PhD Studies

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June 27, 2024

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Group (Cohomology) Theory

Group Ring Let G be a group, written multiplicatively. Let $\mathbb{Z}G$ be the free $\mathbb{Z}$-module generated by the elements of G. The multiplication in G extends uniquely to a $\mathbb{Z}$-bilinear product $\mathbb{ZG} \times \mathbb{ZG} \to \mathbb{ZG}$; this makes $\mathbb{ZG}$ a ring, called the **integral group ring** of G.

Note that G is a subgroup of the group $(\mathbb{Z}G)^*$ of units of $\mathbb{ZG}$

Theorem 1 (Universal property)**.** *Given a ring R and a group homomorphism $f : G \to R^*$, there is a unique extension of f to a ring homomorphism $\mathbb{ZG} \to R$. Thus we have the "adjunction formula"*

$$\mathrm{Hom}_{\textit{(rings)}}(\mathbb{Z}G, R) \approx \mathrm{Hom}_{\textit{(groups)}}\left(G, R^*\right).$$

A **(left)** $\mathbb{Z}G$**-module**, or G-module, consists of an abelian group A together with a homomorphism from $\mathbb{Z}G$ to the ring of endomorphisms of A. By the universal property, G-module is simply an abelian group A together with an action of G on A. For example, one has for any A the trivial module structure, with $ga = a$ for $g \in G, a \in A$.

One way of constructing G-modules is by linearizing permutation representations. More precisely, if X is a G-set (i.e., a set with G-action), then one forms the free abelian group $\mathbb{ZX}$ (also denoted $\mathbb{Z}[X]$) generated by X and one extends the action of G on X to a $\mathbb{Z}$-linear action of G on $\mathbb{Z}X$. The resulting G-module is called a permutation module. In particular, one has a permutation module $\mathbb{Z}[G/H]$ for every subgroup H of G, where G/H is the set of cosets gH and G acts on G/H by left translation.

Proposition 1. *Let X be a free G-set and let E be a set of representatives for the G-orbits in X. Then $\mathbb{ZX}$ is a free $\mathbb{ZG}$-module with basis E.*

3.1 An spectral sequence for group cohomology

Suppose that X is a simplicial set and x_i are simplicial subsets such that $X = UX_i$. Then, setting $X_{ij} = X_i \cap X_j$ (etc.) we'11 obviously have for the realisations: $|x| = U|x_i|, |x_i| \cap \left|x_j\right| = \left|x_{ij}\right|, \ldots$ Let's suppose that the set of indices is linearly ordered. Consider the following bicomplex:

$$K = \longrightarrow \bigoplus_{i<j<k} C_*\left(x_{ijk}\right) \longrightarrow \bigoplus_{i<j} C_*\left(x_{ij}\right) \longrightarrow \bigoplus_{i} C_*\left(x_i\right)$$

Here by a bicomplex we understand a bicomplex in the sense of Grothendieck [9] i.e. the differentials d_1 and d_2 commute. (The sign in this approach appears in the definition of the total differentials). The vertical arrows of the bicomplex map $C_*\left(x_i \cdots{}_i\right)$ into $\underset{k=0}{q}\ C_*\left(x_{i_0} \ldots \hat{i}_k \ldots i_q\right)$, the mapping into the kth summand differing $k = 0$ by a sign $(-1)^k$ from the natural embedding.

The first spectral sequence of this bicomplex degenerates and yields an isomorphism $H_\star(K) \cong H_\star(X)$. (Moreover this isomorphism is induced by the canonical map $K \to C_*(X)$). The second spectral sequence gives us a functorial spectral sequence of the first quadrant, whose limit equals $H_*(X)$, while its differential dr has bidegree $(r-1, -r)$ and its E^1-term looks as follows:

$$E^1_{pqq} = \bigotimes_{i_0 < \ldots < i_q} H_p\left(x_{i_0} \ldots i_q\right)$$

Suppose G is a group. Let X_G denote the simplicial set (and its geometric realisation), whose p-simplices are sequences $\left(g_0, \ldots, g_p\right)$ of elements of G, with the usual faces and degeneracies. This space X_G is contractible by (1.2). The group G

acts from the right on X_G and this action is obviously free, hence $BG = X_G/G$ is a classifying space of G. The complex $C_*(BG) = C_*(G)$ coincides with the usual complex associated with G. Moreover $C_*(G) = C_*(X_G) \otimes_G Z$.

If H is a subgroup of G, then X_G/H is a classifying space for H and hence $BH = X_H/H \to X_G/H$ is a homotopy equivalence. In particular $C_*(H) + C_*(X_G) \otimes_H \mathbb{Z} = C_*(X_G) \otimes_G Z|G/H|$ is a homotopy equivalence.

(2.3) The spectral sequence associated with a family of subgroups.

