PhD Studies

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Contents

IV   K-theory   37

V   Homological stability   43

Part I

Topics of Algebra

Chapter 1

Category Theory

Basic concepts

1. For a topological spaces, the category of open sets with inclusions as morphisms. The opposite of this category, denoted by $\mathfrak{U}$, is used in sheaf theory.

2. If $\mathscr{A}$ and $\mathscr{B}$ are preordered sets, then functors between them are monotone maps.

Let $F : \mathscr{A} \to \mathscr{B}$ be a functor.

1. F is **faithful** provided that all the **hom-set restrictions**

$$F : \hom_{\mathscr{A}}\left(A, A'\right) \to \hom_{\mathscr{B}}\left(FA, FA'\right)$$

   are injective.

2. F is **full** if all hom-set restrictions are surjective.

3. F is an **embedding** if and it is faithful and injective on the class of objects.

4. F is **essentially surjective** if for every object B of $\mathscr{B}$, there is an object A of $\mathscr{A}$ such that FA is isomorphic to B.

5. If F is essentially surjective and fully faithful, it is called an **equivalence of categories**, and $\mathscr{A}$ and $\mathscr{B}$ are said to be **equivalent**.

1.1 Natural transformations

Let $F, G : \mathbf{A} \to \mathbf{B}$ be functors. A natural transformation τ from F to G (denoted by $\tau : F \to G$ or $F \xrightarrow{\tau} G$) is a function that assigns to each **A**-object A a **B**-morphism $\tau_A : FA \to GA$ in such a way that the following naturality condition holds: for each A-morphism $A \xrightarrow{f} A'$, the diagram

$$\begin{array}{ccc} FA & \xrightarrow{\tau_A} & GA \\ {\scriptstyle Ff}\downarrow & & \downarrow{\scriptstyle Gf} \\ FA' & \xrightarrow{\tau_{A'}} & GA' \end{array}$$

commutes.

A natural transformation $F \xrightarrow{\tau} G$ whose components τ_A are isomorphisms is called a **natural isomorphism** from F to G. F and G are said to be **naturally isomorphic**, and we write $F \cong G$.

Example 1. *1. Let $U : \mathrm{Grp} \to Set$ be the forgetful functor, and let $S : \mathrm{Grp} \to Set$ be the "squaring-functor", defined by $S(G \xrightarrow{f} H) = G^2 \xrightarrow{f^2} H^2$. For each group G, its multiplication is a function $\tau_G : G^2 \to G$. The family $\tau = (\tau_G)$ is a natural transformation from S to U. The naturality condition simply means that $f(x \cdot y) = f(x) \cdot f(y)$ for any group homomorphism $G \xrightarrow{f} H$ and any $x, y \in G$. Thus "multiplication" in groups can be regarded as a natural transformation. Similar for other structures.*

2. *Let $(\hat{\imath}): Vec \to Vec$ be the second-dual functor for vector spaces, then $\tau_V : V \to \hat{\hat{V}}$, defined by $(\tau_V(x))(f) = f(x)$, yield a natural transformation $id_{Vec} \xrightarrow{\tau} (\hat{\wedge})$. It becomes a natural isomorphism when restricted to finite-dimensional vector spaces.*

3. *The assignment of the Hurewicz homomorphism $\pi_n(X) \to H_n(X)$ to each topological space X is a natural transformation from the n-th homotopy functor $\pi_n : Top \to Grp$ to the n-th homology functor $H_n : Top \to Grp$.*

4. *If $B \xrightarrow{f} C$ is an **A**-morphism, then $\hom_{\mathbf{A}}(C,-) \xrightarrow{\tau_f} \hom_{\mathbf{A}}(B,-)$, defined by $\tau_f(g) = g \circ f$, and $\hom_{\mathbf{A}}(-,B) \xrightarrow{\sigma_f} \hom_{\mathbf{A}}(-,C)$, defined by $\sigma_f(g) = f \circ g$, are natural transformations.*

5. *(Good definitions of extension) Let $F : Set \to Vec$ be a functor that assigns to each set X a vector space FX with basis X, and to each function $X \xrightarrow{f} Y$ the unique linear extension $FX \xrightarrow{Ff} FY$ of f. This actually is not a correct definition of a functor, since there are many different vector spaces with the same basis. However, the definition is "correct up to natural isomorphism". Whenever we choose, for each set X, a specific vector space FX with basis X, we do obtain a functor $F : Set \to Vec$ (since the above condition determines the action of F on functions uniquely). Furthermore, any two functors that are obtained in this way are naturally isomorphic.*

6. *For any 2-element set A, $\hom(A,-)$ is naturally isomorphic to the squaring functor S^2[3.20(10)] and $\hom(-,A)$ is naturally isomorphic to the contravariant power-set functor $\mathscr{Q}$[3.20(9)]. If B is isomorphic to A, then $\hom(A,-)$ and $\hom(B,-)$ are naturally isomorphic with those functors, the converse is true.*

Proposition 1. *A functor $\mathbf{A} \xrightarrow{F} \mathbf{B}$ is an equivalence if and only if there exists a functor $\mathbf{B} \xrightarrow{G} \mathbf{A}$ such that $id_{\mathbf{A}} \cong G \circ F$ and $F \circ G \cong id_{\mathbf{B}}$.*

1.2 Limits and colimits

Let $A \underset{g}{\overset{f}{\rightrightarrows}} B$ be a pair of morphisms. A morphism $E \xrightarrow{e} A$ is called an equalizer of f and g provided that the following conditions hold: (1) $f \circ e = g \circ e$, (2) for any morphism $e' : E' \to A$ with $f \circ e' = g \circ e'$, there exists a unique morphism $\bar{e} : E' \to E$ such that $e' = e \circ \bar{e}$, i.e., such that the triangle

$$\begin{array}{c} E' \\ \bar{e}\downarrow \quad \searrow^{e'} \\ E \xrightarrow{e} A \underset{g}{\overset{f}{\rightrightarrows}} B \end{array}$$

commutes.

A **source** is a pair $\big(A, (f_i)_{i \in I}\big)$ consisting of an object A and a family of morphisms $f_i : A \to A_i$ with domain A, indexed by some class I.

Proposition 2. *A category has finite products if and only if it has terminal objects and products of pairs of objects. A category that has products for all class-indexed families must be thin. A small category has products if and only if it is equivalent to a complete lattice.*

A **diagram** in a category **A** is a functor $D : \mathbf{I} \to \mathbf{A}$ with codomain **A**. The domain, **I**, is called the **scheme** of the diagram. A diagram with a small (or finite) scheme is said to be **small** (or finite).

1. An A-source $\Big(A \xrightarrow{f_i} D_i\Big)_{i \in Ob(\mathbf{I})}$ is said to be natural for D provided that for each I-morphism $i \xrightarrow{d} j$, the triangle

$$\begin{array}{c} A \\ f_i\downarrow \quad \searrow^{f_j} \\ D_i \xrightarrow{Dd} D_j \end{array}$$

commutes.

2. A **limit** of D is a natural source $\Big(L \xrightarrow{\ell_i} D_i\Big)$ for D with the **universal property** that for each natural source $\Big(A \xrightarrow{f_i} D_i\Big)$ there exists a unique morphism $f : A \to L$ with $f_i = \ell_i \circ f$ for each $i \in Ob(\mathbf{I})$.

3. For a diagram with discrete scheme, every source is natural. A source is a limit of if it is a product . Expressed briefly: products are limits of diagrams with discrete schemes. In particular, an object, considered as an empty source, is a limit of the empty diagram (i.e., the one with empty scheme) if and only if it is a terminal object.

For a pair of A-morphisms $A \underset{g}{\overset{f}{\rightrightarrows}} B$, considered as a diagram D with scheme $\bullet \Rightarrow \bullet$, a source $(A \overset{e}{\leftarrow} C \overset{h}{\rightarrow} B)$ is natural provided that $g \circ e = h = f \circ e$. Observe that in this case h is determined by e. Hence, $C \overset{e}{\rightarrow} A$ is an equalizer of $A \underset{g}{\overset{f}{\longrightarrow}} B$ if and only if the source $(A \overset{e}{\leftarrow} C \overset{f \circ e}{\longrightarrow} B)$ is a limit of D. Thus we may say (imprecisely) that equalizers are limits of diagrams with scheme $\bullet \Rightarrow \bullet$. If, in the above scheme, the two arrows are replaced by an arbitrary set of arrows, then limits of diagrams with such schemes are called multiple equalizers. If $\mathbf{I}$ is a down-directed [57] poset (considered as a category), then limits of diagrams with scheme $\mathbf{I}$ are called projective (or inverse) limits. If, e.g., $\mathbf{I} = \mathbb{N}^{op}$ is the poset of all non-negative integers with the opposite of the usual ordering, a diagram $D : \mathbf{I} \to \mathbf{A}$ with this scheme is essentially a sequence

$$\ldots \overset{d_2}{\longrightarrow} D_2 \overset{d_1}{\longrightarrow} D_1 \overset{d_0}{\longrightarrow} D_0$$

