PhD Studies

Abraham Rojas Vega

June 20, 2024

Contents

Topics of Algebraic Topology

Chapter 1

Simplical sets and complexes

Simplicial complexes are more intuitive, and are the foundation of algebraic topology. Simplicial sets were also called simplicial schemes and semi-simplicial complexes.

1.1 Simplical complexes

1.2 Simplical sets

Let $\mathbf{\Delta}$ be the category of finite ordinal numbers, with order-preserving maps between them. More precisely, the objects for Δ consist of elements $\mathbf{n}, n \geq 0$, where $\mathbf{n}$ is a string of relations

$$0 \to 1 \to 2 \to \cdots \to n$$

(in other words $\mathbf{n}$ is a totally ordered set with $n+1$ elements). A morphism $\theta : \mathbf{m} \to \mathbf{n}$ is an order-preserving set function, or alternatively a functor. We usually commit the abuse of saying that $\mathbf{\Delta}$ is the ordinal number category.

A simplicial set is a contravariant functor $X : \Delta^{op} \to$ Sets, where Sets is the category of sets.

Example 1. *Let $\mathbf{\Delta}$ be the category of finite ordinal numbers, with order-preserving maps between them. More precisely, the objects for Δ consist of elements $\mathbf{n}, n \geq 0$, where $\mathbf{n}$ is a string of relations*

$$0 \to 1 \to 2 \to \cdots \to n$$

(in other words $\mathbf{n}$ is a totally ordered set with $n+1$ elements). A morphism $\theta : \mathbf{m} \to \mathbf{n}$ is an order-preserving set function, or alternatively a functor. We usually commit the abuse of saying that $\mathbf{\Delta}$ is the ordinal number category.

A simplicial set is a contravariant functor $X : \Delta^{op} \to$ Sets, where Sets is the category of sets.

1.3 CW-complexes

1.4 Homotopy theory

Chapter 2

K-theory constructions

2.1 Volodin's K-theory

Let G be a group and $\{G_i\}_{i\varepsilon I}$ a family of subgroups. Define $V(G, \{G_i\})$ to be the simplicial scheme, alias simplicial complex, (and also its geometric realization, i.e., RV $(G, \{G_i\})$ in the notation of, Ch. V, Prop. 7.16), whose vertices are the elements of G, where $g_0, \ldots, g_p\,(g_i \neq g_j)$ form a p-simplex if for some G_i all the elements $g_j g_k^{-1}$ lie in G_i. We'11 often shorten the notation to $V(G)$. If H is another group with a family of subgroups $\{H_j\}$ and $\phi : G \to H$ is a homomorphism sending each G_i into some H_j, then ϕ induces a simplicial map $V(\phi) : V(G) \to V(H)$.

In many situations the space $V(G)$ is not convenient from a technical point of view and it is more convenient to use simplicial sets ([8], Ch. II) instead of simplicial schemes: Denote by $W(G, \{G_i\})$ the geometric realization ([8], Ch. III §3) of the (semi)simplicial set whose p-simplices are the sequences $(g_0, \ldots, g_p)$ of elements of G (not necessarily distinct) such that for some G_i al1 $g_j g_k^{-1}$ lie in G_i, the r-th face (resp. degeneracy) of this simplex being obtained by omitting g_r (resp., repeating g_r). Associating with any p-simplex $(g_0, \ldots, g_p)$ the linear singular simplex of the space $V(G)$ which sends the i-th vertex of the standard simplex to g_j, we obtain a map of simplicial sets from $W(G)$ to the simplicial set of singular simplices of $V(G)$ and hence a cellular map (linear on any simplex) from $W(G)$ to $V(G)$. This map is a homotopy equivalence as one sees from the following lemmas.

2.2 Milnor's K-theory

2.3 Whitehead's K-theory

2.4 Quillen's K-theory

Chapter 3

Homological stability