Suppose G is a group and $G_1, \ldots, G_n$ are subgroups. Then BG_i may be viewed as a simplicial subset of BG and $BG_i \cap BG_j = B\left(G_i \cap G_j\right)$.. Denote UBG_i by X and consider the spectral sequence of the covering $X = UBG_i$. Along with the bicomplex K introduced in (2.1) we also consider the following bicomplex:

$$K' = \underset{i<j<k}{\oplus} C_*(X_G) \otimes_G Z\left[G/G_{ijk}\right] \longrightarrow \underset{i<j}{\oplus} C_*(X_G) \otimes_G Z\left[G/G_{ij}\right] \longrightarrow \underset{i<j}{\oplus} C_*(X_G) \otimes_G Z\left[G/G_i\right]$$

There is a natural mapping of bicomplexes $K + K'$ and because of (2.2) this mapping induces an isomorphism of second spectral sequences so that $H_\star(X) = H_\star(K) = H_*\left(K'\right)$. The first spectral sequence of K' looks as follows: $E^1_{*,q} = C_*(X_G) \otimes_G H_q(L)$, where L is the following complex of left G-modules:

$$\oplus \mathbb{Z}\left[G/G_i\right] + \oplus \mathbb{Z}\left[G/G_{ij}\right] + \oplus \mathbb{Z}\left[G/G_{ijk}\right] + \ldots$$

Proposition 2. *If $G_1, \ldots, G_n$ are subgroups of G, there exists a fuctorial spectral sequence of the first quadrart, the E^2 term of which looks like: $E^2_{pq} = H_p\left(G, H_q(L)\right)$, where L is the complex defined above. It converges to $H_\star\left(UBG_j\right)$ and the differential d^r has bidegree $(-r, r-1)$.*

(2.5) In the notations of (2.3), let $Z(G, \{G\})$ be the simplicial set whose non-degenerate p-simplices are sequences $\left(\bar{g}_0, \ldots, \bar{g}_p\right)$, where $\bar{g}_i \varepsilon G/G_{k_i}, k_0 < \ldots < k_p$, and the $\bar{g}_i$ are such that there is $g \in G$ with $\vec{g}_i = g \bmod G_{k_i}$ for all i. (If one covers G by the right cosets of the G_i, then $Z\left(G_g\{G_i\}\right)$ is the nerve of this covering.) It is easy to see that the geometric realization of this simplicial set is an ordered simplicial space and that the complex $L = L(G, \{G_i\})$ equals the (ordered) simplicial complex [7] of this simplicial space, or in other words, the complex L equals the normalised complex of the simplicial set $Z(G, \{G_i\})$. In particular, $H_*(L) = H_*(Z(G, \{G_i\}))$.

(2.6) Remark. It may be shown easily that the space $Z(G, \{G_i\})$, is homotopy equivalent to Volodin's space $V(G, \{G_i\})$, but we will not need this fact.

Chapter 4

(General) Module Theory

4.1 Linear Algebra

Part II

Topics of Algebraic Topology

Chapter 5

Simplical sets and complexes

[1] Simplicial complexes are more intuitive, and are the foundation of algebraic topology. Simplicial complexes were also called *simplicial schemes* and simplicial sets, *semi-simplicial* complexes.

5.1 (Abstract) simplical complexes

A set (of **vertices**) together with a family of finite subsets (**simplexes**) such that every subset of every simplex is a simplex and every subset consisting of a single vertex is a simplex.

Example 1.

1. *The* ***standard n-simplex*** Δ^n *is the set of all* $(n+1)$*-tuples* $(t_0, \ldots, t_n)$ *of non-negative real numbers such that* $t_0 + \cdots + t_n = 1$*. The standard 0-simplex is a point, the standard 1-simplex is a line segment, the standard 2-simplex is a triangle, and so on.*

2. *The* ***boundary*** *of the standard n-simplex* Δ^n *is the set of all* $(n+1)$*-tuples* $(t_0, \ldots, t_n)$ *of non-negative real numbers such that* $t_0 + \cdots + t_n = 1$ *and at least one of the* t_i *is zero. The boundary of the standard 0-simplex is empty, the boundary of the standard 1-simplex is the set of its two endpoints, the boundary of the standard 2-simplex is the set of its three edges, and so on.*

3. *(**Concrete simplicial complexes**) It is subset of* $\mathbb{R}^n$ *that is a union of standard simplices, that satisfies the previous conditions.*

4. *If* Y *is a subset of the vertex set of a simplicial scheme* S*, then we can introduce on it the induced simplicial scheme structure* $Y \cap S$*, by defining the simplexes as the subsets of* Y *that are simplexes of* S*.*

5. *Let* X *be a set and let* $\{p(y) : y \in Y\}$ *be a covering of* X*. Then we can consider two simplicial complexes.*

   (a) *The nerve* $\mathrm{Nerv}(p)$ *of the covering is the simplicial scheme with the vertex set* Y*, and a subset* Z *of* Y *is counted as a simplex if the intersection* $\bigcap_Z p(y)$ *is non-empty.*

   (b) *The simplicial complex* $V(p)$ *is the simplicial scheme with the vertex set* X*, and a subset* Z *of* X *is counted as a simplex if* Z *is contained in some* $p(y)$*.*

Geometric realization

The construction goes as follows. First, define $|K|$ as a subset of $[0,1]^S$ consisting of functions $t : S \to [0,1]$ satisfying the two conditions: □

$$\{s \in S : t_s > 0\} \in K$$
$$\sum_{s \in S} t_s = 1$$

Now think of the set of elements of $[0,1]^S$ with finite support as the direct limit of $[0,1]^A$ where A ranges over finite subsets of S, and give that direct limit the induced topology. Now give $|K|$ the subspace topology. *It is always Hausdorff.* We will identify an abstract simplicial complex with its geometric realization.