of A-morphisms (where $D(n+1 \to n) = D_{n+1} \overset{d_n}{\longrightarrow} D_n, D(n+2 \to n) = d_n \circ d_{n+1}$, etc.). A natural source for D is a source $\left(A \overset{f_n}{\longrightarrow} D_n\right)_{n \in \mathbb{N}}$ with $f_n = d_n \circ f_{n+1}$ for each n. In Set a projective limit of a diagram D with scheme $\mathbb{N}^{op}$ is a source $\left(L \overset{\ell_n}{\longrightarrow} D_n\right)_{n \in \mathbb{N}}$, where L is the set of all sequences $(x_n)_{n \in \mathbb{N}}$ with $x_n \in D_n$ and $d_n(x_{n+1}) = x_n$ for each $n \in \mathbb{N}$; and where each ℓ_m is a restriction of the m th projection $\pi_m : \prod_{n \in \mathbb{N}} D_n \to D_m$ If $D : \mathbf{A} \to \mathbf{A}$ is the identity functor, then a source $\mathscr{L} = \left(L \overset{\ell_A}{\longrightarrow} A\right)_{A \in Ob\mathbf{A}}$ is a limit of D if and only if L is an initial object of $\mathbf{A}$. The sufficiency is obvious. For the necessity let $\mathscr{L}$ be a limit of D and let $L \overset{f}{\rightarrow} A$ be a morphism. By the naturality of $\mathscr{L}$ for D we obtain $f \circ \ell_L = \ell_A$. Application of this to $f = \ell_A$ yields $\ell_A \circ \ell_L = \ell_A = \ell_A \circ id_L$ for each object A. Hence, by the uniqueness requirement

1.3 Adjoint functors

1.4 Concrete categories

A way to talk of *low level structures* present on the objects of a category. Often it is easier to work with less structures, and there results like Yoneda's lemma that show us that it is possible to restrict our study to them.

Let $\mathscr{C}$ be a category. A **concrete category** over $\mathscr{C}$ is a category $\mathbf{A}$ together wih a faithful functor $U : \mathbf{A} \to \mathscr{C}$, called the **forgetful** (or underlying) functor of the concrete category. $\mathscr{C}$ is called the **base category**. A concrete category over Set is called a **construct**.
The category of groups (or topological spaces, rings, etc.), with the forgetful functor to Set, is a construct.

A functor $F : \mathbf{A} \to$ Set is called representable (by an $\mathbf{A}$-object A) provided that F is naturally isomorphic to the hom-functor $\hom(A, -) : \mathbf{A} \to$ Set. Note that objects that represents the same functor are isomorphic.

Example 2. *1. Forgetful functors are often representable. For example, (a) Vec $\to$ Set is represented by the vector space $\mathbb{R}$, (b) Grp $\to$ Set is represented by the group of integers $\mathbb{Z}$, (c) Top $\to$ Set is represented by any one-point topological space.*

2. *The underlying functor U for the construct Ban [5.2(3)] is not representable (see Exercise 10J). However, the faithful unit ball functor $O :$ Ban $\to$ Set is represented in the complex case by the Banach space $\mathbb{C}$ of complex numbers.*

Proposition 3 (Representative of Constructs). *For constructs $(\mathbf{A}, U)$ the forgetful functor is represented by an object A if and only if A is a free object over a singleton set [see Definition 8.22(2)]. This provides many additional examples of representations.*

For small categories $\mathbf{A}$ and $\mathbf{B}$ the **functor category** $[\mathbf{A}, \mathbf{B}]$ has as objects all functors from $\mathbf{A}$ to $\mathbf{B}$, as morphisms from F to G all natural transformations from F to G, as identities the identity natural transformations, and as composition the (horizontal) composition of natural transformations.

Theorem 1 (uniqueness of representations). *For any functor $F : \mathbf{A} \to$ Set, any $\mathbf{A}$-object A and any element $a \in F(A)$, there exists a unique natural transformation $\tau : \hom(A, -) \to F$ with $\tau_A(id_A) = a$.*

Corollary 1 (Yoneda Lemma)**.** *If $F : \mathbf{A} \to Set$ is a functor and A is an $\mathbf{A}$-object, then the following function*

$$Y : [\hom(A, -), F] \to F(A) \text{ defined by } Y(\sigma) = \sigma_A(id_A),$$

is a bijection (where $[\hom(A, -), F]$ is the set of all natural transformations from hom $(A, -)$ to F).

Corollary 2 (Yoneda Embedding)**.** *For any category $\mathbf{A}$, the functor $E : \mathbf{A} \to [\mathbf{A}^{\mathrm{op}} Set]$, defined by*

$$E(A \xrightarrow{f} B) = \hom(-, A) \xrightarrow{\sigma_f} \hom(-, B),$$

where $\sigma_f(g) = f \circ g$, is a full embedding.

Chapter 2

Homological Algebra

In homological algebra one constructs homological invariants of algebraic objects by the following process, or some variant of it:

Let R be a ring and T a covariant additive functor from R-modules to abelian groups. Thus the map $\operatorname{Hom}_R(M,N) \to \operatorname{Hom}_{\mathbf{z}}(TM,TN)$ defined by T is a homomorphism of abelian groups for all R-modules M,N. For any R module M, choose a free (or projective) resolution $\varepsilon : F \to M$ and consider the chain complex TF of abelian groups obtained by applying T to F termwise. Now T, being additive, preserves chain homotopies; so we can apply the uniqueness theorem for resolutions (I.7.5) to deduce that the complex TF is independent, up to canonical homotopy equivalence, of the choice of resolution. Passing to homology, we obtain groups $H_n(TF)$ which depend only on T and M (up to canonical isomorphism).

This construction is of no interest, of course, if T is an exact functor; for then the augmented complex

$$\cdots \to TF_1 \to TF_0 \to TM \to 0$$

is acyclic, so that $H_n(TF) = 0$ for $n > 0$ and $H_0(TF) = TM$. Thus we can regard the groups $H_n(TF)$ in the general case as a measure of the failure of T to be exact.

2.1   Spectral Sequences

2.2   Abelian categories

2.3   Derived functors

2.4   Derived categories

Chapter 3

Group (Cohomology) Theory

3.1 Actions

3.2 Representations

Group Ring   Let G be a group, written multiplicatively. Let $\mathbb{Z}G$ be the free $\mathbb{Z}$-module generated by the elements of G. The multiplication in G extends uniquely to a $\mathbb{Z}$-bilinear product $\mathbb{Z}\mathbb{G} \times \mathbb{Z}\mathbb{G} \to \mathbb{Z}\mathbb{G}$; this makes $\mathbb{Z}\mathbb{G}$ a ring, called the **integral group ring** of G.

Note that G is a subgroup of the group $(\mathbb{Z}G)^*$ of units of $\mathbb{Z}\mathbb{G}$

Theorem 2 (Universal property). *Given a ring R and a group homomorphism $f : G \to R^*$, there is a unique extension of f to a ring homomorphism $\mathbb{Z}\mathbb{G} \to R$. Thus we have the "adjunction formula"*

$$\mathrm{Hom}_{(rings)}\,(\mathbb{Z}G, R) \approx \mathrm{Hom}_{(groups)}\,\left(G, R^*\right).$$

A (**left**) $\mathbb{Z}G$-**module**, or G-module, consists of an abelian group A together with a homomorphism from $\mathbb{Z}G$ to the ring of endomorphisms of A. By the universal property, G-module is simply an abelian group A together with an action of G on A. For example, one has for any A the trivial module structure, with $ga = a$ for $g \in G, a \in A$.

One way of constructing G-modules is by linearizing permutation representations. More precisely, if X is a G-set (i.e., a set with G-action), then one forms the free abelian group $\mathbb{Z}\mathbb{X}$ (also denoted $\mathbb{Z}[X]$) generated by X and one extends the action of G on X to a $\mathbb{Z}$-linear action of G on $\mathbb{Z}X$. The resulting G-module is called a permutation module. In particular, one has a permutation module $\mathbb{Z}[G/H]$ for every subgroup H of G, where G/H is the set of cosets gH and G acts on G/H by left translation.

Proposition 4. *Let X be a free G-set and let E be a set of representatives for the G-orbits in X. Then $\mathbb{Z}X$ is a free $\mathbb{Z}G$-module with basis E.*

3.3 Co-invariants

If G is a group and M is a G-module, then the group of co-invariants of M, denoted M_G, is defined to be the quotient of M by the additive subgroup generated by the elements of the form $gm - m\,(g \in G, m \in M$). Thus M_G is obtained from M by "dividing out" by the G-action. (The name "co-invariants" comes from the fact that M_G is the largest quotient of M on which G acts trivially, whereas M^G, the group of invariants, is the largest submodule of M on which G acts trivially.) In view of exercise 1a of \$$I$.2, we can also describe M_G as M/IM, where I is the augmentation ideal of $\mathbb{Z}G$ and IM denotes the set of all finite sums $\sum a_i b_i\,(a_i \in I, b_i \in M)$. Still another description of M_G is given by:

$$M_G \approx \mathbb{Z} \otimes_{\mathbb{Z}G} M.$$

Here, in order for the tensor product to make sense, we regard $\mathbb{Z}$ as a right $\mathbb{Z}G$-module (with trivial G-action). To prove 2.1 , note that in $\mathbb{Z} \otimes_{\mathbb{Z}G} M$ we have the identity $1 \otimes gm = 1 \cdot g \otimes m = 1 \otimes m$; hence there is a map $M_G \to \mathbb{Z} \otimes_{\mathbb{Z}G} M$ given by $\bar{m} \mapsto 1 \otimes m$, where $\bar{m}$ denotes the image in M_G of an element $m \in M$. On the other hand, using the universal property of the tensor product, we can define a map $\mathbb{Z} \otimes_{\mathbb{Z}G} M \to M_G$ by $a \otimes m \mapsto a\bar{m}$. These two maps are inverses of one another.

