PhD Studies

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Contents

Part I

Topics of Algebra

Chapter 1

Category Theory

Reference [1, 5]

Example 1. *1. For a topological spaces, the category of open sets with inclusions as morphisms. The opposite of this category, denoted by $\mathfrak{U}$, is used in sheaf theory.*

2. If $\mathcal{A}$and $\mathcal{B}$are preordered sets, then functors between them are monotone maps.

3. $f : \mathbb{Z} \to \mathbb{Q}$ is a monomorphism and epimorphism, but not an isomorphism.

Let $F : \mathcal{A} \to \mathcal{B}$ be a functor.

1. F is **faithful** provided that all the **hom-set restrictions**

$$F : \hom_{\mathcal{A}}(A, A') \to \hom_{\mathcal{B}}(FA, FA')$$

   are injective.

2. F is **full** if all hom-set restrictions are surjective.

3. F is an **embedding** if and it is faithful and injective on the class of objects.

4. F is **essentially surjective** if for every object B of $\mathcal{B}$, there is an object A of $\mathcal{A}$such that FA is isomorphic to B.

5. If F is essentially surjective and fully faithful, it is called an **equivalence of categories**, and $\mathcal{A}$and $\mathcal{B}$are said to be **equivalent**.

Let $F, G : \mathcal{A} \to \mathcal{B}$ be functors. A natural transformation τ from F to G (denoted by $\tau : F \to G$ or $F \xrightarrow{\tau} G$) is a function that assigns to each $\mathcal{A}$-object A a $\mathcal{B}$-morphism $\tau_A : FA \to GA$ in such a way that the following naturality condition holds: for each A-morphism $A \xrightarrow{f} A'$, the diagram $\begin{array}{ccc} FA & \xrightarrow{\tau_A} & GA \\ {\scriptstyle Ff}\downarrow & & \downarrow{\scriptstyle Gf} \\ FA' & \xrightarrow[\tau_{A'}]{} & GA' \end{array}$ commutes.

Example 2. *A typical example of a natural transformation is the connecting homomorphism for singular homology. Consider, for instance, the nth singular homology group of a pair of spaces (X, A). There is a connecting homomorphism*

$$\delta : H_n(X, A) \to H_{n-1}(A).$$

This morphism is natural for morphisms of pairs of topological spaces: if $f : (X, A) \to (Y, B)$ is a continuous map $f : X \to Y$ with $f(A) \subset B$, then the following diagram commutes:

$$\begin{array}{l} H_n(X, A) \xrightarrow{\delta} H_{n-1}(A) \\ H_n(f) \underset{\downarrow}{H_n(Y, B)} \xrightarrow{\delta} H_{n-1}(B) \end{array}$$

Thus, δ is a morphism between functors from the category of pairs of topological spaces to the category of abelian groups, the functor $(X, A) \mapsto H_n(X, A)$ and the functor $(X, A) \mapsto H_{n-1}(A)$.

Anoter example. Let $\mathcal{E}_G$ be the translation category of a discrete group G, as in Example 1.2.3. Then, $\mathcal{E}_G$ is equivalent to the category $[0]$ with one object and one identity morphism. There is a unique functor $P : \mathcal{E}_G \to [0]$, sending every object to 0 and every morphism to the identity morphism on 0 . We define $F : [0] \to \mathcal{E}_G$ via $F(0) = e$, where e denotes the neutral element in G, and we set $F(1_0) = e$. The composite $P \circ F$ is the identity functor on the category [0], whereas the composite $F \circ P$ sends any morphism $h : g \to hg$ in $\mathcal{E}_G$ to $e : e \to e$. We define $\eta : F \circ P \Rightarrow \mathrm{Id}_{\mathcal{E}_G}$ by setting

$$\eta_g : F \circ P(g) = e \to g = \mathrm{Id}_{\mathcal{E}_G}(g)$$

to be the morphism $g : e \to g$ in the translation category. As the diagram

$$\begin{array}{l} \eta_g : F \circ P(g) = e \longrightarrow g \\ F \circ P(h) = e \downarrow \quad \backslash \mathrm{Id}(h) = h \\ \eta_{hg} : F \circ P(hg) = e \xrightarrow{hg} hg \end{array}$$

commutes for all $h, g \in G$ and as $\mathcal{E}_G$ is a groupoid, this defines a natural isomorphism.

A natural transformation $F \xrightarrow{\tau} G$ whose components τ_A are isomorphisms is called a **natural isomorphism** from F to G, and F and G are said to be **naturally isomorphic**, denoted by $F \cong G$.

Example 3.

1. *Let $U : \mathrm{Grp} \to$ Set be the forgetful functor, and let $S : \mathrm{Grp} \to$ Set be the "squaring-functor", defined by $S(G \xrightarrow{f} H) = G^2 \xrightarrow{f^2} H^2$. For each group G, its multiplication is a function $\tau_G : G^2 \to G$. The family $\tau = (\tau_G)$ is a natural transformation from S to U. The naturality condition simply means that $f(x \cdot y) = f(x) \cdot f(y)$ for any group homomorphism $G \xrightarrow{f} H$ and any $x, y \in G$. Thus "multiplication" in groups can be regarded as a natural transformation. Similar for other structures.*

2. *Let $(\hat{\ })$: Vec $\to$ Vec be the second-dual functor for vector spaces, then $\tau_V : V \to \hat{\hat{V}}$, defined by $(\tau_V(x))(f) = f(x)$, yield a natural transformation $id_{\mathrm{Vec}} \xrightarrow{\tau} (\hat{\ })$. It becomes a natural isomorphism when restricted to finite-dimensional vector spaces.*

3. *The assignment of the Hurewicz homomorphism $\pi_n(X) \to H_n(X)$ to each topological space X is a natural transformation from the n-th homotopy functor π_n : Top $\to$ Grp to the n-th homology functor H_n : Top $\to$ Grp.*

4. *If $B \xrightarrow{f} C$ is an $\mathcal{A}$-morphism, then $\hom_{\mathcal{A}}(C,-) \xrightarrow{\tau_f} \hom_{\mathcal{A}}(B,-)$, defined by $\tau_f(g) = g \circ f$, and $\hom_{\mathcal{A}}(-,B) \xrightarrow{\sigma_f} \hom_{\mathcal{A}}(-,C)$, defined by $\sigma_f(g) = f \circ g$, are natural transformations.*

5. *(Good definitions of extension) Let F : Set $\to$ Vec be a functor that assigns to each set X a vector space FX with basis X, and to each function $X \xrightarrow{f} Y$ the unique linear extension $FX \xrightarrow{Ff} FY$ of f. This actually is not a correct definition of a functor, since there are many different vector spaces with the same basis. However, the definition is "correct up to natural isomorphism". Whenever we choose, for each set X, a specific vector space FX with basis X, we do obtain a functor F : Set $\to$ Vec (since the above condition determines the action of F on functions uniquely). Furthermore, any two functors that are obtained in this way are naturally isomorphic.*

6. *For any 2-element set A, hom $(A,-)$ is naturally isomorphic to the squaring functor $S^2[3.20(10)]$ and hom $(-,A)$ is naturally isomorphic to the contravariant power-set functor $\mathcal{Q}[3.20(9)]$. If B is isomorphic to A, then hom $(A,-)$ and hom $(B,-)$ are naturally isomorphic with those functors, the converse is true.*

1.1 Limits and colimits

An object P in a category $\mathcal{C}$ is called projective if for every epimorphism $f : M \to Q$ in $\mathcal{C}$ and every $p : P \to Q$, there is a $\xi \in \mathrm{Hom}(P, M)$ with $f \circ \xi = p$, called the **lift** of p to M.
Dually, an object I in a category $\mathcal{C}$ is called injective if for every monomorphism $f : U \to M$ in $\mathcal{C}$ and every $j : U \to I$, there is a $\zeta \in \mathrm{Hom}(M, I)$ with $\zeta \circ f = j$, called and **extension** of j to M.
$\mathcal{C}$ with $[0], \mathcal{C} * [0]$, has 0 as a terminal object and that $[0] * \mathcal{C}$ has 0 as an initial object. The category $\mathcal{C} * [0]$ is the **inductive cone** with base $\mathcal{C}$, and **[0]*** $\mathcal{C}$ is the **projective cone** with base $\mathcal{C}$.

Example 4. 1. *In the category of sets, every set is injective and projective.*

2. *In the category of left R-modules, a module is projective if and only if it is a direct summand of a free module. A module M is injective if and only if the functor $\mathrm{Hom}_R(-, M)$ is exact.*

Proposition 1. 1. *If P is a projective object of a category $\mathcal{C}$ and if $i : U \to P$ is a monomorphism in $\mathcal{C}$ with a retraction $r : P \to U$, then U is projective. Similarly, if $i : J \to I$ is a monomorphism with retraction $r : I \to J$ and I is injective, then J is injective.*

2. *If $q : Q \to P$ is an epimorphism and if P is projective, then q has a section. Dually, if $j : I \to J$ is a monomorphism and I is injective, then j has a retraction.*

3. *The object C is projective if and only if $\mathcal{C}(C,-) : \mathcal{C} \to$ Sets preserves epimorphisms.*

4. *The object C is injective if and only if $\mathcal{C}(-,C) : \mathcal{C}^o \to$ Sets sends monomorphisms to epimorphisms.*

Let $A \overset{\mathrm{f}}{\underset{g}{\rightrightarrows}} B$ be a pair of morphisms. A morphism $E \xrightarrow{e} A$ is called an equalizer of f and g provided that the following conditions hold: (1) $f \circ e = g \circ e$, (2) for any morphism $e' : E' \to A$ with $f \circ e' = g \circ e'$, there exists a unique morphism $\bar{e} : E' \to E$ such that $e' = e \circ \bar{e}$, i.e., such that the triangle

$$\begin{tikzcd} E' \arrow[d, "\bar{e}"'] \arrow[dr, "e'"] & & \\ E \arrow[r, "e"'] & A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \end{tikzcd}$$

commutes.

A **source** is a pair $\left(A, (f_i)_{i \in I}\right)$ consisting of an object A and a family of morphisms $f_i : A \to A_i$ with domain A, indexed by some class I.

Proposition 2. *A category has finite products if and only if it has terminal objects and products of pairs of objects. A category that has products for all class-indexed families must be thin. A small category has products if and only if it is equivalent to a complete lattice.*

A **diagram** in a category $\mathcal{A}$ is a functor $D : \mathbf{I} \to \mathcal{A}$ with codomain $\mathcal{A}$. The domain, $\mathbf{I}$, is called the **scheme** of the diagram. A diagram with a small (or finite) scheme is said to be **small** (or finite).
An A-source $\left(A \xrightarrow{f_i} D_i\right)_{i \in Ob(\mathrm{I})}$ is said to be **natural for** D provided that for each I-morphism $i \xrightarrow{d} j$, the triangle

$$\begin{tikzcd} A \arrow[d, "f_i"'] \arrow[dr, "f_j"] & \\ D_i \arrow[r, "Dd"'] & D_j \end{tikzcd}$$

commutes.
Equivalently, natural sources can be regarded as natural transformations from constant functors $C : \mathbf{I} \to \mathbf{A}$ to the functor D.
A **limit** of D is a natural source $\left(L \xrightarrow{\ell_i} D_i\right)$ for D with the **universal property** that for each natural source $\left(A \xrightarrow{f_i} D_i\right)$ there exists a unique morphism $f : A \to L$ with $f_i = \ell_i \circ f$ for each $i \in Ob(\mathbf{I})$.

1. Every source is natural for a diagram with discrete scheme. Products are limits of diagrams with discrete scheme. An object, considered as an empty source, is a limit of the empty diagram if and only if it is a terminal object.

2. For A-morphisms $A \overset{f}{\underset{g}{\rightrightarrows}} B$, considered as a diagram D with scheme $\bullet \Rightarrow \bullet$, a source $(A \xleftarrow{e} C \xrightarrow{h} B)$ is natural provided that $g \circ e = h = f \circ e$.
   $C \xrightarrow{e} A$ is an equalizer of $A \underset{g}{\overset{f}{\rightrightarrows}} B$ if and only if the source $(A \xleftarrow{e} C \xrightarrow{f \circ e} B)$ is a limit of D.

3. A poset $\mathbf{I}$ is **down-directed** if every pair of elements has a lower bound. Limits of diagrams with scheme $\mathbf{I}$ are called **projective** (or **inverse**) limits.

Proposition 3. *If* $\mathcal{L} = \left(L \xrightarrow{\ell_i} D_i\right)_{i \in Ob(\mathbf{I})}$ *is a limit of* $D : \mathbf{I} \to \mathbf{A}$, *then*

1. *for each limit* $\mathcal{K} = \left(K \xrightarrow{k_i} D_i\right)_{i \in Ob(\mathrm{I})}$ *of* D, *there exist an isomorphism* $K \xrightarrow{h} L$ *with* $\mathcal{K} = \mathcal{L} \circ h$,

2. *for each isomorphism $A \xrightarrow{h} L$, the source $\mathcal{L} \circ h$ is a limit of D.*

Proposition 4. *If $G : \mathcal{D} \to \mathcal{C}$ is another functor and if $\alpha : F \Rightarrow G$ is a natural transformation, then α induces a morphism $\operatorname{colim}_{\mathcal{D}} \alpha \in \mathcal{C}(\operatorname{colim}_{D} F, \operatorname{colim}_{\mathcal{D}} G)$. Prove that this turns $\operatorname{colim}_{\mathcal{D}}$ into a functor from $\operatorname{Fun}(\mathcal{D}, \mathcal{C})$ to $\mathcal{C}$.*

Proposition 5. *If the colimit ($\operatorname{colim}_{\mathcal{D}} F, \tau$) exists for all functors $F : \mathcal{D} \to \mathcal{C}$, then the functor $\operatorname{colim}_{\mathcal{D}} : \operatorname{Fun}(\mathcal{D}, \mathcal{C}) \to \mathcal{C}$ is left adjoint to the diagonal functor $\Delta : \mathcal{C} \to \operatorname{Fun}(\mathcal{D}, \mathcal{C})$, that is, there are natural isomorphisms*

$$\mathcal{C}(\operatorname{colim}_{\mathcal{D}} F, C) \cong \operatorname{Fun}(\mathcal{D}, \mathcal{C})(F, \Delta(C))$$

for all functors F and all object C of $\mathcal{C}$.

If you build the colimit over a discrete diagram category (small category $\mathcal{D}$ that has only identity morphisms), then the colimit of a functor $F : \mathcal{D} \to \mathcal{C}$ is called the **coproduct** of the $F(D)$ for D an object of $\mathcal{D}$, denoted by $\bigsqcup_{\mathcal{D}} F(D)$. Coproducts in the category of sets and in the category of topological spaces are the disjoint unions Every coproduct comes with canonical structure maps, called **inclusions**.
Pushouts are colimits over a diagram category $\mathcal{D}$ of the form $D_1 \leftarrow D_0 \to D_2$..
Another important class of examples is **coequalizers**. These are colimits of diagrams of the form

$$F(D_0) \underset{\alpha}{\overset{\beta}{\rightrightarrows}} F(D_1).$$

Example 5 (Colimits)**.**

1. *Colimits exist in the category of Sets:*

$$\operatorname{colim}_{\mathcal{D}} F = \bigsqcup_{D \text{ object of } \mathcal{D}} F(D)/\sim,$$

where we declare that an $x \in F(D)$ is equivalent to a $y \in F(D')$ if there is a morphism $f \in \mathcal{D}(D, D')$, such that $F(f)(x) = y$. This relation is not symmetric, so one has to consider the equivalence relation generated by this relation.