5.2 Simplical sets

Let Δ be the category of finite ordinal numbers, with order-preserving maps between them. More precisely, the objects for Δ consist of elements $\mathbf{n}$, $n \geq 0$, where $\mathbf{n}$ is a string of relations

$$0 \to 1 \to 2 \to \cdots \to n$$

(in other words $\mathbf{n}$ is a totally ordered set with $n+1$ elements). A morphism $\theta : \mathbf{m} \to \mathbf{n}$ is an order-preserving set function, or alternatively a functor. We usually commit the abuse of saying that Δ is the ordinal number category.

A simplicial set is a contravariant functor $X : \Delta^{op} \to \text{Sets}$, where Sets is the category of sets.

Remark 1. *The standard covariant functor:* $\mathbf{n} \mapsto |\Delta^n|$ *from* Δ *to* ***Top***. *The singular set* $S(T)$ *is the simplicial set given by*

$$\mathbf{n} \mapsto \hom\left(\left|\Delta^n\right|, T\right).$$

This is the object that gives the singular homology of the space T.

The standard n*-simplex, simplicial* Δ^n *in the simplicial set category* $\mathbf{S}$ *is defined by*

$$\Delta^n = \hom_\Delta(, \mathbf{n}).$$

In other words, Δ^n *is the contravariant functor on* Δ *which is represented by* n.

A map $f : X \to Y$ of simplicial sets (or, more simply, a simplicial map) is a natural transformation of contravariant set-valued functors defined on $\mathbf{\Delta}$. We shall use $\mathbf{S}$ to denote the resulting category of simplicial sets and simplicial maps.

From a simplicial set Y, one may construct a simplicial abelian group $\mathbb{Z}Y$ (ie. a contravariant functor $\mathbf{\Delta}^{op} \to \mathbf{Ab}$), with $\mathbb{Z}Y_n$ set equal to the free abelian group on Y_n. The simplicial abelian group $\mathbb{Z}Y$ has associated to it a chain complex, called its Moore complex and also written $\mathbb{Z}Y$, with

$$\mathbb{Z}Y_0 \xleftarrow{\partial} \mathbb{Z}Y_1 \xleftarrow{\partial} \mathbb{Z}Y_2 \leftarrow \dots \quad \text{and}$$
$$\partial = \sum_{i=0}^{n} (-1)^i d_i$$

in degree n. Recall that the integral singular homology groups $H_*(X;\mathbb{Z})$ of the space X are defined to be the homology groups of the chain complex $\mathbb{Z}SX$. The homology groups $H_n(Y, A)$ of a simplicial set Y with coefficients in an abelian group A are defined to be the homology groups $H_n(\mathbb{Z}Y \otimes A)$ of the chain complex $\mathbb{Z}Y \otimes A$.

Classifying space

Suppose that $\mathscr{C}$ is a (small) category. The classifying space (or nerve) $B\mathscr{C}$ of $\mathscr{C}$ is the simplicial set with

$$B\mathscr{C}_n = \hom_{\text{cat}}(\mathbf{n}, \mathscr{C}),$$

n-simplex is a string

$$a_0 \xrightarrow{\alpha_1} a_1 \xrightarrow{\alpha_2} \dots \xrightarrow{\alpha_n} a_n$$

of composeable arrows of length n in $\mathscr{C}$.

If G is a group, then G can be identified with a category (or groupoid) with one object $*$ and one morphism $g : * \to *$ for each element g of G, and so the classifying space BG of G is defined. Moreover $|BG|$ is an Eilenberg-Mac Lane space of the form $K(G, 1)$, as the notation suggests; this is now the standard construction.

Geometric realization

The simplex category: $\Delta \downarrow X$ of a simplicial set X. The objects of $\Delta \downarrow X$ are the maps $\sigma : \Delta^n \to X$, or simplices of X. An arrow of $\Delta \downarrow X$ is a commutative diagram of simplicial maps .....

Observe that θ is induced by a unique ordinal number map $\theta : \mathbf{m} \to \mathbf{n}$.

Lemma 1. *There is an isomorphism*

$$X \cong \varinjlim_{\substack{\Delta^n \longrightarrow X \\ \text{in } \Delta \downarrow X}} \Delta^n.$$

The realization $|X|$ of a simplicial set X is defined by the colimit

$$|X| = \varinjlim_{\substack{\Delta^n \to X \\ \text{in } \Delta \downarrow X}} \left|\Delta^n\right|.$$

in the category of topological spaces. The construction $X \mapsto |X|$ is seen to be functorial in simplicial sets X, by using the fact that any simplicial map $f : X \to Y$ induces a functor $f_* : \Delta \downarrow X \to \Delta \downarrow Y$ by composition with f.

