\square$



22. With C, D as above (in Problem 21), define $h : D - C \to D - C$ by

$$h(0) = 0$$
$$h(re^{i\theta}) = r\exp\left[i\left(\theta + \frac{2\pi r}{1-r}\right)\right]$$

We show that h is a homeomorphism, but that h cannot be extended to a homeomorphism from D to D and draw a picture which shows the effect of h on a diameter of D:

Proof. The function h restricted to a circle of radius r acts by rotation of $2\pi r/(1-r)$ radians. Now $2\pi r/(1-r) \to \infty$ as $r \to 1$, so as the circle radius grows towards one, it gets rotated to greater and greater angles approaching infinity. Thus intuitively it's pretty obvious this could not be extended to the boundary. See figure **??**.

Now, we can think of (r, θ) as polar coordinates in $\mathbb{R}^2$. And the topology on $\mathbb{C}$ is the same as that on $\mathbb{R}^2$. Thus as a function of two variables r and θ this is just a combination of continuous functions by sums, products, quotients and composition. Since the only denominator involved does not vanish for $|r| < 1$, this is a continuous function of r and θ on D which is clearly onto. Since it is a simple rotation on each circle of radius r, it is also clearly one-to-one. The inverse is evidently

$$h^{-1}(0) = 0$$
$$h^{-1}(re^{i\theta}) = r\exp\left[i\left(\theta - \frac{2\pi r}{1-r}\right)\right]$$

Now, let $r_n = \frac{n}{n+2}$ for n odd and $r_n = \frac{n-1}{n}$ for n even. Then for n odd, $\frac{r}{1-r} = \frac{n}{2}$, and for n even $\frac{r}{1-r} = n-1$. Therefore $\exp\left[i\left(\frac{2\pi r_n}{1-r_n}\right)\right]$ equals 1 if r is even and -1 if r is odd. Now, $r_n \to 1$. So if h could be extended to all of D we must have $h(1) = \lim h(r_n)$. But $h(r_n)$ does not converge, it alternates between 1 and -1. Therefore, there is no way to extend h to C to be continuous. ∎

23. Using the intuitive notion of connectedness, we argue that a circle and a circle with a spike attached cannot be homeomorphic (Fig. 1.26):

Proof. In the circle, if we remove any one point what remains is still connected. However in the circle with a spike attached there is one point we can remove that renders the space not-connected. Since this property of being able to remove a point and retain connectedness must be a topological property preserved by homeomorphism, the two spaces cannot be homeomorphic. □

24. Let X, Y be the subspace of the plane shown in Fig. 1.27. Under the assumption that any homeomorphism from the annulus to itself must send the points of the two boundary circles among themselves, we argue that X and Y cannot be homeomorphic:

    Proof. The two points that connect the two spikes to the two boundary circles in X must go to the two points that connect the two spikes to the boundary circles in Y, because those are the only two points on the boundary circles that can be removed to result in a disconnected space, and because by assumption the circles go to the circles. Since the two points lie on the same circle in Y but on different circles in X, some part of the outer circle in Y must go to the outer circle in X and the rest must go to the inner circle in X. But then some part of the outer circle in Y must go to the interior of X. I'm not sure exactly how Armstrong expects us to prove this but it basically follows from the intermediate value theorem, applied to the two coordinates thinking of these shapes as embedded in $\mathbb{R}^2$. □

25. With X and Y as above, consider the following two subspaces of $\mathbb{E}^3$:

$$X \times [0,1] = \{(x,y,z) \mid (x,y) \in X, 0 \leqslant z \leqslant 1\},$$

$$Y \times [0,1] = \{(x,y,z) \mid (x,y) \in Y, 0 \leqslant z \leqslant 1\}.$$

    Convince yourself that if these spaces are made of rubber then they can be deformed into one another, and hence that they are homeomorphic:

    Proof. With the extra dimension, the squareness can be continuously deformed so that it is a solid torus, with two flat rectangular shapes sticking off. One has both rectangles pointing out and one has one pointing out and the other pointing in. Since the torus is round, the first space made from X can be rotated at the location where the inner rectangle is a full half turn to point the rectangle out, and as parallel slices (discs) of the torus move away from where the rectangle is attached, the rotation gradually gets less and less until it becomes zero before reaching the other rectangle. In this way the inner rectangle can be rotated to point out without affecting the other rectangle and with a gradual change in rotation angle between them guaranteeing the operation is continuous. □

26. This is left to the reader as an exercise.

27. This is left to the reader as an exercise.

3 Continuity

3.1 Open and Closed Sets

(1) We verify each of the following for arbitrary subsets A, B of a space X:

- $\overline{A \cup B} = \overline{A} \cup \overline{B}$:

  Proof. $\overline{A}$ and $\overline{B}$ are closed by theorem (2.3). Thus $\overline{A} \cup \overline{B}$ is closed. Now $A \subseteq \overline{A}$ and $B \subseteq \overline{B}$. Therefore $\overline{A} \cup \overline{B}$ is a closed set containing $A \cup B$. By Theorem 2.3 $\overline{A \cup B}$ is the smallest closed set containing $A \cup B$, thus it must be that $\overline{A \cup B} \subseteq \overline{A} \cup \overline{B}$. Conversely, $\overline{A \cup B}$ is a closed set that contains A, so $\overline{A \cup B} \supseteq \overline{A}$. Similarly $\overline{A \cup B} \supseteq \overline{B}$. Thus $\overline{A \cup B} \supseteq \overline{A} \cup \overline{B}$. Thus $\overline{A \cup B} = \overline{A} \cup \overline{B}$. □

- $\overline{A \cap B} \subseteq \overline{A} \cap \overline{B}$:

  Proof. $\overline{A}$ is a closed set that contains $A \cap B$, so $\overline{A \cap B} \subseteq \overline{A}$. Likewise $\overline{A \cap B} \subseteq \overline{B}$. Thus $\overline{A \cap B} \subseteq \overline{A} \cap \overline{B}$.
  To see that equality does not hold, let $A = \mathbb{Q}$ and let $B = \mathbb{R} - \mathbb{Q}$. Then $A \cap B = \varnothing$, so $\overline{A \cap B} = \varnothing$. But $\overline{A} = \mathbb{R}$ and $\overline{B} = \mathbb{R}$, so $\overline{A} \cap \overline{B} = \mathbb{R}$. □

- $\overline{\overline{A}} = \overline{A}$:

  Proof. $\overline{\overline{A}}$ is the smallest closed set containing $\overline{A}$ by corollary (2.4), and $\overline{A}$ is closed by theorem (2.3), that contains $\overline{A}$. Thus $\overline{A} = \overline{\overline{A}}$. □

- $(A \cup B)^\circ \supseteq \mathring{A} \cup \mathring{B}$:

  Proof. Let $x \in \mathring{A} \cup \mathring{B}$. Assume, without loss of generality, that $x \in \mathring{A}$. Then there is an open set $U \subseteq A$ such that $x \in A$. But then $x \in U \subseteq A \cup B$. So $x \in \mathring{A} \cup \mathring{B}$.
  Thus
  $$\mathring{A} \cup \mathring{B} \subseteq (A \cup B)^\circ$$
  To see that equality does not hold, let $A = \mathbb{Q}$ and let $B = \mathbb{R} - \mathbb{Q}$. Then $\mathring{A} = \varnothing$ and $\mathring{B} = \varnothing$. And $A \cup B = \mathbb{R}$, so $(A \cup B)^\circ = \mathbb{R}$. Therefore $(A \cup B)^\circ = \mathbb{R}$ but $\mathring{A} \cap \mathring{B} = \varnothing$. □

- $(A \cap B)^\circ = \mathring{A} \cap \mathring{B}$:

  Proof. Let U be an open set in $A \cap B$. Then $U \subseteq A$ and $U \subseteq B$. Thus $(A \cap B)^\circ \subseteq \mathring{A} \cap \mathring{B}$. Conversely suppose $x \in \mathring{A} \cap \mathring{B}$. Then $\exists$ open sets U and V s.t. $x \in U \subseteq A$ and $x \in V \subseteq B$. Then $U \cap V$ is open and $x \in U \cap V \subseteq A \cap B$. Thus $(A \cap B)^\circ \supseteq \mathring{A} \cap \mathring{B}$.

Thus

$$(A \cap B)^\circ = \overset{\circ}{A} \cap \overset{\circ}{B}$$

□

- $(\overset{\circ}{A})^\circ = \overset{\circ}{A}$:

  Proof. Clearly

  $$(\overset{\circ}{A})^\circ \subseteq \overset{\circ}{A}$$

  Let $x \in \overset{\circ}{A}$. Then there exists an open set U, such that $x \in U \subseteq A$. Now $\overset{\circ}{A}$ is a union of open sets so is open. Let $V = \overset{\circ}{A} \cap U$. Then $x \in V \subseteq \overset{\circ}{A}$. Therefore $x \in (\overset{\circ}{A})^\circ$ and so,

  $$\overset{\circ}{A} \subseteq (\overset{\circ}{A})^\circ$$

  Thus,

  $$\overset{\circ}{A} = (\overset{\circ}{A})^\circ$$

  □

(2) This is left to the reader as an exercise.

(3) We specify the interior, closure and frontier of the following subsets.

$$\{(x, y) : 1 < x^2 + y^2 \leqslant 2\} \tag{1}$$

$$\mathbb{E}^2 - \{(0, t), (t, 0) : t \in \mathbb{R}\} \tag{2}$$

$$\mathbb{E}^2 - \{(x, sin(1/x) : x > 0\} \tag{3}$$

- $\{(x, y) : 1 < x^2 + y^2 \leqslant 2\} = A$:

  $$\begin{aligned} \mathrm{Fr}\, A &= \overline{A} \cap \overline{(\mathbb{E} - A)} \\ &= \overline{A} \cap \big\{\{(x, y) : 0 \leqslant x^2 + y^2 \leqslant 1\} \cup \{(x, y) : 2 \leqslant x^2 + y^2\}\big\} \\ &= \{(x, y) : x^2 + y^2 = 1 \vee x^2 + y^2 = 2\} \end{aligned}$$

  The rest are left to the reader.
- $\mathbb{E}^2 - \{(0, t), (t, 0) : t \in \mathbb{R}\} = B$:
  From the fact that $\overline{\mathbb{E}^2 - B} = \mathbb{E}^2$, we have

  $$\mathrm{Fr}\, B = B$$

  The rest are left to the reader.

- $\mathbb{E}^2 - \{(x, sin(1/x) : x > 0\} = C$:
  From the fact that $\overline{\mathbb{E}^2 - C} = \mathbb{E}^2$, we have

$$\operatorname{Fr} C = C$$

  The rest are left to the reader as an exercise.

(4) This is left to the reader as an exercise.

(5) We show that if A is a dense subset of a space (X, τ), and if $\mathcal{O} \in \tau$, that $\mathcal{O} \subset \overline{A \cap \mathcal{O}}$:

Proof. Suppose, to the contrary, that $\mathcal{O} \not\subset \overline{A \cap \mathcal{O}}$. Then, there exist some $x \in \mathcal{O}$, such that $x \notin \overline{A \cap \mathcal{O}}$.

As $\overline{A \cap \mathcal{O}}$ is closed, $x \in (\overline{A \cap \mathcal{O}})^c$ and so, there exists some $\mathcal{O}_x \in \tau$ such that $x \in \mathcal{O}_x$, and

$$\overline{A \cap \mathcal{O}} \cap (\underset{x}{\mathcal{O}} - \{x\}) = \varnothing$$

But, as $x \notin \overline{A \cap \mathcal{O}}$, we have

$$\overline{A \cap \mathcal{O}} \cap \mathcal{O}_x = \varnothing$$

and consequently, $A \cap \mathcal{O} \cap \mathcal{O}_x = \varnothing$. But then, setting $B = \mathcal{O} \cap \mathcal{O}_x$, we have $x \in B$, $B \in \tau$, but $A \cap B = \varnothing$, contrary to extra lemma; $\overline{A} \neq X$, a contradiction. □

(6) This is left to the reader as an exercise.

(7) Suppose that Y is a subspace of (X, τ). We show that a subset A of Y is closed in Y if it is the intersection of Y with a closed set in X. Further, we show that we get the same result if we take the closer in Y or X:

Proof. If $A \subset Y$ is closed in Y, then $Y - A$ is open in Y. But then, by the definition of subspace topology, $Y - A = Y \cap \mathcal{O}_y$, for some open $\mathcal{O}_y \in X$. Consequently, we have

$$\begin{aligned} A &= Y - (Y \cap \mathcal{O}_y) \\ &= Y \cap (Y \cap \mathcal{O}_y)^c \\ &= Y \cap (X - \mathcal{O}_y) \end{aligned}$$

And, as $X - \mathcal{O}_y$ is closed in X, this proves the result. We note that a similar case holds in the case where A is open.

Letting $\overline{A_y}$ and $\overline{A_x}$ denote the closure of $A \subset Y$ in Y and X respectively, we show that $\overline{A_y} = \overline{A_x}$:

From the previous part of this proof, we have that $\overline{A_y} = Y \cap C$, where C is closed in X. Now, since C is a closed set in X containing A, we have $\overline{A_x} \subset C$, and so $Y \cap \overline{A_x} \subset \overline{A_y} = Y \cap C$.

Again by the first problem, we have $Y \cap \overline{A_x}$ is closed in Y and contains A. So,

$$\overline{A_y} \subset Y \cap \overline{A_x} \subset \overline{A_x}$$

Thus, $\overline{A_y} = \overline{A_x}$. □

(8) Let Y be a subspace of (X, τ). Given $A \subset Y$, we show that $A_X^\circ \subset A_Y^\circ$, and give an example when the two may not be equal:

Proof. Let $x \in A_x^\circ$. Then, there exists and open $\mathcal{O}_x \subset X$, with $x \in \mathcal{O}_x$ and $\mathcal{O}_x \subset A$. Now, since $\mathcal{O}_x \subset A \subset Y$, $x \in \mathcal{O}_x \cap Y \subset A$ and, $\mathcal{O}_x \cap Y$ is open in Y, by definition. Thus, $x \in A_y^\circ$.

An example when they might not be equal is in the following case: Let $X = \mathbb{R}$, $Y = \mathbb{Z}$, and $A = \{0\}$. Then, $A_x^\circ = \varnothing$. But, every point of $\mathbb{Z}$ is open in the subspace topology, and so $A_y^\circ = \{0\}$. □

(9) This is left to the reader as an exercise. However, here is a counter example to this exercise:

(10) We show that the frontier of a set always contains the frontier of its interior, and describe the relationship between $\mathrm{Fr}(A \cup B)$ and $\mathrm{Fr}\, A$, $\mathrm{Fr}\, B$:

Proof. Let (X, τ) be a topological space, and let $A \subset X$. We want to show that $\mathrm{Fr}\, A^\circ \subset \mathrm{Fr}\, A$.

Let $x \in \mathrm{Fr}\, A^\circ$. Then,

$$x \in \overline{A^\circ} \cap \overline{(X - A^\circ)} = \overline{A^\circ} \cup \overline{(X - A) \cup (A - A^\circ)}$$

Now, if $x \in \overline{A^\circ}$ and $x \in \overline{X - A}$, we are done. So suppose that $x \in \overline{A^\circ}$ and $x \in \overline{(A - A^\circ)}$. But then, $x \in \overline{A^\circ} \cup \overline{(A - A^\circ)} = \overline{A}$. Thus, the result follows.

The inclusion $\mathrm{Fr}(A \cup B) \subset \mathrm{Fr}\, A \cup \mathrm{Fr}\, B$ always holds, and is left to the reader. To show that the reverse inclusion does not always hold, consider $X = \mathbb{R}$, and $A = \mathbb{Q}$. Then,

$$\begin{aligned} \mathrm{Fr}(A \cup A^c) &= \mathrm{Fr}\, \mathbb{R} = \varnothing \\ &\neq \mathrm{Fr}\, A \cup \mathrm{Fr}\, A^c = \mathbb{R} \end{aligned}$$

□

(11) This main part of this exercise is left to the reader. However, we do show that this topology does not have a countable base:

Proof. Let $B = \tau$ be the topology specified. Suppose, to the contrary, that $\{B_n\}_{n=1}^{\infty}$ is a countable base for the topology τ. Define the function $f : \mathbb{R} \to \mathbb{N}$ as follows: for each $x \in \mathbb{R}$, let $f(x) = n$, such that $B_n \subset [x, 1+x)$. Now, we show that f is 1-1 to arrive at a contradiction:

Indeed; Suppose to the contrary, without loss of generality, that $x < y$. Then, if $f(x) = f(y)$, $f(x) = [x, x+1) \subset B_{f(y)} = [y, y+1)$, which is impossible. Thus, $x = y$. ☐

(12) We show that if a topological space (X, τ) has a countable base for its topology, then X contains a countable dense subset. I.e. A second countable space is separable:

Proof. Let $\{B_n\}_n$ be a countable base for τ. By the Axiom of Choice, let A be the collection of elements $\{a_i\}_i$ such that $a_i \in B_i$. The claim is that $\overline{A} = X$.

