

# PhD Studies

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June 22, 2024



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## Part I

# Topics of Algebraic Topology



# Chapter 1

## Simplicial sets and complexes

Simplicial complexes are more intuitive, and are the foundation of algebraic topology. Simplicial complexes were also called *simplicial schemes* and simplicial sets, *semi-simplicial* complexes.

### 1.1 (Abstract) simplicial complexes

A set (of **vertices**) together with a family of finite subsets (**simplexes**) such that every subset of every simplex is a simplex and every subset consisting of a single vertex is a simplex.

**Example 1.** 1. The **standard  $n$ -simplex**  $\Delta^n$  is the set of all  $(n+1)$ -tuples  $(t_0, \dots, t_n)$  of non-negative real numbers such that  $t_0 + \dots + t_n = 1$ . The standard 0-simplex is a point, the standard 1-simplex is a line segment, the standard 2-simplex is a triangle, and so on.

2. The **boundary** of the standard  $n$ -simplex  $\Delta^n$  is the set of all  $(n+1)$ -tuples  $(t_0, \dots, t_n)$  of non-negative real numbers such that  $t_0 + \dots + t_n = 1$  and at least one of the  $t_i$  is zero. The boundary of the standard 0-simplex is empty, the boundary of the standard 1-simplex is the set of its two endpoints, the boundary of the standard 2-simplex is the set of its three edges, and so on.

3. (**Concrete simplicial complexes**) It is subset of  $\mathbb{R}^n$  that is a union of standard simplices, that satisfies the previous conditions.

4. If  $Y$  is a subset of the vertex set of a simplicial scheme  $S$ , then we can introduce on it the induced simplicial scheme structure  $Y \cap S$ , by defining the simplexes as the subsets of  $Y$  that are simplexes of  $S$ .

5. Let  $X$  be a set and let  $\{p(y) : y \in Y\}$  be a covering of  $X$ . Then we can consider two simplicial complexes.

- (a) The nerve  $\text{Nerv}(p)$  of the covering is the simplicial scheme with the vertex set  $Y$ , and a subset  $Z$  of  $Y$  is counted as a simplex if the intersection  $\bigcap_Z p(y)$  is non-empty.
- (b) The simplicial complex  $V(p)$  is the simplicial scheme with the vertex set  $X$ , and a subset  $Z$  of  $X$  is counted as a simplex if  $Z$  is contained in some  $p(y)$ .

## Geometric realization

The construction goes as follows. First, define  $|K|$  as a subset of  $[0, 1]^S$  consisting of functions  $t : S \rightarrow [0, 1]$  satisfying the two conditions:  $\square$

$$\begin{aligned} \{s \in S : t_s > 0\} &\in K \\ \sum_{s \in S} t_s &= 1 \end{aligned}$$

Now think of the set of elements of  $[0, 1]^S$  with finite support as the direct limit of  $[0, 1]^A$  where  $A$  ranges over finite subsets of  $S$ , and give that direct limit the induced topology. Now give  $|K|$  the subspace topology. *It is always Hausdorff.* We will identify an abstract simplicial complex with its geometric realization.

## 1.2 Simplicial sets

Let  $\Delta$  be the category of finite ordinal numbers, with order-preserving maps between them. More precisely, the objects for  $\Delta$  consist of elements  $\mathbf{n}, n \geq 0$ , where  $\mathbf{n}$  is a string of relations

$$0 \rightarrow 1 \rightarrow 2 \rightarrow \cdots \rightarrow n$$

(in other words  $\mathbf{n}$  is a totally ordered set with  $n + 1$  elements). A morphism  $\theta : \mathbf{m} \rightarrow \mathbf{n}$  is an order-preserving set function, or alternatively a functor. We usually commit the abuse of saying that  $\Delta$  is the ordinal number category.

A simplicial set is a contravariant functor  $X : \Delta^{op} \rightarrow \text{Sets}$ , where Sets is the category of sets.

**Remark 1.** The standard covariant functor:  $\mathbf{n} \mapsto |\Delta^n|$  from  $\Delta$  to **Top**. The singular set  $S(T)$  is the simplicial set given by

$$\mathbf{n} \mapsto \text{hom}(|\Delta^n|, T).$$

This is the object that gives the singular homology of the space  $T$ .

The standard  $n$ -simplex, simplicial  $\Delta^n$  in the simplicial set category **S** is defined by

$$\Delta^n = \text{hom}_\Delta(\mathbf{n}).$$

In other words,  $\Delta^n$  is the contravariant functor on  $\Delta$  which is represented by  $\mathbf{n}$ .