Proposition 3. *The realization functor is left adjoint to the singular functor in the sense that there is an isomorphism*

$$\hom_{Top}(|X|, Y) \cong \hom_{\mathbf{S}}(X, SY)$$

which is natural in simplicial sets X and topological spaces Y. In particular, since **S** *has all colimits and the realization functor, $||$ preserves them.*

Proposition 4. *$|X|$ is a CW-complex for each simplicial set X. In particular it is a compactly generated Hausdorff space.*

5.3 CW-complexes

They can be defined in an inductive way:

1. Start with a discrete set X^0, whose points are regarded as 0-cells.
2. Inductively, form the ***n***-skeleton X^n from X^{n-1} by attaching n-cells e^n_α via maps $\varphi_\alpha : S^{n-1} \to X^{n-1}$. This means that X^n is the quotient space of the disjoint union $X^{n-1} \amalg_\alpha D^n_\alpha$ of X^{n-1} with a collection of n-disks D^n_α under the identifications $x \sim \varphi_\alpha(x)$ for $x \in \partial D^n_\alpha$. Thus as a set, $X^n = X^{n-1} \amalg_\alpha e^n_\alpha$ where each e^n_α is an open n-disk.
3. One can either stop this inductive process at a finite stage, setting $X = X^n$ for some $n < \infty$, or one can continue indefinitely, setting $X = \cup_n X^n$. In the latter case X is given the weak topology: A set $A \subset X$ is open (or closed) iff $A \cap X^n$ is open (or closed) in X^n for each n.

Example 2.

1. *A 1-dimensional cell complex $X = X^1$ is what is called a graph in algebraic topology. It consists of vertices (the 0-cells) to which edges (the 1-cells) are attached. The two ends of an edge can be attached to the same vertex.*
2. *The sphere S^n has the structure of a cell complex with just two cells, e^0 and e^n, the n-cell being attached by the constant map $S^{n-1} \to e^0$. This is equivalent to regarding S^n as the quotient space $D^n/\partial D^n$.*
3. ***Real projective n-space** $\mathbb{R}\mathrm{P}^n$**.** It is equivalent to the quotient space of a hemisphere D^n with antipodal points of ∂D^n identified. Since ∂D^n with antipodal points identified is just $\mathbb{R}\mathbb{P}\mathrm{P}^{n-1}$, we see that $\mathbb{R}\mathrm{P}^n$ is obtained from $\mathbb{R}\mathrm{P}^{n-1}$ by attaching an n-cell, with the quotient projection $S^{n-1} \to \mathbb{R}P^{n-1}$ as the attaching map. It follows by induction on n that $\mathbb{R}\mathbb{P}^n$ has a cell complex structure $e^0 \cup e^1 \cup \cdots \cup e^n$ with one cell e^i in each dimension $i \le n$.*
The infinite union $\mathbb{R}P^\infty = U_n \mathbb{R}P^n$ becomes a cell complex with one cell in each dimension. We can view $\mathbb{R}P^\infty$ as the space of lines through the origin in $\mathbb{R}^\infty = \bigcup_n \mathbb{R}^n$.
4. ***Complex projective space** $\mathbb{C}P^n$**.** It is equivalent to the quotient of the unit sphere $S^{2n+1} \subset \mathbb{C}^{n+1}$ with $v \sim \lambda v$ for $|\lambda| = 1$. It is also possible to obtain $\mathbb{C}\mathbb{P}^n$ as a quotient space of the disk D^{2n} under the identifications $v \sim \lambda v$ for $v \in \partial D^{2n}$, in the following way. The vectors in $S^{2n+1} \subset \mathbb{C}^{n+1}$ with last coordinate real and nonnegative are precisely the vectors of the form $\left(w, \sqrt{1-|w|^2}\right) \in \mathbb{C}^n \times \mathbb{C}$ with $|w| \le 1$. Such vectors form the graph of the function $w \mapsto \sqrt{1-|w|^2}$. This is a disk D^{2n}_+ bounded by the sphere $S^{2n-1} \subset S^{2n+1}$ consisting of vectors $(w, 0) \in \mathbb{C}^n \times \mathbb{C}$ with $|w| = 1$. Each vector in S^{2n+1} is equivalent under the identifications $v \sim \lambda v$ to a vector in D^{2n}_+, and the latter vector is unique if its last coordinate is nonzero. If the last coordinate is zero, we have just the identifications $v \sim \lambda v$ for $v \in S^{2n-1}$.*
It follows that $\mathbb{P}^n$ is obtained from $\mathbb{C}\mathrm{P}^{n-1}$ by attaching a cell e^{2n} via the quotient map $S^{2n-1} \to \mathbb{C}\mathbb{P}^{n-1}$. So by induction on n we obtain a cell structure $\mathbb{C}\mathbb{P}^n = e^0 \cup e^2 \cup \cdots \cup e^{2n}$ with cells only in even dimensions. Similarly, $\mathbb{C}\mathbb{P}^\infty$ has a cell structure with one cell in each even dimension.

Each cell e_α^n in a cell complex X has a **characteristic map** $\Phi_\alpha : D_\alpha^n \to X$ which extends the attaching map φ_α and is a homeomorphism from the interior of D_α^n onto e_α^n. Namely, we can take Φ_α to be the composition $D_\alpha^n \hookrightarrow X^{n-1} \coprod_\alpha D_\alpha^n \to X^n \hookrightarrow X$ where the middle map is the quotient map defining X^n.

Chapter 6

Geometric Group Theory

By a G-**complex** we will mean a CW-complex X together with an action of G on X which permutes the cells. Thus we have for each $g \in G$ a homeomorphism $x \mapsto gx$ of X such that the image go of any cell σ of X is again a cell. For example, if X is a simplicial complex on which G acts simplicially, then X is a G-complex.

