Armstrong Topology Solutions

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Contents

1 Extra Proofs and Lemmas **2**
- 1.1 Limit Points, Closure & Density 2
- 1.2 Separation 3
- 1.3 Compactness 3
- 1.4 Quotient Maps & Quotient Topology 4
- 1.5 Topological Groups 6
- 1.6 Homotopy Type 7

2 Introduction **9**
- 2.1 Topological Invariants 9

3 Continuity **13**

4 Homotopy Type **14**

5 Brouwer Fixed-Point Theorem **17**

6 Separation of the Plane **22**

1 Extra Proofs and Lemmas

1.1 Limit Points, Closure & Density

(1) Let (X, τ) be a topological space, and $A \subset X$. We show that $\bar{A} = A \cup L_A$, where L_A is the set of all accumulation points of A:

Proof. If $\bar{A} = A$, then by Theorem 2.2, pg. 29, $L_A \subset A$ and we are done. So, suppose that A is not closed. Then, A does not contain all of its limit points; consequently, $L_A \neq \varnothing$. Further, the set $B = A \cup L_A$ is closed by Theorem 2.2, pg. 29.

To conclude the proof, we show that any closed set C containing A, contains B. Indeed; Let $C \subset X$ be closed, such that $A \subset C$. Now, for $a \in L_A$ and open set O_a containing a, we have that $A \cap (O_a - \{a\}) \neq \varnothing$. But, as $A \subset C$, we have $C \cap (O_a - \{a\}) \neq \varnothing$. Therefore, $a \subset C$. So, $L_A \subset C$, implying $A \cup L_A \subset C$. By Theorem 2.3, pg. 30, $\bar{A} = B = A \cup L_A$. □

(2) Let (X, τ) be a topological space, and $A \subset X$. We prove that L_A, defined above, contains all limit points of sequences contained in A:[1]

Proof. Let $\{a_n\}_{n=1}^{\infty} \subset A$, such that $a_n \to a$. Then, $a \in L_{\{a_n : n \in \mathbb{N}\}}$ and every neighbourhood of a contains a point of $\{a_n : n \in \mathbb{N}\} - \{a\} = B$. As $\{a_n : n \in \mathbb{N}\} \subset A$, every neighbourhood of a has a point in $A - \{a\}$; $a \in L_A$. □

(3) Let (X, τ) be a topological space, and $A \subset X$ such that $\bar{A} = X$. We prove that $A \cap O \neq \varnothing$, for all $O \neq \varnothing$, $O \in \tau$:

Proof. Suppose that for some $O \in \tau$, $O \neq \varnothing$, $A \cap O = \varnothing$. By a previous lemma, we have $\bar{A} = (L_A - A) \cup O^c$, implying $O^c = A$. But then we have $X = O^c = \bar{O}^c = \bar{A}$, as O is open. But, contrarily, this implies that $(O^c)^c = O = X^c = \varnothing$. □

(4) We prove that the intersection of a closed set and a compact set is always compact:

Proof. Let (X, τ) be a topological space. Let $H, K \subset X$, such that H is closed and K is compact. Consider $H \cap K$. Now, if $\{O_\alpha\}_\alpha$ is an open cover of $H \cap K$, then $K \subset \bigcup_{\alpha \in N} O_\alpha \cup (X - H)$. But, since H is closed, $X - H$ is open. In conclusion, as

$$H \cap K \subset \bigcup_{\alpha \in N} O_\alpha \cup (X - H)$$

such a finite subcover of $H \cap K$ exists. □

[1] This assumes the general definition of limit points of a set

(5) We prove that if (X, d) is a metric space with the induced topology, then $C \subset X$ is closed if, and only if, whenever $\{a_n\}_{n=1}^{\infty}$ is a sequence in C, with $\{a_n\} \to L$, we have $L \in C$:

Proof.

($\Longrightarrow$): Suppose, to the contrary, that $\bar{C} = C$, but $L \notin C$. Thus, there exists some $\epsilon > 0$, such that $B_\epsilon(L) \cap C = \varnothing$, as L is not a limit point of C. But then, $a_n \notin B_\epsilon(L)$, for all $n \geqslant N$, $N \in \mathbb{N}$ is sufficiently large; A contradiction.

($\Longleftarrow$): Suppose, to the contrary, that $\bar{C} \neq C$. Then, by extra lemma[2], there is some $l \in L_C$, such that $L \notin C$. Thus, for each $n \in \mathbb{N}$, we pick $a_n \in B_{1/n}(L) \cap C \neq \varnothing$. Consequently, $\{a_n\}_{n=1}^{\infty}$ is a sequence in C such that $a_n \to L \in C$, by hypothesis; A contradiction.

□

1.2 Separation

(1) We show that a compact T_2 space T_3. Consequently, we show that it is T_4:

Proof. Let X be the compact T_2 space. We first show that X is T_3:

Let $A \subset X$, such that A is closed. Then A is compact. Further, let $b \in X$ such that $b \notin A$. Now, for each $a \in A$, there exists an open set O_a, and some open set O_b^a, such that $O_a \cap O_b^a = \varnothing$. It follows that $A \subset \bigcup_{a \in N} O_a \subset \bigcup_a O_a$ and that $\bigcup_{a \in N} O_a = O_A$ is open. Further, $b \in \bigcap_{a \in N} O_b^a = O_b$, is open. And by construction, we have $O_A \cap O_b = \varnothing$. So, X is T_3.

