PhD Studies

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Part I

Topics of Algebra

Chapter 1

Category Theory

References [1, 13].
In general, categories and functors will be denoted with calligraphic letters (except for the classical categories Gr, Ab, Top, Sets, ...) and objects with capital letters.

1.1 Some facts

Example 1. *1. On a topological space, the category of open sets with inclusions as morphisms. The opposite of this category, denoted by $\mathcal{U}$, is essential in sheaf theory.*

2. *If $\mathcal{A}$ and $\mathcal{B}$ are preordered sets, then the functors between them are the monotone maps.*

3. *$i : \mathbb{Z} \hookrightarrow \mathbb{Q}$ is a monomorphism and epimorphism, but not an isomorphism.*

4. *A **grupoid** is a category with whose morphisms are isomorphisms. In particular, a group can be seen as a grupoid with one element.*

5. *For small categories $\mathcal{A}$ and $\mathcal{B}$ the **functor category** $[\mathcal{A}, \mathcal{B}]$ has as objects all functors from $\mathcal{A}$ to $\mathcal{B}$, as morphisms from F to G all natural transformations from F to G, as identities the identity natural transformations, and as composition the (horizontal) composition of natural transformations.*

Let $F : \mathcal{A} \to \mathcal{B}$ be a functor.

1. F is **faithful** provided that all the **hom-set restrictions**
$$F : \hom_{\mathcal{A}}\left(A, A'\right) \to \hom_{\mathcal{B}}\left(FA, FA'\right)$$
are injective.

2. F is **full** if all hom-set restrictions are surjective.

3. F is an **embedding** if and it is faithful and injective on the class of objects.

4. F is **essentially surjective** if for every object B of $\mathcal{B}$, there is an object A of $\mathcal{A}$ such that FA is isomorphic to B.

5. If F is essentially surjective and fully faithful, it is called an **equivalence of categories**, and $\mathcal{A}$ and $\mathcal{B}$ are said to be **equivalent**.

Let $F, G : \mathcal{A} \to \mathcal{B}$ be functors. A **natural transformation** $\tau : F \to G$ is a function that assigns to each $\mathcal{A}$-object A a $\mathcal{B}$-morphism $\tau_A : FA \to GA$ in such a way that the following *natural* condition holds: for each A-morphism $A \xrightarrow{f} A'$, the following diagram commutes

$$\begin{tikzcd} FA \arrow[r, "\tau_A"] \arrow[d, "Ff"'] & GA \arrow[d, "Gf"] \\ FA' \arrow[r, "\tau_{A'}"'] & GA' \end{tikzcd}$$

A natural transformation $F \xrightarrow{\tau} G$ whose components τ_A are isomorphisms is called a **natural isomorphism** from F to G, and F and G are said to be **naturally isomorphic**, denoted by $F \cong G$.

Example 2. 1. *Consider the n-th singular homology group of a pair of topological spaces (X, A). The long exact sequence of the pair contains the group morphisms*

$$\delta : H_n(X, A) \to H_{n-1}(A).$$

This forms a natural transformation between $(X, A) \mapsto H_n(X, A)$ and $(X, A) \mapsto H_{n-1}(A)$, both being from the category of pairs of topological spaces to the category of abelian groups.

2. *The assignment of the Hurewicz homomorphism $\pi_n(X) \to H_n(X)$ to each topological space X is a natural transformation between the functors π_n and H_n.*

3. *If $B \xrightarrow{f} C$ is an $\mathcal{A}$-morphism, then $\hom_{\mathcal{A}}(C, -) \xrightarrow{\tau_f} \hom_{\mathcal{A}}(B, -)$, defined by $\tau_f(g) = g \circ f$, and $\hom_{\mathcal{A}}(-, B) \xrightarrow{\sigma_f} \hom_{\mathcal{A}}(-, C)$, defined by $\sigma_f(g) = f \circ g$, are natural transformations.*

4. *(Good definitions of extension) Let $F : Set \to Vec$ be a functor that assigns to each set X a vector space FX with basis X, and to each function $X \xrightarrow{f} Y$ the unique linear extension $FX \xrightarrow{Ff} FY$ of f. This actually is not a correct definition of a functor, since there are many different vector spaces with the same basis. However, the definition is "correct up to natural isomorphism". Whenever we choose, for each set X, a specific vector space FX with basis X, we do obtain a functor $F : Set \to Vec$ (since the above condition determines the action of F on functions uniquely). Furthermore, any two functors that are obtained in this way are naturally isomorphic.*

1.2 Limits and colimits

An object P in a category $\mathcal{C}$ is called **projective** if, for every epimorphism $f : M \to Q$ in $\mathcal{C}$ and every $p : P \to Q$, there is a $\xi \in \mathrm{Hom}(P, M)$ with $f \circ \xi = p$, called the **lift** of p to M.
Dually, an object I in a category $\mathcal{C}$ is called **injective** if for every monomorphism $f : U \to M$ in $\mathcal{C}$ and every $j : U \to I$, there is a $\zeta \in \mathrm{Hom}(M, I)$ with $\zeta \circ f = j$, called and **extension** of j to M.

$$\begin{tikzcd} & M \arrow[d, two heads, "f"] \\ P \arrow[ur, dashed, "\xi"] \arrow[r, "p"'] & Q \end{tikzcd} \qquad \begin{tikzcd} & M \arrow[dl, dashed, "\eta"'] \\ I & U \arrow[l, "j"] \arrow[u, hook, "f"'] \end{tikzcd}$$

Example 3. 1. *In* Sets, *every object is injective and projective.*

2. *In* $R -$ Mod *(left), a module is projective iff it is a direct summand of a free module. A module M is injective if and only if the functor* $\mathrm{Hom}_R(-, M)$ *is exact.*

Proposition 1. *1. A is projective if and only if $\mathrm{Hom}_{\mathcal{C}}(A, -) : \mathcal{C} \to$ Sets preserves epimorphisms.*

2. A is injective if and only if $\mathrm{Hom}_{\mathcal{C}}(-, A) : \mathcal{C}^o \to$ Sets sends monomorphisms to epimorphisms.

Let $A \underset{g}{\overset{f}{\rightrightarrows}} B$ be a pair of morphisms. A morphism $E \xrightarrow{e} A$ is called an **equalizer** of f and g provided that the following conditions hold: (1) $f \circ e = g \circ e$, (2) for any morphism $e' : E' \to A$ with $f \circ e' = g \circ e'$, there exists a unique morphism $\bar{e} : E' \to E$ such that $e' = e \circ \bar{e}$, i.e., such that the triangle

$$\begin{tikzcd} E' \arrow[d, "\bar{e}"'] \arrow[dr, "e'"] & & \\ E \arrow[r, "e"'] & A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \end{tikzcd}$$

commutes.

A **source** is a pair $(A, (f_i)_{i \in I})$ consisting of an object A and a family of morphisms $f_i : A \to A_i$ with domain A, indexed by some class I.
A source $\mathcal{P} = \left(P \xrightarrow{p_i} A_i\right)_I$ is called a **product** provided that for every source $\mathcal{S} = \left(A \xrightarrow{f_i} A_i\right)_I$ with the same codomain as $\mathcal{P}$ there exists a unique morphism $A \xrightarrow{f} P$ with $\mathcal{S} = \mathcal{P} \circ f$. A product with codomain $(A_i)_I$ is called a **product of the family** $(A_i)_I$.

A **diagram** in a category $\mathcal{A}$ is a functor $D : \mathbf{I} \to \mathcal{A}$, where $\mathbf{I}$ is called the **scheme** of the diagram. A diagram with a small (or finite) scheme is said to be **small** (or finite).
An A-source $\left(A \xrightarrow{f_i} D_i\right)_{i \in Ob(\mathrm{I})}$ is said to be **natural** for the diagram D provided that for each I-morphism $i \xrightarrow{d} j$, the triangle

$$\begin{tikzcd} A \arrow[d, "f_i"'] \arrow[dr, "f_j"] & \\ D_i \arrow[r, "Dd"'] & D_j \end{tikzcd}$$

commutes. Equivalently, natural sources can be regarded as natural transformations from constant functors $C : \mathbf{I} \to \mathcal{A}$ to the functor D.
A **limit** of a diagram D is a natural source $\left(L \xrightarrow{\ell_i} D_i\right)$ for D with the **universal property** that for each natural source $\left(A \xrightarrow{f_i} D_i\right)$ there exists a unique morphism $f : A \to L$ with $f_i = \ell_i \circ f$ for each $i \in Ob(\mathbf{I})$.
A poset $\mathbf{I}$ is **down-directed** if every pair of elements has a lower bound. Limits of diagrams with this king of scheme are called **projective** (or **inverse**) limits.

Proposition 2. *1. For A-morphisms $A \underset{g}{\overset{f}{\rightrightarrows}} B$, considered as a diagram D with scheme $\bullet \Rightarrow \bullet$, a source $(A \xleftarrow{e} C \xrightarrow{h} B)$ is natural provided that $g \circ e = h = f \circ e$.*

$C \xrightarrow{e} A$ is an equalizer of $A \underset{g}{\overset{f}{\rightrightarrows}} B$ if and only if the source $(A \xleftarrow{e} C \xrightarrow{f \circ e} B)$ is a limit of D.

Proposition 3 (Uniqueness). *If $\mathcal{L} = \left(L \xrightarrow{\ell_i} D_i\right)_{i \in Ob(\mathbf{I})}$ is a limit of $D : \mathbf{I} \to \mathcal{A}$, then*

1. *for each limit $\mathcal{K} = \left(K \xrightarrow{k_i} D_i\right)_{i \in Ob(\mathrm{I})}$ of D, there exist an isomorphism $K \xrightarrow{h} L$ with $\mathcal{K} = \mathcal{L} \circ h$,*

2. *for each isomorphism $A \xrightarrow{h} L$, the source $\mathcal{L} \circ h$ is a limit of D.*

Example 4 (Limits). *1. Let $(X_n)_{n\in\mathbb{N}_0}$ be a family of sets with $X_{n+1} \subset X_n$. Then, the limit of the system $\ldots \subset X_{n+1} \subset X_n \subset \ldots \subset X_1 \subset X_0$ is the intersection of the sets X_n.*

2. *Let p be a fixed prime. The inverse limit of the diagram is the ring of p-adic integers, $\mathbb{Z}_p$. Here, the maps p_i are the canonical projection maps. An explicit model of the limit is*

$$\left\{(x_1, x_2, x_3, \ldots) \in \prod_{n\geq 1} \mathbb{Z}/p^n\mathbb{Z} \mid p_i(x_i) = x_{i-1} \text{ for all } i \geq 2\right\}.$$

This carries a ring structure, where addition and multiplication are defined coordinatewise.

3. *Kernels in the category of abelian groups are limits of diagrams of the form $A \underset{f}{\overset{0}{\longrightarrow}} B$.*

4. *The presheaf F is a sheaf if for every $U \in \mathfrak{U}(X)$ and for every open covering $(U_i)_{i\in I}$ of U, the following diagram is an equalizer:*

$$F(U) \longrightarrow \prod_{i\in I} F(U_i) \Longrightarrow \prod_{i,j\in I} F(U_i \cap U_j).$$

Here, the first map is induced by the restriction maps res $U_U^{U_i}$, and the second pair of arrows is induced by two sets of restriction maps. $U_i \cap U_j$ is a subset of U_i and of U_j. Sheaves form a category as a full subcategory of the category of presheaves.

5. *Fiber products in the category of sets are limits of diagrams of the form $A \underset{f}{\overset{g}{\longrightarrow}} C$. A concrete model for this pullback in these categories is*

$$f^*(p) := Z \times_Y X := \{(z, x) \in Z \times X \mid f(z) = p(x)\}$$

Dually (inverting the arrow) we define colimit, coproducts, coequalizers...

If you build the colimit over a discrete diagram category (small category $\mathcal{D}$ that has only identity morphisms), then the colimit of a functor $F : \mathcal{D} \to \mathcal{C}$ is called the **coproduct** of the $F(D)$ for D an object of $\mathcal{D}$, denoted by $\bigsqcup_{\mathcal{D}} F(D)$. Coproducts in the category of sets and in the category of topological spaces are the disjoint unions. Every coproduct comes with canonical structure maps, called **inclusions**.
Pushouts are colimits over a diagram category $\mathcal{D}$ of the form $D_1 \leftarrow D_0 \to D_2$..
Another important class of examples is **coequalizers**. These are colimits of diagrams of the form $F(D_0) \underset{\alpha}{\overset{\beta}{\Longrightarrow}} F(D_1)$.

Example 5 (Colimits). *1. Colimits exist in the category of Sets:*

$$\operatorname{colim}_{\mathcal{D}} F = \bigsqcup_{D \text{ object of } \mathcal{D}} F(D)/\sim,$$

where we declare that an $x \in F(D)$ is equivalent to a $y \in F(D')$ if there is a morphism $f \in \mathcal{D}(D, D')$, such that $F(f)(x) = y$. This relation is not symmetric, so one has to consider the equivalence relation generated by this relation.

2. *If all structure maps $F(i < j)$ are monomorphisms, then we might interpret the colimit $\operatorname{colim}_{\mathcal{D}} F$ as the union of the $F(i)$ s. Typical examples are increasing sequences of sets or topological spaces*

$$X_0 \subset X_1 \subset X_2 \subset \ldots$$

or increasing sequences of abelian groups, vector spaces, and other algebraic objects.

3. *An important class of examples is CW complexes. These are the colimits of their skeleta.*

4. *In stable homotopy theory, the stable homotopy groups of spheres are a central object of study. Let $\mathbb{S}^n$ denote the unit sphere in $\mathbb{R}^{n+1}$. As the smash product of spheres satisfies $\mathbb{S}^1 \wedge \mathbb{S}^n \cong \mathbb{S}^{n+1}$ we have stabilization maps*

$$\pi_n\left(\mathbb{S}^m\right) = \left[\mathbb{S}^n, \mathbb{S}^{m+1}\right]_* \to \left[\mathbb{S}^{n+1}, \mathbb{S}^{m+1}\right]_* = \pi_{n+1}\left(\mathbb{S}^m\right)$$

that send a homotopy class $[f]$ to the homotopy class of $\mathbb{S}^1 \wedge f$. Therefore, for every m, we get a sequential colimit and as $\pi_n\left(\mathbb{S}^m\right) = 0$ for $n < m$, we can express $\pi_n\left(\mathbb{S}^m\right)$ as $\pi_{k+m}\left(\mathbb{S}^m\right)$ in the nontrivial cases, with $k \geq 0$, and get the k th stable homotopy group of spheres as

$$\pi_k^s = \operatorname{colim}\left(\pi_{k+m}\left(\mathbb{S}^m\right) \to \pi_{k+m+1}\left(\mathbb{S}^{m+1}\right) \to \pi_{k+m+2}\left(\mathbb{S}^{m+2}\right) \to \dots\right)$$

5. *The first groups are $\pi_0^s = \mathbb{Z}$, $\pi_1^s = \mathbb{Z}/2\mathbb{Z}$ generated by the stabilization of the Hopf* map *$\eta : \mathbb{S}^3 \to \mathbb{S}^2$, $\pi_2^s = \mathbb{Z}/2\mathbb{Z}$, $\pi_3^s = \mathbb{Z}/24\mathbb{Z}$, and so on.*

6. *In the category of pointed topological spaces the pointed sum (also known as the bouquet of spaces) is the coproduct.*

7. *Coproducts in the category of abelian groups are given by the direct sum. Coproducts in the category of general groups is the free product.*

8. *If A is a topological space, together with continuous maps $f : A \to X$ and $g : A \to Y$, the pushout of $X \leftarrow A \to Y$ is the quotient space of the disjoint union $X \sqcup Y$ by the equivalence relation that identifies $f(a)$ with $g(a)$ for all $a \in A$.*

9. *Pushouts of groups are given by amalgamated products, given by $G_1 *_{G_0} G_2$, which is the quotient of the free product $G_1 * G_2$ by the normal subgroup generated by words of the form $f\left(g_0\right) h\left(g_0\right)^{-1}$ for $g_0 \in G_0$.*

10. *The cokernel of a homomorphism f is the coequalizer of the diagram $A \underset{f}{\overset{0}{\rightrightarrows}} B$ in the category Ab.*

1.3 Adjoint functors

Let $\mathcal{C}$ and $\mathcal{C}'$ be categories. An **adjunction** between $\mathcal{C}$ and $\mathcal{C}'$ is a pair of functors $L : \mathcal{C} \to \mathcal{C}'$, $R : \mathcal{C}' \to \mathcal{C}$, such that for each pair of objects C of $\mathcal{C}$ and C' of $\mathcal{C}'$, there is a bijection of sets

$$\varphi_{C,C'} : \mathcal{C}'\left(L(C), C'\right) \cong \mathcal{C}\left(C, R\left(C'\right)\right),$$

which is natural in C and C'. The functor L is then left adjoint to R, and R is right adjoint to L. We call (L, R) an adjoint pair of functors.
The naturality condition on the bijections $\varphi_{C,C'}$ can be spelled out explicitly as follows: For all morphisms $f : C \to D$ in $\mathcal{C}$ and $g : C' \to D'$ in $\mathcal{C}'$, the following diagram commutes:

$$\begin{tikzcd}
\mathcal{C}'(L(D), C') \arrow[r, "\mathcal{C}'(Lf{,}C')"] \arrow[d, "\varphi_{D,C'}"] & \mathcal{C}'(L(C), C') \arrow[r, "\mathcal{C}'(L(C){,}g)"] \arrow[d, "\varphi_{C,C'}"] & \mathcal{C}'(L(C), D') \arrow[d, "\varphi_{C,D'}"] \\
\mathcal{C}(D, R(C')) \arrow[r, "\mathcal{C}(f{,}R(C'))"] & \mathcal{C}(C, R(C')) \arrow[r, "\mathcal{C}(C{,}R(g))"] & \mathcal{C}(C, R(D'))
\end{tikzcd}$$

Example 6. *A prototypical example of an adjunction is a forgetful functor and a 'free' functor: if $R = U$ is a forgetful functor and if a left adjoint of U exists, then the defining property means that for each morphism from C to $U(C')$ in the underlying category, there is a unique corresponding morphism from $L(C)$ to C', so, in this sense, $L(C)$ is the free object associated with C. For topological spaces, the free topological space on a set is the set with discrete topology.*

Proposition 4.
1. *The functor L is left adjoint to R iff there are natural transformations $\eta : Id \Rightarrow R \circ L$ and $\varepsilon : L \circ R \Rightarrow Id$ with the properties that*

$$\varepsilon_L \circ L(\eta) = \mathrm{Id}_L \text{ and } R(\varepsilon) \circ \eta_R = \mathrm{Id}_R$$

2. *Adjunction can be composed.*
3. *Each of the functors L and R determines the other functor uniquely up to isomorphism.*

The transformation η is called the **unit of the adjunction** and ε is the **counit**.

Theorem 1. *Let $F : \mathcal{C} \to \mathcal{D}$ be an arbitrary functor. Then the following are equivalent.*

1. *The functor F possesses a left adjoint L, and the corresponding natural transformations $\varepsilon : LF \Rightarrow \mathrm{Id}$ and $\eta : Id \Rightarrow FL$ are natural isomorphisms.*
2. *There is a functor $L : \mathcal{D} \to \mathcal{C}$ and two arbitrary natural isomorphisms $\mathrm{Id} \cong FL$ and $LF \cong \mathrm{Id}$.*
3. *The functor F is fully faithful and essentially surjective.*

1.4 Concrete categories

The language of concrete categories is a way to refer to *low level structures* present on the objects of a category. Conversely, often it is easier to work with less structures, and there results like Yoneda's lemma that show us that it is possible to restrict our study to them.

Let $\mathcal{C}$ be a category. A **concrete category** over $\mathcal{C}$ is a category $\mathcal{A}$ together wih a faithful functor $U : \mathcal{A} \to \mathcal{C}$, called the **forgetful** (or underlying) functor of the concrete category. $\mathcal{C}$ is called the **base category**. A concrete category over Set is called a **construct**.
The category of groups (or topological spaces, rings, etc.), with the forgetful functor to Set, is a construct.
In this section $|A|$ will denote the underlying object after applying the forgetful functor.

Let $\mathcal{A}$ be a concrete category over $\mathcal{C}$

1. An A-morphism $A \xrightarrow{f} B$ is called **initial** provided that for any A-object C an $\mathcal{C}$ morphism $|C| \xrightarrow{g} |A|$ is an A-morphism whenever $|C| \xrightarrow{f \circ g} |B|$ is an A-morphism.
2. An initial morphism $A \xrightarrow{f} B$ that has a monomorphic underlying X-morphism $|A| \xrightarrow{f} |B|$ is called an **embedding**.
3. If $A \xrightarrow{f} B$ is an embedding, then (f, B) is called an **extension** of A and (A, f) is called an **initial subobject** of B.

4. A **structured arrow** with domain C is a pair (f, A) consisting of an $\mathcal{A}-$object A and an $\mathcal{C}-$morphism $C \xrightarrow{f} |A|$. It is **generating** provided that for any pair of $\mathcal{A}$ -morphisms $r, s : A \to B$ the equality $r \circ f = s \circ f$ implies that $r = s$; and it is called **extremally generating** (resp. **concretely generating**) provided that each $\mathcal{A}$ -monomorphism (resp. $\mathcal{A}$ -embedding) $m : A' \to A$, through which f factors (i.e., $f = m \circ g$ for some $\mathcal{C}$-morphism g), is an $\mathcal{A}$ -isomorphism.

5. In a construct, an object A is (**extremally** resp. **concretely**) generated by a subset X of $|A|$ provided that the inclusion map $X \hookrightarrow |A|$ is (**extremally** resp. **concretely**) generating.

Proposition 5. *In a concrete category $\mathcal{A}$ over $\mathcal{C}$ the following hold for each structured arrow $f : X \to |A|$:*

1. *If (f, A) is extremally generating, then (f, A) is concretely generating.*
2. *If (f, A) is concretely generating, then (f, A) is generating.*
3. *If $X \xrightarrow{f} |A|$ is an $\mathcal{C}$-epimorphism, then (f, A) is generating.*
4. *If $X \xrightarrow{f} |A|$ is an extremal epimorphism in $\mathcal{C}$, and if $|\cdot|$ preserves monomorphisms, then (f, A) is extremally generating.*

Example 7. 1. *If an abstract category $\mathcal{A}$ is considered to be concrete over itself via the identity functor, then an A-morphism $A \xrightarrow{f} B$, considered as a structured arrow (f, B), is generating (resp. extremally or concretely generating) if and only if f is an epimorphism (resp. an extremal epimorphism). That is,*

$$\mathrm{Gen}(\mathcal{A}) = \mathrm{Epi}(\mathcal{A}) \textit{ and } \mathrm{ExtrGen}(\mathcal{A}) = \mathrm{ConcGen}(\mathcal{A}) = \mathrm{ExtrEpi}(\mathcal{A})$$

(a) *In Vec, Grp, Sgr, Rng, and other algebraic constructs, the concepts of concrete generation and of extremal generation coincide with the familiar (non-categorical) concept of generation. In the constructs Sgr and Rng the inclusion map $\mathbb{Z} \hookrightarrow \mathbb{Q}$ is generating, but is not concretely generating [cf. 7.40(5)].*

(b) *In the construct $\mathcal{A} = Top$ we have*

$$\begin{aligned} \textit{ConcGen(A)} &= \mathrm{Gen}(\mathcal{A}) = \textit{ Surjective maps, and} \\ \mathrm{ExtrGen}(\mathcal{A}) &= \textit{ Surjective maps with discrete codomain.} \end{aligned}$$

(c) *In the construct $\mathcal{A} = Haus$ we have*

$$\begin{aligned} \mathrm{Gen}(\mathcal{A}) &= \textit{ Dense maps} \\ \textit{ConcGen(A)} &= \textit{ Surjective maps, and} \\ \textit{ExtrGen(A)} &= \textit{ Surjective maps with discrete codomain.} \end{aligned}$$

(d) *$A \xrightarrow{f} B$ is an epimorphism if and only if (f, B) is generating.*

(e) *If (f, B) is extremally generating and the forgetful functor preserves monomorphisms, then $A \xrightarrow{f} B$ is an extremal epimorphism.*

(f) *If $A \xrightarrow{f} B$ is an extremal epimorphism, then (f, B) is concretely generating.*

Free objects

A **universal arrow** over an $\mathcal{C}$-object X is a structured arrow $X \xrightarrow{u} |A|$ with domain X such that, for each structured arrow $X \xrightarrow{f} |B|$ with domain X, there exists a unique A-morphism $\hat{f} : A \to B$ such that the triangle

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[dr, "f"'] & {|A|} \arrow[d, "\overline{f}"] \\ & {|B|} \end{tikzcd}$$

commutes. The pair (u, A) is called a **free object**.

Example 8. *1. In a construct, an object A is a free object over the empty set if and only if A is an initial object, and over a singleton set if and only if A represents the forgetful functor.*

2. *In the construct Vec each object is a free object over any basis for it.*

3. *In the constructs Top and Pos the free objects are precisely the discrete ones.*

4. *In the construct* **Ab** *free objects over X are the free abelian groups generated by X. Similarly, the familiar free group generated by a set X is a free object over X in the construct Grp.*

5. *To construct a universal arrow in (Ban, O) over a set X, let $\ell_1(X)$ be the subspace of the vector space K^X consisting of all $r = (r_x)_{x \in X}$ in K^X whose norm $\|r\| = \sum_{x \in X} |r_x|$ is finite. Then $\ell_1(X)$ is a Banach space. Define $X \xrightarrow{u} O(\ell_1(X))$ at y by the Dirac function $u(y) = \left(\delta_{yx}\right)_{x \in X}$. Then $(u, \ell_1(X))$ is a universal arrow over X. Observe, for comparison, that for the construct (Ban, U) the only set having a universal arrow is the empty set, and that for the construct Ban* $\mathrm{B_b}$ *the only sets having universal arrows are the finite ones.*

Proposition 6. *1. Every universal arrow is extremally generating.*

2. *Any two universal arrows with domain X are isomorphic. Conversely, if $X \xrightarrow{u} |A|$ is a universal arrow and $A \xrightarrow{k} A'$ is an $\mathcal{A}$-isomorphism, then $X \xrightarrow{kou} |A'|$ is also universal.*

3. *If a concrete category $\mathcal{A}$ over $\mathcal{C}$ has free objects, then an $\mathcal{A}$-morphism is an $\mathcal{A}$-monomorphism if and only if it is an $\mathcal{C}$-monomorphism.*

4. *If a construct $\mathcal{A}$ has a free object over a singleton set, then the monomorphisms in $\mathcal{A}$ are precisely those morphisms that are injective functions.*

A concrete category over $\mathcal{C}$ is said to have free objects provided that for each $\mathcal{C}$-object X there exists a universal arrow over X.
The constructs Vec, Grp, Ab, Mon, Sgr, Alg (Ω), Top, Pos, and (Ban,O) have free objects.

Representable functors

A functor $F : \mathcal{A} \to$ Set is called representable (by an $\mathcal{A}$-object A) provided that F is naturally isomorphic to the hom-functor $\hom(A, -) : \mathcal{A} \to$ Set. Note that objects that represents the same functor are isomorphic.

Example 9. *1. Forgetful functors are often representable. For example, (a) Vec $\to$ Set is represented by the vector space $\mathbb{R}$, (b)* Grp $\to$ *Set is represented by the group of integers $\mathbb{Z}$, (c) Top $\to$ Set is represented by any one-point topological space.*

2. *The underlying functor U for the construct Ban [5.2(3)] is not representable (see Exercise 10J). However, the faithful unit ball functor O :* Ban $\to$ *Set is represented in the complex case by the Banach space $\mathbb{C}$ of complex numbers.*

Proposition 7 (Representative of Constructs)**.** *For constructs $(\mathcal{A}, U)$ the forgetful functor is represented by an object A if and only if A is a free object over a singleton set. This provides many additional examples of representations.*

Theorem 2 (uniqueness of representations)**.** *For any functor $F : \mathcal{A} \to Set$, any $\mathcal{A}$-object A and any element $a \in F(A)$, there exists a unique natural transformation $\tau : \hom(A, -) \to F$ with $\tau_A\,(id_A) = a$.*

Corollary 1 (Yoneda Lemma)**.** *If $F : \mathcal{A} \to Set$ is a functor and A is an $\mathcal{A}$-object, then the following function*

$$Y : [\hom(A, -), F] \to F(A) \text{ defined by } Y(\sigma) = \sigma_A\,(id_A)\,,$$

is a bijection (where $[\hom(A, -), F]$ is the set of all natural transformations from hom $(A, -)$ to F).

Corollary 2 (Yoneda Embedding)**.** *For any category $\mathcal{A}$, the functor $E : \mathcal{A} \to [\mathcal{A}^{\mathrm{op}} Set]$, defined by*

$$E(A \xrightarrow{f} B) = \hom(-, A) \xrightarrow{\sigma_f} \hom(-, B),$$

where $\sigma_f(g) = f \circ g$, is a full embedding.

Proposition 8. *G has a left-adjoint F if and only if $\mathrm{Hom}_C(X, G-)$ is representable for all X in C. The natural isomorphism $\Phi_X : \mathrm{Hom}_D(FX, -) \to \mathrm{Hom}_C(X, G-)$ yields the adjointness; that is*

$$\Phi_{X,Y} : \mathrm{Hom}_{\mathcal{D}}(FX, Y) \to \mathrm{Hom}_{\mathcal{C}}(X, GY)$$

is a bijection for all X and Y.

1.5 Kan extensions

Kan extensions take a given functor and extend it to a different category. There are two ways of doing that, via colimits and via limits. These extensions does not have to exist, and even if they exist, they might not have nice properties. But in controlled situations, they are extremely useful and they are actually ubiquitous.

Let $G : \mathcal{C} \to \mathcal{D}$ and $F : \mathcal{C} \to \mathcal{E}$ be functors. The **left Kan extension** of F along G is a pair (K, α), where

- $K : \mathcal{D} \to \mathcal{E}$ is a functor, and
- $\alpha : F \Rightarrow K \circ G$ is a natural transformation.
- for all pairs (H, β), where $H : \mathcal{D} \to \mathcal{E}$ is a functor and $\beta : F \Rightarrow H \circ G$ is a natural transformation, there is a unique natural transformation $\gamma : K \Rightarrow H$ with the property that $\gamma_G \circ \alpha = \beta$.

Theorem 3. *Let $G : \mathcal{C} \to \mathcal{D}$ and $F : \mathcal{C} \to \mathcal{E}$ be functors. Assume that the category $\mathcal{C}$ is small and that $\mathcal{E}$ is cocomplete. Then, the left Kan extension of F along G exists.*

Theorem 4. *For small categories $\mathcal{C}, \mathcal{D}$ and $G : \mathcal{C} \to \mathcal{D}$ and a cocomplete category $\mathcal{E}$ the functor,*

$$G^* : [\mathcal{D}, \mathcal{E}] \to [\mathcal{C}, \mathcal{E}]$$

has a left adjoint, and this adjoint is given by the left Kan extension.