A map  $f : X \rightarrow Y$  of simplicial sets (or, more simply, a simplicial map) is a natural transformation of contravariant set-valued functors defined on  $\Delta$ . We shall use  $\mathbf{S}$  to denote the resulting category of simplicial sets and simplicial maps.

From a simplicial set  $Y$ , one may construct a simplicial abelian group  $\mathbb{Z}Y$  (ie. a contravariant functor  $\Delta^{op} \rightarrow \mathbf{Ab}$ ), with  $\mathbb{Z}Y_n$  set equal to the free abelian group on  $Y_n$ . The simplicial abelian group  $\mathbb{Z}Y$  has associated to it a chain complex, called its Moore complex and also written  $\mathbb{Z}Y$ , with

$$\mathbb{Z}Y_0 \xleftarrow{\partial} \mathbb{Z}Y_1 \xleftarrow{\partial} \mathbb{Z}Y_2 \leftarrow \dots \quad \text{and}$$

$$\partial = \sum_{i=0}^n (-1)^i d_i$$

in degree  $n$ . Recall that the integral singular homology groups  $H_*(X; \mathbb{Z})$  of the space  $X$  are defined to be the homology groups of the chain complex  $\mathbb{Z}SX$ . The homology groups  $H_n(Y, A)$  of a simplicial set  $Y$  with coefficients in an abelian group  $A$  are defined to be the homology groups  $H_n(\mathbb{Z}Y \otimes A)$  of the chain complex  $\mathbb{Z}Y \otimes A$ .

## Classifying space

Suppose that  $\mathcal{C}$  is a (small) category. The classifying space (or nerve)  $BC$  of  $\mathcal{C}$  is the simplicial set with

$$BC_n = \text{hom}_{\text{cat}}(\mathbf{n}, \mathcal{C}),$$

$n$ -simplex is a string

$$a_0 \xrightarrow{\alpha_1} a_1 \xrightarrow{\alpha_2} \dots \xrightarrow{\alpha_n} a_n$$

of composable arrows of length  $n$  in  $\mathcal{C}$ .

If  $G$  is a group, then  $G$  can be identified with a category (or groupoid) with one object  $*$  and one morphism  $g : * \rightarrow *$  for each element  $g$  of  $G$ , and so the classifying space  $BG$  of  $G$  is defined. Moreover  $|BG|$  is an Eilenberg-Mac Lane space of the form  $K(G, 1)$ , as the notation suggests; this is now the standard construction.

## Geometric realization

**The simplex category:**  $\Delta \downarrow X$  of a simplicial set  $X$ . The objects of  $\Delta \downarrow X$  are the maps  $\sigma : \Delta^n \rightarrow X$ , or simplices of  $X$ . An arrow of  $\Delta \downarrow X$  is a commutative diagram of simplicial maps .....

Observe that  $\theta$  is induced by a unique ordinal number map  $\theta : \mathbf{m} \rightarrow \mathbf{n}$ .

**Lemma 1.** *There is an isomorphism*

$$X \cong \varinjlim_{\Delta^n \rightarrow X} \Delta^n.$$

The realization  $|X|$  of a simplicial set  $X$  is defined by the colimit

$$|X| = \varinjlim_{\Delta^n \rightarrow X} |\Delta^n|.$$

in the category of topological spaces. The construction  $X \mapsto |X|$  is seen to be functorial in simplicial sets  $X$ , by using the fact that any simplicial map  $f : X \rightarrow Y$  induces a functor  $f_* : \Delta \downarrow X \rightarrow \Delta \downarrow Y$  by composition with  $f$ .

**Proposition 1.** *The realization functor is left adjoint to the singular functor in the sense that there is an isomorphism*

$$\text{hom}_{\text{Top}}(|X|, Y) \cong \text{hom}_{\mathbf{S}}(X, SY)$$

which is natural in simplicial sets  $X$  and topological spaces  $Y$ . In particular, since  $\mathbf{S}$  has all colimits and the realization functor,  $| \cdot |$  preserves them.