To show that X is T_4, let $A, B \subset X$, be disjoint and closed. Then, arguing as above, we have two disjoint open subsets $O_B \in \tau$, $O_A \in \tau$ with $O_A \cap O_B = \varnothing$. □

1.3 Compactness

(1) We show that a homeomorphism between locally compact T_2 spaces, X, Y, extends to a homeomorphism between the Alexandroff compactifications; in other-words, locally compact homeomorphic T_2 spaces have homeomorphic one point compactifications.

Proof. Suppose that $f : X \to Y$ is the homeomorphism. Define $g : X \cup \{\infty_1\} \to Y \cup \{\infty_2\}$ as follows:

$$g(x) = \begin{cases} f(x) & x \neq \infty_1 \\ \infty_2 & x = \infty_1 \end{cases}$$

[2] reference this

It is clear that as f is a bijection, so is g. We show that g is a homeomorphism:

Suppose that U is open in Y. Then, $g^{-1}[U] = f^{-1}[U]$, which is open in $X \cup \{\infty_1\}$. Now, if $\mathcal{O}$ is open in $Y \cup \{\infty_2\}$, we have $\mathcal{O} = K \cup \{\infty_2\}$, where $K \subset Y$ is compact. Then,

$$g^{-1}[\mathcal{O}] = g^{-1}[K \cup \{\infty_2\}] = g^{-1}[K] \cup g^{-1}[\{\infty_2\}] = g^{-1}[K] \cup \{\infty_1\}$$

which is open in $X \cup \{\infty_1\}$.

Consequently, as g is one-to-one and onto, since f is, g is a homeomorphism by Theorem 3.7, pg. 48. □

(2) We show that if (X, τ) is a topological space, and $S \subset X$ is compact, then if W is a neighbourhood of S, there exists another neighbourhood G, such that

$$S \subset G \subset W$$

Proof. By exercise 21, section 3 pg. 55, and the fact that $X \cup \{y\} \cong X$, the result follows. □

1.4 Quotient Maps & Quotient Topology

(1) We show that if $q : X \to Y$ is a quotient map[3], then the topology of Y is the largest which makes q continuous:

Proof. Suppose that τ was some other topology on Y, such that q was continuous. We conclude by showing that $\tau \subset \tau_Y$:

Indeed; if $\mathcal{O} \in \tau$, then $q^{-1}[\mathcal{O}]$ is open in X. But then, $q[q^{-1}[\mathcal{O}]] = \mathcal{O} \in \tau_Y$. Thus, $\tau \subset \tau_Y$. □

(2) Suppose that (X, τ) is a topological space, and A a non-empty set. We show that if $f : X \to A$, then the quotient topology on A, τ_A is indeed a topology:

Proof. By definition, as noted above, $\mathcal{O}$ is open in A, if $f^{-1}[\mathcal{O}]$ is open in X:

(a) As $f^{-1}[\varnothing] = \varnothing$, and $f^{-1}[A] = X$, $\varnothing, A \in \tau_A$.

[3] Armstrong does not do a good job describing what the topology on Y is. A simple exercise shows that by letting $\mathcal{O}$ be open in Y whenever $q^{-1}[\mathcal{O}]$ is open in X, we have a topology on Y; call this τ_Y. Further, Armstrong does not do an adequate job describing what a quotient map is: $q : X \to Y$ is a quotient map if it is onto, continuous with respect to τ_Y, and such that $g^{-1}[\mathcal{O}]$ is open in X implies $\mathcal{O}$ open in Y. We can summarize by saying that q is onto, and such that $\mathcal{O}$ is open in Y iff $g^{-1}[\mathcal{O}]$ is open in X.

(b) As

$$f^{-1}[\bigcup_{j\in J}\mathcal{O}_j] = \bigcup_{j\in J} f^{-1}[\mathcal{O}_j],$$

we see that $\bigcup_{j\in J}\mathcal{O}_j \in \tau_A$.

(c) Lastly,

$$f^{-1}[\bigcap_{j=1}^{\infty}\mathcal{O}_j] = \bigcap_{j=1}^{\infty} f^{-1}[\mathcal{O}_j]$$

So, $\bigcap_{j=1}^{\infty}\mathcal{O}_j \in \tau_A$.

□

(3) We show that projective real n-space, $\mathbb{P}^n$, is Hausdorff[4]:

Proof. Let $q : S^n \to \mathbb{P}^n$ be the quotient map, and suppose that $u, v \in \mathbb{P}^n$, $u \neq v$; there are such $x, y \in S^n$, such that $q^{-1}[\{u\}] = \{x, -x\}$, and $q^{-1}[\{v\}] = \{y, -y\}$. We exhibit an $\epsilon \in \mathbb{R}$ such that the open neighbourhoods of radius ϵ, centered at u, v are disjoint:

Let

$$\epsilon = \frac{1}{2}\min\{||x-y||, ||x+y||\},$$

and set $U = B(x, \epsilon)$, and $V = B(y, \epsilon)$. It follows form the construction of U, V, that $U, V, -U, -V$ are pairwise disjoint, and open neighbourhoods of $x, y, -x, -y \in S^n$. Moreover, as noted above, $q^{-1}(q[U]) = U \cup -U$, and $q^{-1}(q[V]) = V \cup -V$. To conclude, we show that $q[U] \cap q[V]$ are disjoint neighbourhoods for $u, v \in \mathbb{P}^n$.