Example 10. *1. Let G be a finite group and let H be a subgroup of G. Consider the inclusion of the category $\mathcal{C}_H$ with one object and morphisms H into the category $\mathcal{C}_G$, $i : \mathcal{C}_H \to \mathcal{C}_G$. A functor $F : \mathcal{C}_H \to \mathrm{Ab}$ is nothing but a $\mathbb{Z}[H]$-module. $M = F(*)$ carries a linear H-action. What is the left Kan extension of a given F along i ?*

2. *Assume that $f: X \rightarrow Y$ is a continuous map between topological spaces and $\mathcal{F}$ is a presheaf on Y. One could try to pull $\mathcal{F}$ back via f by defining $f^{-1}\mathcal{F}(U)=\mathcal{F}(f(U))$, but, of course, $f(U)$ doesn't have to be open, so instead, one defines the inverse image presheaf as the left Kan extension*

$$f^{-1}\mathcal{F}(U)=\operatorname{colim}_{f(U)\subset V \text{ open}} \mathcal{F}(V).$$

Even if $\mathcal{F}$ was a sheaf, $f^{-1}\mathcal{F}$ might not be one, so for sheaves, $f^{-1}\mathcal{F}$ is defined as the sheafification.

The functor H preserves the left Kan extension (K, α) of F along G if $(H\circ K, H\alpha)$ is a left Kan extension of $H\circ F$ along G.

Theorem 5. *Let $G: \mathcal{C} \rightarrow \mathcal{D}$ be a functor between small categories. Left adjoint functors $L: \mathcal{E} \rightarrow \mathcal{F}$ preserve left Kan extensions of functors $F: \mathcal{D} \rightarrow \mathcal{E}$.*

A right Kan extension of $F: \mathcal{C} \rightarrow \mathcal{E}$ is pointwise if and only if it is preserved by all representable functors $\mathcal{E}(E,-): \mathcal{E} \rightarrow$ Sets.
The dual statement is also true, but in that case, we have to consider the representable functors $\mathcal{E}(-, E)$ which transform colimits to limits in Sets $^\circ$.

Let $\mathcal{D}$ and $\mathcal{E}$ be categories. Assume that $H_1: \mathcal{D}^\circ \times \mathcal{D} \rightarrow \mathcal{E}$ and $H_2: \mathcal{D}^o \times \mathcal{D} \rightarrow \mathcal{E}$ are functors, and let

$$\tau_D: H_1(D,D) \rightarrow H_2(D,D)$$

be a family (indexed over the objects of $\mathcal{D}$) of morphisms $\tau_D \in \mathcal{E}\left(H_1(D,D), H_2(D,D)\right)$. Then, $(\tau_D)_D$ is called a dinatural transformation if for all morphisms $f \in \mathcal{D}\left(D, D'\right)$, the diagram

$$H_1\left(D', D\right) \xrightarrow{H_1(f,D)} H_1(D,D)$$

commutes.

1. An important example of a functor $H: \mathcal{D}^o \times \mathcal{D} \rightarrow \mathcal{E}$ is a natural evaluation map. Fix a K-vector space W, and denote by $L(V,W)$ the vector space of K-linear maps from V to W. Consider the functor

$$L(-,W)\otimes \mathrm{Id} \rightarrow \text{ vect}^{\,o} \times \text{ vect } \rightarrow \text{ vect}, \quad (V_1, V_2) \mapsto L\left(V_1, W\right)\otimes V_2.$$

   A dinatural transformation from this functor to the constant functor on W, κ_W, consists of a family of linear maps

$$\tau_V: L(V,W)\otimes V \rightarrow W$$

   which transform naturally in V.

2. Let V and W be K-vector spaces, and denote by Iso (V,W) the vector space of K-linear isomorphisms from V to W. Then,

$$\mathrm{Iso}: \text{vect}^{\,0} \times \text{ vect } \rightarrow \text{ vect}$$

   is a functor, and Iso (V,V) is the group of automorphisms of V. For instance, if $K=\mathbb{R}$, we can consider the orientation preserving automorphisms of $V, \mathrm{Aut}^+(V)$. The inclusion of $\mathrm{Aut}^+(V)$ into Aut (V) is then a τ_V where τ is a dinatural transformation.

3. In fact, the preceding example generalizes to any category. For two objects C_1 and C_2 of a category $\mathcal{C}$, we can always consider the set of isomorphisms from C_1 to C_2, Iso (C_1, C_2), and Aut $(C_1)=$ Iso (C_1, C_1), the group of automorphisms of the object C_1. If this group has interesting subgroups that transform naturally in C_1, then the inclusion of such a subgroup into Aut (C_1) gives rise to a dinatural transformation. Last but not least, we fix an object E of $\mathcal{E}$ and consider the constant functor on E, κ_E, as a functor

$$\kappa_E: \mathcal{D}^o \times \mathcal{D} \rightarrow \mathcal{E}.$$

Let $H : \mathcal{D}^\circ \times \mathcal{D} \to \mathcal{E}$ be a functor. An end of H is a pair (E, τ), where E is an object of $\mathcal{E}$ and τ is a dinatural transformation from κ_E to H, with the property that for all other objects E' of $\mathcal{E}$ with a dinatural transformation ν from $\kappa_{E'}$ to H, there is a unique $\xi \in \mathcal{E}(E', E)$, such that $\nu_D = \tau_D \circ \xi$ for all D.

Example 11. 1. *Let $\mathcal{D}$ be a small category, let $\mathcal{E}$ be an arbitrary category, and assume F and G are functors from $\mathcal{D}$ to $\mathcal{E}$. We consider*

$$\mathcal{E}(F(-), G(-)) : \mathcal{D}^o \times \mathcal{D} \to \textit{ Sets}$$

as a functor. An end of this functor is a set X, together with a universal dinatural transformation

$$\varepsilon_D : X \to \mathcal{E}(F(D), G(D))$$

for all objects D of $\mathcal{D}$, which satisfies the coherence condition, as illustrated in the diagram (4.4.1). It is clear that the set of all natural transformations satisfies this condition: If X' is another set with a dinatural transformation ν from $\kappa_{X'}$ to $\mathcal{E}(F(-), G(-))$, then for every element $x \in X', \nu_D(x)$ is actually a natural transformation because of the naturality of ν, but then, we obtain a function $f : X' \to X = \mathrm{nat}(F, G)$, with $f(x)_D = \nu_D(x)$.

As a special case, we obtain that the abelian group of R-module homomorphism between two left R-modules M and N is an end.

2. *Example 4.4.7. Let $\mathcal{D}$ be a small category and let $F : \mathcal{D}^o \to k$-mod and $G : \mathcal{D} \to k$-mod be functors. Here, k is an arbitrary commutative ring with unit, and k-mod denotes the category of k-modules and k-linear maps. Then, we can build the tensor product of F and G as*

$$F \otimes_{\mathcal{D}} G := \bigoplus_D F(D) \otimes_k G(D) / \sim,$$

where the sum is indexed by all objects D of $\mathcal{D}$ and where we divide out by the k-submodule of $\oplus_D F(D) \otimes_k G(D)$ generated by

$$F(f)(x) \otimes y - x \otimes G(f)(y), \quad x \in F\left(D'\right), y \in G(D), f \in \mathcal{D}\left(D, D'\right).$$

We claim that $F \otimes_{\mathcal{D}} G$, together with the dinatural transformation τ that sends $F(D) \otimes_k G(D)$ to the class of the summand in $F \otimes_{\mathcal{D}} G$, is the coend of the functor $F \otimes_k G : \mathcal{D}^\circ \times \mathcal{D} \to k$-mod that sends (D_1, D_2) to $F(D_1) \otimes_k G(D_2)$ and $(f, g) \in \mathcal{D}(D_1, D_2) \times \mathcal{D}(D_3, D_4)$ to $F(f) \otimes_k G(g)$

Let $\mathcal{D}$ and $\mathcal{E}$ be categories and let $H : \mathcal{D}^\circ \times \mathcal{D} \to \mathcal{E}$ be a functor. We denote by $\int_{\mathcal{D}} H$ the end of the functor H; and by $\int^{\mathcal{D}} H$ the coend of the functor H.

Proposition 9 (Fubini theorem for ends)**.** *Let $H : (\mathcal{D} \times \mathcal{D}')^o \times (\mathcal{D} \times \mathcal{D}') \to \mathcal{E}$ be a functor. If the ends $\int_{\mathcal{D}} H\left(D, D_1', D, D_2'\right)$ exist for all objects D_1', D_2' of $\mathcal{D}'$ and if the ends $\int_{\mathcal{D}'} H\left(D_1, D', D_2, D'\right)$ exist for all objects D_1 and D_2 of $\mathcal{D}$, then*

$$\int_{\mathcal{D}} \int_{\mathcal{D}'} H\left(D, D', D, D'\right) \cong \int_{\mathcal{D}'} \int_{\mathcal{D}} H\left(D, D', D, D'\right) \cong \int_{\mathcal{D} \times \mathcal{D}'} H\left(D, D', D, D'\right),$$

and if one of them exists, then the others do as well.

1.6 Grupoids

If we want a limited amount of interaction between $\mathcal{C}$ and $\mathcal{D}$, we can form the join of $\mathcal{C}$ and $\mathcal{D}$, denoted by $\mathcal{C} * \mathcal{D}$. The objects of $\mathcal{C} * \mathcal{D}$ are the disjoint union of the objects of $\mathcal{C}$ and the objects of $\mathcal{D}$ and as morphism we have

$$(\mathcal{C} * \mathcal{D})(X,Y) = \begin{cases} \mathcal{C}(X,Y), \text{ if } X \text{ and } Y \text{ are objects of } \mathcal{C} \\ \mathcal{D}(X,Y), \text{ if } X \text{ and } Y \text{ are objects of } \mathcal{D} \\ \{*\}, \text{ if } X \text{ is an object of } \mathcal{C} \text{ and } Y \text{ is an object of } \mathcal{D} \\ \varnothing, \text{ otherwise.} \end{cases}$$

A category is a grupoid if all morphisms are isomorphisms.

Example 12. *1. If G is a group, then we denote by $\mathcal{C}_G$ the category with one object $*$ and $\mathcal{C}_G(*,*) = G$ with group multiplication as composition of maps. Then, $\mathcal{C}_G$ is a groupoid. Hence every group gives rise to a groupoid. Vice versa, a groupoid can be thought of as a group with many objects.*

2. *Let X be a topological space. The fundamental groupoid of $X, \Pi(X)$, is the category whose objects are the points of X, and $\Pi(X)(x,y)$ is the set of homotopy classes of paths from x to y :*

$$\Pi(X)(x,y) = [[0,1],0,1;X,x,y].$$

The endomorphisms $\Pi(x,x)$ of $x \in X$ constitute the fundamental group of X with respect to the basepoint $x, \pi_1(X,x)$.

3. *Another important example of a groupoid is the translation category of a group. If G is a discrete group, then we denote by $\mathcal{E}_G$ the category whose objects are the elements of the group and*

$$\mathcal{E}_G(g,h) = \left\{hg^{-1}\right\}, g \xrightarrow{hg^{-1}} h.$$

This category has the important feature that there is precisely one morphism from one object to any other object, so every object has equal rights.

Chapter 2

Homological Algebra

References [13, 19]

2.1 Abelian Categories

A **preadditítve category** is a category $\mathcal{A}$, such that for every pair of objects A_1, A_2, there is an abelian group of morphisms from A_1 to A_2 and the composition of morphisms is a bilinear map.

Example 13. *A preadditive category with only one object is nothing but a ring. The endomorphisms of that object are an abelian group, and the composition of morphisms defines the multiplicative structure. Thus, a preadditive category can be thought of as a ring with many objects. A group with many objects in this sense is a groupoid, so one might call a preadditive category a ringoid.*

Let $\mathcal{A}$ and $\mathcal{A}'$ be preadditive categories. A functor $F : \mathcal{A} \to \mathcal{A}'$ is **additive** if for any two objects A_1, A_2 of $\mathcal{A}$, the map $F : \mathcal{A}(A_1, A_2) \to \mathcal{A}'(F(A_1), F(A_2))$ is a group homomorphism.

Assume that a category $\mathcal{C}$ has zero morphisms. Then, the **kernel** of a morphism $f \in \mathcal{C}(C_1, C_2)$ is the equalizer of the morphisms $f, 0 : C_1 \to C_2$. Dually, the **cokernel** of a morphism $f \in \mathcal{C}(C_1, C_2)$ is the coequalizer of the morphisms $f, 0 : C_1 \to C_2$.

Proposition 10.

1. *In a preadditive categoty, all equalizers are kernels.*
2. *Initial object exists if and only if zero object exists.*
3. *A finite product exists if and only if the finite coproduct exists, called **biproduct**.*

A preadditive category is called **additive** if it has all finite biproducts.

Proposition 11. *A functor between additive categories is additive if and only if it preserves biproducts or just products.*

A preadditive category is an **abelian** category if it satisfies the following properties:

- There exists a zero object in $\mathcal{A}$.
- The category $\mathcal{A}$ has finite biproducts.
- Every morphism $f \in \mathcal{A}(A, B)$ has a cokernel and a kernel.
- Every monomorphism is a kernel, and every epimorphism is a cokernel.

Theorem 6. *Let $\mathcal{A}$ be an abelian category:*

1. *A morphism is an isomorphism if and only if it is both a monomorphism and an epimorphism.*
2. *A morphism is a monomorphism if and only if its kernel is zero.*
3. *Let f be a morphism. Then, we can factor f as $f = i \circ p$, where p is an epimorphism and i is a monomorphism. Here, i is the kernel of the cokernel of f and p is the cokernel of the kernel of f.*
4. *A monomorphism is the kernel of its cokernel, and an epimorphism is the cokernel of its kernel.*

Proposition 12. *Let $\mathcal{D}$ be a small category and let $\mathcal{A}$ be abelian. Then, the functor category $[\mathcal{D}, \mathcal{A}]$ is abelian.*

In homological algebra one constructs homological invariants of algebraic objects by the following process, or some variant of it:
Let R be a ring and T a covariant additive functor from R-modules to abelian groups. Thus the map $\mathrm{Hom}_R(M, N) \to \mathrm{Hom}_{\mathbf{Z}}(TM, TN)$ defined by T is a homomorphism of abelian groups for all R-modules M, N. For any R module M, choose a free (or projective) resolution $\varepsilon : F \to M$ and consider the chain complex TF of abelian groups obtained by applying T to F termwise. Now T, being additive, preserves chain homotopies; so we can apply the *uniqueness theorem for resolutions* to deduce that the complex TF is independent, up to canonical homotopy equivalence, of the choice of resolution. Passing to homology, we obtain groups $H_n(TF)$ which depend only on T and M (up to canonical isomorphism).
This construction is of no interest, of course, if T is an exact functor, for then the augmented complex $\cdots \to TF_1 \to TF_0 \to TM \to 0$ is acyclic, so that $H_n(TF) = 0$ for $n > 0$ and $H_0(TF) = TM$. Thus we can regard the groups $H_n(TF)$ in the general case as a measure of the failure of T to be exact.

2.2 Chain complexes

Here are some important constructions on chain complexes. A chain complex B is called a **subcomplex** of C if each B_n is a submodule of C_n and the differential on B is the restriction of the differential on C, that is, when the inclusions $i_n : B_n \subseteq C_n$ constitute a chain map $B \to C$. In this case we can assemble the quotient modules C_n / B_n into a chain complex

$$\cdots \to C_{n+1}/B_{n+1} \xrightarrow{d} C_n/B_n \xrightarrow{d} C_{n-1}/B_{n-1} \xrightarrow{d} \cdots$$

denoted C/B and called the quotient complex. If $f : B \to C$ is a chain map, the kernels $\{\ker(f_n)\}$ assemble to form a subcomplex of B denoted $\ker(f)$, and the cokernels $\{\mathrm{coker}\ (f_n)\}$ assemble to form a quotient complex of C denoted $\mathrm{coker}(f)$. *This definitions coincides with the usual one on Ch.*

Theorem 7. *The category* $\mathrm{Ch} = \mathbf{Ch}(\mathcal{A})$ *of chain complexes is an abelian category.*

If C is a chain complex and n is an integer, we let $\tau_{\geq n}C$ denote the subcomplex of C defined by

$$(\tau_{\geq n}C)_i = \begin{cases} 0 & \text{if } i < n \\ Z_n & \text{if } i = n \\ C_i & \text{if } i > n. \end{cases}$$

Clearly $H_i(\tau_{\geq n}C) = 0$ for $i < n$ and $H_i(\tau_{\geq n}C) = H_i(C)$ for $i \geq n$. The complex $\tau_{\geq n}C$ is called the **(good) truncation of** C **below** n, and the quotient complex $\tau_{<n}C = C/(\tau_{\geq n}C)$ is called the

(good) truncation of C **above** n; $H_i(\tau_{<n}C)$ is $H_i(C)$ for $i<n$ and 0 for $i \geq n$.

If C is a complex and p an integer, we form a new complex $C[p]$ as follows:

$$C[p]_n = C_{n+p} \quad \left(\text{resp. } C[p]^n = C^{n-p}\right)$$

with differential $(-1)^p d$. We call $C[p]$ the p^{th} translate of C. Note that translation shifts homology:

$$H_n(C[p]) = H_{n+p}(C) \quad \left(\text{resp. } H^n(C[p]) = H^{n-p}(C)\right).$$

We make translation into a functor by shifting indices on chain maps. That is, if $f : C \to D$ is a chain map, then $f[p]$ is the chain map given by the formula

$$f[p]_n = f_{n+p} \quad \left(\text{resp. } f[p]^n = f^{n-p}\right)$$

Proposition 13. *1. If C is a complex, there are exact sequences of complexes:*

$$0 \longrightarrow Z(C) \longrightarrow C \xrightarrow{d} B(C)[-1] \longrightarrow 0;$$
$$0 \longrightarrow H(C) \longrightarrow C/B(C) \xrightarrow{d} Z(C)[-1] \longrightarrow H(C)[-1] \longrightarrow 0.$$

2. *(Mapping cone) Let $f : B \to C$ be a morphism of chain complexes. Form a double chain complex D out of f by thinking of f as a chain complex in* **Ch** *and using the sign trick, putting $B[-1]$ in the row $q = 1$ and C in the row $q = 0$. Thinking of C and $B[-1]$ as double complexes in the obvious way, show that there is a short exact sequence of double complexes*

$$0 \longrightarrow C \longrightarrow D \xrightarrow{\delta} B[-1] \longrightarrow 0.$$

The total complex of D is cone (f'), the mapping cone (see section 1.5) of a map f', which differs from f only by some $\pm$ signs and is isomorphic to f.

Proposition 14. *The following are proved first for chain complexes of R-modules, but they hold in any abelian category, by the Freyd-Mitchell embedding theorem.*

1. *(3 Lemma) Consider the conmutative diagram of $R-$modules*

$$\begin{tikzcd} & A' \arrow[r] \arrow[d, "f"] & B' \arrow[r] \arrow[d, "g"] & C' \arrow[r] \arrow[d, "h"] & 0 \\ 0 \arrow[r] & A \arrow[r, "i"] & B \arrow[r] & C & \end{tikzcd}$$

If the rows are exact, there is an exact sequence

$$\ker(f) \to \ker(g) \to \ker(h) \xrightarrow{\partial} \operatorname{coker}(f) \to \operatorname{coker}(g) \to \operatorname{coker}(h)$$

with ∂ defined by the formula

$$\partial\left(c'\right) = i^{-1} g p^{-1}\left(c'\right), \quad c' \in \ker(h)$$

Moreover, if $A' \to B'$ is monic, then so is $\ker(f) \to \ker(g)$, and if $B \to C$ is onto, then so is coker $(f) \to$ coker (g).

2. ***5-lemma*** *In any commutative diagram*

$$\begin{tikzcd}
A' \arrow[r] \arrow[d, "a\simeq"] & B' \arrow[r] \arrow[d, "b\simeq"] & C' \arrow[r] \arrow[d, "c\simeq"] & D' \arrow[r] \arrow[d, "d\simeq"] & E' \arrow[d, "e\simeq"] \\
A \arrow[r] & B \arrow[r] & C \arrow[r] & D \arrow[r] & E
\end{tikzcd}$$

with exact rows in any abelian category, show that if a, b, d, and e are isomorphisms, then c is also an isomorphism. More precisely, show that if b and d are monic and a is an epi, then c is monic. Dually, show that if b and d are epis and e is monic, then c is an epi.

Theorem 8. *Let $0 \to A. \xrightarrow{f} B. \xrightarrow{g} C. \to 0$ be a short exact sequence of chain complexes. Then there are natural maps $\partial : H_n(C) \to H_{n-1}(A)$, called* ***connecting homomorphisms****, such that*

$$\cdots \xrightarrow{g} H_{n+1}(C) \xrightarrow{\partial} H_n(A) \xrightarrow{f} H_n(B) \xrightarrow{g} H_n(C) \xrightarrow{\partial} H_{n-1}(A) \xrightarrow{f} \cdots$$

is an exact sequence. The long exact sequence is a functor from $\mathcal{S}$ to $\mathcal{L}$. That is, for every short exact sequence there is a long exact sequence, and for every map (*) of short exact sequences there is a commutative ladder diagram

When one computes with modules, it is useful to be able to push elements around. By decoding the above proof, we obtain the following formula for the connecting homomorphism: Let $z \in H_n(C)$, and represent it by a cycle $c \in C_n$. Lift the cycle to $b \in B_n$ and apply d. The element db of B_{n-1} actually belongs to the submodule $Z_{n-1}(A)$ and represents $\partial(z) \in H_{n-1}(A)$.

The data of the long exact sequence is sometimes organized into the mnemonic shape

$$\begin{tikzcd}
H_*(A) \arrow[rr] & & H_*(B) \arrow[dl] \\
& H_*(C) \arrow[ul, "\delta"] &
\end{tikzcd}$$

This is called an exact triangle for obvious reasons. This mnemonic shape is responsible for the term "triangulated category," which we will discuss in Chapter 10. The category **K** of chain equivalence classes of complexes and maps is an example of a triangulated category.

Now suppose that we are given two chain complexes C and D, together with randomly chosen maps $s_n : C_n \to D_{n+1}$. Let f_n be the map from C_n to D_n defined by the formula

$$f_n = d_{n+1}s_n + s_{n-1}d_n.$$

$$\begin{tikzcd}
C_{n+1} \arrow[r, "d"] & C_n \arrow[r, "d"] \arrow[dl, "s"'] \arrow[d, "f"] & C_{n-1} \arrow[dl, "s"'] \\
D_{n+1} \arrow[r, "d"] & D_n \arrow[r, "d"] & D_{n-1}
\end{tikzcd}$$

Note that f is, in fact, a chain map.

We say that two chain maps f and g from C to D are chain homotopic if their difference $f - g$ is null homotopic, that is, if

$$f - g = sd + ds.$$

The maps $\{s_n\}$ are called a chain homotopy from f to g. Finally, we say that $f : C \to D$ is a chain homotopy equivalence (Bourbaki uses homotopism) if there is a map $g : D \to C$ such that gf and fg are chain homotopic to the respective identity maps of C and D.

Proposition 15. *1. If $f : C \to D$ is null homotopic, then every map $f_* : H_n(C) \to H_n(D)$ is zero. If f and g are chain homotopic, then they induce the same* maps $H_n(C) \to H_n(D)$.

2. *Consider the homology $H_*(C)$ of C as a chain complex with zero differentials. Show that if the complex C is split, then there is a chain homotopy equivalence between C and $H_*(C)$. Give an example in which the converse fails.*

2.3 Derived functors

2.4 Derived categories

2.5 Spectral sequences

References [**cohen**, 18]

A spectral sequence is the algebraic machinery for studying sequences of long exact sequences that are interelated in a particular way. For instance, we can study a filtration of a chain complex C_* by subcomplexes,

$$0 = F_0\left(C_*\right) \hookrightarrow F_1\left(C_*\right) \hookrightarrow \cdots \hookrightarrow F_k\left(C_*\right) \hookrightarrow F_{k+1}\left(C_*\right) \hookrightarrow \cdots \hookrightarrow C_* = \bigcup_k F_k\left(C_*\right)$$

Let D_*^k be the subquotient complex $D_*^k = F_k\left(C_*\right)/F_{k-1}\left(C_*\right)$ and so for each k there is a long exact sequence in homology

$$\longrightarrow H_{q+1}\left(D_*^k\right) \longrightarrow H_q\left(F_{k-1}\left(C_*\right)\right) \longrightarrow H_q\left(F_k\left(C_*\right)\right) \longrightarrow H_q\left(D_*^k\right) \longrightarrow \cdots$$

By putting these long exact sequences together, in principle one should be able to use information about $\oplus_k H_*\left(D_*^k\right)$ in order to obtain information about

$$H_*\left(C_*\right) = \lim_k H_*\left(F_k\left(C_*\right)\right)$$

A spectral sequence is the bookkeeping device that allows one to do this.

2.5.1 Double complexes

A **double complex** (or bicomplex) in $\mathcal{A}$ is a family $\left\{C_{p,q}\right\}$ of objects of $\mathcal{A}$, together with maps

$$d^h : C_{p,q} \to C_{p-1,q} \quad \text{and} \quad d^v : C_{p,q} \to C_{p,q-1}$$

such that $d^h \circ d^h = d^v \circ d^v = d^v d^h + d^h d^v = 0$. It is useful to picture the bicomplex $C_{\cdot,\cdot}$ as a lattice

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
\cdots \arrow[r] & C_{p-1,q+1} \arrow[d, "d^v"'] & C_{p,q+1} \arrow[l, "d^h"'] \arrow[d, "d^v"'] & C_{p+1,q+1} \arrow[l, "d^h"'] \arrow[d, "d^v"'] & \cdots \arrow[l] \\
\cdots \arrow[r] & C_{p-1,q} \arrow[d, "d^v"'] & C_{p,q} \arrow[l, "d^h"'] \arrow[d, "d^v"'] & C_{p+1,q} \arrow[l, "d^h"'] \arrow[d, "d^v"'] & \cdots \arrow[l] \\
\cdots \arrow[r] & C_{p-1,q-1} \arrow[d] & C_{p,q-1} \arrow[l, "d^h"'] \arrow[d] & C_{p+1,q-1} \arrow[l, "d^h"'] & \cdots \arrow[l] \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}$$

in which the maps d^h go horizontally, the maps d^v go vertically, and each square anticommutes. Each row C_{*q} and each column C_{p*} is a chain complex.

We say that a double complex C is **bounded** if C has only finitely many nonzero terms along each diagonal line $p+q=n$; for example, if C is concentrated in the first quadrant of the plane (a first quadrant double complex).

Because of the anticommutivity, the maps d^v are not maps in Ch, but chain maps f_{*q} from C_{*q} to $C_{*,q-1}$ can be defined by introducing $\pm$ signs:

$$f_{p,q}=(-1)^p d^v_{p,q}: C_{p,q}\to C_{p,q-1}.$$

Using this sign trick, we can identify the category of double complexes with the category $\mathbf{Ch}(\mathbf{Ch})$ of chain complexes in the abelian category $\mathbf{Ch}$.

To see why the anticommutative condition $d^v d^h + d^h d^v = 0$ is useful, define the **total complexes** $\mathrm{Tot}(C)=\mathrm{Tot}^{\Pi}(C)$ and $\mathrm{Tot}^{\oplus}(C)$ by

$$\mathrm{Tot}^{\Pi}(C)_n=\prod_{p+q=n} C_{p,q} \quad \text{and} \quad \mathrm{Tot}^{\oplus}(C)_n=\bigoplus_{p+q=n} C_{p,q}.$$

The formula $d=d^h+d^v$ defines maps

$$d:\mathrm{Tot}^{\Pi}(C)_n\to\mathrm{Tot}^{\Pi}(C)_{n-1} \quad \text{and} \quad d:\mathrm{Tot}^{\oplus}(C)_n\to\mathrm{Tot}^{\oplus}(C)_{n-1}$$

such that $d\circ d=0$, making $\mathrm{Tot}^{\Pi}(C)$ and $\mathrm{Tot}^{\oplus}(C)$ into chain complexes. Note that $\mathrm{Tot}^{\oplus}(C)=\mathrm{Tot}^{\Pi}(C)$ if C is bounded, and especially if C is a first quadrant double complex.

$\mathrm{Tot}^{\Pi}(C)$ and $\mathrm{Tot}^{\oplus}(C)$ do not exist in all abelian categories, like the category of finite abelian groups.