In view of 2.1 and standard properties of the tensor product, we immediately obtain the following two properties of the co-invariants functor:

3.4   An spectral sequence for group cohomology

Suppose that X is a simplicial set and x_i are simplicial subsets such that $X = UX_i$. Then, setting $X_{ij} = X_i \cap X_j$ (etc.) we'11 obviously have for the realisations: $|x| = U|x_i|, |x_i| \cap |x_j| = |x_{ij}|, \ldots$ Let's suppose that the set of indices is linearly ordered. Consider the following bicomplex:

$$K = \longrightarrow \underset{i<j<k}{\oplus} C_* \left(x_{ijk} \right) \longrightarrow \underset{i<j}{\oplus} C_* \left(x_{ij} \right) \longrightarrow \underset{i}{\oplus} C_* \left(x_i \right)$$

Here by a bicomplex we understand a bicomplex in the sense of Grothendieck [9] i.e. the differentials d_1 and d_2 commute. (The sign in this approach appears in the definition of the total differentials). The vertical arrows of the bicomplex map $C_*(x_i \cdots _i)$ into $\underset{k=0}{q} C_* \left(x_{i_0} \ldots \hat{i}_k \ldots i_q \right)$, the mapping into the kth summand differing $k = 0$ by a sign $(-1)^k$ from the natural embedding.

The first spectral sequence of this bicomplex degenerates and yields an isomorphism $H_\star(K) \cong H_\star(X)$. (Moreover this isomorphism is induced by the canonical map $K \to C_*(X)$). The second spectral sequence gives us a functorial spectral sequence of the first quadrant, whose limit equals $H_*(X)$, while its differential dr has bidegree $(r-1, -r)$ and its E^1-term looks as follows:

$$E^1_{pqq} = \underset{i_0 < \ldots < i_q}{\otimes} H_p \left(x_{i_0} \ldots i_q \right)$$

Suppose G is a group. Let X_G denote the simplicial set (and its geometric realisation), whose p-simplices are sequences $(g_0, \ldots, g_p)$ of elements of G, with the usual faces and degeneracies. This space X_G is contractible by (1.2). The group G acts from the right on X_G and this action is obviously free, hence $BG = X_G/G$ is a classifying space of G. The complex $C_*(BG) = C_*(G)$ coincides with the usual complex associated with G. Moreover $C_*(G) = C_*(X_G) \otimes_G Z$.

If H is a subgroup of G, then X_G/H is a classifying space for H and hence $BH = X_H/H \to X_G/H$ is a homotopy equivalence. In particular $C_*(H) + C_*(X_G) \otimes_H Z = C_*(X_G) \otimes_G Z|G/H|$ is a homotopy equivalence.

(2.3) The spectral sequence associated with a family of subgroups.

Suppose G is a group and $G_1, \ldots, G_n$ are subgroups. Then BG_i may be viewed as a simplicial subset of BG and $BG_i \cap BG_j = B\left(G_i \cap G_j \right)$.. Denote UBG_i by X and consider the spectral sequence of the covering $X = UBG_i$. Along with the bicomplex K introduced in (2.1) we also consider the following bicomplex:

$$K' = \underset{i<j<k}{\oplus} C_*(X_G) \otimes_G Z \left[G/G_{ijk} \right] \longrightarrow \underset{i<j}{\oplus} C_*(X_G) \otimes_G Z \left[G/G_{ij} \right] \longrightarrow \underset{i<j}{\oplus} C_*(X_G) \otimes_G Z[G/G_i]$$

There is a natural mapping of bicomplexes $K + K'$ and because of (2.2) this mapping induces an isomorphism of second spectral sequences so that $H_\star(X) = H_\star(K) = H_*\left(K' \right)$. The first spectral sequence of K' looks as follows: $E^1_{*,q} = C_*(X_G) \otimes_G H_q(L)$, where L is the following complex of left G-modules:

$$\oplus \mathbb{Z}[G/G_i] + \oplus \mathbb{Z} \left[G/G_{ij} \right] + \oplus \mathbb{Z} \left[G/G_{ijk} \right] + \ldots$$

Proposition 5. *If $G_1, \ldots, G_n$ are subgroups of G, there exists a fuctorial spectral sequence of the first quadrart, the E^2 term of which looks like: $E^2_{pq} = H_p\left(G, H_q(L) \right)$, where L is the complex defined above. It converges to $H_\star\left(UBG_j \right)$ and the differential d^r has bidegree $(-r, r-1)$.*

(2.5) In the notations of (2.3), let $Z(G, \{G\})$ be the simplicial set whose non-degenerate p-simplices are sequences $(\bar{g}_0, \ldots, \bar{g}_p)$, where $\bar{g}_i \varepsilon G/G_{k_i}, k_0 < \ldots < k_p$, and the $\bar{g}_i$ are such that there is $g \in G$ with $\vec{g}_i = g \bmod G_{k_i}$ for all i. (If one covers G by the right cosets of the G_i, then $Z\left(G_g \{G_i\} \right)$ is the nerve of this covering.) It is easy to see that the geometric realization of this simplicial set is an ordered simplicial space and that the complex $L = L(G, \{G_i\})$ equals the (ordered) simplicial complex [7] of this simplicial space, or in other words, the complex L equals the normalised complex of the simplicial set $Z(G, \{G_i\})$. In particular, $H_*(L) = H_*(Z(G, \{G_i\}))$.

(2.6) Remark. It may be shown easily that the space $Z(G, \{G_i\})$, is homotopy equivalent to Volodin's space $V(G, \{G_i\})$, but we will not need this fact.

Chapter 4

(General) Module Theory

4.1 Linear Algebra

Part II

Topics of Algebraic Topology

Chapter 5

Simplical sets and complexes

Simplicial complexes are more intuitive, and are the foundation of algebraic topology. Simplicial complexes were also called *simplicial schemes* and simplicial sets, *semi-simplicial* complexes.

5.1   (Abstract) simplical complexes

A set (of **vertices**) together with a family of finite subsets (**simplexes**) such that every subset of every simplex is a simplex and every subset consisting of a single vertex is a simplex.

Example 3. *1. The **standard n-simplex** Δ^n is the set of all $(n+1)$-tuples $(t_0,\ldots,t_n)$ of non-negative real numbers such that $t_0 + \cdots + t_n = 1$. The standard 0-simplex is a point, the standard 1-simplex is a line segment, the standard 2-simplex is a triangle, and so on.*

*2. The **boundary** of the standard n-simplex Δ^n is the set of all $(n+1)$-tuples $(t_0,\ldots,t_n)$ of non-negative real numbers such that $t_0 + \cdots + t_n = 1$ and at least one of the t_i is zero. The boundary of the standard 0-simplex is empty, the boundary of the standard 1-simplex is the set of its two endpoints, the boundary of the standard 2-simplex is the set of its three edges, and so on.*

*3. (**Concrete simplicial complexes**) It is subset of $\mathbb{R}^n$ that is a union of standard simplices, that satisfies the previous conditions.*

4. If Y is a subset of the vertex set of a simplicial scheme S, then we can introduce on it the induced simplicial scheme structure $Y \cap S$, by defining the simplexes as the subsets of Y that are simplexes of S.

5. Let X be a set and let $\{p(y) : y \in Y\}$ be a covering of X. Then we can consider two simplicial complexes.

   (a) The nerve $\mathrm{Nerv}(p)$ of the covering is the simplicial scheme with the vertex set Y, and a subset Z of Y is counted as a simplex if the intersection $\underset{Z}{\cap}p(y)$ is non-empty.

   (b) The simplicial complex $V(p)$ is the simplicial scheme with the vertex set X, and a subset Z of X is counted as a simplex if Z is contained in some $p(y)$.

Geometric realization

The construction goes as follows. First, define $|K|$ as a subset of $[0,1]^S$ consisting of functions $t : S \to [0,1]$ satisfying the two conditions: $\square$

$$\{s \in S : t_s > 0\} \in K$$
$$\sum_{s \in S} t_s = 1$$

Now think of the set of elements of $[0,1]^S$ with finite support as the direct limit of $[0,1]^A$ where A ranges over finite subsets of S, and give that direct limit the induced topology. Now give $|K|$ the subspace topology. *It is always Hausdorff.* We will identify an abstract simplicial complex with its geometric realization.

5.2   Simplical sets

Let Δ be the category of finite ordinal numbers, with order-preserving maps between them. More precisely, the objects for Δ consist of elements $\mathbf{n}, n \geq 0$, where $\mathbf{n}$ is a string of relations

$$0 \to 1 \to 2 \to \cdots \to n$$

(in other words $\mathbf{n}$ is a totally ordered set with $n+1$ elements). A morphism $\theta : \mathbf{m} \to \mathbf{n}$ is an order-preserving set function, or alternatively a functor. We usually commit the abuse of saying that Δ is the ordinal number category.