If X is a G-complex then the action of G on X induces an action of G on the cellular chain complex $C_*(X)$, which thereby becomes a chain complex of G-modules. Moreover, the canonical augmentation $\varepsilon : C_0(X) \to \mathbb{Z}$ (defined by $\varepsilon(v) = 1$ for every 0 -cell v of X) is a map of G-modules.

We will say that X is a free G-complex if the action of G freely permutes the cells of X (i.e., $g\sigma \neq \sigma$ for all σ if $g \neq 1$). In this case each chain module $C_n(X)$ has a $\mathbb{Z}$-basis which is freely permuted by G, hence by $3.1 C_n(X)$ is a free $\mathbb{Z}G$-module with one basis element for every G-orbit of cells. (Note that to obtain a specific basis we must choose a representative cell from each orbit and we must choose an orientation of each such representative.)

Finally, if X is contractible, then $H_*(X) \approx H_*$ (pt.); in other words, the sequence

$$\cdots \to C_n(X) \xrightarrow{\partial} C_{n-1}(X) \to \cdots \to C_0(X) \xrightarrow{\varepsilon} \mathbb{Z} \to 0$$

is exact. We have, therefore:

Proposition 5. *Let X be a contractible free G-complex. Then the augmented cellular chain complex of X is a free resolution of $\mathbb{Z}$ over $\mathbb{Z}G$.*

Chapter 7

Homotopy theory

Let I^n be the n-dimensional unit cube, the product of n copies of the interval $[0,1]$. The boundary ∂I^n of I^n is the subspace consisting of points with at least one coordinate equal to 0 or 1 . For a space X with basepoint $x_0 \in X$, define $\pi_n(X, x_0)$ to be the set of homotopy classes of maps $f : (I^n, \partial I^n) \to (X, x_0)$, where homotopies f_t are required to satisfy $f_t(\partial I^n) = x_0$ for all t. The definition extends to the case $n = 0$ by taking I^0 to be a point and ∂I^0 to be empty, so $\pi_0(X, x_0)$ is just the set of path-components of X.

When $n \geq 2$, a sum operation in $\pi_n(X, x_0)$, generalizing the composition operation in π_1, is defined by

$$(f+g)(s_1, s_2, \cdots, s_n) = \begin{cases} f(2s_1, s_2, \cdots, s_n), & s_1 \in [0, 1/2] \\ g(2s_1 - 1, s_2, \cdots, s_n), & s_1 \in [1/2, 1] \end{cases}$$

It is evident that this sum is well-defined on homotopy classes. Since only the first coordinate is involved in the sum operation, the same arguments as for π_1 show that $\pi_n(X, x_0)$ is a group, with identity element the constant map sending I^n to x_0 and with inverses given by $-f(s_1, s_2, \cdots, s_n) = f(1 - s_1, s_2, \cdots, s_n)$.

Proposition 6. *If $n \geq 2$, then $\pi_n(X, x_0)$ is abelian.*

7.1 Covering spaces

The reader who has studied covering spaces has, of course, seen many examples of free G-complexes. Indeed, suppose $p : Y \to Y$ is a regular covering map with G as group of deck transformations. (See the appendix to this chapter for a review of regular covers.) If Y is a CW-complex, then it is an elementary fact that $\boldsymbol{Y}$ inherits a CW-structure such that the G-action permutes the cells, cf. Schubert [1968], III.6.9. Explicitly, the open cells of $\boldsymbol{Y}$ lying over an open cell σ of Y are simply the connected components of $p^{-1}\sigma$; these cells are permuted freely and transitively by G, and each is mapped homeomorphically onto σ under p. Thus $\tilde{Y}$ is a free G-complex and $C_* \tilde{Y}$ is a complex of free $\mathbb{Z}\mathbb{G}$-modules with one basis element for each cell of Y.

In view of 4.1, it is natural now to consider CW-complexes Y satisfying the following three conditions:

1. Y is connected.
2. $\pi_1(Y)$ is isomorphic to G.
3. The universal covering space X of Y is contractible.

Part III

Topics of Geometry

Part IV

K-theory

Chapter 8

K-theory constructions

8.1 Milnor's K-theory

For $n \geq 3$ the **Steinberg group** $\mathrm{St}_n(R)$ of a ring R is the group defined by generators $x_{ij}(r)$, with i, j a pair of distinct integers between 1 and n and $r \in R$, subject to the following "Steinberg relations":

$$x_{ij}(r)x_{ij}(s) = x_{ij}(r+s),$$
$$\left[x_{ij}(r), x_{k\ell}(s)\right] = \begin{cases} 1 & \text{if } j \neq k \text{ and } i \neq \ell, \\ x_{i\ell}(rs) & \text{if } j = k \text{ and } i \neq \ell, \\ x_{kj}(-sr) & \text{if } j \neq k \text{ and } i = \ell. \end{cases}$$

As observed in (1.3.1), the Steinberg relations are also satisfied by the elementary matrices $e_{ij}(r)$ which generate the subgroup $E_n(R)$ of $GL_n(R)$. Hence there is a canonical group surjection $\phi_n : St_n(R) \to E_n(R)$ sending $x_{ij}(r)$ to $e_{ij}(r)$.

The Steinberg relations for $n+1$ include the Steinberg relations for n, so there is an obvious map $St_n(R) \to St_{n+1}(R)$. We write $St(R)$ for $\lim_{\longrightarrow} St_n(R)$ and observe that by stabilizing, the ϕ_n induce a surjection $\phi : St(R) \to E(R)$.