Proposition 16. *Let $0\to A\to B\to C\to 0$ be a short exact sequence of double complexes of modules. Show that there is a short exact sequence of total complexes, and conclude that if $\mathrm{Tot}(C)$ is acyclic, then $\mathrm{Tot}(A)\to\mathrm{Tot}(B)$ is a quasi-isomorphism.*

2.5.2 Terminology

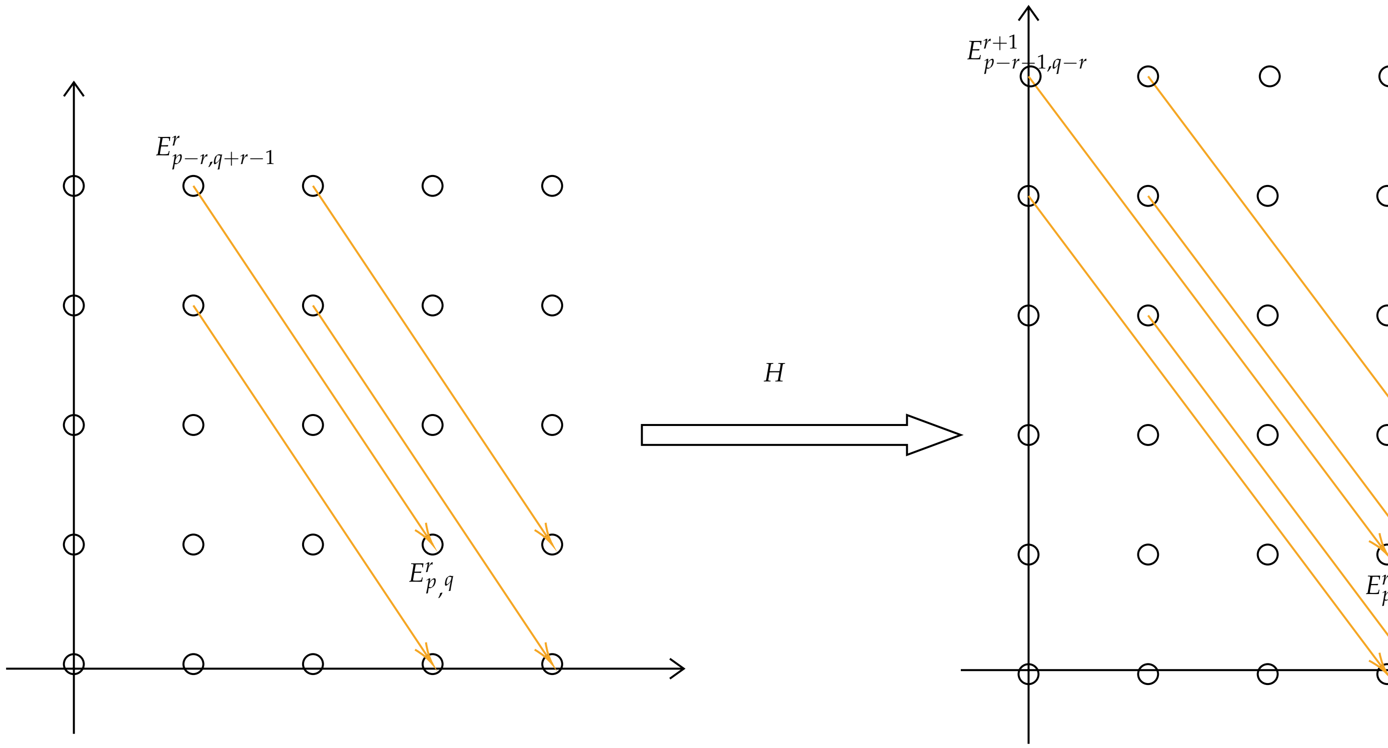
A **homology spectral sequence** (starting with E^a) in an abelian category $\mathcal{A}$ consists of the following data:

1. A family $\left\{E^r_{pq}\right\}$ of objects of $\mathcal{A}$ defined for all integers p,q, and $r\geq a$
2. Maps $d^r_{pq}: E^r_{pq}\to E^r_{p-r,q+r-1}$ that are differentials in the sense that $d^r d^r=0$, so that the "lines of slope $-\left(\dfrac{r+1}{r}\right)$" in the lattice E^r_{**} form chain complexes (we say the differentials go "to the left")
3. Isomorphisms between E^{r+1}_{pq} and the homology of E^r_{**} at the spot E^r_{pq} :

$$E^{r+1}_{pq}\cong \ker\left(d^r_{pq}\right) / \text{ image } \left(d^r_{p+r,q-r+1}\right)$$

(E^{r+1}_{pq} is a subquotient of E^r_{pq}). The total degree of the term E^r_{pq} is $n=p+q$; the terms of total degree n lie on a line of slope -1 , and each differential d^r_{pq} decreases the total degree by one.

A **first quadrant** (homology) spectral sequence is one with $E^r_{pq}=0$ unless $p\geq 0$ and $q\geq 0$.



Bounded convergence

A homology spectral sequence is said to be **bounded** if for each n there are only finitely many nonzero terms of total degree n in E^a_{**}. If so, then for each p and q there is an r_0 such that $E^r_{pq} = E^{r+1}_{pq}$ for all $r \geq r_0$. We write E^∞_{pq} for this stable value of E^r_{pq}.
To see this, consider a first quadrant spectral sequence E^a_{pq}. If we fix p and q, then $E^r_{pq} = E^{r+1}_{pq}$ for all large $r(r > \max\{p, q+1\}$ will do), because the d^r landing in the (p, q) spot come from the fourth quadrant, while the d^r leaving E^r_{pq} land in the second quadrant.

We say that a bounded spectral sequence **converges** to H_* if we are given a family of objects H_n of $\mathcal{A}$, each having a finite filtration

$$0 = F_s H_n \subseteq \cdots \subseteq F_{p-1} H_n \subseteq F_p H_n \subseteq F_{p+1} H_n \subseteq \cdots \subseteq F_t H_n = H_n,$$

and we are given isomorphisms $E^\infty_{pq} \cong F_p H_{p+q} / F_{p-1} H_{p+q}$. The traditional symbolic way of describing such a bounded convergence is like this:

$$E^a_{pq} \Rightarrow H_{p+q}$$

A (homology) spectral sequence **collapses at** $E^r (r \geq 1)$ if there is exactly one nonzero row or column in the lattice $\left\{E^r_{pq}\right\}$.
If a collapsing spectral sequence converges to H_*, we can read the H_n off: H_n is the unique nonzero E^r_{pq} with $p+q = n$. The overwhelming majority of all applications of spectral sequences involve spectral sequences that collapse at E^1 or E^2.

You could stop after reading this part! Just take a look on the Lerray-Serre spectral sequence.

General case

We are assuming axioms (Ab4) and (Ab4*)!

Given a homology spectral sequence, we see that each E^{r+1}_{pq} is a subquotient of the previous term E^r_{pq}. By induction on r, we see that there is a nested family of subobjects of E^a_{pq} :

$$0 = B^a_{pq} \subseteq \cdots \subseteq B^r_{pq} \subseteq B^{r+1}_{pq} \subseteq \cdots \subseteq Z^{r+1}_{pq} \subseteq Z^r_{pq} \subseteq \cdots \subseteq Z^a_{pq} = E^a_{pq}$$

such that $E^r_{pq} \cong Z^r_{pq}/B^r_{pq}$. We introduce the intermediate objects

$$B^\infty_{pq} = \bigcup_{r=a}^{\infty} B^r_{pq} \quad \text{and} \quad Z^\infty_{pq} = \bigcap_{r=a}^{\infty} Z^r_{pq}$$

and define $E^\infty_{pq} = Z^\infty_{pq}/B^\infty_{pq}$. In a bounded spectral sequence both the union and intersection are finite, so $B^\infty_{pq} = B^r_{pq}$ and $Z^\infty_{pq} = Z^r_{pq}$ for large r. Thus this definition agrees with the previous one. A homology spectral sequence is said to be **bounded below** if for each n there is an integer $s = s(n)$ such that the terms E^a_{pq} of total degree n vanish for all $p < s$. These spectral sequences have good convergence properties. Bounded spectral sequences are bounded below. Right half-plane homology spectral sequences are bounded below but not bounded.

We say the spectral sequence **weakly converges** to H_* if we are given objects H_n of $\mathcal{A}$, each having a filtration

$$\cdots \subseteq F_{p-1}H_n \subseteq F_pH_n \subseteq F_{p+1}H_n \subseteq \cdots \subseteq H_n,$$

together with isomorphisms $\beta_{pq} : E^\infty_{pq} \cong F_pH_{p+q}/F_{p-1}H_{p+q}$ for all p and q. Note that a weakly convergent spectral sequence cannot detect elements of $\cap F_pH_n$, nor can it detect elements in H_n that are not in $\cup F_pH_n$.
We say that the spectral sequence $\left\{E^r_{pq}\right\}$ approaches H_* (or **abuts** to H_*) if it weakly converges to H_* and we also have $H_n = \cup F_pH_n$ and $\cap F_pH_n = 0$ for all n. Every weakly convergent spectral sequence approaches $\cup F_pH_*/\cap F_pH_*$.

We say that a spectral sequence is **regular** if for each p and q the differentials d^r_{pq} (or d^{pq}_r) leaving E^r_{pq} (or E^{pq}_r) are zero for all large r.
Regularity is the most useful general condition for convergence used in practice; bounded below spectral sequences are also regular. Note that a spectral sequence is regular iff for each p and $q : Z^\infty_{pq} = Z^r_{pq}$ for all large r.

We say that the spectral sequence **converges** to H_* if it approaches H_*, it is regular, and $H_n = \lim \left(H_n/F_pH_n\right)$ for each n.
A bounded below spectral sequence converges to H_* whenever it approaches H_*, because the inverse limit condition is always satisfied in a bounded below spectral sequence.
We say that a map $h : H_* \to H'_*$ is compatible with a morphism $f : E \to E'$ if h maps F_pH_n to $F_pH'_n$ and the associated maps $F_pH_n/F_{p-1}H_n \to F_pH'_n/F_{p-1}H'_n$ correspond under β and β' to $f^\infty_{pq} : E^\infty_{pq} \to E'_{pq} \quad (q = n - p)$

Theorem 9 (Comparison Theorem)**.** *Let $\left\{E^r_{pq}\right\}$ and $\left\{E'^r_{pq}\right\}$ converge to H_* and H'_*, respectively. Suppose given a map $h : H_* \to H'_*$ compatible with a morphism $f : E \to E'$ of spectral sequences. If*

$f^r : E^r_{pq} \cong E'^r_{pq}$ *is an isomorphism for all* p *and* q *and some* r *(hence for* $r = \infty$ *by the Mapping Lemma), then* $h : H_* \to H'_*$ *is an isomorphism.*

Filtered Chains

A filtration F on a chain complex C is an ordered family of chain subcomplexes $\cdots \subseteq F_{p-1}C \subseteq F_pC \subseteq \cdots$ of C. The filtration is **exhaustive** if $C = \cup F_pC$.
A filtration on a chain complex C is called **bounded** if for each n there are integers $s < t$ such that $F_sC_n = 0$ and $F_tC_n = C_n$. In this case, there are only finitely many nonzero terms of total degree n in E^0_{**}, so the spectral sequence is bounded.
The filtration is called **bounded below** if for each n there is an integer s so that $F_sC_n = 0$, and it is called **bounded above** if for each n there is a t so that $F_tC_n = C_n$. Bounded filtrations are bounded above and below. Being bounded above is merely an easy way to ensure that a filtration is exhaustive.

Example 14. *We call the filtration* ***canonically bounded*** *if* $F_{-1}C = 0$ *and* $F_nC_n = C_n$ *for each* n*. As* $E^0_{pq} = F_pC_{p+q}/F_{p-1}C_{p+q}$*, every canonically bounded filtration gives rise to a first quadrant spectral sequence (converging to* $H_*(C)$ *). For example, the Leray- Serre spectral sequence arises from a canonically bounded filtration of the singular chain complex* $S_*(E)$*.*

Theorem 10 (Construction of a spectral sequence)**.** *A filtration* F *of a chain complex* C *naturally determines a spectral sequence starting with* $E^0_{pq} = F_pC_{p+q}/F_{p-1}C_{p+q}$ *and* $E^1_{pq} = H_{p+q}\left(E^0_{p*}\right)$*.*

A filtration on a chain complex C is called **Hausdorff** if $\cap F_pC = 0$. It will be clear from the construction that both C and its Hausdorff quotient $C^h = C/\cap F_pC$ give rise to the same spectral sequence.
A filtration on C is called **complete** if $C = \lim C/F_pC$. Complete filtrations are Hausdorff because $\cap F_pC$ is the kernel of the map from C to its completion $\widehat{C} = \lim C/F_pC$ (which is also a filtered complex: $F_n\widehat{C} = \underleftarrow{\lim} F_nC/F_pC\Big)$.
Bounded below filtrations are complete, and hence Hausdorff, because $F_sH_n(C) = 0$ for each n.

Corollary 3. *The two spectral sequences arising from* C *and* $\widehat{C}$ *are the same.*

A filtration on a chain complex C induces a filtration on the homology of $C : F_pH_n(C)$ is the image of the map $H_n\left(F_pC\right) \to H_n(C)$. If the filtration on C is exhaustive, then the filtration on H_n is also exhaustive $\left(H_n = \cup F_pH_n\right)$, because every element of H_n is represented by an element c of some F_pC_n such that $d(c) = 0$. If the filtration on C is bounded below then the filtration on each $H_n(C)$ is also bounded below, since $F_pC = 0$ implies that $F_pH_n(C) = 0$. But this not happen with Hausdorff condition.

Theorem 11 (Classical convergence)**.**

1. *Suppose that the filtration on* C *is bounded. Then the spectral sequence is bounded and converges to* $H_*(C)$ *:*

$$E^1_{pq} = H_{p+q}\left(F_pC/F_{p-1}C\right) \Rightarrow H_{p+q}(C).$$

2. *Suppose that the filtration on* C *is bounded below and exhaustive. Then the spectral sequence is bounded below and also converges to* $H_*(C)$*. Moreover, the convergence is natural in the sense that if* $f : C \to C'$ *is a map of filtered complexes, then the map* $f_* : H_*(C) \to H_*(C')$ *is compatible with the corresponding map of spectral sequences.*

Theorem 12 (Complete convergence)**.** *Suppose that the filtration on* C *is complete and exhaustive and the spectral sequence is regular (5.2.10). Then:*

1. 1. *The spectral sequence weakly converges to $H_*(C)$.*
2. *If the spectral sequence is bounded above, it converges to $H_*(C)$.*

2.5.3 Spectral sequences from double complexes

There are two filtrations associated to every double complex C (seen as a complex of complexes), resulting in two spectral sequences related to the homology of $\mathrm{Tot}(C)$, each one with interesting properties. The interplay between them is the key of many calculations.

Filtration by columns. If $C = C_{**}$ is a double complex, we may filter the (product or direct sum) total complex $\mathrm{Tot}(C)$ by the columns of C, letting ${}^{I}F_n \mathrm{Tot}(C)$ be the total complex of the double subcomplex $\left({}^{I}\tau_{\leq n}C\right)_{pq} = \begin{cases} C_{pq} & \text{if } p \leq n \\ 0 & \text{if } p > n \end{cases}$ of C. This gives rise to a spectral sequence $\left\{{}^{I}E^r_{pq}\right\}$, starting with ${}^{I}E^0_{pq} = C_{pq}$. The maps d^0 are just the vertical differentials d^v of C, so

$$ {}^{I}E^1_{pq} = H^v_q\left(C_{p*}\right) $$

The maps $d^1 : H^v_q\left(C_{p*}\right) \to H^v_q\left(C_{p-1,*}\right)$ are induced on homology from the horizontal differentials d^h of C, so we may use the suggestive notation:

$$ {}^{I}E^2_{pq} = H^h_p H^v_q(C) $$

If C is a first quadrant double complex, the filtration is canonically bounded, and we have the convergent spectral sequence as in the previous section:

$$ {}^{I}E^2_{pq} = H^h_p H^v_q(C) \Rightarrow H_{p+q}(\mathrm{Tot}(C)) $$

Filtration by rows. If C is a double complex, we may also filter $\mathrm{Tot}(C)$ by the rows of C, letting ${}^{II}F_n \mathrm{Tot}(C)$ be the total complex of $\left({}^{II}\tau_{\leq n}C\right)_{pq} = \begin{cases} C_{pq} & \text{if } q \leq n \\ 0 & \text{if } q > n \end{cases}$.
Since $F_p \mathrm{Tot}(C)/F_{p-1}\mathrm{Tot}(C)$ is the row C_{*p}, ${}^{I}E^0_{pq} = C_{qp}$ and ${}^{II}E^1_{pq} = H^h_q\left(C_{*p}\right)$. (Beware the interchange of p and q in the notation!) The maps d^1 are induced from the vertical differentials d^v of C, so we may use the suggestive notation

$$ {}^{II}E^2_{pq} = H^v_p H^h_q(C). $$

Of course, this should not be surprising, since interchanging the roles of p and q converts the filtration by rows into the filtration by columns, and interchanges the spectral sequences ${}^{I}E$ and ${}^{I1}E$.

As before, if C is a first quadrant double complex, this filtration is canonically bounded, and the spectral sequence converges to $H_* \mathrm{Tot}(C)$.

We can prove the balancing property of Tor using both spectral sequences. We can also prove the Künneth formula, the Univeral Coefficient Theorem and the Acyclic Assembly Lemma from the following result:

Theorem 13 (Künneth spectral sequence)**.** *Let P be a bounded below complex of flat R-modules and M an R-module. Then there is a boundedly converging right half-plane spectral sequence*

$$ E^2_{pq} = \mathrm{Tor}^R_p\left(H_q(P), M\right) \Rightarrow H_{p+q}\left(P \otimes_R M\right) $$

Hypercohomology

Let $\mathcal{A}$ be an abelian category that has enough projectives. A **(left) Cartan-Eilenberg resolution** P_{**} of a chain complex A_* in $\mathcal{A}$ is an upper half-plane double complex ($P_{pq} = 0$ if $q < 0$), consisting of projective objects of $\mathcal{A}$, together with a chain map ("augmentation") $P_{*0} \xrightarrow{\epsilon} A_*$ such that for every p

1. If $A_p = 0$, the column P_{p*} is zero.
2. The maps on boundaries and homology

$$B_p(\epsilon) : B_p\left(P, d^h\right) \to B_p(A)$$
$$H_p(\epsilon) : H_p\left(P, d^h\right) \to H_p(A)$$

   are projective resolutions in $\mathcal{A}$. Here $B_p\left(P, d^h\right)$ denotes the horizontal boundaries in the (p,q) spot, that is, the chain complex whose q^{th} term is $d^h\left(P_{p+1,q}\right)$. The chain complexes $Z_p\left(P, d^h\right)$ and $H_p\left(P, d^h\right) = Z_p\left(P, d^h\right) / B_p\left(P, d^h\right)$ are defined similarly.

Lemma 1. *Every chain complex has a Cartan-Eilenberg resolution.*

Let $f, g : D \to E$ be two maps of double complexes. A **chain homotopy** from f to g consists of maps $s^h_{pq} : D_{pq} \to E_{p+1,q}$ and $s^v_{pq} : D_{pq} \to E_{p,q+1}$ so that

$$g - f = (d^h s^h + s^h d^h) + (d^v s^v + s^v d^v)$$
$$s^v d^h + d^h s^v = s^h d^v + d^v s^h = 0.$$

This definition is set up so that $\left\{s^h + s^v \ : \ \mathrm{Tot}(D)_n \to \mathrm{Tot}(E)_{n+1}\right\}$ forms an ordinary chain homotopy between the maps $\mathrm{Tot}(f)$ and $\mathrm{Tot}(g)$ from $\mathrm{Tot}^{\oplus}(D)$ to $\mathrm{Tot}^{\oplus}(E)$.

Proposition 17. 1. *If $f, g : A \to B$ are homotopic maps of chain complexes, and $\tilde{f}, \tilde{g} : P \to Q$ are maps of Cartan-Eilenberg resolutions lying over them, show that $\tilde{f}$ is chain homotopic to $\tilde{g}$.*

2. *Show that any two Cartan-Eilenberg resolutions P, Q of A are chain homotopy equivalent. Conclude that for any additive functor F the chain complexes $\mathrm{Tot}^{\oplus}(F(P))$ and $\mathrm{Tot}^{\oplus}(F(Q))$ are chain homotopy equivalent.*

Let $F : \mathcal{A} \to \mathcal{B}$ be a right exact functor, and assume that $\mathcal{A}$ has enough projectives. If A is a chain complex in $\mathcal{A}$ and $P \to A$ is a Cartan-Eilenberg resolution, define $\mathbb{L}_i F(A)$ to be $H_i \,\mathrm{Tot}^{\oplus}(F(P))$. The Proposition shows that $\mathbb{L}_i F(A)$ is independent of the choice of P.
If $f : A \to B$ is a chain map and $\tilde{f} : P \to Q$ is a map of Cartan-Eilenberg resolutions over f, define $\mathbb{L}_i F(f)$ to be the map $H_i(\mathrm{Tot}(\tilde{f}))$ from $\mathbb{L}_i F(A)$ to $\mathbb{L}_i F(B)$. The Proposition implies that $\mathbb{L}_i F$ is a functor from $\mathrm{Ch}(\mathcal{A})$ to $\mathcal{B}$, at least when $\mathcal{B}$ is cocomplete. The $\mathbb{L}_i F$ are called the left hyper-derived functors of F.
If $\mathcal{B}$ is not cocomplete, $\mathrm{Tot}^{\oplus}(F(P))$ and $\mathbb{L}_i F(A)$ may not exist for all chain complexes A. In this case we restrict to the category $\mathrm{Ch}_+(\mathcal{A})$ of all chain complexes A which are bounded below in the sense that there is a p_0 such that $A_p = 0$ for $p < p_0$. Since $P_{pq} = 0$ if $p < p_0$ or $q < 0$, $\mathrm{Tot}^{\oplus}(F(P))$ exists in $\mathbf{Ch}(\mathcal{B})$ and we may consider $\mathbb{L}_i F$ to be a functor from $\mathrm{Ch}_+(\mathcal{A})$ to $\mathcal{B}$.

Lemma 2. *If $0 \to A \to B \to C \to 0$ is a short exact sequence of bounded below complexes, there is a long exact sequence*

$$\cdots \mathbb{L}_{i+1} F(C) \xrightarrow{\delta} \mathbb{L}_i F(A) \to \mathbb{L}_i F(B) \to \mathbb{L}_i F(C) \xrightarrow{\delta} \cdots$$

Proposition 18. *There is always a convergent spectral sequence*

$$ {}^{II}E^2_{pq} = \left(L_pF\right)\left(H_q(A)\right) \Rightarrow \mathbb{L}_{p+q}F(A). $$

If A is bounded below, there is a convergent spectral sequence

$$ {}^{I}E^2_{pq} = H_p\left(L_qF(A)\right) \Rightarrow \mathbb{L}_{p+q}F(A) $$

Corollary 4.

1. *If A is exact, $\mathbb{L}_iF(A) = 0$ for all i.*
2. *Any quasi-isomorphism $f : A \to B$ induces isomorphisms*

$$ \mathbb{L}_*F(A) \cong \mathbb{L}_*F(B) $$

3. *If each A_p is F-acyclic (2.4.3), that is, $L_qF\left(A_p\right) = 0$ for $q \neq 0$, and A is bounded below, then*

$$ \mathbb{L}_pF(A) = H_p(F(A)) \text{ for all } p $$

we can understand all these result in the more general context of derived categories and functors.

Example 15. *Let X be a topological space and $\mathcal{F}^*$ a cochain complex of sheaves on X. The hypercohomology $\mathbb{N}^i\left(X, \mathcal{F}^*\right)$ is $\mathbb{R}^i\Gamma\left(\mathcal{F}^*\right)$, where Γ is the global sections functor. This generalizes sheaf cohomology to complexes of sheaves, and if $\mathcal{F}^*$ is a bounded below complex of injective sheaves, then $\mathbb{H}^i\left(X, \mathcal{F}^*\right) = \boldsymbol{H}^i\left(\Gamma\left(\mathcal{F}^*\right)\right)$. The hypercohomology spectral sequence is ${}^{11}E_2^{pq} = H^p\left(X, H^q\left(\mathcal{F}^*\right)\right) \Rightarrow \mathbb{H}^{p+q}\left(X, \mathcal{F}^*\right)$.*

Grothenidieck spectral sequence

Cohomological Setup. Let $\mathcal{A}, \mathcal{B}$, and $\mathcal{C}$ be abelian categories such that both $\mathcal{A}$ and $\mathcal{B}$ have enough injectives. We are given left exact functors $G : \mathcal{A} \to \mathcal{B}$ and $F : \mathcal{B} \to \mathcal{C}$.

Let $F : B \to C$ be a left exact functor. An object B of $\mathcal{B}$ is called F**-acyclic** if the derived functors of F vanish on B, that is, if $R^iF(B) = 0$ for $i \neq 0$. (Compare with 2.4.3.)

Theorem 14 (Grothendieck Spectral Sequence Theorem)**.** *Let $\mathcal{A}, \mathcal{B}$, and $\mathcal{C}$ be abelian categories such that both $\mathcal{A}$ and $\mathcal{B}$ have enough projectives. Suppose given right exact functors $G : \mathcal{A} \to \mathcal{B}$ and $F : \mathcal{B} \to \mathcal{C}$ such that G sends projective objects of $\mathcal{A}$ to F-acyclic objects of $\mathcal{B}$. Then there is a convergent first quadrant homology spectral sequence for each A in $\mathcal{A}$:*

$$ E^2_{pq} = \left(L_pF\right)\left(L_qG\right)(A) \Rightarrow L_{p+q}(FG)(A). $$

The exact sequence of low degree terms is

$$ L_2(FG)A \to \left(L_2F\right)(GA) \to F\left(L_1G(A)\right) \to L_1(FG)A \to \left(L_1F\right)(GA) \to 0. $$

Example 16 (Leray Spectral Sequence)**.** *Let $f : X \to Y$ be a continuous map of topological spaces. The direct image sheaf functor f_* (2.6.6) has the exact functor f^{-1} as its left adjoint (exercise 2.6.2), so f_* is left exact and preserves injectives by 2.3.10. If $\mathcal{F}$ is a sheaf of abelian groups on X, the global sections of $f_*\mathcal{F}$ is the group $\left(f_*\mathcal{F}\right)(Y) = \mathcal{F}\left(f^{-1}Y\right) = \mathcal{F}(X)$. Thus we are in the situation*

The Grothendieck spectral sequence in this case is called the Leray spectral sequence: Since $R^p\Gamma$ is sheaf cohomology (2.5.4), it is usually written as

$$ E_2^{pq} = H^p\left(Y; R^qf_*\mathcal{F}\right) \Rightarrow H^{p+q}(X; \mathcal{F}) $$

Chapter 3

Group (Cohomology) Theory

A **semigroup** is a nonempty set G together with a binary operation on G which is associative. A **monoid** is a semigroup G which contains a (two-sided) identity element. A **group** is a monoid G such that for every element there exists a (two-sided) inverse element.

Theorem 15. *Let G be a finitely generated abelian group.*

1. *There is a unique nonnegative integer s such that the number of infinite cyclic summands in any decomposition of G as a direct sum of cyclic groups is precisely s;*

2. *either G is free abelian or there is a unique list of (not necessarily distinct) positive integers* $m_1, \ldots, m_t$ *such that* $m_1 > 1,\ m_1 \mid m_2 | \cdots \mid m_t$ *and*

$$G \cong \mathbf{Z}_{m_1} \oplus \cdots \oplus \mathbf{Z}_{m_t} \oplus F$$

with F free abelian;

3. *either G is free abelian or there is a list of positive integers* $p_1^{s_1}, \ldots, p_k^{s_k}$*, which is unique except for the order of its members, such that* $p_1, \ldots, p_k$ *are (not necessarily distinct) primes,* $s_1, \ldots, s_k$ *are (not necessarily distinct) positive integers and*

$$G \cong Z_{p_1} s_1 \oplus \ldots \oplus Z_{p_k} s_k \oplus F$$

with F free abelian.

3.1 Actions

[2] An action of a group G on a set S is a function $G \times S \to S$ (usually denoted by $(g, x) \mapsto gx$) such that for all $x \in S$ and $g_1,\ g_2 \in G$:

$$ex = x \quad \text{and} \quad (g_1\, g_2)\, x = g_1\, (\, g_2 x)\,.$$

When such an action is given, we say that G acts on the set S. The **orbit of** $x \in X$ is $Gx = \{gx \mid g \in G\}$ and its **stabilizer** (or isotropy group) is $G_x = \{g \in G \mid gx = x\}$.

Theorem 16. 1. *Orbits have cardinality equal to the index of the corresponding stabilizer.*

2. *The number of elements in the conjugacy class of* $x \in G$ *is* $[G : C_G(x)]$*, which divides* $|G|$*;*

3. ***(Class equation)*** *If* $\overline{\mathrm{x}}_1, \ldots, \overline{\mathrm{x}}_{\mathrm{n}}\ (\mathrm{x_i} \in \mathrm{G})$ *are the distinct conjugacy classes of G , then*

$$|\mathrm{G}| = \sum_{i=1}^{n} [\mathrm{G} : \mathrm{C_G}(\mathrm{x_i})]$$

In particular, we can take G acting on itself by conjugation, so that the conjugacy classes are the orbits of this action.

4. *The number of subgroups of G conjugate to K is* $[\mathrm{G} : \mathrm{N_G}(\mathrm{K})]$, *which divides* $|\mathrm{G}|$.

Let G and H be groups and $\theta : H \to \operatorname{Aut} G$ a homomorphism. Let $G \times_\theta H$ be the set $G \times H$ with the following binary operation: $(g,h)(g',h') = (g[\theta(h)(g')], hh')$. Show that $G \times_\theta H$ is a group with identity element (e,e) and $(g,h)^{-1} = (\theta(h^{-1})(g^{-1}), h^{-1})$.$G \times_\theta H$ is called the of G and H.

Group rings Let G be a (multiplicative) group. Let $\mathbb{Z}G$ be the free $\mathbb{Z}$-module generated by the elements of G. The multiplication in G extends uniquely to a $\mathbb{Z}$-bilinear product $\mathbb{Z}\mathbb{G} \times \mathbb{Z}\mathbb{G} \to \mathbb{Z}\mathbb{G}$; this makes $\mathbb{Z}\mathbb{G}$ a ring, called the **(integral) group ring** of G.

Note that G is a subgroup of the group $(\mathbb{Z}G)^*$ of units of $\mathbb{Z}\mathbb{G}$

Theorem 17 (Universal property)**.** *Given a ring R and a group homomorphism* $f : G \to R^*$, *there is a unique extension of f to a ring homomorphism* $\mathbb{Z}\mathbb{G} \to R$. *Thus we have the "adjunction formula"*

$$\mathrm{Hom}_{(rings)}(\mathbb{Z}G, R) \approx \mathrm{Hom}_{(groups)}(G, R^*).$$

A **(left)** $\mathbb{Z}G$**-module**, or G-module, consists of an abelian group A together with a homomorphism from $\mathbb{Z}G$ to the ring of endomorphisms of A. By the universal property, G-module is simply an abelian group A together with an action of G on A. For example, one has for any A the trivial module structure, with $ga = a$ for $g \in G, a \in A$.

One way of constructing G-modules is by linearizing permutation representations. More precisely, if X is a G-set (i.e., a set with G-action), then one forms the free abelian group $\mathbb{Z}\mathbb{X}$ (also denoted $\mathbb{Z}[X]$) generated by X and one extends the action of G on X to a $\mathbb{Z}$-linear action of G on $\mathbb{Z}X$. The resulting G-module is called a permutation module. In particular, one has a permutation module $\mathbb{Z}[G/H]$ for every subgroup H of G, where G/H is the set of cosets gH and G acts on G/H by left translation.

