UM201: Homework 1

Sirjan Hansda, SR No- 22188

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Problem 1. A fair coin is tossed n times. Show that the probability that there is exactly one run of heads and two runs of tails is $\frac{(n-1)(n-2)}{2^{n+1}}$. [Note: A run is a consecutive sequence of tosses givingthe same result. For example, 001101000 has three runs of tails and two runs of heads]

Answer. For calculating this probability, we take the probability space as:

$\Omega = \{(x_1, x_2, x_3 \ldots, x_n) | x_i \in \{0, 1\}, \forall\, i\}$

And, we take the event **A** as :

$A = B_1 \cup B_2 \cup B_3 \cup B_4 \ldots B_{n-2}$, where $B_i = \{(x_1, x_2, x_3 \ldots, x_n) | x_j = 1\, \forall j; m \leq j \leq k\}$ and , $2 \leq m$ and $k \leq n-1$, and $m - k + 1 = i$

Clearly, $\#\mathbf{\Omega} = \mathbf{2^n}$. This is because, have n coins and each can have 2 possible outcomes.
Now, for B_i , m can take any values from 2 to $n-i$. Thus, $\#B_i = n - i - 1$.
Thus, $\#B_1 = n-2$; $\#B_2 = n-3$; $\#B_3 = n-4$; $\ldots$; $\#B_{n-2} = 1$

Therefore, B_i's being disjoint sets, $\#A = \sum_{i=1}^{n-2} i = \frac{(n-2)(n-1)}{2}$
Thus,

$$P(A) = \frac{\#A}{\#\Omega} = \frac{(n-2)(n-1)}{2^{n+1}}$$

Hence Proved