The group $K_2(R)$ is the kernel of $\phi : St(R) \to E(R)$. Thus there is an exact sequence of groups

$$1 \to K_2(R) \to St(R) \xrightarrow{\phi} GL(R) \to K_1(R) \to 1.$$

It will follow from Theorem 5.2.1 below that $K_2(R)$ is an abelian group. Moreover, it is clear that St and K_2 are both covariant functors from rings to groups, just as GL and K_1 are.

Theorem 2. *$K_2(R)$ is an abelian group. In fact it is precisely the center of* $\mathrm{St}(R)$.

We'll define right actions of the symmetric group S_n on $_{GL}(R)$ and on $St_n(R)$ by setting

$$\left(\alpha^s\right)_{k,\ell} = \alpha_s(k), s(\ell); \quad x_{k\ell}(a)^s = x_s^{-1}(k), s^{-1}(\ell)(a).$$

These actions are compatible with the projections $St_n(R) \to E_n(R)$ and with the homomorphisms $St_n(R) + St_{n+1}(R)$ and $GL_n(R) + GL_{n+1}(R)$. In particular, they induce an action on $\overline{St}_n(R)$.

Lemma 2. *For any $s \in S_{n+1}$ the embeddings u_n and u_n^s are homotopic.*

8.2 Volodin's K-theory

Let G be a group and $\{G_i\}_{i\varepsilon I}$ a family of subgroups. Define $V(G, \{G_i\})$, or just $V(G)$ to be the simplicial complex, whose vertices are the elements of G, where $g_0, \ldots, g_p\left(g_i \neq g_j\right)$ form a p-simplex if for some G_i all the elements $g_j g_k^{-1}$ lie in G_i. If H is another group with a family of subgroups $\left\{H_j\right\}$ and $\phi : G \to H$ is a homomorphism sending each G_i into some H_j, then ϕ induces a simplicial map $V(\phi) : V(G) \to V(H)$.

In many situations it is more convenient to use simplicial sets instead of simplicial complexes: Denote by $W(G, \{G_i\})$ the geometric realization of the simplicial set whose p-simplices are the sequences $\left(g_0, \ldots, g_p\right)$ of elements of G (not necessarily distinct) such that for some G_i al1 $g_j g_k^{-1}$ lie in G_i, the r-th face (resp. degeneracy) of this simplex being obtained by omitting g_r (resp., repeating g_r). Associating with any p-simplex $\left(g_0, \ldots, g_p\right)$ the linear singular simplex of the space

$V(G)$ which sends the i-th vertex of the standard simplex to g_j, we obtain a map of simplicial sets from $W(G)$ to the simplicial set of singular simplices of $V(G)$ and hence a cellular map (linear on any simplex) from $W(G)$ to $V(G)$. This map is a homotopy equivalence ....

Suppose that R is a ring, n a natural number and σ a partial ordering of $\{1,\ldots,n\}$. Define $T_n^\sigma(R)$ to be the subgroup of $GL_n(R)$ consisting of the α with $\alpha_{ij}=1$ and $\alpha_{ij}=0$ if $i\&j$. Subgroups of this form will be called triangular subgroups of $GL_n(R)$. The space $V\left(GL_n(R),\left\{T_n^\sigma(R)\right\}\right)$ will be denoted by $V_n(R)$. Since any partial ordering may be extended to a linear ordering, it suffices to consider linear orderings when defining $V_n(R)$. The natural embedding $GL_n \hookrightarrow GL_{n+1}(R)$ defines an embedding $V_n(R) \longleftrightarrow V_{n+1}(R)$ and we'1l define $V_\infty(R)$ as $\lim_{\substack{\rightarrow}n} V_n(R)$.

Finally for $i\geq 1$, put

$$k_{i,n}(R)=\pi_{i-1}\left(V_n(R)\right)$$

and $k_i(R)=k_{i,\infty}(R)=\lim_{\rightarrow} k_{i,n}(R)$ (compare [26], [27]). Evidently $K_{1,n}(R)=GL_n(R)/E_n(R)$ and $K_{i,n}(R)$ is a group if $i\geq 2$, and this group is abelian if $i\geq 3$. Moreover the $K_i(R)$ are abelian groups for all $i\geq 1$ (see [26], [27]). The connected component of $V_n(R)$ passing through T_n equals $V\left(E_n(R),\left\{T_n^\sigma(R)\right\}\right)$. It is easy to show that the universal covering space of $V_n\left(E_n(R),\left\{T_n^\sigma(R)\right\}\right)$ equals $V\left(St(R),\left\{T_n^\sigma(R)\right\}\right)$, where T_n^σ is identified with the subgroup of $St_n(R)$ generated by the $x_{ij}(a)$ with a $\varepsilon R, i \overset{\sigma}{<} j (n\geq 3)$. Hence

Lemma 3. *$K_{2,n}(R)=\ker\left(St_n(R)+E_n(R)\right)$, and $K_{i,n}(R)=\pi_{i-1}\left(V\left(St_n(R)\right)\right)=\pi_{i-1}\left(W\left(St_n(R)\right)\right)$ if $i\geq 3$ $(n\geq 3)$.*

Let's define $\overline{St}_n(R)$ to be the inverse image of $GL_n(R)$ under the projection $St(R)\to E(R)$. There is a canonical homomorphism $St_n(R)\to \overline{st}_n(R)$ and stability for K_1, k_2 ([10], [20], [22]) shows that this homomorphism is surjective if $n\geq s.r.R+1$ and bijective if $n\geq s.r.R+2$. The spaces $W\left(St_n(R)\right)$ and $W\left(\overline{St}_n(R)\right)$ will play an essential role in the sequel. We'll denote them by $W_n(R), \bar{W}_n(R)$, resp. (So $W_n(R)=\bar{W}_n(R)$ if $n\geq$ s.r. $R+2$.)