Indeed;

$$\begin{aligned} q[U] \cap q[V] &= q[B(x,\epsilon) \cap S^n] \cap q[B(y,\epsilon) \cap S^n] \\ &= q[B(x,\epsilon)] \cap q[B(y,\epsilon)] \cap \mathbb{P}^n \\ &= B(u, q(\epsilon)) \cap B(v, q(\epsilon)) \cap \mathbb{P}^n \\ &= B_1 \cap B_2 \cap \mathbb{P}^n \end{aligned}$$

Now, if $B_1 \cap B_2 \neq \varnothing$, then we would have $q^{-1}[B_1 \cap B_2] = U \cap V \neq \varnothing$, contary to assumption. Thus,

$$q[U] \cap q[V] = \varnothing \cap \mathbb{P}^n = \varnothing$$

□

[4]Credit for the initial idea of the proof goes to Brain M. Scott. In addtion, we use the notation $B(z, z')$ for the open ball of radius z', centered at z.

1.5 Topological Groups

(1) Suppose that $(G, \tau, *)$ is a topological group. Fixing $x \in G$, we show that $x\,\mathcal{O} \in \tau$ if, and only if, $\mathcal{O} \in \tau$:

Proof.

($\implies$): Suppose that $x\,\mathcal{O} \in \tau$. It follows from pg. 75, that $L_{x^{-1}} : G \to G$ is a homeomorphism and an open map: thus, $L_{x^{-1}}[\mathcal{O}] \in \tau$.

($\impliedby$): Suppose that $\mathcal{O} \in \tau$. Then, as noted on pg. 75, $L_x : G \to G$ is a homeomorphism and an open map: thus, $L_x[\mathcal{O}] = x\,\mathcal{O} \in \tau$.

□

(2) Suppose that $(G, \tau, *)$ is a topological group. We show that $\mathcal{O}$ is a neighbourhood of $x \in G$, if, and only if, $x^{-1}\,\mathcal{O}$ is a neighbourhood of e:

Proof. Suppose that $x \in G$, and without loss of generality, $\mathcal{O}$ is an open set containing x. Then, by a previous lemma[5], $L_{x^{-1}}[\mathcal{O}] = x^{-1}\,\mathcal{O} \in \tau$, and contains $x^{-1}x = e$.

By considering L_x, similar logic shows the reverse implication. □

(3) We show that $\mathbb{R}$, with the Euclidean topology and addition is a topological group:

Proof. The fact that $(\mathbb{R}, +)$ is a group is clear. To conclude, we show that $+ \equiv m : \mathbb{R} \to \mathbb{R}$, and $-1 \equiv i : \mathbb{R} \to \mathbb{R}$ are continuous:

Now, $i(x) = -x$, which is polynomial, and so continuous. And, as $m(x, y) = x + y = P_1(x, y) + P_2(x, y)$, where P_i is the projection mapping from $\mathbb{R}^2$, m is continuous; it is the sum of two continuous functions.

A similar argument show that $(\mathbb{R}^n, \tau, +)$ is a topological group. □

(4) Let (G, τ, m) be a topological group. We show that the topological automorphisms of (G, τ, m), for a subgroup of $(Aut)(G)$. We denote the set of all topological automorphism of G by $\mathrm{Aut}_\tau(G)$.

Proof. It is well known that $(\mathrm{Aut}(G), \circ)$ is a group. We show that $(\mathrm{Aut}_\tau(G), \circ)$ is a group:

As the composition of automorphism/homeomorphisms is another automorphism/homeomorphism, the fact that $(\mathrm{Aut}_\tau(G), \circ)$ is closed is clear. To conclude, as $f \in \mathrm{Aut}_\tau(G)$ is a homeomorphism, $f^{-1} \in \mathrm{Aut}_\tau(G)$. Furthermore, $e : (G, \tau) \to (G, \tau)$ given by $e(x) = x$ is a homeomorphism. So, $e \in \mathrm{Aut}_\tau(G)$.

[5] reference this

By the subgroup test,

$$\operatorname*{Aut}_{\tau}(G) \leqslant \operatorname{Aut}(G)$$

□

(5) Suppose that (G, τ, m) is a T_2, finite, topological group. We claim $\tau = \mathcal{P}(G)$:

Proof. As G is T_2, $\{x\}$ is open for each $x \in G$. Thus, as the union of open sets is open, it follows that $\tau = \mathcal{P}(G)$. □

(6) We claim that all topological groups of order 2 are topologically isomorphic.

Proof. Let (G, τ_G, m) be the topological group of order 2. Consider the topological group $(Z_2, \tau_{Z_2}, +)$ and the map $\phi : G \to Z_2$, given by $\phi(e_G) = 0$, and $\phi(g) = 1$. The fact that ϕ is a group isomorphism is well-known. Further, ϕ is 1-1 and onto, and ϕ^{-1} is continuous, as $\tau_G = \mathcal{P}(G)$ and $\tau_{Z_2} = \mathcal{P}(Z_2)$. This concludes the proof. □

1.6 Homotopy Type

(1) We prove that if A is a subspace of a topological space X and $G : X \times I \to X$ is deformation retract relative to A, then X and A have the same homotopy type[6]:

Proof. Indeed: Let $f : A \to X$ be the inclusion map and $g : X \to A$ be defined by $g(x) = G(x, 1)$. We have previously shown that f and g are continuous. It is only left to show that $f \circ g \simeq 1_X$ and $g \circ f \simeq 1_A$:

Direct computation shows that

$$(g \circ f)(a) = g(f(a)) = g(a) = G(a, 1) = a,$$

while $f \circ g \simeq_G 1_X$. This completes the proof. □

(2) We show that any non-empty convex subset, X, of a euclidean space is homotopy equivalent to a point:

Proof. Without loss of generality, we assume that $X \subset \mathbb{E}^n$. We first show the existence of a deformation retract, $G : X \times I \hookrightarrow \{x\}$, where $x \in X$ is fixed:

Let G be defined as $G(y, t) = (1 - t)y + tx$, for all $y \in X$. Then, clearly $G(y, 0) = y$, and $G(y, 1) = x \in \{x\}$ for all $y \in X$. So, G is deformation retract. By the above, X and A are of the same homotopy type: null-homotopic. □

[6] Armstrong does not do a good job of defining a deformation retract. We use the following definition. "Strong" deformation retract: A is a 'strong' deformation retract of X iff there exists a map $D : X \times I \to X$ such that $D(a, t) = a$ for every $a \in A$, $D(x, 0) = x$ and $D(x, 1) \in A$ for all $x \in X$. See this post on SE.