**Proposition 2.**  *$|X|$  is a CW-complex for each simplicial set  $X$ . In particular it is a compactly generated Hausdorff space.*

### 1.3 CW-complexes

They can be defined in an inductive way:

1. Start with a discrete set  $X^0$ , whose points are regarded as 0-cells.
2. Inductively, form the  $n$ -skeleton  $X^n$  from  $X^{n-1}$  by attaching  $n$ -cells  $e_\alpha^n$  via maps  $\varphi_\alpha : S^{n-1} \rightarrow X^{n-1}$ . This means that  $X^n$  is the quotient space of the disjoint union  $X^{n-1} \amalg_\alpha D_\alpha^n$  of  $X^{n-1}$  with a collection of  $n$ -disks  $D_\alpha^n$  under the identifications  $x \sim \varphi_\alpha(x)$  for  $x \in \partial D_\alpha^n$ . Thus as a set,  $X^n = X^{n-1} \amalg_\alpha e_\alpha^n$  where each  $e_\alpha^n$  is an open  $n$ -disk.
3. One can either stop this inductive process at a finite stage, setting  $X = X^n$  for some  $n < \infty$ , or one can continue indefinitely, setting  $X = \cup_n X^n$ . In the latter case  $X$  is given the weak topology: A set  $A \subset X$  is open (or closed) iff  $A \cap X^n$  is open (or closed) in  $X^n$  for each  $n$ .

**Example 2.** 1. A 1-dimensional cell complex  $X = X^1$  is what is called a graph in algebraic topology. It consists of vertices (the 0-cells) to which edges (the 1-cells) are attached. The two ends of an edge can be attached to the same vertex.

2. The sphere  $S^n$  has the structure of a cell complex with just two cells,  $e^0$  and  $e^n$ , the  $n$ -cell being attached by the constant map  $S^{n-1} \rightarrow e^0$ . This is equivalent to regarding  $S^n$  as the quotient space  $D^n/\partial D^n$ .

3. **Real projective  $n$ -space  $\mathbb{RP}^n$ .** It is equivalent to the quotient space of a hemisphere  $D^n$  with antipodal points of  $\partial D^n$  identified. Since  $\partial D^n$  with antipodal points identified is just  $\mathbb{RP}^{n-1}$ , we see that  $\mathbb{RP}^n$  is obtained from  $\mathbb{RP}^{n-1}$  by attaching an  $n$ -cell, with the quotient projection  $S^{n-1} \rightarrow \mathbb{RP}^{n-1}$  as the attaching map. It follows by induction on  $n$  that  $\mathbb{RP}^n$  has a cell complex structure  $e^0 \cup e^1 \cup \dots \cup e^n$  with one cell  $e^i$  in each dimension  $i \leq n$ .

The infinite union  $\mathbb{RP}^\infty = \bigcup_n \mathbb{RP}^n$  becomes a cell complex with one cell in each dimension. We can view  $\mathbb{RP}^\infty$  as the space of lines through the origin in  $\mathbb{R}^\infty = \bigcup_n \mathbb{R}^n$ .

4. **Complex projective space  $\mathbb{CP}^n$ .** It is equivalent to the quotient of the unit sphere  $S^{2n+1} \subset \mathbb{C}^{n+1}$  with  $v \sim \lambda v$  for  $|\lambda| = 1$ .

It is also possible to obtain  $\mathbb{CP}^n$  as a quotient space of the disk  $D^{2n}$  under the identifications  $v \sim \lambda v$  for  $v \in \partial D^{2n}$ , in the following way. The vectors in  $S^{2n+1} \subset \mathbb{C}^{n+1}$  with last coordinate real and nonnegative are precisely the vectors of the form  $(w, \sqrt{1-|w|^2}) \in \mathbb{C}^n \times \mathbb{C}$  with  $|w| \leq 1$ . Such vectors form the graph of the function  $w \mapsto \sqrt{1-|w|^2}$ . This is a disk  $D_+^{2n}$  bounded by the sphere  $S^{2n-1} \subset S^{2n+1}$  consisting of vectors  $(w, 0) \in \mathbb{C}^n \times \mathbb{C}$  with  $|w| = 1$ . Each vector in  $S^{2n+1}$  is equivalent under the identifications  $v \sim \lambda v$  to a vector in  $D_+^{2n}$ , and the latter vector is unique if its last coordinate is nonzero. If the last coordinate is zero, we have just the identifications  $v \sim \lambda v$  for  $v \in S^{2n-1}$ .

It follows that  $\mathbb{P}^n$  is obtained from  $\mathbb{CP}^{n-1}$  by attaching a cell  $e^{2n}$  via the quotient map  $S^{2n-1} \rightarrow \mathbb{CP}^{n-1}$ . So by induction on  $n$  we obtain a cell structure  $\mathbb{CP}^n = e^0 \cup e^2 \cup \dots \cup e^{2n}$  with cells only in even dimensions. Similarly,  $\mathbb{CP}^\infty$  has a cell structure with one cell in each even dimension.