2. *If all structure maps $F(i < j)$ are monomorphisms, then we might interpret the colimit $\operatorname{colim}_{\mathcal{D}} F$ as the union of the $F(i)$ s. Typical examples are increasing sequences of sets or topological spaces*

$$X_0 \subset X_1 \subset X_2 \subset \ldots$$

or increasing sequences of abelian groups, vector spaces, and other algebraic objects.

3. *An important class of examples is CW complexes. These are the colimits of their skeleta.*

4. *In stable homotopy theory, the stable homotopy groups of spheres are a central object of study. Let $\mathbb{S}^n$ denote the unit sphere in $\mathbb{R}^{n+1}$. As the smash product of spheres satisfies $\mathbb{S}^1 \wedge \mathbb{S}^n \cong \mathbb{S}^{n+1}$ we have stabilization maps*

$$\pi_n(\mathbb{S}^m) = \left[\mathbb{S}^n, \mathbb{S}^{m+1}\right]_* \to \left[\mathbb{S}^{n+1}, \mathbb{S}^{m+1}\right]_* = \pi_{n+1}(\mathbb{S}^m)$$

that send a homotopy class $[f]$ to the homotopy class of $\mathbb{S}^1 \wedge f$. Therefore, for every m, we get a sequential colimit and as $\pi_n(\mathbb{S}^m) = 0$ for $n < m$, we can express $\pi_n(\mathbb{S}^m)$ as $\pi_{k+m}(\mathbb{S}^m)$ in the nontrivial cases, with $k \geq 0$, and get the k th stable homotopy group of spheres as

$$\pi_k^s = \operatorname{colim}\left(\pi_{k+m}(\mathbb{S}^m) \to \pi_{k+m+1}\left(\mathbb{S}^{m+1}\right) \to \pi_{k+m+2}\left(\mathbb{S}^{m+2}\right) \to \ldots\right)$$

5. *The first groups are* $\pi_0^s = \mathbb{Z}, \pi_1^s = \mathbb{Z}/2\mathbb{Z}$ *generated by the stabilization of the Hopf* map $\eta : \mathbb{S}^3 \to \mathbb{S}^2, \pi_2^s = \mathbb{Z}/2\mathbb{Z}, \pi_3^s = \mathbb{Z}/24\mathbb{Z}$*, and so on.*

6. *In the category of pointed topological spaces the pointed sum (also known as the bouquet of spaces) is the coproduct.*

7. *Coproducts in the category of abelian groups are given by the direct sum. Coproducts in the category of general groups is the free product.*

8. *If A is a topological space, together with continuous maps $f : A \to X$ and $g : A \to Y$, the pushout of $X \leftarrow A \to Y$ is the quotient space of the disjoint union $X \sqcup Y$ by the equivalence relation that identifies $f(a)$ with $g(a)$ for all $a \in A$.*

9. *Pushouts of groups are given by amalgamated products, given by $G_1 *_{G_0} G_2$, which is the quotient of the free product $G_1 * G_2$ by the normal subgroup generated by words of the form $f(g_0)\, h(g_0)^{-1}$ for $g_0 \in G_0$.*

10. *The cokernel of a homomorphism f is the coequalizer of the diagram $A \underset{f}{\overset{0}{\longrightarrow}} B$ in the category Ab.*

Limits, products... are defined dually

Example 6 (Limits)**.** 1. *Let $(X_n)_{n\in\mathbb{N}_0}$ be a family of sets with $X_{n+1} \subset X_n$. Then, the limit of the system*

$$\ldots \subset X_{n+1} \subset X_n \subset \ldots \subset X_1 \subset X_0$$

is the intersection of the sets X_n.

2. *Let p be a fixed prime. The inverse limit of the diagram is the ring of p-adic integers, $\mathbb{Z}_p$. Here, the maps p_i are the canonical projection maps. An explicit model of the limit is*

$$\left\{ (x_1, x_2, x_3, \ldots) \in \prod_{n\geq 1} \mathbb{Z}/p^n\mathbb{Z} \mid p_i(x_i) = x_{i-1} \text{ for all } i \geq 2 \right\}.$$

This carries a ring structure, where addition and multiplication are defined coordinatewise.

3. *Kernels in the category of abelian groups are limits of diagrams of the form $A \underset{f}{\overset{0}{\longrightarrow}} B$.*

4. *The presheaf F is a sheaf if for every $U \in \mathfrak{U}(X)$ and for every open covering $(U_i)_{i\in I}$ of U, the following diagram is an equalizer:*

$$F(U) \longrightarrow \prod_{i\in I} F(U_i) \Longrightarrow \prod_{i,j\in I} F(U_i \cap U_j).$$

Here, the first map is induced by the restriction maps res $U_U^{U_i}$, and the second pair of arrows is induced by two sets of restriction maps. $U_i \cap U_j$ is a subset of U_i and of U_j. Sheaves form a category as a full subcategory of the category of presheaves.

5. *Fiber products in the category of sets are limits of diagrams of the form $A \underset{f}{\overset{g}{\longrightarrow}} C$. A concrete model for this pullback in these categories is $f^*(p) := Z \times_Y X := \{(z, x) \in Z \times X \mid f(z) = p(x)\}$*

1.2 Adjoint functors

Let $\mathcal{C}$ and $\mathcal{C}'$ be categories. An **adjunction** between $\mathcal{C}$ and $\mathcal{C}'$ is a pair of functors $L : \mathcal{C} \to \mathcal{C}', R : \mathcal{C}' \to \mathcal{C}$, such that for each pair of objects C of $\mathcal{C}$ and C' of $\mathcal{C}'$, there is a bijection of sets

$$\varphi_{C,C'} : \mathcal{C}'\left(L(C), C'\right) \cong \mathcal{C}\left(C, R\left(C'\right)\right),$$

which is natural in C and C'. The functor L is then left adjoint to R, and R is right adjoint to L. We call (L, R) an adjoint pair of functors.

The naturality condidition on the bijections $\varphi_{C,C'}$ can be spelled out explicitly as follows: For all morphisms $f : C \to D$ in $\mathcal{C}$ and $g : C' \to D'$ in $\mathcal{C}'$, the diagram commutes.

$$\mathcal{C}'\left(L(D), C'\right) \xrightarrow{\mathcal{C}'\left(Lf, C'\right)} \mathcal{C}'\left(L(C), C'\right) \xrightarrow{\mathcal{C}'(L(C), g)} \mathcal{C}'\left(L(C), D'\right)$$
$$\varphi_{D,C'} \downarrow\downarrow \varphi_{C,C'} \downarrow \varphi_{C,D'}$$
$$\mathcal{C}\left(D, R\left(C'\right)\right) \xrightarrow{\mathcal{C}\left(f, R\left(C'\right)\right)} \mathcal{C}\left(C, R\left(C'\right)\right) \xrightarrow{\mathcal{C}(C, R(g))} \mathcal{C}\left(C, R\left(D'\right)\right)$$

Example 7. *A prototypical example of an adjunction is a forgetful functor and a 'free' functor: if $R = U$ is a forgetful functor and if a left adjoint of U exists, then the defining property means that for each morphism from C to $U\left(C'\right)$ in the underlying category, there is a unique corresponding morphism from $L(C)$ to C', so, in this sense, $L(C)$ is the free object associated with C. For topological spaces, the free topological space on a set is the set with discrete topology.*

Proposition 6. 1. *The functor L is left adjoint to R iff there are natural transformations η : Id $\Rightarrow R \circ L$ and $\varepsilon : L \circ R \Rightarrow$ Id with the properties that*

$$\varepsilon_L \circ L(\eta) = \mathrm{Id}_L \text{ and } R(\varepsilon) \circ \eta_R = \mathrm{Id}_R$$

hence, the diagrams

$$L(C) \xrightarrow{L(\eta)} LRL(C) \text{ and } R\left(C'\right) \xrightarrow{\eta_{R(C')}} RLR\left(C'\right)$$

commute for all objects C of $\mathcal{C}$ and C'' of $\mathcal{C}'$.

2. *Adjunction can be composed.*

3. *Each of the functors L and R determines the other functor uniquely up to isomorphism.*

4. *G has a left-adjoint F if and only if $\mathrm{Hom}_C(X, G-)$ is representable for all X in C. The natural isomorphism $\Phi_X : \mathrm{Hom}_D(FX, -) \to \mathrm{Hom}_C(X, G-)$ yields the adjointness; that is*

$$\Phi_{X,Y} : \mathrm{Hom}_{\mathcal{D}}(FX, Y) \to \mathrm{Hom}_{\mathcal{C}}(X, GY)$$

is a bijection for all X and Y.

The transformation η is called the **unit of the adjunction** and ε is the **counit**.

Theorem 1. *Let $F : \mathcal{C} \to \mathcal{D}$ be an arbitrary functor. Then the following are equivalent.*

1. *The functor F possesses a left adjoint L, and the corresponding natural transformations $\varepsilon : LF \Rightarrow \mathrm{Id}$ and $\eta : Id \Rightarrow FL$ are natural isomorphisms.*

2. *There is a functor $L : \mathcal{D} \to \mathcal{C}$ and two arbitrary natural isomorphisms $\mathrm{Id} \cong FL$ and $LF \cong \mathrm{Id}$.*

3. *The functor F is fully faithful and essentially surjective.*

Skeleta of categories

Example 8. *1. A category is called* ***reduced*** *if isomorphic objects are identical. A subcategory $\mathcal{S}$ of a category $\mathcal{C}$ is a* ***skeleton*** *if $\mathcal{S}$ is reduced and if the inclusion $\mathcal{S} \hookrightarrow \mathcal{C}$ is an equivalence of categories.*

2. *Consider the category of finite sets and functions. It contains the full subcategory whose objects are the sets of the form $\{1, \ldots, n\}$ for $n \geq 0$. Here, we use the convention that the empty set is encoded by $n = 0$. The inclusion functor is full and faithful. As every finite set is in bijection with a standardized set of the form $\{1, \ldots, n\}$ as above, the inclusion functor is also essentially surjective. Therefore, these finite sets build a skeleton.*

3. *A similar example is the category of finite-dimensional K-vector spaces. This has as a skeleton the full subcategory of vector spaces of the form K^n for some finite natural number n. Here, $n = 0$ encodes the zero vector space.*

Proposition 7. *Every category has an skeleton*

1.3 Concrete categories and representable functors

A way to talk of *low level structures* present on the objects of a category. Often it is easier to work with less structures, and there results like Yoneda's lemma that show us that it is possible to restrict our study to them.

Let $\mathcal{C}$be a category. A **concrete category** over $\mathcal{C}$is a category $\mathcal{A}$ together wih a faithful functor $U : \mathcal{A} \to \mathcal{C}$, called the **forgetful** (or underlying) functor of the concrete category. $\mathcal{C}$ is called the **base category**. A concrete category over Set is called a **construct**.
The category of groups (or topological spaces, rings, etc.), with the forgetful functor to Set, is a construct.

1. A **structured arrow** with domain X is a pair (f, A) consisting of an A-object A and an X-morphism $X \xrightarrow{f} |A|$,
2. if (f, A) is **generating** provided that for any pair of A-morphisms $r, s : A \to B$ the equality $r \circ f = s \circ f$ implies that $r = s$,
3. and this (f, A) is called **extremally generating** (resp. **concretely generating**) provided that each A-monomorphism (resp. A-embedding) $m : A' \to A$, through which f factors (i.e., $f = m \circ g$ for some **X**-morphism g), is an **A**-isomorphism.
4. In a construct, an object A is (extremally resp. concretely) generated by a subset X of $|A|$ provided that the inclusion map $X \hookrightarrow |A|$ is (extremally resp. concretely) generating.

Proposition 8. *In a concrete category* **A** *over* **X** *the following hold for each structured arrow $f : X \to |A|$:*

1. *If (f, A) is extremally generating, then (f, A) is concretely generating.*
2. *If (f, A) is concretely generating, then (f, A) is generating.*

3. *If $X \xrightarrow{f} |A|$ is an* $\mathbf{X}$*-epimorphism, then (f, A) is generating.*

4. *If $X \xrightarrow{f} |A|$ is an extremal epimorphism in* $\mathbf{X}$*, and if $||$ preserves monomorphisms, then (f, A) is extremally generating.*

Example 9. 1. *If an abstract category* $\mathbf{A}$ *is considered to be concrete over itself via the identity functor, then an A-morphism $A \xrightarrow{f} B$, considered as a structured arrow (f, B), is generating (resp. extremally or concretely generating) if and only if f is an epimorphism (resp. an extremal epimorphism). That is,*

$$\mathrm{Gen}(\mathbf{A}) = \mathrm{Epi}(\mathbf{A}) \textit{ and } \mathrm{ExtrGen}(\mathbf{A}) = \mathrm{ConcGen}(\mathbf{A}) = \mathrm{ExtrEpi}(\mathbf{A})$$

(a) *In Vec, Grp, Sgr, Rng, and other algebraic constructs, the concepts of concrete generation and of extremal generation coincide with the familiar (non-categorical) concept of generation. In the constructs Sgr and Rng the inclusion map $\mathbb{Z} \hookrightarrow \mathbb{Q}$ is generating, but is not concretely generating [cf. 7.40(5)].*

(b) *In the construct* $\mathbf{A} = $ *Top we have*

$$\begin{aligned} \textit{ConcGen(A)} &= \mathrm{Gen}(\mathbf{A}) = \textit{ Surjective maps, and} \\ \mathrm{ExtrGen}(\mathbf{A}) &= \textit{ Surjective maps with discrete codomain.} \end{aligned}$$

(c) *In the construct* $\mathbf{A} = $ *Haus we have*

$$\begin{aligned} \mathrm{Gen}(\mathbf{A}) &= \textit{ Dense maps} \\ \textit{ConcGen(A)} &= \textit{ Surjective maps, and} \\ \textit{ExtrGen(A)} &= \textit{ Surjective maps with discrete codomain.} \end{aligned}$$

(d) *$A \xrightarrow{f} B$ is an epimorphism if and only if (f, B) is generating.*

(e) *If (f, B) is extremally generating and the forgetful functor preserves monomorphisms, then $A \xrightarrow{f} B$ is an extremal epimorphism.*

(f) *If $A \xrightarrow{f} B$ is an extremal epimorphism, then (f, B) is concretely generating.*

A **universal arrow** over an $\mathbf{X}$-object X is a structured arrow $X \xrightarrow{u} |A|$ with domain X such that, for each structured arrow $X \xrightarrow{f} |B|$ with domain X, there exists a unique A-morphism $\hat{f} : A \to B$ such that the triangle

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[dr, "f"'] & {|A|} \arrow[d, "\overline{f}"] \\ & {|B|} \end{tikzcd}$$

commutes. The pair (u, A) is called a **free object**.