Proposition 19. *Let X be a free G-set and let E be a set of representatives for the G-orbits in X. Then* $\mathbb{Z}X$ *is a free* $\mathbb{Z}G$*-module with basis E.*

3.2 Co-invariants

If G is a group and M is a G-module, then the group of co-invariants of M, denoted M_G, is defined to be the quotient of M by the additive subgroup generated by the elements of the form $gm - m\ (g \in G, m \in M\ )$. Thus M_G is obtained from M by "dividing out" by the G-action. (The name "co-invariants" comes from the fact that M_G is the largest quotient of M on which G acts trivially, whereas M^G, the group of invariants, is the largest submodule of M on which G acts trivially.) In view of exercise 1a of \$$I$.2, we can also describe M_G as M/IM, where I is the augmentation ideal of $\mathbb{Z}G$ and IM denotes the set of all finite sums $\sum a_i b_i\ (a_i \in I, b_i \in M)$. Still another description of M_G is given by:

$$M_G \approx \mathbb{Z} \otimes_{\mathbb{Z}G} M.$$

Here, in order for the tensor product to make sense, we regard $\mathbb{Z}$ as a right $\mathbb{Z}G$-module (with trivial G-action). To prove 2.1 , note that in $\mathbb{Z} \otimes_{\mathbb{Z}G} M$ we have the identity $1 \otimes gm = 1 \cdot g \otimes m =$

$1 \otimes m$; hence there is a map $M_G \to \mathbb{Z} \otimes_{\mathbb{Z}G} M$ given by $\bar{m} \mapsto 1 \otimes m$, where $\bar{m}$ denotes the image in M_G of an element $m \in M$. On the other hand, using the universal property of the tensor product, we can define a map $\mathbb{Z} \otimes_{\mathbb{Z}G} M \to M_G$ by $a \otimes m \mapsto a\bar{m}$. These two maps are inverses of one another.

$$\text{Also} M^G \simeq \text{Hom}_G(\mathbb{Z}, M)$$

In view of 2.1 and standard properties of the tensor product, we immediately obtain the following two properties of the co-invariants functor:

1. Right-exactness: Given an exact sequence $M' \to M \to M'' \to 0$ of G-modules, the induced sequence $M'_G \to M_G \to M''_G \to 0$ is exact.

2. If F is a free $\mathbb{Z}G$-module with basis (e_i), then F_G is a free $\mathbb{Z}$-module with basis $(\bar{e}_i)$.

Proposition 20. *Let X be a free G-complex and let Y be the orbit complex X/G. Then $C_*(Y) \approx C_*(X)_G$.*

3.3 Cohomology

References [18]. Let A be a G-module. We write $H_*(G; A)$ for the left derived functors $L_*\left(-_G\right)(A)$ and call them the homology groups of G with cO efficients in A; by the lemma above,

$$H_*(G; A) \cong \text{Tor}_*^{\mathbb{Z}G}(\mathbb{Z}, A)$$

By definition, $H_0(G; A) = A_G$. Similarly, we write $H^*(G; A)$ for the right derived functors $R^*\left(^G\right)(A)$ and call them the cohomology groups of G with coefficients in A; by the lemma above,

$$H^*(G; A) \cong \text{Ext}^*_{\mathbb{Z}G}(\mathbb{Z}, A)$$

By definition, $H^0(G; A) = A^G$

Example 17. 1. *If $G = 1$ is the trivial group, $A_G = A^G = A$. Since the higher derived functors of an exact functor vanish, $H_*(1; A) = H^*(1; A) = 0$ for $* \neq 0$.*

2. *Let G be the infinite cyclic group T with generator t. We may identify $\mathbb{Z}T$ with the Laurent polynomial ring $\mathbb{Z}\left[t, t^{-1}\right]$. Since the sequence*

$$0 \to \mathbb{Z}T \xrightarrow{t-1} \mathbb{Z}T \to \mathbb{Z} \to 0$$

is exact,

$$H_n(T; A) = H^n(T; A) = 0 \text{ for } n \neq 0, 1, \text{ and}$$
$$H_1(T; A) \cong H^0(T; A) = A^T, H^1(T; A) \cong H_0(T; A) = A_T$$

In particular, $H_1(T; \mathbb{Z}) = H^1(T; \mathbb{Z}) = \mathbb{Z}$. We will see in the next section that all free groups display similar behavior, because $pd_G(\mathbb{Z}) = 1$.

The **augmentation ideal** of $\mathbb{Z}G$ is the kernel $\mathfrak{I}$ of the ring map $\mathbb{Z}G \xrightarrow{\epsilon} \mathbb{Z}$ which sends $\sum n_g g$ to $\sum n_g$. Because $\{1\} \cup \{g - 1 : g \in G, g \neq 1\}$ is a basis for $\mathbb{Z}G$ as a free $\mathbb{Z}$-module, it follows that $\mathfrak{I}$ is a free $\mathbb{Z}$ module with basis $\{g - 1 : g \in G, g \neq 1\}$.

Example 18. 1. *Since the trivial G-module $\mathbb{Z}$ is $\mathbb{Z}G/\mathfrak{I}$, $H_0(G; A) = A_G$ is isomorphic to $\mathbb{Z} \otimes_{\mathbb{Z}G} A = \mathbb{Z}G/\mathcal{I} \otimes_{\mathbb{Z}G} A \cong A/\mathcal{I}A$ for every G-module A. For example, $H_0(G; \mathbb{Z}) = \mathbb{Z}/\mathfrak{I}\mathbb{Z} = \mathbb{Z}$, $H_0(G; \mathbb{Z}G) = \mathbb{Z}G/\mathfrak{I} \cong \mathbb{Z}$, and $H_0(G; \mathfrak{I}) = \mathfrak{I}/\mathfrak{I}^2$*

2. *Because $\mathbb{Z}G$ is a projective object in $\mathbb{Z}G$-mod, $H_*(G;\mathbb{Z}G) = 0$ for $* \neq 0$ and $H_0(G;\mathbb{Z}G) = \mathbb{Z}$. When G is a finite group, Shapiro's Lemma (6.3.2 below) implies that $H^*(G;\mathbb{Z}G) = 0$ for $* \neq 0$. This fails when G is infinite; for example, we saw in 6.1 .4 that $H^1(T;\mathbb{Z}T) \cong \mathbb{Z}$ for the infinite cyclic group T. If G is finite, then $H^0(G;\mathbb{Z}G) \cong \mathbb{Z}$, but $H^0(G;\mathbb{Z}G) = 0$ if G is infinite.*

Theorem 18 (H_1). *For any group G, $H_1(G;\mathbb{Z}) \cong \mathfrak{I}/\mathfrak{I}^2 \cong G/[G,G]$.*

Theorem 19 (Trivial $G-$module). *If A is any trivial G-module, $H_0(G;A) \cong A$, $H_1(G;A) \cong G/[G,G] \otimes_{\mathbb{Z}} A$, and for $n \geq 2$ there are (noncanonical) isomorphisms:*

$$H_n(G;A) \cong H_n(G;\mathbb{Z}) \otimes_{\mathbb{Z}} A \oplus \mathrm{Tor}_1^{\mathbb{Z}}\left(H_{n-1}(G;\mathbb{Z}),A\right)$$

Spectral sequence

If A_* is a chain complex of G-modules, the hyperderived functors $\mathbb{L}_i(-G)\left(A_*\right)$ of 5.7.4 are written as $\mathbb{H}_i\left(G;A_*\right)$ and called the hyperhomology groups of G. Similarly, if A^* is a cochain complex of G modules, the hypercohomology groups $\mathbb{H}^i\left(G;A^*\right)$ are just the hyper-derived functors $\mathbb{R}^i\left(-^G\right)\left(A^*\right)$. The generalities of Chapter 5 , section 7 become the following facts in this case. The hyperhomology spectral sequences are

$$\begin{aligned} &{}^{II}E^2_{pq} = H_p\left(G;H_q\left(A_*\right)\right) \Rightarrow \mathbb{H}_{p+q}\left(G;A_*\right);\ \text{and} \\ &{}^{I}E^2_{pq} = H_p\left(H_q\left(G;A_*\right)\right) \Rightarrow \mathbb{H}_{p+q}\left(G;A_*\right)\ \text{ when } A_* \text{ is bounded below,} \end{aligned}$$

In particular, suppose that A is bounded below. If each A_i is a flat $\mathbb{Z}G$-module, then $\mathbb{H}_i\left(G;A_*\right) = H_i\left(\left(A_*\right)_G\right)$; if each A^i is a projective $\mathbb{Z}G$-module, then $H^i\left(G;A^*\right) = H^i\left(\left(A^*\right)^G\right)$.

3.4 Cyclic and Free Groups Cohomology

Theorem 20. *If A is a module for the cyclic group $G = C_m$, then*

$$H_n\left(C_m;A\right) = \left\{\begin{array}{ll} A/(\sigma-1)A & \textit{if } n=0 \\ A^G/NA & \textit{if } n=1,3,5,7,\ldots\}; \\ \{a\in A: Na=0\}/(\sigma-1)A & \textit{if } n=2,4,6,8,\ldots \end{array}\right\};$$

$$H^n\left(C_m;A\right) = \left\{\begin{array}{ll} A^G & \textit{if } n=0 \\ \{a\in A: Na=0\}/(\sigma-1)A & \textit{if } n=1,3,5,7,\ldots \\ A^G/NA & \textit{if } n=2,4,6,8,\ldots \end{array}\right\}.$$

$$H_n\left(C_m;\mathbb{Z}\right) = \left\{\begin{array}{ll} \mathbb{Z} & \text{if } n=0 \\ \mathbb{Z}/m & \text{if } n=1,3,5,7,\ldots \\ 0 & \text{if } n=2,4,6,8,\ldots \end{array}\right\};$$

$$H^n\left(C_m;\mathbb{Z}\right) = \left\{\begin{array}{ll} \mathbb{Z} & \text{if } n=0 \\ 0 & \text{if } n=1,3,5,7,\ldots \\ \mathbb{Z}/m & \text{if } n=2,4,6,8,\ldots \end{array}\right\}$$

Tate Cohomology. Taking full advantage of this periodicity, we set

$$\hat{H}^n\left(C_m;A\right) = \left\{\begin{array}{ll} A^G/NA & \text{if } n\in\mathbb{Z} \text{ is even} \\ \{a\in A: NA=0\}/(\sigma-1)A & \text{if } n\in\mathbb{Z} \text{ is odd} \end{array}\right\}$$

More generally, if G is a finite group and A is a G-module, we define the Tate cohomology groups of G to be the groups

$$\hat{H}^n(G;A) = \left\{ \begin{array}{ll} H^n(G;A) & \text{if } n \geq 1 \\ A^G/NA & \text{if } n = 0 \\ \{a \in A : Na = 0\}/\mathcal{J}A & \text{if } n = -1 \\ H_{1-n}(G;A) & \text{if } n \leq -2 \end{array} \right\}$$

Example 19 (Dimension-shifting)**.** *Given a G-module A (G finite), choose a short exact sequence $0 \to K \to P \to A \to 0$ with P projective. Shapiro's Lemma (below) implies that $\hat{H}^*(G,P) = 0$ for all $* \in \mathbb{Z}$. Therefore $\hat{H}^n(G;A) \cong \hat{H}^{n+1}(G;K)$. This shows that every Tate cohomology group $\hat{H}^n(G;A)$ determines the entire theory.*

Proposition 21. *Let G be the free group on the set X. Then the augmentation ideal $\mathfrak{I}$ is a free $\mathbb{Z}G$-module with basis the set $X - 1 = \{x - 1 : x \in X\}$. Also, $0 \to \mathfrak{I} \to \mathbb{Z}G \to \mathbb{Z} \to 0$ is a free resolution of $\mathbb{Z}$. Consequently, $\mathrm{pd}_G(\mathbb{Z}) = 1$, that is, $H_n(G;A) = H^n(G;A) = 0$ for $n \neq 0, 1$. Moreover, $H_0(G;\mathbb{Z}) \cong H^0(G;\mathbb{Z}) \cong \mathbb{Z}$, while*

$$H_1(G;\mathbb{Z}) \cong \bigoplus_{x \in X} \mathbb{Z} \quad \textit{and} \quad H^1(G;\mathbb{Z}) \cong \prod_{x \in X} \mathbb{Z}$$

Stallings and Swan proved the converse!

3.5 Calculations with Shapiro's Lemma

If H is a subgroups of G, $\mathbb{Z}G \otimes_{\mathbb{Z}H} A$ is called the **induced G-module** and is written $\mathrm{Ind}_H^G(A)$. Similarly, $\mathrm{Hom}_H(\mathbb{Z}G, A)$ is called the **coinduced G-module** and is written $\mathrm{Coind}_H^G(A)$.

Theorem 21 (Shapiro's Lemma)**.** *Let H be a subgroup of G and A an H-module. Then*

$$H_*\left(G; \mathrm{Ind}_H^G(A)\right) \cong H_*(H;A); \textit{ and } H^*\left(G; \mathrm{Coind}_H^G(A)\right) \cong H^*(H;A)$$

Corollary 5. *1. If A is an abelian group, then*

$$H_*\left(G; \mathbb{Z}G \otimes_{\mathbb{Z}} A\right) = H^*\left(G; \mathrm{Hom}_{\mathrm{Ab}}(\mathbb{Z}G, A)\right) = \left\{ \begin{array}{ll} A & \textit{if } * = 0 \\ 0 & \textit{if } * \neq 0 \end{array} \right\}$$

2. If G is a finite group, then $H^*\left(G; \mathbb{Z}G \otimes_{\mathbb{Z}} A\right) = 0$ for $* \neq 0$ and all A.

3. If G is finite and P is a projective G module,

$$\widehat{H}^*(G;P) = 0 \textit{ for all } *.$$

Theorem 22 (Hilbert 90, additive version)**.** *Let $K \subset L$ be a finite Galois extension of fields, with Galois group G. Then L is a G-module, $L^G \cong L_G \cong K$, and*

$$H^*(G;L) = H_*(G;L) = 0 \textit{ for } * \neq 0$$

Example 20 (Cyclic Galois extensions)**.** *Suppose that G is cyclic of order m, generated by σ. The trace $\mathrm{tr}(x)$ of an element $x \in L$ is the element $x + \sigma x + \cdots + \sigma^{m-1}x$ of K. In this case, Hilbert's Theorem 90 states that there is an exact sequence*

$$0 \to K \to L \xrightarrow{\sigma - 1} L \xrightarrow{tr} K \to 0.$$

Indeed, we saw in the last section that for $* \neq 0$ every group $H_*(G;L)$ and $H^*(G;L)$ is either $K/\operatorname{tr}(L)$ or $\ker(\operatorname{tr})/(\sigma-1)K$.

As an application, suppose that $\operatorname{char}(K) = p$ and that $[L:K] = p$. Since $\operatorname{tr}(1) = p \cdot 1 = 0$, there is an $x \in L$ such that $(\sigma - 1)x = 1$, that is, $\sigma x = x + 1$. Hence $L = K(x)$ and $x^p - x \in K$ because

$$\sigma\left(x^p - x\right) = (x+1)^p - (x+1) = x^p - x$$

3.6 Universal Central Extensions

A **central extension** of G is an extension $0 \to A \to X \xrightarrow{\pi} G \to 1$ such that A is in the center of X. (If π and A are clear from the context, we will just say that X is a central extension of G.) A homomorphism over G from X to another central extension $0 \to B \to Y \xrightarrow{\tau} G \to 1$ of G is a map $f : X \to Y$ such that $\pi = \tau f$. X is called a universal central extension of G if for every central extension $0 \to B \to Y \xrightarrow{\tau} G \to 1$ of G there exists a unique homomorphism f from X to Y over G.

Clearly, a universal central extension is unique up to isomorphism over G, provided that it exists.

A group G is **perfect** if it equals its commutator gorup $[G,G]$, or equivalently, if $H_1(G;\mathbb{Z}) = 0$.

Proposition 22. *1. Universal central extensions of perfect groups are perfect.*

2. If $0 \to A \to X \to G \to 1$ is any central extension in which G and X are perfect groups, show that $H_1(X;\mathbb{Z}) = 0$ and that there is an exact sequence

$$\mathrm{H}_2(X;\mathbb{Z}) \xrightarrow{cor} H_2(G;\mathbb{Z}) \to A \to 0$$

3. Show that if G is perfect then central extensions $0 \to A \to X \to G \to 1$ are classified by $\operatorname{Hom}\left(\mathrm{H}_2(G;\mathbb{Z}), A\right)$.

Theorem 23. *A group G has a universal central extension if and only if G is perfect. In this case, the universal central extension is*

$$0 \to \mathrm{H}_2(G;\mathbb{Z}) \to \frac{[F,F]}{[R,F]} \xrightarrow{\pi} G \to 1.$$

Here $1 \to R \to F \to G \to 1$ is any presentation of G.

Proposition 23 (Recognition Criterion)**.** *A central extension $0 \to A \to X \xrightarrow{\pi} G \to 1$ is universal if and only if X is perfect and every central extension of X splits as a direct product of X with an abelian group.*

Corollary 6. *1. If $0 \to A \to X \to G \to 1$ is a universal central extension, then*

$$H_1(X;\mathbb{Z}) = H_2(X;\mathbb{Z}) = 0.$$

2. If G is a perfect group and $H_2(G;\mathbb{Z}) = 0$, then every central extension of G is a direct product of G with an abelian group.

Example 21. *The smallest perfect group is A_5. The universal central extension of A_5 describes A_5 as the quotient $PSL_2(\mathbb{F}_5)$ of the binary icosahedral group $X = SL_2(\mathbb{F}_5)$ by the center of order 2, $A = \pm \begin{pmatrix} (0) \\ 01 \end{pmatrix}$) [Suz, 2.9].*

$$0 \longrightarrow \mathbb{Z}/2 \xrightarrow{\begin{pmatrix} -1 & 0 \\ 0-1 & \end{pmatrix}} SL_2\left(\mathbb{F}_5\right) \longrightarrow PSL_2\left(\mathbb{F}_5\right) \longrightarrow 1$$

Example 6.9.10 (Alternating groups) It is well known that the alternating groups A_n are perfect if $n \geq 5$. From [Suz, 3.2] we see that

$$H_2\left(A_n ; \mathbb{Z}\right) \cong \left\{\begin{array}{ll} \mathbb{Z}/6 & \text{if } n=6,7 \\ \mathbb{Z}/2 & \text{if } n=4,5 \text{ or } n \geq 8 \\ 0 & \text{if } n=1,2,3 \end{array}\right\}$$

We have already mentioned (6.9.1) the universal central extension of A_5. In general, the regular representation $A_n \rightarrow SO_{n-1}$ gives rise to a central extension

$$0 \rightarrow \mathbb{Z}/2 \rightarrow \widetilde{A}_n \rightarrow A_n \rightarrow 1$$

by restricting the central extension

$$0 \rightarrow \mathbb{Z}/2 \rightarrow \operatorname{Spin}_{n-1}(\mathbb{R}) \rightarrow SO_{n-1} \rightarrow 1$$

If $n \neq 6,7$, $\tilde{A}_n$ must be the universal central extension of A_n.

Example 22. *It is known [Suz, 1.9] that if F is a field, then the special linear group $SL_n(F)$ is perfect, with the exception of $SL_2\left(\mathbb{F}_2\right) \cong D_6$ and $SL_2\left(\mathbb{F}_3\right)$, which is a group of order 24 . The center of $SL_n(F)$ is the group $\mu_n(F)$ of n^{th} roots of unity in F (times the identity matrix I), and the quotient of $SL_n(F)$ by $\mu_n(F)$ is the projective special linear group $PSL_n(F)$. When $F=\mathbb{F}_q$ is a finite field, we know that $H_2\left(SL_n\left(\mathbb{F}_q\right) ; \mathbb{Z}\right)=0$ [Suz, 2.9]. It follows, again with two exceptions, that*

$$0 \rightarrow \mu_n\left(\mathbb{F}_q\right) \xrightarrow{I} SL_n\left(\mathbb{F}_q\right) \rightarrow PSL_n\left(\mathbb{F}_q\right) \rightarrow 1$$

is the universal central extension of the finite group $PSL_n\left(\mathbb{F}_q\right)$.

3.7 An spectral sequence for group cohomology

Suppose that X is a simplicial set and x_i are simplicial subsets such that $X=UX_i$. Then, setting $X_{ij}=X_i \cap X_j$ (etc.) we'11 obviously have for the realisations: $|x|=U\left|x_i\right|,\left|x_i\right| \cap\left|x_j\right|=\left|x_{ij}\right|, \ldots$ Let's suppose that the set of indices is linearly ordered. Consider the following bicomplex:

$$K=\longrightarrow \underset{i<j<k}{\oplus} C_*\left(x_{ijk}\right) \longrightarrow \underset{i<j}{\oplus} C_*\left(x_{ij}\right) \longrightarrow \underset{i}{\oplus} C_*\left(x_i\right)$$

Here by a bicomplex we understand a bicomplex in the sense of Grothendieck [9] i.e. the differentials d_1 and d_2 commute. (The sign in this approach appears in the definition of the total differentials). The vertical arrows of the bicomplex map $C_*\left(x_i \cdots{}_i\right)$ into $\underset{k=0}{q} C_*\left(x_{i_0} \ldots \hat{i}_k \ldots i_q\right)$, the mapping into the kth summand differing $k=0$ by a sign $(-1)^k$ from the natural embedding.

The first spectral sequence of this bicomplex degenerates and yields an isomorphism $H_\star(K) \cong H_\star(X)$. (Moreover this isomorphism is induced by the canonical map $K \rightarrow C_*(X)$). The second spectral sequence gives us a functorial spectral sequence of the first quadrant, whose limit equals $H_*(X)$, while its differential dr has bidegree $(r-1,-r)$ and its E^1-term looks as follows:

$$E_{pqq}^1=\underset{i_0<\ldots<i_q}{\otimes} H_p\left(x_{i_0} \ldots i_q\right)$$

Suppose G is a group. Let X_G denote the simplicial set (and its geometric realisation), whose p-simplices are sequences $\left(g_0, \ldots, g_p\right)$ of elements of G, with the usual faces and degeneracies. This space X_G is contractible by (1.2). The group G acts from the right on X_G and this action is obviously free, hence $BG = X_G/G$ is a classifying space of G. The complex $C_*(BG) = C_*(G)$ coincides with the usual complex associated with G. Moreover $C_*(G) = C_*\left(X_G\right) \otimes_G Z$.

If H is a subgroup of G, then X_G/H is a classifying space for H and hence $BH = X_H/H \rightarrow X_G/H$ is a homotopy equivalence. In particular $C_*(H) + C_*\left(X_G\right) \otimes_H \mathbb{Z} = C_*\left(X_G\right) \otimes_G Z|G/H|$ is a homotopy equivalence.

(2.3) The spectral sequence associated with a family of subgroups.

Suppose G is a group and $G_1, \ldots, G_n$ are subgroups. Then BG_i may be viewed as a simplicial subset of BG and $BG_i \cap BG_j = B\left(G_i \cap G_j\right)$.. Denote UBG_i by X and consider the spectral sequence of the covering $X = UBG_i$. Along with the bicomplex K introduced in (2.1) we also consider the following bicomplex:

$$K' = \underset{i<j<k}{\oplus} C_*\left(X_G\right) \otimes_G Z\left[G/G_{ijk}\right] \longrightarrow \underset{i<j}{\oplus} C_*\left(X_G\right) \otimes_G Z\left[G/G_{ij}\right] \longrightarrow \underset{i<j}{\oplus} C_*\left(X_G\right) \otimes_G Z\left[G/G_i\right]$$

There is a natural mapping of bicomplexes $K + K'$ and because of (2.2) this mapping induces an isomorphism of second spectral sequences so that $H_\star(X) = H_\star(K) = H_*\left(K'\right)$. The first spectral sequence of K' looks as follows: $E^1_{*,q} = C_*\left(X_G\right) \otimes_G H_q(L)$, where L is the following complex of left G-modules:

$$\oplus\mathbb{Z}\left[G/G_i\right] + \oplus\mathbb{Z}\left[G/G_{ij}\right] + \oplus\mathbb{Z}\left[G/G_{ijk}\right] + \ldots$$

Proposition 24. *If $G_1, \ldots, G_n$ are subgroups of G, there exists a fuctorial spectral sequence of the first quadrart, the E^2 term of which looks like: $E^2_{pq} = H_p\left(G, H_q(L)\right)$, where L is the complex defined above. It converges to $H_\star\left(UBG_j\right)$ and the differential d^r has bidegree $(-r, r-1)$.*

Chapter 4

Rings (with identity)

Let R be a ring and S a nonempty subset of R that is closed under the operations of addition and multiplication in R . If S is itself a ring under these operations then S is called a subring of R . A subring I of a ring R is a **left ideal** provided that $r \in R$ and $x \in I$ implies $rx \in I$. I is a **right ideal** provided $r \in R$ and $x \in I$ implies $xr \in I$.

I is an **ideal** if it is both a left and right ideal. Note that proper ideals does not contain any unit. We denote by (X) the ideal generated by the subset X of R , i.e., the smallest ideal containing X.

Theorem 24. 1. *$(a) = \left\{\sum_{i=1}^{n} \mathrm{r_j a s_i} \mid \mathrm{r_i, s_i} \in \mathrm{R}; n \in \mathbf{N}^*\right\}$ (principal ideal).*

2. *If a is in the center of R , then $\mathrm{Ra} = (\mathrm{a}) = \mathrm{aR}$.*

3. *If X is in the center of R , then the ideal (X) consists of all finite sums*

$$\mathrm{r_1 a_1} + \cdots + \mathrm{r_n a_n}\,(\mathrm{n} \in \mathbf{N}^*; \mathrm{r_i} \in \mathrm{R}; \mathrm{a_i} \in \mathrm{X})\,.$$

4. *For ideals, multiplication and addition are distributive and associative.*

An ideal P in a ring R is said to be prime if $\mathrm{P} \neq \mathrm{R}$ and for any ideals A, B in R

$$\mathrm{AB} \subset \mathrm{P} \Rightarrow \mathrm{A} \subset \mathrm{P} \text{ or } \mathrm{B} \subset \mathrm{P}.$$

Theorem 25. *If P is an ideal in a ring R such that $\mathrm{P} \neq \mathrm{R}$ and for all $\mathrm{a, b} \in \mathbf{R}$*

$$\mathrm{ab} \in \mathrm{P} \Rightarrow \mathrm{a} \in \mathrm{P} \textit{ or } \mathrm{b} \in \mathrm{P},$$

then P is prime. Conversely if P is prime and R is commutative, then P satisfies condition (1).

An ideal [resp. left ideal] M in a ring R is said to be **maximal** if $\mathbf{M} \neq \mathrm{R}$ and for every ideal [resp. left ideal] $\mathbf{N}$ such that $\mathbf{M} \subset \mathbf{N} \subset \mathrm{R}$, either $\mathbf{N} = \mathbf{M}$ or $\mathbf{N} = \mathbf{R}$.

Theorem 26. 1. *In a nonzero ring R with identity maximal [left] ideals always exist. In fact every $[left]$ ideal in R (except R itself) is contained in a maximal [left] ideal.*

2. *(In general, for $R^2 = R$) Every maximal ideal is prime.*

3. *If M is maximal and R is conmutative, then the R/M is a field. the converse is true in general, even when R/M is noncommutative.*

4. *R is a field if and only if the (0) is a maximal ideal.*

4.1 Modules

Every module we consider are unitary, i.e., the action of the neutral multiplicative element is trivial on the module

Let I be a left ideal of the ring R, A an R-module and S a nonempty subset of A. Then

$$IS = \left\{ \sum_{i=1}^{n} r_i a_i \mid r_i \in I; a_i \in S; n \in \mathbf{N}^* \right\}$$

is a submodule of A. Similarly if $a \in A$, then $Ia = \{ra \mid r \in I\}$ is a submodule of A.

If X is a subset of a module A over a ring R , then the intersection of all submodules of A containing X is called the submodule generated by X (or spanned by X). We have

$$(A) = RX = \left\{ \sum_{i=1}^{s} r_i a_i \mid s \in N^*; a_i \in X; r_i \in R \right\}$$

Theorem 27 (Free-modules)**.** *Let* **R** *be a ring with identity. The following conditions on a unitary R -module F are equivalent:*

1. *F has a nonempty basis;*
2. *F is the internal direct sum of a family of cyclic R -modules, each of which is isomorphic as a left* **R***-module to* **R***;*
3. *F is R -module isomorphic to a direct sum of copies of the left R -module R ;*
4. *There exists a nonempty set X and a function $\iota : \mathrm{X} \to \mathrm{F}$ with the following property: given any unitary R -module A and function $\mathrm{f} : \mathrm{X} \to \mathrm{A}$, there exists a unique R -module homomorphism $\bar{\mathrm{f}} : \mathrm{F} \to \mathrm{A}$ such that $\bar{\mathrm{f}}\iota = \mathrm{f}$. In other words, F is a free object in the category of unitary R -modules.*

Theorem 28. *Let R be a ring with identity and F a free R-module with an infinite basis X. Then every basis of F has the same cardinality as X.*

Let R be a ring with identity such that for every free R-module F, any two bases of F have the same cardinality. Then R is said to have the **invariant dimension property** and the cardinal number of any basis of F is called the dimension (or rank) of F over R.

Note that, in this case, two free modules are isomorphic if and only if they have the same rank. Also, R does not satisfy the invariant dimension property iff $R^n \simeq R^m$ for $n \neq m$

Theorem 29. *For ring with identity:*

1. *Every lineary independent subset of a vector space over a division ring can be extended to a basis.*
2. *Every division ring has the invariant dimension property.*
3. *Any finite-dimensional algebra over a division ring has the invariant dimension property.*
4. *Every conmutative ring (and so group rings) has the invariant dimension property.*

Example 23. $\mathrm{End}_F(F^\infty)$ *does not have the invariant dimension property.*

If R is a ring, for each n > 0, $\mathrm{Mat}_n\, R$ is a ring. We denote the identity matrix by I_n.

If M has a fixed ordered basis, we call M a based free module and define the **rank** of the based free module M to be the cardinality of its given basis. If R has the invariant dimension property, then the rank of a free module is constant.

Theorem 30. *Homomorphisms between based free modules are naturally identified with matrices over R.*

An R-module P is called **stably free** if $P \oplus R^m \cong R^n$ for some m and n.
By the fundamental theorem of Linear algebra, the kernel of any surjective linear map $\sigma : R^n \to R^m$ is a stably free module (split on finite dimension).

Lemma 3. 1. *If $P \oplus R^m \simeq R^\infty$ then $P \simeq R^\infty$.*

2. *Free and stably free modules are projective.*

When are stably free modules free? The most important special case, at least for inductive purposes, is when $m = 1$, i.e., $P \oplus R \cong R^n$.

A row vector $\sigma = (r_1, \dots, r_n)$ in R is called **unimodular** if they generate R^n. The following condition are equivalent:

- σ is unimodular.
- $R^n \cong P \oplus R$, where σ is identified with the projection $R^n \to R$ and $P = \ker(\sigma)$.
- $1 = r_1 s_1 + \cdots + r_n s_n$ for some $s_i \in R$.

Note that P is a free module if and only if σ may be completed to an invertible transformation.