(3) We show that (X, τ) is contractible[7] if, and only if, every map $f : X \to Y$, (Y, τ_Y) a topological space, is null-homotopic:

Proof.

($\implies$): Suppose that (X, τ) is contractible to $x_0 \in X$. Let F be the contraction, and $f : X \to Y$ any map. The claim is that $f \circ F : X \times I \to Y$ is a homotopy between f, and $f(x_0)$.

Indeed; We note that the composition of two continuous functions is continuous and

$$(f \circ F)(x, 0) = f(F(x, 0)) = f(x)$$

As well as,

$$(f \circ F)(x, 1) = f(F(x, 1)) = f(x_0)$$

($\impliedby$): Suppose that every map $f : X \to Y$ is null-homotopic. Then, in particular, $i : X \to X$ is null-homotopic. Thus, there exists a map $F : X \times I \to X$, such that $F(x, 0) = x = i(x)$, and $F(x, 1) = x_0 = i(x_0)$.

□

(4) Let (X, τ) be a topological space. We show that the cone on X, CX is contractible. We consider the "cone tip" as $X \times \{0\}$:

Proof. Consider the function $\bar{H} : (X \times I) \times I \to X \times I$, given by $\bar{H}((x, v), s) = (x, v(1 - s))$. By the product topology, it follows that $\bar{H}$ is continuous. [8] Let $\pi : X \times I \to CX = {}^{X \times I}/_{X \times \{0\}}$ be the canonical map. The claim is that $\pi \circ \bar{H} : (X \times I) \times I \to CX$ is the desired homotopy.

Direct computation shows that, for all $x \in X$, $v \in V$,

$$\pi(\bar{H}((x, v), 0)) = \pi((x, v)) = (x, v),$$

and

$$\pi(\bar{H}((x, v), 1)) = \pi((x, 0)) = (x, 0)$$

Thus, $H \equiv \pi \circ \bar{H}$ is the desired homotopy. □

[7] A space (X, τ) is contractible (to a point $x_0 \in X$) provided that there exists a map $F : X \times I \to X$ such that $F(x, 0) = x$ and $F(x, 1) = x_0$.

[8] For every $t \in I$, we can associate $\pi(\bar{H}((x, v), t))$ with $(c, v(1 - t), t)$, showing that the product topology of $CX \times I$ agrees with that of $(X \times I) \times I$.

2 Introduction

2.1 Topological Invariants

1. We prove that $v(T) - e(T) = 1$ for any tree T:

   Proof. We proceed by induction on the number of edges in T:

   - Basis: If $e(T) = 1$, then $v(T) = 2$, and so, $v(T) - e(T) = 1$.
   - Hypothesis: Suppose that $v(T) - e(T) = 1$ for any tree T, such that $e(T) = n$, for some $n \in \mathbb{N}$.
   - Step: Suppose that T is a tree, such that $e(T) = n + 1$. Now, removing any edge will disconnect the tree, since by definition, T contains no loops. Say that we remove and edge $e_0 \in T$, breaking up T into T_1, T_2. Now, clearly, $v(T) = v(T_1) + v(T_1)$, and $e(T) = n + 1 = e(T_1) + e(T_2) + 1$. Thus, by the inductive hypothesis,

   $$\begin{aligned} v(T) - e(T) &= v(T_1) + v(T_2) - e(T_1) - e(T_2) - 1 \\ &= 1 + 1 - 1 \\ &= 1 \end{aligned}$$

   □

2. We show that inside any graph we can always find a tree which contains all the vertexes:

   Proof. Let G be a finite graph. If G is already a tree, we are done. So, suppose that G is not a tree. Then, by the comments on pg. 3, G contains a (without loss of generality, minimal) loop, L. Now, we remove some edge $e_L \in L$. Now, as L is a loop, e_L does not disconnect L, and so $L - e_L$ is a tree. We continue in this way, as long as the new graph formed by removing an edge is not a tree.