Example 10. 1. *In a construct, an object A is a free object over the empty set if and only if A is an initial object, and over a singleton set if and only if A represents the forgetful functor.*

2. *In the construct Vec each object is a free object over any basis for it.*

3. *In the constructs Top and Pos the free objects are precisely the discrete ones.*

4. *In the construct* $\mathbf{Ab}$ *free objects over* X *are the free abelian groups generated by* X*. Similarly, the familiar free group generated by a set* X *is a free object over* X *in the construct Grp.*

5. *To construct a universal arrow in (Ban, O) over a set* X*, let* $\ell_1(X)$ *be the subspace of the vector space* K^X *consisting of all* $r = (r_x)_{x \in X}$ *in* K^X *whose norm* $\|r\| = \sum_{x \in X} |r_x|$ *is finite. Then* $\ell_1(X)$ *is a Banach space. Define* $X \xrightarrow{u} O(\ell_1(X))$ *at* y *by the Dirac function* $u(y) = (\delta_{yx})_{x \in X}$*. Then* $(u, \ell_1(X))$ *is a universal arrow over* X*. Observe, for comparison, that for the construct (Ban, U) the only set having a universal arrow is the empty set, and that for the construct Ban* $\mathrm{B_b}$ *the only sets having universal arrows are the finite ones.*

Proposition 9. 1. *Every universal arrow is extremally generating.*

2. *Any two universal arrows with domain* X *are isomorphic. Conversely, if* $X \xrightarrow{u} |A|$ *is a universal arrow and* $A \xrightarrow{k} A'$ *is an* $\mathbf{A}$*-isomorphism, then* $X \xrightarrow{kou} |A'|$ *is also universal.*

3. *If a concrete category* $\mathbf{A}$ *over* $\mathbf{X}$ *has free objects, then an* $\mathbf{A}$*-morphism is an* $\mathbf{A}$*-monomorphism if and only if it is an* $\mathbf{X}$*-monomorphism.*

4. *If a construct* $\mathbf{A}$ *has a free object over a singleton set, then the monomorphisms in* $\mathbf{A}$ *are precisely those morphisms that are injective functions.*

A concrete category over $\mathbf{X}$ is said to have free objects provided that for each $\mathbf{X}$-object X there exists a universal arrow over X.
The constructs Vec, Grp, Ab, Mon, Sgr, Alg (Ω), Top, Pos, and (Ban,O) have free objects; but the constructs Ban$_b$.

A functor $F : \mathcal{A} \to$ Set is called representable (by an $\mathcal{A}$-object A) provided that F is naturally isomorphic to the hom-functor $\hom(A, -) : \mathcal{A} \to$ Set. Note that objects that represents the same functor are isomorphic.

Example 11. 1. *Forgetful functors are often representable. For example, (a) Vec* $\to$ *Set is represented by the vector space* $\mathbb{R}$*, (b)* $\mathrm{Grp} \to$ *Set is represented by the group of integers* $\mathbb{Z}$*, (c) Top* $\to$ *Set is represented by any one-point topological space.*

2. *The underlying functor U for the construct Ban [5.2(3)] is not representable (see Exercise 10J). However, the faithful unit ball functor* $O : \mathrm{Ban} \to$ *Set is represented in the complex case by the Banach space* $\mathbb{C}$ *of complex numbers.*

Proposition 10 (Representative of Constructs)**.** *For constructs* $(\mathcal{A}, U)$ *the forgetful functor is represented by an object* A *if and only if* A *is a free object over a singleton set [see Definition 8.22(2)]. This provides many additional examples of representations.*

For small categories $\mathcal{A}$ and $\mathcal{B}$ the **functor category** $[\mathcal{A}, \mathcal{B}]$ has as objects all functors from $\mathcal{A}$ to $\mathcal{B}$, as morphisms from F to G all natural transformations from F to G, as identities the identity natural transformations, and as composition the (horizontal) composition of natural transformations.

Theorem 2 (uniqueness of representations)**.** *For any functor* $F : \mathcal{A} \to$ *Set, any* $\mathcal{A}$*-object* A *and any element* $a \in F(A)$*, there exists a unique natural transformation* $\tau : \hom(A, -) \to F$ *with* $\tau_A(id_A) = a$*.*

Corollary 1 (Yoneda Lemma)**.** *If $F : \mathcal{A} \to Set$ is a functor and A is an $\mathcal{A}$-object, then the following function*

$$Y : [\hom(A, -), F] \to F(A) \text{ defined by } Y(\sigma) = \sigma_A(id_A),$$

is a bijection (where $[\hom(A,-), F]$ is the set of all natural transformations from hom $(A, -)$ to F).

Corollary 2 (Yoneda Embedding)**.** *For any category $\mathcal{A}$, the functor $E : \mathcal{A} \to [\mathcal{A}^{\mathrm{op}} Set]$, defined by*

$$E(A \xrightarrow{f} B) = \hom(-, A) \xrightarrow{\sigma_f} \hom(-, B),$$

where $\sigma_f(g) = f \circ g$, is a full embedding.

Proposition 11. *Consider the representable functor $\mathcal{D}(D, -) : \mathcal{D} \to$ Sets for some object D of $\mathcal{D}$. A useful fact is that*

$$\mathrm{colim}_{\mathcal{D}}\, \mathcal{D}(D, -) \cong \{*\}.$$

1.4 Grupoids

If we want a limited amount of interaction between $\mathcal{C}$ and $\mathcal{D}$, we can form the join of $\mathcal{C}$ and $\mathcal{D}$, denoted by $\mathcal{C} * \mathcal{D}$. The objects of $\mathcal{C} * \mathcal{D}$ are the disjoint union of the objects of $\mathcal{C}$ and the objects of $\mathcal{D}$ and as morphism we have

$$(\mathcal{C} * \mathcal{D})(X, Y) = \begin{cases} \mathcal{C}(X, Y), \text{ if } X \text{ and } Y \text{ are objects of } \mathcal{C} \\ \mathcal{D}(X, Y), \text{ if } X \text{ and } Y \text{ are objects of } \mathcal{D} \\ \{*\}, \text{ if } X \text{ is an object of } \mathcal{C} \text{ and } Y \text{ is an object of } \mathcal{D} \\ \varnothing, \text{ otherwise.} \end{cases}$$

A category is a grupoid if all morphisms are isomorphisms.

Example 12. *1. If G is a group, then we denote by $\mathcal{C}_G$ the category with one object $*$ and $\mathcal{C}_G(*, *) = G$ with group multiplication as composition of maps. Then, $\mathcal{C}_G$ is a groupoid. Hence every group gives rise to a groupoid. Vice versa, a groupoid can be thought of as a group with many objects.*

2. Let X be a topological space. The fundamental groupoid of $X, \Pi(X)$, is the category whose objects are the points of X, and $\Pi(X)(x, y)$ is the set of homotopy classes of paths from x to y :

$$\Pi(X)(x, y) = [[0, 1], 0, 1; X, x, y].$$

The endomorphisms $\Pi(x, x)$ of $x \in X$ constitute the fundamental group of X with respect to the basepoint $x, \pi_1(X, x)$.

3. Another important example of a groupoid is the translation category of a group. If G is a discrete group, then we denote by $\mathcal{E}_G$ the category whose objects are the elements of the group and

$$\mathcal{E}_G(g, h) = \left\{hg^{-1}\right\}, g \xrightarrow{hg^{-1}} h.$$

This category has the important feature that there is precisely one morphism from one object to any other object, so every object has equal rights.

Chapter 2

Homological Algebra

References [5]

2.1 Preadditive categories

A **preadditive category** is a category $\mathcal{A}$, such that for every pair of objects A_1, A_2, there is an abelian group of morphisms from A_1 to A_2 and the composition of morphisms is a bilinear map.
A preadditive category with only one object is nothing but a ring. The endomorphisms of that object are an abelian group, and the composition of morphisms defines the multiplicative structure. Thus, a preadditive category can be thought of as a ring with many objects. A group with many objects in this sense is a groupoid, so one might call a preadditive category a ringoid.
Let $\mathcal{A}$ and $\mathcal{A}'$ be preadditive categories. A functor $F : \mathcal{A} \to \mathcal{A}'$ is additive if for any two objects A_1, A_2 of $\mathcal{A}$, the map $F : \mathcal{A}(A_1, A_2) \to \mathcal{A}'(F(A_1), F(A_2))$ is a group homomorphism.

Assume that a category $\mathcal{C}$ has zero morphisms. Then, the kernel of a morphism $f \in \mathcal{C}(C_1, C_2)$ is the equalizer of the morphisms $f, 0 : C_1 \to C_2$. Dually, the cokernel of a morphism $f \in \mathcal{C}(C_1, C_2)$ is the coequalizer of the morphisms $f, 0 : C_1 \to C_2$.

Proposition 12.
1. *In a preadditive categoty, all equalizers are kernels.*
2. *Initial object exists if and only if zero object exists.*
3. *A finite product exists if and only if the finite coproduct exists, called* ***biproduct****.*

A preadditive category is called **additive** if it has all finite biproducts.

Proposition 13. *A functor between additive categories is additive if and only if it preserves biproducts or just products.*

A preadditive category is an **abelian** category if it satisfies the following:

- There exists a zero object in $\mathcal{A}$.
- The category $\mathcal{A}$ has finite biproducts.

- Every morphism $f \in \mathcal{A}(A, B)$ has a cokernel and a kernel.
- Every monomorphism is a kernel, and every epimorphism is a cokernel.

Theorem 3. *Let $\mathcal{A}$be an abelian category:*

1. *a morphism is an isomorphism if and only if it is both a monomorphism and an epimorphism.*
2. *A morphism is a monomorphism if and only if its kernel is zero.*
3. *Let f be a morphism. Then, we can factor f as $f = i \circ p$, where p is an epimorphism and i is a monomorphism. Here, i is the kernel of the cokernel of f and p is the cokernel of the kernel of f.*
4. *A monomorphism is the kernel of its cokernel, and an epimorphism is the cokernel of its kernel.*

Proposition 14. *Let $\mathcal{D}$ be a small category and let $\mathcal{A}$ be abelian. Then, the functor category $\mathrm{Fun}(\mathcal{D}, \mathcal{A})$ is abelian.*

In homological algebra one constructs homological invariants of algebraic objects by the following process, or some variant of it:

Let R be a ring and T a covariant additive functor from R-modules to abelian groups. Thus the map $\mathrm{Hom}_R(M, N) \to \mathrm{Hom}_{\mathbf{z}}(TM, TN)$ defined by T is a homomorphism of abelian groups for all R-modules M, N. For any R module M, choose a free (or projective) resolution $\varepsilon : F \to M$ and consider the chain complex TF of abelian groups obtained by applying T to F termwise. Now T, being additive, preserves chain homotopies; so we can apply the uniqueness theorem for resolutions (I.7.5) to deduce that the complex TF is independent, up to canonical homotopy equivalence, of the choice of resolution. Passing to homology, we obtain groups $H_n(TF)$ which depend only on T and M (up to canonical isomorphism).

This construction is of no interest, of course, if T is an exact functor; for then the augmented complex

$$\cdots \to TF_1 \to TF_0 \to TM \to 0$$

is acyclic, so that $H_n(TF) = 0$ for $n > 0$ and $H_0(TF) = TM$. Thus we can regard the groups $H_n(TF)$ in the general case as a measure of the failure of T to be exact.

2.2 Spectral Sequences

2.3 Abelian categories

2.4 Derived functors

2.5 Derived categories

Chapter 3

Group (Cohomology) Theory

A **semigroup** is a nonempty set G together with a binary operation on G which is associative. A **monoid** is a semigroup G which contains a (two-sided) identity element. A **group** is a monoid G such that for every element there exists a (two-sided) inverse element.

Theorem 4. *Let G be a finitely generated abelian group.*

1. *There is a unique nonnegative integer s such that the number of infinite cyclic summands in any decomposition of G as a direct sum of cyclic groups is precisely s;*

2. *either G is free abelian or there is a unique list of (not necessarily distinct) positive integers* $\mathrm{m}_1, \ldots, \mathrm{m}_\mathrm{t}$ *such that* $\mathrm{m}_1 > 1,\ \mathrm{m}_1 \mid \mathrm{m}_2 | \cdots \mid \mathrm{m}_\mathrm{t}$ *and*

$$\mathrm{G} \cong \mathbf{Z}_{m_1} \oplus \cdots \oplus \mathbf{Z}_{m_t} \oplus \mathrm{F}$$

   with F free abelian;

3. *either G is free abelian or there is a list of positive integers* ${\mathrm{p}_1}^{s_1}, \ldots, {\mathrm{p}_\mathrm{k}}^{8k}$*, which is unique except for the order of its members, such that* $\mathrm{p}_1, \ldots, \mathrm{p}_\mathrm{k}$ *are (not necessarily distinct) primes,* $\mathrm{s}_1, \ldots, \mathrm{s}_\mathrm{k}$ *are (not necessarily distinct) positive integers and*

$$\mathrm{G} \cong \mathrm{Z}_{\mathrm{p1}}\mathrm{st}^1 \oplus \ldots \oplus \mathrm{Z}_{\mathrm{pk}}\mathrm{sk}_\mathrm{k} \oplus \mathrm{F}$$

   with F free abelian.

3.1 Actions

An action of a group G on a set S is a function $\mathrm{G} \times \mathrm{S} \to \mathrm{S}$ (usually denoted by $(\mathrm{g}, \mathrm{x}) \mapsto \mathrm{gx}$) such that for all $\mathrm{x} \varepsilon \mathrm{S}$ and $\mathrm{g}_1,\ \mathrm{g}_2 \in \mathrm{G}$:

$$\mathrm{ex} = \mathrm{x} \quad \text{and} \quad (\mathrm{g}_1\ \mathrm{g}_2)\,\mathrm{x} = \mathrm{g}_1\,(\ \mathrm{g}_2\mathrm{x})\,.$$

When such an action is given, we say that G acts on the set S. Gx denotes the orbit of x and G_x denotes its stabilizer (or isotropy group).

Theorem 5. 1. *Orbits have cardinality equal to the index of the corresponding stabilizer.*

2. *The number of elements in the conjugacy class of* $\mathrm{x}\varepsilon\mathrm{G}$ *is* $[\mathrm{G} : \mathrm{C_G(x)}]$, *which divides* $|\mathrm{G}|$;

3. ***(Class equation)*** *if* $\overline{\mathrm{x}}_1, \ldots, \overline{\mathrm{x}}_\mathrm{n}\ (\mathrm{x_i} \in \mathrm{G})$ *are the distinct conjugacy classes of G , then*

$$|\mathrm{G}| = \sum_{i=1}^{n} [\mathrm{G} : \mathrm{C_G}\,(\mathrm{x_i})]$$

   In particular, we can take G acting on itself by conjugation, so that the conjugacy classes are the orbits of this action.

4. *the number of subgroups of G conjugate to K is* $[\mathrm{G} : \mathrm{N_G(K)}]$, *which divides* $|\mathrm{G}|$.

Let G and H be groups and $\theta : H \to \mathrm{Aut}\ G$ a homomorphism. Let $G \times_\theta H$ be the set $G \times H$ with the following binary operation: $(g, h)\,(g', h') = (g\,[\theta(h)\,(g')]\,, hh')$. Show that $G \times_\theta H$ is a group with identity element (e, e) and $(g, h)^{-1} = \left(\theta\left(h^{-1}\right)\left(g^{-1}\right), h^{-1}\right).G \times_\theta H$ is called the semidirect product of G and H.