We say that a ring R satisfies condition (S_n) (**stable base condition on dimension** n) if for every unimodular row $(r_0, r_1, \dots, r_n)$ in R^{n+1}, there is a unimodular vector $\left(r'_1, \dots, r'_n\right)$ in R^n with $r'_i = r_i - r_0 t_i$ for some $t_1, \dots, t_n$ in R.
The **stable range** of R, $\operatorname{sr}(R)$, is defined to be the smallest n such that R satisfies condition (S_n).

Proposition 25. 1. *(Vaserstein) (S_n) holds for all $n \geq \operatorname{sr}(R)$.*

2. *If $\operatorname{sr}(R) = n$ then all stably free projective modules of rank $\geq n$ are free.*

3. $\operatorname{sr}(R) = 1$ *for every artinian ring R, and stably free projective modules are free over artinian rings.*

4. *If I is an ideal of R, then $\operatorname{sr}(R) \geq \operatorname{sr}(R/I)$.*

5. *(Veldkamp) If $\operatorname{sr}(R) = n$ for some n then R has the invariant dimension property.*

We will focus most of our attention on the category $\mathbf{P}(R)$ of finitely generated projective R-modules; the morphisms are the R-module maps. Since the direct sum of projectives is projective, $\mathbf{P}(R)$ is an additive category. We may regard $\mathbf{P}$ as a covariant functor on rings, since if $R \to S$ is a ring map, then up to coherence there is an additive functor $\mathbf{P}(R) \to \mathbf{P}(S)$ sending P to $P \otimes_R S$. (Formally, there is an additive functor $\mathbf{P}'(R) \to \mathbf{P}(S)$ and an equivalence $\mathbf{P}'(R) \to \mathbf{P}(R)$; see Ex. 2.16.)

Lemma 4 (Some properties of projective modules)**.** 1. *A module is projective [generated by n elements] if and only if it is a direct summand of a free module [of rank n].*

2. *Every finitely generated projective R-module arises from an idempotent element in a matrix ring $M_n(R)$.*

3. *If R is a principal ideal domain, then every projective module is free. If R is a local ring, then every finitely generated projective R-module P is free of rank $\dim_{R/\mathrm{m}}(P/\mathrm{m}P)$.*

Some results for conmutative rings

Theorem 31 (Bass' Cancelation Theorem). *If R is a conmutative noetherian ring of Krull dimension d, or more generally, if* $\operatorname{SpecMax}(R)$ *is a finite union of spaces of dimension $\leq d$, then $sr(R) \leq d+1$*

Theorem 32. *Let R be a $d-$dimensional commutative noetherian ring. Then every stably free R-module of rank $> d$ is a free module. Equivalently, every unimodular row of length $n \geq d+2$ may be completed to an invertible matrix.*

Let R be a commutative ring. The **rank of a finitely generated R-module M at a prime ideal** p, $\operatorname{rank}_{\mathfrak{p}}(M)$ is the minimal number of generators of M_p.

Lemma 5. *Let R be a commutative ring, P a finitely generated projective R-module, the functions* $\operatorname{rank} P : \operatorname{Spec}(R) \to \mathbb{N}, \mathbb{Z}$ *are continuous.*

If a module M is not projective, $\operatorname{rank}(M)$ need not be a continuous function on $\operatorname{Spec}(R)$, as the example $R = \mathbb{Z}, M = \mathbb{Z}/p$ shows.

We say that P has constant rank n if $n = \operatorname{rank}_p(P)$ is independent of p.
If $\operatorname{Spec}(R)$ is topologically connected, every finitely generated projective R-module has constant rank.

We say that two R-modules M, M' are **stably isomorphic** if $M \oplus R^m \cong M' \oplus R^m$ for some $m \geq 0$.

Theorem 33 (Bass-Serre cancellation). *Theorem 2.3 (Bass-Serre cancellation). Let R be a $d-$dimensional commutative noetherian ring, and let P be a projective R-module of constant rank $n > d$.*

1. *(Serre) $P \cong P_0 \oplus R^{n-d}$ for some projective R-module P_0 of constant rank d.*
2. *(Bass) If P is stably isomorphic to P', then $P \cong P'$.*
3. *(Bass) For all M, M', if $P \oplus M$ is stably isomorphic to M', then $P \oplus M \cong M'$.*

Milnor squares

It is sometimes useful to be able to build projective modules by patching free modules. The following data suffices. Suppose that $s_1, \ldots, s_c \in R$ form a unimodular row, i.e., $s_1 R + \cdots + s_c R = R$. Then $\operatorname{Spec}(R)$ is covered by the open sets $D(s_i) \cong \operatorname{Spec}\left(R\left[\frac{1}{s_i}\right]\right)$. Suppose we are given $g_{ij} \in GL_n\left(R\left[\frac{1}{s_i s_j}\right]\right)$ with $g_{ii} = 1$ and $g_{ij} g_{jk} = g_{ik}$ in $GL_n\left(R\left[\frac{1}{s_i s_j s_k}\right]\right)$ for every i, j, k. Then

$$P = \left\{(x_1, \ldots, x_c) \in \prod_{i=1}^{c}\left(R\left[\frac{1}{s_i}\right]\right)^n : g_{ij}\left(x_j\right) = x_i \text{ in } R\left[\frac{1}{s_i s_j}\right]^n \text{ for all } i, j\right\}$$

is a finitely generated projective R-module by , because each $P\left[\frac{1}{s_i}\right]$ is isomorphic to $R\left[\frac{1}{s_i}\right]^n$.

Another type of patching arises from an ideal I in R and a ring map $f : R \to S$ such that I is mapped isomorphically onto an ideal of S, which we also call I. In this case R is the "pullback" of S and R/I :

$$R = \{(\bar{r}, s) \in (R/I) \times S : \bar{f}(\bar{r}) = s \text{ modulo } I\}$$

the square $\begin{array}{ccc} R & \xrightarrow{f} & S \\ \downarrow & & \downarrow \\ R/I & \xrightarrow{\bar{f}} & S/I \end{array}$ is called a **Milnor square**.

Example 24 (Conductor square)**.** *This arises when R is commutative and S is a finite extension of R with the same total ring of fractions. (S is often the integral closure of R.) The ideal I is chosen to be the conductor ideal, i.e., the largest ideal of S contained in R, which is just $I = \{x \in R : xS \subset R\} = \operatorname{ann}_R(S/R)$. If S is reduced, then I cannot lie in any minimal prime of R or S, so the rings R/I and S/I have lower Krull dimension.*

Given a Milnor square, we can construct an R-module $M = (M_1, g, M_2)$ from the following "descent data": an S-module M_1, an R/I-module M_2, and an S/I-module isomorphism $g : M_2 \otimes_{R/I} S/I \cong M_1/IM_1$. In fact M is the kernel of the R-module map

$$M_1 \times M_2 \to M_1/IM_1, \quad (m_1, m_2) \mapsto \bar{m}_1 - g\left(\bar{f}\left(m_2\right)\right).$$

We call M the R-module obtained by patching M_1 and M_2 together along g. An important special case is when we patch S^n and $(R/I)^n$ together along a matrix $g \in GL_n(S/I)$. For example, R is obtained by patching S and R/I together along $g = 1$. We will return to this point when we study $K_1(R)$ and $K_0(R)$.

Theorem 34. *In a Milnor square;*

1. *If P is obtained by patching together a finitely generated projective S-module P_1 and a finitely generated projective R/I-module P_2, then P is a finitely generated projective R-module.*
2. *$P \otimes_R S \cong P_1$ and $P/IP \cong P_2$.*
3. *Every finitely generated projective R-module arises in this way.*
4. *If P is obtained by patching free modules along $g \in GL_n(S/I)$ and Q is obtained by patching free modules along g^{-1}, then $P \oplus Q \cong R^{2n}$.*

Deternimant and elementary matrices

Theorem 35 (Existence of determinant)**.** *Let $(R, \mathcal{M})$ be a (possibly non-commutative) local ring. There is a well-defined determinant homomorphism* $\det : GL(R) \to \overline{R}^*$*, where $\bar{R}^* = (R^*)^{ab}$, satisfying:*

1. $\det(AB) = \det A \cdot \det B$,
2. $\det A = 1$ *for all $A \in E(R)$,*
3. *the composite $R^* = GL_1(R) \longrightarrow GL(R) \xrightarrow{det} \bar{R}^*$ is the natural quotient map.*

Let $E_n(R)$ be the subgroup of **elementary matrices**, defined to be the group generated by the matrices $e_{ij}^{(n)}(\lambda), 1 \le i \ne j \le n, \lambda \in R$, where $e_{ij}^{(n)}(\lambda)$ is the unipotent matrix whose only nontrivial off-diagonal entry is λ in the (i,j) th position. Thus, if $i < j$, then $e_{ij}^{(n)}(\lambda)$ has the form

$$e_{ij}^{(n)}(\lambda) = \begin{pmatrix} 1 & 0 & \cdots & 0 & \cdots & 0 & \\ 0 & 1 & \cdots & 0 & \cdots & 0 & \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \\ 0 & 0 & \cdots & 1 & \lambda & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 & 1 & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & \cdots & 0 & 1 \end{pmatrix}$$

Let $GL_n(R) \hookrightarrow GL_{n+1}(R)$ by $A \longmapsto \begin{bmatrix} A & 0 \\ 0 & 1 \end{bmatrix}$ and let $GL(R) = \lim_{\to} GL_n(R)$. Similarly, let $E(R) = \lim_{\to} E_n(R)$. Since $e_{ij}^{(n)}(\lambda) \longmapsto e_{ij}^{(n+1)}(\lambda)$ under $E_n(R) \hookrightarrow E_{n+1}(R)$, we obtain matrices

$e_{ij}(\lambda) \in E(R)$ as the common image of all the $e_{ij}^{(n)}(\lambda)$ for $n \geq i, j$, and $E(R)$ is the subgroup of $GL(R)$ generated by the $e_{ij}(\lambda)$. The $e_{ij}(\lambda)$ satisfy the following identities:

1. $e_{ij}(\lambda) \cdot e_{ij}(\mu) = e_{ij}(\lambda + \mu), \forall \lambda, \mu \in R$
2. $[e_{ij}(\lambda), e_{k\ell}(\mu)] = 1$ for $j \neq k, i \neq \ell$
3. $\left[e_{ij}(\lambda), e_{jk}(\mu)\right] = e_{ik}(\lambda\mu)$ for $i \neq k, \forall \lambda, \mu \in R$.

Lemma 6. *1. $E_n(R)$, $n \geq 3$, and $E(R)$ are perfect groups.*

2. (Whitehead) $E(R) = [E(R), E(R)] = [GL(R), GL(R)]$.

Part II

Topics of Algebraic Topology

Chapter 5

Ordinary homology

5.1 CW-complexes

They can be defined in an inductive way:

1. Start with a discrete set X^0, whose points are regarded as 0 -cells.
2. Inductively, form the ***n***-skeleton X^n from X^{n-1} by attaching n-cells e_α^n via maps $\varphi_\alpha : S^{n-1} \to X^{n-1}$. This means that X^n is the quotient space of the disjoint union $X^{n-1} \amalg_\alpha D_\alpha^n$ of X^{n-1} with a collection of n-disks D_α^n under the identifications $x \sim \varphi_\alpha(x)$ for $x \in \partial D_\alpha^n$. Thus as a set, $X^n = X^{n-1} \amalg_\alpha e_\alpha^n$ where each e_α^n is an open n-disk.
3. One can either stop this inductive process at a finite stage, setting $X = X^n$ for some $n < \infty$, or one can continue indefinitely, setting $X = \cup_n X^n$. In the latter case X is given the weak topology: A set $A \subset X$ is open (or closed) iff $A \cap X^n$ is open (or closed) in X^n for each n.

Note that a subspace is closed in X iff it meets each X^n in a closed set.

Example 25. *1. A 1-dimensional cell complex $X = X^1$ is what is called a graph in algebraic topology. It consists of vertices (the 0 -cells) to which edges (the 1-cells) are attached. The two ends of an edge can be attached to the same vertex.*

2. The sphere S^n has the structure of a cell complex with just two cells, e^0 and e^n, the n-cell being attached by the constant map $S^{n-1} \to e^0$. This is equivalent to regarding S^n as the quotient space $D^n / \partial D^n$.

*3. **Real projective n-space** $\mathbb{R}\mathrm{P}^n$. It is equivalent to the quotient space of a hemisphere D^n with antipodal points of ∂D^n identified. Since ∂D^n with antipodal points identified is just $\mathbb{R}\mathbb{P}\mathrm{P}^{n-1}$, we see that $\mathbb{R}\mathrm{P}^n$ is obtained from $\mathbb{R}\mathrm{P}^{n-1}$ by attaching an n-cell, with the quotient projection $S^{n-1} \to \mathbb{R}P^{n-1}$ as the attaching map. It follows by induction on n that $\mathbb{R}\mathbb{P}^n$ has a cell complex structure $e^0 \cup e^1 \cup \cdots \cup e^n$ with one cell e^i in each dimension $i \leq n$.*
The infinite union $\mathbb{R}P^\infty = U_n \mathbb{R}P^n$ becomes a cell complex with one cell in each dimension. We can view $\mathbb{R}P^\infty$ as the space of lines through the origin in $\mathbb{R}^\infty = \bigcup_n \mathbb{R}^n$.

*4. **Complex projective space** $\mathbb{C}P^n$. It is equivalent to the quotient of the unit sphere $S^{2n+1} \subset \mathbb{C}^{n+1}$ with $v \sim \lambda v$ for $|\lambda| = 1$.*
It is also possible to obtain $\mathbb{C}\mathbb{P}^n$ as a quotient space of the disk D^{2n} under the identifications $v \sim \lambda v$ for $v \in \partial D^{2n}$, in the following way. The vectors in $S^{2n+1} \subset \mathbb{C}^{n+1}$ with last coordinate real and nonnegative are precisely the vectors of the form $\left(w, \sqrt{1-|w|^2}\right) \in \mathbb{C}^n \times \mathbb{C}$ with $|w| \leq 1$. Such

vectors form the graph of the function $w \mapsto \sqrt{1-|w|^2}$. This is a disk D_+^{2n} bounded by the sphere $S^{2n-1} \subset S^{2n+1}$ consisting of vectors $(w, 0) \in \mathbb{C}^n \times \mathbb{C}$ with $|w| = 1$. Each vector in S^{2n+1} is equivalent under the identifications $v \sim \lambda v$ to a vector in D_+^{2n}, and the latter vector is unique if its last coordinate is nonzero. If the last coordinate is zero, we have just the identifications $v \sim \lambda v$ for $v \in S^{2n-1}$.

It follows that $\mathbb{P}^n$ is obtained from $\mathbb{CP}^{n-1}$ by attaching a cell e^{2n} via the quotient map $S^{2n-1} \to \mathbb{CP}^{n-1}$. So by induction on n we obtain a cell structure $\mathbb{CP}^n = e^0 \cup e^2 \cup \cdots \cup e^{2n}$ with cells only in even dimensions. Similarly, $\mathbb{CP}^\infty$ has a cell structure with one cell in each even dimension.

Each cell e_α^n in a cell complex X has a **characteristic map** $\Phi_\alpha : D_\alpha^n \to X$ which extends the attaching map φ_α and is a homeomorphism from the interior of D_α^n onto e_α^n. Namely, we can take Φ_α to be the composition $D_\alpha^n \hookrightarrow X^{n-1} \coprod_\alpha D_\alpha^n \to X^n \hookrightarrow X$ where the middle map is the quotient map defining X^n.

5.2 (Abstract) simplical complexes

A set (of **vertices**) together with a family of finite subsets (**simplexes**) such that every subset of every simplex is a simplex and every subset consisting of a single vertex is a simplex.

Example 26. 1. *The **standard n-simplex** Δ^n is the set of all $(n+1)$-tuples $(t_0, \ldots, t_n)$ of non-negative real numbers such that $t_0 + \cdots + t_n = 1$. The standard 0-simplex is a point, the standard 1-simplex is a line segment, the standard 2-simplex is a triangle, and so on.*

2. *The **boundary** of the standard n-simplex Δ^n is the set of all $(n+1)$-tuples $(t_0, \ldots, t_n)$ of non-negative real numbers such that $t_0 + \cdots + t_n = 1$ and at least one of the t_i is zero. The boundary of the standard 0-simplex is empty, the boundary of the standard 1-simplex is the set of its two endpoints, the boundary of the standard 2-simplex is the set of its three edges, and so on.*

3. *(**Concrete simplicial complexes**) It is subset of $\mathbb{R}^n$ that is a union of standard simplices, that satisfies the previous conditions.*

4. *If Y is a subset of the vertex set of a simplicial scheme S, then we can introduce on it the induced simplicial scheme structure $Y \cap S$, by defining the simplexes as the subsets of Y that are simplexes of S.*

5. *Let X be a set and let $\{p(y) : y \in Y\}$ be a covering of X. Then we can consider two simplicial complexes.*

   (a) *The nerve $\mathrm{Nerv}(p)$ of the covering is the simplicial scheme with the vertex set Y, and a subset Z of Y is counted as a simplex if the intersection $\bigcap_Z p(y)$ is non-empty.*

   (b) *The simplicial complex $V(p)$ is the simplicial scheme with the vertex set X, and a subset Z of X is counted as a simplex if Z is contained in some $p(y)$.*

Geometric realization

cellular chain complexes. We define a space ΓX, called the "geometric realization of the total singular complex of X," as follows. As a set

$$\Gamma X = \coprod_{n \geq 0} (S_n X \times \Delta_n) / (\sim)$$

where the equivalence relation $\sim$ is generated by

$$(f, \delta_i u) \sim (d_i(f), u) \text{ for } f : \Delta_n \longrightarrow X \quad \text{and} \quad u \in \Delta_{n-1}$$

and

$$(f, \sigma_i v) \sim (s_i(f), v) \text{ for } f : \Delta_n \longrightarrow X \quad \text{and} \quad v \in \Delta_{n+1}$$

Topologize ΓX by giving

$$\coprod_{0 \le n \le q} (S_n X \times \Delta_n) / (\sim)$$

the quotient topology and then giving ΓX the topology of the union. Define $\gamma : \Gamma X \longrightarrow X$ by

$$\gamma |f, u| = f(u) \text{ for } f : \Delta_n \longrightarrow X \quad \text{and} \quad u \in \Delta_n$$

where $|f, u|$ denotes the equivalence class of (f, u). Now the following two theorems imply that that this construction provides a canonical way of realizing our original construction of homology.

Theorem 36. *For any space $X, \Gamma X$ is a CW complex with one n-cell for each nondegenerate singular n-simplex.*

5.3 Singular homology

The standard topological n-simplex is the subspace

$$\Delta_n = \left\{ (t_0, \dots, t_n) \mid 0 \le t_i \le 1, \sum t_i = 1 \right\}$$

of $\mathbb{R}^{n+1}$. There are "face maps"

$$\delta_i : \Delta_{n-1} \longrightarrow \Delta_n, \quad 0 \le i \le n$$

specified by

$$\delta_i (t_0, \dots, t_{n-1}) = (t_0, \dots, t_{i-1}, 0, t_i, \dots, t_{n-1})$$

and "degeneracy maps"

$$\sigma_i : \Delta_{n+1} \longrightarrow \Delta_n, \quad 0 \le i \le n$$

specified by

$$\sigma_i (t_0, \dots, t_{n+1}) = (t_0, \dots, t_{i-1}, t_i + t_{i+1}, \dots, t_{n+1})$$

For a space X, define $S_n X$ to be the set of continuous maps $f : \Delta_n \longrightarrow X$. In particular, regarding a point of X as the map that sends 1 to x, we may identify the underlying set of X with $S_0 X$. Define the i th face operator

$$d_i : S_n X \longrightarrow S_{n-1} X, \quad 0 \le i \le n$$

by

$$d_i(f)(u) = f(\delta_i(u))$$

where $u \in \Delta_{n-1}$, and define the i th degeneracy operator

$$s_i : S_n X \longrightarrow S_{n+1} X, \quad 0 \le i \le n$$

by

$$s_i(f)(v) = f\left(\sigma_i(v)\right)$$

where $v \in \Delta_{n+1}$. The following identities are easily checked:

$$d_i \circ d_j = d_{j-1} \circ d_i \text{ if } i < j$$

$$d_i \circ s_j = \begin{cases} s_{j-1} \circ d_i & \text{if } i < j \\ \text{id} & \text{if } i = j \text{ or } i = j+1 \\ s_j \circ d_{i-1} & \text{if } i > j+1 \\ s_i \circ s_j = s_{j+1} \circ s_i \text{ if } i \le j. & \end{cases}$$

A map $f : \Delta_n \longrightarrow X$ is called a singular n-simplex. It is said to be nondegenerate if it is not of the form $s_i(g)$ for any i and g. Let $C_n(X)$ be the free Abelian group generated by the nondegenerate n-simplexes, and think of $C_n(X)$ as the quotient of the free Abelian group generated by all singular n-simplexes by the subgroup generated by the degenerate n-simplexes. Define

$$d = \sum_{i=0}^{n} (-1)^i d_i : C_n(X) \longrightarrow C_{n-1}(X)$$

The identities ensure that $C_*(X)$ is then a well defined chain complex. In fact,

$$d \circ d = \sum_{i=0}^{n-1} \sum_{j=0}^{n} (-1)^{i+j} d_i \circ d_j$$

and, for $i < j$, the (i, j) th and $(j-1, i)$ th summands add to zero. This gives that $d \circ d = 0$ before quotienting out the degenerate simplexes, and the degenerate simplexes span a subcomplex.

The singular homology of X is usually defined in terms of this chain complex:

$$H_*(X; \pi) = H_*\left(C_*(X) \otimes \pi\right)$$

Chapter 6

Simpliciality and Classifying Spaces

References [13, 3, 4, 10].

A simplicial set K_* is a sequence of sets $K_n, n \geq 0$, connected by face and degeneracy operators $d_i : K_n \longrightarrow K_{n-1}$ and $s_i : K_n \longrightarrow K_{n+1}, 0 \leq i \leq n$, that satisfy the commutation relations that we displayed for the total singular complex $S_*X = \{S_nX\}$ of a space X. Thus S_* is a functor from spaces to simplicial sets.

We may define the geometric realization $|K_*|$ of general simplicial sets exactly as we defined the geometric realization $\Gamma X = |S_*X|$ of the total singular complex of a topological space. In fact, the total singular complex and geometric realization functors are adjoint.

Simplicial sets were originally used to give precise and convenient descriptions of classifying spaces of groups. This idea was vastly extended by Grothendieck's idea of considering classifying spaces of categories, and in particular by Quillen's work of algebraic K-theory. In this work, which earned him a Fields Medal, Quillen developed surprisingly efficient methods for manipulating infinite simplicial sets. These methods were used in other areas on the border between algebraic geometry and topology. For instance, the André-Quillen homology of a ring is a "non-abelian homology", defined and studied in this way.

Both the algebraic K-theory and the André-Quillen homology are defined using algebraic data to write down a simplicial set, and then taking the homotopy groups of this simplicial set.

In recent years, simplicial sets have been used in higher category theory and derived algebraic geometry. Quasi-categories can be thought of as categories in which the composition of morphisms is defined only up to homotopy, and information about the composition of higher homotopies is also retained. Quasi-categories are defined as simplicial sets satisfying one additional condition, the weak Kan condition.

As $\mathcal{C}$ is an arbitrary category, we can consider simplicial R-modules, simplicial sets, simplicial rings, simplicial topological spaces, and many more. Simplicial sets are particularly important because they model topological spaces. Simplicial objects in an abelian category $\mathcal{A}$ model non-negatively graded chain complexes over $\mathcal{A}$.

Theorem 37. *The normalized chain complex is part of an equivalence of categories between the simplicial objects in $\mathcal{A}$ and the non-negatively graded chain complexes over $\mathcal{A}$.*

Simplicial complexes are more intuitive, and are the foundation of algebraic topology. Δ-complexes are usueful for computations. Simplicial sets are more suitable to high level concepts.

6.1 Simplicial objects in a category

We consider the finite set $\{0, 1, \ldots, n\}$ with its natural ordering $0 < 1 < \ldots < n$ and call this ordered set $[n]$ for all $n \geq 0$.

The **simplicial category**, Δ, has as objects the ordered sets $[n],\ n \geq 0$, and the morphisms in Δ are the order-preserving functions, that is, functions $f : [n] \to [m]$, such that $f(i) \leq f(j)$ for all $i < j$.
Let $\mathcal{C}$ be an arbitrary category. A **simplicial object** in $\mathcal{C}$ is a contravariant functor from Δ to $\mathcal{C}$. A cosimplicial object in $\mathcal{C}$ is a covariant functor from Δ to $\mathcal{C}$.
Simplicial objects in a category $\mathcal{C}$ form a category, where the morphisms are natural transformations of functors. We denote this category by $s\mathcal{C}$, note that $\mathcal{C}$ can be embedded in $\mathcal{C}$, considering constant simplicial objects.

Assume that we have a functor $X : \Delta^{op} \to \mathcal{C}$. Then, for every object $[n] \in \Delta$, we have an object $X([n]) =: X_n$ in $\mathcal{C}$. As all morphisms in Δ can be described as a composite of δ_i s and σ_j s, it suffices to know what the maps $X(\delta_i) =: d_i : X_n \to X_{n-1}$ and $X(\sigma_j) =: s_j$ do. Hence, if you want to describe a simplicial object, then you have to understand the sequence of objects $X_0, X_1, \ldots$ and the morphisms d_i, s_j in $\mathcal{C}$. These maps satisfy the dual relations:

$$\begin{aligned}
&d_i \circ d_j = d_{j-1} \circ d_i, \quad i < j, \\
&s_i \circ s_j = s_{j+1} \circ s_i, \quad i \leq j, \text{ and} \\
&d_i \circ s_j = \begin{cases} s_{j-1} \circ d_i, & i < j, \\ \quad 1_{[n]}, & i = j, j+1, \\ s_j \circ d_{i-1}, & i > j+1. \end{cases}
\end{aligned}$$

Thus a simplicial object can be visualized as a diagram of the form

$$X_0 \;\substack{\longleftarrow \\ \longrightarrow \\ \longleftarrow}\; X_1 \;\substack{\longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow}\; X_2 \cdots$$

where the morphisms $\leftarrow$ correspond to the d_i s, whereas the morphisms $\rightarrow$ are given by the s_j s. Note that on X_n, you have $n+1$ maps going out to the left and to the right.
The d_i s are called **face** maps and the s_j s are called **degeneracy maps**. For a concrete category $\mathcal{C}$ with a faithful functor $U : \mathcal{C} \to$ Sets the elements $x \in U(X_n)$ are the n-simplices of X. We will omit the functor U from the notation. Elements of the form $x = s_i y \in X_n$ for a $y \in X_{n-1}$ are called degenerate n-simplices.

Let $\Delta_n : \Delta^{op} \to$ Sets be the functor given by $[m] \mapsto \Delta([m], [n])$. The Yoneda lemma identifies the set X_n with the set of natural transformations from Δ_n to X for every simplicial set X :

$$X_n \cong s\,\mathrm{Sets}\,(\triangle_n, X)$$

The **category of elements of a simplicial set** X, $\mathrm{el}(X)$, is the category $X \backslash \Delta^\circ$ associated with the functor $X : \Delta^\circ \to$ Sets. Explicitly, the objects of el (X) are the $x \in X_n$ for some n. The morphisms in el (X) from $x \in X_n$ to $y \in X_m$ are all $f \in \Delta([n], [m])$, with $X(f)(y) = x$.

Proposition 26 (consequence of density theorem)**.** *For every simplicial set X there is an isomorphism of simplicial sets*

$$\operatorname{colim}_{\mathrm{el}(X)} \Delta_n \cong X$$

6.2 Geometric realization

The geometric realization of a simplicial set was introduced by Milnor [Mi57]. Definition 10.6.1. Let X be a simplicial set. The geometric realization of X, $|X|$, is the topological space

$$|X| = \bigsqcup_{n\geq 0} X_n \times \triangle^n / \sim .$$

Here, we consider the sets X_n as discrete topological spaces, and $\triangle^n$ denotes the topological n-simplex

$$\triangle^n = \left\{ (t_0, \ldots, t_n) \in \mathbb{R}^{n+1} \mid 0 \leq t_i \leq 1, \sum t_i = 1 \right\}.$$

The spaces $\triangle^n, n \geq 0$ form a cosimplicial topological space with structure maps

$$\delta_i \left(t_0, \ldots, t_n\right) = \left(t_0, \ldots, t_{i-1}, 0, t_i, \ldots, t_n\right) \text{ for } 0 \leq i \leq n$$

and

$$\sigma_j \left(t_0, \ldots, t_n\right) = \left(t_0, \ldots, t_j + t_{j+1}, \ldots, t_n\right) \text{ for } 0 \leq i \leq n.$$

The quotient in the geometric realization is generated by the relations

$$\left(d_i(x), (t_0, \ldots, t_n)\right) \sim \left(x, \delta_i \left(t_0, \ldots, t_n\right)\right), \quad \left(s_j(x), (t_0, \ldots, t_n)\right) \sim \left(x, \sigma_j \left(t_0, \ldots, t_n\right)\right).$$

Remark 1. *The geometric realization of a simplicial set X is nothing but the coend of the functor*

$$H : \Delta^o \times \Delta \to \textit{Top},$$

with $H([n], [m]) = X_n \times \triangle^m$, using that $[n] \mapsto X_n$ is a contravariant functor from Δ to Sets and that $[m] \mapsto \triangle^m$ is a covariant functor from the category Δ to the category Top. Here, we use the embedding of Sets into Top.

If $f : X \to Y$ is a morphism of simplicial sets, that is, a natural transformation from X to Y, then f induces a continuous map of topological spaces

$$|f| : |X| \to |Y|,$$

where an equivalence class $[(x, t_0, \ldots, t_n)] \in |X|$ is sent to the class $[(f(x), t_0, \ldots, t_n)] \in |Y|$. This turns the geometric realization into a functor from the category of simplicial sets to the category of topological spaces.

Elements of the form $s_j(x)$ are identified with something of a lower degree in the geometric realization, because of the relation

$$\left(s_j(x), (t_0, \ldots, t_n)\right) \sim \left(x, \sigma_j \left(t_0, \ldots, t_n\right)\right).$$

Hence, these elements do not contribute any geometric information to $|X|$. This might justify the name degenerate for such elements. Note that elements in X_0 are never degenerate.

An element $(y, (t_0, \ldots, t_m)) \in X_m \times \triangle^m$ is called **nondegenerate**, if y is not of the form $s_j(x)$ for any x and j and if $(t_0, \ldots, t_m) \in \Delta^m$ is not a point on the boundary of the topological m-simplex.