Each cell  $e_\alpha^n$  in a cell complex  $X$  has a **characteristic map**  $\Phi_\alpha : D_\alpha^n \rightarrow X$  which extends the attaching map  $\varphi_\alpha$  and is a homeomorphism from the interior of  $D_\alpha^n$  onto  $e_\alpha^n$ . Namely, we can take  $\Phi_\alpha$  to be the composition  $D_\alpha^n \hookrightarrow X^{n-1} \coprod_\alpha D_\alpha^n \rightarrow X^n \hookrightarrow X$  where the middle map is the quotient map defining  $X^n$ .



## Chapter 2

# Homotopy theory

Let  $I^n$  be the  $n$ -dimensional unit cube, the product of  $n$  copies of the interval  $[0, 1]$ . The boundary  $\partial I^n$  of  $I^n$  is the subspace consisting of points with at least one coordinate equal to 0 or 1. For a space  $X$  with basepoint  $x_0 \in X$ , define  $\pi_n(X, x_0)$  to be the set of homotopy classes of maps  $f : (I^n, \partial I^n) \rightarrow (X, x_0)$ , where homotopies  $f_t$  are required to satisfy  $f_t(\partial I^n) = x_0$  for all  $t$ . The definition extends to the case  $n = 0$  by taking  $I^0$  to be a point and  $\partial I^0$  to be empty, so  $\pi_0(X, x_0)$  is just the set of path-components of  $X$ .

When  $n \geq 2$ , a sum operation in  $\pi_n(X, x_0)$ , generalizing the composition operation in  $\pi_1$ , is defined by

$$(f + g)(s_1, s_2, \dots, s_n) = \begin{cases} f(2s_1, s_2, \dots, s_n), & s_1 \in [0, 1/2] \\ g(2s_1 - 1, s_2, \dots, s_n), & s_1 \in [1/2, 1] \end{cases}$$

It is evident that this sum is well-defined on homotopy classes. Since only the first coordinate is involved in the sum operation, the same arguments as for  $\pi_1$  show that  $\pi_n(X, x_0)$  is a group, with identity element the constant map sending  $I^n$  to  $x_0$  and with inverses given by  $-f(s_1, s_2, \dots, s_n) = f(1 - s_1, s_2, \dots, s_n)$ .

**Proposition 3.** *If  $n \geq 2$ , then  $\pi_n(X, x_0)$  is abelian.*



Part II

K-theory





## Chapter 3

# K-theory constructions

### 3.1 Volodin's K-theory

Let  $G$  be a group and  $\{G_i\}_{i \in I}$  a family of subgroups. Define  $V(G, \{G_i\})$ , or just  $V(G)$  to be the simplicial complex, whose vertices are the elements of  $G$ , where  $g_0, \dots, g_p$  ( $g_i \neq g_j$ ) form a  $p$ -simplex if for some  $G_i$  all the elements  $g_j g_k^{-1}$  lie in  $G_i$ . If  $H$  is another group with a family of subgroups  $\{H_j\}$  and  $\phi : G \rightarrow H$  is a homomorphism sending each  $G_i$  into some  $H_j$ , then  $\phi$  induces a simplicial map  $V(\phi) : V(G) \rightarrow V(H)$ .

In many situations it is more convenient to use simplicial sets instead of simplicial complexes: Denote by  $W(G, \{G_i\})$  the geometric realization of the simplicial set whose  $p$ -simplices are the sequences  $(g_0, \dots, g_p)$  of elements of  $G$  (not necessarily distinct) such that for some  $G_i$  all  $g_j g_k^{-1}$  lie in  $G_i$ , the  $r$ -th face (resp. degeneracy) of this simplex being obtained by omitting  $g_r$  (resp., repeating  $g_r$ ). Associating with any  $p$ -simplex  $(g_0, \dots, g_p)$  the linear singular simplex of the space  $V(G)$  which sends the  $i$ -th vertex of the standard simplex to  $g_j$ , we obtain a map of simplicial sets from  $W(G)$  to the simplicial set of singular simplices of  $V(G)$  and hence a cellular map (linear on any simplex) from  $W(G)$  to  $V(G)$ . This map is a homotopy equivalence ....

Suppose that  $R$  is a ring,  $n$  a natural number and  $\sigma$  a partial ordering of  $\{1, \dots, n\}$ . Define  $T_n^\sigma(R)$  to be the subgroup of  $GL_n(R)$  consisting of the  $\alpha$  with  $\alpha_{ij} = 1$  and  $\alpha_{ij} = 0$  if  $i \& j$ . Subgroups of this form will be called triangular subgroups of  $GL_n(R)$ . The space  $V(GL_n(R), \{T_n^\sigma(R)\})$  will be denoted by  $V_n(R)$ . Since any partial ordering may be extended to a linear ordering, it suffices to consider linear orderings when defining  $V_n(R)$ . The natural embedding  $GL_n \hookrightarrow GL_{n+1}(R)$  defines an embedding  $V_n(R) \hookrightarrow V_{n+1}(R)$  and we'll define  $V_\infty(R)$  as  $\lim_{\substack{\longrightarrow \\ n}} V_n(R)$ .