Group Ring Let G be a group, written multiplicatively. Let $\mathbb{Z}G$ be the free $\mathbb{Z}$-module generated by the elements of G. The multiplication in G extends uniquely to a $\mathbb{Z}$-bilinear product $\mathbb{ZG} \times \mathbb{ZG} \to \mathbb{ZG}$; this makes $\mathbb{ZG}$ a ring, called the **integral group ring** of G.

Note that G is a subgroup of the group $(\mathbb{Z}G)^*$ of units of $\mathbb{ZG}$

Theorem 6 (Universal property)**.** *Given a ring R and a group homomorphism $f : G \to R^*$, there is a unique extension of f to a ring homomorphism $\mathbb{ZG} \to R$. Thus we have the "adjunction formula"*

$$\mathrm{Hom}_{\textit{(rings)}}\,(\mathbb{Z}G, R) \approx \mathrm{Hom}_{\textit{(groups)}}\,(G, R^*)\,.$$

A **(left)** $\mathbb{Z}G$**-module**, or G-module, consists of an abelian group A together with a homomorphism from $\mathbb{Z}G$ to the ring of endomorphisms of A. By the universal property, G-module is simply an abelian group A together with an action of G on A. For example, one has for any A the trivial module structure, with $ga = a$ for $g \in G, a \in A$.

One way of constructing G-modules is by linearizing permutation representations. More precisely, if X is a G-set (i.e., a set with G-action), then one forms the free abelian group $\mathbb{ZX}$ (also denoted $\mathbb{Z}[X]$) generated by X and one extends the action of G on X to a $\mathbb{Z}$-linear action of G on $\mathbb{Z}X$. The resulting G-module is called a permutation module. In particular, one has a permutation module $\mathbb{Z}[G/H]$ for every subgroup H of G, where G/H is the set of cosets gH and G acts on G/H by left translation.

Proposition 15. *Let X be a free G-set and let E be a set of representatives for the G-orbits in X. Then $\mathbb{Z}X$ is a free $\mathbb{Z}G$-module with basis E.*

3.2 Co-invariants

If G is a group and M is a G-module, then the group of co-invariants of M, denoted M_G, is defined to be the quotient of M by the additive subgroup generated by the elements of the form $gm - m\,(g \in G, m \in M\ )$. Thus M_G is obtained from M by "dividing out" by the G-action. (The name "co-invariants" comes from the fact that M_G is the largest quotient of M on which G acts trivially, whereas M^G, the group of invariants, is the largest submodule of M on which G acts trivially.) In view of exercise $1\mathrm{a}$ of $\$I.2$, we can also describe M_G as

M/IM, where I is the augmentation ideal of $\mathbb{Z}G$ and IM denotes the set of all finite sums $\sum a_i b_i \, (a_i \in I, b_i \in M)$. Still another description of M_G is given by:

$$M_G \approx \mathbb{Z} \otimes_{\mathbb{Z}G} M.$$

Here, in order for the tensor product to make sense, we regard $\mathbb{Z}$ as a right $\mathbb{Z}G$-module (with trivial G-action). To prove 2.1 , note that in $\mathbb{Z} \otimes_{\mathbb{Z}G} M$ we have the identity $1 \otimes gm = 1 \cdot g \otimes m = 1 \otimes m$; hence there is a map $M_G \to \mathbb{Z} \otimes_{\mathbb{Z}G} M$ given by $\bar{m} \mapsto 1 \otimes m$, where $\bar{m}$ denotes the image in M_G of an element $m \in M$. On the other hand, using the universal property of the tensor product, we can define a map $\mathbb{Z} \otimes_{\mathbb{Z}G} M \to M_G$ by $a \otimes m \mapsto a\bar{m}$. These two maps are inverses of one another.

In view of 2.1 and standard properties of the tensor product, we immediately obtain the following two properties of the co-invariants functor:

3.3 An spectral sequence for group cohomology

Suppose that X is a simplicial set and x_i are simplicial subsets such that $X = UX_i$. Then, setting $X_{ij} = X_i \cap X_j$ (etc.) we'11 obviously have for the realisations: $|x| = U\,|x_i|\,, |x_i| \cap |x_j| = |x_{ij}|\,, \ldots$ Let's suppose that the set of indices is linearly ordered. Consider the following bicomplex:

$$K = \longrightarrow \underset{i<j<k}{\oplus} C_*\left(x_{ijk}\right) \longrightarrow \underset{i<j}{\oplus} C_*\left(x_{ij}\right) \longrightarrow \underset{i}{\oplus} C_*\left(x_i\right)$$

Here by a bicomplex we understand a bicomplex in the sense of Grothendieck [9] i.e. the differentials d_1 and d_2 commute. (The sign in this approach appears in the definition of the total differentials). The vertical arrows of the bicomplex map $C_*\left(x_i \cdots {}_i\right)$ into $\underset{k=0}{q}\, C_*\left(x_{i_0} \ldots \hat{i}_k \ldots i_q\right)$, the mapping into the kth summand differing $k = 0$ by a sign $(-1)^k$ from the natural embedding.

The first spectral sequence of this bicomplex degenerates and yields an isomorphism $H_\star(K) \cong H_\star(X)$. (Moreover this isomorphism is induced by the canonical map $K \to C_*(X)$). The second spectral sequence gives us a functorial spectral sequence of the first quadrant, whose limit equals $H_*(X)$, while its differential dr has bidegree $(r-1, -r)$ and its E^1-term looks as follows:

$$E^1_{pqq} = \underset{i_0 < \ldots < i_q}{\otimes} H_p\left(x_{i_0} \ldots i_q\right)$$

Suppose G is a group. Let X_G denote the simplicial set (and its geometric realisation), whose p-simplices are sequences $(g_0, \ldots, g_p)$ of elements of G, with the usual faces and degeneracies. This space X_G is contractible by (1.2). The group G acts from the right on X_G and this action is obviously free, hence $BG = X_G/G$ is a classifying space of G. The complex $C_*(BG) = C_*(G)$ coincides with the usual complex associated with G. Moreover $C_*(G) = C_*\left(X_G\right) \otimes_G Z$.

If H is a subgroup of G, then X_G/H is a classifying space for H and hence $BH = X_H/H \to X_G/H$ is a homotopy equivalence. In particular $C_*(H) + C_*\left(X_G\right) \otimes_H \mathbb{Z} = C_*\left(X_G\right) \otimes_G Z|G/H|$ is a homotopy equivalence.

(2.3) The spectral sequence associated with a family of subgroups.

Suppose G is a group and $G_1, \ldots, G_n$ are subgroups. Then BG_i may be viewed as a simplicial subset of BG and $BG_i \cap BG_j = B(G_i \cap G_j)$.. Denote UBG_i by X and consider the spectral sequence of the covering $X = UBG_i$. Along with the bicomplex K introduced in (2.1) we also consider the following bicomplex:

$$K' = \underset{i<j<k}{\oplus} C_*(X_G) \otimes_G Z[G/G_{ijk}] \longrightarrow \underset{i<j}{\oplus} C_*(X_G) \otimes_G Z[G/G_{ij}] \longrightarrow \underset{i<j}{\oplus} C_*(X_G) \otimes_G Z[G/G_i]$$

There is a natural mapping of bicomplexes $K + K'$ and because of (2.2) this mapping induces an isomorphism of second spectral sequences so that $H_\star(X) = H_\star(K) = H_*(K')$. The first spectral sequence of K' looks as follows: $E^1_{*,q} = C_*(X_G) \otimes_G H_q(L)$, where L is the following complex of left G-modules:

$$\oplus \mathbb{Z}[G/G_i] + \oplus \mathbb{Z}[G/G_{ij}] + \oplus \mathbb{Z}[G/G_{ijk}] + \ldots$$

Proposition 16. *If $G_1, \ldots, G_n$ are subgroups of G, there exists a fuctorial spectral sequence of the first quadrart, the E^2 term of which looks like: $E^2_{pq} = H_p(G, H_q(L))$, where L is the complex defined above. It converges to $H_\star(UBG_j)$ and the differential d^r has bidegree $(-r, r-1)$.*

(2.5) In the notations of (2.3), let $Z(G, \{G\})$ be the simplicial set whose non-degenerate p-simplices are sequences $(\bar{g}_0, \ldots, \bar{g}_p)$, where $\bar{g}_i \varepsilon G/G_{k_i}, k_0 < \ldots < k_p$, and the $\bar{g}_i$ are such that there is $g \in G$ with $\vec{g}_i = g \bmod G_{k_i}$ for all i. (If one covers G by the right cosets of the G_i, then $Z(G_g\{G_i\})$ is the nerve of this covering.) It is easy to see that the geometric realization of this simplicial set is an ordered simplicial space and that the complex $L = L(G, \{G_i\})$ equals the (ordered) simplicial complex [7] of this simplicial space, or in other words, the complex L equals the normalised complex of the simplicial set $Z(G, \{G_i\})$. In particular, $H_*(L) = H_*(Z(G, \{G_i\}))$.

(2.6) Remark. It may be shown easily that the space $Z(G, \{G_i\})$, is homotopy equivalent to Volodin's space $V(G, \{G_i\})$, but we will not need this fact.

Chapter 4

Rings with identity

Let R be a ring and S a nonempty subset of R that is closed under the operations of addition and multiplication in R . If S is itself a ring under these operations then S is called a subring of R . A subring I of a ring R is a **left ideal** provided

$$\mathrm{r}\varepsilon\mathrm{R} \text{ and } \mathrm{x} \in \mathrm{I} \quad \Rightarrow \quad \mathrm{rx} \in \mathrm{I};$$

I is a **right ideal** provided

$$\mathrm{r}\varepsilon\mathrm{R} \text{ and } \mathrm{x} \in \mathrm{I} \quad \Rightarrow \quad \mathrm{xr}\varepsilon\mathrm{I};$$

I is an **ideal** if it is both a left and right ideal. Note that proper ideals does not contain any unit. We denote by (X) the ideal generated by the subset X of R , i.e., the smallest ideal containing X.

Theorem 7. *1. (a)* $= \{\sum_{i=1}^{n} \mathrm{r_j a s_i} \mid \mathrm{r_i}, \mathrm{s_i} \in \mathrm{R}; n \in \mathbf{N}^*\}$ *(principal ideal).*

2. *If a is in the center of R , then* $\mathrm{Ra} = (\mathrm{a}) = \mathrm{aR}$.

3. *If X is in the center of R , then the ideal* (X) *consists of all finite sums* $\mathrm{r_1 a_1 + \cdots + r_n a_n}$ $(\mathrm{n} \in \mathbf{N}^*; \mathrm{r_i} \in \mathrm{R}; \mathrm{a_i}\varepsilon\mathrm{X})$.

4. *for ideals, multiplication and addition are distributive and associative.*

Theorem 8. *If P is an ideal in a ring R such that* $\mathrm{P} \neq \mathrm{R}$ *and for all* $\mathrm{a}, \mathrm{b}\varepsilon\mathbf{R}$

$$\mathrm{ab}\varepsilon\mathrm{P} \Rightarrow \mathrm{a}\varepsilon\mathrm{P} \text{ or } \mathrm{b}\varepsilon\mathrm{P},$$

then P is prime. Conversely if P is prime and R is commutative, then P satisfies condition (1).

4.1 Linear Algebra

Part II

Topics of Algebraic Topology

Chapter 5

Simpliciality and Classifying Spaces

References [5, 2, 3].

Simplicial sets were originally used to give precise and convenient descriptions of classifying spaces of groups. This idea was vastly extended by Grothendieck's idea of considering classifying spaces of categories, and in particular by Quillen's work of algebraic K-theory. In this work, which earned him a Fields Medal, Quillen developed surprisingly efficient methods for manipulating infinite simplicial sets. These methods were used in other areas on the border between algebraic geometry and topology. For instance, the André-Quillen homology of a ring is a "non-abelian homology", defined and studied in this way.

Both the algebraic K-theory and the André-Quillen homology are defined using algebraic data to write down a simplicial set, and then taking the homotopy groups of this simplicial set.

Simplicial methods are often useful when one wants to prove that a space is a loop space. The basic idea is that if G is a group with classifying space BG, then G is homotopy equivalent to the loop space ΩBG. If BG itself is a group, we can iterate the procedure, and G is homotopy equivalent to the double loop space $\Omega^2 B(BG)$. In case G is an abelian group, we can actually iterate this infinitely many times, and obtain that G is an infinite loop space.

Even if X is not an abelian group, it can happen that it has a composition which is sufficiently commutative so that one can use the above idea to prove that X is an infinite loop space. In this way, one can prove that the algebraic K-theory of a ring, considered as a topological space, is an infinite loop space.

In recent years, simplicial sets have been used in higher category theory and derived algebraic geometry. Quasi-categories can be thought of as categories in which the composition of morphisms is defined only up to homotopy, and information about the composition of higher homotopies is also retained. Quasi-categories are defined as simplicial sets satisfying one additional condition, the weak Kan condition.

As $\mathcal{C}$ is an arbitrary category, we can consider simplicial R-modules, simplicial sets, simplicial rings, simplicial topological spaces, and many more. Simplicial sets are particularly important because they model topological spaces (see 10.6.1). Simplicial objects in an abelian category $\mathcal{A}$ model non-negatively graded chain complexes over $\mathcal{A}$. The famous Dold-Kan correspondence (see Theorem 10.11.2) is an equivalence of categories between $s\mathcal{A}$ and $\mathrm{Ch}_{\geq 0}(\mathcal{A})$.

Theorem 9. *The normalized chain complex is part of an equivalence of categories between the simplicial objects in $\mathcal{A}$ and the non-negatively graded chain complexes over $\mathcal{A}$.*

Simplicial complexes are more intuitive, and are the foundation of algebraic topology. Δ-complexes are usueful for computations. Simplicial sets are more suitable to high level concepts.

5.1 Simplicial objects in a category

We consider the finite set $\{0, 1, \ldots, n\}$ with its natural ordering $0 < 1 < \ldots < n$ and call this ordered set $[n]$ for all $n \geq 0$. The **simplicial category**, Δ, has as objects the ordered sets $[n], n \geq 0$, and the morphisms in Δ are the order-preserving functions, that is, functions $f : [n] \to [m]$, such that $f(i) \leq f(j)$ for all $i < j$.
Let $\mathcal{C}$ be an arbitrary category. A simplicial object in $\mathcal{C}$ is a contravariant functor from Δ to $\mathcal{C}$. A cosimplicial object in $\mathcal{C}$ is a covariant functor from Δ to $\mathcal{C}$.
Simplicial objects in a category $\mathcal{C}$ form a category, where the morphisms are natural transformations of functors. We denote this category by $s\mathcal{C}$, note that $\mathcal{C}$can be embedded in $\mathcal{C}$, considering constant simplicial objects.