A simplicial set is a contravariant functor $X : \Delta^{op} \to \text{Sets}$, where Sets is the category of sets.

Remark 1. *The standard covariant functor:* $\mathbf{n} \mapsto |\Delta^n|$ *from* Δ *to* **Top***. The singular set* $S(T)$ *is the simplicial set given by*

$$\mathbf{n} \mapsto \hom\left(\left|\Delta^n\right|, T\right).$$

This is the object that gives the singular homology of the space T.

The standard n-simplex, simplicial Δ^n *in the simplicial set category* **S** *is defined by*

$$\Delta^n = \hom_\Delta(, \mathbf{n}).$$

In other words, Δ^n *is the contravariant functor on* Δ *which is represented by n.*

A map $f : X \to Y$ of simplicial sets (or, more simply, a simplicial map) is a natural transformation of contravariant set-valued functors defined on $\boldsymbol{\Delta}$. We shall use **S** to denote the resulting category of simplicial sets and simplicial maps.

From a simplicial set Y, one may construct a simplicial abelian group $\mathbb{Z}Y$ (ie. a contravariant functor $\boldsymbol{\Delta}^{op} \to \mathbf{Ab}$), with $\mathbb{Z}Y_n$ set equal to the free abelian group on Y_n. The simplicial abelian group $\mathbb{Z}Y$ has associated to it a chain complex, called its Moore complex and also written $\mathbb{Z}Y$, with

$$\mathbb{Z}Y_0 \xleftarrow{\partial} \mathbb{Z}Y_1 \xleftarrow{\partial} \mathbb{Z}Y_2 \leftarrow \ldots \quad \text{and}$$

$$\partial = \sum_{i=0}^{n} (-1)^i d_i$$

in degree n. Recall that the integral singular homology groups $H_*(X; \mathbb{Z})$ of the space X are defined to be the homology groups of the chain complex $\mathbb{Z}SX$. The homology groups $H_n(Y, A)$ of a simplicial set Y with coefficients in an abelian group A are defined to be the homology groups $H_n(\mathbb{Z}Y \otimes A)$ of the chain complex $\mathbb{Z}Y \otimes A$.

Geometric realization

The simplex category: $\Delta \downarrow X$ of a simplicial set X. The objects of $\Delta \downarrow X$ are the maps $\sigma : \Delta^n \to X$, or simplices of X. An arrow of $\Delta \downarrow X$ is a commutative diagram of simplicial maps .....

Observe that θ is induced by a unique ordinal number map $\theta : \mathbf{m} \to \mathbf{n}$.

Lemma 1. *There is an isomorphism*

$$X \cong \varinjlim_{\Delta^n \to X} \Delta^n.$$

in $\Delta \downarrow X$

The realization $|X|$ of a simplicial set X is defined by the colimit

$$|X| = \varinjlim_{\substack{\Delta^n \to X \\ \text{in } \Delta \downarrow X}} \left|\Delta^n\right|.$$

in the category of topological spaces. The construction $X \mapsto |X|$ is seen to be functorial in simplicial sets X, by using the fact that any simplicial map $f : X \to Y$ induces a functor $f_* : \Delta \downarrow X \to \Delta \downarrow Y$ by composition with f.

Proposition 6. *The realization functor is left adjoint to the singular functor in the sense that there is an isomorphism*

$$\hom_{Top}(|X|, Y) \cong \hom_{\mathbf{S}}(X, SY)$$

which is natural in simplicial sets X and topological spaces Y. In particular, since $\mathbf{S}$ has all colimits and the realization functor, $| \, |$ preserves them.

Proposition 7. *$|X|$ is a CW-complex for each simplicial set X. In particular it is a compactly generated Hausdorff space.*

5.3 Semi-simplicial sets

It will suffice for our purposes to keep track of less structure, and replace Δ by its subcategory Δ_{inj} with the same objects but morphisms only injective order-preserving maps. A semi-simplicial set is a functor $\Delta_{\text{inj}}^{\text{op}} \to$ Set and a semi-simplicial space is a functor $\Delta_{\text{inj}}^{\text{op}} \to$ Top. That, it is the analogue of a simplicial space which only face maps and no degeneracy maps. We can restrict $\Delta^{\bullet}$ to Δ_{inj} and once more take the coend to get a geometric realisation

$$\|X_{\bullet}\| := \Delta^{\bullet} \otimes_{\Delta_{\text{inj}}} X_{\bullet} = \left(\bigsqcup_{p \geq 0} \Delta^p \times X_p \right) / \sim$$

with $\sim$ the equivalence relation generated by $(\delta_i t, x) \sim (t, d_i x)$. A reference for semisimplicial spaces and their properties is [ERW19].

We now define the semi-simpicial set with Σ_n-action which will replace $*$. Let FI be the category whose objects are finite sets and whose morphisms are injections.

$W_n(\underline{1})\bullet$ is the **semi-simplicial set with p-simplices** given by

$$W_n(\underline{1})\bullet = \text{Hom}_{\text{FI}}([p], \underline{n})$$

and face maps d_i given by precomposition with $\delta_i : [p-1] \to [p]$. That is, $W_n(\underline{1})_p$ has as p-simplices the ordered words $\left(m_0 m_1 \cdots m_p \right)$ of elements of $\underline{n}$ and no letter duplicated. The i th face map forgets the i th letter m_i. This explains why we call this the semi-simplicial set of injective words. The notation is rather complicated, but will become clear in the next lecture.

The group Σ_n acts on $W_n(1)\bullet$ by post-composition, and hence on the geometric realisation. We have that:

Proposition 8.   *1. $\|W_n(\mathbf{1})\cdot\|$ is homologically $\frac{n-1}{2}$-connected.*

(ii) $W_n(\underline{1})_p$ is a transitive Σ_n-set, and the stabiliser of $x \in W_n(1)_p$ is the group of permutations of $\underline{n} \setminus \text{im}(x)$.

A filtration $F_0 X \subset F_1 X \subset \dots$ on a space X makes the singular chains $C_*(X)$ into a filtered chain complex by setting $F_r C_*(X) := \text{im}(C_*(F_r X) \to C_*(X))$. Assuming that $F_{r-1} X \to F_r X$ is a cofibration and $F_r X / F_{r-1} X$ is at least $(r-1)$-connected, this gives a strongly-convergent first-quadrant spectral sequence

$$E_{p,q}^1 = \widetilde{H}_{p+q}\left(F_p X / F_{p-1} X; \mathbb{Z} \right) \Longrightarrow H_{p+q}(X; \mathbb{Z})$$

with differentials given by $d^r : E_{p,q}^r \to E_{p-r,q+r-1}^r$. Remark 2.3.1. If you are unfamiliar with these notions, I recommend you look at [McC01, Hat]. Roughly, a spectral sequence is an algebraic object that conveniently packages all long exact sequences in homology for the pairs $(F_s X, F_r X)$ with $s \geq r$ with the goal of compute the homology of X. We can in particular apply this to the geometric realisation $\|X_{\bullet}\|$. This has a filtration

$$F_r \|X_{\bullet}\| := \left(\bigsqcup_{0 \leq p \leq r} \Delta^p \times X_p \right) / \sim$$

with equivalence relation $\sim$ as before, all of whose maps are cofibrations under a mild condition on $X_{\bullet}$ that will be satisfied in examples in these notes. The associated graded is given by

$$\frac{F_r \|X_{\bullet}\|}{F_{r-1} \|X_{\bullet}\|} = \frac{\Delta^r}{\partial \Delta^r} \wedge (X_r)_+$$

so is at least $(r-1)$-connected. Thus we get [Seg68] (see also [ERW19, Section 1.4]):

Theorem 3. *There is a strongly convergent first quadrant spectral sequence*

$$E_{p,q}^1 = H_q\left(X_p; \mathbb{Z} \right) \Longrightarrow H_{p+q}\left(\|X_{\bullet}\|; \mathbb{Z} \right)$$

with differentials given by $d^r : E_{p,q}^r \to E_{p-r,q+r-1}^r$. Moreover $d^1 : E_{p,q}^1 \to E_{p-1,q}^1$ is given by $\sum_{i=0}^{p} (-1)^i (d_i)_*$, and the edge homomorphism $E_{0,q}^1 \to E_{0,q}^\infty \to H_q(X; \mathbb{Z})$ is equal to the map induced on homology by the inclusion $X_0 \to \|X_{\bullet}\|$.

5.4   CW-complexes

They can be defined in an inductive way:

1. Start with a discrete set X^0, whose points are regarded as 0 -cells.

2. Inductively, form the **n-skeleton** X^n from X^{n-1} by attaching n-cells e_α^n via maps $\varphi_\alpha : S^{n-1} \to X^{n-1}$. This means that X^n is the quotient space of the disjoint union $X^{n-1} \amalg_\alpha D_\alpha^n$ of X^{n-1} with a collection of n-disks D_α^n under the identifications $x \sim \varphi_\alpha(x)$ for $x \in \partial D_\alpha^n$. Thus as a set, $X^n = X^{n-1} \amalg_\alpha e_\alpha^n$ where each e_α^n is an open n-disk.