Indeed; let $\mathcal{O} \in \tau$. Then, $\mathcal{O} = \bigcup_j B_j$, where $B_j \in B$. Now, as $A = \bigcup_i x_i$, where $x_i \in B_i$, we have $A \cap \mathcal{O} \neq \varnothing$. Thus, by extra lemma, $\overline{A} = X$. ☐

3.2 Continuous Functions

(1) We show that if $f : \mathbb{R} \to \mathbb{R}$ is a map, then the set of points left fixed by f is closed. Further, the kernel of f is closed.

Proof. Define $f_0(x) = f(x) - x$. The rest is left to the reader. ☐

(2) This is left to the reader as an exercise.

(3) Let $f : \mathbb{E} \to \mathbb{R}$ be a map and define $\Gamma_f : \mathbb{E} \to \mathbb{E}^2$ by $\Gamma_f(x) = (x, f(x))$. Whe show that Γ_f is continuous and that its image, with the induced topology, is homeomorphic to $\mathbb{E}$:

Proof. We show that Γ_f is continuous. To do this, we use the sequential criterion for continuity in $\mathbb{E}^n$. Let $\{x_n\}_n$ be a sequence in $\mathbb{E}$, such that $x_n \to x \in \mathbb{E}$. Then Γ_f is clearly continuous by the fact that f is, and component wise comparison. That is,

$$\Gamma_f(x_n) = (x_n, f(x_n)) \to (x, f(x)) = \Gamma_f(x)$$

Next, we show that $\operatorname{Im} \Gamma_f$ is homeomorphic to $\mathbb{E}$:

We claim that the function $p_1 : \operatorname{Im} \Gamma_f \to \mathbb{E}$, defined by $p_1((x, f(x)) = x$ is the desired homeomorphism.

Continuity: We have previously shown that the projection map is continuous.

1-1: This is true, by construction.

Onto: This is clear, as f a map from $\mathbb{E}$ to $\mathbb{E}$.

Cont. Inverse: This is clear, as $p_1^{-1} \equiv \Gamma_f$, and we have shown Γ_f is continuous.

This proves the result.[9] $\square$

(4) This is left to the reader as an exercise. Hint: Show that the discrete topology is the smallest topology on X for which this is the case.

(5) Consider $X = \mathbb{R}$ with the co-finite topology, $(\mathbb{R}, CO)$. We show that $f : \mathbb{E} \to X$ defined by $f(x) = x$ is a map, but not a homeomorphism:

Proof. To see that f is continuous, let $\mathcal{O} \in CO$. Then, $X - \mathcal{O} = \mathbb{R} - \mathcal{O}$ is finite. But then, $f^{-1}[\mathcal{O}] = \mathcal{O}$, and $\mathcal{O}^c$ is finite in $\mathbb{E}$. This implies that $\mathcal{O} = \overline{\mathcal{O}}$. Thus, $\mathcal{O}^c = \mathcal{O}$ is open in $\mathbb{E}$. The fact that f is 1-1 is clear.

Consider the inverse function of f. Note that (a, b), $a < b$ is open in $\mathbb{E}$. But, $f^{-1}[(a, b)] = (a, b) = f[(a, b)]$, which is not open in X. Thus, f^{-1} is not continuous. $\square$

(6) This is left to the reader as an exercise.

(7) Let (X, τ) be a topological space, $A \subset X$, and χ_A its characteristic function. We describe the frontier of A in terms of χ:

Proof. We note that if $\overline{(X - A)} \cap \overline{A} \neq \varnothing$, then there exists $a \in \overline{(X - A)}$, $a \in \overline{A}$. With this in mind, we claim that χ_A is continuous at $a \in X$ if, and only if, $a \notin \operatorname{Fr} A$.

Indeed;

$(\implies)$: Suppose, to the contrary, that χ_A is continuous at $a \in X$, but that $a \in \operatorname{Fr} A$. Then, $a \in \overline{X - A}$, and $a \in \overline{A}$. Now, let $\mathcal{O} = (.99, 1.99) \subset \mathbb{R}$, without loss of generality. Then, $\chi_A(a) = 1 \in \mathcal{O}$. Now, since χ_A is continuous, $\chi_A^{-1}[\mathcal{O}]$ is open, and such that $a \in \chi_A^{-1}[\mathcal{O}]$. But then, $\overline{X - A} \cap \chi_A^{-1}[\mathcal{O}] \neq \varnothing$. Further, this implies that $X - A \cap \chi_A^{-1}[\mathcal{O}] \neq \varnothing$. But then, $\chi_A(a) = 0 \in \mathcal{O}$, a contradiction.

$(\impliedby)$:

$\square$

(8) This is left to the reader as an exercise.

(9) See section section 1.

[9] A similar argument applies to $\mathbb{E}^n$.

3.3 Space-Filling Curves

(1) We find a Peano curve which fills out the unit square in $\mathbb{E}^2$:

Proof. We apply lemma 2.10. By a previous exercise, we have that $\partial[0,1]^2$ is homeomorphic to ∂S^1. Further, the boundary of the unit triangle is homeomorphic to ∂S^1. Therefore, by lemma 2.10, there is a homeomorphism, f, from the unit triangle to the unit disc. Let $h : [0,1] \to T$ be the space filling curve of the triangle as mentioned in Armstrong. Then, $f \circ h$ is a continuous mapping from $[0,1] \to [0,1]^2$. Further, it is space filling. □

(2) We find an onto, continuous function from $[0,1]$ to S^2:

Proof. From a previous exercise, we have that $\mathbb{E}^2 \cong_f (0,1) \times (0,1)$. Further, we have shown that $\mathbb{E}^2 \cong_\pi S^2 - (0,0,1)$. Thus, we extend f to $g : [0,1]^2 \to S^2$ by

$$g(x) = \begin{cases} f(x) & x \notin \partial[0,1]^2 \\ (0,0,1) & x \in \partial[0,1]^2 \end{cases}$$

Then, if $\mathcal{O} \subset S^2$ is open, and contains $(0,0,1)$, then $g^{-1}[\mathcal{O}]$ is the entire square via homeomorphism. Now, if $\mathcal{O} \subset S^2$ does not contain $(0,0,1)$ and is open, then $g^{-1}[\mathcal{O}] = f^{-1}[\mathcal{O}]$ which is clearly open.

Thus, $g \circ h$, where h is the triangle space-filling curve, is the desired function. □

(3) This is left to the reader as an exercise. Hint: Use the fact that $[0,1]$ is compact.

(4) This is left to the reader as an exercise. Hint: Use component wise definitions.

(5) To proved a rigorous proof of this, at this point, is out of the question. However, via theorems 3.3, and 3.7, it is not true.

3.4 The Tietze Extension Theorem

Throughout these exercises, we assume that (X, d) is a metric space, and that if $A \subset X$, it has the subspace metric. Further, the topology on X is the induced topology, if not stated otherwise.

(1) We show that $d(x, A) = 0$ if, and only if, $x \in \overline{A}$:

Proof.

Suppose that $d(x, A) = 0 = \inf_{a \in A} d(x, a)$. Then, for every $\epsilon > 0$, there exists $a \in A$, such that

$$|d(x, a) - 0| = d(x, a) < \epsilon$$

Now, let $\mathcal{O}$ be an open subset of X, $x \in \mathcal{O}$. Choose $a \in A$ such that, $d(x, a) < \epsilon$. Then, $a \in \mathcal{O}$, and $\mathcal{O} \cap A \neq \varnothing$. Thus, $x \in \overline{A}$.

Suppose that $x \in \overline{A}$. Then, we do the canonical ball construction and pick a sequence $\{a_n\}_n$ of elements of A, such that $d(x, a_n) \to 0$, $n \to \infty$.

□

(2) We show that if $A, B \subset X$ are disjoint and closed, there exists disjoint open sets U, V such that $A \subset U$ and $B \subset V$:

Proof. By lemma 2.13, there exists a continuous function $f : X \to [-1, 1]$, such that $f[A] \equiv 1$, and $f[B] \equiv -1$. Let $O_1 = [-1, 0]$, and $O_2 = [0, 1]$. Then, O_1, O_2 are open in $[-1, 1]$. Thus, $f^{-1}[O_i]$ is open, as f is continuous. Further, $f^{-1}[O_1] \cap f^{-1}[O_2] = \varnothing$, and $A \subset f^{-1}[O_1]$, $B \subset f^{-1}[O_2]$. □

(3) This is left to the reader as an exercise.

(4) We show that every closed subset of a metric space is the intersection of a countable number of open sets:

Proof. Let A be a closed subset of X. Define

$$A_n = \{x \in X : d(x, A) < \frac{1}{n}\}$$

Then, A_n is open, for each $n \in \mathbb{N}$. The claim is that $\bigcap_n A_n = A$:

Clearly, $A \subset A_n$, for each n so, $A \subset \bigcap_n A_n$. Next, suppose that $x \in \bigcap_n A_n$, but $x \notin A$. Then,

$$1 > \inf_{a \in \bigcap_n A_n} d(x, a) = \epsilon > 0$$

By the Archimedian Principal, there exists n_0 large enough so that $1/n_0 < \epsilon$. But then,

$$x \notin \bigcap_n A_n$$

Thus, $\bigcap_n A_n \subset A$.

This concludes the proof. □

(5) This is left to the reader as an exercise.

(6) We show that if A is a closed subset of X, then any map $f : A \to \mathbb{E}^n$ can be extended over X:

Proof. The is the Tietze extension theorem applied component-wise. □

(7) This is left to the reader as an exercise.

(8) This is left to the reader as an exercise.

(9) This is left to the reader as an exercise.

(10) This is left to the reader as an exercise.

4 Compactness and Connectedness

4.1 Closed and Bounded Subsets of $\mathbb{E}^n$

There are not exercises for this section.

4.2 The Heine-Borel Theorem

(1) This is left to the reader as an exercise.

(2) This is left to the reader as an exercise.

(3) We use the Heine-Borel theorem to show that an infinite subset of a closed interval must have a limit point:

Proof. By 'infinite', we assume that the author means that the cardinality of the set is non-finite and countable. We proceed by contradiction:

Let C be the closed interval. Suppose that such an infinite subset, A, of C does not have any limit points. Then, for each $x \in C$, there is an open set $\mathcal{O}_x$ in C, such that $x \in \mathcal{O}_x$ and $\mathcal{O}_x \cap (A - \{x\}) = \varnothing$. In addition,

$$C \subset \bigcup_{x \in C} \mathcal{O}_x$$

So, $\bigcup_{x \in C} \mathcal{O}_x$ is an open cover of C, and consequently, has a finite subcover:

$$\mathcal{F} = \bigcup_{\substack{x_i \\ i \in \{1,2,\ldots,N\}}} \mathcal{O}_{x_i}$$

Then, $A \subset \mathcal{F}$ and $\mathcal{O}_{x_i} \cap A = x_i$ for each i. Thus,

$$A = \mathcal{F} \cap A = \{x_1, x_2, \ldots, x_N\}$$

A contradiction. □

(4) We rephrase the definition of compactness in terms of closed sets:[10]

Proof. We claim that X is compact if, and only if, for every collection $\{C_i\}_{i \in I}$ of closed sets in X with the FIP, $\bigcap_i C_i \neq \varnothing$. Indeed:

($\Longrightarrow$) We proceed by contradiction: Let X be compact, and let $\{C_i\}_{i \in I}$ be a collection of closed sets with the FIP, such that, $\bigcap_i C_i = \varnothing$.

Then, by De-Morgan's Laws, as C_i^c is open,

$$\bigcup_i C_i^c = \Big(\bigcap_i C_i\Big)^c = \varnothing^c = X$$

[10]Definition; The finite intersection propert (FIP): Let $\mathcal{F}$ be a collection of sets. Then, $\mathcal{F}$ has the finite intersection property if whenever $F_1, F_2, \ldots, F_n \in \mathcal{F}$, $F_1 \cap F_2 \cap \ldots \cap F_n \neq \varnothing$.

Thus, there exists a finite subcover,

$$\{C_{i_1}^c, C_{i_2}^c, ..., C_{i_n}^c\},$$

with $X = \bigcup_k C_{i_k}^c$. And, so,

$$X^c = \varnothing = \left(\bigcup_k C_{i_k}^c\right)^c = \bigcap_k C_{i_k} \neq \varnothing$$

However, this is clearly a contradiction. Thus, $\bigcap_i C_i \neq \varnothing$.

($\Longleftarrow$) Let $\mathcal{F} = \{\mathcal{O}_i\}_{i \in I}$ be an open cover of X. Then, $\{\mathcal{O}_i^c\}_{i \in I}$ is a collection of closed sets, such that

$$\bigcap_i \mathcal{O}_i^c = \left(\bigcup_i \mathcal{O}_i\right)^c = X^c = \varnothing$$

Thus, there exists a finite set $\{\mathcal{O}_1^c, \mathcal{O}_2^c, ..., \mathcal{O}_n^c\}$, such that

$$\bigcap_{k=1,2,\ldots,n} \mathcal{O}_k^c = \varnothing$$

But then,

$$\left(\bigcap_{k=1,2,\ldots,n} \mathcal{O}_k^c\right)^c = \bigcup_{k=1,2,\ldots,n} \mathcal{O}_k = (X^c)^c = X$$

So, $\mathcal{F}$ contains a finite subcover.

$\square$

4.3 Properties of Compact Spaces

(1) This is left to the reader as an exercise.

(2) This is left to the reader as an exercise. Hint: Consider the identity map on $\mathbb{R}$, and $\mathbb{R}$, with the discrete and indiscreet topologies, respectively.

(3) We show that Lebesgue's Lemma fails for $\mathbb{E}^2$:

Proof. Let $B(0,1)$ be the open ball of radius one, centered around the origin. And, for each $p \neq 0$ in $\mathbb{E}^2$, let $B(p, {}^1\!/\!_{||p||_2})$ be the open ball centered at p and radius ${}^1\!/\!_{||p||_2}$. Now, let $\delta > 0$, and let $n \in \mathbb{N}$ be such that

$$\frac{1}{n} < \frac{\delta}{3}$$

Then, the open ball of radius ${}^{2\delta}\!/\!_3$ around the point $(n,0)$ is not contained in any $B(p, {}^1\!/\!_{||p||_2})$. $\square$

(4) *Lindelöf's Theorem*: We show that if X has a countable base for its topology, τ, then any open cover of X contains a countable subcover:

Proof. Suppose that $\mathcal{F}$ is a countable base for the topology on (X, τ). Then, $\mathcal{F} = \{F_1, F_2, ..., F_n, ...\}$. Let $\{\mathcal{O}_\alpha\}_\alpha$ be an open cover of X. Then, as $\mathcal{F}$ is a base for τ,

$$\bigcup_\alpha \underset{\alpha}{\mathcal{O}} = \bigcup_{i \in I} F_i$$

In addition, for each $x \in X$ there exists $n \in I$, and α_0, such that

$$x \in F_n \subset \underset{\alpha_0}{\mathcal{O}}$$

Thus, let

$$\{F_{n_k}\}_k$$

be the collection of all such F_{n_k}, as described above. Then, clearly

$$\bigcup_k F_{n_k} = X$$

and, for each $F_{n_k} \in \{F_{n_k}\}_k$, there is an α_{n_k}, such that

$$F_{n_k} \subset \underset{\alpha_{n_k}}{\mathcal{O}}$$

Consequently,

$$\{\underset{\alpha_{n_k}}{\mathcal{O}}\}_{n_k}$$

is a subcover of $\{\mathcal{O}_\alpha\}_\alpha$. Finally, as I is countable, so is $\{F_{n_k}\}$ and $\{\mathcal{O}_{\alpha_{n_k}}\}$. □

(5) This is shown by an extra lemma, and the fact that compact subsets of T_2 spaces are closed.