How can we describe simplicial objects in an explicit manner? Assume that we have a functor $X : \Delta^{op} \to \mathcal{C}$. Then, for every object $[n] \in \Delta$, we have an object $X([n]) =: X_n$ in $\mathcal{C}$. As all morphisms in Δ can be described as a composite of δ_i s and σ_j s, it suffices to know what the maps $X(\delta_i) =: d_i : X_n \to X_{n-1}$ and $X(\sigma_j) =: s_j$ do. Hence, if you want to describe a simplicial object, then you have to understand the sequence of objects $X_0, X_1, \ldots$ and the morphisms d_i, s_j in $\mathcal{C}$. These maps satisfy the dual relations:

$$\begin{aligned}
&d_i \circ d_j = d_{j-1} \circ d_i, \quad i < j, \\
&s_i \circ s_j = s_{j+1} \circ s_i, \quad i \leq j, \text{ and} \\
&d_i \circ s_j = \begin{cases} s_{j-1} \circ d_i, & i < j, \\ 1_{[n]}, & i = j, j+1, \\ s_j \circ d_{i-1}, & i > j+1. \end{cases}
\end{aligned}$$

Thus a simplicial object can be visualized as a diagram of the form

$$X_0 \overset{X_1}{\overset{\rightleftarrows}{\rightleftarrows}} X_2 \quad \ldots,$$

where the morphisms $\leftarrow$ correspond to the d_i s, whereas the morphisms $\to$ are given by the s_j s. Note that on X_n, you have $n+1$ maps going out to the left and to the right.
The d_i s are called **face** maps and the s_j s are called **degeneracy maps**. For a concrete category $\mathcal{C}$ with a faithful functor $U : \mathcal{C} \to$ Sets the elements $x \in U(X_n)$ are the n-simplices of X. We will omit the functor U from the notation. Elements of the form $x = s_i y \in X_n$ for a

$y \in X_{n-1}$ are called degenerate n-simplices.

Let $\Delta_n : \Delta^{op} \to \text{Sets}$ be the functor given by $[m] \mapsto \Delta([m],[n])$. The Yoneda lemma identifies the set X_n with the set of natural transformations from Δ_n to X for every simplicial set X :

$$X_n \cong s\,\text{Sets}\,(\triangle_n, X)$$

The **category of elements of a simplicial set** $X, \text{el}(X)$, is the category $X \backslash \Delta^\circ$ associated with the functor $X : \Delta^\circ \to \text{Sets}$. Explicitly, the objects of el (X) are the $x \in X_n$ for some n. The morphisms in el (X) from $x \in X_n$ to $y \in X_m$ are all $f \in \Delta([n],[m])$, with $X(f)(y) = x$.

Proposition 17 (consequence of density theorem)**.** *For every simplicial set X there is an isomorphism of simplicial sets*

$$\operatorname{colim}_{\text{el}(X)} \Delta_n \cong X$$

5.2 Geometric realization

The geometric realization of a simplicial set was introduced by Milnor [Mi57]. Definition 10.6.1. Let X be a simplicial set. The geometric realization of $X, |X|$, is the topological space

$$|X| = \bigsqcup_{n \geq 0} X_n \times \triangle^n / \sim .$$

Here, we consider the sets X_n as discrete topological spaces, and $\triangle^n$ denotes the topological n-simplex

$$\triangle^n = \left\{ (t_0, \ldots, t_n) \in \mathbb{R}^{n+1} \mid 0 \leq t_i \leq 1, \sum t_i = 1 \right\}.$$

The spaces $\triangle^n, n \geq 0$ form a cosimplicial topological space with structure maps

$$\delta_i\,(t_0, \ldots, t_n) = (t_0, \ldots, t_{i-1}, 0, t_i, \ldots, t_n) \text{ for } 0 \leq i \leq n$$

and

$$\sigma_j\,(t_0, \ldots, t_n) = (t_0, \ldots, t_j + t_{j+1}, \ldots, t_n) \text{ for } 0 \leq i \leq n.$$

The quotient in the geometric realization is generated by the relations

$$(d_i(x), (t_0, \ldots, t_n)) \sim (x, \delta_i\,(t_0, \ldots, t_n)), \quad (s_j(x), (t_0, \ldots, t_n)) \sim (x, \sigma_j\,(t_0, \ldots, t_n)).$$

REMARK 10.6.2. The geometric realization of a simplicial set X is nothing but the coend of the functor

$$H : \Delta^o \times \Delta \to \text{ Top},$$

with $H([n],[m]) = X_n \times \triangle^m$, using that $[n] \mapsto X_n$ is a contravariant functor from Δ to Sets and that $[m] \mapsto \triangle^m$ is a covariant functor from the category Δ to the category Top. Here, we use the embedding of Sets into Top.

If $f : X \to Y$ is a morphism of simplicial sets, that is, a natural transformation from X to Y, then f induces a continuous map of topological spaces

$$|f| : |X| \to |Y|,$$

where an equivalence class $[(x, t_0, \ldots, t_n)] \in |X|$ is sent to the class $[(f(x), t_0, \ldots, t_n)] \in |Y|$. This turns the geometric realization into a functor from the category of simplicial sets to the category of topological spaces.

Elements of the form $s_j(x)$ are identified with something of a lower degree in the geometric realization, because of the relation

$$(s_j(x), (t_0, \ldots, t_n)) \sim (x, \sigma_j (t_0, \ldots, t_n)) .$$

Hence, these elements do not contribute any geometric information to $|X|$. This might justify the name degenerate for such elements. Note that elements in X_0 are never degenerate.

An element $(y, (t_0, \ldots, t_m)) \in X_m \times \triangle^m$ is called **nondegenerate**, if y is not of the form $s_j(x)$ for any x and j and if $(t_0, \ldots, t_m) \in \Delta^m$ is not a point on the boundary of the topological m-simplex.

Proposition 18. *The geometric realization of a simplicial ser is a CW complex, such that every nondegenerate $n-$simplex corresponds to a $n-$cell.*

Example 13.

1. *The topological 1 -sphere is the quotient space $[0,1]/0 \sim 1$. If we want to find a simplicial model for the 1-sphere, such that the geometric realization has the desired cell structure, then we should define a simplicial set $\mathbb{S}^1$ with one 0 -simplex, 0 , and one nondegenerate 1 -simplex, 1 . The simplicial identities force the existence of a 1-simplex $s_0(0)$, so we get two 1-simplices. For the cell structure we do not need any further maps, so we just take these simplices and all the resulting elements that are given due to the simplicial structure maps. We then get $\mathbb{S}^1_n \cong [n]$ with face and degeneracy maps as follows:*

   $$[0] \overset{\rightleftarrows}{\rightleftarrows} [1] \underset{\rightleftarrows}{\rightleftarrows} [2] \ldots,$$

   The map $s_i : [n] \to [n+1]$ is the unique monotone injection, whose image does not contain $i+1$, while $d_i : [n] \to [n-1]$ is given by $d_i(j) = j$ if $j < i$, $d_i(i) = i$ if $i < n$, and $d_n(n) = 0$ and $d_i(j) = j - 1$ if $j > i$.

   The face maps glue the only nondegenerate 1-simplex 1 to the zero simplex $0 \in [0]$, and we obtain that the geometric realization, $\left|\mathbb{S}^1\right|$, is the topological 1sphere.

2. *The geometric realization of the representable simplicial set Δ_n is $|\Delta_n| = \triangle^n$. This is a general fact about tensor products of functors and representable objects 15.1.5.*

3. *Let X and Y be two simplicial sets. We already saw the product, $X \times Y$, which is the simplicial set with $(X \times Y)_n = X_n \times Y_n$. The simplicial structure maps d_i and s_j are defined coordinatewise. Be careful, an n-simplex $(x, y) \in X_n \times Y_n$ of the form $(s_i x', s_j y')$ for $i \neq j$ might not be degenerate in $X_n \times Y_n$, despite the fact that both coordinates are degenerate.*

Proposition 19.

1. *Assume that X and Y are two simplicial sets, such that $|X| \times |Y|$ is a CW complex, with the CW structure induced by the one on $|X|$ and $|Y|$. Then,*

   $$|X \times Y| \cong |X| \times |Y|.$$

2. *If $f, g : X \to Y$ are maps of simplicial sets that are homotopic, then $|f|$ is homotopic to $|g|$.*

We consider the full subcategory $\Delta_{\leq n}$ of Δ with objects $[0], \ldots, [n]$. The inclusion functor

$$\iota_n : \Delta_{\leq n} \to \Delta$$

allows us to restrict simplicial sets X to $\Delta_{\leq n}$ by considering $X \circ \iota_n : \Delta^o_{\leq n} \to$ Sets. The n**-skeleton** of a simplicial set X, $sk_n X$, is the left Kan extension of $X \circ \iota_n$ along ι_n. It is easy to see that

$$|sk_n X| \cong sk_n |X| =: X^{(n)},$$

where $sk_n|X| = X^{(n)}$ denotes the n-skeleton of the CW complex $|X|$.

Fat realization of a Semi-simplicial set

Sometimes, you might want to use a variant of the geometric realization functor. An obvious reason is, that there are sequences of objects $X_0, X_1, \ldots$ that are only connected via face maps, but there are no degeneracy maps. Such functors are often called semisimplicial objects. In that situation, you cannot perform the geometric realization. The other situation that makes an alternative desirable is the situation, where you want to perform the geometric realization of a simplicial space and this space has bad point set behavior.

Let X be a simplicial set (or space), then the fat realization of X, $\|X\|$, is

$$\|X\| = \bigsqcup_{n \geq 0} X_n \times \triangle^n / \sim,$$

where the quotient in the fat geometric realization is generated by the relations

$$(d_i(x), (t_0, \ldots, t_n)) \sim (x, \delta_i (t_0, \ldots, t_n)).$$

There are several alternative descriptions of $\|X\|$. One is to consider the semisimplicial category, Δ, whose objects are the objects of Δ, but we restrict to injective order-preserving maps. These are dual to the face maps used in the identifications in fat geometric realization. Thus, we can describe $\|X\|$ as the coend of the functor

$$H : \Delta^o \times \Delta \to \text{ Top},$$

with $H([p], [q]) = X_p \times \triangle^q$. There is yet another description of the fat realization of a simplicial set or simplicial topological space (see, for instance, [We05, Proof of Proposition 1.3] or [Se74, p. 308]) as the ordinary geometric realization of a "fattened up" simplicial set. Of course, $\|X\|$ also makes sense, if you start with a semisimplicial object, that is, a functor $X : \Delta^o \to$ Sets.

As we do not collapse degenerate simplices, the fat realization of a simplicial set is larger than the geometric realization.

Proposition 20.
1. *If all the X_n are spaces of the homotopy type of a CW complex, then so is $\|X\|$.*
2. *If $f : X \to Y$ is a morphism of simplicial topological spaces, such that all $f_n : X_n \to Y_n$ are homotopy equivalences, then $\|f\|$ is a homotopy equivalence.*
3. *Fat realization commutes with finite products.*

5.3 Classifying spaces

To any small category, you can associate a topological space that takes the data of the category (objects, morphisms, and composition of morphisms) and translates it into a CW complex. This is done in a two-stage process: First you construct a simplicial set out of your category, and then, you form its geometric realization.

1. For a small category $\mathcal{C}$, let $M_n(\mathcal{C})$ be the set

$$\left\{C_0 \xrightarrow{f_1} C_1 \xrightarrow{f_2} \dots \xrightarrow{f_n} C_n \mid C_i \text{ object of } \mathcal{C}, f_i \text{ morphism in } \mathcal{C}\right\}$$

   of the n-tuples of composable morphisms in $\mathcal{C}$. We denote an element, as earlier, as $[f_n|\dots|f_1]$.

2. The **nerve** of the category $\mathcal{C}$ is the simplicial set $N\mathcal{C} : \Delta^{op} \to$ Sets, which sends $[n]$ to the set $M_n(\mathcal{C})$. The degeneracies insert identity morphisms

$$s_i\,[f_n|\dots|f_1] = [f_n|\dots|f_{i+1}\,|1_{C_i}|\,f_i|\dots|f_1]\,,\quad 0 \le i \le n,$$

   and the face maps compose morphisms:

$$d_i\,[f_n|\dots|f_1] = \begin{cases} [f_n|\dots|f_2]\,, & i = 0, \\ [f_n|\dots|f_{i+1} \circ f_i|\dots|f_1]\,, & 0 < i < n, \\ [f_{n-1}|\dots|f_1]\,, & i = n. \end{cases}$$

3. The **classifying space** of the category $\mathcal{C}$ is the geometric realization of the nerve of $\mathcal{C}$: $B\mathcal{C} = |N\mathcal{C}|$.

You can also interpret the i th face map as omitting the object C_i. If i is 0 or n, then the morphism next to C_i just dies with the object, whereas for $0 < i < n$, deleting the object causes the composition of the adjacent morphisms.

The objects C of $\mathcal{C}$ give zero cells in $B\mathcal{C}$, and a nonidentity morphism from C to C'' gives rise to an edge whose endpoints correspond to the objects C and C'. If $g \circ f$ is a composition of morphisms in $\mathcal{C}$, then in the classifying space, you will find a triangle, with edges corresponding to f, g, and $g \circ f$. Threefold compositions give rise to tetrahedra and so on.
The topological space $B\mathcal{C}$ is always a CW complex, and a functor $F : \mathcal{C} \to \mathcal{D}$ induces a continuous and cellular map of topological spaces: $BF : B\mathcal{C} \to B\mathcal{D}$. .

Hence, B is a functor from the category cat to the category Top of topological spaces.

Example 14. *1. If X is a set and $\mathcal{C}$ is the corresponding discrete category, then the classifying space BC is X with the discrete topology.*

2. If G is a group and we consider the small category $\mathcal{C}_G$ associated with G, then the classifying space $B\left(\mathcal{C}_G\right)$ is called the classifying space of the group G and is denoted by BG. If the group G is abelian, then the group composition is a group homomorphism, and it induces a functor $\mathcal{C}_G \times \mathcal{C}_G \to \mathcal{C}_G$. We therefore obtain a map

$$BG \times BG \to B\left(\mathcal{C}_G \times \mathcal{C}_G\right) \to B\left(\mathcal{C}_G\right) = BG,$$

and for abelian groups G, BG is a topological group. For instance, $B\mathbb{Z} \simeq \mathbb{S}^1$. If G is a topological group, then we can implement the topology into the construction of BG by endowing $G^n \times \triangle^n$ with the product topology.

For instance, $B\mathbb{S}^1 \simeq \mathbb{C}P^\infty$, and this is an Eilenberg-Mac Lane space of type $(\mathbb{Z}, 2)$, $K(\mathbb{Z}, 2) = \mathbb{C}P^\infty \simeq B(B\mathbb{Z})$. In general, if A is a finitely generated abelian group, then the n-fold iterated classifying space construction is a model of the Eilenberg-Mac Lane space $K(A, n)$. why?

If G is a discrete group, then the homology of the group G (with coefficients in $\mathbb{Z}$) is the singular homology $H_*(BG; \mathbb{Z})$.