3. One can either stop this inductive process at a finite stage, setting $X = X^n$ for some $n < \infty$, or one can continue indefinitely, setting $X = \cup_n X^n$. In the latter case X is given the weak topology: A set $A \subset X$ is open (or closed) iff $A \cap X^n$ is open (or closed) in X^n for each n.

Example 4.   *1. A 1-dimensional cell complex $X = X^1$ is what is called a graph in algebraic topology. It consists of vertices (the 0 -cells) to which edges (the 1-cells) are attached. The two ends of an edge can be attached to the same vertex.*

   2. The sphere S^n has the structure of a cell complex with just two cells, e^0 and e^n, the n-cell being attached by the constant map $S^{n-1} \to e^0$. This is equivalent to regarding S^n as the quotient space $D^n / \partial D^n$.

   *3. **Real projective n-space** $\mathbb{R}\mathrm{P}^n$. It is equivalent to the quotient space of a hemisphere D^n with antipodal points of ∂D^n identified. Since ∂D^n with antipodal points identified is just $\mathbb{R}\mathbb{P}\mathrm{P}^{n-1}$, we see that $\mathbb{R}\mathrm{P}^n$ is obtained from $\mathbb{R}\mathrm{P}^{n-1}$ by attaching an n-cell, with the quotient projection $S^{n-1} \to \mathbb{R}P^{n-1}$ as the attaching map. It follows by induction on n that $\mathbb{R}\mathrm{P}^n$ has a cell complex structure $e^0 \cup e^1 \cup \cdots \cup e^n$ with one cell e^i in each dimension $i \leq n$. The infinite union $\mathbb{R}P^\infty = U_n \mathbb{R}P^n$ becomes a cell complex with one cell in each dimension. We can view $\mathbb{R}P^\infty$ as the space of lines through the origin in $\mathbb{R}^\infty = \bigcup_n \mathbb{R}^n$.*

   *4. **Complex projective space** $\mathbb{C}P^n$. It is equivalent to the quotient of the unit sphere $S^{2n+1} \subset \mathbb{C}^{n+1}$ with $v \sim \lambda v$ for $|\lambda| = 1$.*
   It is also possible to obtain $\mathbb{C}P^n$ as a quotient space of the disk D^{2n} under the identifications $v \sim \lambda v$ for $v \in \partial D^{2n}$, in the following way. The vectors in $S^{2n+1} \subset \mathbb{C}^{n+1}$ with last coordinate real and nonnegative are precisely the vectors of the form $\left(w, \sqrt{1 - |w|^2}\right) \in \mathbb{C}^n \times \mathbb{C}$ with $|w| \leq 1$. Such vectors form the graph of the function $w \mapsto \sqrt{1 - |w|^2}$. This is a disk D_+^{2n} bounded by the sphere $S^{2n-1} \subset S^{2n+1}$ consisting of vectors $(w, 0) \in \mathbb{C}^n \times \mathbb{C}$ with $|w| = 1$. Each vector in S^{2n+1} is equivalent under the identifications $v \sim \lambda v$ to a vector in D_+^{2n}, and the latter vector is unique if its last coordinate is nonzero. If the last coordinate is zero, we have just the identifications $v \sim \lambda v$ for $v \in S^{2n-1}$.
   It follows that $\mathbb{P}^n$ is obtained from $\mathbb{C}P^{n-1}$ by attaching a cell e^{2n} via the quotient map $S^{2n-1} \to \mathbb{C}P^{n-1}$. So by induction on n we obtain a cell structure $\mathbb{C}P^n = e^0 \cup e^2 \cup \cdots \cup e^{2n}$ with cells only in even dimensions. Similarly, $\mathbb{C}P^\infty$ has a cell structure with one cell in each even dimension.

Each cell e_α^n in a cell complex X has a **characteristic map** $\Phi_\alpha : D_\alpha^n \to X$ which extends the attaching map φ_α and is a homeomorphism from the interior of D_α^n onto e_α^n. Namely, we can take Φ_α to be the composition $D_\alpha^n \hookrightarrow X^{n-1} \amalg_\alpha D_\alpha^n \to X^n \hookrightarrow X$ where the middle map is the quotient map defining X^n.

Chapter 6

Geometric Group Theory

By a G-**complex** we will mean a CW-complex X together with an action of G on X which permutes the cells. Thus we have for each $g \in G$ a homeomorphism $x \mapsto gx$ of X such that the image $g\sigma$ of any cell σ of X is again a cell. For example, if X is a simplicial complex on which G acts simplicially, then X is a G-complex.

If X is a G-complex then the action of G on X induces an action of G on the cellular chain complex $C_*(X)$, which thereby becomes a chain complex of G-modules. Moreover, the canonical augmentation $\varepsilon : C_0(X) \to \mathbb{Z}$ (defined by $\varepsilon(v) = 1$ for every 0-cell v of X) is a map of G-modules.

We will say that X is a free G-complex if the action of G freely permutes the cells of X (i.e., $g\sigma \neq \sigma$ for all σ if $g \neq 1$). In this case each chain module $C_n(X)$ has a $\mathbb{Z}$-basis which is freely permuted by G, hence by $3.1 C_n(X)$ is a free $\mathbb{Z}G$-module with one basis element for every G-orbit of cells. (Note that to obtain a specific basis we must choose a representative cell from each orbit and we must choose an orientation of each such representative.)

Finally, if X is contractible, then $H_*(X) \approx H_*$ (pt.); in other words, the sequence

$$\cdots \to C_n(X) \xrightarrow{\partial} C_{n-1}(X) \to \cdots \to C_0(X) \xrightarrow{\varepsilon} \mathbb{Z} \to 0$$

is exact. We have, therefore:

Proposition 9. *Let X be a contractible free G-complex. Then the augmented cellular chain complex of X is a free resolution of $\mathbb{Z}$ over $\mathbb{Z}G$.*

6.1 Classifying space

Suppose that $\mathscr{C}$ is a (small) category. The classifying space (or nerve) $B\mathscr{C}$ of $\mathscr{C}$ is the simplicial set with

$$B\mathscr{C}_n = \hom_{\mathrm{cat}}(\mathbf{n}, \mathscr{C}),$$

n-simplex is a string

$$a_0 \xrightarrow{a_1} a_1 \xrightarrow{a_2} \ldots \xrightarrow{a_n} a_n$$

of composeable arrows of length n in $\mathscr{C}$.

If G is a group, then G can be identified with a category (or groupoid) with one object $*$ and one morphism $g : * \to *$ for each element g of G, and so the classifying space BG of G is defined. Moreover $|BG|$ is an Eilenberg-Mac Lane space of the form $K(G, 1)$, as the notation suggests; this is now the standard construction.

Recall that we constructed BG as the geometric realisation of the nerve of a category $* // G$. As the notation suggests, this can be interpreted as a quotient, or more precisely a homotopy quotient. One can construct the homotopy quotient $X//G$ of any space X with G-action by a group G, and here we just take $X = *$. By abuse of notation $* // G = |N(*//G)|^2$ A reference for its construction and properties is [Rie14], but we will only need the following facts:

1. Homotopy quotients are natural. If $X \to Y$ is an equivariant map between G-spaces then there is an induced map $X//G \to Y//G$.

2. Homotopy quotients preserve homological connectivity. If $X \to Y$ is an equivariant map between G-spaces which is homologically d-connected then $X//G \to Y//G$ is also homologically d-connected. (Recall that a map is homologically d-connected if it is an isomorphism on H_i for $i < d$ and surjection on H_d.)

3. Homotopy quotients commute with geometric realisation. If $X_\bullet$ is a semi-simplicial G-space, then $\|X_\bullet\|/\!/G \simeq \|X_\bullet/\!/G\|$. (We will explain the terminology and notation later.)

4. Homotopy quotients of transitive G-sets. If S is a transitive G-set, then $S/\!/G \simeq B\,\mathrm{Stab}_G(s)$ for any $s \in S$.

Chapter 7

Homotopy theory

Let I^n be the n-dimensional unit cube, the product of n copies of the interval $[0,1]$. The boundary ∂I^n of I^n is the subspace consisting of points with at least one coordinate equal to 0 or 1 . For a space X with basepoint $x_0 \in X$, define $\pi_n(X, x_0)$ to be the set of homotopy classes of maps $f : (I^n, \partial I^n) \to (X, x_0)$, where homotopies f_t are required to satisfy $f_t(\partial I^n) = x_0$ for all t. The definition extends to the case $n = 0$ by taking I^0 to be a point and ∂I^0 to be empty, so $\pi_0(X, x_0)$ is just the set of path-components of X.

When $n \geq 2$, a sum operation in $\pi_n(X, x_0)$, generalizing the composition operation in π_1, is defined by

$$(f + g)(s_1, s_2, \cdots, s_n) = \begin{cases} f(2s_1, s_2, \cdots, s_n), & s_1 \in [0, 1/2] \\ g(2s_1 - 1, s_2, \cdots, s_n), & s_1 \in [1/2, 1] \end{cases}$$

It is evident that this sum is well-defined on homotopy classes. Since only the first coordinate is involved in the sum operation, the same arguments as for π_1 show that $\pi_n(X, x_0)$ is a group, with identity element the constant map sending I^n to x_0 and with inverses given by $-f(s_1, s_2, \cdots, s_n) = f(1 - s_1, s_2, \cdots, s_n)$.