Proposition 27. *The geometric realization of a simplicial ser is a CW complex, such that every nondegenerate $n-$simplex corresponds to a $n-$cell.*

Example 27. 1. *The topological 1 -sphere is the quotient space $[0,1]/0 \sim 1$. If we want to find a simplicial model for the 1-sphere, such that the geometric realization has the desired cell structure, then we should define a simplicial set $\mathbb{S}^1$ with one 0 -simplex, 0 , and one nondegenerate 1 -simplex, 1 . The simplicial identities force the existence of a 1-simplex $s_0(0)$, so we get two 1-simplices. For the cell structure we do not need any further maps, so we just take these simplices and all the resulting elements that are given due to the simplicial structure maps. We then get $\mathbb{S}^1_n \cong [n]$ with face and degeneracy maps as follows:*

$$[0] \;\substack{\longleftarrow\\ \longrightarrow\\ \longleftarrow}\; [1] \;\substack{\longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow}\; [2] \cdots$$

The map $s_i : [n] \to [n+1]$ is the unique monotone injection, whose image does not contain $i+1$, while $d_i : [n] \to [n-1]$ is given by $d_i(j) = j$ if $j < i, d_i(i) = i$ if $i < n$, and $d_n(n) = 0$ and $d_i(j) = j-1$ if $j > i$.

The face maps glue the only nondegenerate 1-simplex 1 to the zero simplex $0 \in [0]$, and we obtain that the geometric realization, $|\mathbb{S}^1|$, is the topological 1sphere.

2. *The geometric realization of the representable simplicial set Δ_n is $|\Delta_n| = \triangle^n$. This is a general fact about tensor products of functors and representable objects 15.1.5.*

3. *Let X and Y be two simplicial sets. We already saw the product, $X \times Y$, which is the simplicial set with $(X \times Y)_n = X_n \times Y_n$. The simplicial structure maps d_i and s_j are defined coordinatewise. Be careful, an n-simplex $(x, y) \in X_n \times Y_n$ of the form $(s_i x', s_j y')$ for $i \neq j$ might not be degenerate in $X_n \times Y_n$, despite the fact that both coordinates are degenerate.*

Proposition 28. 1. *Assume that X and Y are two simplicial sets, such that $|X| \times |Y|$ is a CW complex, with the CW structure induced by the one on $|X|$ and $|Y|$. Then,*

$$|X \times Y| \cong |X| \times |Y|.$$

2. *If $f, g : X \to Y$ are maps of simplicial sets that are homotopic, then $|f|$ is homotopic to $|g|$.*

We consider the full subcategory $\Delta_{\leq n}$ of Δ with objects $[0], \ldots, [n]$. The inclusion functor

$$\iota_n : \Delta_{\leq n} \to \Delta$$

allows us to restrict simplicial sets X to $\Delta_{\leq n}$ by considering $X \circ \iota_n : \Delta^o_{\leq n} \to$ Sets. The ***n*-skeleton** of a simplicial set $X, sk_n X$, is the left Kan extension of $X \circ \iota_n$ along ι_n. It is easy to see that

$$|sk_n X| \cong sk_n |X| =: X^{(n)},$$

where $sk_n|X| = X^{(n)}$ denotes the n-skeleton of the CW complex $|X|$.

Fat realization of a Semi-simplicial set

Sometimes, you might want to use a variant of the geometric realization functor. An obvious reason is, that there are sequences of objects $X_0, X_1, \ldots$ that are only connected via face maps, but there are no degeneracy maps. Such functors are often called semisimplicial objects. In that situation, you cannot perform the geometric realization. The other situation that makes an alternative desirable is the situation, where you want to perform the geometric realization of a simplicial space and this space has bad point set behavior.

Let X be a simplicial set (or space), then the fat realization of X, $\|X\|$, is

$$\|X\| = \bigsqcup_{n\geq 0} X_n \times \triangle^n / \sim,$$

where the quotient in the fat geometric realization is generated by the relations

$$(d_i(x), (t_0, \dots, t_n)) \sim (x, \delta_i(t_0, \dots, t_n)).$$

There are several alternative descriptions of $\|X\|$. One is to consider the semisimplicial category, Δ, whose objects are the objects of Δ, but we restrict to injective order-preserving maps. These are dual to the face maps used in the identifications in fat geometric realization. Thus, we can describe $\|X\|$ as the coend of the functor

$$H : \Delta^o \times \Delta \to \text{ Top},$$

with $H([p],[q]) = X_p \times \triangle^q$. There is yet another description of the fat realization of a simplicial set or simplicial topological space (see, for instance, [We05, Proof of Proposition 1.3] or [Se74, p. 308]) as the ordinary geometric realization of a "fattened up" simplicial set.
Of course, $\|X\|$ also makes sense, if you start with a semisimplicial object, that is, a functor $X : \Delta^o \to \text{Sets}$.

As we do not collapse degenerate simplices, the fat realization of a simplicial set is larger than the geometric realization.

Proposition 29. *1. If all the X_n are spaces of the homotopy type of a CW complex, then so is $\|X\|$.*

2. If $f : X \to Y$ is a morphism of simplicial topological spaces, such that all $f_n : X_n \to Y_n$ are homotopy equivalences, then $\|f\|$ is a homotopy equivalence.

3. Fat realization commutes with finite products.

6.3 Classifying spaces of small categories

To any small category, you can associate a topological space that takes the data of the category (objects, morphisms, and composition of morphisms) and translates it into a CW complex. This is done in a two-stage process: First you construct a simplicial set out of your category, and then, you form its geometric realization.

1. For a small category $\mathcal{C}$, let $M_n(\mathcal{C})$ be the set

$$\left\{ C_0 \xrightarrow{f_1} C_1 \xrightarrow{f_2} \dots \xrightarrow{f_n} C_n \mid C_i \text{ object of } \mathcal{C}, f_i \text{ morphism in } \mathcal{C} \right\}$$

of the n-tuples of composable morphisms in $\mathcal{C}$. We denote an element, as earlier, as $[f_n|\dots|f_1]$.

2. The **nerve** of the category $\mathcal{C}$ is the simplicial set $N\mathcal{C} : \Delta^{op} \to \text{Sets}$, which sends $[n]$ to the set $M_n(\mathcal{C})$. The degeneracies insert identity morphisms

$$s_i[f_n|\dots|f_1] = [f_n|\dots|f_{i+1}|1_{C_i}|f_i|\dots|f_1], \quad 0 \leq i \leq n,$$

and the face maps compose morphisms:

$$d_i[f_n|\dots|f_1] = \begin{cases} [f_n|\dots|f_2], & i = 0, \\ [f_n|\dots|f_{i+1} \circ f_i|\dots|f_1], & 0 < i < n, \\ [f_{n-1}|\dots|f_1], & i = n. \end{cases}$$

3. The **classifying space** of the category $\mathcal{C}$ is the geometric realization of the nerve of $\mathcal{C}$: $B\mathcal{C} = |N\mathcal{C}|$.

The objects C of $\mathcal{C}$ give zero cells in $B\mathcal{C}$, and a nonidentity morphism from C to C'' gives rise to an edge whose endpoints correspond to the objects C and C'. If $g \circ f$ is a composition of morphisms in $\mathcal{C}$, then in the classifying space, you will find a triangle, with edges corresponding to f, g, and $g \circ f$. Threefold compositions give rise to tetrahedra and so on.

The topological space $B\mathcal{C}$ is always a CW complex, and a functor $F : \mathcal{C} \to \mathcal{D}$ induces a continuous and cellular map of topological spaces: $BF : B\mathcal{C} \to B\mathcal{D}$. .

Hence, B is a functor from the category cat to the category Top of topological spaces.

Example 28. *1. If X is a set and $\mathcal{C}$ is the corresponding discrete category, then the classifying space $B\mathcal{C}$ is X with the discrete topology.*

2. *If G is a group and we consider the small category $\mathcal{C}_G$ associated with G, then the classifying space $B(\mathcal{C}_G)$ is called the classifying space of the group G and is denoted by BG.*

   If the group G is abelian, we have a new model construction of BG. The group composition is a group homomorphism, and it induces a functor $\mathcal{C}_G \times \mathcal{C}_G \to \mathcal{C}_G$. We therefore obtain a map $BG \times BG \to B(\mathcal{C}_G \times \mathcal{C}_G) \to B(\mathcal{C}_G) = BG$, and for abelian groups G, BG is a topological group. For instance, $B\mathbb{Z} \simeq \mathbb{S}^1$.

   If G is a topological group, then we can implement the topology into the construction of BG by endowing $G^n \times \triangle^n$ with the product topology. For instance, $B\mathbb{S}^1 \simeq \mathbb{C}P^\infty$, and this is an Eilenberg-Mac Lane space of type $(\mathbb{Z}, 2)$, $K(\mathbb{Z}, 2) = \mathbb{C}P^\infty \simeq B(B\mathbb{Z})$. In general, if A is a finitely generated abelian group, then the n-fold iterated classifying space construction is a model of the Eilenberg-Mac Lane space $K(A, n)$.

   If G is a discrete group, then the homology of the group G (with coefficients in $\mathbb{Z}$) is the singular homology $H_*(BG; \mathbb{Z})$.

3. *Let us consider the category Σ. This has as objects the natural numbers (including zero), and the only morphisms are automorphisms with $\Sigma([n], [n]) = \Sigma_n$. Therefore, the classifying space of Σ has one component for every natural number, because the different objects are not connected by morphisms. Thus,*

$$B\Sigma = \bigsqcup_{n \geq 0} B\Sigma_n.$$

Theorem 38. *1. For two functors $F, F' : \mathcal{C} \to \mathcal{D}$, a natural transformation $\tau : F \Rightarrow F'$ induces a homotopy between BF and BF'.*

2. *If $\mathcal{C} \underset{R}{\overset{L}{\rightleftarrows}} \mathcal{D}$ is an adjoint pair of functors, then $B\mathcal{C}$ is homotopy equivalent to $B\mathcal{D}$.*

3. *In particular, an equivalence of categories gives rise to a homotopy equivalence of classifying spaces.*

4. *If a small category $\mathcal{C}$ has an initial or terminal object, then its classifying space is contractible.*

Chapter 7

Homotopy theory

References: [**cohen**, 10]

A **homotopy** $h : p \simeq q$ between maps $p, q : X \longrightarrow Y$ is a continuous map $h : X \times I \longrightarrow Y$ such that $h(x, 0) = p(x)$ and $h(x, 1) = q(x)$, where I is the unit interval $[0, 1]$.
A map $f : X \longrightarrow Y$ is a **homotopy equivalence** if there is a map $g : Y \longrightarrow X$ such that both $g \circ f \simeq \mathrm{id}$ and $f \circ g \simeq \mathrm{id}$.

Top_* denotes the **category of pointed topological spaces**, whose objects are pairs (X, x_0), where X is a topological space and $x_0 \in X$ (**basepoint**), with morphisms the continuous functions that preserve the basepoints.
$h\mathrm{Top}_*$ denotes the category of pointed topological spaces, with morphisms the based homotopy classes of based maps. These set of morphisms are denoted by $[X, Y]$. Its isomorphisms are the based homotopy equivalences.
The product in this category is the **smash product**

$$X \wedge Y = X \times Y / X \vee Y$$

, where $X \vee Y$ is the subspace of $X \times Y$ consisting of pair containing at least on basepoint.

For a based space X define its **suspension**

$$\Sigma X = X \wedge S^1 = X \times S^1 / \left(\{*\} \times S^1 \cup X \times \{1\} \right)$$

We define the loop space of X to be $\Omega X = F\left(S^1, X\right)$. Its points are the loops at the basepoint. Composition of loops defines a multiplication on this set. Explicitly, for $f, g : \Sigma X \longrightarrow Y$, we write

$$(g + f)(x \wedge t) = (g(x) \cdot f(x))(t) = \begin{cases} f(x \wedge 2t) & \text{if } 0 \leq t \leq 1/2 \\ g(x \wedge (2t - 1)) & \text{if } 1/2 \leq t \leq 1 \end{cases}$$

Lemma 7. *1.* $[\Sigma X, Y] \cong [X, \Omega Y]$

2. $[\Sigma X, Y]$ *is a group and* $\left[\Sigma^2 X, Y\right]$ *is an Abelian group.*

For a based topological space (X, x_0), define

$$\pi_n (X, x_0) = [S^n, X]_*, \quad \text{for } n \geq 0.$$

the set of homotopy classes of based maps $S^n \longrightarrow X$. This is a group if $n \geq 1$ and an Abelian group if $n \geq 2$. When $n = 0$ and $n = 1$, this agrees with our previous definitions. Observe that

$$\pi_n(X) = \pi_{n-1}(\Omega X) = \cdots = \pi_0\left(\Omega^n X\right)$$

$\pi_1(X, x_0)$ is called the **fundamental group** of X with base point x_0. If X is path-connected, then $\pi_1(X, x_0)$ is independent of the choice of base point x_0. Many important facts are known about this group. For instance, it induces a functor from Top_0 to Gr, which factors through a functor $h\mathrm{Top}_0 \to$Gr.

Hurewicz isomorphism

A space X is said to be n-connected if $\pi_q(X, x) = 0$ for $0 \leq q \leq n$ and all x.

For based spaces X, define the **Hurewicz homomorphism**

$$h : \pi_n(X) \longrightarrow \tilde{H}_n(X)$$

by

$$h([f]) = f_*\left(i_n\right)$$

where i_n is a generator of $\tilde{H}_n(S^n)$.

Proposition 30. *The Hurewicz homomorphism is natural.*

Theorem 39 (Hurewicz Theorem)**.** *If X is $(n-1)$-connected, then the Hurewicz homomorphism $h : \pi_n(X) \longrightarrow \tilde{H}_n(X)$ is an isomorphism for $n \geq 2$. For $n = 1$, it is the abelianization homomorphism.*

This isomorphism is a consequence of the axiomatic definition of homology theory.

7.1 Fundamental groupoid and covering spaces

The **fundamental groupoid** $\Pi_1(X)$ of a space X is the grupoid whose objects are the points of X and whose morphisms are the homotopy classes of paths in X with fixed endpoints. Then, Π_1 is a functor Top$\to$Grd.

Theorem 40 (grupoid version)**.** *Let $\mathscr{O} = \{U\}$ be a cover of a space X by path connected open subsets such that the intersection of finitely many subsets in $\mathscr{O}$ is again in $\mathscr{O}$. Regard $\mathscr{O}$ as a category whose morphisms are the inclusions of subsets and observe that the functor Π, restricted to the spaces and maps in $\mathscr{O}$, gives a diagram*

$$\Pi \mid \mathscr{O} : \mathscr{O} \longrightarrow \mathscr{G}\mathscr{P}$$

of groupoids. The groupoid $\Pi(X)$ is the colimit of this diagram. In symbols,

$$\Pi(X) \cong \operatorname{colim}_{U \in \mathscr{O}} \Pi(U)$$

Theorem 41 (Van Kampen)**.** *Let X be path connected and choose a basepoint $x \in X$. Let $\mathscr{O}$ be a cover of X by path connected open subsets such that the intersection of finitely many subsets in $\mathscr{O}$ is again in $\mathscr{O}$ and x is in each $U \in \mathscr{O}$. Regard $\mathscr{O}$ as a category whose morphisms are the inclusions of subsets and observe that the functor $\pi_1(-, x)$, restricted to the spaces and maps in $\mathcal{O}$, gives a diagram*

$$\pi_1 \mid \mathscr{O} : \mathscr{O} \longrightarrow \mathscr{G}$$

of groups. The group $\pi_1(X, x)$ is the colimit of this diagram. In symbols,

$$\pi_1(X, x) \cong \operatorname{colim}_{U \in \mathscr{O}} \pi_1(U, x)$$

Corollary 7. 1. *Let X be the wedge of a set of path connected based spaces X_i, each of which contains a contractible neighborhood V_i of its basepoint. Then $\pi_1(X)$ is the free product of the groups $\pi_1(X_i)$.*

2. *For based spaces: $\pi_1(X \times Y) \cong \pi_1(X) \times \pi_1(Y)$.*

3. *Let $X = U \cup V$, where U, V, and $U \cap V$ are path connected open neighborhoods of the basepoint of X and V is simply connected. Then $\pi_1(U) \longrightarrow \pi_1(X)$ is an epimorphism whose kernel is the smallest normal subgroup of $\pi_1(U)$ that contains the image of $\pi_1(U \cap V)$.*

A space X is said to be **locally path connected** if for any $x \in X$ and any neighborhood U of x, there is a smaller neighborhood V of x each of whose points can be connected to x by a path in U. Equivalently, the topology of X have a basis consisting of path connected open sets. If X is connected and locally path connected, then it is path connected. Throughout this section, we assume that all given spaces are connected and locally path connected.

A map $p : E \longrightarrow B$ is a covering space if it is surjective and if each point $b \in B$ has an open neighborhood V such that each component of $p^{-1}(V)$ is open in E and is mapped homeomorphically onto V by p. We call E the **total space**, B the **base space**, and $F_b = p^{-1}(b)$ a **fiber of the covering** p.

7.2 Eilenberg-Mac Lane spaces

Theorem 42 (Construction)**.** 1. *Let π be any group. There is a connected CW complex $K(\pi, 1)$ such that $\pi_1(K(\pi, 1)) = \pi$ and $\pi_q(K(\pi, 1)) = 0$ for $q \neq 1$.*

2. *Let $n \geq 1$ and let π be an Abelian group. There is a connected CW complex $K(\pi, n)$ such that $\pi_n(X) = \pi$ and $\pi_q(X) = 0$ for $q \neq n$, called the **Eilenberg-Mac Lane spaces**.*

3. *Eilenberg-Mac Lane spaces are unique up to homotopy equivalence.*

There is a beautiful construction of the Eilenberg-Mac Lane spaces for discrete abelian topological groups. We define the "classifying spaces" and "universal bundles" associated to topological groups G.

We define a map $p_* : E_*(G) \longrightarrow B_*(G)$ of simplicial topological spaces. Let $E_n(G) = G^{n+1}$ and $B_n(G) = G^n$, and let $p_n : G^{n+1} \longrightarrow G^n$ be the projection on the first n coordinates. The faces and degeneracies are defined on $E_n(G)$ by

$$d_i(g_1, \ldots, g_{n+1}) = \begin{cases} (g_2, \ldots, g_{n+1}) & \text{if } i = 0 \\ (g_1, \ldots, g_{i-1}, g_i g_{i+1}, g_{i+2}, \ldots, g_{n+1}) & \text{if } 1 \leq i \leq n \end{cases}$$

and

$$s_i(g_1, \ldots, g_{n+1}) = (g_1, \ldots, g_{i-1}, e, g_i, \ldots, g_{n+1}) \text{ if } 0 \leq i \leq n$$

The faces and degeneracies on $B_n(G)$ are defined in the same way, except that the last coordinate g_{n+1} is omitted and the last face operation d_n takes the form

$$d_n(g_1, \ldots, g_n) = (g_1, \ldots, g_{n-1})$$

Certainly p_* is a map of simplicial spaces. If we let G act from the right on $E_n(G)$ by multiplication on the last coordinate,

$$(g_1, \dots, g_n, g_{n+1})\, g = (g_1, \dots, g_n, g_{n+1}g)$$

then $E_*(G)$ is a simplicial G-space. That is, the action of G commutes with the face and degeneracy maps. We may view $B_n(G)$ as the orbit space $E_n(G)/G$. We define

$$E(G) = |E_*(G)|\,, \quad B(G) = |B_*(G)|\,, \quad \text{and} \quad p = |p_*(G)| : E(G) \longrightarrow B(G)$$

Then $E(G)$ inherits a free right action by G, and $B(G)$ is the orbit space $E(G)/G$. The space BG is called the classifying space of G.

The space $E(G)$ is the union of the images $E(G)^n$ of the spaces $\coprod_{m\le n} G^{m+1}\times \Delta_m$, and

$$E(G)^n - E(G)^{n-1} = (G^n - W) \times G \times (\Delta_n - \partial\Delta_n)$$

where $W \subset G^n$ is the "fat wedge" consisting of those points at least one of whose coordinates is the identity element e. Similarly, we have subspaces $B(G)^n$ such that

$$B(G)^n - B(G)^{n-1} = (G^n - W) \times (\Delta_n - \partial\Delta_n)$$

The map p restricts to the projection between these subspaces. Intuitively, it looks as if p should be a bundle with fiber G, and this is indeed the case if the identity element of G is a nondegenerate basepoint. This condition is enough to ensure local triviality as we glue together over the filtration $\{B(G)^n\}$. It is less intuitive, but true, that the space $E(G)$ is contractible. By the long exact homotopy sequence, these facts imply that

$$\pi_{q+1}(BG) \cong \pi_q(G)$$

for all $q \ge 0$. For topological groups G and H, the obvious shuffle homeomorphisms

$$(G \times H)^n \cong G^n \times H^n$$

specify isomorphisms of simplicial spaces

$$E_*(G \times H) \cong E_*(G) \times E_*(H) \quad \text{and} \quad B_*(G \times H) \cong B_*(G) \times B_*(H)$$

that are compatible with the projections. Since geometric realization commutes with products, we conclude that $B(G \times H)$ is homeomorphic to $B(G) \times B(H)$. Thus B is a product-preserving functor on the category of topological groups.

Now suppose that G is a commutative topological group. Then its multiplication $G \times G \longrightarrow G$ and inverse map $G \longrightarrow G$ are homomorphisms. We conclude that $B(G)$ and $E(G)$ are again commutative topological groups. The multiplication on $B(G)$ is determined by the multiplication on G as the composite

$$B(G) \times B(G) \cong B(G \times G) \longrightarrow B(G)$$

Moreover, the map $p : E(G) \longrightarrow B(G)$ and the inclusion of G in $E(G)$ as the fiber over the basepoint (the unique point in $B_0(G)$) are homomorphisms. This allows us to iterate the construction, setting $B^0(G) = G$ and $B^n(G) = B\left(B^{n-1}(G)\right)$ for $n \ge 1$. Specializing to a discrete Abelian group π, we define

$$K(\pi, n) = B^n(\pi)$$

As promised, we have

$$\pi_q(K(\pi,n)) = \pi_{q-1}(K(\pi,n-1)) = \cdots = \pi_{q-n}(K(\pi,0)) = \begin{cases} \pi & \text{if } q = n \\ 0 & \text{if } q \neq n \end{cases}$$

Chapter 8

Fibrations

A surjective map $p : E \longrightarrow B$ is a **(Hurewicz) fibration** if it satisfies the **covering homotopy property** (CHP). This means that if $h \circ i_0 = p \circ f$ in the diagram

$$\begin{tikzcd} Y \arrow[r, "f"] \arrow[d, "i_0"'] & E \arrow[d, "p"] \\ Y \times I \arrow[r, "h"] \arrow[ur, dashed, "\tilde{h}"] & B \end{tikzcd}$$

then there exists $\tilde{h}$ that makes the diagram commute.

Proposition 31. *1. If $p : E \longrightarrow B$ is a covering, then p is a fibration with a unique path lifting function s. In this case*

$$p_* : \pi_1(B, b) \xrightarrow{\cong} \pi_1(E, s(b))$$

2. *Every bundle is a fibration.*

3. $\pi_n(X \times Y) \simeq \pi_n(X) \to \pi_n(Y)$.

4. $\pi_n(S^n) \simeq \mathbb{Z}$. *If $i < n$, then $\pi_i(S^n) = 0$.*

5. *If X is the colimit of a sequence of inclusions $X_i \longrightarrow X_{i+1}$ of based spaces, then the natural map*

$$\operatorname{colim}_i \pi_n(X_i) \longrightarrow \pi_n(X)$$

is an isomorphism for each n.

6. *For path connected spaces, change of basepoint determines a natural isomorphism on homotopy groups.*

7. *Homotopy equivalences of spaces induce isomorphisms on homotopy groups.*

Theorem 43 (Homotopy long exact sequence of a fibration)**.** *Let $p : E \longrightarrow B$ be a fibration with fiber F. Then there is a long exact sequence of homotopy groups*

$$\cdots \longrightarrow \pi_{n+1}(F) \longrightarrow \pi_n(E) \longrightarrow \pi_n(B) \longrightarrow \pi_n(F) \longrightarrow \cdots$$

8.1 Serre-Leray spectral sequence

Theorem 44. *Theorem 4.28. Let $p : E \to B$ be a fibration with fiber F. Assume that F is connected and B is simply connected. Then there are chain complexes $C_*(E)$ and $C_*(B)$ computing the homology of E and B respectively, and a filtation of $C_*(E)$ leading to a spectral sequence converging to $H_*(E)$ with the following properties:*

1. $E_2^{r,s} = H_r\left(B; H_s(F)\right)$
2. *The inclusion of the fiber into the total space induces a homomorphism*

   $$i_* : H_n(F) \to H_n(E)$$

   which can be computed as follows:

   $$i_* : H_n(F) = E_2^{0,n} \to E_\infty^{0,n} \subset H_n(E)$$

   where $E_2^{0,n} \to E_\infty^{0,n}$ is the projection map which exists because all the differentials d_j are zero on $E_j^{0,n}$.
3. *The projection map induces a homomorphism*

   $$p_* : H_n(E) \to H_n(B)$$

   which can be computed as follows:

   $$H_n(E) \to E_\infty^{n,0} \subset E_2^{n,0} = H_n(B)$$

   where $E_\infty^{n,0}$ includes into $E_2^{n,0}$ as the subspace consisting of those classes on which all differentials are zero. This is well defined because no class in $E_j^{n,0}$ can be a boundary for any j.

The theorem holds when the base space is not simply connected also, and in the more general context of *Serre fibrations*. Hurewicz theorem can be obtained as a corollary of this theorem.

Theorem 45 (Homology long exact sequence)**.** *Let $p : E \to B$ be a fibration with connected fiber F, where B is simply connected and $H_i(B) = 0$ for $0 < i < n$, and $H_i(F) = 0$ for $i < i < m$. Then there is an exact sequence*

$$H_{n+m-1}(F) \xrightarrow{i_*} H_{n+m-1}(E) \xrightarrow{p_*} H_{n+m-1}(B) \xrightarrow{\tau} H_{n+m-2}(F) \to \cdots \to H_1(E) \to 0$$

Corollary 8. *Corollary 4.32. Suppose X and Y are simply connected CW - complexes and $f : X \to Y$ a continuous map that induces an isomorphism in homology groups,*

$$f_* : H_k(X) \xrightarrow{\cong} H_k(Y) \quad \textit{for all } k \geq 0$$

Then $f : X \to Y$ is a homotopy equivalence.

Chapter 9

Geometric Group Theory

By a G**-complex** we will mean a CW-complex X together with an action of G on X which permutes the cells. Thus we have for each $g \in G$ a homeomorphism $x \mapsto gx$ of X such that the image go of any cell σ of X is again a cell. For example, if X is a simplicial complex on which G acts simplicially, then X is a G-complex.

If X is a G-complex then the action of G on X induces an action of G on the cellular chain complex $C_*(X)$, which thereby becomes a chain complex of G-modules. Moreover, the canonical augmentation $\varepsilon : C_0(X) \to \mathbb{Z}$ (defined by $\varepsilon(v) = 1$ for every 0 -cell v of X) is a map of G-modules.

We will say that X is a free G-complex if the action of G freely permutes the cells of X (i.e., $g\sigma \neq \sigma$ for all σ if $g \neq 1$). In this case each chain module $C_n(X)$ has a $\mathbb{Z}$-basis which is freely permuted by G, hence by $3.1 C_n(X)$ is a free $\mathbb{Z}G$-module with one basis element for every G-orbit of cells. (Note that to obtain a specific basis we must choose a representative cell from each orbit and we must choose an orientation of each such representative.)

Finally, if X is contractible, then $H_*(X) \approx H_*$ (pt.); in other words, the sequence

$$\cdots \to C_n(X) \xrightarrow{\partial} C_{n-1}(X) \to \cdots \to C_0(X) \xrightarrow{\varepsilon} \mathbb{Z} \to 0$$

is exact. We have, therefore:

Proposition 32. *Let X be a contractible free G-complex. Then the augmented cellular chain complex of X is a free resolution of $\mathbb{Z}$ over $\mathbb{Z}G$.*

9.1 Classifying space of groups

Suppose that $\mathcal{C}$ is a (small) category. The classifying space (or nerve) $B\mathcal{C}$ of $\mathcal{C}$ is the simplicial set with

$$B\mathcal{C}_n = \hom_{\text{cat}}(\mathbf{n}, \mathcal{C}),$$

n-simplex is a string

$$a_0 \xrightarrow{\alpha_1} a_1 \xrightarrow{\alpha_2} \ldots \xrightarrow{\alpha_n} a_n$$

of composeable arrows of length n in $\mathcal{C}$.

If G is a group, then G can be identified with a category (or groupoid) with one object $*$ and one morphism $g : * \to *$ for each element g of G, and so the classifying space BG of G is defined. Moreover $|BG|$ is an Eilenberg-Mac Lane space of the form $K(G, 1)$, as the notation suggests; this is now the standard construction.

Recall that we constructed BG as the geometric realisation of the nerve of a category * // G. As the notation suggests, this can be interpreted as a quotient, or more precisely a homotopy quotient. One can construct the homotopy quotient $X//G$ of any space X with G-action by a group G, and here we just take $X = *$. By abuse of notation * // $G = |N(*//G)|$[2] A reference for its construction and properties is [Rie14], but we will only need the following facts:

1. Homotopy quotients are natural. If $X \to Y$ is an equivariant map between G-spaces then there is an induced map $X//G \to Y//G$.

2. Homotopy quotients preserve homological connectivity. If $X \to Y$ is an equivariant map between G-spaces which is homologically d-connected then $X//G \to Y//G$ is also homologically d-connected. (Recall that a map is homologically d-connected if it is an isomorphism on H_i for $i < d$ and surjection on H_d.)

3. Homotopy quotients commute with geometric realisation. If $X_\bullet$ is a semi-simplicial G-space, then $\|X_\bullet\|//G \simeq \|X_\bullet//G\|$. (We will explain the terminology and notation later.)

4. Homotopy quotients of transitive G-sets. If S is a transitive G-set, then $S//G \simeq B\,\mathrm{Stab}_G(s)$ for any $s \in S$.

9.2 Acyclic spaces and acyclic groups

References [6, 14, 19].

We call a topological space E **acyclic** if it has the homology of a point.

