***Abstract*—**

1 Problem

Find the foot of the perpendicular for a point on the line of intersection of planes $9x^2 - 4y^2 + z^2 - 6xz - 4y - 1 = 0$ on to the plane containing the point $(-1, -4, 3)$.

2 Solution

Given the equation of two intersecting planes is

$$9x^2 - 4y^2 + z^2 - 6xz - 4y - 1 = 0 \tag{2.0.1}$$

1) **Second order equation in terms of matrices:** The general equation of a second order algebraic surface is given by

$$ax^2 + by^2 + cz^2 + 2dxy + 2exz$$
$$+ 2fyz + 2lx + 2my + 2nz + q = 0. \tag{2.0.2}$$

This equation can be written as:-

$$\mathbf{x}^T\mathbf{V}\mathbf{x} + 2\mathbf{u}^T\mathbf{x} + f = 0 \tag{2.0.3}$$

$$\mathbf{V} = \begin{pmatrix} a & d & e \\ d & b & f \\ e & f & c \end{pmatrix} \quad \mathbf{u} = \begin{pmatrix} l \\ m \\ n \end{pmatrix} \tag{2.0.4}$$

Substituting values from the given equation (2.0.1), we get,

$$\mathbf{x}^T \begin{pmatrix} 9 & 0 & -6 \\ 0 & -4 & 0 \\ -6 & 0 & 1 \end{pmatrix} \mathbf{x} + 2 \begin{pmatrix} 0 \\ -2 \\ 0 \end{pmatrix}^T \mathbf{x} - 1 = 0 \tag{2.0.5}$$

$$\mathbf{V} = \begin{pmatrix} 9 & 0 & -6 \\ 0 & -4 & 0 \\ -6 & 0 & 1 \end{pmatrix} \quad \mathbf{u} = \begin{pmatrix} 0 \\ -2 \\ 0 \end{pmatrix} \quad f = -1 \tag{2.0.6}$$

a) The eigen values of matrix $\mathbf{V}$ can be calculated as:-

$$\left|\mathbf{V} - \lambda\mathbf{I}\right| = 0 \tag{2.0.7}$$

$$\begin{vmatrix} 9-\lambda & 0 & -6 \\ 0 & -4-\lambda & 0 \\ -6 & 0 & 1-\lambda \end{vmatrix} = 0 \tag{2.0.8}$$

$$\implies -\lambda^3 + 6\lambda^2 + 67\lambda + 108 = 0 \tag{2.0.9}$$

$$(-\lambda - 4).(\lambda^2 - 10\lambda - 27) = 0 \tag{2.0.10}$$

From above, we get,

$$\lambda_1 = -4 \tag{2.0.11}$$

$$\lambda_2 = -2\sqrt{13} + 5 \tag{2.0.12}$$

$$\lambda_3 = 2\sqrt{13} + 5 \tag{2.0.13}$$

The corresponding eigen vectors for these eigen values are:-

$$\text{for } \lambda_1 = -4, \mathbf{v_1} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \tag{2.0.14}$$

$$\text{for } \lambda_2 = -2\sqrt{13} + 5, \mathbf{v_2} = \begin{pmatrix} \frac{\sqrt{13}-2}{3} \\ 0 \\ 1 \end{pmatrix} \tag{2.0.15}$$

$$\text{for } \lambda_3 = 2\sqrt{13} + 5, \mathbf{v_3} = \begin{pmatrix} \frac{-\sqrt{13}-2}{3} \\ 0 \\ 1 \end{pmatrix} \tag{2.0.16}$$

Eigen vectors matrix is

$$\begin{pmatrix} 0 & \frac{\sqrt{13}-2}{3} & -\frac{\sqrt{13}-2}{3} \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} \tag{2.0.17}$$