Proposition 10. *If $n \geq 2$, then $\pi_n(X, x_0)$ is abelian.*

7.1   Covering spaces

In view of 4.1, it is natural now to consider CW-complexes Y satisfying the following three conditions:

1. Y is connected.

2. $\pi_1(Y)$ is isomorphic to G.

3. The universal covering space X of Y is contractible.

Part III

Topics of Geometry

Chapter 8

Smooth and complex manifolds

Chapter 9

Symplectic manifolds

Chapter 10

Sheaf theory

Chapter 11

Algebraic geometry

Part IV

K-theory

Chapter 12

K-theory constructions

12.1 Milnor's K-theory

For $n \geq 3$ the **Steinberg group** $\mathrm{St}_n(R)$ of a ring R is the group defined by generators $x_{ij}(r)$, with i,j a pair of distinct integers between 1 and n and $r \in R$, subject to the following "Steinberg relations":

$$x_{ij}(r)x_{ij}(s) = x_{ij}(r+s),$$

$$[x_{ij}(r), x_{k\ell}(s)] = \begin{cases} 1 & \text{if } j \neq k \text{ and } i \neq \ell, \\ x_{i\ell}(rs) & \text{if } j = k \text{ and } i \neq \ell, \\ x_{kj}(-sr) & \text{if } j \neq k \text{ and } i = \ell. \end{cases}$$

As observed in (1.3.1), the Steinberg relations are also satisfied by the elementary matrices $e_{ij}(r)$ which generate the subgroup $E_n(R)$ of $GL_n(R)$. Hence there is a canonical group surjection $\phi_n : St_n(R) \to E_n(R)$ sending $x_{ij}(r)$ to $e_{ij}(r)$.

The Steinberg relations for $n+1$ include the Steinberg relations for n, so there is an obvious map $St_n(R) \to St_{n+1}(R)$. We write $St(R)$ for $\varinjlim St_n(R)$ and observe that by stabilizing, the ϕ_n induce a surjection $\phi : St(R) \to E(R)$.

The group $K_2(R)$ is the kernel of $\phi : St(R) \to E(R)$. Thus there is an exact sequence of groups

$$1 \to K_2(R) \to St(R) \xrightarrow{\phi} GL(R) \to K_1(R) \to 1.$$

It will follow from Theorem 5.2.1 below that $K_2(R)$ is an abelian group. Moreover, it is clear that St and K_2 are both covariant functors from rings to groups, just as GL and K_1 are.

Theorem 4. $K_2(R)$ *is an abelian group. In fact it is precisely the center of* $\mathrm{St}(R)$.

We'll define right actions of the symmetric group S_n on $_{GL}(R)$ and on $St_n(R)$ by setting

$$\left(\alpha^s\right)_{k,\ell} = \alpha_{s(k),s(\ell)}; \quad x_{k\ell}(a)^s = x_{s^{-1}(k), s^{-1}(\ell)}(a).$$

These actions are compatible with the projections $St_n(R) \to E_n(R)$ and with the homomorphisms $St_n(R) + St_{n+1}(R)$ and $GL_n(R) + GL_{n+1}(R)$. In particular, they induce an action on $\overline{St}_n(R)$.

Lemma 2. *For any $s \in S_{n+1}$ the embeddings u_n and u_n^s are homotopic.*

12.2 Volodin's K-theory

Let G be a group and $\{G_i\}_{i \varepsilon I}$ a family of subgroups. Define $V(G, \{G_i\})$, or just $V(G)$ to be the simplicial complex, whose vertices are the elements of G, where $g_0, \ldots, g_p$ $(g_i \neq g_j)$ form a p-simplex if for some G_i all the elements $g_j g_k^{-1}$ lie in G_i. If H is another group with a family of subgroups $\{H_j\}$ and $\phi : G \to H$ is a homomorphism sending each G_i into some H_j, then ϕ induces a simplicial map $V(\phi) : V(G) \to V(H)$.

In many situations it is more convenient to use simplicial sets instead of simplicial complexes: Denote by $W(G, \{G_i\})$ the geometric realization of the simplicial set whose p-simplices are the sequences $(g_0, \ldots, g_p)$ of elements of G (not

necessarily distinct) such that for some G_i all $g_j g_k^{-1}$ lie in G_i, the r-th face (resp. degeneracy) of this simplex being obtained by omitting g_r (resp., repeating g_r). Associating with any p-simplex $(g_0, \ldots, g_p)$ the linear singular simplex of the space $V(G)$ which sends the i-th vertex of the standard simplex to g_j, we obtain a map of simplicial sets from $W(G)$ to the simplicial set of singular simplices of $V(G)$ and hence a cellular map (linear on any simplex) from $W(G)$ to $V(G)$. This map is a homotopy equivalence ....

Suppose that R is a ring, n a natural number and σ a partial ordering of $\{1, \ldots, n\}$. Define $T_n^\sigma(R)$ to be the subgroup of $GL_n(R)$ consisting of the α with $\alpha_{ij} = 1$ and $\alpha_{ij} = 0$ if $i \& j$. Subgroups of this form will be called triangular subgroups of $GL_n(R)$. The space $V\left(GL_n(R), \{T_n^\sigma(R)\}\right)$ will be denoted by $V_n(R)$. Since any partial ordering may be extended to a linear ordering, it suffices to consider linear orderings when defining $V_n(R)$. The natural embedding $GL_n \hookrightarrow GL_{n+1}(R)$ defines an embedding $V_n(R) \longleftrightarrow V_{n+1}(R)$ and we'll define $V_\infty(R)$ as $\lim_{\rightarrow n} V_n(R)$.

Finally for $i \geq 1$, put

$$k_{i,n}(R) = \pi_{i-1}(V_n(R))$$

and $k_i(R) = k_{i,\infty}(R) = \lim_{\rightarrow} k_{i,n}(R)$ (compare [26], [27]). Evidently $K_{1,n}(R) = GL_n(R)/E_n(R)$ and $K_{i,n}(R)$ is a group if $i \geq 2$, and this group is abelian if $i \geq 3$. Moreover the $K_i(R)$ are abelian groups for all $i \geq 1$ (see [26], [27]). The connected component of $V_n(R)$ passing through T_n equals $V\left(E_n(R), \{T_n^\sigma(R)\}\right)$. It is easy to show that the universal covering space of $V_n\left(E_n(R), \{T_n^\sigma(R)\}\right)$ equals $V\left(St(R), \{T_n^\sigma(R)\}\right)$, where T_n^σ is identified with the subgroup of $St_n(R)$ generated by the $x_{ij}(a)$ with a $\varepsilon R, i \overset{\sigma}{<} j (n \geq 3)$. Hence

Lemma 3. $K_{2,n}(R) = \ker(St_n(R) + E_n(R))$, and $K_{i,n}(R) = \pi_{i-1}(V(St_n(R))) = \pi_{i-1}(W(St_n(R)))$ $\quad$ if $\quad$ $i \geq 3$ $(n \geq 3)$.

Let's define $\overline{St}_n(R)$ to be the inverse image of $GL_n(R)$ under the projection $St(R) \to E(R)$. There is a canonical homomorphism $St_n(R) \to \overline{st}_n(R)$ and stability for K_1, k_2 ([10], [20], [22]) shows that this homomorphism is surjective if $n \geq s.r.R + 1$ and bijective if $n \geq s.r.R + 2$. The spaces $W(St_n(R))$ and $W\left(\overline{St}_n(R)\right)$ will play an essential role in the sequel. We'll denote them by $W_n(R), \bar{W}_n(R)$, resp. (So $W_n(R) = \bar{W}_n(R)$ if $n \geq$ s.r. $R + 2$.)

Lemma 4. Denote the canonical embedding $\bar{W}_n(R) \longleftrightarrow \bar{W}_{n+1}(R)$ by u_n. If $n \geq s \cdot r.R$ and $x \in \overline{St}_{n+1}(R)$, then u_n and $u_n \cdot x$ are homotopic. (Here $(u_n \cdot x)(g) = (u_n(g)) \cdot x \cdot)$

Lemma 5. For any $s \in S_{n+1}$ the embeddings u_n and u_n^s are homotopic.