   To conclude, we note that this process must stop at some loop, because $e(G) < \infty$; further, we have not removed any vertexes. □

3. We prove that $v(\Gamma) - e(\Gamma) \leqslant 1$ for any graph Γ, with equality, precisely when Γ is a tree:

   Proof. We have previously shown the equality condition. Using the above proof, we select a subtree, with the same vertexes as Γ. Call this tree Γ'. Then, $v(\Gamma') - e(\Gamma') = 1$. As Γ' was created by removing edges, and not vertexes, we have $v(\Gamma) = v(\Gamma')$, and $e(\Gamma) \leqslant e(\Gamma')$. So,

   $$e(\Gamma) - e(\Gamma) \leqslant v(\Gamma') - e(\Gamma') = 1$$

   □

4. This is left to the reader as an exercise.

5. This is left to the reader as an exercise.

6. Let P be a regular polyhedron in which each face has p edges and for which q faces meet at each vertex. We use Euler's formula to prove that,

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{2} + \frac{1}{e}$$

   Proof. We count the number of faces of P as follows: Since each face of P has p edges, $pf = 2e$. We count the number of vertexes as follows: Since each vertex has q faces which meet on it, $qv = 2e$. In all, $f = {}^{2e}\!/_{p}$, and $v = {}^{2e}\!/_{q}$. Assuming that Euler's formula holds for P, we have

$$v - e + f = \frac{2e}{q} - e + \frac{2e}{p} = 2$$

   And the result follows. □

7. We deduce that there are only 5 regular (convex) polyhedra.

   Proof. If P is a polyhedra, it must satisfy the above criterion.Further, we assume that $p \geqslant 3$, since if not, we cannot construct a polygon. We use Shlüt notation. By checking routinley, for values up to $p, q \leqslant 5$ we have the following set of polyhedra:

$$\{\{3,3\},\{3,5\},\{5,3\},\{3,4\},\{4,3\}\}$$

   Now, if $p \geqslant 6$, then we must have

$$\frac{1}{6} + \frac{1}{q} = \frac{1}{2} + \frac{1}{e}$$

   Further, as $0 \leqslant {}^{1}\!/_{q} \leqslant {}^{1}\!/_{5}$, we have

$$\begin{aligned} \frac{1}{6} + \frac{1}{q} &= \frac{q+6}{6q} \\ &< \frac{12}{6q} = \frac{2}{q} \\ &< \frac{1}{2} + \frac{1}{e} \end{aligned}$$

   A contradiction. □

8. This is left to the reader as an exercise.

9. This is left to the reader as an exercise.

10. We find a homeomorphism from the real xline to the open interval $(0, 1)$, and show that any two open intervals are homeomorphic:

    Proof. We show that the real line, $\mathbb{R}$ is homeomorphic to $(-\pi/2, \pi/2)$.This amount to showing that $arctan : \mathbb{R} \to (-\pi/2, \pi/2)$ is continuous, 1-1, onto and that $tan \equiv arctan^{-1}$ is continuous. Thus, $\mathbb{R} \simeq (-\pi/2, \pi/2)$.

    To conclude the proof, we show that any open subset, $(a, b) \subset \mathbb{R}$ is homeomorphic to $(-\pi/2, \pi/2)$, $a \neq b$:

    Indeed, under the mapping $f(x) = (b - a)(\frac{x}{\pi} + \frac{1}{2}) + a$, we have

    $$(-\pi/2, \pi/2) \to (a, b)$$

    $\square$

11. This is left to the reader as an exercise.

12. We find a homeomorphism from S^2 to $\mathbb{E}^2$:

    Proof. $\square$

13. Let $x, y \in S^2$. We find a homeomorphism from S^2 to S^2 which takes x to y. We do the same for the plane, and torus:

    Proof. $\square$

14. This is left to the reader as an exercise.

15. This is left to the reader as an exercise.

16. This is left to the reader as an exercise.

17. Define $f : [0, 1) \to C$, by $f(x) = e^{2\pi ix}$. We prove that f is a continuous bijection. In addition, we find a point $x \in [0, 1)$ and a neighbourhood N, of x in $[0, 1)$, such that $f[N]$ is not a neighbourhood of $f(x)$ in C; consequently, this means f is not a homeomorphism.

    Proof. $\square$

18. This is left to the reader as an exercise.

19. This is left to the reader as an exercise.

20. We prove that the radial projection of the tetrahedron to the sphere is a homeomorphism:

    Proof. $\square$

21. Let $\partial D = S^1 \subset \mathbb{C}$. Given two points $x, y \in \operatorname{Int} D$, we find a homeomorphism from D to D which interchanges x and y and leaves all the points of C fixed:

    Proof. □

22. See accompanying solution.

23. This is left to the reader as an exercise.

24. This is left to the reader as an exercise.

25. This is left to the reader as an exercise.

26. This is left to the reader as an exercise.

27. This is left to the reader as an exercise.

3 Continuity

4 Homotopy Type

Throughout this section, $\simeq$ will typically denote homotopy equivalence.

(1) If $X \simeq Y$ and $X' \simeq Y'$, we show that $X \times X' \simeq Y \times Y'$:

Proof. We are given maps $f : X \to Y$, $g : Y \to X$, $f' : X' \to Y'$ and $g' : Y' \to X'$, such that $g \circ f \simeq_{\gamma'} 1_X$, $f \circ g \simeq_{\gamma} 1_Y$, $g' \circ f' \simeq_{\delta'} 1_{X'}$ and $f' \circ g' \simeq_{\delta} 1_{Y'}$.

From the component-wise definition of the product topology, $F : X \times X' \to Y \times Y'$, defined by $F(x, x') = (f(x), f'(x'))$, and $G : Y \times Y' \to X \times X'$, defined by $G(y, y') = (g(y), g'(y'))$ are continuous. Further, $F \circ G \simeq 1_{Y \times Y'}$, and $G \circ F \simeq 1_{X \times X'}$, via the canonical maps, component-wise defined: $\mathcal{F} : (X \times X') \times I \to X \times X'$, $\mathcal{F} \equiv (\gamma', \delta')$, and $\mathcal{G} : (Y \times Y') \times I \to Y \times Y'$, $\mathcal{G} \equiv (\gamma, \delta)$, respectively.