(6) Let A be a compact subset of a metric space (X, d). We show,

- that the diameter of A, $\operatorname{Diam} A$, is equal to $d(x, y)$ for some $x, y \in A$:

  Proof. By lemma (2.13), for a fixed $y \in A$, the function defined by $f(x) = d(x, \{y\})$ is continuous. By theorem 3.10, f is bounded, and obtains its bounds on A; for some $x_0 \in A$, $d(x, \{y\}) \leqslant d(x_0, \{y\})$ for all $x \in A$. Further, ranging over $y \in A$, we see that

  $$\operatorname{Diam} A = d(x_0, y_0), \quad x_0, y_0 \in A.$$

  □

- that given $x \in X$, $d(x, A) = d(x, y)$, for some $y \in A$:

Proof. This is lemma (2.13), and the fact that A is compact. $\square$

- that given a closed subset B, disjoint from A, that $d(A, B) > 0$:

  Proof. Suppose, to the contrary, that $d(A, B) = 0$. Then, by definition, we have

  $$\inf_{\substack{x \in A \\ y \in B}} d(x, y) = d(A, B) = 0$$

  An application of the above shows that, actually, $d(A, B) = 0 = d(a, b)$ for some $a \in A$, and $b \in B$. But then, by properties of metrics spaces, we have $a = b$; a contraction, as $A \cap B = \varnothing$. $\square$

(7) We find an example of a topological space (X, τ) and a compact subset whose closure is not compact:

Proof. Consider $\mathbb{R} = X$, and $\tau = \{\mathbb{R}, \varnothing, (-a, a) : a > 0\}$. Then, $\{0\}$ is clearly compact in X, and it is closed. But, $\overline{\{0\}} = \mathbb{R}$ as each $(-a, a)^c = (-\infty, -a] \cup [a, \infty)$ and $0 \notin (-\infty, -a] \cup [a, \infty)$. $\square$

(8) This is left to the reader as an exercise.

(9) Let $f : X \to Y$ be a closed map such that $f^{-1}(y)$, $y \in Y$ is a compact subset of X. We show that $f^{-1}[K]$ is compact whenever K is compact in Y:

Proof. This follows from the fact that

$$f^{-1}[K] = f^{-1}[\bigcup_{k \in K} \{k\}] = \bigcup_{k \in K} f^{-1}[\{k\}]$$

and the fact that the union of finite sets is finite. $\square$

(10) This is left to the reader as an exercise. Hint: Use the fact that a subset of a T_2 space is T_2, and theorem 3.7. Show that $f : X \to f(X)$ is onto.

(11) The proof of the first part is left to the reader. However, we show

- that any closed subset of a locally compact space is locally compact:

  Proof. Let (X, τ) be the locally compact space, and $A \subset X$, with $\overline{A} = A$. Let $p \in A$. Then, as X is locally compact, there exists a compact neighbourhood

  $$K_X \subset X, \quad p \in K_X$$

  Then, via the subspace topology, $K_X \cap A$ is a neighbourhood of p in A.

Next, let $\mathcal{F}$ be an open cover of $K_X \cap A$. Then, we see that $\mathcal{F} \cup \{A^c\}$ is an open cover of K_X. Further, there exists a finite subcover $\mathcal{F}_1 \subset \mathcal{F}$, such that $\mathcal{F}_1 - \{A^c\}$ is a finite subcover of

$$K_X \cap A \subset \underset{1}{\mathcal{F}} - \{A^c\}$$

and so $K_X \cap A$ is a compact neighbourhood of p in A. □

- that $\mathbb{Q}$ is not locally compact (as a subset of $\mathbb{R}$):

  Proof. Suppose, to the contrary, that $\mathbb{Q}$ is locally compact. Then, let $A \subset \mathbb{Q}$ be a compact neighbourhood. Then, there exists an open interval $I \subset \mathbb{R}$, with

  $$I \cap \mathbb{Q} \subset A$$

  Now, let $x \in I$, where x is irrational. As $\mathbb{Q}$ is dense in $\mathbb{R}$, there exists a sequence of rationals in $\mathbb{Q} \cap A$ with,

  $$\{q_n\} \to x \notin A, \quad n \to \infty$$

  But then, A is not closed. Thus, by theorem (3.5), it cannot be compact. □

- that local compactness is preserved under homeomorphism:

  Proof. Suppose that $f : X \to Y$ is a homeomorphism, and that X is locally compact. We want to show that Y is locally compact.
  Let $p \in Y$. Then, there is a locally compact neighbourhood K_p of $f^{-1}(p)$ in X. Now, let $\mathcal{F}$ be an open cover of $f(\{K_p\})$. Then,

  $$\underset{1}{\mathcal{F}} = \{f^{-1}(\{F\}) : F \in \mathcal{F}\}$$

  is an open cover of K_p in X. It follows that $\mathcal{F}_1$ has a finite subcover, $\mathcal{F}_2$, and that

  $$\{f(\{F\}) : F \in \underset{2}{\mathcal{F}}\}$$

  is a finite subcover for $f(\{K_p\}) \subset Y$. □

(12) Suppose that X is locally compact and Hausdorff. Given $x \in X$, and a neighbourhood $\mathcal{O}$ of x, we find a compact neighbourhood K of x which is contained in $\mathcal{O}$:

Proof. Let $\mathcal{O}$ be as given. Then, since X is locally compact, there exists a compact neighbourhood V, such that $x \in V$. By the subspace topology, it follows that $x \in V \cap \mathcal{O}$ and $\mathcal{O} \cap V$ is open in V. Further, as V is compact, it is closed, by theorem 3.6. □

(13) This is left to the reader as an exercise.

(14) We prove that $\mathbb{E}^n \cup \{\infty\}$ is homeomorphic to S^n:

Proof. As before [11], there exists a homeomorphism

$$h : \mathbb{E}^n \to S^n - \{p\}, \quad p \in S^n$$

Clearly, the one point compactification of $\mathbb{E}^n$ is $\mathbb{E}^n \cup \{\infty\}$. And further, the one point compactification of $S^n - \{p\}$ is S^n.

As both these spaces are T_2, it follows by an extra lemma that

$$\mathbb{E}^n \cup \{\infty\} \simeq S^n$$

□

(15) Let X, and Y be locally compact Hausdorff spaces, and let $f : X \to Y$ be an onto map. We show that f extends to a map from $X \cup \{\infty\}$ onto $Y \cup \{\infty\}$, if and only if, $f^{-1}[K]$ is compact for each compact subset K of Y. Further, we deduce that if X and Y are homeomorphic spaces, then so are their one-point compactifications and find two spaces which are not homeomorphic, but have homeomorphic one-point compactifications:

Proof. This problem was first shown as an extra lemma. Thus, we find two spaces which are note homeomorphic but have homeomorphic one point compactifications:

Consider

$$X_1 = [0, 1) \cup (1/2, 1] \quad X_2 = [0, 1)$$

and the result follows. □

4.4 Product Spaces

(1) This is left to the reader as an exercise.

(2) Suppose that A, B are compact in X, Y, respectively. We show that if W is a neighbourhood of $A \times B$ in $X \times Y$, that there exists a neighbourhood U of A in X, and a neighbourhood V of B in Y, such that

$$U \times V \subset W$$

Proof. Since A, B are compact, by theorem (3.15), $A \times B$ is compact. As this is the case, there exists a finite set of (a, b), $a \in A$, $b \in B$, with neighbourhoods

$$U_{a_1} \times V_{b_1}, U_{a_2} \times V_{b_2}, \ldots, U_{a_n} \times V_{b_n}$$

such that

$$A \times B \subset \bigcup_i U_{a_i} \times V_{b_i}$$

[11] reference this

Now, for each $a \in A$, let

$$E_a = \bigcap_{\substack{U_{a_i} \\ a \in U_{a_i}}} U_{a_i},$$

and

$$F_b = \bigcap_{\substack{V_{b_i} \\ b \in V_{b_i}}} V_{b_i}$$

for each $b \in B$. Then, as $\{U_{a_i} \times V_{b_i}\}_i$ is finite,

$$\{E_x \times F_y : x \in A, y \in B\}$$

is finite. Further, we see that

$$A \times B \subset \bigcup_{a,b} E_a \times F_b = U \times V$$

Consequently, $U \times V \subset W$, by construction. □

(3) We prove that

- the product of two second-countable spaces is second-countable:

  Proof. Let X, Y be second countable spaces. Then, as the finite union of countable sets is countable, it follows that the topology on $X \times Y$ has a countable base. □

- the product of two separable spaces is separable:

  Proof. Suppose that X, Y are separable. Then, they contain a countable dense subset, $A \subset X$, $B \subset Y$. From set theory, it follows that $A \times B$ is countable. Thus, we show that $\overline{A \times B} = X \times Y$:
  Indeed; from exercise twenty on page fifty-five, we have

  $$\overline{A \times B} = \overline{A} \times \overline{B} = X \times Y$$

  □

(4) We prove that $[0, 1) \times [0, 1) \simeq [0, 1] \times [0, 1)$:

Proof. As previously show, there exists some $h : [0, 1) \times [0, 1) \to D_1^2$,

$$D_1^2 = D^2 - \{x = (x_1, x_2) \in S^1 : x_1 > 0 \wedge x_2 > 0\},$$

a homeomorphism. Similarly, under the same map, $h : [0, 1] \times [0, 1) \to D_2^2$,

$$D_2^2 = D^2 - \{x = (x_1, x_2) \in S^1 : x_2 > \sqrt{2}/2\},$$

is a homeomorphism. But, $\{x = (x_1, x_2) \in S^1 : x_2 > \sqrt{2}/2\} = B$, $\{x = (x_1, x_2) \in S^1 : x_1 > 0 \wedge x_2 > 0\} = A$ are open in S^1, and so homeomorphic via, say, h'.

Consequently,
$$S^1 - A \simeq_{h'} S^1 - B$$

But then,

$$D_1^2 = \operatorname{Int} D^2 \cup (S^1 - A) \simeq_{h' \circ i} \operatorname{Int} D^2 \cup (S^1 - B) = D_2^2$$

It follows that

$$[0,1) \times [0,1) \simeq_h D_1^2 \simeq D_2^2 \simeq_h [0,1] \times [0,1)$$

□

(5) This is left to the reader as an exercise.

(6) The first part is trivial. So, we show that X is Hausdorff if, and only if, $\Delta(\{X\})$ is closed in $X \times X$:

Proof. ($\Longrightarrow$) Suppose that X is T_2. We show that $\Delta(\{X\})^c$ is open in $X \times X$. Now, for $(x, y) \notin \Delta(\{X\})$, we have that there exists open sets, U_x, U_y, such that

$$x \in U_x, \quad y \in U_y, \quad U_x \cap U_y = \varnothing$$

Now, let $U_x \times U_y = W$, which is open in $X \times X$. If $p \in W \cap \Delta(\{X\})$, then $p \in W$, and so $U_x \cap U_y \neq \varnothing$. Thus,

$$W \subset \Delta(\{X\})^c$$

and so $\Delta(\{X\})$ is closed.

($\Longleftarrow$) Suppose that $\Delta(\{X\})$ is closed in $X \times X$. Now, if $(x, y) \notin \Delta(\{X\})$, then $(x, y) \in \Delta(\{X\})^c$, which is open. And so, by definition of the topology on $X \times X$, there exist open sets U, V, such that

$$(x, y) \in U \times V \subset \Delta(\{X\})^c$$

But, then
$$(U \times V) \cap \Delta(\{X\}) = \varnothing,$$
implies $U \cap V = \varnothing$, by set theory.

□

(7) This is left to the reader as an exercise. Hint: Consider $f : \mathbb{R}^+ \to \mathbb{R}$, given by $f(x) = \frac{1}{x}$ and use the sequential criterion for closure.

(8) This is left to the reader as an exercise. Hint: This is the obvious topology.

(9) This is left to the reader as an exercise. Hint: Consider the function $f : \mathbb{R}^+ \to \mathbb{R}$ given by $f(x) = \frac{x}{1+x}$.

(10) We show that

- the box topology contains the product topology:

  Proof. As X_i is open in (X_i, τ_i), the result follows. ☐

- the box topology and the product topology are equal if, and only if, X_i is an indiscreet space for all but finitely man values of i:

  Proof. If $U_i = \varnothing$, for any i, then

  $$\prod X_i = \varnothing$$

  And, so the only sets of the form

  $$U_1 \times U_2 \times ...$$

  which are non-empty, are those which have $U_i = X_i$ for all but finitely many i. ☐

5 Connectedness

(1) Let $X = \{(a, b) \in \mathbb{E}^2 : a \in \mathbb{Q} \vee b \in \mathbb{Q}\}$. We show that X with the induced topology is connected:

Proof. We seek to apply theorem (3.25), on page fifty-eight. Indeed, for each $q \in Q$, let

$$X_q = \{(a, b) \in \overset{2}{\mathbb{E}} : a = q \vee b = q\}$$

Then clearly

$$X_q = A_q \cup B_q,$$

where $A_q = \{(a, b) \in \mathbb{E}^2 : a = q, b \in \mathbb{E}\}$, $B_q = \{(a, b) \in \mathbb{E}^2 : a \in \mathbb{E} \vee b = q\}$. Further, for each $q \in \mathbb{Q}$,

$$A_q \cong \mathbb{E} \cong B_q$$

So, A_q, B_q are connected, and as $A_q \cap B_q = \{(q, q)\}$, X_q is connect by theorem (3.25).

For $t, q \in \mathbb{Q}$, we have

$$X_t \cap X_q = \{(q, t), (t, q)\} \neq \varnothing$$

and so, another application of 3.25 shows that

$$X = \bigcup_{q \in \mathbb{Q}} X_q$$

is connected. ☐

(2) This is left to the reader as an exercise. Hint: Use exercise eleven, chapter two, page thirty two.

(3) This is left to the reader as an exercise. Hint: Use theorem (3.27), page sixty.

(4) (Intermediate Value Theorem) We show that if $f : [a, b] \to \mathbb{E}$ is a map such that $f(a) < 0$ and $f(b) > 0$, then there exists some $c \in [a, b]$ for which $f(c) = 0$:

Proof. By theorem (3.21), page fifty-eight, $f([a, b]) \subset \mathbb{E}$ is connected. By theorem (3.19), page fifty-seven, $f([a, b])$ is an interval. In particular, $f([a, b]) = [f(a'), f(b')]$, where $f(a')$, $f(b')$, are the minimum and maximum of f on $[a, b]$. Thus, $[f(a), f(b)] \subset [f(a'), f(b')]$, and $0 \in [f(a), f(b)]$. Thus, there exists $c \in [a, b]$, $f(c) = 0$. □

(5) We show that

- $\mathbb{E}^n$ is locally connected:

  Proof. As has been shown,

  $$\{I_1 \times I_2 \times ... \times I_n : I_i \in \tau_{\mathbb{E}}\}$$

  is a base for the topology in $\mathbb{E}^n$. Thus, if $x \in \mathcal{O}_x$ for some open neighbourhood $\mathcal{O}_x$ in $\mathbb{E}^n$, there exists

  $$I = I_1 \times I_2 \times ... \times I_n \subset \underset{x}{\mathcal{O}}$$

  And, as each I_i is connected, it follows by exercise 3.20, that I is connected. □

- $X = \{0\} \cup \{1/n : n \in \mathbb{N}\}$ is *not* locally connected:

  Proof. Let $\mathcal{O}$ be an open neighbourhood of $0 \in X$, which is connected. But then, by theorem (3.19), $\mathcal{O}$ is an interval. However, X clearly contains no intervals. □

(6) We show that local connectedness is preserved by a homeomorphism, but need not be preserved by a continuous function:

Proof. The first part of this proof is direct. For the second part, consider $Y = \{0\} \cup \{1/n : n \in \mathbb{N}\}$ as above, and $X = \mathbb{Z}^+$ as subspaces of $\mathbb{E}$. Let $f : X \hookrightarrow\!\!\!\!\rightarrow Y$ be the canonical map. Then, f is clearly continuous, but we have previously show that Y is not locally compact. □

(7) We show that (X, τ) is locally connected, if, and only if every component of each open subset of X is an open set:

Proof. ($\Longrightarrow$): Suppose that X is connected, and let $\mathcal{O}$ be an open set in X, and C a (maximal) connected component of $\mathcal{O}$. Then, for some $x \in C$, there exists a connected open set V_x, such that

$$x \in V_x \subset \mathcal{O}$$

But, since C is maximally connected, $V_x \subset C$. Then,

$$C = \bigcup_{x \in C} V_x$$

and so C is open.

($\Longleftarrow$): Now, let $x \in X$, and $\mathcal{O}$ an open set containing x. Then, the components of $\mathcal{O}$ are open. Thus, let V be a component such that $x \in V \subset \mathcal{O}$. But then, V is open, closed, and so connected. Thus, X is locally connected.

$\square$

6 Joining Points by Paths

(1) We show that the continuous image of a path-connected space is path-connected:

Proof. Without loss of generality, suppose that $f : X \to Y = f(X)$ is onto, and X is path-connected. Then, for

$$y_1 = f^{-1}(x_1), \quad y_2 = f^{-1}(x_2)$$

in Y, there exists a path h from x_1 to x_2. Thus, $f \circ h$ is the desired path. $\square$

(2) This is left to the reader as an exercise. Hint: Consider the case $x \neq -y$, and use the straight-line homotopy composed with the canonical map. In the case where $x = -y$, pick a third point which is not equal and apply the above technique.