For any simplicial set X we'll denote by $C_*(X)$ its chain complex, i.e., the complex of abelian groups with $C_p(x)$ equal to the free abelian group generated by the p-simplices of X and each differential equal to an alternating sum of homomorphisms induced by taking faces. It is well known that $C_*(X)$ is homotopy equivalent to the singular complex of the geometric realization of X. In view of (1.5) the maps of complexes $C_*(u_n), C_*(u_n(n,n+1)) : C_*\left(\bar{W}_n(R)\right) + C_*\left(\bar{W}_{n+1}(R)\right)$ are homotopic. Looking through the proof of (1.5) one sees that the corresponding homotopy operator $\phi_{n+1}^k : C_p\left(\bar{W}_n(R)\right) + C_{p+1}\left(\bar{W}_{n+1}(R)\right)$ may be taken in the following form: (We denote $x_{k,n+1}(1)$ by x_k and

$$
\begin{aligned}
x_{n+1,k}&(-1) \text{ by } y_k) \\
\phi_{n+1}^k\left(\alpha_0, \ldots, \alpha_p\right) &= \sum_{i=0}^p (-1)^{i+1}\left[\left(\alpha_0^{x_k y_k}, \ldots, \alpha_i x_k y_k, \alpha_i^{(k,n+1)}, \ldots, \alpha_p^{(k,n+1)}\right)\right. \\
&- \left(\alpha_0^{x_k y_k}, \ldots, \alpha_i^{x} y_k, \alpha_i x_k y_k, \ldots, \alpha_p^{x_k y_k}\right) \\
&+ \left(\alpha_0^{x_k} \cdot y_k, \ldots, \alpha_i^{x_k} \cdot y_k, \alpha_i^{x_k y_k}, \ldots, \alpha_{pk} y_k\right) - \left(\alpha_0 y_k, \ldots, \alpha_i y_k, \alpha_i, \ldots, \alpha_p\right) \\
&+ \left.\left(\alpha_0 y_k, \ldots, \alpha_i y_k, \alpha_i^{x_k} \cdot y_k, \ldots, \alpha_p^{x_k} \cdot y_k\right) - \left(\alpha_0 y_k, \ldots, \alpha_i y_k, \alpha_i y_k, \ldots, \alpha_p y_k\right)\right]
\end{aligned}
$$

Lemma 6. The homotopy operators ϕ_{n+1}^k have the following properties:

1. $(\partial - \alpha(k, n+1)) = d\phi_{n+1}^k(\alpha) + \phi_{n+1}^k(d\alpha)$, where $\alpha = \left(\alpha_0, \ldots, \alpha_p\right)$ is a p-simplex of $\bar{W}_n(R)$.

2. $\phi_{n+1}^n \mid C_*\left(\bar{W}_{n-1}(R)\right) = 0$.

3. For any $s \in S_n$ the following formula is valid:

$$\phi_{n+1}^k\left(\alpha^s\right) = \left[\phi_{n+1}^s(k)(\alpha)\right]^s$$

4. $\phi_{n+1}^k \mid C_*\left(\bar{W}_{n-1}(R)\right) = \left(\phi_n^k\right)(n+1, n)$

Lemma 7. *Suppose $c \in C_p\left(\bar{W}_{n-q}(R)\right)$, dc $\&C_{p-1}\left(\bar{W}_{n-q-1}(R)\right)$. Set*

$$c_0 = c, c_1 = \phi^{n-q}_{n-q+1}(c_0) \& c_{p+1}\left(\bar{W}_{n-q+1}(R)\right), \ldots, c_k$$

$$= \phi^{n-q+k-1}_{n-q+k}(c_{k-1}) \varepsilon c_{p+k}\left(\bar{w}_{n-q+k}(R)\right). \text{ Then, if } k \geq 1, \text{ we have:}$$

$$dc_k = c_{k-1} - c^{(n-q+k,n-q+k-1)}_{k-1} + \ldots + (-1)^k c^{(n-q+k,\ldots,n-q)}_{k-1}.$$

12.2.1   The Aciclicity Theorem

If X is an arbitrary set, we'll denote by $F_m(X)$ the partially ordered set of functions defined on non-empty subsets of $\{1, \ldots, m\}$ and taking values in X. The partial ordering is defined as follows:

$$f \leq g \Leftrightarrow \operatorname{dom} f \subset \operatorname{dom} g, \, g|_{\operatorname{dom}} f = f.$$

(Here dom f is the subset of $\{1, \ldots, m\}$ where f is defined). Following van der Kallen [11] we'll say that $F \subset F_m(X)$ satisfies the chain condition if F contains with any function all its restrictions (to non-empty subsets of its domain). It is clear that f and g have a common restriction if and only if there exists i $\varepsilon\{1, \ldots, m\}$ such that f and g are defined at i and equal at i. In this case there obviously exists a maximal common restriction inf(f, g).

If $F \subset F_m(X)$ satisfies the chain condition, then by F_* we'11 denote the geometric realization of the semi-simplicial set, whose non-degenerate p-simplices are the functions $f \varepsilon F$ with $|\operatorname{dom} f| = p + 1$, and whose faces are defined by the formulas $d_j(f) = f|_{\{i_0, \ldots, \hat{i}_j, \ldots, i_p\}}$ where $\{i_0, \ldots, i_p\} = \operatorname{dom} f, (i_0 < \ldots < i_p)$. If $f \varepsilon F, |\operatorname{dom} f| = p + 1$, then by $|f|$ we'll denote the corresponding p-simplex of F_*. It is clear that $|f| \cap |g|$ is either empty or else equals $|\inf(f, g)|$. In particular, F_* is a simplicial space [7].

Let R be a ring (associative with identity), R^∞ the free left R-module on the basis $e_1, \ldots, e_n, \ldots$, and R^n its sub-module generated by $e_1, \ldots, e_n$. If X is any subset of R^∞, then by $U_m(X)$ we' 11 denote the subset of $F_m(X)$ consisting of those functions f for which $f(i_0), \ldots, f(i_p)$ is a unimodular frame (i.e., a basis of a free direct summand of R^∞), where $\{i_0, \ldots, i_p\} = \operatorname{dom}(f)$.

Theorem 5. *Suppose R is a ring, $r = s.r.R$ and m, n are natural numbers. Then $U_m(R^n)$ is $\min(m-2, n-r-1)$-acyclic.*

Corollary 3. $U_n(R^n)$ *is $(n-r-1)-\operatorname{acyc} 1 ic$.*

Corollary 4. *Consider in $\operatorname{St}_{n+1}(\Lambda)$ the following subgroups: $A^i = \{\alpha : e_i \cdot \pi(\alpha) = e_i\}(i = 1, \ldots, n+1)$ and consider the simplicial set $Z'\left(St_{n+1}(R), A^i\right)$ constructed as in (2.5), but using left cosets instead of right cosets. This simplicial set is $(n-r)$-acyclic.*

12.3   Whitehead's K-theory

12.4   Quillen's K-theory

Part V

Homological stability

Chapter 13

Motivation

[1]

The symmetric group Σ_n is the group of bijections of the finite set $\underline{n} = \{1, \dots, n\}$, under composition. The classifying space BG of a discrete group G, such as Σ_n, is the connected space determined uniquely up to weak homotopy equivalence by the property

$$\pi_*(BG) = \begin{cases} G & \text{if } * = 1, \\ 0 & \text{otherwise} \end{cases}$$

It can be constructed by extracting from G the groupoid $*//G$ given by: - a single object $*$, - morphisms given by $* \xrightarrow{g} *$ for $g \in G$, and - composition given by multiplication.

We then take its nerve to obtain a simplicial set, and take the geometric realisation to get a topological space $|N(*//G)|$; this is a model for BG. Exercise 1.3.1 proves it indeed has the desired property.

Proposition 11. *$H_*(B\Sigma_n; \mathbb{Z})$ is the same as computing the group homology of Σ_n with coefficients in $\mathbb{Z}$.*

Let us compute these groups and the homology of their classifying spaces for the first few values of n.

Example 5. *1. For $n = 0, 1$, the group Σ_n is trivial so its classifying space is weakly contractible and hence has trivial homology.*

2. Example 1.1.4. For $n = 2, \Sigma_2$ is isomorphic to the cyclic abelian group $\mathbb{Z}/2$. Then $B\mathbb{Z}/2$, as constructed above, is homotopy equivalent to $\mathbb{R}P^\infty$. We conclude that

$$H_*(B\mathbb{Z}/2; \mathbb{Z}) = H_*\left(\mathbb{R}P^\infty; \mathbb{Z}\right) = \begin{cases} \mathbb{Z} & \text{if } * = 0 \\ \mathbb{Z}/2 & \text{if } * > 0 \text{ is odd,} \\ 0 & \text{if } * > 0 \text{ is even.} \end{cases}$$

3. Example 1.1.5. For $n = 3$, the group Σ_3 is the dihedral group D_3 with 6 elements (i.e. the symmetries of a triangle). A more complicated computation given in Exercise 1.3.5 yields the homology of D_3 :

$$H_*(BD_3; \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } * = 0 \\ \mathbb{Z}/2 & \text{if } * > 0 \text{ and } * \equiv 1 \pmod 4 \\ \mathbb{Z}/6 & \text{if } * > 0 \text{ and } * \equiv 3 \pmod 4, \\ 0 & \text{otherwise} \end{cases}$$