3. *For a monoid, one can build $B(\mathcal{C}_M)$. We will learn more about this space later (see, for instance, Theorem 13.4.6 and Proposition 13.4.4).*

4. *Let us consider the category Σ. This has as objects the natural numbers (including zero), and the only morphisms are automorphisms with $\Sigma([n], [n]) = \Sigma_n$. Therefore, the classifying space of Σ has one component for every natural number, because the different objects are not connected by morphisms. Thus,*
$$B\Sigma = \bigsqcup_{n\geq 0} B\Sigma_n.$$

5. *If we consider the poset $[n]$ as a category, then the nerve of $[n]$ is isomorphic to the representable functor Δ_n and $B[n] \cong \triangle^n$.*

Theorem 10. 1. *For two functors $F, F' : \mathcal{C} \to \mathcal{D}$, a natural transformation $\tau : F \Rightarrow F'$ induces a homotopy between BF and BF'.*

2. *If $\mathcal{C} \underset{R}{\overset{L}{\rightleftarrows}} \mathcal{D}$ is an adjoint pair of functors, then $B\mathcal{C}$ is homotopy equivalent to $B\mathcal{D}$.*

3. *In particular, an equivalence of categories gives rise to a homotopy equivalence of classifying spaces.*

4. *If a small category $\mathcal{C}$has an initial or terminal object, then its classifying space is contractible.*

5. *Let G be a discrete group. We saw that in its translation category, $\mathcal{E}_G$, every object is terminal and initial, and thus, $B\mathcal{E}_G$ is contractible. In fact, $B\mathcal{E}_G$ is a model for the universal space for G-bundles, EG, and this is, in general, not homeomorphic to a point. For instance. $E\mathbb{Z}/2\mathbb{Z}$ is $\mathbb{S}^\infty = \operatorname{colim}_n \mathbb{S}^n$. The simplicial structure on $N\mathcal{E}_G$ is as follows: An element in $(N\mathcal{E}_G)$ is a string*
$$g_0 \xrightarrow{g_1 g_0^{-1}} g_1 \xrightarrow{g_2 g_1^{-1}} \ldots \xrightarrow{g_q g_{q-1}^{-1}} g_q,$$
but this can be simplified by just remembering the $(q+1)$-tuple of group elements $(g_0, \ldots, g_q)$. With this identification, the face maps omit a g_i, and the degeneracies double a g_i.

Chapter 6

Simplicial Complexes

6.1 (Abstract) simplical complexes

A set (of **vertices**) together with a family of finite subsets (**simplexes**) such that every subset of every simplex is a simplex and every subset consisting of a single vertex is a simplex.

Example 15. *1. The **standard n-simplex** Δ^n is the set of all $(n+1)$-tuples $(t_0, \ldots, t_n)$ of non-negative real numbers such that $t_0 + \cdots + t_n = 1$. The standard 0-simplex is a point, the standard 1-simplex is a line segment, the standard 2-simplex is a triangle, and so on.*

2. *The **boundary** of the standard n-simplex Δ^n is the set of all $(n+1)$-tuples $(t_0, \ldots, t_n)$ of non-negative real numbers such that $t_0 + \cdots + t_n = 1$ and at least one of the t_i is zero. The boundary of the standard 0-simplex is empty, the boundary of the standard 1-simplex is the set of its two endpoints, the boundary of the standard 2-simplex is the set of its three edges, and so on.*

3. *(**Concrete simplicial complexes**) It is subset of $\mathbb{R}^n$ that is a union of standard simplices, that satisfies the previous conditions.*

4. *If Y is a subset of the vertex set of a simplicial scheme S, then we can introduce on it the induced simplicial scheme structure $Y \cap S$, by defining the simplexes as the subsets of Y that are simplexes of S.*

5. *Let X be a set and let $\{p(y) : y \in Y\}$ be a covering of X. Then we can consider two simplicial complexes.*

   (a) *The nerve $\mathrm{Nerv}(p)$ of the covering is the simplicial scheme with the vertex set Y, and a subset Z of Y is counted as a simplex if the intersection $\bigcap_Z p(y)$ is non-empty.*

   (b) *The simplicial complex $V(p)$ is the simplicial scheme with the vertex set X, and a subset Z of X is counted as a simplex if Z is contained in some $p(y)$.*

Geometric realization

The construction goes as follows. First, define $|K|$ as a subset of $[0,1]^S$ consisting of functions $t : S \to [0,1]$ satisfying the two conditions: $\square$

$$\{s \in S : t_s > 0\} \in K$$
$$\sum_{s \in S} t_s = 1$$

Now think of the set of elements of $[0,1]^S$ with finite support as the direct limit of $[0,1]^A$ where A ranges over finite subsets of S, and give that direct limit the induced topology. Now give $|K|$ the subspace topology. *It is always Hausdorff.* We will identify an abstract simplicial complex with its geometric realization.

6.2 CW-complexes

They can be defined in an inductive way:

1. Start with a discrete set X^0, whose points are regarded as 0 -cells.
2. Inductively, form the $\boldsymbol{n}$-skeleton X^n from X^{n-1} by attaching n-cells e^n_α via maps $\varphi_\alpha : S^{n-1} \to X^{n-1}$. This means that X^n is the quotient space of the disjoint union $X^{n-1} \amalg_\alpha D^n_\alpha$ of X^{n-1} with a collection of n-disks D^n_α under the identifications $x \sim \varphi_\alpha(x)$ for $x \in \partial D^n_\alpha$. Thus as a set, $X^n = X^{n-1} \amalg_\alpha e^n_\alpha$ where each e^n_α is an open n-disk.
3. One can either stop this inductive process at a finite stage, setting $X = X^n$ for some $n < \infty$, or one can continue indefinitely, setting $X = \cup_n X^n$. In the latter case X is given the weak topology: A set $A \subset X$ is open (or closed) iff $A \cap X^n$ is open (or closed) in X^n for each n.

Example 16. *1. A 1-dimensional cell complex $X = X^1$ is what is called a graph in algebraic topology. It consists of vertices (the 0 -cells) to which edges (the 1-cells) are attached. The two ends of an edge can be attached to the same vertex.*

2. The sphere S^n has the structure of a cell complex with just two cells, e^0 and e^n, the n-cell being attached by the constant map $S^{n-1} \to e^0$. This is equivalent to regarding S^n as the quotient space $D^n/\partial D^n$.

*3. **Real projective $\boldsymbol{n}$-space** $\mathbb{R}\mathrm{P}^n$. It is equivalent to the quotient space of a hemisphere D^n with antipodal points of ∂D^n identified. Since ∂D^n with antipodal points identified is just $\mathbb{R}\mathbb{P}\mathrm{P}^{n-1}$, we see that $\mathbb{R}\mathrm{P}^n$ is obtained from $\mathbb{R}\mathrm{P}^{n-1}$ by attaching an n-cell, with the quotient projection $S^{n-1} \to \mathbb{R}P^{n-1}$ as the attaching map. It follows by induction on n that $\mathbb{R}\mathbb{P}^n$ has a cell complex structure $e^0 \cup e^1 \cup \cdots \cup e^n$ with one cell e^i in each dimension $i \leq n$.*
The infinite union $\mathbb{R}P^\infty = U_n \mathbb{R}P^n$ becomes a cell complex with one cell in each dimension. We can view $\mathbb{R}P^\infty$ as the space of lines through the origin in $\mathbb{R}^\infty = \bigcup_n \mathbb{R}^n$.

*4. **Complex projective space** $\mathbb{C}P^n$. It is equivalent to the quotient of the unit sphere $S^{2n+1} \subset \mathbb{C}^{n+1}$ with $v \sim \lambda v$ for $|\lambda| = 1$.*
It is also possible to obtain $\mathbb{C}\mathbb{P}^n$ as a quotient space of the disk D^{2n} under the identifications

$v \sim \lambda v$ for $v \in \partial D^{2n}$, in the following way. The vectors in $S^{2n+1} \subset \mathbb{C}^{n+1}$ with last coordinate real and nonnegative are precisely the vectors of the form $\left(w, \sqrt{1-|w|^2}\right) \in \mathbb{C}^n \times \mathbb{C}$ with $|w| \leq 1$. Such vectors form the graph of the function $w \mapsto \sqrt{1-|w|^2}$. This is a disk D^{2n}_+ bounded by the sphere $S^{2n-1} \subset S^{2n+1}$ consisting of vectors $(w, 0) \in \mathbb{C}^n \times \mathbb{C}$ with $|w| = 1$. Each vector in S^{2n+1} is equivalent under the identifications $v \sim \lambda v$ to a vector in D^{2n}_+, and the latter vector is unique if its last coordinate is nonzero. If the last coordinate is zero, we have just the identifications $v \sim \lambda v$ for $v \in S^{2n-1}$.

It follows that $\mathbb{P}^n$ is obtained from $\mathbb{C}\mathrm{P}^{n-1}$ by attaching a cell e^{2n} via the quotient map $S^{2n-1} \to \mathbb{CP}^{n-1}$. So by induction on n we obtain a cell structure $\mathbb{CP}^n = e^0 \cup e^2 \cup \cdots \cup e^{2n}$ with cells only in even dimensions. Similarly, $\mathbb{CP}^\infty$ has a cell structure with one cell in each even dimension.

Each cell e^n_α in a cell complex X has a **characteristic map** $\Phi_\alpha : D^n_\alpha \to X$ which extends the attaching map φ_α and is a homeomorphism from the interior of D^n_α onto e^n_α. Namely, we can take Φ_α to be the composition $D^n_\alpha \hookrightarrow X^{n-1} \coprod_\alpha D^n_\alpha \to X^n \hookrightarrow X$ where the middle map is the quotient map defining X^n.

Chapter 7

Geometric Group Theory

By a G**-complex** we will mean a CW-complex X together with an action of G on X which permutes the cells. Thus we have for each $g \in G$ a homeomorphism $x \mapsto gx$ of X such that the image go of any cell σ of X is again a cell. For example, if X is a simplicial complex on which G acts simplicially, then X is a G-complex.

If X is a G-complex then the action of G on X induces an action of G on the cellular chain complex $C_*(X)$, which thereby becomes a chain complex of G-modules. Moreover, the canonical augmentation $\varepsilon : C_0(X) \to \mathbb{Z}$ (defined by $\varepsilon(v) = 1$ for every 0 -cell v of X) is a map of G-modules.

We will say that X is a free G-complex if the action of G freely permutes the cells of X (i.e., $g\sigma \neq \sigma$ for all σ if $g \neq 1$). In this case each chain module $C_n(X)$ has a $\mathbb{Z}$-basis which is freely permuted by G, hence by $3.1 C_n(X)$ is a free $\mathbb{Z}G$-module with one basis element for every G-orbit of cells. (Note that to obtain a specific basis we must choose a representative cell from each orbit and we must choose an orientation of each such representative.)

Finally, if X is contractible, then $H_*(X) \approx H_*$ (pt.); in other words, the sequence

$$\cdots \to C_n(X) \xrightarrow{\partial} C_{n-1}(X) \to \cdots \to C_0(X) \xrightarrow{\varepsilon} \mathbb{Z} \to 0$$

is exact. We have, therefore:

Proposition 21. *Let X be a contractible free G-complex. Then the augmented cellular chain complex of X is a free resolution of $\mathbb{Z}$ over $\mathbb{Z}G$.*

7.1 Classifying space of groups

Suppose that $\mathcal{C}$ is a (small) category. The classifying space (or nerve) $B\mathcal{C}$ of $\mathcal{C}$ is the simplicial set with

$$B\mathcal{C}_n = \hom_{\text{cat}}(\mathbf{n}, \mathcal{C}),$$

n-simplex is a string

$$a_0 \xrightarrow{\alpha_1} a_1 \xrightarrow{\alpha_2} \dots \xrightarrow{\alpha_n} a_n$$

of composeable arrows of length n in $\mathcal{C}$.

If G is a group, then G can be identified with a category (or groupoid) with one object $*$ and one morphism $g : * \to *$ for each element g of G, and so the classifying space BG of G is defined. Moreover $|BG|$ is an Eilenberg-Mac Lane space of the form $K(G, 1)$, as the notation suggests; this is now the standard construction.

Recall that we constructed BG as the geometric realisation of the nerve of a category * // G. As the notation suggests, this can be interpreted as a quotient, or more precisely a homotopy quotient. One can construct the homotopy quotient $X//G$ of any space X with G-action by a group G, and here we just take $X = *$. By abuse of notation * // $G = |N(*//G)|$[2] A reference for its construction and properties is [Rie14], but we will only need the following facts:

1. Homotopy quotients are natural. If $X \to Y$ is an equivariant map between G-spaces then there is an induced map $X//G \to Y//G$.
2. Homotopy quotients preserve homological connectivity. If $X \to Y$ is an equivariant map between G-spaces which is homologically d-connected then $X//G \to Y//G$ is also homologically d-connected. (Recall that a map is homologically d-connected if it is an isomorphism on H_i for $i < d$ and surjection on H_d.)
3. Homotopy quotients commute with geometric realisation. If $X_\bullet$ is a semi-simplicial G-space, then $\|X_\bullet\| //G \simeq \|X_\bullet//G\|$. (We will explain the terminology and notation later.)
4. Homotopy quotients of transitive G-sets. If S is a transitive G-set, then $S//G \simeq B\,\mathrm{Stab}_G(s)$ for any $s \in S$.

Chapter 8

Homotopy theory

Let I^n be the n-dimensional unit cube, the product of n copies of the interval $[0,1]$. The boundary ∂I^n of I^n is the subspace consisting of points with at least one coordinate equal to 0 or 1 . For a space X with basepoint $x_0 \in X$, define $\pi_n(X, x_0)$ to be the set of homotopy classes of maps $f : (I^n, \partial I^n) \to (X, x_0)$, where homotopies f_t are required to satisfy $f_t(\partial I^n) = x_0$ for all t. The definition extends to the case $n = 0$ by taking I^0 to be a point and ∂I^0 to be empty, so $\pi_0(X, x_0)$ is just the set of path-components of X.

When $n \geq 2$, a sum operation in $\pi_n(X, x_0)$, generalizing the composition operation in π_1, is defined by

$$(f+g)(s_1, s_2, \cdots, s_n) = \begin{cases} f(2s_1, s_2, \cdots, s_n), & s_1 \in [0, 1/2] \\ g(2s_1 - 1, s_2, \cdots, s_n), & s_1 \in [1/2, 1] \end{cases}$$

It is evident that this sum is well-defined on homotopy classes. Since only the first coordinate is involved in the sum operation, the same arguments as for π_1 show that $\pi_n(X, x_0)$ is a group, with identity element the constant map sending I^n to x_0 and with inverses given by $-f(s_1, s_2, \cdots, s_n) = f(1 - s_1, s_2, \cdots, s_n)$.

Proposition 22. *If $n \geq 2$, then $\pi_n(X, x_0)$ is abelian.*

8.1 Covering spaces

In view of 4.1, it is natural now to consider CW-complexes Y satisfying the following three conditions:

1. Y is connected.
2. $\pi_1(Y)$ is isomorphic to G.
3. The universal covering space X of Y is contractible.

8.2 Eilenberg-Mac Lane spaces

The nth singular cohomology group of a space X with coefficients in an abelian group A is isomorphic to the homotopy classes of maps from X to an Eilenberg-Mac Lane space, denoted by $K(A,n)$ of the homotopy type of a CW space, such that

$$\pi_i K(A,n) = \begin{cases} A, & \text{if } i = n \\ 0, & \text{otherwise} \end{cases}$$

The $K(A,n)$ are infinite loop spaces, and hence, the set of homotopy classes of maps $[X, K(A,n)]$ is actually an abelian group for all $n \geq 0$ and

$$H^n(X;A) \cong [X, K(A,n)]$$

is an isomorphism of abelian groups that is natural in the space X. Thus the functor $X \mapsto H^n(X;A)$ is representable. A cohomology operation $\varphi_{(A,n),(B,m)} : H^n(X;A) \to H^m(X;B)$, which is natural in X, can hence be identified with a natural transformation between the functors $X \mapsto [X, K(A,n)]$ and $X \mapsto [X, K(B,m)]$, and these in turn are in bijection with $[K(A,n), K(B,m)] \cong H^m(K(A,n);B)$. Here, we actually get an isomorphism of abelian groups!! As $K(A,n)$ doesn't have nontrivial cohomology groups below degree n (due to the Hurewicz theorem), these operations are trivial for $m < n$. For $A = B = \mathbb{F}_p$, a prime field, the collection of all such cohomology operations constitutes the Steenrod algebra.