Thus, $X \times X' \simeq Y \times Y'$. $\square$

(2) We show that the cone, CX, is contractible for any space (X, τ_X):[9]

Proof. By definition, we have $CX = {}^{X \times I}/_{X \times \{1\}}$. Define $H : CX \times I \to CX$ by

$$H((x, t), s) = (x, t(1 - s))$$

Then, we have

$$H((x, t), 0) = (x, t) = 1_{CX}$$
$$H((x, t), 1) = (x, 0)$$

Now, if $\mathcal{O}_{CX} \in \tau_{CX}$, then $H^{-1}(\mathcal{O}_{CX}) = (\mathcal{O}_{CX}, I) \in CX \times I$. Thus, H is continuous. This proves the result. $\square$

(3) We show that the punctured torus deformation retracts onto the one-point union of two circles.

Proof. We consider the torus as the identification space of a square, X, and assume, without loss of generality, that the point $(0, 0)$ is removed.

We show that $F : T \times I \to T$ defined by

$$F(x, t) = (1 - t)x + \frac{x}{||x||}$$

is a deformation retract onto the one point union of two circles:

[9] We proved another proof for this. The first of which is in the extra lemmas section.

As

$$F(x,0) = x$$
$$F(x,1) = \frac{x}{||x||}$$

F is a deformation retract onto ∂X. The function $f : \partial X \to S^1_1 \bigvee S^1_2$, given by

$$f(x) = \begin{cases} g(x) & x \in (-1, \pm 1)t + (1, \pm 1)(1-t) \\ s(x) & x \in (\pm 1, 1)t + (\pm, -1)(1-t) \end{cases}$$

for all $t \in I$, where g, s are the guaranteed homeomorphisms from $[a, b]$ to S^1, is a homeomorphism by the Gluing Lemma. $\square$

(4) For each of the following cases, we choose as base point in C and describe the generators for the fundamental groups of C and S. Further, we write down the homomorphism, in terms of these generators, the fundamental groups induced by the inclusion of C in S.

Consider the following examples of a circle C embedded in a surface S:

(a) $S =$ Möbius Strip and $C = \partial S$:

(b) $S = S^1 \times S^1 = T^1$ and $C = \{(x.y) \in T^1 : x = y\}$:

(c) $S = S^1 \times I$ and $C = S^1 \times 1$:

(5) Suppose that $f, g : S^1 \to X$ are homotopic maps. We prove that the spaces formed from X by attaching a disc, using f and using g are homotopy equivalent; in other words, we prove that $X \cup_f D \simeq X \cup_g D$:

Proof. We aim to find maps r, s', such that $r : X \cup_f D \to X \cup_g D$, $s : X \cup_g D \to X \cup_f D$, $s \circ r \simeq 1_{X \cup_f D}$ and $r \circ s \simeq 1_{X \cup_g D}$. We note that

$$X \sqcup D/_{f(A) \sim A} = X \cup_f D, \quad X \sqcup D/_{g(A) \sim A} = X \cup_g D$$

As $f \simeq_F g$, F is a map such that $F : S^1 \times I \to X$, and $F(x,0) = f(x)$, as well as, $F(x,1) = g(x)$. Using F, we construct the function $r : X \cup_f D \to X \cup_g D$ as follows:

$$r(x) = \begin{cases} \{x, F(x,1)\} & x \in \{x, f(x)\} \\ x & \text{otherwise} \end{cases}$$

Similarly, define $s : X \cup_g D \to X \cup_f D$ by

$$r(x) = \begin{cases} \{x, F(x,0)\} & x \in \{x, f(x)\} \\ x & \text{otherwise} \end{cases}$$

$\square$

(6) We use the previous problem, and the third example of homotopy given in section 5.1, to show that the 'dunce hat' has the homotopy type of a disc, and is therefore contractible:

(7) We show that the 'house with two rooms' is contractible:

(8) We give a detailed proof to show that the cylinder and the Möbius strip have the homotopy type of the circle:

(9) Let X be the comb space. We prove that the identity map of X is not homotopic rel$\{p\}$, to the constant map, p.

(10) FTA: We prove the fundamental theorem of algebra:

5 Brouwer Fixed-Point Theorem

Throughout, we say a topological space (X, τ_X) has the fixed-point property if every continuous function from X to itself has a fixed point.

(1) We determine which of the following have the fixed point property:

(a) The 2-Sphere: S^2 does not exhibit the fixed point property.

Proof. We have previously shown that the map $f : S^n \to S^n$, given by $f(x) = -x$ is a homeomorphism. $\square$

(b) The Torus: $T^1 = S^1 \times S^1$ does not exhibit the fixed point property.

Proof. As $T^1 = S^1 \times S^1$, and the antipodal map, f, on S^1 is a homeomorphism, we have that $F : T^1 \to T^1$, defined by $F((x, y)) = (f(x), f(y))$, is continuous and doesn't exhibit the fixed point property. $\square$

(c) The interior of the unit disc: $\mathrm{Int}D^1$ does not exhibit the fixed point property:

Proof. Note that the interior of the unit disc is homeomorphic to the euclidean plane by a homeomorphism $h : \mathrm{Int}D^1 \to \mathbb{E}^2$. We define a function g from the euclidean plane to the euclidean plane by $g((x, y)) = (x + 1, y)$. Then, $h^{-1} \circ g \circ h$ is a continuous function from the disc to itself that has no fixed point. $\square$

(d) The one point union of two circles: $X \vee Y = {}^{X \amalg Y}/_{\{p\}}$ does not exhibit the fixed point property:

Proof. Define a function f as follows: If $x \in X$ then we map e^{ix} to $e^{i(x+\pi)}$. Also if $y \in Y$, we map y to $f(p) = e^{i\pi/2}$. This function f, as defined on the one point union, leaves no points fixed. $\square$

(2) Suppose X and Y are of the same homotopy type and X has the fixed-point property. We prove that Y does not necessarily have the fixed point property:

Proof. Let Y be the subspace $(0, 1) \subset \mathbb{E}^n$, and let $X = \{1/2\}$. Note that Y is homotopic to X by the straight line homotopy, and every map from X to itself has a fixed point. Yet, the function $f : Y \to Y$, defined by $f(y) = y^2$ has no fixed point. $\square$

(3) Suppose that X retracts onto the subspace $A \subset X$, and that A has the fixed point property. We show that X may not exhibit the fixed point property:

Proof. Take X and Y, as above. The straight-line homotopy proves the assertion. $\square$

(4) We show that if X retracts onto the subspace A, and X has the fixed-point property, then A also has it:

Proof. Let $f : A \to X$ be a continuous function. Since X retracts onto A, there exists a map $g : X \to A$ such that $g \restriction A \equiv 1_A$. Then, $f \circ g$ is a continuous function from X to X, and so has a fixed point. Hence, there exists an $a \in A$ such that $f(g(a)) = f(a) = a$. This completes the proof. $\square$

(5) We deduce that the fixed-point property holds for the 'house with two rooms', X:

Proof. As was previously show, X is contractible to some x_0. Thus, there exists some map $F : X \times I \to X$ such that $F(x, 0) = x$, and $F(x, 1) = x_0$. Thus, by extra lemma [10], every map $f : X \to Y$ is null-homotopic. In particular, this includes that maps $g : X \to X$. Thus, X has the fixed point property[11].

To use the previous problems hints, we could think about starting with the unit cylinder, and pushing in the areas from the top and bottom. However, this method is not rigorous. $\square$

(6) Let f be a fixed-point-free map from a compact metric space (X, d) to itself. We prove there is a positive number ϵ such that $d(x, f(x)) > \epsilon$, $\forall x \in X$:

Proof. We show the contrapositive; Suppose that for all $\epsilon > 0$ there exists an element $x \in X$ such that $d(x, f(x)) \leqslant \epsilon$. Pick $x_1 \in X$ such that $d(x_1, f(x)) \leqslant \frac{1}{2} < 1$.

It follows that there exists a set, $\{x_1, ..., x_n\} \subset X$ such that $d(x_i, f(x_i)) < d(x_k, f(x_k))$, where $i < k$ and $d(x_k, f(x_k)) < \dfrac{1}{k}$, for all $1 \leqslant k \leqslant n$.

Now, let

$$\epsilon_0 = \frac{1}{2} \min \left\{ d(x_1, f(x_1)), ..., d(x_n, f(x_n), \frac{1}{n+1} \right\}$$

Then there exists an element $x \in X$ such that $d(x, f(x)) \leqslant \epsilon_0$. Pick $x \in X$ that satisfies this property, and call this x_{n+1}. Note here that $d(x_n, f(x_n)) < d(x_{n+1}, f(x_{n+1}))$ by construction. So that for all $i < n+1$, $d(x_i, f(x_i)) \leqslant d(x_n, f(x_n)) < d(x_{n+1}, f(x_{n+1}))$. Thus by induction, we have created an infinite sequence $\{x_n\}_{n=1}^{\infty}$ of distinct points such that $d(x_k, f(x_k)) < 1/k$ for each positive integer k. And, $\{d(x_n, f(x_n))\}_n$ is a monotone decreasing sequence.

[10] include this

[11] If this is not totally clear, see extra lemma.

Since X is compact, every infinite subset has a limit point. Therefore $\{x_n\}_{n=1}^{\infty}$ has a limit point $x \in X$.

Let $\epsilon > 0$ be given. Since f is continuous at x, then there exists a $\delta > 0$ such that $d(x, a) < \delta \implies d(f(x), f(a) < \epsilon/3$ for all $a \in X$. By the Archimedian Property, we can find a positive integer k, such that

$$\frac{1}{k} < \frac{\epsilon}{3}$$

Set $r = \min\{\frac{\epsilon}{3}, \delta\}$. Then, since x is a limit point of $\{x_n\}_{n=1}^{\infty}$, there are infinitely many points of the sequence, such that $d(x_N, x) < r$. Thus, there exists a point x_N such that $N > k$ and $d(x_N, x) < r$. Since $d(x_N, x) < \delta$, then $d(f(x_N), f(x)) < \epsilon/3$. Since $N > k$,

$$d(x_N, f(x_N)) < d(x_k, f(x_k)) < 1/k < \frac{\epsilon}{3}$$

Therefore

$$d(x, f(x)) \leqslant d(x, x_N) + d(x_N, f(x_N)) + d(f(x_N), f(x)) < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon$$

Thus, for all $\epsilon > 0$, we have that $d(x, f(x)) < \epsilon$. Thus, $d(x, f(x)) \leqslant 0$, and hence $d(x, f(x)) = 0$. Consequently, $f(x) = x$, and so f is not a fixed-point-free map. $\square$

(7) We show that the unit ball, B^n, in $\mathbb{E}^n$, with $(1, 0, ..., 0)$ removed does not exhibit the fixed point property:

Proof. Let $p = (1, ..., 0)$. Consider the function $f : B^n \backslash \{p\} \to B^n \backslash \{p\}$, defined by $f(x) = \frac{x+p}{2}$. We show that f is continuous:

Let $\mathcal{O}$ be an open set in $B^n\{p\}$. Let $x \in f^{-1}(\mathcal{O})$. Then $f(x) \in \mathcal{O}$, and since $\mathcal{O}$ is open there exists an $r > 0$ such that $B_r(f(x)) \subset \mathcal{O}$. Let $y \in B_{2r}(x)$. Then $|x - y| < 2r$. Then,

$$|f(x) - f(y)| = |\frac{x+p}{2} - \frac{y+p}{2}| = \frac{1}{2}|x - y| < r$$

Thus, $y \in B_r(f(x))$. Hence, $y \in f^{-1}(\mathcal{O})$. Thus, $B_{2r}(x) \subset f^{-1}(\mathcal{O})$. Therefore, $f^{-1}(\mathcal{O})$ is open and f is continuous.