(3) We prove that the product of two path-connected spaces is path-connected:

Proof. This follows directly from the definition of the product topology and set theory. $\square$

(4) If A and B are path-connected subsets of a space (X, τ), and $A \cap B$ is non-empty, we show that $A \cup B$ is path-connected:

Proof. Suppose, without loss of generality, that

$$x, y \in A \cup B, \quad x \in A, \quad x \in B$$

Pick $c \in A \cap B$, and by assumption, there exists paths γ_1 in A, γ_2 in B, which connect x to c, and c to y, respectively.

It follows that $\gamma_1 \circ \gamma_2$ is the desired path. $\square$

(5) We find a path-connected subset of a space whose closure is not path-connected:

Proof. As the comments on page sixty-two point out, letting

$$Z = \{(x, \sin(\pi/x) : x \in \overset{+}{\mathbb{R}}\}$$

we see

$$\overline{Z} = Y \cup Z = X$$

□

(6) We show that any indiscreet space is path-connected:

Proof. As any function on an indiscreet space is continuous, the result follows. □

(7) We determine whether or not the space shown in fig. 3.4, page sixty-three, is locally path-connected, and covert $X = \{0\} \cup \{1/n : n = 1, 2, ...\}$ into a subspace off $\mathbb{E}^2$ which is path-connected but not locally path-connected:

Proof. It is not the case that fig. 3.4 is locally path-connect. To see this, we note that any path at the origin must contain points from Y and Z. As the comments on page sixty-two point out, the result follows.

For the second part of the proof, let

$$X_0 = \{(0, y) : y \in [0, 1]\}, \quad X_n = \{(1/n, y) : y \in [0, 1]\}$$

Further, let $Y = \{(x, 0) : x \in [0, 1]\}$ and

$$X = Y \cup \Big(\bigcup_n X_n\Big)$$

Then, X is path connected as $Y \cap X_n \neq \varnothing$, for all n.

To see that X is not path connected, let $p \in (0, 1)$, and consider $\mathcal{O} \subset B(p, 1)$ be open. Then, $\mathcal{O} \cap Y = \varnothing$, so $\mathcal{O} \cap X$ is a collection of line segments separate from each other. So, $V \cap X$ is not path connected. □

(8) We prove that a space which is connected, and locally path-connected is path-connected:

Proof. Let $x \in X$. And let $\mathcal{O}_x$ be the set of all $y \in X$ such that y is path-connected to x; then, $\mathcal{O}_x \neq \varnothing$. We claim that $\mathcal{O}_x = X$.

Indeed; as $\mathcal{O}_x$ is a maximally connected component it is open, by extra lemma[12]. Similarly, $\mathcal{O}_x^c$ is open. Thus, $\mathcal{O}_x$ is both open and closed. Therefore, as X is connected, $\mathcal{O}_x = X$. This completes the proof. □

[12]cite this

7 Identification Spaces

7.1 Constructing the Möbius Strip

There are no exercises listed for this section.

7.2 The Identification Topology

(1) We check that the three descriptions of $\mathbb{R}\,\mathrm{P}^n$ all lead to the same space:

Proof. We first note that the canonical construction of $\mathbb{R}\,\mathrm{P}^n$ is formed via the relation

$$x \sim_{\mathcal{R}} \lambda x \quad \forall \lambda \neq 0, \quad \lambda \in \mathbb{R}$$

As λ can always be chosen so that $||x \cdot \lambda|| = 1$, we consider such λ as equivalence class representatives.

To see that (a), (b) are equivalent, let $h : \mathbb{E}^{n+1} \to \mathbb{E}/\sim_{\mathcal{R}}$ be the aforementioned quotient map and $i : S^n \to \mathbb{E}^{n+1}$ be the natural embedding. Then, as S^n is compact, $\mathbb{E}/\sim_{\mathcal{R}}$ is Hausdorff, it follows by an extra lemma, and corollary (4.4), that $h \circ i$ is an identification map. So, (a), (b) are equivalent.

To see that (b) and (c) are equivalent, let $i : S^n \to B^n$ be the natural embedding, and g is quotient map given in (c). Then, again, by corollary (4.4), $g \circ i$ is an identification map. Thus, (b) and (c) are equivalent. $\square$

(2) This is left to the reader as an exercise. Hint: It is $\mathbb{R}\,\mathrm{P}^2$.

(3) This is left to the reader as an exercise. Hint: Consider the canonical map from $[0, 1]$ to S^1, and $[0, 1) \subset [0, 1]$.

(4) With the terminology from the previous problem, we show that if A is open in X, and if f is an open map then $f|_A : A \to f(A)$ is an identification map:

Proof. This is theorem (4.3), page sixty-seven. $\square$

(5) Let X denote the union of all circles of the form

$$\left(x - \frac{1}{n}\right)^2 + y^2 = \left(\frac{1}{n}\right)^2, \quad n \in \mathbb{N}$$

with the induced topology. Let Y denote the identification space obtained from the real line b identifying all the integers to a single point. We show that $X \not\cong Y$:

Proof. We claim that X is compact, but Y is not.

Indeed; To show that X is compact, first let $\{\mathcal{O}_\alpha\}_\alpha$ be an open cover of X. Then, there exists some α_0, such that

$$\{(0, 0)\} \subset \underset{\alpha_0}{\mathcal{O}}$$

Thus, as $\mathcal{O}_{\alpha_0}$ is an open set containing $\{(0,0)\}$, there exists some $n_0 \in \mathbb{N}$ for which

$$\left(x - \frac{1}{n}\right)^2 + y^2 \subset B(\underset{\alpha_0}{\mathcal{O}}), \quad n \geqslant n_0$$

where $B(\mathcal{O}_{\alpha_0})$ denotes the solid ball with boundary $\mathcal{O}_{\alpha_0}$. Consequently, only finitely many circles are outside $B(\mathcal{O}_{\alpha_0})$. Thus, compactness follows.

To show that Y is not compact, consider the open cover of $\mathbb{R}$ given by

$$\{(n - 1/2, n + 1 + 1/2)\}_{n \in \mathbb{Z}}$$

There is clearly no finite subcover of $\mathbb{R}$. This completes the proof. $\square$

(6) We given an example of an identification map which is neither open nor closed:

Proof. Let $\pi_1 : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be projection onto the first coordinate and consider

$$A = \{(x,y) : x \geqslant 0 \vee y = 0\} \subset \mathbb{R} \times \mathbb{R}$$

We show that $\pi_1|_A$ is a quotient map which is neither open nor closed:

Indeed; We know that π_1 is a quotient map, as it send open sets to open sets. In addition, $\pi_1|_A$ is obviously saturated and so a quotient map. To see that it is neither open nor closed, consider

$$C = \{(x, 1/x) : x \in \mathbb{R} - \{0\}\}$$

which we have previously shown is closed. Then, clearly $C \cap A$ is clearly closed in A, but

$$\pi_1|_A[C \cap A] = (0, \infty)$$

which is open. And,

$$V = A \cap (\mathbb{R} \times (-1, \infty))$$

is open in A, but

$$\pi_1|_A[V] = [0, \infty)$$

is closed in $\mathbb{E}$. This proves the result. $\square$

(7) This is left to the reader as an exercise.

(8) Suppose that X is a compact T_2 space. We show that

- the cone on X is homeomorphism to the one-point compactification of $X \times [0,1)$:

  Proof. Note that, as X is compact, $X \times [0,1]$ is compact. Further,

  $$X \times [0,1] = X \times [0,1) \cup X \times \{1\}$$

  As the one point compactification of $X \times [0,1)$ is $X \times [0,1]$ and $X \times \{1\}$ is compact in $X \times [0,1]$, the result follows. $\square$

- if A is closed in X, then X/A is homeomorphic to the one-point compactification of $X - A$:

  Proof. Let $Y = X - A \cup \{\infty\}$ be the one point compactification of $X - A$. Further, let P be the image of A in X/A, under the identification map

  $$i : X \to X/A$$

  We show that the function $h : X/A \to Y$ given by

  $$h(x) = \begin{cases} x & x \in X - A \\ \infty & x \in P \end{cases}$$

  is a homeomorphism. We apply theorem (3.7) and claim that $h \circ i$ is continuous.
  Indeed; If $\mathcal{O}$ is open in Y and $\infty \notin \mathcal{O}$, then $\mathcal{O} \subset X - A$, and so,

  $$(h \circ i)^{-1}(\mathcal{O}) = \mathcal{O} \subset X$$

  is open. Now, if $\infty \in \mathcal{O}$, then $\mathcal{O}^c$ is compact in $X - A$ by definition and is closed. Thus, $(h \circ i)^{-1}(\mathcal{O}^c)$ is closed in X. Consequently,

  $$(h \circ i)^{-1}(\overset{c}{\mathcal{O}})^c = (h \circ i)^{-1}(\mathcal{O})$$

  is open in X. So, h is continuous.
  As $X - A \cup \{\infty\}$ is T_2 and $X - A$ compact (it is the continuous image under i of X), h is a homeomorphism by theorem (3.7), page forty-eight. This completes the proof. ☐

(9) Let $f : X \to X'$ be a continuous function and suppose that we have partitions $\mathscr{P}$, $\mathscr{P}'$ of X and X' respectively, such that if two points of X lie in the same member of $\mathcal{P}$, their images under f lie in the same member of $\mathscr{P}'$. We show that if Y, Y' are the identification spaces given by these partitions, that f induces a map $f' : Y \to Y'$, and that if f is an identification map then so is f':

Proof. Let $\pi : X \to Y$, $\pi' : X' \to Y'$ be the given identification maps. We show that f induces a map $f' : Y \to Y'$:

Define $f' : Y \to Y'$ by $f'(\pi(\mathcal{P})) = \pi'(f(\mathcal{P}))$, for all $\mathcal{P} \in \mathscr{P}$. To show that f' is a map, we first show that it is well-defined. For all $\mathcal{P} \in \mathscr{P}$, there exists $\mathcal{P}' \in \mathscr{P}'$, such that $f(\mathcal{P}) \subset \mathcal{P}'$, by assumption. The fact that f' is well-defined follows and everywhere defined is clear.

To show that f' is a map, let $\mathcal{O}$ be open in Y'. Then, we have

$$f'^{-1}(\mathcal{O}) = (\pi' \circ f)^{-1}(\mathcal{O}) = f^{-1}(\pi'^{-1}(\mathcal{O})),$$

which is open as f is continuous. Thus, f' is a map.

To conclude the proof, we show that if f is an identification map, then so is f': Indeed; Let $\mathcal{O}$ be open in Y. We claim that $f'(\mathcal{O})$ is open in Y'. By definition, if $\mathcal{O}$ is open in Y, then as π is an identification map, $\pi^{-1}(\mathcal{O})$ is open in X. Further,

$$(\pi' \circ f \circ \pi^{-1})(\mathcal{O}) = \pi'(f(\pi^{-1}(\mathcal{O}))) = f'(\pi(\pi^{-1}(\mathcal{O}))) = f'(\mathcal{O})$$

and,

$$f^{-1}(\pi'^{-1}(f'(\mathcal{O}))) = f^{-1}(\pi'^{-1}(\pi'(f(\pi^{-1}(\mathcal{O}))))) = \pi^{-1}(\mathcal{O})$$

Thus, by definition of identification map, $f'(\mathcal{O})$ is open in Y'. This concludes the proof. □

(10) This is left to the reader as an exercise. Hint: See attached solution.

(11) This is left to the reader as an exercise. See attached solution.

(12) This is left to the reader as an exercise. See attached solution.

7.3 Topological Groups

Throughout this section, and the rest of the book, the author assumes that such groups are T_2. Unless otherwise stated, this will be the case throughout the solutions.

(1) This is left to the reader as an exercise.

(2) Suppose that (G, m, τ) is a topological group. We show that

- if H is a subgroup of G, then its closure $\overline{H}$ is also a subgroup:

  Proof. Let $a, b \in \overline{H}$. We claim that $a + b \in \overline{H}$. Now, as $H \subset \overline{H}$, $m^{-1}(H) \subset m^{-1}(\overline{H})$. It follows that as

  $$m^{-1}(H) = \{(a, b) : a, b \in H\} = H \times H$$

  we have $H \times H \subset \overline{H} \times \overline{H} \subset m^{-1}(\overline{H})$. But, as $\overline{H}$ is closed, and m is continuous, we have

  $$m(\overline{H} \times \overline{H}) \subset \overline{H}$$

  Similarly we can show this for the inverse function. □

- if H is normal, then so is $\overline{H}$:

  Proof. We use the characterization for normality give by left and right co-set equality. As previously show, $\overline{H} \leqslant G$. As $gH = Hg$, we have that

  - $gH = Hg \subset g\overline{H}$, and so $\overline{H}g \subset g\overline{H}$.
  - Similarly, $g\overline{H} \subset \overline{H}g$.

□

(3) Let G be a compact Hausdorff space which has the structure of a group. We show that G is a topological group if the multiplication $m : G \times G \to G$ is continuous:

Proof. Let π be the canonical homeomorphism between G and $\{e\} \times G$, and p_2 projection onto the second coordinate. From the fact that compositions of homeomorphisms are homeomorphisms, and the comments on page seventy-five, it follows that $L_{g^{-2}}$ is a homeomorphism and

$$p_2 \circ (\pi \circ L_{g^{-2}}) \equiv i : G \to G^{-1}$$

is continuous. □

(4) We prove that $O(n)$ is homeomorphic to $SO(n)$ and that they are isomorphic as topological groups:

Proof. Consider the determinate function restricted to $O(n)$. It follows that the function $g : O(n) \to SO(n) \times \mathbb{Z}_2$ given by

$$g(X) = (\det(X)X, \det X)$$

is 1-1 and onto. We show that g is continuous. Let $\mathcal{O}$ be open in $SO(n) \times \mathbb{Z}_2$. Then, $\mathcal{O} = U \times \{1\}$, or $\mathcal{O} = U \times \{0\}$ for some open U in $SO(n)$. But, as $SO(n)$ is a subgroup of $O(n)$, U is open in $O(n)$. Thus, as $U, -U$ are open by extra lemma, it follows that

$$g^{-1}(U \times \{1\}) = -U, \quad g^{-1}(U \times \{0\}) = U$$

are open. Thus, g is continuous. As $O(n)$ is compat, and $SO(n) \times \mathbb{Z}_2$ is Hausdorff, it follows from theorem (3.7), page forty-eight, that g is a homeomorphism.

To conclude the proof, we note that $\det XY = \det X \det Y$. □

(5) Let A, B be compact subsets of a topological group. We show that the product set AB is compact:

Proof. As $m : G \times G \to G$ is continuous, and $m(A \times B) = AB$, compactness follows from theorem (3.4), page forty-seven. □

(6) We show that if U is a neighbourhood of e in a topological group, there is a neighbourhood V of e fow which $VV^{-1} \subset U$:

Proof. Consider $m^{-1}(U)$. Then, $m^{-1}(U)$ is an open neighbourhood of (e, e) in $G \times G$. It follows from the definition of product topology that $m^{-1}(U) = V \times V$ for some open set V in G. Via $L_{v^{-2}}$, we have

$$V \cong_{L_{v^{-2}}} V^{-1}$$

and so,

$$V \times V \cong V \times V^{-1}$$

And, as m is continuous, we have

$$m(V \times V^{-1}) = VV^{-1} \subset U = m(m^{-1}(U))$$

□

(7) Let H be a discrete subgroup of a topological group (G, m, τ). We find a neighbourhood N of e in G such that the translates $hN = L_h(N)$, $h \in H$ are all disjoint:

Proof. As H is disrete, for some open cover $\{\mathcal{O}_\alpha\}_\alpha$, each $\mathcal{O}_\alpha$ contains one element of H. In particular, there exits some $\mathcal{O}_{\alpha_0}$, such that

$$\{e\} \subset \underset{\alpha_0}{\mathcal{O}}$$

Thus, $\mathcal{O}_{\alpha_0} \cap H \neq \{e\}$. By an exercise above, there exists some open N in G, for which $NN^{-1} \subset \mathcal{O}_{\alpha_0}$. Now, suppose to the contrary that for some $h, g \in H$, $h \neq g$, $hN \cap gN = \varnothing$.

Then, there exits some $n_1, n_2 \in N$, for which $hn_1 = gn_2$, implying that

$$g^{-1}h = n_2 n_1^{-1} \in NN^{-1} \subset \underset{\alpha_0}{\mathcal{O}}$$

But then, $g^{-1}h = n_2 n_1^{-1} \in \mathcal{O}_{\alpha_0}$. This implies that

$$g^{-1}h \in \underset{\alpha_0}{\mathcal{O}} \cap H = \{e\};$$

a contradiction. □

(8) We show that if C is a compact subset of a topological group (G, m, τ), and if H is a discrete subgroup of G that $H \cap C$ is finite:

Proof. By extra lemma, $\overline{H} = H$, and as C is compact in a T_2 space, C is closed. This implies that $H \cap C \subset C$ is closed, and compact by theorem (3.5), page forty-seven.