More generally, Brown's representability theorem states that every generalized cohomology theory can be represented by an Omega spectrum ([Ad74], [Sw75, chapter 9]).

Part III

K-theory

The subject can be said to begin with Alexander Grothendieck (1957), who used it to formulate his Grothendieck-Riemann-Roch theorem. It takes its name from the German Klasse, meaning "class". [4] Grothendieck needed to work with coherent sheaves on an algebraic variety X. Rather than working directly with the sheaves, he defined a group using isomorphism classes of sheaves as generators of the group, subject to a relation that identifies any extension of two sheaves with their sum. The resulting group is called $K(X)$ when only locally free sheaves are used, or $G(X)$ when all are coherent sheaves. Either of these two constructions is referred to as the Grothendieck group; $K(X)$ has cohomological behavior and $G(X)$ has homological behavior.

If X is a smooth variety, the two groups are the same. If it is a smooth affine variety, then all extensions of locally free sheaves split, so the group has an alternative definition.

In topology, by applying the same construction to vector bundles, Michael Atiyah and Friedrich Hirzebruch defined $K(X)$ for a topological space X in 1959, and using the Bott periodicity theorem they made it the basis of an extraordinary cohomology theory. It played a major role in the second proof of the Atiyah-Singer index theorem (circa 1962). Furthermore, this approach led to a noncommutative K-theory for C^*-algebras.

Already in 1955, Jean-Pierre Serre had used the analogy of vector bundles with projective modules to formulate Serre's conjecture, which states that every finitely generated projective module over a polynomial ring is free; this assertion is correct, but was not settled until 20 years later. (Swan's theorem is another aspect of this analogy.)

The other historical origin of algebraic K-theory was the work of J. H. C. Whitehead and others on what later became known as Whitehead torsion.

There followed a period in which there were various partial definitions of higher K-theory functors. Finally, two useful and equivalent definitions were given by Daniel Quillen using homotopy theory in 1969 and 1972. A variant was also given by Friedhelm Waldhausen in order to study the algebraic K-theory of spaces, which is related to the study of pseudo-isotopies. Much modern research on higher K-theory is related to algebraic geometry and the study of motivic cohomology.

The corresponding constructions involving an auxiliary quadratic form received the general name L-theory. It is a major tool of surgery theory.

In string theory, the K-theory classification of Ramond-Ramond field strengths and the charges of stable Dbranes was first proposed in 1997.[5]

Chapter 9

Milnor's K-theory

References [6]

Let R be an associative ring (with 1), and let $\mathcal{P}(R)$ denote the category of finitely generated projective R-modules. We define the Grothendieck group $K_0(R)$ to be the quotient

$$K_0(R) = \mathcal{F}/\mathcal{R},$$

$\mathcal{F}$ = free Abelian group on the isomorphism classes of projective modules in $\mathcal{P}(R)$, $\mathcal{R}$ = subgroup generated by elements

$$[P \oplus Q] - [P] - [Q], \text{ for all } P, Q \in \mathcal{P}(R).$$

Thus, for any $P, Q \in \mathcal{P}(R), [P] = [Q]$ in $K_0(R) \iff P \oplus P' \cong Q \oplus P'$ for some $P' \in \mathcal{P}(R) \iff P \oplus R^n \cong Q \oplus R^n$ for some $n \geq 0$ Further, we can find $Q' \in \mathcal{P}(R)$ such that $P' \oplus Q' \cong R^n$ for some n, since P' is a quotient of some R^n (P' is finitely generated) and P' is projective. Hence $P \oplus P' \cong Q \oplus P' \Longrightarrow P \oplus R^n \cong Q \oplus R^n$.
If $f : R \to S$ is a homomorphism of rings, f induces a functor $\mathcal{P}(R) \to \mathcal{P}(S)$ given by $P \longmapsto S \otimes_R P$. This preserves direct sums, and hence induces a homomorphism $f_* : K_0(R) \to K_0(S)$.

For $n \geq 3$ the **Steinberg group** $\mathrm{St}_n(R)$ of a ring R is the group defined by generators $x_{ij}(r)$, with i, j a pair of distinct integers between 1 and n and $r \in R$, subject to the following "Steinberg relations":

$$x_{ij}(r)x_{ij}(s) = x_{ij}(r+s),$$
$$[x_{ij}(r), x_{k\ell}(s)] = \begin{cases} 1 & \text{if } j \neq k \text{ and } i \neq \ell, \\ x_{i\ell}(rs) & \text{if } j = k \text{ and } i \neq \ell, \\ x_{kj}(-sr) & \text{if } j \neq k \text{ and } i = \ell. \end{cases}$$

As observed in (1.3.1), the Steinberg relations are also satisfied by the elementary matrices $e_{ij}(r)$ which generate the subgroup $E_n(R)$ of $GL_n(R)$. Hence there is a canonical group surjection $\phi_n : St_n(R) \to E_n(R)$ sending $x_{ij}(r)$ to $e_{ij}(r)$.

The Steinberg relations for $n+1$ include the Steinberg relations for n, so there is an obvious map $St_n(R) \to St_{n+1}(R)$. We write $St(R)$ for $\lim_{\longrightarrow} St_n(R)$ and observe that by stabilizing, the ϕ_n induce a surjection $\phi : St(R) \to E(R)$.

The group $K_2(R)$ is the kernel of $\phi : St(R) \to E(R)$. Thus there is an exact sequence of groups

$$1 \to K_2(R) \to St(R) \xrightarrow{\phi} GL(R) \to K_1(R) \to 1.$$

It will follow from Theorem 5.2.1 below that $K_2(R)$ is an abelian group. Moreover, it is clear that St and K_2 are both covariant functors from rings to groups, just as GL and K_1 are.

Theorem 11. *$K_2(R)$ is an abelian group. In fact it is precisely the center of $\mathrm{St}(R)$.*

We'll define right actions of the symmetric group S_n on ${}_{GL}(R)$ and on $St_n(R)$ by setting

$$(\alpha^s)_{k,\ell} = \alpha_s(k), s(\ell); \quad x_{k\ell}(a)^s = x_s^{-1}(k), s^{-1}(\ell)(a).$$

These actions are compatible with the projections $St_n(R) \to E_n(R)$ and with the homomorphisms $St_n(R) + St_{n+1}(R)$ and $GL_n(R) + GL_{n+1}(R)$. In particular, they induce an action on $\overline{St}_n(R)$.

Lemma 1. *For any $s \in S_{n+1}$ the embeddings u_n and u_n^s are homotopic.*

Chapter 10

Grothendieck’s K-theory

Chapter 11

Whitehead's K-theory

Chapter 12

Quillen's K-theory

Chapter 13

Volodin's K-theory

Let G be a group and $\{G_i\}_{i\varepsilon I}$ a family of subgroups. Define $V(G, \{G_i\})$, or just $V(G)$ to be the simplicial complex, whose vertices are the elements of G, where $g_0, \ldots, g_p\ (g_i \neq g_j)$ form a p-simplex if for some G_i all the elements $g_j g_k^{-1}$ lie in G_i. If H is another group with a family of subgroups $\{H_j\}$ and $\phi : G \to H$ is a homomorphism sending each G_i into some H_j, then ϕ induces a simplicial map $V(\phi) : V(G) \to V(H)$.

In many situations it is more convenient to use simplicial sets instead of simplicial complexes: Denote by $W(G, \{G_i\})$ the geometric realization of the simplicial set whose p-simplices are the sequences $(g_0, \ldots, g_p)$ of elements of G (not necessarily distinct) such that for some G_i al1 $g_j g_k^{-1}$ lie in G_i, the r-th face (resp. degeneracy) of this simplex being obtained by omitting g_r (resp., repeating g_r). Associating with any p-simplex $(g_0, \ldots, g_p)$ the linear singular simplex of the space $V(G)$ which sends the i-th vertex of the standard simplex to g_j, we obtain a map of simplicial sets from $W(G)$ to the simplicial set of singular simplices of $V(G)$ and hence a cellular map (linear on any simplex) from $W(G)$ to $V(G)$. This map is a homotopy equivalence ....

Suppose that R is a ring, n a natural number and σ a partial ordering of $\{1, \ldots, n\}$. Define $T_n^\sigma(R)$ to be the subgroup of $GL_n(R)$ consisting of the α with $\alpha_{ij} = 1$ and $\alpha_{ij} = 0$ if $i \& j$. Subgroups of this form will be called triangular subgroups of $GL_n(R)$. The space $V(GL_n(R), \{T_n^\sigma(R)\})$ will be denoted by $V_n(R)$. Since any partial ordering may be extended to a linear ordering, it suffices to consider linear orderings when defining $V_n(R)$. The natural embedding $GL_n \hookrightarrow GL_{n+1}(R)$ defines an embedding $V_n(R) \longleftrightarrow V_{n+1}(R)$ and we'l1 define $V_\infty(R)$ as $\lim_{\substack{\rightarrow}} {}_n V_n(R)$.
Finally for $i \geq 1$, put

$$k_{i,n}(R) = \pi_{i-1}(V_n(R))$$

and $k_i(R) = k_{i,\infty}(R) = \lim_\rightarrow k_{i,n}(R)$ (compare [26], [27]). Evidently $K_{1,n}(R) = GL_n(R)/E_n(R)$ and $K_{i,n}(R)$ is a group if $i \geq 2$, and this group is abelian if $i \geq 3$. Moreover the $K_i(R)$ are abelian groups for all $i \geq 1$ (see [26], [27]). The connected component of $V_n(R)$ passing through T_n equals $V(E_n(R), \{T_n^\sigma(R)\})$. It is easy to show that the universal covering space of $V_n(E_n(R), \{T_n^\sigma(R)\})$ equals $V(St(R), \{T_n^\sigma(R)\})$, where T_n^σ is identified with the subgroup of $St_n(R)$ generated by the $x_{ij}(a)$ with a $\varepsilon R, i \overset{\sigma}{<} j (n \geq 3)$. Hence

Lemma 2. *$K_{2,n}(R) = \ker(St_n(R) + E_n(R))$, and $K_{i,n}(R) = \pi_{i-1}(V(St_n(R))) = \pi_{i-1}(W(St_n(R)))$ if $\quad i \geq 3 \quad (n \geq 3)$.*

Let's define $\overline{St}_n(R)$ to be the inverse image of $GL_n(R)$ under the projection $St(R) \to E(R)$. There is a canonical homomorphism $St_n(R) \to \overline{st}_n(R)$ and stability for K_1, k_2 ([10], [20], [22]) shows that this homomorphism is surjective if $n \geq s.r.R + 1$ and bijective if $n \geq s.r.R + 2$. The spaces $W\left(St_n(R)\right)$ and $W\left(\overline{St}_n(R)\right)$ will play an essential role in the sequel. We'll denote them by $W_n(R), \bar{W}_n(R)$, resp. (So $W_n(R) = \bar{W}_n(R)$ if $n \geq$ s.r. $R+2$.)

Lemma 3. *Denote the canonical embedding $\bar{W}_n(R) \longleftrightarrow \bar{W}_{n+1}(R)$ by u_n. If $n \geq s \cdot r.R$ and $x \in \overline{St}_{n+1}(R)$, then u_n and $u_n \cdot x$ are homotopic. (Here $\left(u_n \cdot x\right)(g) = \left(u_n(g)\right) \cdot x \cdot$))*

Lemma 4. *For any $s \in S_{n+1}$ the embeddings u_n and u_n^s are homotopic.*

For any simplicial set X we'll denote by $C_*(X)$ its chain complex, i.e., the complex of abelian groups with $C_p(x)$ equal to the free abelian group generated by the p-simplices of X and each differential equal to an alternating sum of homomorphisms induced by taking faces. It is well known that $C_*(X)$ is homotopy equivalent to the singular complex of the geometric realization of X. In view of (1.5) the maps of complexes $C_*\left(u_n\right), C_*\left(u_n(n, n+1)\right):$ $C_*\left(\bar{W}_n(R)\right) + C_*\left(\bar{W}_{n+1}(R)\right)$ are homotopic. Looking through the proof of (1.5) one sees that the corresponding homotopy operator $\phi_{n+1}^k : C_p\left(\bar{W}_n(R)\right) + C_{p+1}\left(\bar{W}_{n+1}(R)\right)$ may be taken in the following form: (We denote $x_{k,n+1}(1)$ by x_k and

$$
\begin{aligned}
& x_{n+1,k}(-1) \text{ by } y_k) \\
& \phi_{n+1}^k\left(\alpha_0, \ldots, \alpha_p\right) = \sum_{i=0}^p (-1)^{i+1}\left[\left(\alpha_0^{x_k y_k}, \ldots, \alpha_i x_k y_k, \alpha_i^{(k, n+1)}, \ldots, \alpha_p^{(k, n+1)}\right)\right. \\
& \quad - \left(\alpha_0^{x_k y_k}, \ldots, \alpha_i{}^x y_k, \alpha_i x_k y_k, \ldots, \alpha_p^{x_k y_k}\right) \\
& \quad + \left(\alpha_0^{x_k} \cdot y_k, \ldots, \alpha_i^{x_k} \cdot y_k, \alpha_i{}^{x_k y_k}, \ldots, \alpha_{pk} y_k\right) - \left(\alpha_0 y_k, \ldots, \alpha_i y_k, \alpha_i, \ldots, \alpha_p\right) \\
& \quad \left.+ \left(\alpha_0 y_k, \ldots, \alpha_i y_k, \alpha_i^{x_k} \cdot y_k, \ldots, \alpha_p^{x_k} \cdot y_k\right) - \left(\alpha_0 y_k, \ldots, \alpha_i y_k, \alpha_i y_k, \ldots, \alpha_p y_k\right)\right]
\end{aligned}
$$

Lemma 5. *The homotopy operators ϕ_{n+1}^k have the following properties:*

1. *$(\partial - \alpha(k, n+1)) = d\phi_{n+1}^k(\alpha) + \phi_{n+1}^k(d\alpha)$, where $\alpha = \left(\alpha_0, \ldots, \alpha_p\right)$ is a p-simplex of $\bar{W}_n(R)$.*
2. *$\phi_{n+1}^n \mid C_*\left(\bar{W}_{n-1}(R)\right) = 0$.*
3. *For any $s \in S_n$ the following formula is valid:*

$$\phi_{n+1}^k\left(\alpha^s\right) = \left[\phi_{n+1}^s(k)(\alpha)\right]^s$$

4. *$\phi_{n+1}^k \mid C_*\left(\bar{W}_{n-1}(R)\right) = \left(\phi_n^k\right)(n+1, n)$*

Lemma 6. *Suppose $c \in C_p\left(\bar{W}_{n-q}(R)\right)$, $dc \,\& C_{p-1}\left(\bar{W}_{n-q-1}(R)\right)$. Set*

$$
\begin{aligned}
& c_0 = c, c_1 = \phi_{n-q+1}^{n-q}\left(c_0\right) \& c_{p+1}\left(\bar{W}_{n-q+1}(R)\right), \ldots, c_k \\
& = \phi_{n-q+k}^{n-q+k-1}\left(c_{k-1}\right) \varepsilon c_{p+k}\left(\bar{w}_{n-q+k}(R)\right). \text{ Then, if } k \geq 1, \text{ we have:} \\
& dc_k = c_{k-1} - c_{k-1}^{(n-q+k, n-q+k-1)} + \ldots + (-1)^k c_{k-1}^{(n-q+k, \ldots, n-q)}.
\end{aligned}
$$

13.0.1 The Aciclicity Theorem

If X is an arbitrary set, we'll denote by $F_m(X)$ the partially ordered set of functions defined on non-empty subsets of $\{1, \ldots, m\}$ and taking values in X. The partial ordering is defined as follows:

$$f \leq g \Leftrightarrow \operatorname{dom} f \subset \operatorname{dom} g,\, g|_{\text{dom}}\ f = f.$$

(Here dom f is the subset of $\{1, \ldots, m\}$ where f is defined). Following van der Kallen [11] we'll say that $F \subset F_m(X)$ satisfies the chain condition if F contains with any function all its restrictions (to non-empty subsets of its domain). It is clear that f and g have a common restriction if and only if there exists i $\varepsilon\{1, \ldots, m\}$ such that f and g are defined at i and equal at i. In this case there obviously exists a maximal common restriction inf(f, g).