Lemma 4. *Denote the canonical embedding $\bar{W}_n(R)\longleftrightarrow \bar{W}_{n+1}(R)$ by u_n. If $n\geq s\cdot r.R$ and $x\in\overline{St}_{n+1}(R)$, then u_n and $u_n\cdot x$ are homotopic. (Here $\left(u_n\cdot x\right)(g)=\left(u_n(g)\right)\cdot x\cdot)$)*

Lemma 5. *For any $s\in S_{n+1}$ the embeddings u_n and u_n^s are homotopic.*

For any simplicial set X we'll denote by $C_*(X)$ its chain complex, i.e., the complex of abelian groups with $C_p(x)$ equal to the free abelian group generated by the p-simplices of X and each differential equal to an alternating sum of homomorphisms induced by taking faces. It is well known that $C_*(X)$ is homotopy equivalent to the singular complex of the geometric realization of X. In view of (1.5) the maps of complexes $C_*\left(u_n\right), C_*\left(u_n(n,n+1)\right): C_*\left(\bar{W}_n(R)\right)+C_*\left(\bar{W}_{n+1}(R)\right)$ are homotopic. Looking through the proof of (1.5) one sees that the corresponding homotopy operator $\phi_{n+1}^k: C_p\left(\bar{W}_n(R)\right)+C_{p+1}\left(\bar{W}_{n+1}(R)\right)$ may be taken in the following form: (We denote $x_{k,n+1}(1)$ by x_k and

$x_{n+1,k}(-1)$ by y_k)

$$\begin{aligned}
\phi_{n+1}^k\left(\alpha_0,\ldots,\alpha_p\right)=\sum_{i=0}^p(-1)^{i+1}&\left[\left(\alpha_0^{x_ky_k},\ldots,\alpha_i x_k y_k,\alpha_i^{(k,n+1)},\ldots,\alpha_p^{(k,n+1)}\right)\right.\\
&-\left(\alpha_0^{x_ky_k},\ldots,\alpha_i{}^{x}y_k,\alpha_i x_k y_k,\ldots,\alpha_p^{x_ky_k}\right)\\
&+\left(\alpha_0^{x_k}\cdot y_k,\ldots,\alpha_i^{x_k}\cdot y_k,\alpha_i{}^{x_ky_k},\ldots,\alpha_{pk}y_k\right)-\left(\alpha_0 y_k,\ldots,\alpha_i y_k,\alpha_i,\ldots,\alpha_p\right)\\
&\left.+\left(\alpha_0 y_k,\ldots,\alpha_i y_k,\alpha_i^{x_k}\cdot y_k,\ldots,\alpha_p^{x_k}\cdot y_k\right)-\left(\alpha_0 y_k,\ldots,\alpha_i y_k,\alpha_i y_k,\ldots,\alpha_p y_k\right)\right]
\end{aligned}$$

Lemma 6. *The homotopy operators ϕ_{n+1}^k have the following properties:*

1. $(\partial-\alpha(k,n+1))=d\phi_{n+1}^k(\alpha)+\phi_{n+1}^k(d\alpha)$, *where $\alpha=\left(\alpha_0,\ldots,\alpha_p\right)$ is a p-simplex of $\bar{W}_n(R)$.*
2. $\phi_{n+1}^n \mid C_*\left(\bar{W}_{n-1}(R)\right)=0$.
3. *For any $s\in S_n$ the following formula is valid:*

$$\phi_{n+1}^k\left(\alpha^s\right)=\left[\phi_{n+1}^s(k)(\alpha)\right]^s$$

4. $\phi_{n+1}^k \mid C_*\left(\bar{W}_{n-1}(R)\right)=\left(\phi_n^k\right)(n+1,n)$

Lemma 7. *Suppose $c\in C_p\left(\bar{W}_{n-q}(R)\right)$, $dc\,\&C_{p-1}\left(\bar{W}_{n-q-1}(R)\right)$. Set*

$$\begin{aligned}
&c_0=c, c_1=\phi_{n-q+1}^{n-q}\left(c_0\right)\&c_{p+1}\left(\bar{W}_{n-q+1}(R)\right),\ldots,c_k\\
&=\phi_{n-q+k}^{n-q+k-1}\left(c_{k-1}\right)\varepsilon c_{p+k}\left(\bar{w}_{n-q+k}(R)\right). \textit{ Then, if } k\geq 1, \textit{ we have:}\\
&dc_k=c_{k-1}-c_{k-1}^{(n-q+k,n-q+k-1)}+\ldots+(-1)^k c_{k-1}^{(n-q+k,\ldots,n-q)}.
\end{aligned}$$