The normalizing these values, we obtain

$$\begin{pmatrix} 0 & \frac{\sqrt{13}-2}{\sqrt{26-4\sqrt{13}}} & \frac{-\sqrt{13}-2}{\sqrt{26+4\sqrt{13}}} \\ 1 & 0 & 0 \\ 0 & \frac{3}{\sqrt{26-4\sqrt{13}}} & \frac{3}{\sqrt{26+4\sqrt{13}}} \end{pmatrix} \tag{2.0.18}$$

b) Affine transformation:
We can obtain the affine transformation for equation at (2.0.3) by taking the value of $\mathbf{x}$ as:

$$\mathbf{x} = \mathbf{P}\mathbf{y} + c \tag{2.0.19}$$

such that

$$\mathbf{P}^T\mathbf{V}\mathbf{P} = \mathbf{D} \tag{2.0.20}$$

$$\mathbf{D} = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} \tag{2.0.21}$$

and $\mathbf{P}$ is the transformation matrix and can be given by the normalized eigen vectors of matrix $\mathbf{V}$. Therefore,

$$\mathbf{P} = \begin{pmatrix} 0 & \frac{\sqrt{13}-2}{\sqrt{26-4\sqrt{13}}} & \frac{-\sqrt{13}-2}{\sqrt{26+4\sqrt{13}}} \\ 1 & 0 & 0 \\ 0 & \frac{3}{\sqrt{26-4\sqrt{13}}} & \frac{3}{\sqrt{26+4\sqrt{13}}} \end{pmatrix} \tag{2.0.22}$$

Also, $\mathbf{P}^T = \mathbf{P}^{-1}$

$$\mathbf{P}^T\mathbf{V}\mathbf{P} = \begin{pmatrix} 0 & 1 & 0 \\ \frac{\sqrt{13}-2}{\sqrt{26-4\sqrt{13}}} & 0 & \frac{3}{\sqrt{26-4\sqrt{13}}} \\ \frac{-\sqrt{13}-2}{\sqrt{26+4\sqrt{13}}} & 0 & \frac{3}{\sqrt{26+4\sqrt{13}}} \end{pmatrix}$$
$$\begin{pmatrix} 9 & 0 & -6 \\ 0 & -4 & 0 \\ -6 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & \frac{\sqrt{13}-2}{\sqrt{26-4\sqrt{13}}} & \frac{-\sqrt{13}-2}{\sqrt{26+4\sqrt{13}}} \\ 1 & 0 & 0 \\ 0 & \frac{3}{\sqrt{26-4\sqrt{13}}} & \frac{3}{\sqrt{26+4\sqrt{13}}} \end{pmatrix} \tag{2.0.23}$$

$$= \begin{pmatrix} -4 & 0 & 0 \\ 0 & -2\sqrt{13}+5 & 0 \\ 0 & 0 & 2\sqrt{13}+5 \end{pmatrix} = \mathbf{D} \tag{2.0.24}$$

(2.0.3) can be expressed as:

$$\mathbf{y}^T\mathbf{D}\mathbf{y} = \mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} - f \quad \therefore |\mathbf{V}| \neq 0 \tag{2.0.25}$$

$$\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u}$$
$$= \begin{pmatrix} 0 & -2 & 0 \end{pmatrix} \begin{pmatrix} \frac{-1}{27} & 0 & \frac{-2}{9} \\ 0 & \frac{-1}{4} & 0 \\ \frac{-2}{9} & 0 & \frac{-1}{3} \end{pmatrix} \begin{pmatrix} 0 \\ -2 \\ 0 \end{pmatrix}$$
$$= -1 \tag{2.0.26}$$

Therefore, we obtain

$$\mathbf{y}^T \begin{pmatrix} -4 & 0 & 0 \\ 0 & -2\sqrt{13}+5 & 0 \\ 0 & 0 & 2\sqrt{13}+5 \end{pmatrix} \mathbf{y} = 0 \tag{2.0.27}$$

Expressing in terms of individual plane equations, we get

$$(3x - 2y - z - 1)(3x + 2y - z + 1) = 0 \tag{2.0.28}$$