A Comprehensive Introduction to Simplicial Sets

From Foundations to Applications

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Outline

What Are Simplicial Sets?

- Simplicial sets generalize simplicial complexes.
- Provide a combinatorial framework for:
  - Algebraic topology
  - Higher category theory
- Key features:
  - Flexibility in modeling homotopy types
  - Strong categorical properties
  - Applications in modern mathematics

Historical Context

- Originated in algebraic topology (1940s-50s)
- Developed to model homotopy theory
- Key milestones:
  - Simplicial objects (Eilenberg-Zilber)
  - Kan complexes (1950s)
  - Quillen model categories (1960s)
  - Higher category theory (2000s)
- Modern applications in derived geometry and higher topos theory

Category Theory Basics

- A category $\mathcal{C}$ consists of:
  - Objects ob $\mathcal{C}$
  - Morphisms mor $\mathcal{C}$
  - Composition $(f \circ g)$ and identities
- Functors: $F : \mathcal{C} \to \mathcal{D}$
- Natural transformations: $\alpha : F \Rightarrow G$

Yoneda Lemma

Theorem (Yoneda Lemma)

For any small category $\mathcal{C}$, object $c \in \mathcal{C}$, and functor $F : \mathcal{C}^{op} \to \mathbf{Set}$:

$$Nat(\mathcal{C}(-, c), F) \cong Fc$$

- Yoneda embedding:

$$y : \mathcal{C} \to \mathbf{Set}^{\mathcal{C}^{op}}, \quad c \mapsto \mathcal{C}(-, c)$$

- Full and faithful functor
- Foundation for simplicial sets

The Category Δ

Definition

Δ is the category whose:

- Objects are finite, non-empty, totally ordered sets

$$[n] = \{0, 1, \ldots, n\}$$

- Morphisms are order-preserving maps

Structure of Δ

- Generating morphisms:
  - Coface maps $d_i : [n-1] \to [n]$
  - Codegeneracy maps $s_i : [n+1] \to [n]$
- Relations among morphisms:

$$d_j d_i = d_i d_{j-1}, \quad i < j$$

$$s_j s_i = s_i s_{j+1}, \quad i \leq j$$

$$s_j d_i = \begin{cases} 1 & i = j, j+1 \\ d_i s_{j-1} & i < j \\ d_{i-1} s_j & i > j+1 \end{cases}$$

Definition of Simplicial Sets

Definition

A simplicial set is a functor:

$$X : \Delta^{op} \to \mathbf{Set}$$

- Equivalent to:
  - Sets X_n for $n \geq 0$
  - Face maps $d_i : X_n \to X_{n-1}$
  - Degeneracy maps $s_i : X_n \to X_{n+1}$
  - Satisfying dual relations to Δ

Key Properties of Simplicial Sets

- X_n: Set of n-simplices
- Face maps d_i:

$$d_i(x) = \text{remove the } i\text{-th vertex of } x$$

- Degeneracy maps s_i:

$$s_i(x) = \text{collapse edge between } i \text{ and } i+1$$

- Degenerate simplices: images of degeneracy maps
- Non-degenerate simplices: not degenerate

Standard Simplex Δ^n

- Represented functor:
$$\Delta^n = \Delta(-, [n])$$
- k-simplices:
$$(\Delta^n)_k = \Delta([k], [n])$$
- Unique non-degenerate n-simplex: identity map
- Non-degenerate k-simplices: injective maps $[k] \to [n]$

Nerve of a Category

- For category $\mathcal{C}$, the nerve $N\mathcal{C}$:
  - $N\mathcal{C}_0 = \text{ob } \mathcal{C}$
  - $N\mathcal{C}_1 = \text{mor } \mathcal{C}$
  - $N\mathcal{C}_2 = \{f, g \mid \text{cod}(f) = \text{dom}(g)\}$
- Face maps:
  - d_0, d_2: drop outer arrows
  - d_1: compose inner arrows
- Degeneracy maps: insert identity morphisms

Singular Complex

- For topological space Y, the singular complex SY:

$$(SY)_n = \mathrm{Top}(\Delta^n, Y)$$

- n-simplices are continuous maps from Δ^n to Y
- Face and degeneracy maps defined by precomposition

Kan Complexes

Definition

A Kan complex is a simplicial set X where every horn has a filler:

$$\Lambda^n_k \to X \quad \text{extends to} \quad \Delta^n \to X$$

- Models for homotopy types
- Example: Singular complex SY of a space Y

Quasi-Categories

Definition

A quasi-category is a simplicial set where every inner horn has a filler.

- Models for $(\infty, 1)$-categories
- Example: Nerve of any ordinary category

Applications in Algebraic Topology

- Singular homology:

$$H_n(X) = H_n(S_*X)$$

- Classifying spaces via nerve functor
- Homotopy theory: Kan complexes as models

Applications in Higher Category Theory

- $(\infty, 1)$-categories as quasi-categories
- Higher categorical limits and colimits
- Derived functors and higher topos theory

Recommended Texts

- Classical References:
  - May: "Simplicial Objects in Algebraic Topology"
  - Gabriel-Zisman: "Calculus of Fractions and Homotopy Theory"
- Modern Developments:
  - Lurie: "Higher Topos Theory"
  - Goerss-Jardine: "Simplicial Homotopy Theory"