Suppose, to the contrary, that $H \cap C$ was not finite. As

$$\bigcup_{x \in C} \{x\}$$

is an open cover of $H \cap C$, no finite set of $\{x : x \in C\}$ could cover $H \cap C$, contradicting the compactness of $H \cap C$. □

(9) We prove that every nontrivial discrete subgroup of $\mathbb{R}$ is infinite cyclic:

Proof. Let G be a non-trivial discrete subgroup of $\mathbb{R}$. Let $g \in G$, $g \neq 0$. Then, as (G, m, τ) is discrete, $\{0, g\}$ is a neighbourhood of 0, and so, g generates G (theorem (4.11), page seventy-five). The proof that $G = \langle g \rangle$ is clear from the division algorithm. To conclude, we show that $|G| = \infty$:

Suppose, to the contrary, that $|G| = p$, for some $p \in \mathbb{N}$. Then, as $\overline{G} = G$, $\inf G \in G$. But then, by the Archimedian Principle, there exists some $n \in \mathbb{N}$, for which $ng > \inf G$, $hg \in \langle g \rangle$. Thus, $|G| = \infty$. $\square$

(10) We prove that every non-trivial discrete subgroup of the circle is finite and cyclic:

Proof. Let G be a discrete subgroup of S^1. Define $f : [0, 1] \to S^1$ by $f(t) = e^{2\pi i t}$. Then, G is finite, since it is discrete, and compact. It is easy to see that $f^{-1}(G)$ is a discrete subgroup of $\mathbb{R}$, so is cyclic. Thus, $G = f(f^{-1}(G))$ is cyclic. $\square$

(11) Suppose that $A, B \in O(2)$, such that $\det A = 1$, $\det B = -1$. We show that $B^2 = I$, and $BAB^{-1} = A^{-1}$, and deduce that every discrete subgroup of $O(2)$ is either cyclic or dihedral:

Proof. By exhaustion,

$$B = \begin{bmatrix} \pm 1 & 0 \\ 0 & \mp 1 \end{bmatrix}$$

Thus, either way, $B^2 = I$. The fact that $BAB^{-1} = A^{-1}$ is routine. To complete the proof, we show that every discrete subgroup of $O(2)$ is either cyclic or dihedral:

From the comments on page seventy-seven, and exercise sixteen, we have that

$$O(2) \cong SO(2) \times \underset{2}{\mathbb{Z}} \cong S^1 \times \underset{2}{\mathbb{Z}}$$

Now, let G be a discrete subgroup of $S^1 \times \mathbb{Z}_2$. Then, for every $X \in O(2)$, $\det X \neq \pm 1$, we have

$$f(X) \cong SO(2) \times \{1\} \cong S^1 \times \{1\} \cong S^1$$

as in the previous exercise. By the exercise a above, we have that $f(X)$ must be finite cyclic. In other-words, $f^{-1}(f(X)) = X$ is finite cyclic.

To finish the proof, suppose that $X \in G$ was such that $\det X = -1$, without loss of generality. Now, let $K = G \cap SO(2)$. Then, K is cyclic and K is the set of $Z \in G$, such that $\det Z = 1$. Let M be the generator of K. Likewise, let N be the generator of $G - K$. Then, as $N^2 = I$, $NMN^{-1} = N^{-1}$, $\langle M, N \rangle$ is dihedral.

We claim that $G = \langle M, N\rangle$; Let $L \in G - \langle M, N\rangle$. Then, $L \notin K$, and so $\det L = -1$. Consequently, $\det LMN = 1$, and so, $LMN \in K$. But then, $L \in G$; a contradiction. This completes the proof. $\square$

(12) We show that

- if T is an automorphism of the topological group $\mathbb{R}$, that $T(r) = rT(1)$, for any rational r:

  Proof. If $n \in \mathbb{N}$, then we must have $T(n + x) = T(n) + T(x)$. Further, as T sends generators to generators, we must have

  $$T(n) = n, \quad T(n + x) = n + T(x)$$

  It follows from the equality

  $$T(1) = \sum_{i=1}^{n} T(1/n) = n + T(1/n)$$

  that

  $$T(m/n) = mT(1/n) = \frac{m}{n}T(1)$$

  Note the change in notation. Thus, as $\mathbb{Q}$ is dense in $\mathbb{R}$, it follows from the sequential criterion for continuity, that $xT(1) = T(x)$ for all $x \in \mathbb{R}$. $\square$

- the automorphism group of $\mathbb{R}$ is isomorphic to $\mathbb{R} \times \mathbb{Z}_2$:

  Proof. We first show that $\operatorname{Aut}\mathbb{R} \cong \mathbb{R} \times \mathbb{Z}_2$; The map $T_x : \mathbb{R} \to \mathbb{R}$, given by $T_x(y) = yx$, $x \neq 0$ is an automorphism of $\mathbb{R}$, such that $T_x(1) = x$. Thus, consider $f : \mathbb{R} - \{0\} \to \operatorname{Aut}\mathbb{R}$ given by $f(x) = x = T_x(1)$, for each $T_x \in \operatorname{Aut}\mathbb{R}$, as described above. Then, clearly, f is 1-1 and onto. Further,

  $$f(xy) = T_x(1)T_y(1) = xy = f(x)f(y)$$

  To conclude, we show that $\mathbb{R} - \{0\} \cong \mathbb{R} \times \mathbb{Z}_2$; Consider the map

  $$g : \mathbb{R} \times \underset{2}{\mathbb{Z}} \to \mathbb{R}, \quad g(x, y) = \begin{cases} e^x & y = 0 \\ -e^x & y = 1 \end{cases}$$

  We claim that g is a homeomorphic isomorphism. As $e^x : \mathbb{R} \to \mathbb{R}^+$ is a homeomorphism from $\mathbb{R}$ to $\mathbb{R}^+$, it follows that g is a homeomorphism. To show that it is a isomorhism, we show the homomorphism property: Without loss of generality, suppose that $(x, y), (x', y') \in \mathbb{R} \times \mathbb{Z}_2$ is such that $y = 0$, $y' = 1$; then,

  $$g(x, y)g(x', y') = e^x(-e^{x'}) = -e^{x+x'} = g(x + x', y + y')$$

  This completes the proof. $\square$

(13) We show that the automorphism group of the circle group is isomorphic to $\mathbb{Z}_2$:

Proof. We first note that the identity map $i : S^1 \to S^1$ is the trivial automorphism. In addition, we have show that the conjugate map given by $\bar{i}(x) = \bar{x}$ is an automorphism. We show that $|\operatorname{Aut} S^1| = 2$:

We show that if f is any other automorphism, the $f \equiv i$, or $f \equiv \bar{i}$. Indeed; first note that, as

$$S^1 \cong U(1) = \bigcup_n U_n, \quad U_n = \{c \in \mathbb{C} : c^n = 1\},$$

if $f \in \operatorname{Aut} S^1$, then f must send generators to generators. So, consider the generators of $U(1)$, given by U_2, and U_3. Then,

$$U_2 \cong \underset{2}{\mathbb{Z}}, \quad U_3 \cong \underset{3}{\mathbb{Z}}$$

Further, for $f \in \operatorname{Aut} S^1$, we must have $f(-1) = -1$, and thusly,

$$f(\frac{-1 + i\sqrt{3}}{2}) = \frac{-1 \pm i\sqrt{3}}{2}$$

In which case, $f \cong i$ and $f \cong \bar{i}$, respectively. Thus, by extra lemma, we have that $|\operatorname{Aut} S^1| = 2$ implies $\operatorname{Aut} U(1) \cong \mathbb{Z}_2$. $\square$

7.4 Orbit Spaces

(1) We give an action of $\mathbb{Z}$ on $\mathbb{E} \times [0,1]$ which has the Möbius Strip as an orbit space:

Proof. Define $\pi : \mathbb{Z} \times (\mathbb{E} \times [0,1]) \to \mathbb{E}^2$, by

$$\pi(z,(x,y)) = \begin{cases} (x+z, y) & z \in 2\,\mathbb{Z} \\ (x+z, 1-y) & z \in 2\,\mathbb{Z}+1 \end{cases}$$

We show that π is a topological group action:

- Clear by construction.
- As, $\pi(0,(x,y)) = (x+0,y) = (x,y)$, the result follows.
- As the components are continuous, π is continuous.

$\square$

(2) We find an action of $\mathbb{Z}_2$ on the torus with oribit space the cylinder:

Proof. Consider the cylinder on page eighty. Define $\pi : \mathbb{Z}_2 \times T \to T$, by

$$\pi(g,(x,y,z)) = \begin{cases} (x,-y,z) & g = 1 \\ (x,y,z) & g = 0 \end{cases}$$

We check that π is a T group action:

- By definition.
- As $0 \equiv e \in \mathbb{Z}_2$, clearly $\pi(0,(x,y,z)) = (x,y,z)$.
- As the components are continuous, π is continuous.

To show that $\pi(\mathbb{Z}_2 \times T) \cong C$, we note that $\{(x,y,z) : y \in \mathbb{R}\}$ is the canonical cylinder. $\square$

(3) We describe the orbits of the natural action of $SO(n)$ on $\mathbb{E}^n$ as a group of linear transformation and identify the orbit space:

Proof. Let $r \in \mathbb{R}$, $r \geqslant 0$, let

$$S_r = \{p \in \overset{n}{\mathbb{E}} \mid \|p\| = r\}$$

Then since SO(n) preserves distances, $SO(n)$ must take S_r to itself. Furthermore, the action on S_r is transitive, because it is transitive on $S^{n-1} \subset \mathbb{E}^n$ and $S_r = r \cdot S^{n-1}$. To see the action on S^{n-1} is transitive, for any vector $\mathbf{v} \in S^{n-1}$, it can be put into an orthonormal basis $\mathbf{B}$. Then there is a change of coordinates matrix M from the standard basis $\mathbf{e_1}, \ldots, \mathbf{e_n}$ to $\mathbf{B}$ that takes $\mathbf{e_1}$ to v. Since both bases are orthonormal, $M \in O(n)$. Clearly M can be chosen to be in $SO(n)$ such that $M(\mathbf{e_1}) = v$ (if it's not already in $SO(n)$, just multiply one of the other $\mathbf{e_i}$'s by -1). Since any element of S^{n-1} can be taken to $\mathbf{e_1}$, the action must be transitive. Now if $\mathbf{v} \in \mathbb{E}^n$ is arbitrary ($v \neq 0$) then just scale v to be in S^{n-1}, transform within S^{n-1} and scale back. The two scaling operations assure that the resulting transformation has determinant equal to one, and therefore is in $SO(n)$.

Now let $r, r' \in \mathbb{R}$, $r, r' > 0$, $r \neq r'$. Since things in $SO(n)$ are length preserving, an element of $SO(n)$ cannot take an element of S_r to an element of $S_{r'}$. Thus each S_r is exactly one orbit (true also if $r = 0$ since S_0 consists of one point).

Let $f : \mathbb{E}^n \to [0,\infty)$ be given by $f(\mathbf{v}) = \|\mathbf{v}\|$. Then f is a continuous function that identifies each orbit of the action to a single point. Let B be an open ball in $\mathbb{E}^n$. Then clearly $f(B)$ is an interval, open in $[0,\infty)$. Since functions respect unions, it follows that f is an open map. By Corollary 4.4 f is an identification map. Thus the identification space is homeomorphic to the image of f, which is $[0,\infty)$. Thus the orbit space of $SO(n)$ on $\mathbb{E}^n$ is homeomorphic to $[0,\infty)$. $\square$

(4) Suppose that $\pi : X \to X/G$ is the natural identification map. We show that

- if $\mathcal{O}$ is open in X, then $\pi^{-1}(\pi(\mathcal{O}))$ is the union of the sets $g(\mathcal{O})$, $g \in G$:

  Proof. Let $a \in \mathcal{O}$ and $g \in G$. Since ga is in the same orbit as a, $\pi(ga) = \pi(a)$. Thus $\pi(g(\mathcal{O})) = \pi(\mathcal{O})$, for all $g \in G$,

  $$g(\mathcal{O}) \subset \pi^{-1}(\pi(\mathcal{O})) \quad \forall g \in G$$

  and,

  $$\bigcup_{g \in G} g(\mathcal{O}) \subset \pi^{-1}(\pi(\mathcal{O}))$$

  Now suppose $x \in \pi^{-1}(\pi(\mathcal{O}))$; Then $\pi(x) \in \pi(\mathcal{O})$. Thus x is in the same orbit as some element of $\mathcal{O}$, i.e. $x = ga$ for some $g \in G$ and $a \in \mathcal{O}$. Consequently, $x \in g(\mathcal{O})$ implies

  $$\pi^{-1}(\pi(\mathcal{O})) \subset \bigcup_{g \in G} g(\mathcal{O})$$

  Since we have containment in both directions, we can conclude that

  $$\pi^{-1}(\pi(\mathcal{O})) = \bigcup_{g \in G} g(\mathcal{O})$$

  □

- π is an open map:

  Proof. Now suppose $\mathcal{O}$ is open in X. Recall a set $U \subset X/G$ is open in X/G if, and only if, $\pi^{-1}(U)$ is open in X. Now, $\pi^{-1}(\pi(\mathcal{O}))$ is a union of sets of the form $g(\mathcal{O})$ and (since each g induces a homeomorphism of X) $g(\mathcal{O})$ is open in X, for all $g \in G$. Thus $\pi^{-1}(\pi(\mathcal{O}))$ is open. Consequently, $\pi(O)$ is open in X/G. □

- π is not necessarily a closed map:

  Proof. We will show by counter-example that π is not a closed map. Let $\mathbb{Z}$ act on $\mathbb{R}$ by translation: $x \mapsto x + z$. This orbit space is S^1. For each $n = 0, 1, 2, \ldots$, let

  $$A_n = [n + \frac{1}{n+3}, n + \frac{1}{n+2}]$$

  So,

  $$\begin{aligned} A_0 &= [1/3, 1/2] \\ A_1 &= [1 + \frac{1}{4}, 1 + \frac{1}{3}] \\ A_2 &= [2 + \frac{1}{5}, 2 + \frac{1}{4}] \\ &\vdots \end{aligned}$$

Let $A = \bigcup_n A_n$. Then A is closed in $\mathbb{E}$, but $\pi(A) = \pi((0, 1/2])$ which is not a closed subset of S^1. $\square$

(5) We show that

- if X is Hausdorff, it is not necessarily the case that X/G be Hausdorff:

  Proof. Consider $X = S^1 \times S^1$. We claim that

  $$X/\mathbb{R}$$

  given in example seven, page eighty-three is not T_2. Clearly, $S^1 \times S^1 = T$ is T_2. Suppose to the contrary, that for some $x, y \in X/\mathbb{R}$, there exists disjoint neighbourhoods $\mathcal{O}_x$, $\mathcal{O}_y$. But then, as the comments on page eighty-three point out, $\operatorname{Orb} x$ is dense in T. But then,

  $$\underset{x}{\mathcal{O}} \cap \underset{y}{\mathcal{O}} \neq \varnothing$$

  $\square$

- if (X, m, τ) is a topological group and G is a closed subgroup acting on X by left translation, that X/G is Hausdorff:

  Proof. Let
  $$C = \{(x, y) \in X \times X \mid x^{-1}y \in G\}$$
  and $h : X \times X \to X$ be the map $h(x, y) = x^{-1}y$. Then $h^{-1}(G) = C$. Since G is closed and h is continuous, it follows that C is closed. Let $f : X \to X/G$ be the identification map. Let $g : X \times X \to X/G \times X/G$ be the map $g(x, y) = (f(x), f(y))$. By Problem 29 f is an open map. It follows that g is an open map. Thus by Theorem 4.3 g is an identification map. Let Δ be the diagonal in $X/G \times X/G$. Then $g^{-1}(\Delta) = C$. Since g is an identification map and C is closed in $X \times X$, it follows that Δ is closed in $X/G \times X/G$. By Chapter 3, Problem 25 (page 55) it follows that X/G is Hausdorff.

  $\square$

(6) We show that the stabilizer of any point is closed subgroup of G when X is Hausdorff, and that points in the same orbit have conjugate stabilizers for any X:

Proof. If g and g' are in the stabilizer of x, then $gg'x = gx = x$ so gg' is in the stabilizer of x. And $g^{-1}gx = 1 \cdot x = x$, but also $g^{-1}gx = g^{-1}x$. Thus $g^{-1}x = x$ so g^{-1} is in the stabilizer of x. It follows that the stabilizer of x is a subgroup of G. Now, let $f : G \to X$ be given by $f(g) = gx$. Then f is continuous. Since X is Hausdorff, by Theorem 3.6 points are closed (finite sets are always compact). Thus $f^{-1}(x)$ is closed in X. But $f^{-1}(x)$ is exactly the stabilizer of x. Thus the stabilizer of x is closed in X.