Finally for  $i \geq 1$ , put

$$k_{i,n}(R) = \pi_{i-1}(V_n(R))$$

and  $k_i(R) = k_{i,\infty}(R) = \lim_{\substack{\longrightarrow \\ n}} k_{i,n}(R)$  (compare [26], [27]). Evidently  $K_{1,n}(R) = GL_n(R)/E_n(R)$  and  $K_{i,n}(R)$  is a group if  $i \geq 2$ , and this group is abelian if

$i \geq 3$ . Moreover the  $K_i(R)$  are abelian groups for all  $i \geq 1$  (see [26], [27]). The connected component of  $V_n(R)$  passing through  $T_n$  equals  $V(E_n(R), \{T_n^\sigma(R)\})$ . It is easy to show that the universal covering space of  $V_n(E_n(R), \{T_n^\sigma(R)\})$  equals  $V(St(R), \{T_n^\sigma(R)\})$ , where  $T_n^\sigma$  is identified with the subgroup of  $St_n(R)$  generated by the  $x_{ij}(a)$  with  $a \in R, i <^\sigma j (n \geq 3)$ . Hence

**Lemma 2.**  $K_{2,n}(R) = \ker(St_n(R) + E_n(R))$ , and  $K_{i,n}(R) = \pi_{i-1}(V(St_n(R))) = \pi_{i-1}(W(St_n(R)))$  if  $i \geq 3$  ( $n \geq 3$ ).

Let's define  $\overline{St}_n(R)$  to be the inverse image of  $GL_n(R)$  under the projection  $St(R) \rightarrow E(R)$ . There is a canonical homomorphism  $St_n(R) \rightarrow \overline{st}_n(R)$  and stability for  $K_1, k_2$  ([10], [20], [22]) shows that this homomorphism is surjective if  $n \geq s.r.R + 1$  and bijective if  $n \geq s.r.R + 2$ . The spaces  $W(St_n(R))$  and  $W(\overline{St}_n(R))$  will play an essential role in the sequel. We'll denote them by  $W_n(R), \overline{W}_n(R)$ , resp. (So  $W_n(R) = \overline{W}_n(R)$  if  $n \geq s.r. R + 2$ .)

### 3.2 Milnor's K-theory

### 3.3 Whitehead's K-theory

### 3.4 Quillen's K-theory

## Chapter 4

# Homological stability

### 4.1 Motivation

The symmetric group  $\Sigma_n$  is the group of bijections of the finite set  $\underline{n} = \{1, \dots, n\}$ , under composition. The classifying space  $BG$  of a discrete group  $G$ , such as  $\Sigma_n$ , is the connected space determined uniquely up to weak homotopy equivalence by the property

$$\pi_*(BG) = \begin{cases} G & \text{if } * = 1, \\ 0 & \text{otherwise} \end{cases}$$

It can be constructed by extracting from  $G$  the groupoid  $*/G$  given by: - a single object  $*$ , - morphisms given by  $* \xrightarrow{g} *$  for  $g \in G$ , and - composition given by multiplication.

We then take its nerve to obtain a simplicial set, and take the geometric realisation to get a topological space  $|N(*//G)|$ ; this is a model for  $BG$ . Exercise 1.3.1 proves it indeed has the desired property.

**Proposition 4.**  $H_*(B\Sigma_n; \mathbb{Z})$  is the same as computing the group homology of  $\Sigma_n$  with coefficients in  $\mathbb{Z}$ .

Let us compute these groups and the homology of their classifying spaces for the first few values of  $n$ .

**Example 3.** 1. For  $n = 0, 1$ , the group  $\Sigma_n$  is trivial so its classifying space is weakly contractible and hence has trivial homology.

2. Example 1.1.4. For  $n = 2$ ,  $\Sigma_2$  is isomorphic to the cyclic abelian group  $\mathbb{Z}/2$ . Then  $B\mathbb{Z}/2$ , as constructed above, is homotopy equivalent to  $\mathbb{R}P^\infty$ . We conclude that

$$H_*(B\mathbb{Z}/2; \mathbb{Z}) = H_*(\mathbb{R}P^\infty; \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } * = 0 \\ \mathbb{Z}/2 & \text{if } * > 0 \text{ is odd,} \\ 0 & \text{if } * > 0 \text{ is even.} \end{cases}$$