8.2.1 The Aciclicity Theorem

If X is an arbitrary set, we'll denote by $F_m(X)$ the partially ordered set of functions defined on non-empty subsets of $\{1,\ldots,m\}$ and taking values in X. The partial ordering is defined as follows:

$$f \le g \Leftrightarrow \operatorname{dom} f \subset \operatorname{dom} g,\ g\big|_{\operatorname{dom}} f = f.$$

(Here dom f is the subset of $\{1,\ldots,m\}$ where f is defined). Following van der Kallen [11] we'll say that $F \subset F_m(X)$ satisfies the chain condition if F contains with any function all its restrictions (to non-empty subsets of its domain). It is clear that f and g have a common restriction if and only if there exists i $\varepsilon\{1,\ldots,m\}$ such that f and g are defined at i and equal at i. In this case there obviously exists a maximal common restriction $\inf(f,g)$.

If $F \subset F_m(X)$ satisfies the chain condition, then by F_* we'11 denote the geometric realization of the semi-simplicial set, whose non-degenerate p-simplices are the functions $f\varepsilon F$ with $|\operatorname{dom} f| = p+1$, and whose faces are defined by the formulas $d_j(f) = f\big|_{\{i_0,\ldots,\hat{i}_j,\ldots,i_p\}}$ where $\{i_0,\ldots,i_p\} = \operatorname{dom} f, (i_0 < \ldots < i_p)$. If $f\varepsilon F, |\operatorname{dom} f| = p+1$, then by $|f|$ we'll denote the corresponding p-simplex of F_*. It is clear that $|f| \cap |g|$ is either empty or else equals $|\inf(f,g)|$. In particular, F_* is a simplicial space [7].

Let R be a ring (associative with identity), R^∞ the free left R-module on the basis $e_1,\ldots,e_n,\ldots$, and R^n its submodule generated by $e_1,\ldots,e_n$. If X is any subset of R^∞, then by $U_m(X)$ we' 11 denote the subset of $F_m(X)$ consisting of those functions f for which $f(i_0),\ldots,f(i_p)$ is a unimodular frame (i.e., a basis of a free direct summand of R^∞), where $\{i_0,\ldots,i_p\} = \operatorname{dom}(f)$.

Theorem 3. *Suppose R is a ring, $r =$ s.r.R and m, n are natural numbers. Then $U_m(R^n)$ is $\min(m-2, n-r-1)$-acyclic.*

Corollary 1. $U_n(R^n)$ *is* $(n-r-1)-$acyc1 *ic.*

Corollary 2. *Consider in* $\mathrm{St}_{n+1}(\Lambda)$ *the following subgroups: $A^i = \{\alpha : e_i \cdot \pi(\alpha) = e_i\}$ $(i = 1,\ldots,n+1)$ and consider the simplicial set $Z'(St_{n+1}(R), A^i)$ constructed as in (2.5), but using left cosets instead of right cosets. This simplicial set is $(n-r)$-acyclic.*

8.3 Whitehead's K-theory

8.4 Quillen's K-theory

Chapter 9

Homological stability

9.1 Motivation

The symmetric group Σ_n is the group of bijections of the finite set $\underline{n} = \{1, \dots, n\}$, under composition. The classifying space BG of a discrete group G, such as Σ_n, is the connected space determined uniquely up to weak homotopy equivalence by the property

$$\pi_*(BG) = \begin{cases} G & \text{if } * = 1, \\ 0 & \text{otherwise} \end{cases}$$

It can be constructed by extracting from G the groupoid $*//G$ given by: - a single object *, - morphisms given by $* \xrightarrow{g} *$ for $g \in G$, and - composition given by multiplication.

We then take its nerve to obtain a simplicial set, and take the geometric realisation to get a topological space $|N(*//G)|$; this is a model for BG. Exercise 1.3.1 proves it indeed has the desired property.

Proposition 7. *$H_*(B\Sigma_n; \mathbb{Z})$ is the same as computing the group homology of Σ_n with coefficients in $\mathbb{Z}$.*

Let us compute these groups and the homology of their classifying spaces for the first few values of n.

Example 3. *1. For $n = 0, 1$, the group Σ_n is trivial so its classifying space is weakly contractible and hence has trivial homology.*

2. Example 1.1.4. For $n = 2, \Sigma_2$ is isomorphic to the cyclic abelian group $\mathbb{Z}/2$. Then $B\mathbb{Z}/2$, as constructed above, is homotopy equivalent to $\mathbb{R}P^\infty$. We conclude that

$$H_*(B\mathbb{Z}/2; \mathbb{Z}) = H_*\left(\mathbb{R}P^\infty; \mathbb{Z}\right) = \begin{cases} \mathbb{Z} & \textit{if } * = 0 \\ \mathbb{Z}/2 & \textit{if } * > 0 \textit{ is odd,} \\ 0 & \textit{if } * > 0 \textit{ is even.} \end{cases}$$

Bibliography

[1] Charles A. Weibel. *An Introduction to Homological Algebra*. en. 1st ed. Cambridge University Press, Apr. 1994. ISBN: 978-0-521-43500-0 978-0-521-55987-4 978-1-139-64413-6. DOI: 10.1017/CBO9781139644136. URL: https://www.cambridge.org/core/product/identifier/9781139644136/type/book (visited on 05/02/2024).