It remains to show points in the same orbit have conjugate stabilizers. Let x, y be in the same orbit, so $x = gy$ for some $g \in G$ and, let $a \in \operatorname{stab} x$. Then

$$\begin{aligned} g^{-1}agy = g^{-1}ax = g^{-1}x = y \\ \implies g^{-1}ag \in \operatorname{stab} y \\ \implies g^{-1}(\operatorname{stab} x)g \subset \operatorname{stab} y \end{aligned}$$

Now let $a \in \operatorname{stab} y$. Then,

$$gag^{-1}x = gay = gy = x$$

So,

$$gag^{-1} \in \operatorname{stab} x \implies \operatorname{stab} y \subset g^{-1}(\operatorname{stab} x)g$$

Since we have set containment in both directions it follows that

$$g^{-1}(\operatorname{stab} x)g = \operatorname{stab} y$$

□

(7) Suppose that G is compact, X is Hausdorff and that G acts transitively on X. We show that X is homeomorphic to the orbit space $G/$(stabilizer of x) for any $x \in X$:

Proof. Let $x \in X$, $f : G \to X$ be given by $f(g) = gx$. Since G acts transitively, f is onto. Since G is compact and X is Hausdorff, f is an identification map by corollary (4.4). So G^*, the identification space associated to f, is homeomorphic to X.

Now, suppose $f(g_1) = f(g_2)$. Then $g_2^{-1}g_1 \in \operatorname{stab} x$. Consequently, g_1 is in the same coset as g_2 with respect to the subgroup (stabilizer of x). Thus G^* is exactly $G/$(stabilizer of x). □

(8) We prove that the resulting space is homeomorphic to the Lens space $L(p, q)$:

Proof. □

(9) We show that $L(2, 1)$ is homeomorphic to $\mathbb{R}\mathrm{P}^3$ and that if p divides $q - q'$, that $L(p, q)$ is homeomorphic to $L(p, q')$:

Proof. □

8 Homotopic Maps

Throughout this section i will typically denote the identity map, unless otherwise stated.

(1) Let C denote the unit circle in the plane. Suppose that $f : C \to C$ is a map which is not homotopic to the identity. We show that $f(x) = -x$ for some $x \in C$:

Proof. Suppose, to the contrary, that for any $x \in C$, $f(x) \neq -x$. Then, we claim that $f \simeq i$. Indeed; as the example on page eighty-nine points out,

$$F(x,t) = \frac{(1-t)f(x) + tg(x)}{||(1-t)f(x) + tg(x)||}$$

is such that

$$f \simeq_F i \equiv g;$$

a contradiction. □

(2) With C as above, we show that the map which takes each point of C to its antipodal is homotopic to the identity:

Proof. Consider the matrix

$$A_\theta = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

Then, $A_\pi \equiv f(x)$. Thus, define $F : C \times I \to C$ by

$$F(x,t) = \begin{bmatrix} \cos(\pi(1-t)) & -\sin(\pi(1-t)) \\ \sin(\pi(1-t)) & \cos(\pi(1-t)) \end{bmatrix} x$$

We show that F is homotopy from f to i:

Clearly,

$$F(x,0) = A_\pi x \equiv f(x)$$
$$F(x,1) = A_0 x = Ix \equiv i$$

Further, as F is the sum of continuous functions, it is continuous. Thus,

$$f \simeq_F i$$

□

(3) Let D be the disc bounded by C, and parametrize D using polar coordinates. Let $h : D \to D$ be the homeomorphism defined by $h(0) = 0$, $h(r,\theta) = (r, \theta + 2\pi r)$. We find a homotopy F, from h to the identity, such that the functions

$$F|_{D\times\{t\}} : D \times \{t\} \to D, \quad t \in [0,1]$$

are all homeomorphisms:

Proof. Let $i(r,\theta) = (r,\theta)$ the identity map. Define $F : D \times I \to D$ by $F((r,\theta),t) = (r, \theta + 2\pi r(1-t))$. F is given by polynomials in r, θ and t, so F is continuous. Further,

$$F((r,\theta),0) = h(r,\theta)$$
$$F((r,\theta),1) = (r,\theta) = i(r,\theta)$$

Thus,

$$h \simeq_F i$$

Since $F|_{D\times\{t\}} \to D$ is a one-to-one continuous map from a compact space to a Hausdorff space, theorem (3.7) implies that $F|_{D\times\{t\}} \to D$ is a homeomorphism. □

(4) With the above terminology, we show that h is homotopic to the identity relative to C:

Proof. As we are consider h relative to C, $r = 1$ for all $x \in C$. Thus,

$$h = h' = (1, \theta + 2\pi)$$

Define $F : \partial D \times I \to \partial D$ by

$$F(x,t) = (1, \theta + 2\pi(1-t))$$

Then,

$$F(x,0) = (1,\theta + 2\pi) = h'(x)$$
$$F(x,1) = (1,\theta) = i(x),$$

where i is the identity on C. Likewise, F is continuous as it is polynomial in θ, t. Consquently,

$$h' \equiv h|_{\partial D} \simeq_F i$$

□

(5) Let $f : X \to S^n$ be a map that is not onto. We show that f is null-homotopic:

Proof. Let p be a point in S^n such that its antipodal point $-p$ is not in the image of f. Now let $g : X \to S^n$ be the constant function $g(x) = p$. Then $g(x)$ and $f(x)$ never give a pair of antipodal points for any $x \in X$. By example two on page eighty-nine, f and g are homotopic. □

(6) Let CY denote the cone on Y. We show that any two maps $f, g : X \to CY$ are homotopic:

Proof. By extra lemma [13] f, g are null-homotopic. □

(7) We show that a map from X to Y is null homotopic if, and only, if it extends to a map from the cone on X to Y:

Proof. ($\Longrightarrow$): Suppose that $f : X \to Y$ is null-homotopic, via F. Thus,

$$F : CX \to Y, \quad \text{as} \quad CX = X/X \times \{1\}$$

Thus, we need to show that F is an extension of f.
Indeed, $X \times \{0\} \subset X \times I$, and

$$\{f(x) : x \in X\} \cup \{0\} \subset Y$$

So,

$$F|_{X \times \{0\}}(x,t) = F(x,0) = f(x), \quad \forall x \in X \times \{0\} \cong X.$$

As f is continuous, it follows that $F|_{X\times\{0\}}$ is.
($\Longleftarrow$): Suppose that $f : X \times Y$ is a map and that $F : CX \to Y$ is an extension of f. Then, consider the map $H(x,t) = F(x,t)$. As

$$F(x,1) \equiv X \times \{1\},$$

we have

$$F(x,1) = H(x,1) = c, \quad \text{for some } c \in Y.$$

In addition,

$$H(x,0) \equiv F|_{X\times\{0\}} \equiv f$$

and we have

$$f \simeq_H i$$

□

(8) Let A denote the annulus $\{(r,\theta) : r \in [1,2], \theta \in [0,2\pi]\}$, and let h be a homeomorphism of A defined by $h(r,\theta) = (r, \theta + 2\pi(r-1))$. We show that h is homotopic to the identity map:

Proof. Consider the function $F : A \times I \to A$ given by

$$F((r,\theta),t) = (r, \theta + 2\pi t(r-1))$$

Then,

$$F((r,\theta),0) = (r,\theta) = i(r,\theta)$$
$$F((r,\theta),1) = (r, \theta + 2\pi(r-1))$$

As F is polynomial in θ, t, r, it follows that F is continuous. Thus,

$$h \simeq_F i$$

□

[13] reference this

9 Construction of the Fundamental Group

(1) Let α, β, γ be loops in a space X, all based at p. We write out the formulae for $(\alpha.\beta).\gamma$, and $\alpha.(\beta.\gamma)$ and show that they are homotopic loops relative $\{0,1\}$:

Proof. By definition, we have

$$((\alpha.\beta).\gamma)(t) = \begin{cases} \alpha(4t) & t \in [0, 1/4] \\ \beta(4t-1) & t \in [1/4, 1/2] \\ \gamma(2t-1) & t \in [1/2, 1] \end{cases}$$

and,

$$(\alpha.(\beta.\gamma))(t) = \begin{cases} \alpha(2t) & t \in [0, 1/2] \\ \beta(4t-2) & t \in [1/2, 3/4] \\ \gamma(4t-3) & t \in [3/4, 1] \end{cases}$$

Considering the canonical diagram,we have claim that $H : X \times I \to X$, defined by

$$H(t,s) = \begin{cases} \alpha({}^{4t}\!/_{s+1}) & t \in [0, \frac{s+1}{4}] \\ \beta(4t-1-s) & t \in [\frac{s+1}{4}, \frac{s+2}{4}] \\ \gamma({}^{4t-2-s}\!/_{2-s}) & t \in [\frac{s+2}{4}, 1] \end{cases}$$

is a homotopy. Indeed;

$$H(t,0) = \begin{cases} \alpha(4t) & t \in [0, 1/4] \\ \beta(4t-1) & t \in [1/4, 1/2] \\ \gamma(2t-1) & t \in [1/2, 1] \end{cases}$$

and, similarly,

$$(H(t,1) = \alpha.(\beta.\gamma))(t)$$

By two applications of the Gluing Lemma, H is continuous. So,

$$(\alpha.\beta).\gamma \simeq_H \alpha.(\beta.\gamma) \text{ rel}\{0,1\}$$

□

(2) Let γ, σ be two paths in the space X which start at p and end at q. We show that σ_* is the composition of γ_* and the inner automorphism of $\pi_1(X,q)$ induced by the element $\langle \sigma^{-1}\gamma \rangle$:

Proof. The isomorphism defined by σ_* and γ_* are

$$\sigma_*(\langle \alpha \rangle) = \sigma^{-1}.\alpha.\sigma$$

and,

$$\gamma_*(\langle \alpha \rangle) = \gamma^{-1}.\alpha.\gamma$$

where α is a loop based at p. The claim is that

$$\sigma_*(\langle\alpha\rangle) = \gamma_*(\langle\gamma.\sigma^{-1}\rangle\langle\alpha\rangle\langle\sigma.\gamma^{-1}\rangle)$$

Indeed;

$$\begin{aligned}\gamma_*(\langle\gamma.\sigma^{-1}\rangle\langle\alpha\rangle\langle\sigma.\gamma^{-1}\rangle) &= \gamma_*(\langle\gamma.\sigma^{-1}\rangle)\gamma_*(\langle\alpha\rangle)\gamma_*(\langle\sigma.\gamma^{-1}\rangle)\\ &= (\gamma^{-1}.\gamma.\sigma^{-1}.\gamma).(\gamma^{-1}.\sigma.\gamma).(\gamma^{-1}.\sigma.\gamma^{-1}.\gamma)\\ &= \sigma^{-1}.\alpha.\sigma\\ &= \sigma_*(\langle\alpha\rangle)\end{aligned}$$

□

(3) Let X be a path connected space. We describe when it is true that for any two points $p, q \in X$, all paths from p, q induce the same isomorphism between $\pi_1(X, p)$ and $\pi_1(X, q)$:

Proof. As the above proof shows, if γ, σ are paths from p to q, then

$$\sigma_*(\langle\alpha\rangle) = \sigma^{-1}.\alpha.\sigma$$

Thus, $\sigma_* = \gamma_*$ if, and only, if

$$\gamma_*(\langle\gamma.\sigma^{-1}\rangle\langle\alpha\rangle\langle\sigma.\gamma^{-1}\rangle) = \gamma_*(\langle\alpha\rangle),$$

i.e. $\langle\alpha\rangle.\langle\beta\rangle = \langle\beta\rangle.\langle\alpha\rangle$. Thus, $\pi_1(X)$ must be abelian. □

(4) We show that any indiscreet space has trivial fundamental group:

Proof. By a previous problem, if (X, τ) is indiscreet, then X is path connected as $\tau = \{X, \varnothing\}$. The idea here is that there are no open sets (holes) for a path to get caught on. We claim that X is contractible:
Consider $\pi_1(X, x_0)$, for some fixed $x_0 \in X$. Define $i \in \pi_1(X, x_0)$ by

$$i(x) = x_0, \quad \forall x \in X$$

Now, for an arbitrary $f \in \pi_1(X, x_0)$, define $H : X \times I \to X$ by

$$H(x,t) = \begin{cases} i(x) & t \in [0, 1/2] \\ f(x) & t \in [1/2, 1] \end{cases}$$

Then, H is the desired homotopy as

$$\begin{aligned} H(x, 0) &= i(x) \\ H(x, 1) &= f(x) \end{aligned}$$

and it is continuous as the only non-trivial open sets in τ is X and $H^{-1}(X) = X \times I$, which is open in $X \times I$. Therefore,

$$f \simeq_H i$$

□

(5) Let (G, m, τ) be a path-connected topological group. Given two loops α, β based at e in G, define a map $F : [0,1]^2 \to G$ by

$$F(s,t) = \alpha(s).\beta(t)$$

We show the effect of this map on the square, and prove that the fundamental group of G is abelian:

Proof. We want to show that for any two loops α, β that

$$\langle\alpha\rangle\langle\beta\rangle = \langle\beta\rangle\langle\alpha\rangle$$

As

$$F(s,0) = F(s,1) = \alpha(s), \quad F(0,t) = F(1,t) = \beta(t),$$

we have that

$$\alpha \simeq_F \alpha(s), \quad \beta \simeq_F \beta(t).$$

We first show that $\alpha.\beta \simeq m(\alpha(t), \beta(t))$:

Define $P : [0,1]^2 \to G$ by

$$P(s,t) = \begin{cases} \alpha(\frac{2t}{1-s}) & t \in [0, \frac{1+s}{2}] \\ e & t \in (\frac{1+s}{2}, 1] \end{cases}$$

and, $Q : [0,1]^2 \to G$ by

$$Q(s,t) = \begin{cases} e & t \in [0, \frac{1-s}{2}] \\ \beta(\frac{2t-1+2}{1+s}) & t \in (\frac{1-s}{2}, 1] \end{cases}$$

Then, let $H(s,t) = m(P(s,t), Q(s,t))$. Note that H is a homotopy between $\alpha.\beta$ and $m(\alpha(t), \beta(t))$. Similarly, we can show that

$$m(\alpha(t), \beta(t)) \simeq_{H'} \beta.\alpha,$$

where $H' = QP$. Therefore,

$$\alpha.\beta \text{ rel}\{0,1\} \simeq_H m(\alpha(t), \beta(t)) \simeq_{H'} \beta.\alpha \text{ rel}\{0,1\}$$

I.e. $\pi_1(G)$ is abelian.

$\square$

(6) We show that the space $\mathbb{E}^n - B$, where $B = \{(x,y,z) : y = 0 \wedge 0 \leqslant z \leqslant 1\}$ has trivial fundamental group:

Proof. As B is homeomorphic to a point, it follows by extra lemma that $\mathbb{E}^n - B$ has trivial fundamental group. $\square$

10 Calculations

(1) We use theorem (5.13) to show that the Möbius strip and the cylinder both have fundamental group $\mathbb{Z}$:

Proof. Letting $X = \mathbb{E}^1 \times I$, it is clear that as $\mathbb{E}$ and $[0,1]$ are convex, that X is simply connected. Consider the action of $\mathbb{Z}$ on the identification square which yields the Möbius loop. It follows that

$$\pi_1(\mathbb{E} \times [0,1]/\mathbb{Z}) = \pi_1(M) \cong \mathbb{Z}$$

This is similar for the torus. $\square$

(2) Consider $S^n \subset \mathbb{E}^{n+1}$. Given a loop α in S^n, we find a loop β in $\mathbb{E}^{n+1}$ which is based at the same point as α, and is made up of a finite number of straight line segments, and satisfy

$$||\alpha(s) - \beta(s)|| < 1, \quad s \in [0,1]$$

And, we use this to deduce that S^n is simply connected when $n \geqslant 2$:

Proof. Note that for each n, S^n is compact. Likewise, as $[0,1]$ is compact, $\alpha([0,1])$ is compact. Now, either α contains antipodal points, or it doesn't. Let

$$\{\underset{b}{\mathcal{O}}\}_b$$

be an open cover of $\alpha([0,1])$. Then, it has an finite subcover; say,

$$\{\underset{b_i}{\mathcal{O}}\}_{i=1}^N$$

If α has no antipodal points, then we are done as the path β which connects one point form each $\mathcal{O}_{b_i}$ at the same value is the desired path.