Further, f does not have the fixed-point property; If $f(x) = x = (x_1, ..., x_n)$, then we have $2x = x + p$. Implying that $x_i = 2x_i$, $2 \leqslant i \leqslant n$. Thus, $x_i = 0$. However, then the only solution to $x_1 + 1 = 2x_1$ is $x_1 = 1$. This is a contradiction since we must have $x = p \notin B^n \backslash \{p\}$. $\square$

(8) We show that the one-point union of X and Y, $X \vee Y$, has the fixed-point property if, and only if, both X and Y have it:

Proof.

($\Longrightarrow$): Suppose that the one point union $X \vee Y$ has the fixed-point-property. Let $f : X \to X$ be a map, and let p be the point glued together in the one point union. Then, define $g : X \vee Y \to X \vee Y$, as $g(x) = f(x)$, if $x \in X$, and $g(x) = f(p)$, if $x \in Y$. By the gluing lemma, g is a continuous map and so, by hypothesis, has a fixed point x_0. Note that g is a map into X. Thus the fixed point must be in X. Hence, by construction, $x_0 = g(x_0) = f(x_0)$, so that f has a fixed point. Similarly, any continuous function from Y to itself has a fixed point.

($\Longleftarrow$): Suppose X and Y have the fixed point property and let $f : X \vee Y \to X \vee Y$ be a map. Then, suppose $f(p) \in X$, and define a map $g : X \to X$ such that $g(x) = f(x)$, if $f(x) \in X$, and $g(x) = p$, if $f(x) \notin X$. Then, since g is continuous, it has a fixed point. By construction, the fixed point must be one $x = g(x) = f(x)$. Thus, f has a fixed point. Next, suppose $f(p) \in Y$, and define a map $g : Y \to Y$ such that $g(y) = f(y)$, if $f(y) \in Y$, and $g(y) = p$, if $f(y) \notin Y$. Then, since g is continuous, it has a fixed point. By construction, the fixed point must be one $y = g(y) = f(y)$. Consequently, f has a fixed point.

□

(9) How does changing 'continuous function' to 'homeomorphism' in the definition of the fixed-point property affect problem 33, 37?

Proof. We first examine problem 33:

S^2: This topological space would not exhibit the fixed point property. We know that the antipodal map is a homeomorphism which leaves no points fixed.

$S^1 \times S^1$: Likewise, the antipodal map on each component, S^1, provides another counter example.

$\operatorname{Int} D$: Neither does this space have the homeomorphic fixed point property. To see this, we note that $\operatorname{Int} D \simeq \mathbb{E}$, and the map $g : \mathbb{E} \to \mathbb{E}$, given by $g(x) = x + 1$ is a homeomorphism which leaves no points fixed.

$X \bigvee Y$: First, consider the map $E_1 : X \bigvee Y \to X \bigvee Y$, given by $E_1(x) = x$, if $x \in X - \{p\}$. And, $E_2(y) = e^{yi\pi/2}$. Then, E_1 is a homeomorphism, which leaves X fixed, except for p. Similarly, define E_2 for Y. Then, it follows that $E_2 \circ E_1 : X \bigvee Y \to X \bigvee Y$ is a homeomorphism that leaves no points of $X \bigvee Y$ fixed. Thus, $X \bigvee Y$ does not exhibit the fixed point property[12].

[12]We make use of the previous proof; that is $X \bigvee Y$ has the fixed point propert iff X and Y both have it.

Now, we examine problem 37:

This shape does not exhibit the homeomorphic fixed point property. Consider $\partial B^n = S^{n-1}$, and let $p = (1, 0, ..., 0)$. Then, $B^n - \{p\} = \operatorname{Int} B^n \cup S^{n-1} - \{p\}$. Furthermore, without loss of generality, $S^{n-1} - \{p\} \simeq \{-p\}$.[13] It follows that $\operatorname{Int} B^n \cup S^{n-1} - \{p\} \simeq \operatorname{Int} B^n \cup \{-p\}$. To conclude, we exhibit a homeomorphism on $\operatorname{Int} B^n \cup \{-p\}$ which does not have a fixed point:

Consider $f : \operatorname{Int} B^n \cup \{-p\} \to \operatorname{Int} B^n \cup \{-p\}$ defined by

$$f(x) = \begin{cases} -x & x \notin \{-p, 0\} \\ 0 & x = -p \\ -p & x = 0 \end{cases}$$

The fact that f is 1-1 and onto is clear by construction. Further, f is continuous, by the Gluing Lemma[14]. It follows that f is a homeomorphism, as $f^{-1} \equiv -f$. However, f clearly leaves no points fixed. $\square$

[13] See previous exercises.

[14] Armstrong, pg. 69.

6 Separation of the Plane