Lemma 8. *Let E be an acyclic space. Then E is connected, its fundamental group $G = \pi_1(E)$ is a perfect group, and $H_2(G;\mathbb{Z}) = 0$.*

Proof. E must be connected, as $H_0(E) = \mathbb{Z}$. Since $G^{ab} = G/[G,G] = H_1(E;\mathbb{Z}) = 0$, we have G is perfect. To calculate $H_2(G)$, note that $H_1(\tilde{E};\mathbb{Z}) = 0$. Moreover, the homotopy fiber of the canonical map $E \to BG$ is homotopy equivalent to $\tilde{E}$. In fact, the Serre spectral sequence for this homotopy fibration is $E^2_{pq} = H_p\left(G; H_q(\tilde{E};\mathbb{Z})\right) \Rightarrow H_{p+q}(E;\mathbb{Z})$ and the conclusion that $H_2(G;\mathbb{Z}) = 0$ follows from the associated exact sequence of low degree terms:

$$H_2(E;\mathbb{Z}) \to H_2(G;\mathbb{Z}) \xrightarrow{d^2} H_1(\tilde{E};\mathbb{Z})^G \to H_1(E;\mathbb{Z}) \to H_1(G;\mathbb{Z})$$

This implies that $F(f)$ is connected and $\pi_1 F(f)$ is a perfect group. □

Let X and Y be based connected CW complexes. A **map** $f : X \to Y$ is called **acyclic** if the homotopy fiber $F(f)$ of f is acyclic (has the homology of a point).

From the exact sequence $\pi_1 F(f) \to \pi_1(X) \to \pi_1(Y) \to \pi_0 F(f)$ we also have that $\pi_1(X) \to \pi_1(Y)$ is onto, and its kernel P is a perfect normal subgroup of $\pi_1(X)$.

Let P be a perfect normal subgroup of $\pi_1(X)$, where X is a based connected CW complex. An acyclic map $f : X \to Y$ is called a **+-construction** on X relative to P if P is the kernel of $\pi_1(X) \to \pi_1(Y)$.

Lemma 9. *If X is acyclic, the map $X \to *$ is a + -construction.*

When Quillen introduced the notion of acyclic maps in 1969, during his construction of higher $K-$theory, he observed that both Y and the map f are determined up to homotopy by the subgroup P.

Theorem 46 (Quillen). *Let (X, x) be a connected CW complex, $N \lhd \pi_1(X, x)$ a perfect normal subgroup. Then there exists a continuous map of pairs $f : (X, x) \longrightarrow (X^+, x^+)$ such that*

1. *There is an exact sequence*

$$0 \longrightarrow N \longrightarrow \pi_1(X, x) \xrightarrow{f_*} \pi_1\left(X^+, x^+\right) \longrightarrow 0$$

2. *For any local coefficient system L on X^+,*

$$f_* : H_n\left(X, f^*L\right) \longrightarrow H_n\left(X^+, L\right)$$

is an isomorphism for any $n \geq 0$.

3. *If $g : (X, x) \longrightarrow (Y, y)$ is a continuous map such that*

$$N \subset \ker\left(g_* : \pi_1(X, x) \longrightarrow \pi_1(Y, y)\right)$$

then there exists a continuous map $h : (X^+, x^+) \longrightarrow (Y, y)$, unique up to homotopy, making the diagram commute. In particular, if g is another +-construction relative to P, then the map h above is a homotopy equivalence: $h : Y \xrightarrow{\sim} Z$.

Proof. asdfasdf □

Every group G has a unique largest perfect subgroup P, called the perfect radical of G, and it is a normal subgroup of G. If no mention is made to the contrary, the notation X^+will always denote the +-construction relative to the perfect radical of $\pi_1(X)$.

Proposition 33. 1. *Let X and Y be connected CW complexes. A map $f : X \to Y$ is acyclic if and only if $H_*(X, M) \cong H_*(Y, M)$ for every $\pi_1(Y)$-module M.*

2. *Let P be a perfect normal subgroup of a group G, with corresponding +-construction $f : BG \to BG^+$. If $F(f)$ is the homotopy fiber of f, then $\pi_1 F(f)$ is the universal central extension of P, and $\pi_2\left(BG^+\right) \cong H_2(P; \mathbb{Z})$*

Proposition 34. *Let $(\hat{X}, \hat{x}) \longrightarrow (X, x)$ be the covering of X corresponding to the subgroup $N \lhd \pi_1(X, x)$, and let $\left(\tilde{X}^+, \tilde{x}^+\right)$be the universal covering of (X^+, x^+). Then $\left(\tilde{X}^+, \tilde{x}^+\right)$is the result of applying the plus construction to $(\hat{X}, \hat{x})$.*

Part III

Topics of Geometry

Chapter 10

Fibre bundles over paracompact spaces

10.1 Bundles

References [5, 10]

A **paracompact space** is a topological space in which every open cover has an open refinement that is locally finite.

Theorem 47. *Compact spaces, CW-complexes and metrizable spaces are paracompact.*

Every space in the following will be paracompact.

A **bundle** is a triple (E, p, B), where $p : E \to B$ is a map. The space B is called the **base space**, the space E is called the **total space**, and the map p is called the **projection of the bundle**. For each $b \in B$, the space $p^{-1}(b)$ is called the **fibre** of the bundle over $b \in B$.
Bundle form a category, denoted by Bun. The subcategory of bundles over B is denoted by Bun_B

Let (E, p, B) and (E', p', B') be two bundles. A bundle morphism$(u, f) : (E, p, B) \to (E', p', B')$ is a pair of maps $u : E \to E'$ and $f : B \to B'$ such that $p'u = fp$.

Let $\xi = (E, p, B)$ be a bundle, and let $f : B_1 \to B$ be a map. The induced bundle of ξ under f, denoted $f^*(\xi)$, has as base space B_1, as total space E_1 which is the subspace of all pairs $(b_1, x) \in B_1 \times E$ with $f(b_1) = p(x)$, and as projection p_1 the map $(b_1, x) \mapsto b_1$. *The induced bundle is the pullback on Bun*.

10.2 Fibre bundles

Let G be a topological group. Every G-space X determines a bundle $\alpha(X) = (X, \pi, X/G)$. If $h : X \to Y$ is a G-space morphism, we have $h(xG) \subset h(x)G$ for each $x \in X$. The **quotient map of** h is the map $f : X/G \to Y \bmod G$, where $f(xG) = h(x)G$. Let $\alpha(h)$ denote the bundle morphism $(h, f) : \alpha(X) \to \alpha(Y)$.

A bundle (X, p, B) is called a G**-bundle** provided (X, p, B) and $\alpha(X)$ are isomorphic for some G-space structure on X by an isomorphism $(1, f) : \alpha(X) \to (X, p, B)$, where $f : X \bmod G \to B$ is a homeomorphism.

Let X be a free $G-$space. Let X^* be the subspace of all $(x, xs) \in X \times X$, where $x \in X, s \in G$ for a free G-space X. There is a function $\tau : X^* \to G$ such that $x\tau(x, x') = x'$ for all $(x, x') \in X^*$.
A G-space X is called **principal** provided X is a free G-space with a continuous τ. A **principal G-bundle** is a G-bundle (X, p, B), where X is a principal G-space.
Principal $G-$bundles form a category, denoted by $\mathrm{Bun}(G)$. The subcategory of principal $G-$bundles over B is denoted by $\mathrm{Bun}_B(G)$.

Example 29. 1. *The product principal G-bundle, $B \times G$.*

2. *Let G be a closed subgroup of a topological group Γ. Then G acts on the right of Γ by multiplication. The base space of the corresponding principal G-bundle is the space of left cosets Γ/G.*

3. *Let $S^{\mathbf{n}}$ be the $\mathbb{Z}_2$-space with action given by the relation $x(\pm 1) = \pm x$. Then $(S^n)^*$ is the subspace of $(x, \pm x) \in S^n \times S^n$. This principal Z_2-space defines a principal Z_2- bundle with base space RP^n.*

Proposition 35. 1. *Let $\xi = (X, p, B)$ be a principal G-bundle. Then ξ is a bundle with fibre G.*

2. *Morphism on $Bun_B(G)$ are isomorphisms.*

3. *$f^* : \mathrm{Bun}_B(G) \to \mathrm{Bun}_{B_1}(G)$ is a functor.*

Let $\xi = (X, p, B)$ be a principal G-bundle, and let F be a left G-space. The relation $(x, y)s = (xs, s^{-1}y)$ defines a right G-space structure on $X \times F$. Let X_F denote the quotient space $(X \times F)$ mod G, and let $p_F : X_F \to B$ be the factorization of the composition of $X \times F \xrightarrow{p_X} X \xrightarrow{p} B$ by the projection $X \times F \to X_F$. Explicitly, we have $p_F((x, y)G) = p(x)$ for $(x, y) \in X \times F$.
The bundle (X_F, p_F, B), denoted $\xi[F]$, is called the **fibre bundle over** B **with fibre** F (viewed as a G-space) and **associated principal bundle** ξ. The group G is called the **structure group** of the fibre bundle $\xi[F]$.
In general, the total space of $\xi[F]$ reflects the "twist" in the topology of the total space X and the "twist" in the action of G on F. In the next proposition we prove that $\xi[F]$ is a bundle with fibre F.

Let $(u, f) : (X, p, B) \to (X', p', B')$ be a principal bundle morphism, and let F be a left G-space. The morphism (u, f) defines a G-morphism $u \times 1_F : X \times F \to X' \times F$, and by passing to quotients, we have a map $u_F : X_F \to X'_F$ such that $(u_F, f) : \xi[F] \to \xi'[F]$, where $\xi = (X, p, B)$ and $\xi' = (X', p', B')$. A **fibre bundle morphism** from $\xi[F]$ to $\xi'[F]$ is a bundle morphism of the form $(u_F, f) : \xi[F] \to \xi'[F]$, where $(u, f) : \xi \to \xi'$ is a principal bundle morphism. If $B = B'$ and $f = 1_B$, then $u_F : \xi[F] \to \xi'[F]$ is called a fibre bundle morphism over B.

Let ξ be the product principal G-bundle $(B \times G, p, B)$. For each left G-space F, the fibre bundle $\xi[F] = (Y, q, B)$ is B-isomorphic over B to the product bundle $(B \times F, p, B)$. Let $g : Y \to B \times F$ be defined by $g((b, s, y)G) = (b, sy)$. Then g is a B-isomorphism.
Two principal G-bundles ξ and η over B are locally isomorphic provided each $b \in B$ has an open neighborhood U such that $\xi \mid U$ and $\eta \mid U$ are U-isomorphic (as principal bundles). Two fibre bundles $\xi[F]$ and $\eta[F]$ are locally isomorphic provided ξ and η are locally isomorphic.
A principal G-bundle ξ over B is trivial or locally trivial provided ξ is a principal G-bundle that is isomorphic or locally isomorphic to the product principal G-bundle. A fibre bundle $\xi[F]$ is trivial or locally trivial provided ξ is trivial or locally trivial, respectively.

Proposition 36. *Let $\xi[F] = \times(X_F, p_F, B)$ be the fibre bundle with associated principal G-bundle $\xi = (X, p, B)$ and fibre F. For each $b \in B$, the fibre F is homeomorphic to $p_F^{-1}(b)$.*

Let $\xi = (X, p, B)$ be a principal G-bundle, and let H be a closed subgroup of G. Then the relation on X defined by the action of the group H is compatible with the projection $p : X \to B$. Therefore, there is a bundle $\xi \bmod H = (X \bmod H, q, B)$, where q is the result of factoring p by the canonical map $X \to X \bmod H$.

Theorem 48 (Restriction of structure group)**.** *Let $\xi = (X, p, B)$ be a principal G-bundle, and let H be a closed subgroup of G. Then there is a canonical B-isomorphism of bundles $\xi \bmod H \to \xi[G \bmod H]$, where the fibre $G \bmod H$ is the homogeneous space of right cosets of H in G.*

10.3 Classifying space of a group

For each paracompact space B, let $k_G(B)$ denote the set of isomorphism classes of principal G-bundles over B. Let $\{\xi\}$ denote the isomorphism class of the principal G-bundle ξ over B. For a homotopy class $[f] : X \to Y$ we define a function $k_G([f]) : k_G(Y) \to k_G(X)$ by the relation $k_G([f])\{\xi\} = \{f^*(\xi)\}$. Let **H** denote the category of all spaces and homotopy classes of maps.

Theorem 49.

1. *k_G are well defined functions.*
2. *The collection of functions $k_G : \mathbf{H} \to Set$ is a cofunctor.*
3. *If $f : X \to Y$ is a homotopy equivalence, $k_G([f]) : k_G(Y) \to k_G(X)$ is a bijection.*
4. *If X is contractible, each numerable principal G-bundle over X is trivial.*

Let $\omega = (E_0, p_0, B_0)$ be a fixed principal G-bundle. For each space X we define

$$\phi_\omega(X) : [X, B_0] \to k_G(X) \quad \text{defined by} \quad \phi_\omega(X)[u] = \{u^*(\omega)\}$$

.

Proposition 37. *$\phi_\omega : [-, B_0] \to k_G$ are functions and they define a natural transformation $\mathbf{H} \to Set$.*

A principal G-bundle $\omega = (E_0, p_0, B_0)$ is **universal** provided ω is numerable and $\phi_\omega : [-, B_0] \to k_G$ is an isomorphism. The space B_0 is called a **classifying space** of G.

Theorem 50. *A principal G-bundle $\omega = (E_0, p_0, B_0)$, where B_0 paracompact, is universal if and only if the following are true.*

1. *For each numerable principal G-bundle ξ over X there exists a map $f : X \to B_0$ such that ξ and $f^*(\omega)$ are isomorphic over X.*
2. *If $f, g : X \to B_0$ are two maps such that $f^*(\omega)$ and $g^*(\omega)$ are isomorphic over X, then f and g are homotopic.*

This is also equivalent that X is contractible.

Theorem 51 (Milnor)**.** *Let G be a topological group. Then a classifying space B_0 of G exists. (always paracompact??)*

Let H be a closed subgroup of G, let $\omega_H = (Y_0, q_0, B_H)$ be a universal bundle for H, and let $\omega_G = (X_0, p_0, B_G)$ be a universal bundle for G, which is a numerable principal G-bundle over B_H. By the classification theorem 4(12.2), there is a principal G-bundle morphism $(h_0, f_0) : \omega_H[G] \to \omega_G$, where $f_0^*(\omega_G)$ and $\omega_H[G]$ are isomorphic over B_H.

Theorem 52. *With the above notations, let $\xi = (X, p, B)$ be a numerable principal G-bundle over B with classifying map $f : B \to B_G$; that is, $f^*(\omega_G)$ and ξ are B-isomorphic. Then the restrictions $\eta = (Y, q, B)$ of ξ are in bijective correspondence with homotopy classes of maps $g : B \to B_H$ such that $f_0 g$ and f are homotopic. We have the following diagram:*

10.4 Vector bundles

A **vector bundle** over X is a locally trivial family of vector spaces $\eta : E \to X$,i .e., $x \in X$ has a neighborhood U such that $\eta|U : E|U \to U$ is trivial.

The **orthogonal group** in k dimensions, denoted $O(k)$, is the subgroup of $u \in \mathbf{GL}(k, \mathbf{R})$ such that $(u(x) \mid u(y)) = (x \mid y)$ for each $x, y \in \mathbf{R}^k$. The **unitary group** in k dimensions, denoted $U(k)$, is the subgroup of $u \in \mathbf{GL}(k, \mathbf{C})$ such that $(u(x) \mid u(y)) = (x \mid y)$ for each $x, y \in \mathbf{C}^k$. The **symplectic group** in k dimensions, denoted $Sp(k)$, is the subgroup of $u \in \mathbf{GL}(k, \mathbf{H})$ such that $(u(x) \mid u(y)) = (x \mid y)$ for each $x, y \in \mathbf{H}^k$.
These groups are closed and bounded subsets of the space of matrices. Therefore, they are compact (topological) groups.

The **special orthogonal group** in k dimensions, denoted $SO(k)$, is the closed subgroup of $u \in O(k)$ with det $u = +1$. The **special unitary group** in k dimensions, denoted $SU(k)$, is the closed subgroup of $u \in U(k)$ with det $u = +1$. The **special symplectic group** in k dimensions, denoted $Sp(k)$, is the closed subgroup of $u \in Sp(k)$ with det $u = +1$.
These groups and the previous ones are referred to as the **classical groups.**

Theorem 53. *Let ξ be a vector bundle over B paracompact. Then ξ has an atlas whose transition functors $\{g_{i,j}\}$ have their values in $O(n)$, the real case with $F = \mathbf{R}$; $U(n)$, the complex case with $F = \mathbf{C}$; and $Sp(n)$, the quaternionic case with $F = \mathbf{H}$.*

10.5 Characteristic classes

Chapter 11

Smooth and complex manifolds

Chapter 12

Symplectic manifolds

Chapter 13

Sheaf theory

13.1 Sheaves

13.1.1 Cěch complexes

Chapter 14

Algebraic geometry

Part IV

K-theory

The subject can be said to begin with Alexander Grothendieck (1957), who used it to formulate his Grothendieck-Riemann-Roch theorem. It takes its name from the German Klasse, meaning "class". [4] Grothendieck needed to work with coherent sheaves on an algebraic variety X. Rather than working directly with the sheaves, he defined a group using isomorphism classes of sheaves as generators of the group, subject to a relation that identifies any extension of two sheaves with their sum. The resulting group is called $K(X)$ when only locally free sheaves are used, or $G(X)$ when all are coherent sheaves. Either of these two constructions is referred to as the Grothendieck group; $K(X)$ has cohomological behavior and $G(X)$ has homological behavior.

If X is a smooth variety, the two groups are the same. If it is a smooth affine variety, then all extensions of locally free sheaves split, so the group has an alternative definition.

In topology, by applying the same construction to vector bundles, Michael Atiyah and Friedrich Hirzebruch defined $K(X)$ for a topological space X in 1959, and using the Bott periodicity theorem they made it the basis of an extraordinary cohomology theory. It played a major role in the second proof of the Atiyah-Singer index theorem (circa 1962). Furthermore, this approach led to a noncommutative K-theory for C^*-algebras.

Already in 1955, Jean-Pierre Serre had used the analogy of vector bundles with projective modules to formulate Serre's conjecture, which states that every finitely generated projective module over a polynomial ring is free; this assertion is correct, but was not settled until 20 years later. (Swan's theorem is another aspect of this analogy.)

The other historical origin of algebraic K-theory was the work of J. H. C. Whitehead and others on what later became known as Whitehead torsion.

There followed a period in which there were various partial definitions of higher K-theory functors. Finally, two useful and equivalent definitions were given by Daniel Quillen using homotopy theory in 1969 and 1972. A variant was also given by Friedhelm Waldhausen in order to study the algebraic K-theory of spaces, which is related to the study of pseudo-isotopies. Much modern research on higher K-theory is related to algebraic geometry and the study of motivic cohomology.

The corresponding constructions involving an auxiliary quadratic form received the general name L-theory. It is a major tool of surgery theory.

In string theory, the K-theory classification of Ramond-Ramond field strengths and the charges of stable Dbranes was first proposed in 1997.[5]

Chapter 15

Grothendieck's K-theory

Chapter 16

Topological K-Theory

Chapter 17

Milnor's K-theory

References [14]

Let R be an associative ring (with 1), and let $\mathcal{P}(R)$ denote the category of finitely generated projective R-modules. We define the Grothendieck group $K_0(R)$ to be the quotient

$$K_0(R) = \mathcal{F}/\mathcal{R},$$

$\mathcal{F}$ = free Abelian group on the isomorphism classes of projective modules in $\mathcal{P}(R)$, $\mathcal{R}$ = subgroup generated by elements

$$[P \oplus Q] - [P] - [Q], \text{ for all } P, Q \in \mathcal{P}(R).$$

Thus, for any $P, Q \in \mathcal{P}(R)$, $[P] = [Q]$ in $K_0(R) \Longleftrightarrow P \oplus P' \cong Q \oplus P'$ for some $P' \in \mathcal{P}(R) \Longleftrightarrow P \oplus R^n \cong Q \oplus R^n$ for some $n \geq 0$ Further, we can find $Q' \in \mathcal{P}(R)$ such that $P' \oplus Q' \cong R^n$ for some n, since P' is a quotient of some R^n (P' is finitely generated) and P' is projective. Hence $P \oplus P' \cong Q \oplus P' \Longrightarrow P \oplus R^n \cong Q \oplus R^n$.
If $f : R \to S$ is a homomorphism of rings, f induces a functor $\mathcal{P}(R) \to \mathcal{P}(S)$ given by $P \longmapsto S \otimes_R P$. This preserves direct sums, and hence induces a homomorphism $f_* : K_0(R) \to K_0(S)$.

Proposition 38. 1. *Let $(R, \mathcal{M})$ be a local ring, i.e., $\mathcal{M}$ is a 2 -sided maximal ideal, and $R - \mathcal{M} = R^*$. Then $K_0(R) = \mathbb{Z}$, with a generator given by the class of the free R-module of* rank 1.

2. *Let R be a Dedekind domain, i.e., a commutative Noetherian integrally closed domain such that every non-zero prime ideal of R is maximal. Then $K_0(R) \cong \mathbb{Z} \oplus C\ell(R)$ where $C\ell(R)$ is the ideal class group of R, the group of isomorphism classes of invertible ideals (with tensor product as the group operation).*

17.1 K_1

$$\begin{aligned} K_1(R) &= GL(R)/[GL(R), GL(R)] \\ &= GL(R)/E(R) = H_1(GL(R), \mathbb{Z}) \end{aligned}$$

Proposition 39. *Let $(R, \mathcal{M})$ be a (possibly non-commutative) local ring. Then the natural* map $GL_1(R) \to K_1(R)$ *induces an isomorphism*

$$R^* / [R^*, R^*] \cong K_1(R)$$

Corollary 9 (Dieudonné)**.** . *If D is a division ring, then $K_1(D) \cong D^* / [D^*, D^*]$, induced by the Dieudonné determinant $GL(D) \to (D^*)^{ab}$.*

17.2 K_2

Let R be a ring with identity. The n**th Steinberg group** $St_n(R)$ is defined to be the quotient of the free group on symbols $x_{ij}^{(n)}(\lambda)$ for $1 \leq i, j \leq n, i \neq j$, and for all $\lambda \in R$, modulo the normal subgroup generated by the words:

1. $x_{ij}^{(n)}(\lambda) \cdot x_{ij}^{(n)}(\mu) \cdot x_{ij}^{(n)}(\lambda + \mu)^{-1}$ for all i, j, for all $\lambda, \mu \in R$
2. $\left[x_{ij}^{(n)}(\lambda), x_{k\ell}^{(n)}(\mu)\right]$ for $i \neq \ell, k \neq j$, for all $\lambda, \mu \in R$
3. $\left[x_{ij}^{(n)}(\lambda), x_{jk}^{(n)}(\mu)\right] \cdot x_{ik}^{(n)}(\lambda\mu)^{-1}$ for $i \neq k$, for all $\lambda, \mu \in R$.

By properties of elementary matrices, we have a natural surjection $\phi_n : St_n(R) \to E_n(R)$, given by $\phi_n\left(x_{ij}^{(n)}(\lambda)\right) = e_{ij}^{(n)}(\lambda)$. We also have natural homomorphisms $\mathrm{St}_n(R) \to St_{n+1}(R)$ (which need not be injective), and so we obtain the infinite Steinberg group $St(R) = \lim_{\to} St_n(R)$, and the surjection $\phi : \mathrm{St}(R) \longrightarrow E(R)$. Let

$$K_2(R) := \ker \phi.$$

Proposition 40. *1. $St(R)$ and $St_n(R), n \geq 3$ are perfect.*

2. $\mathrm{St}(R)$ and $S_n(R), n \geq 5$ have no non-split central extensions.

Corollary 10. *The extension*

$$0 \longrightarrow K_2(R) \longrightarrow St(R) \longrightarrow E(R) \longrightarrow 0$$

is a universal central extension of $E(R)$. In particular,

$$K_2(R) = H_2(E(R), \mathbb{Z})$$

Chapter 18

Quillen's K-theories

18.1 The $+-$Construction

References [14, 19].
Inspiration was given by [7]

The higher algebraic K-groups of a ring R will be defined to be the homotopy groups $K_n(R) = \pi_n K(R)$ of a certain topological space $K(R)$. Since $\pi_1(BGL(R)) \cong GL(R), \pi_1(BGL(R))$ has a perfect normal subgroup isomorphic to $E(R)$.

We construct a CW complex $BGL(R)^+$ with distinguished inclusion $i : BGL(R) \to BGL(R)^+$ such that:

1. $i_* : \pi_1(BGL(R)) \longrightarrow \pi_1\left(BGL(R)^+\right)$ is the quotient map $GL(R) \longrightarrow GL(R)/E(R) = K_1(R)$
2. for any local coefficient system L on $BGL(R)^+$,

$$i_* : H_n\left(BGL(R), i^*L\right) \longrightarrow H_n\left(BGL(R)^+, L\right)$$

   is an isomorphism for all $n \geq 0$.

$BGL(R)^+$ is unique up to homotopy. Hence, the homotopy groups $K_n(R)$ of $BGL(R)^+$ are well-defined up to a canonical isomorphism, and the *Quillen's $K-$groups* are defined as

$$K_i(R) = \pi BGL(R)^+, \quad \forall n \geq 1$$

In this situation, take $K(R) = K_0(R) \times BGL(R)^+$. By construction, $K_0(R) = \pi_0 K(R)$. Moreover, it is clear that $\pi_n K(R) = \pi_n BGL(R)^+ = K_n(R)$ for $n \geq 1$, as desired. This K is not functorial, but it is possible to modify the components of this $K(R)$, up to homotopy equivalence, to gain functoriality.

Proposition 41. *Let $F(R)$ be the homotopy fiber of $BGL(R) \longrightarrow BGL(R)^+$. Then:*

1. *$F(R)$ is acyclic, i.e., $\tilde{H}_n(F(R), \mathbb{Z}) = 0$ for all $n \geq 0$.*
2. *$\pi_1(F(R)) \cong St(R)$, the Steinberg group.*
3. *$\pi_1(F(R))$ acts trivially on $\pi_i(F(R)), i \geq 2$ i.e., $F(R)$ is simple in dimensions ≥ 2.*

Proof. (a) If we replace $BGL(R)^+$by its universal cover, and $BGL(R)$ by the induced covering, this does not change the homotopy type of $F(f)$ (see (A.27)); we again use $F(R)$ to denote the homotopy fiber of

$$\widehat{BGL}(R) \to \widetilde{BGL}(R)^+$$

where $\widetilde{BGL}(R)^+$is the universal cover of $BGL(R)^+$, and $\widehat{BGL}(R)$ is the induced covering of $BGL(R)$, which is just the covering space associated to the subgroup $E(R) \subset GL(R) = \pi_1(BGL(R)$) (thus $\widehat{BGL}(R)$ has the homotopy type of $BE(R)$, the Eilenberg-MacLane space with $\pi_1(BE(R)) \cong E(R), \pi_i(BE(R)) = 0$

We have a spectral sequence (see (A.27))

$$E^2_{p,q} = H_p\left(\widetilde{BGL}(R)^+, H_q(F(R), \mathbb{Z})\right) \Longrightarrow H_{p+q}(\widehat{BGL}(R), \mathbb{Z})$$

(where the E^2-term is the usual homology group with coefficients in $H_q(F(R), \mathbf{Z})$, since the local coefficient system associated to $H_q(F(R), \mathbf{Z})$ is trivial on the simply connected space $\widetilde{BGL}(R)^+\Big)$. Further, from Proposition (2.3) and Theorem (2.1)(b), the edge homomorphisms

$$H_n(\widehat{BGL}(R), \mathbf{Z}) \to E^\infty_{n,0} \longrightarrow E^2_{n,0} = H_n\left(\widetilde{BGL}(R)^+, \mathbb{Z}\right)$$

are isomorphisms, i.e., $E^2_{n,0} = E^\infty_{n,0}$ and $E^\infty_{p,q} = 0$ for $q \neq 0$. Now suppose $F(R)$ is not acyclic; since $\widetilde{BGL}(R)^+$is simply connected and $\widehat{BGL}(R)$ is connected, $F(R)$ is path connected. Thus if q is the smallest integer such that $\tilde{H}_q(F(R), \mathbf{Z}) \neq 0$, then $q > 0$. Then $E^2_{p,q'} = E^r_{p,q'} = 0$ for all p, and all q' with $0 < q' < q$. Since $d_r : E^r_{p,q} \longrightarrow E^r_{p-r,q+r-1}$, we see that $E^2_{0,q} \cong E^{q+1}_{0,q}, E^{q+2}_{0,q} \cong E^\infty_{0,q}$, and $E^{q+2}_{0,q}$ is the cokernel of $d_{q+1} : E^{q+1}_{q+1,0} \longrightarrow E^{q+1}_{0,q}$. But $E^2_{n,0} = E^\infty_{n,0}$, so that $d_{q+1} = 0$ also. Hence $E^2_{0,q} = E^\infty_{0,q} = 0$ as seen above. But on the other hand,

$$E^2_{0,q} = H_0\left(BGL(R)^+, H_q(F(R), \mathbf{Z}) \neq 0\right.$$

as we assumed $H_q(F(R), \mathbb{Z}) = \bar{H}_q(F(R), \mathbb{Z}) \neq 0$. This contradiction proves that $F(R)$ is acyclic.

(b), (c): Let $G = \pi_1(F(R)\ )$, and consider the spectral sequence (see (A.28)) for the universal covering $\tilde{F}(R) \longrightarrow F(R)$. This has the form

$$E^2_{p,q} = H_p\left(G, H_q(\tilde{F}(R), \mathbb{Z})\right) \Longrightarrow H_{p+q}(F(R), \mathbb{Z})$$

where by (a) we have $\tilde{H}_n(F(R), \mathbf{Z}) = 0 \forall n$. Also, $\tilde{F}(R)$ is simply connected, so $H_1(\tilde{F}(R), \mathbb{Z}) = 0$. This forces

$$E^2_{1,0} = E^\infty_{1,0} = 0, E^2_{2,0} = E^\infty_{2,0} = 0$$

further $E^2_{3,0} \cong E^3_{3,0} \xrightarrow{d_3} E^3_{0,2} \cong E^2_{0,2}$ must be an isomorphism, since the kernel and cokernel are respectively $E^\infty_{3,0} = 0$ and $E^\infty_{0,2} = 0$. Thus

$$H_1(G, \mathbb{Z}) = H_2(G, \mathbb{Z}) = 0, \text{and } H_3(G, \mathbb{Z}) \cong H_0\left(G, H_2(\tilde{F}(R)\right)$$

The homotopy exact sequence (see (A.18)) for

$$F(R) \longrightarrow \widehat{BGL}(R) \longrightarrow \widetilde{BGL}(R)^+$$

together with

$$\pi_i\left(\widetilde{BGL}(R)^+\right) \cong \pi_i\left(BGL(R)^+\right), \quad i \geq 2$$
$$\pi_i(\widehat{BGL}(R)) = 0, \quad i \geq 2$$

and $\pi_1(BGL(R)) = E(R)$ yields isomorphisms

$$\pi_{i+1}\left(BGL(R)^+\right) \cong \pi_i(F(R)), \quad i \geq 2$$

and an exact sequence

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \pi_2(BGL(R)^+) & \longrightarrow & \pi_1(F(R)) & \longrightarrow & E(R) & \longrightarrow & 0 \\ & & & & \| & & & & \\ & & & & G & & & & \end{array}$$

By Lemma (A.26), the conjugation action of G on itself is trivial on $\ker(G \longrightarrow E(R))$, and G acts trivially on $\pi_i(F(R)), i \geq 2$ (i.e., $F(R)$ is simple in dimensions $i \geq 2$). Thus the above exact sequence (*) is a central extension of $E(R)$, such that $H_1(G,\mathbb{Z}) = H_2(G,\mathbb{Z}) = 0$, i.e., by Proposition (1.10) is isomorphic to the universal central extension

$$0 \longrightarrow K_2(R) \longrightarrow St(R) \longrightarrow E(R) \longrightarrow 0$$

In particular $G \cong St(R)$.