Now, if α has antipodal points for some $s \in [0,1]$ wew can disjointize neighbourhoods so that $\alpha(s) \in V_i$, and $-\alpha(s) \in V_j$, such that $\alpha(s) \notin V_j$ and $-\alpha(s) \in V_i$, such that

$$\bigcup V_i = \bigcup \underset{b_i}{\mathcal{O}}$$

and the construction of β follows, as above.

To conclude the proof, we show that S^n is simply connected: Indeed; by construction, no line adjoining $\alpha(s)$ and $\beta(s)$ goes through the origin. Thus, the projective straight linke homotopy shows that S^n is simply connected, as $\mathbb{E}^{n+1}$ is for $n \geqslant 2$.

Note, this breaks down in the case where the stereographic projective homotopy fails. This completes the proof. $\square$

(3) Considering the 'proof' of theorem (5.13), we show that for $g_1, g_2 \in G$, $\gamma_1.(g_1 \circ \gamma_2)$ joins x_0 to $g_1 g_2(x_0)$, where γ_1, γ_2 are paths from x_0 to $g_1(x_0)$, $g_2(x_0)$, respectively. In addidtion, we use this to deduce that ϕ is a homomorphism:

Proof. As the statement on page one-hundered-two points out,

$$\gamma_1.(g_1 \circ \gamma_2)$$

is a path from x_0 to $\gamma_1\gamma_2(x_0)$. Thus, as in the proof of (5.13),

$$\begin{aligned}\phi(g_1 g_2) &= \langle \pi \circ (\gamma_1.g_1 \circ \gamma_2) \rangle \\ &= \langle \pi \circ \gamma_1.\pi \circ \gamma_2 \rangle \\ &= \langle \pi \circ \gamma_1 \rangle \langle \pi \circ \gamma_2 \rangle \\ \phi(g_1)&\phi(g_2)\end{aligned}$$

As $\pi \circ \gamma_1.\pi \circ \gamma_2$ is a loop at $\pi(x_0)$ (it follows that they are in the same homotopy class). □

(4) Let $\pi : X \to Y$ be a covering map and α a path in Y. We show that α lifts to a (unique) path in X which begins at any preassigned point of $\pi^{-1}(\alpha(0))$:

Proof. As $X = \bigcup_{y \in Y} \{y\}$, we can form an open cover of Y by evenly covered, i.e. canonical, neighbourhoods. Let $\mathcal{O}$ be the aforementioned open cover, and set

$$\alpha^{-1}(\mathcal{O})$$

to be the family of open sets which cover $[0, 1]$. As $[0, 1] \subset \mathbb{R}$, is compact, and

$$\bigcup \alpha^{-1}(\mathcal{O})$$

is an open cover of $[0, 1]$, $\bigcup \alpha^{-1}(\mathcal{O})$ has a Lebesgue number, δ.

Now, choose $n \in \mathbb{N}$, so that $1/n < \delta$. Then, consider the partition of $[0, 1]$, given by

$$\{0, 1/n, 2/n \ldots, {}^{n-1}/_{n}\}$$

Then, for each $i = 1, 2, \ldots, n$,

$$\alpha\Big(\Big[\frac{i-1}{n}, \frac{i}{n}\Big]\Big)$$

lies inside a canonical neighbourhood of Y. Thus, by setting

$$t_i = \frac{i-1}{n},$$

the result follows. □

(5) Let $\pi : X \to Y$ be a covering map, $p \in Y$, $q \in \pi^{-1}(p)$, and $F : I \times I \to Y$ a map such that

$$F(0,t) = F(1,t) = p, \quad t \in [0,1]$$

We use the argument of lemma (5.11) to find a map $F' : I \times I \to X$ which satisfies

$$\pi \circ F' = F, \quad F'(0,t) = q, \quad t \in [0,1]$$

and is unique:

Proof. As the argument of lemma (5.11) is really quite bad, we proceed with a more technical proof via problem twenty. □

(6) With the terminology above, we note that for each $t \in [0,1]$, we have a path $F_t(s) = F(s,t)$ in Y which begins at p. Let F'_t be its unique lift to a path X which begins at q, and set $F'(s,t) = F'_t(s)$. We show that F' is continuous and lifts F:

Proof. By the homotopy lifting lemma, there exists a unique homotopy lift, B of F. By the proof so mentioned, B is continuous and satisfies

$$\pi \circ B \equiv F$$

But, by the path lifting lemma, as $F'(s,t_0)$, $t_0 \in [0,1]$ is a lift of a path $F(s,t)$ with initial point q,

$$B = F'$$

□

(7) We describe the homomorphism $f_* : \pi_1(S^1, 1) \to \pi_1(S^1, f(1))$ induced by each of the following maps:

- $f(e^{i\theta}) = e^{i(\theta+\pi)}$, $\theta \in [0, 2\pi]$:

  Proof. Note, $f_*(\langle\alpha\rangle) = \langle f(\alpha)\rangle = \langle -\alpha\rangle$. And, as $f(1) = -1$, $f_* \equiv id$. □

- $f(e^{i\theta}) = e^{in\theta}$, $\theta \in [0, 2\pi]$, $n \in \mathbb{Z}$:

  Proof. Note, $f(1) = 1$. Thus, for each loop, we have $f_*(\langle\alpha\rangle) = \langle n\alpha\rangle$. So, f_* sends the loop component to the n-th degree of α. □

- $f(e^{i\theta}) = \begin{cases} e^{i\theta} & \theta \in [0,\pi] \\ e^{i(2\pi-\theta)} & \theta \in [\pi, 2\pi] \end{cases}$:

  Proof. Note, $f(1) = 1$. f_* means that any loop part on the upper half of S^1 is identified to the lower part of S^1. Thus, it returns the straight line for each loop on S^1. □

(8) For each of the three different action of $\mathbb{Z}_2$ on the torus, in section 4.4, we describe the homomorphism from the fundamental group of the torus to that of the orbit space induced by the natural identification map:

Proof. □

(9) As in problem eight, page ninety-one, we show that it is impossible to find a homotopy from h to the identify which is relative to the two boundary circles of A:

Proof. If $h \simeq_F i$ rel $\{c_1, c_2\}$, then

$$\alpha^{-1}\beta \simeq_F c \text{ rel}\{0,1\}$$

However, $\alpha^{-1}\beta$ is homotopic to the loop of constant radius which we know is non-trivial via a deformation retract. □

11 Homotopy Type

Throughout this section, $\simeq$ will typically denote homotopy equivalence.

(1) If $X \simeq Y$ and $X' \simeq Y'$, we show that $X \times X' \simeq Y \times Y'$:

Proof. We are given maps $f : X \to Y$, $g : Y \to X$, $f' : X' \to Y'$ and $g' : Y' \to X'$, such that $g \circ f \simeq_{\gamma'} 1_X$, $f \circ g \simeq_{\gamma} 1_Y$, $g' \circ f' \simeq_{\delta'} 1_{X'}$ and $f' \circ g' \simeq_{\delta} 1_{Y'}$.

From the component-wise definition of the product topology, $F : X \times X' \to Y \times Y'$, defined by $F(x, x') = (f(x), f'(x'))$, and $G : Y \times Y' \to X \times X'$, defined by $G(y, y') = (g(y), g'(y'))$ are continuous. Further, $F \circ G \simeq 1_{Y \times Y'}$, and $G \circ F \simeq 1_{X \times X'}$, via the canonical maps, component-wise defined: $\mathcal{F} : (X \times X') \times I \to X \times X'$, $\mathcal{F} \equiv (\gamma', \delta')$, and $\mathcal{G} : (Y \times Y') \times I \to Y \times Y'$, $\mathcal{G} \equiv (\gamma, \delta)$, respectively.

Thus, $X \times X' \simeq Y \times Y'$. □

(2) We show that the cone, CX, is contractible for any space (X, τ_X):[14]

Proof. By definition, we have $CX = {}^{X \times I}/_{X \times \{1\}}$. Define $H : CX \times I \to CX$ by

$$H((x,t), s) = (x, t(1-s))$$

Then, we have

$$H((x,t), 0) = (x,t) = 1_{CX}$$
$$H((x,t), 1) = (x, 0)$$

Now, if $\mathcal{O}_{CX} \in \tau_{CX}$, then $H^{-1}(\mathcal{O}_{CX}) = (\mathcal{O}_{CX}, I) \in CX \times I$. Thus, H is continuous. This proves the result. □

[14] We proved another proof for this. The first of which is in the extra lemmas section.

(3) We show that the punctured torus deformation retracts onto the one-point union of two circles.

Proof. We consider the torus as the identification space of a square, X, bounded by the box whose edge points are $(-1,-1),(-1,1),(1,1),(1,-1)$. Assume, without loss of generality, that the point $(0,0)$ is removed.

We show that $F: T \times I \to T$ defined by

$$F(x,t) = (1-t)x + t\frac{x}{||x||}$$

is a deformation retract onto the one point union of two circles:

As

$$\begin{aligned} F(x,0) &= x \\ F(x,1) &= \frac{x}{||x||} \end{aligned}$$

F is a deformation retract onto ∂X. The function $f: \partial X \to S_1^1 \bigvee S_2^1$, given by

$$f(x) = \begin{cases} g(x) & x \in (-1,\pm 1)t + (1,\pm 1)(1-t) \\ s(x) & x \in (\pm 1, 1)t + (\pm, -1)(1-t) \end{cases}$$

for all $t \in I$, where g, s are the guaranteed homeomorphisms from $[a,b]$ to S^1, is a homeomorphism by the Gluing Lemma. $\square$

(4) For each of the following cases, we choose as base point in C and describe the generators for the fundamental groups of C and S. Further, we write down the homomorphism, in terms of these generators, the fundamental groups induced by the inclusion of C in S.

Consider the following examples of a circle C embedded in a surface S:

(a) $S =$ Möbius Strip and $C = \partial S$:

(b) $S = S^1 \times S^1 = T^1$ and $C = \{(x.y) \in T^1 : x = y\}$:

(c) $S = S^1 \times I$ and $C = S^1 \times 1$:

(5) Suppose that $f, g: S^1 \to X$ are homotopic maps. We prove that the spaces formed from X by attaching a disc, using f and using g are homotopy equivalent; in other words, we prove that $X \cup_f D \simeq X \cup_g D$:

Proof. We show that this is true for S^{n-1}. To show this, we claim that $X \cup_f D$ and $X \cup_g D$ are deformation retracts of the same space, $X \cup_F (D^n \times I)$, where

$$f \simeq_F g$$

As $F : S^1 \times I \to S^1$ is a homotopy between f and g, we have that

$$F(x, 0) = f(x)$$
$$F(x, 1) = g(x)$$

We claim that the function $r : D^n \times I \to D^n \times \{0\} \cup S^{n-1} \times I$ given by

$$r(x,t) = \begin{cases} \left(\frac{2x}{2-t}, 0\right) & t \in [0, 2(1 - ||x||)] \\ \left(\frac{x}{||x||}, 2 - \frac{2-t}{||x||}\right) & t \in [2(1 - ||x||), 1] \end{cases}$$

is a retraction.

Lemma. r, as defined above is a retraction:

In the first case, suppose that

$$(x, t) \in D^n \times \{0\} \cup S^{n-1} \times I = A$$

is such that $x \in D^n$, and $t = 0$. Then, we have that

$$r(x, t) = r(x, 0) = (x, 0)$$

In the second case, if $t > 0$, and $x \in S^{n-1}$, then $||x|| = 1$ and so $t \in [0, 1]$ and

$$r(x,t) = \left(\frac{x}{1}, 2 - \frac{2-t}{1}\right) = (x, t)$$

Thus, $r|_A = 1_A$. A similar argument shows that $r(D^n \times I) = A$.

To complete the proof of the lemma, we note that as r is polynomial in x and t, it follows that r is continuous. Consequently, r is a retraction. □

We further claim that r is a deformation retraction. To show this, consider the function

$$d : (D^n \times I) \times I \to D^n \times I$$
$$d((x,t), s) = s(r(x,t)) + (1 - s)(x, t)$$

As

$$d((x, t), 0 = (x, t)$$
$$d((x, t), 1) = r(x, t)$$

it follows that d is a homotopy between r and the identity. Consequently, r is a deformation retraction.

Using the fact that r is a deformation retraction we have that $X \cup_F (D^n \times I)$ deformation retracts onto

$$X \cup_F (D^n \times \{0\} \cup S^{n-1} \times I) = X \cup_f (D^n \times \{0\}) \cong D^n$$

(where $X \cup_F (D^n \times \{0\} \cup S^{n-1} \times I) \cong (D^n \times \{0\} \cup_f X)$ via the map sending $D^n \times \{0\}$ and X to itself, and $S^{n-1} \times I$ to $F(S^{n-1} \times I)$.)

A similar argument show a deformation retract of $D^n \times I$ onto

$$D^n \times \{1\} \cup S^{n-1} \times I$$

which gives the identification space with g as the attaching map. And, the result follows. □

(6) We use the previous problem, and the third example of homotopy given in section 5.1, to show that the 'dunce hat' has the homotopy type of a disc, and is therefore contractible:

Proof. This is so horrendously explained by Armstrong. By "the third example of homotopy given in section 5.1," we assume that Armstrong really means "*(to take as the canonical definition of 'dunce cap') that the dunce cap is constructed by* gluing D^2 to S^1 via the map $g : S^1 \to S^1$ given by

$$g(e^{i\theta}) = \begin{cases} e^{4i\theta} & 0 \leqslant \theta \leqslant \pi/2 \\ e^{4i(2\theta-\pi)} & \pi/2 \leqslant \theta \leqslant 3\pi/2 \text{ "} \\ e^{8i(\pi-\theta)} & 3\pi/2 \leqslant \theta \leqslant 2\pi \end{cases}$$

That is, $S^1 \cup_g D^2$ is the dunce hat. With this assumption, we continue with the proof:

Consider the adjunct space given by $i : S^1 \to D^2$; $X \cup_i D^2$. To conclude, we show that g is homotopic to the identity map:

Consider $F : S^1 \times I \to S^1$ given by

$$F(e^{i\theta}, t) = tg(e^{i\theta}) + (1-t)e^{i\theta}$$

This is the straight line homotopy;

$$g \simeq_F i$$

Thus, by exercise twenty-seven, page one-hundred-nine, $i, g : S^1 \to D^2$ are homotopic, so that

$$S^1 \cup_g D^2 \simeq S^1 \cup_i D^2 \cong D^2$$

and the result follows. □

(7) We show that the 'house with two rooms' is contractible:

Proof. A rigorus proof of this theorem is probably beyond the scope of this section. However, a sketch of it is given in Allen Hatcher's *Algebraic Topology*, Chapter 0. text[15]. □

[15] cite this

(8) We give a detailed proof to show that the cylinder and the Möbius strip have the homotopy type of the circle:

Proof. We have previously shown that both the cylinder and the Möbius strip are homotopic to S^1. For a more general argument:

A deformation retract $F : X \times I \to X$ of X onto $A \subset X$ induces a homotopy equivalence by taking $F(-,1) : X \to A$ and the inclusion $\iota : A \to X$. Thus, the composition of the homotopy equivalences $M \to S^1 \to S^1 \times I$ prove the result. $\square$

(9) Let X be the comb space. We prove that the identity map of X is not homotopic rel$\{p\}$, to the constant map, $p = (0, 1/2)$:

Proof. This is a problem in Munker's *Topology.* In his text, the point p is $(0, 1)$. We proceed with this assumption:

Let $p = x_0 = (0, 1)$. Consider a neighborhood U of x_0 which is disjoint from $I \times \{0\}$. Suppose $H : X \times I \to X$ is the homotopy starting with the identity on X and ending with the constant map $X \to \{x_0\}$ such that $H(x_0, t) = x_0$ for all times t. That means $\{x_0\} \times I$ has a neighborhood $H^{-1}(U)$. By the tube lemma (this is the statement that in the product topology, there is an open set containing a compact product), there is an open set V such that $\{x_0\} \times I \subset V \times I \subset H^{-1}(U)$. That means every point in V stays in U during the entire deformation. However a point $y = (a, 1)$ must traverse a path to x_0 and no such path exists within U. $\square$

(10) *FTA*: We prove the fundamental theorem of algebra:

Proof. This is shown in Munker's *Topology*, page three-hundered-fifty. $\square$

12 Brouwer Fixed-Point Theorem

Throughout, we say a topological space (X, τ_X) has the fixed-point property if every continuous function from X to itself has a fixed point.

(1) We determine which of the following have the fixed point property:

- The 2-Sphere: S^2 does not exhibit the fixed point property.

  Proof. We have previously shown that the map $f : S^n \to S^n$, given by $f(x) = -x$ is a homeomorphism. $\square$

- The Torus: $T^1 = S^1 \times S^1$ does not exhibit the fixed point property.