If $F \subset F_m(X)$ satisfies the chain condition, then by F_* we'11 denote the geometric realization of the semi-simplicial set, whose non-degenerate p-simplices are the functions $f \varepsilon F$ with $|$ dom f $|= p+1$, and whose faces are defined by the formulas $d_j(f) = f|_{\{i_0,\ldots,\hat{i}_j,\ldots,i_p\}}$ where $\{i_0, \ldots, i_p\} = \operatorname{dom} f, (i_0 < \ldots < i_p)$. If $f \varepsilon F, |\operatorname{dom} f| = p+1$, then by $|f|$ we'll denote the corresponding p-simplex of F_*. It is clear that $|f| \cap |g|$ is either empty or else equals $|\inf(f, g)|$. In particular, F_* is a simplicial space [7].

Let R be a ring (associative with identity), R^∞ the free left R-module on the basis $e_1, \ldots, e_n, \ldots$, and R^n its submodule generated by $e_1, \ldots, e_n$. If X is any subset of R^∞, then by $U_m(X)$ we' 11 denote the subset of $F_m(X)$ consisting of those functions f for which $f(i_0), \ldots, f(i_p)$ is a unimodular frame (i.e., a basis of a free direct summand of R^∞), where $\{i_0, \ldots, i_p\} = \operatorname{dom}(f)$.

Theorem 12. *Suppose R is a ring, r = s.r.R and m, n are natural numbers. Then $U_m(R^n)$ is $\min(m-2, n-r-1)$-acyclic.*

Corollary 3. $U_n(R^n)$ *is* $(n-r-1) - \text{acyc}\, 1$ *ic.*

Corollary 4. *Consider in $\mathrm{St}_{n+1}(\Lambda)$ the following subgroups: $A^i = \{\alpha : e_i \cdot \pi(\alpha) = e_i\}$ $(i = 1, \ldots, n+1)$ and consider the simplicial set $Z'\left(St_{n+1}(R), A^i\right)$ constructed as in (2.5), but using left cosets instead of right cosets. This simplicial set is $(n-r)$-acyclic.*

Part IV

Homological stability

Chapter 14

Motivation

[4]

The symmetric group Σ_n is the group of bijections of the finite set $\underline{n} = \{1, \ldots, n\}$, under composition. The classifying space BG of a discrete group G, such as Σ_n, is the connected space determined uniquely up to weak homotopy equivalence by the property

$$\pi_*(BG) = \begin{cases} G & \text{if } * = 1, \\ 0 & \text{otherwise} \end{cases}$$

It can be constructed by extracting from G the groupoid $*//G$ given by: - a single object *, - morphisms given by $* \xrightarrow{g} *$ for $g \in G$, and - composition given by multiplication.

We then take its nerve to obtain a simplicial set, and take the geometric realisation to get a topological space $|N(*//G)|$; this is a model for BG. Exercise 1.3.1 proves it indeed has the desired property.

Proposition 23. *$H_*(B\Sigma_n; \mathbb{Z})$ is the same as computing the group homology of Σ_n with coefficients in $\mathbb{Z}$.*

Let us compute these groups and the homology of their classifying spaces for the first few values of n.

Example 17.

1. *For $n = 0, 1$, the group Σ_n is trivial so its classifying space is weakly contractible and hence has trivial homology.*

2. *Example 1.1.4. For $n = 2, \Sigma_2$ is isomorphic to the cyclic abelian group $\mathbb{Z}/2$. Then $B\mathbb{Z}/2$, as constructed above, is homotopy equivalent to $\mathbb{R}P^\infty$. We conclude that*

$$H_*(B\mathbb{Z}/2; \mathbb{Z}) = H_*(\mathbb{R}P^\infty; \mathbb{Z}) = \begin{cases} \mathbb{Z} & \textit{if } * = 0 \\ \mathbb{Z}/2 & \textit{if } * > 0 \textit{ is odd,} \\ 0 & \textit{if } * > 0 \textit{ is even.} \end{cases}$$

3. *Example 1.1.5. For $n = 3$, the group Σ_3 is the dihedral group D_3 with 6 elements (i.e. the symmetries of a triangle). A more complicated computation given in Exercise 1.3.5 yields the*

homology of D_3 :

$$H_*(BD_3;\mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } * = 0 \\ \mathbb{Z}/2 & \text{if } * > 0 \text{ and } * \equiv 1 \pmod 4 \\ \mathbb{Z}/6 & \text{if } * > 0 \text{ and } * \equiv 3 \pmod 4, \\ 0 & \text{otherwise} \end{cases}$$

Conjectures

1. Each reduced homology group $\widetilde{H}_d(B\Sigma_n;\mathbb{Z})$ is finite and has small exponent.
2. The homology in fixed degree $* = d$ becomes independent of n as $n \to \infty$.
3. Before becoming independent of n, the homology only increases in size.
4. The p-power torsion only changes when $p \mid n$.

If we want to attempt to prove (2)-(4), we need a better way to compare the homology groups for different n than just as abstract abelian groups. This is done by observing that the inclusion $\underline{n} \hookrightarrow \underline{n+1}$ of finite sets gives a homomorphism

$$\sigma : \Sigma_n \longrightarrow \Sigma_{n+1},$$

by extending a permutation of $\underline{n}$ by the identity on $n+1 \in \underline{n+1}$ to a permutation of $n+1$. Our construction of BG is natural in groups and homomorphisms, so this homomorphism induces a map

$$\sigma : B\Sigma_n \longrightarrow B\Sigma_{n+1},$$

which in turn induces a map $\sigma_* : H_*(B\Sigma_n;\mathbb{Z}) \to H_*(B\Sigma_{n+1};\mathbb{Z})$ on homology. We can then give sharper formulations of (2)-(4) in terms of these stabilisation maps: (2′) The maps σ_* are isomorphisms in a range increasing with n.
(3′) The maps σ_* are injective.
(4′) The maps σ_* are isomorphisms on p-power torsion unless $p \mid n+1$.
Property (1) holds for all finite groups, and the result which proves it also implies (4′):

Proposition 24. *For a finite group G, $\widetilde{H}_*(BG;\mathbb{Z}[1/|G|]) = 0$. More generally, for $H \subset G$ the map $\iota_* : H_*(BH;\mathbb{Z}[1/[G:H]]) \to H_*(BG;\mathbb{Z}[1/[G:H]])$ admits a right inverse τ (i.e. $\iota_* \circ \tau = \mathrm{id}$).*

To deduce (4′) from Proposition 1.1.6, note that $[\Sigma_{n+1} : \Sigma_n] = n+1$ so by the long exact sequence on homology groups so that $H_*(B\Sigma_n;\mathbb{Z}) \to H_*(B\Sigma_{n+1};\mathbb{Z})$ is surjective after inverting $n+1$. Now set $n+1$ equal to p and invoke (3′). It is phenomenon indicated by (2′) that is the subject of this minicourse:

A sequence $X_0 \xrightarrow{\sigma} X_1 \xrightarrow{\sigma} X_2 \xrightarrow{\sigma} \cdots$ exhibits **homological stability** if the maps $\sigma_* : H_*(X_n;\mathbb{Z}) \to H_*(X_{n+1};\mathbb{Z})$ are isomorphisms in a range of degrees * increasing with n.

In the next two lectures we will prove the following result, due to Nakaoka [Nak60] (though he proved much more):

Theorem 13. *The sequence $B\Sigma_0 \xrightarrow{\sigma} B\Sigma_1 \xrightarrow{\sigma} B\Sigma_2 \xrightarrow{\sigma} \cdots$ exhibits homological stability. More precisely, the induced map*

$$\sigma_* : H_*(B\Sigma_n;\mathbb{Z}) \longrightarrow H_*(B\Sigma_{n+1};\mathbb{Z})$$

is surjective if $* \le \frac{n}{2}$ and an isomorphism if $* \le \frac{n-1}{2}$.

Remark 1.1.9. Of course, if we know property (3') holds then the range in the previous theorem in which σ_* is an isomorphism improves to $* \leq \frac{n}{2}$. However, property (3 ') is rather special—related to the existence of transfer maps-and you should not expect it to hold for general sequences of classifying spaces of groups. We will not comment on it again, but see Exercise 1.3.6.
Remark 1.1.10. The ranges in the previous remark are optimal among those of the form $* \leq an + b$ with $a, b \in \mathbb{Q}$.

14.1 Applications

Homological stability is a structural property of a sequence of groups, or more generally topological spaces, but it is also useful tool. In fact, many homological stability theorems are proven in service of obtaining other mathematical results. To illustrate this, I now want to explain some straightforward applications of Theorem 1.1.8. These concern the transfer of information from low n to high n and vice-versa. They can be obtained by other methods as well, but their generalisations to other sequences of groups often can not.

14.1.1 Altenating groups

Recall that for path-connected X, the Hurewicz map $\pi_1(X) \to H_1(X; \mathbb{Z})$ coincides with abelianisation (we are suppressing the basepoint). In particular, the map $G \to H_1(BG; \mathbb{Z})$ induces an isomorphism $G^{\mathrm{ab}} \to H_1(BG; \mathbb{Z})$ naturally in G. Thus we can understand the abelianisation of Σ_n by computing its first homology group. The sign homomorphism sign: $\Sigma_n \to \mathbb{Z}/2$ yields a map

$$\text{sign: } B\Sigma_n \longrightarrow B\mathbb{Z}/2,$$

which induces a map on homology. This is compatible with stabilisation, in the sense that sign $\circ\sigma =$ sign, so we get a commutative squares

$$\begin{array}{cc} H_1\left(B\Sigma_{n-1}; \mathbb{Z}\right) \xrightarrow{\sigma_*} H_1\left(B\Sigma_n; \mathbb{Z}\right) & \\ \downarrow_{\text{sign}} & \mid \text{ sign} \\ \mathbb{Z}/2 \overset{Z}{=\!=} /2. & \end{array}$$

The map $H_1\left(B\Sigma_2; \mathbb{Z}\right) \to \mathbb{Z}/2$ is an isomorphism because sign: $\Sigma_2 \to \mathbb{Z}/2$ is. By Theorem 1.1.8, in the commutative diagram the right-most top horizontal map is surjective and the other top horizontal maps are isomorphisms. A single diagram chase then deduces from the fact that the left-most vertical map is an isomorphism that all other vertical maps are.

Thus we have used homological stability to prove that

$$\text{sign: } \Sigma_n \longrightarrow \mathbb{Z}/2$$

is the abelianisation for $n \geq 2$, or equivalently that the kernel of the sign homomorphism is exactly the subgroup $[\Sigma_n, \Sigma_n]$ generated by commutators. Recalling that this kernel is exactly the alternating group A_n, we conclude that:

Theorem 14. $[\Sigma_n, \Sigma_n] = A_n$.

Remark 1.2.2. This is a fact you likely knew already, and elementary group-theoretic arguments exist. We could have used this fact instead to give an elementary proof of Theorem 1.1.8 in degree $* = 1$.

14.2 Group Completion

Homological stability implies that for in fixed degree $*$, for n sufficienty large the canonical map

$$H_*\left(B\Sigma_n;\mathbb{Z}\right) \longrightarrow \underset{n\to\infty}{\operatorname{colim}} H_*\left(B\Sigma_n;\mathbb{Z}\right)$$

is an isomorphism; the right hand side is known as the stable homology. This has two somewhat tautological consequences: 1. We can compute the right side from the left side. 2. We can compute the left side from the right side.

This is particularly interesting because the stable homology on the right side has a more familiar description.

When we constructed the stabilisation map, we used that inclusion $\underline{n} \to \underline{n+1}$ yields a homomorphism $\Sigma_n \to \Sigma_{n+1}$. More generally, disjoint union induces a homomorphism $\Sigma_n \times \Sigma_m \to \Sigma_{n+m}$, which yields "multiplication" maps

$$B\Sigma_n \times B\Sigma_m \longrightarrow B\Sigma_{n+m},$$

making the space $\bigsqcup_{n\geq 0} B\Sigma_n$ into a unital topological monoid (these are associative but not commutative, and it is probably better to say E_1-space since that is a homotopy-invariant notion).

Theorem 15 (McDuff-Segal)**.** *If M is a homotopy-commutative unital associative topological monoid, then $H_*(M;\mathbb{Z})\left[\pi_0^{-1}\right] \cong H_*(\Omega BM;\mathbb{Z})$.*

14.3 Serre's finiteness theorem and variations

Let us now use Corollary 1.2.6. By (1) the groups $H_*\left(B\Sigma_n;\mathbb{Z}\right)$ are finite for $* > 0$. By Theorem 1.1.8 the same is true for the stable homology as long as restrict to degrees $* \leq \frac{n}{2}$. Since n is arbitrary, the stable homology is finite in all positive degrees. This has the following consequence:

Theorem 16. *$\pi_*(\mathbb{S})$ is finite for all $* > 0$.*

Exercise 1.3.8 (Using Serre's finiteness theorem). Serre proved that $\pi_*(\mathbb{S})$ is finite for $* > 0$. Combine this with Corollary 1.2.6 and Exercise 1.3.6 to prove that the sequence $B\Sigma_0 \xrightarrow{\sigma} B\Sigma_1 \xrightarrow{\sigma} B\Sigma_2 \xrightarrow{\sigma} \cdots$ exhibits homological stability. (Hint: you will not be able to give an explicit range.)

Remark 1.3.9. See [McD75] for a similar qualitative argument for configuration spaces of manifolds.

Chapter 15

Homological stability for symmetric groups

Theorem 17. *The sequence* $B\Sigma_0 \xrightarrow{\sigma} B\Sigma_1 \xrightarrow{\sigma} B\Sigma_2 \xrightarrow{\sigma} \cdots$ *exhibits homological stability. More precisely,*

$$\sigma_* : H_*(B\Sigma_n;\mathbb{Z}) \longrightarrow H_*(B\Sigma_{n+1};\mathbb{Z})$$

is surjective if $* \leq \frac{n}{2}$ *and an isomorphism if* $* \leq \frac{n-1}{2}$.

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