Conjectures

1. Each reduced homology group $\widetilde{H}_d(B\Sigma_n; \mathbb{Z})$ is finite and has small exponent.

2. The homology in fixed degree $* = d$ becomes independent of n as $n \to \infty$.

3. Before becoming independent of n, the homology only increases in size.

4. The p-power torsion only changes when $p \mid n$.

If we want to attempt to prove (2)-(4), we need a better way to compare the homology groups for different n than just as abstract abelian groups. This is done by observing that the inclusion $\underline{n} \hookrightarrow \underline{n+1}$ of finite sets gives a homomorphism

$$\sigma : \Sigma_n \longrightarrow \Sigma_{n+1},$$

by extending a permutation of $\underline{n}$ by the identity on $n+1 \in \underline{n+1}$ to a permutation of $n+1$. Our construction of BG is natural in groups and homomorphisms, so this homomorphism induces a map

$$\sigma : B\Sigma_n \longrightarrow B\Sigma_{n+1},$$

which in turn induces a map $\sigma_* : H_*(B\Sigma_n; \mathbb{Z}) \to H_*(B\Sigma_{n+1}; \mathbb{Z})$ on homology. We can then give sharper formulations of (2)-(4) in terms of these stabilisation maps: (2') The maps σ_* are isomorphisms in a range increasing with n.
(3') The maps σ_* are injective.
(4') The maps σ_* are isomorphisms on p-power torsion unless $p \mid n+1$.
Property (1) holds for all finite groups, and the result which proves it also implies (4'):

Proposition 12. *For a finite group* $G, \widetilde{H}_*(BG; \mathbb{Z}[1/|G|]) = 0$. *More generally, for* $H \subset G$ *the map* $\iota_* : H_*(BH; \mathbb{Z}[1/[G : H]]) \to H_*(BG; \mathbb{Z}[1/[G : H]])$ *admits a right inverse* τ *(i.e.* $\iota_* \circ \tau = \mathrm{id}$ *).*

To deduce (4') from Proposition 1.1.6, note that $[\Sigma_{n+1} : \Sigma_n] = n+1$ so by the long exact sequence on homology groups so that $H_*(B\Sigma_n; \mathbb{Z}) \to H_*(B\Sigma_{n+1}; \mathbb{Z})$ is surjective after inverting $n+1$. Now set $n+1$ equal to p and invoke (3'). It is phenomenon indicated by (2') that is the subject of this minicourse:

A sequence $X_0 \xrightarrow{\sigma} X_1 \xrightarrow{\sigma} X_2 \xrightarrow{\sigma} \cdots$ exhibits **homological stability** if the maps $\sigma_* : H_*(X_n; \mathbb{Z}) \to H_*(X_{n+1}; \mathbb{Z})$ are isomorphisms in a range of degrees * increasing with n.

In the next two lectures we will prove the following result, due to Nakaoka [Nak60] (though he proved much more):

Theorem 6. *The sequence* $B\Sigma_0 \xrightarrow{\sigma} B\Sigma_1 \xrightarrow{\sigma} B\Sigma_2 \xrightarrow{\sigma} \cdots$ *exhibits homological stability. More precisely, the induced map*

$$\sigma_* : H_*(B\Sigma_n; \mathbb{Z}) \longrightarrow H_*(B\Sigma_{n+1}; \mathbb{Z})$$

is surjective if $* \leq \frac{n}{2}$ *and an isomorphism if* $* \leq \frac{n-1}{2}$.

Remark 1.1.9. Of course, if we know property (3') holds then the range in the previous theorem in which σ_* is an isomorphism improves to $* \leq \frac{n}{2}$. However, property (3 ') is rather special—related to the existence of transfer maps-and you should not expect it to hold for general sequences of classifying spaces of groups. We will not comment on it again, but see Exercise 1.3.6.
Remark 1.1.10. The ranges in the previous remark are optimal among those of the form $* \leq an + b$ with $a, b \in \mathbb{Q}$.

13.1   Applications

Homological stability is a structural property of a sequence of groups, or more generally topological spaces, but it is also useful tool. In fact, many homological stability theorems are proven in service of obtaining other mathematical results. To illustrate this, I now want to explain some straightforward applications of Theorem 1.1.8. These concern the transfer of information from low n to high n and vice-versa. They can be obtained by other methods as well, but their generalisations to other sequences of groups often can not.

13.1.1   Altenating groups

Recall that for path-connected X, the Hurewicz map $\pi_1(X) \to H_1(X; \mathbb{Z})$ coincides with abelianisation (we are suppressing the basepoint). In particular, the map $G \to H_1(BG; \mathbb{Z})$ induces an isomorphism $G^{\mathrm{ab}} \to H_1(BG; \mathbb{Z})$ naturally in G. Thus we can understand the abelianisation of Σ_n by computing its first homology group. The sign homomorphism sign: $\Sigma_n \to \mathbb{Z}/2$ yields a map

$$\mathrm{sign}: B\Sigma_n \longrightarrow B\mathbb{Z}/2,$$

which induces a map on homology. This is compatible with stabilisation, in the sense that sign $\circ \sigma = $ sign, so we get a commutative squares

$$
\begin{array}{ccc}
H_1(B\Sigma_{n-1}; \mathbb{Z}) & \xrightarrow{\sigma_*} & H_1(B\Sigma_n; \mathbb{Z}) \\
\downarrow{\scriptstyle \mathrm{sign}} & & |\; \mathrm{sign} \\
\mathbb{Z}/2 & \xlongequal{Z} & /2.
\end{array}
$$

The map $H_1(B\Sigma_2;\mathbb{Z}) \to \mathbb{Z}/2$ is an isomorphism because sign: $\Sigma_2 \to \mathbb{Z}/2$ is. By Theorem 1.1.8, in the commutative diagram the right-most top horizontal map is surjective and the other top horizontal maps are isomorphisms. A single diagram chase then deduces from the fact that the left-most vertical map is an isomorphism that all other vertical maps are.

Thus we have used homological stability to prove that

$$\text{sign: } \Sigma_n \longrightarrow \mathbb{Z}/2$$

is the abelianisation for $n \geq 2$, or equivalently that the kernel of the sign homomorphism is exactly the subgroup $[\Sigma_n, \Sigma_n]$ generated by commutators. Recalling that this kernel is exactly the alternating group A_n, we conclude that:

Theorem 7. $[\Sigma_n, \Sigma_n] = A_n$.

Remark 1.2.2. This is a fact you likely knew already, and elementary group-theoretic arguments exist. We could have used this fact instead to give an elementary proof of Theorem 1.1.8 in degree $* = 1$.

13.2 Group Completion

Homological stability implies that for in fixed degree $*$, for n sufficienty large the canonical map

$$H_*(B\Sigma_n;\mathbb{Z}) \longrightarrow \operatorname*{colim}_{n\to\infty} H_*(B\Sigma_n;\mathbb{Z})$$

is an isomorphism; the right hand side is known as the stable homology. This has two somewhat tautological consequences: 1. We can compute the right side from the left side. 2. We can compute the left side from the right side.

This is particularly interesting because the stable homology on the right side has a more familiar description.

When we constructed the stabilisation map, we used that inclusion $\underline{n} \to \underline{n+1}$ yields a homomorphism $\Sigma_n \to \Sigma_{n+1}$. More generally, disjoint union induces a homomorphism $\Sigma_n \times \Sigma_m \to \Sigma_{n+m}$, which yields "multiplication" maps

$$B\Sigma_n \times B\Sigma_m \longrightarrow B\Sigma_{n+m},$$

making the space $\bigsqcup_{n\geq 0} B\Sigma_n$ into a unital topological monoid (these are associative but not commutative, and it is probably better to say E_1-space since that is a homotopy-invariant notion).

Theorem 8 (McDuff-Segal). *If M is a homotopy-commutative unital associative topological monoid, then $H_*(M;\mathbb{Z})\left[\pi_0^{-1}\right] \cong H_*(\Omega BM;\mathbb{Z})$.*

13.3 Serre's finiteness theorem and variations

Let us now use Corollary 1.2.6. By (1) the groups $H_*(B\Sigma_n;\mathbb{Z})$ are finite for $* > 0$. By Theorem 1.1.8 the same is true for the stable homology as long as restrict to degrees $* \leq \frac{n}{2}$. Since n is arbitrary, the stable homology is finite in all positive degrees. This has the following consequence:

Theorem 9. $\pi_*(\mathbb{S})$ *is finite for all $* > 0$.*

Exercise 1.3.8 (Using Serre's finiteness theorem). Serre proved that $\pi_*(\mathbb{S})$ is finite for $* > 0$. Combine this with Corollary 1.2.6 and Exercise 1.3.6 to prove that the sequence $B\Sigma_0 \xrightarrow{\sigma} B\Sigma_1 \xrightarrow{\sigma} B\Sigma_2 \xrightarrow{\sigma} \cdots$ exhibits homological stability. (Hint: you will not be able to give an explicit range.)

Remark 1.3.9. See [McD75] for a similar qualitative argument for configuration spaces of manifolds.

Chapter 14

Homological stability for symmetric groups

Theorem 10. *The sequence $B\Sigma_0 \xrightarrow{\sigma} B\Sigma_1 \xrightarrow{\sigma} B\Sigma_2 \xrightarrow{\sigma} \cdots$ exhibits homological stability. More precisely,*

$$\sigma_* : H_*(B\Sigma_n; \mathbb{Z}) \longrightarrow H_*(B\Sigma_{n+1}; \mathbb{Z})$$

is surjective if $* \leq \frac{n}{2}$ and an isomorphism if $* \leq \frac{n-1}{2}$.

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