Proof. As $T^1 = S^1 \times S^1$, and the antipodal map, f, on S^1 is a homeomorphism, we have that $F : T^1 \to T^1$, defined by $F((x, y)) = (f(x), f(y))$, is continuous and doesn't exhibit the fixed point property. $\square$

- The interior of the unit disc: $\mathrm{Int}D^1$ does not exhibit the fixed point property:

  Proof. Note that the interior of the unit disc is homeomorphic to the euclidean plane by a homeomorphism $h : \mathrm{Int}D^1 \to \mathbb{E}^2$. We define a function g from the euclidean plane to the euclidean plane by $g((x, y)) = (x + 1, y)$. Then, $h^{-1} \circ g \circ h$ is a continuous function from the disc to itself that has no fixed point. $\square$

- The one point union of two circles: $X \vee Y = {}^{X \amalg Y}/_{\{p\}}$ does not exhibit the fixed point property:

  Proof. Define a function f as follows: If $x \in X$ then we map e^{ix} to $e^{i(x+\pi)}$. Also if $y \in Y$, we map y to $f(p) = e^{i\pi/2}$. This function f, as defined on the one point union, leaves no points fixed. $\square$

(2) Suppose X and Y are of the same homotopy type and X has the fixed-point property. We prove that Y does not necessarily have the fixed point property:

Proof. Let Y be the subspace $(0, 1) \subset \mathbb{E}^n$, and let $X = \{1/2\}$. Note that Y is homotopic to X by the straight line homotopy, and every map from X to itself has a fixed point. Yet, the function $f : Y \to Y$, defined by $f(y) = y^2$ has no fixed point. $\square$

(3) Suppose that X retracts onto the subspace $A \subset X$, and that A has the fixed point property. We show that X may not exhibit the fixed point property:

Proof. Take X and Y, as above. The straight-line homotopy proves the assertion. $\square$

(4) We show that if X retracts onto the subspace A, and X has the fixed-point property, then A also has it:

Proof. Let $f : A \to X$ be a continuous function. Since X retracts onto A, there exists a map $g : X \to A$ such that $g\restriction A \equiv 1_A$. Then, $f \circ g$ is a continuous function from X to X, and so has a fixed point. Hence, there exists an $a \in A$ such that $f(g(a)) = f(a) = a$. This completes the proof. $\square$

(5) We deduce that the fixed-point property holds for the 'house with two rooms', X:

Proof. As was previously show, X is contractible to some x_0. Thus, there exists some map $F : X \times I \to X$ such that $F(x, 0) = x$, and $F(x, 1) = x_0$. Thus, by extra lemma [16], every map $f : X \to Y$ is null-homotopic. In particular, this includes that maps $g : X \to X$. Thus, X has the fixed point property[17].

To use the previous problems hints, we could think about starting with the unit cylinder, and pushing in the areas from the top and bottom. However, this method is not rigorous. $\square$

(6) Let f be a fixed-point-free map from a compact metric space (X, d) to itself. We prove there is a positive number ϵ such that $d(x, f(x)) > \epsilon$, $\forall x \in X$:

Proof. We show the contrapositive; Suppose that for all $\epsilon > 0$ there exists an element $x \in X$ such that $d(x, f(x)) \leqslant \epsilon$. Pick $x_1 \in X$ such that $d(x_1, f(x)) \leqslant \frac{1}{2} < 1$.

It follows that there exists a set, $\{x_1, ..., x_n\} \subset X$ such that $d(x_i, f(x_i)) < d(x_k, f(x_k))$, where $i < k$ and $d(x_k, f(x_k)) < \dfrac{1}{k}$, for all $1 \leqslant k \leqslant n$.

Now, let

$$\epsilon_0 = \frac{1}{2} \min \left\{ d(x_1, f(x_1)), ..., d(x_n, f(x_n), \frac{1}{n+1} \right\}$$

Then there exists an element $x \in X$ such that $d(x, f(x)) \leqslant \epsilon_0$. Pick $x \in X$ that satisfies this property, and call this x_{n+1}. Note here that $d(x_n, f(x_n)) < d(x_{n+1}, f(x_{n+1}))$ by construction. So that for all $i < n+1$, $d(x_i, f(x_i)) \leqslant d(x_n, f(x_n)) < d(x_{n+1}, f(x_{n+1}))$. Thus by induction, we have created an infinite sequence $\{x_n\}_{n=1}^{\infty}$ of distinct points such that $d(x_k, f(x_k)) < 1/k$ for each positive integer k. And, $\{d(x_n, f(x_n))\}_n$ is a monotone decreasing sequence.

Since X is compact, every infinite subset has a limit point. Therefore $\{x_n\}_{n=1}^{\infty}$ has a limit point $x \in X$.

Let $\epsilon > 0$ be given. Since f is continuous at x, then there exists a $\delta > 0$ such that $d(x, a) < \delta \implies d(f(x), f(a) < \epsilon/3$ for all $a \in X$. By the Archimedian Property, we can find a positive integer k, such that

$$\frac{1}{k} < \frac{\epsilon}{3}$$

Set $r = \min\{\frac{\epsilon}{3}, \delta\}$. Then, since x is a limit point of $\{x_n\}_{n=1}^{\infty}$, there are infinitely many points of the sequence, such that $d(x_N, x) < r$. Thus, there exists a point x_N such that $N > k$ and $d(x_N, x) < r$. Since $d(x_N, x) < \delta$, then $d(f(x_N), f(x)) < \epsilon/3$. Since $N > k$,

$$d(x_N, f(x_N)) < d(x_k, f(x_k)) < 1/k < \frac{\epsilon}{3}$$

[16] include this

[17] If this is not totally clear, see extra lemma.

Therefore

$$d(x, f(x)) \leqslant d(x, x_N) + d(x_N, f(x_N)) + d(f(x_N), f(x)) < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon$$

Thus, for all $\epsilon > 0$, we have that $d(x, f(x)) < \epsilon$. Thus, $d(x, f(x)) \leqslant 0$, and hence $d(x, f(x)) = 0$. Consequently, $f(x) = x$, and so f is not a fixed-point-free map. □

(7) We show that the unit ball, B^n, in $\mathbb{E}^n$, with $(1, 0, ..., 0)$ removed does not exhibit the fixed point property:

Proof. Let $p = (1, ..., 0)$. Consider the function $f : B^n \backslash \{p\} \to B^n \backslash \{p\}$, defined by $f(x) = \frac{x+p}{2}$. We show that f is continuous:

Let $\mathcal{O}$ be an open set in $B^n\{p\}$. Let $x \in f^{-1}(\mathcal{O})$. Then $f(x) \in \mathcal{O}$, and since $\mathcal{O}$ is open there exists an $r > 0$ such that $B_r(f(x)) \subset \mathcal{O}$. Let $y \in B_{2r}(x)$. Then $|x - y| < 2r$. Then,

$$|f(x) - f(y)| = |\frac{x+p}{2} - \frac{y+p}{2}| = \frac{1}{2}|x - y| < r$$

Thus, $y \in B_r(f(x))$. Hence, $y \in f^{-1}(\mathcal{O})$. Thus, $B_{2r}(x) \subset f^{-1}(\mathcal{O})$. Therefore, $f^{-1}(\mathcal{O})$ is open and f is continuous.

Further, f does not have the fixed-point property; If $f(x) = x = (x_1, ..., x_n)$, then we have $2x = x + p$. Implying that $x_i = 2x_i$, $2 \leqslant i \leqslant n$. Thus, $x_i = 0$. However, then the only solution to $x_1 + 1 = 2x_1$ is $x_1 = 1$. This is a contradiction since we must have $x = p \notin B^n \backslash \{p\}$. □

(8) We show that the one-point union of X and Y, $X \vee Y$, has the fixed-point property if, and only if, both X and Y have it:

Proof.

($\Longrightarrow$): Suppose that the one point union $X \vee Y$ has the fixed-point-property. Let $f : X \to X$ be a map, and let p be the point glued together in the one point union. Then, define $g : X \vee Y \to X \vee Y$, as $g(x) = f(x)$, if $x \in X$, and $g(x) = f(p)$, if $x \in Y$. By the gluing lemma, g is a continuous map and so, by hypothesis, has a fixed point x_0. Note that g is a map into X. Thus the fixed point must be in X. Hence, by construction, $x_0 = g(x_0) = f(x_0)$, so that f has a fixed point. Similarly, any continuous function from Y to itself has a fixed point.

($\Longleftarrow$): Suppose X and Y have the fixed point property and let $f : X \vee Y \to X \vee Y$ be a map. Then, suppose $f(p) \in X$, and define a map $g : X \to X$ such that $g(x) = f(x)$, if $f(x) \in X$, and $g(x) = p$, if $f(x) \notin X$. Then, since g is continuous, it has a fixed point. By construction, the fixed point must be one $x = g(x) = f(x)$. Thus, f has a fixed point. Next, suppose $f(p) \in Y$, and define a map $g : Y \to Y$ such that $g(y) = f(y)$, if $f(y) \in Y$, and $g(y) = p$, if $f(y) \notin Y$. Then, since g is continuous, it has a fixed point. By construction, the fixed point must be one $y = g(y) = f(y)$. Consequently, f has a fixed point.

$\square$

(9) How does changing 'continuous function' to 'homeomorphism' in the definition of the fixed-point property affect problem 33, 37?

Proof. We first examine problem 33:

S^2: This topological space would not exhibit the fixed point property. We know that the antipodal map is a homeomorphism which leaves no points fixed.

$S^1 \times S^1$: Likewise, the antipodal map on each component, S^1, provides another counter example.

$\operatorname{Int} D$: Neither does this space have the homeomorphic fixed point property. To see this, we note that $\operatorname{Int} D \simeq \mathbb{E}$, and the map $g : \mathbb{E} \to \mathbb{E}$, given by $g(x) = x + 1$ is a homeomorphism which leaves no points fixed.

$X \bigvee Y$: First, consider the map $E_1 : X \bigvee Y \to X \bigvee Y$, given by $E_1(x) = x$, if $x \in X - \{p\}$. And, $E_2(y) = e^{yi\pi/2}$. Then, E_1 is a homeomorphism, which leaves X fixed, except for p. Similarly, define E_2 for Y. Then, it follows that $E_2 \circ E_1 : X \bigvee Y \to X \bigvee Y$ is a homeomorphism that leaves no points of $X \bigvee Y$ fixed. Thus, $X \bigvee Y$ does not exhibit the fixed point property[18].

Now, we examine problem 37:

This shape does not exhibit the homeomorphic fixed point property. Consider $\partial B^n = S^{n-1}$, and let $p = (1, 0, ..., 0)$. Then, $B^n - \{p\} = \operatorname{Int} B^n \cup S^{n-1} - \{p\}$. Furthermore, without loss of generality, $S^{n-1} - \{p\} \simeq \{-p\}$.[19] It follows that $\operatorname{Int} B^n \cup S^{n-1} - \{p\} \simeq \operatorname{Int} B^n \cup \{-p\}$. To conclude, we exhibit a homeomorphism on $\operatorname{Int} B^n \cup \{-p\}$ which does not have a fixed point:

Consider $f : \operatorname{Int} B^n \cup \{-p\} \to \operatorname{Int} B^n \cup \{-p\}$ defined by

$$f(x) = \begin{cases} -x & x \notin \{-p, 0\} \\ 0 & x = -p \\ -p & x = 0 \end{cases}$$

[18] We make use of the previous proof; that is $X \bigvee Y$ has the fixed point propert iff X and Y both have it.

[19] See previous exercises.

The fact that f is 1-1 and onto is clear by construction. Further, f is continuous, by the Gluing Lemma[20]. It follows that f is a homeomorphism, as $f^{-1} \equiv -f$. However, f clearly leaves no points fixed. □

13 Separation of the Plane

Most of the exercises in the section are too advanced for an undergraduate course in algebraic topology. As a result, we refer to well-known sources, which prove the results.

(1) Let A be a compact subset of $\mathbb{E}^n$. We show that $\mathbb{E}^n - A$ has exactly one unbounded component:

Proof. As A is compact, it is closed and bounded. Further, we identify $\mathbb{E}^n$ with $S^n - \{p\}$ under stereographic projection, π. Then, A is a compact set in S^n such that $\{p\} \not\subset A \cap S^n$. The remainder of this proof is an extension of lemma (61.1), Munkers' *Topology*[21]. □

(2) Let J be a polygonal Jordan curve in the plane. Let p be a point in the unbounded component of $\mathbb{E}^2 - J$ which does not lie on any of the lines produced by extending each of the segments of J in both directions. Given a point x of $\mathbb{E}^2 - J$, say that x is inside (outside) J if the straight line joining p to x cuts across J an odd (even) number of times. We show that the complement of J has exactly two components, namely the set of inside points and the set of outside points:

Proof. See *The Jordan-Schónflies Theorem and the Classification of Surfaces*, by Carsten Thomassen. □

(3) Let J be a polygonal Jordan curve in the plane, and let X denote the closure of the bounded component of J. We show that X is homeomorphic to a disc:

Proof. See *The Jordan-Schónflies Theorem and the Classification of Surfaces*, by Carsten Thomassen. □

(4) We prove Schönflies theorem for polygonal Jordan curves:

Proof. See *The Jordan-Schónflies Theorem and the Classification of Surfaces*, by Carsten Thomassen. □

(5) If J is a Jordan curve in the plane, we use theorem (5.12) to show that the frontier of any component of $\mathbb{E}^2 - J$ is J:

[20] Armstrong, pg. 69.
[21] cite this

Proof. See *The Jordan-Schóflies Theorem and the Classification of Surfaces*, by Carsten Thomassen. □

(6) We give an example of a subspace of the plane which has the homotopy type of a circle, which separates the plane into two components, but which is not the frontier of both these components:

Proof. Consider the standard annulus in the plane,

$$A = \{(x, y) : x^2 + y^2 = r \wedge 1 \leqslant r \leqslant 2\}$$

Then, as we have previously shown, A deformation retracts onto $S^1 \subset A$, and so A has the homotopy type of a circle. Further,

$$\operatorname{Int} A = \{(x, y) : x^2 + y^2 < 1\},$$

with $\partial \operatorname{Int} A = S^1 \neq A$. In addition,

$$\operatorname{Out} A = \{(x, y) : x^2 + y^2 > 2\},$$

and $\partial \operatorname{Out} A = 2S^1 \neq A$. □

(7) We give example of simple closed curves which separate, and fail to separate

- the torus:

  Proof. Considering the torus as the identification space of $[0, 1]^2$, the circle of radius $1/4$ centered at $(1/2, 1/2)$, separates T, while the line $x = 1/2$ does not. □

- $\mathbb{R}\,\mathrm{P}^2$:

  Proof. Note that $\mathbb{R}\,\mathrm{P}^2$ is given by the equivalence relation $x\ \lambda x$, $\lambda \neq 0$. Thus, a curve in $\mathbb{R}\,\mathrm{P}^2$ is a set of equivalence classes. Thus, a circle of radius $1/4$ centered at $(1/2, 1/2)$, which is equivalent to an arc on the first quadrant of S^1 separates $\mathbb{R}\,\mathrm{P}^2$. And, S^1 does not separate $\mathbb{R}\,\mathrm{P}^2$. □

(8) Let X be the subspace of the plane which is homeomorphic to a disc. We generalize the argument of theorem (5.21) to show that X cannot separate the plane:

Proof. See *The Jordan-Schóflies Theorem and the Classification of Surfaces*, by Carsten Thomassen. □

(9) Suppose that X is both connected and locally path-connected. We show that a map $f : X \to S^1$ lifts to a map $f' : X \to \mathbb{R}$ if, and only, if the induced homomorphism $f_* : \pi_1(X) \to \pi_1(S^1)$ is the zero homomorphism:

Proof. See the accompanying solution[22]. □

[22] http://stanford.edu/class/math215b/Sol3.pdf

14 The Boundary of a Surface

(1) We use an argument similar to that of theorem (5.23) to prove that $\mathbb{E}^2$ and $\mathbb{E}^3$ are not homeomorphic:

Proof. We have previously show that $\pi_1(\mathbb{E}^m)$ is trivial for $m \geqslant 3$, while $\pi_1(\mathbb{E}^2)$ is not. The result follows. $\square$

(2) We use the material of this section to show that the spaces X, and Y illustrated in problem twenty-four of chapter one are not homeomorphic:

Proof. Suppose, to the contrary, that X and Y were homeomorphic. Then, by corollary (5.25), they have homeomorphic boundaries. Further, it follows from the proof of theorem (5.24), that the outer boundaries, $\operatorname{Out} X$, of X and $\operatorname{Out} Y$ of Y must be homeomorphic, as well as the inner boundaries. However, $\square$