□

Corollary 11. $\pi_i\left(BGL(R)^+\right) \cong K_i(R), i = 1,2,$ *and* $\pi_3\left(BGL(R)^+\right) \cong H_3(St(R),\mathbb{Z})$.

Theorem 54 (Recognition criteria)**.** *The map* $i : BGL(R) \to BGL(R)^+$*is universal for maps into H-spaces. That is, for each map* $f : BGL(R) \to H$*, where* H *is an H-space, there is a map* $g : BGL(R)^+ \to H$ *so that* $f = g$ *and such that the induced* map $\pi_i\left(BGL(R)^+\right) \to \pi_i(H)$ *is independent of* g*.*

18.2 Exact categories

Chapter 19

Volodin's K-theory

References [19, 16, 15]

Volodin constructed a model for $X(R)$, that we denote by $V(R)$.
For each n, let $T_n(R)$ denote the subgroup of $GL_n(R)$ consisting of upper triangular matrices with 1's on the diagonal. As n varies, the union of these groups forms a subgroup $T(R)$ of $GL(R)$. Similarly we may regard the permutation groups Σ_n as subgroups of $GL_n(R)$ by their representation as permutation matrices, and their union (the infinite permutation group Σ_∞) is a subgroup of $GL(R)$. For each $\sigma \in \Sigma_n$, let $T_n^\sigma(R)$ denote the subgroup of $GL_n(R)$ obtained by conjugating $T_n(R)$ by σ. For example, if $\sigma = (n, \ldots, 1)$, then $T_n^\sigma(R)$ is the subgroup of lower triangular matrices.

Since the classifying spaces $BT_n(R)$ and their conjugates $BT_n(R)^\sigma$ are subspaces of $BGL_n(R)$, and hence of $BGL(R)$, we may form their union over all n and σ : $X(R) = \bigcup_{n,\sigma} BT_n(R)^\sigma$.

Theorem 55. *The space $X(R)$ is acyclic.*

Let G be a group and $\{G_i\}_{i\varepsilon I}$ a family of subgroups. Define $V(G, \{G_i\})$, or just $V(G)$ to be the simplicial complex, whose vertices are the elements of G, where $g_0, \ldots, g_p$ $\left(g_i \neq g_j\right)$ form a p-simplex if for some G_i all the elements $g_j g_k^{-1}$ lie in G_i. If H is another group with a family of subgroups $\left\{H_j\right\}$ and $\phi : G \to H$ is a homomorphism sending each G_i into some H_j, then ϕ induces a simplicial map $V(\phi) : V(G) \to V(H)$.

In many situations it is more convenient to use simplicial sets instead of simplicial complexes: Denote by $W(G, \{G_i\})$ the geometric realization of the simplicial set whose p-simplices are the sequences $\left(g_0, \ldots, g_p\right)$ of elements of G (not necessarily distinct) such that for some G_i al1 $g_j g_k^{-1}$ lie in G_i, the r-th face (resp. degeneracy) of this simplex being obtained by omitting g_r (resp., repeating g_r). Associating with any p-simplex $\left(g_0, \ldots, g_p\right)$ the linear singular simplex of the space $V(G)$ which sends the i-th vertex of the standard simplex to g_j, we obtain a map of simplicial sets from $W(G)$ to the simplicial set of singular simplices of $V(G)$ and hence a cellular map (linear on any simplex) from $W(G)$ to $V(G)$. This map is a homotopy equivalence ....

Suppose that R is a ring, n a natural number and σ a partial ordering of $\{1, \ldots, n\}$. Define $T_n^\sigma(R)$ to be the subgroup of $GL_n(R)$ consisting of the α with $\alpha_{ij} = 1$ and $\alpha_{ij} = 0$ if $i\&j$. Subgroups of this form will be called triangular subgroups of $GL_n(R)$. The space $V\left(GL_n(R), \{T_n^\sigma(R)\}\right)$ will be denoted by $V_n(R)$. Since any partial ordering may be extended to a linear ordering, it suffices to consider linear orderings when defining $V_n(R)$. The natural embedding $GL_n \hookrightarrow GL_{n+1}(R)$ defines an embedding $V_n(R) \longleftrightarrow V_{n+1}(R)$ and we'l1 define $V_\infty(R)$ as $\lim\limits_{\to} {}_n V_n(R)$.

Finally for $i \geq 1$, put

$$k_{i,n}(R) = \pi_{i-1}\left(V_n(R)\right)$$

and $k_i(R) = k_{i,\infty}(R) = \lim_{\to} k_{i,n}(R)$ (compare [26], [27]). Evidently $K_{1,n}(R) = GL_n(R)/E_n(R)$ and $K_{i,n}(R)$ is a group if $i \geq 2$, and this group is abelian if $i \geq 3$. Moreover the $K_i(R)$ are abelian groups for all $i \geq 1$ (see [26], [27]). The connected component of $V_n(R)$ passing through T_n equals $V\left(E_n(R), \{T_n^\sigma(R)\}\right)$. It is easy to show that the universal covering space of $V_n\left(E_n(R), \{T_n^\sigma(R)\}\right)$ equals $V\left(St(R), \{T_n^\sigma(R)\}\right)$, where T_n^σ is identified with the subgroup of $St_n(R)$ generated by the $x_{ij}(a)$ with a $\varepsilon R, i \overset{\sigma}{<} j (n \geq 3)$. Hence

Lemma 10. *$K_{2,n}(R) = \ker\left(St_n(R) + E_n(R)\right)$, and $K_{i,n}(R) = \pi_{i-1}\left(V\left(St_n(R)\right)\right) = \pi_{i-1}\left(W\left(St_n(R)\right)\right)$ if $\quad i \geq 3 \quad (n \geq 3)$.*

Let's define $\overline{St}_n(R)$ to be the inverse image of $GL_n(R)$ under the projection $St(R) \to E(R)$. There is a canonical homomorphism $St_n(R) \to \overline{st}_n(R)$ and stability for K_1, k_2 ([10], [20], [22]) shows that this homomorphism is surjective if $n \geq s.r.R + 1$ and bijective if $n \geq s.r.R + 2$. The spaces $W\left(St_n(R)\right)$ and $W\left(\overline{St}_n(R)\right)$ will play an essential role in the sequel. We'll denote them by $W_n(R), \bar{W}_n(R)$, resp. (So $W_n(R) = \bar{W}_n(R)$ if $n \geq$ s.r. $R + 2$.)

Lemma 11. *Denote the canonical embedding $\bar{W}_n(R) \longleftrightarrow \bar{W}_{n+1}(R)$ by u_n. If $n \geq s \cdot r.R$ and $x \in \overline{St}_{n+1}(R)$, then u_n and $u_n \cdot x$ are homotopic. (Here $\left(u_n \cdot x\right)(g) = \left(u_n(g)\right) \cdot x \cdot$))*

Lemma 12. *For any $s \in S_{n+1}$ the embeddings u_n and u_n^s are homotopic.*

For any simplicial set X we'll denote by $C_*(X)$ its chain complex, i.e., the complex of abelian groups with $C_p(x)$ equal to the free abelian group generated by the p-simplices of X and each differential equal to an alternating sum of homomorphisms induced by taking faces. It is well known that $C_*(X)$ is homotopy equivalent to the singular complex of the geometric realization of X. In view of (1.5) the maps of complexes $C_*\left(u_n\right), C_*\left(u_n(n, n+1)\right) : C_*\left(\bar{W}_n(R)\right) + C_*\left(\bar{W}_{n+1}(R)\right)$ are homotopic. Looking through the proof of (1.5) one sees that the corresponding homotopy operator $\phi_{n+1}^k : C_p\left(\bar{W}_n(R)\right) + C_{p+1}\left(\bar{W}_{n+1}(R)\right)$ may be taken in the following form: (We denote $x_{k,n+1}(1)$ by x_k and

$$
\begin{aligned}
& x_{n+1,k}(-1) \text{ by } y_k) \\
& \phi_{n+1}^k\left(\alpha_0, \ldots, \alpha_p\right) = \sum_{i=0}^{p} (-1)^{i+1} \left[\left(\alpha_0^{x_k y_k}, \ldots, \alpha_i x_k y_k, \alpha_i^{(k,n+1)}, \ldots, \alpha_p^{(k,n+1)}\right)\right. \\
& \quad - \left(\alpha_0^{x_k y_k}, \ldots, \alpha_i{}^x y_k, \alpha_i x_k y_k, \ldots, \alpha_p^{x_k y_k}\right) \\
& \quad + \left(\alpha_0^{x_k} \cdot y_k, \ldots, \alpha_i^{x_k} \cdot y_k, \alpha_i{}^{x_k y_k}, \ldots, \alpha_{pk} y_k\right) - \left(\alpha_0 y_k, \ldots, \alpha_i y_k, \alpha_i, \ldots, \alpha_p\right) \\
& \quad \left. + \left(\alpha_0 y_k, \ldots, \alpha_i y_k, \alpha_i^{x_k} \cdot y_k, \ldots, \alpha_p^{x_k} \cdot y_k\right) - \left(\alpha_0 y_k, \ldots, \alpha_i y_k, \alpha_i y_k, \ldots, \alpha_p y_k\right)\right]
\end{aligned}
$$

Lemma 13. *The homotopy operators ϕ_{n+1}^k have the following properties:*

1. *$(\partial - \alpha(k, n+1)) = d\phi_{n+1}^k(\alpha) + \phi_{n+1}^k(d\alpha)$, where $\alpha = \left(\alpha_0, \ldots, \alpha_p\right)$ is a p-simplex of $\bar{W}_n(R)$.*
2. *$\phi_{n+1}^n \mid C_*\left(\bar{W}_{n-1}(R)\right) = 0$.*
3. *For any $s \in S_n$ the following formula is valid:*

$$\phi_{n+1}^k\left(\alpha^s\right) = \left[\phi_{n+1}^s(k)(\alpha)\right]^s$$

4. *$\phi_{n+1}^k \mid C_*\left(\bar{W}_{n-1}(R)\right) = \left(\phi_n^k\right)(n+1, n)$*

Lemma 14. *Suppose $c \in C_p\left(\bar{W}_{n-q}(R)\right)$, dc &$C_{p-1}\left(\bar{W}_{n-q-1}(R)\right)$. Set*

$$
\begin{aligned}
& c_0 = c, c_1 = \phi_{n-q+1}^{n-q}\left(c_0\right) \& c_{p+1}\left(\bar{W}_{n-q+1}(R)\right), \ldots, c_k \\
& = \phi_{n-q+k}^{n-q+k-1}\left(c_{k-1}\right) \varepsilon c_{p+k}\left(\bar{w}_{n-q+k}(R)\right). \textit{ Then, if } k \geq 1, \textit{ we have:} \\
& dc_k = c_{k-1} - c_{k-1}^{(n-q+k, n-q+k-1)} + \ldots + (-1)^k c_{k-1}^{(n-q+k, \ldots, n-q)}.
\end{aligned}
$$

19.0.1 The Aciclicity Theorem

If X is an arbitrary set, we'll denote by $F_m(X)$ the partially ordered set of functions defined on non-empty subsets of $\{1, \ldots, m\}$ and taking values in X. The partial ordering is defined as follows:

$$f \leq g \Leftrightarrow \operatorname{dom} f \subset \operatorname{dom} g, \, g|_{\operatorname{dom}} \, f = f.$$

(Here dom f is the subset of $\{1, \ldots, m\}$ where f is defined). Following van der Kallen [11] we'll say that $F \subset F_m(X)$ satisfies the chain condition if F contains with any function all its restrictions (to non-empty subsets of its domain). It is clear that f and g have a common restriction if and only if there exists i $\varepsilon\{1, \ldots, m\}$ such that f and g are defined at i and equal at i. In this case there obviously exists a maximal common restriction inf(f, g).

If $F \subset F_m(X)$ satisfies the chain condition, then by F_* we'11 denote the geometric realization of the semi-simplicial set, whose non-degenerate p-simplices are the functions $f \varepsilon F$ with $|\operatorname{dom} f| = p+1$, and whose faces are defined by the formulas $d_j(f) = f|_{\{i_0, \ldots, \hat{i}_j, \ldots, i_p\}}$ where $\{i_0, \ldots, i_p\} = \operatorname{dom} f, (i_0 < \ldots < i_p)$. If $f \varepsilon F, |\operatorname{dom} f| = p+1$, then by $|f|$ we'll denote the corresponding p-simplex of F_*. It is clear that $|f| \cap |g|$ is either empty or else equals $|\inf(f, g)|$. In particular, F_* is a simplicial space [7].

Let R be a ring (associative with identity), R^∞ the free left R-module on the basis $e_1, \ldots, e_n, \ldots$, and R^n its submodule generated by $e_1, \ldots, e_n$. If X is any subset of R^∞, then by $U_m(X)$ we' 11 denote the subset of $F_m(X)$ consisting of those functions f for which $f(i_0), \ldots, f(i_p)$ is a unimodular frame (i.e., a basis of a free direct summand of R^∞), where $\{i_0, \ldots, i_p\} = \operatorname{dom}(f)$.

Theorem 56. *Suppose R is a ring, $r =$ s.r.R and m, n are natural numbers. Then $U_m(R^n)$ is $\min(m-2, n-r-1)$-acyclic.*

Corollary 12. $U_n(R^n)$ *is* $(n-r-1) - \mathrm{acyc}\,1$ *ic.*

Corollary 13. *Consider in* $\mathrm{St}_{n+1}(\Lambda)$ *the following subgroups: $A^i = \{\alpha : e_i \cdot \pi(\alpha) = e_i\}$ $(i = 1, \ldots, n+1)$ and consider the simplicial set $Z'(St_{n+1}(R), A^i)$ constructed as in (2.5), but using left cosets instead of right cosets. This simplicial set is $(n-r)$-acyclic.*

Chapter 20

Other important results

20.1 Whitehead

20.2 Bass

Part V

Homological stability

Chapter 21

Motivation

[8]

The symmetric group Σ_n is the group of bijections of the finite set $\underline{n} = \{1, \ldots, n\}$, under composition. The classifying space BG of a discrete group G, such as Σ_n, is the connected space determined uniquely up to weak homotopy equivalence by the property

$$\pi_*(BG) = \begin{cases} G & \text{if } * = 1, \\ 0 & \text{otherwise} \end{cases}$$

It can be constructed by extracting from G the groupoid $*//G$ given by: - a single object *, - morphisms given by $* \xrightarrow{g} *$ for $g \in G$, and - composition given by multiplication.

We then take its nerve to obtain a simplicial set, and take the geometric realisation to get a topological space $|N(*//G)|$; this is a model for BG. Exercise 1.3.1 proves it indeed has the desired property.

Proposition 42. *$H_*(B\Sigma_n; \mathbb{Z})$ is the same as computing the group homology of Σ_n with coefficients in $\mathbb{Z}$.*

Let us compute these groups and the homology of their classifying spaces for the first few values of n.

Example 30. 1. *For $n = 0, 1$, the group Σ_n is trivial so its classifying space is weakly contractible and hence has trivial homology.*

2. *Example 1.1.4. For $n = 2, \Sigma_2$ is isomorphic to the cyclic abelian group $\mathbb{Z}/2$. Then $B\mathbb{Z}/2$, as constructed above, is homotopy equivalent to $\mathbb{R}P^\infty$. We conclude that*

$$H_*(B\mathbb{Z}/2; \mathbb{Z}) = H_*(\mathbb{R}P^\infty; \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } * = 0 \\ \mathbb{Z}/2 & \text{if } * > 0 \text{ is odd,} \\ 0 & \text{if } * > 0 \text{ is even.} \end{cases}$$

3. *Example 1.1.5. For $n = 3$, the group Σ_3 is the dihedral group D_3 with 6 elements (i.e. the symmetries of a triangle). A more complicated computation given in Exercise 1.3.5 yields the homology of D_3 :*

$$H_*(BD_3; \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } * = 0 \\ \mathbb{Z}/2 & \text{if } * > 0 \text{ and } * \equiv 1 \pmod 4 \\ \mathbb{Z}/6 & \text{if } * > 0 \text{ and } * \equiv 3 \pmod 4, \\ 0 & \text{otherwise} \end{cases}$$

Conjectures

1. Each reduced homology group $\widetilde{H}_d\left(B\Sigma_n;\mathbb{Z}\right)$ is finite and has small exponent.
2. The homology in fixed degree $* = d$ becomes independent of n as $n \to \infty$.
3. Before becoming independent of n, the homology only increases in size.
4. The p-power torsion only changes when $p \mid n$.

If we want to attempt to prove (2)-(4), we need a better way to compare the homology groups for different n than just as abstract abelian groups. This is done by observing that the inclusion $\underline{n} \hookrightarrow \underline{n+1}$ of finite sets gives a homomorphism

$$\sigma : \Sigma_n \longrightarrow \Sigma_{n+1},$$

by extending a permutation of $\underline{n}$ by the identity on $n+1 \in \underline{n+1}$ to a permutation of $n+1$. Our construction of BG is natural in groups and homomorphisms, so this homomorphism induces a map

$$\sigma : B\Sigma_n \longrightarrow B\Sigma_{n+1},$$

which in turn induces a map $\sigma_* : H_*\left(B\Sigma_n;\mathbb{Z}\right) \to H_*\left(B\Sigma_{n+1};\mathbb{Z}\right)$ on homology. We can then give sharper formulations of (2)-(4) in terms of these stabilisation maps: (2′) The maps σ_* are isomorphisms in a range increasing with n.
(3′) The maps σ_* are injective.
(4′) The maps σ_* are isomorphisms on p-power torsion unless $p \mid n+1$.
Property (1) holds for all finite groups, and the result which proves it also implies (4′):

Proposition 43. *For a finite group $G, \widetilde{H}_*(BG;\mathbb{Z}[1/|G|]) = 0$. More generally, for $H \subset G$ the map $\iota_* : H_*(BH;\mathbb{Z}[1/[G:H]]) \to H_*(BG;\mathbb{Z}[1/[G:H]])$ admits a right inverse τ (i.e. $\iota_* \circ \tau = \mathrm{id}$).*

To deduce (4′) from Proposition 1.1.6, note that $[\Sigma_{n+1} : \Sigma_n] = n+1$ so by the long exact sequence on homology groups so that $H_*\left(B\Sigma_n;\mathbb{Z}\right) \to H_*\left(B\Sigma_{n+1};\mathbb{Z}\right)$ is surjective after inverting $n+1$. Now set $n+1$ equal to p and invoke (3′). It is phenomenon indicated by (2′) that is the subject of this minicourse:

A sequence $X_0 \xrightarrow{\sigma} X_1 \xrightarrow{\sigma} X_2 \xrightarrow{\sigma} \cdots$ exhibits **homological stability** if the maps $\sigma_* : H_*\left(X_n;\mathbb{Z}\right) \to H_*\left(X_{n+1};\mathbb{Z}\right)$ are isomorphisms in a range of degrees * increasing with n.

In the next two lectures we will prove the following result, due to Nakaoka [Nak60] (though he proved much more):

Theorem 57. *The sequence $B\Sigma_0 \xrightarrow{\sigma} B\Sigma_1 \xrightarrow{\sigma} B\Sigma_2 \xrightarrow{\sigma} \cdots$ exhibits homological stability. More precisely, the induced map*

$$\sigma_* : H_*\left(B\Sigma_n;\mathbb{Z}\right) \longrightarrow H_*\left(B\Sigma_{n+1};\mathbb{Z}\right)$$

is surjective if $* \leq \frac{n}{2}$ and an isomorphism if $* \leq \frac{n-1}{2}$.

Remark 1.1.9. Of course, if we know property (3′) holds then the range in the previous theorem in which σ_* is an isomorphism improves to $* \leq \frac{n}{2}$. However, property (3 ′) is rather special—related to the existence of transfer maps-and you should not expect it to hold for general sequences of classifying spaces of groups. We will not comment on it again, but see Exercise 1.3.6.
Remark 1.1.10. The ranges in the previous remark are optimal among those of the form $* \leq an+b$ with $a, b \in \mathbb{Q}$.

21.1 Applications

Homological stability is a structural property of a sequence of groups, or more generally topological spaces, but it is also useful tool. In fact, many homological stability theorems are proven in service of obtaining other mathematical results. To illustrate this, I now want to explain some straightforward applications of Theorem 1.1.8. These concern the transfer of information from low n to high n and vice-versa. They can be obtained by other methods as well, but their generalisations to other sequences of groups often can not.

21.1.1 Altenating groups

Recall that for path-connected X, the Hurewicz map $\pi_1(X) \to H_1(X;\mathbb{Z})$ coincides with abelianisation (we are suppressing the basepoint). In particular, the map $G \to H_1(BG;\mathbb{Z})$ induces an isomorphism $G^{\mathrm{ab}} \to H_1(BG;\mathbb{Z})$ naturally in G. Thus we can understand the abelianisation of Σ_n by computing its first homology group. The sign homomorphism sign: $\Sigma_n \to \mathbb{Z}/2$ yields a map

$$\text{sign: } B\Sigma_n \longrightarrow B\mathbb{Z}/2,$$

which induces a map on homology. This is compatible with stabilisation, in the sense that sign $\circ\sigma = \text{sign}$, so we get a commutative squares

$$\begin{array}{cc} H_1\left(B\Sigma_{n-1};\mathbb{Z}\right) \xrightarrow{\sigma_*} H_1\left(B\Sigma_n;\mathbb{Z}\right) & \\ \downarrow_{\text{sign}} & \mid \text{ sign} \\ \mathbb{Z}/2 \overset{\mathbb{Z}}{=\!=} /2. & \end{array}$$

The map $H_1\left(B\Sigma_2;\mathbb{Z}\right) \to \mathbb{Z}/2$ is an isomorphism because sign: $\Sigma_2 \to \mathbb{Z}/2$ is. By Theorem 1.1.8, in the commutative diagram the right-most top horizontal map is surjective and the other top horizontal maps are isomorphisms. A single diagram chase then deduces from the fact that the left-most vertical map is an isomorphism that all other vertical maps are.

Thus we have used homological stability to prove that

$$\text{sign: } \Sigma_n \longrightarrow \mathbb{Z}/2$$

is the abelianisation for $n \geq 2$, or equivalently that the kernel of the sign homomorphism is exactly the subgroup $[\Sigma_n, \Sigma_n]$ generated by commutators. Recalling that this kernel is exactly the alternating group A_n, we conclude that:

Theorem 58. $[\Sigma_n, \Sigma_n] = A_n$.

Remark 1.2.2. This is a fact you likely knew already, and elementary group-theoretic arguments exist. We could have used this fact instead to give an elementary proof of Theorem 1.1.8 in degree $* = 1$.

21.2 Group Completion

Homological stability implies that for in fixed degree $*$, for n sufficienty large the canonical map

$$H_*\left(B\Sigma_n;\mathbb{Z}\right) \longrightarrow \operatorname*{colim}_{n\to\infty} H_*\left(B\Sigma_n;\mathbb{Z}\right)$$

is an isomorphism; the right hand side is known as the stable homology. This has two somewhat tautological consequences: 1. We can compute the right side from the left side. 2. We can compute the left side from the right side.

This is particularly interesting because the stable homology on the right side has a more familiar description.

When we constructed the stabilisation map, we used that inclusion $\underline{n} \to \underline{n+1}$ yields a homomorphism $\Sigma_n \to \Sigma_{n+1}$. More generally, disjoint union induces a homomorphism $\Sigma_n \times \Sigma_m \to \Sigma_{n+m}$, which yields "multiplication" maps

$$B\Sigma_n \times B\Sigma_m \longrightarrow B\Sigma_{n+m},$$

making the space $\bigsqcup_{n \geq 0} B\Sigma_n$ into a unital topological monoid (these are associative but not commutative, and it is probably better to say E_1-space since that is a homotopy-invariant notion).

Theorem 59 (McDuff-Segal)**.** *If M is a homotopy-commutative unital associative topological monoid, then $H_*(M;\mathbb{Z})\left[\pi_0^{-1}\right] \cong H_*(\Omega BM;\mathbb{Z})$.*

21.3 Serre's finiteness theorem and variations

Let us now use Corollary 1.2.6. By (1) the groups $H_*\left(B\Sigma_n;\mathbb{Z}\right)$ are finite for $* > 0$. By Theorem 1.1.8 the same is true for the stable homology as long as restrict to degrees $* \leq \frac{n}{2}$. Since n is arbitrary, the stable homology is finite in all positive degrees. This has the following consequence:

Theorem 60. *$\pi_*(\mathbb{S})$ is finite for all $* > 0$.*

Exercise 1.3.8 (Using Serre's finiteness theorem). Serre proved that $\pi_*(\mathbb{S})$ is finite for $* > 0$. Combine this with Corollary 1.2.6 and Exercise 1.3.6 to prove that the sequence $B\Sigma_0 \xrightarrow{\sigma} B\Sigma_1 \xrightarrow{\sigma} B\Sigma_2 \xrightarrow{\sigma} \cdots$ exhibits homological stability. (Hint: you will not be able to give an explicit range.)

Remark 1.3.9. See [McD75] for a similar qualitative argument for configuration spaces of manifolds.

21.4 Computations with Homological Stability

[17]

Chapter 22

Stability in algebraic K-theory

22.1 Previous work

Theorem 61. *The canonical map $k^{Q}_{i,n}(R) \to k^{Q}_{i,n+1}(R)$ is surjective for $n \geq 2i + \max(s.r.R - 1, 1) - 1$ and bijective for $n \geq 2i + \max(s.r.R - 1, 1) + 1$.*

A common feature in all these papers is the approach to stability problems for higher K-groups through stability for homology of linear groups.

22.2 Suslin's work

[15]

Theorem 62. *Let R be a ring, $r = s.r.R$. The canonical homomorphism $k_{i,n}(R) \to k_{i,n+1}(R)$ is surjective for $n \geq r + i - 1$ and bijective for $n \geq r + i$.*

Proposition 44. *If $q \leq n - r$, then the differentials d^{t}_{pq} are trivial for $t \geq 2$. Moreover $E^{\infty}_{p,q} = 0$ for $0 < q \leq n - r$.*

Corollary 14. *If $n \geq r + i$ then the action of $St_n(R)$ and of S_n on $K_{i,n}(R)$ is trivial.*

22.2.1 Homotopy fiber of Quillen's plus construction

Suppose that G is a group, H a perfect normal subgroup and $BG \to \mathrm{BG}^{+}$Quillen's plus construction relative to H. Let Y be the homotopy fiber of $BG \to BG^{+}$.

Lemma 15. *1. a) Y has the homotopy type of a CW-complex. b) Y is connected, $\pi_1(Y)$ is a universal central extension of the perfect group H (see* [15]*), $\pi_j(Y)$ acts trivially on $\pi_i(Y)(i \geq 2)$. c) $\tilde{H}_\star(Y) = 0$. d) $\pi_i(Y) = \pi_{i+1}(BG^{+})$for $i \geq 2$.*
These properties characterize Y up to homotopy equivalence.

22.3 Aciclicity Theorem II

Theorem 63. *Suppose that $j_j, \ldots, j_r$ are distinct indices. Then the space $Z\left(T; T^{j}1, \ldots, T^{j}r\right)$ (see* §2*) is $\left[\frac{r-3}{2}\right]$-acyc 1 ic.*

Theorem 64. $\tilde{H}_p\left(x_n(R)\right) = 0$ *for* $n \geq 2p + 1$

Corollary 15. $H_p\left(X_{2p+1}(R)\right) = H_p\left(X_{2p+2}(R)\right) = \ldots = H_p\left(X_\infty(R)\right)$.

Corollary 16. *The canonical homomorphism* $H_p\left(X_{2p+1}(R)\right) \to H_p\left(X_\infty(R)\right)$ *equals zero.*

22.4 Stability in Quillen's K-theory

Theorem 65. *If* $n \geq 2i+1$*, then there exists a canonical homomorphism* $k_{i,n}(R) + k^Q_{i,n}$*. This homomorphism is surjective for* $n \geq \max(2i+1, s.r.\ R+i-1)$ *and bijective for* $n \geq \max(2i+1, s.r.\ R+i)$*.*

Theorem 66. *The canonical homomorphism* $k^Q_{i,n}(R) \to k^Q_{i,n+1}(R)$ *is surjective for* $n \geq \max(2i, s.r.R + i-1)$ *and bijective for* $n \geq \max(2i+1, s.r.\ R+i)$*.*

Corollary 17. *The canonical homomorphism* $\mathrm{H}_i\left(\mathrm{GL}_n(R)\right) \to \left(\mathrm{H}_i\left(\mathrm{GL}_{n+1}(R)\right)\right.$ *is surjective for* $n \geq \max(2i, s.r.R+i-1)$ *and bijective for* $n \geq \max(2i+1, s\cdot r.R+i)$

It seems reasonable in view of [20], [12], [21] to suppose that for essentially commutative rings (i.e., rings that are finitely generated as a module over their center) the group $St_n(R)$ acts trivially on $K_{i,n}(R)$ for $n \geq i+2$ and hence $k_{i,n}(R) = k^Q_{i,n}(R)$ for $n \geq 2i+1$.

Chapter 23

Homological stability for general linear groups

References:[9, 15, 20]

Chapter 24

Homological stability for unitary and symplectic groups I

[11, 12]

Chapter 25

Homological stability for symmetric groups

[8]

Theorem 67. *The sequence $B\Sigma_0 \xrightarrow{\sigma} B\Sigma_1 \xrightarrow{\sigma} B\Sigma_2 \xrightarrow{\sigma} \cdots$ exhibits homological stability. More precisely,*

$$\sigma_* : H_*\left(B\Sigma_n; \mathbb{Z}\right) \longrightarrow H_*\left(B\Sigma_{n+1}; \mathbb{Z}\right)$$

is surjective if $* \leq \frac{n}{2}$ and an isomorphism if $* \leq \frac{n-1}{2}$.

Chapter 26

Homological stability for unitary and symplectic groups II

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