PhD Studies

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Contents

Part I

Topics of Algebraic Topology

Chapter 1

Simplical sets and complexes

Simplicial complexes are more intuitive, and are the foundation of algebraic topology. Simplicial complexes were also called *simplicial schemes* and simplicial sets, *semi-simplicial* complexes.

1.1 (Abstract) simplical complexes

A set (of **vertices**) together with a family of finite subsets (**simplexes**) such that every subset of every simplex is a simplex and every subset consisting of a single vertex is a simplex.

Example 1.

1. *The **standard n-simplex** Δ^n is the set of all $(n+1)$-tuples $(t_0, \ldots, t_n)$ of non-negative real numbers such that $t_0 + \cdots + t_n = 1$. The standard 0-simplex is a point, the standard 1-simplex is a line segment, the standard 2-simplex is a triangle, and so on.*

2. *The **boundary** of the standard n-simplex Δ^n is the set of all $(n+1)$-tuples $(t_0, \ldots, t_n)$ of non-negative real numbers such that $t_0 + \cdots + t_n = 1$ and at least one of the t_i is zero. The boundary of the standard 0-simplex is empty, the boundary of the standard 1-simplex is the set of its two endpoints, the boundary of the standard 2-simplex is the set of its three edges, and so on.*

3. *(**Concrete simplicial complexes**) It is subset of $\mathbb{R}^n$ that is a union of standard simplices, that satisfies the previous conditions.*

4. *If Y is a subset of the vertex set of a simplicial scheme S, then we can introduce on it the induced simplicial scheme structure $Y \cap S$, by defining the simplexes as the subsets of Y that are simplexes of S.*

5. *Let X be a set and let $\{p(y) : y \in Y\}$ be a covering of X. Then we can consider two simplicial complexes.*

(a) *The nerve* Nerv(p) *of the covering is the simplicial scheme with the vertex set Y, and a subset Z of Y is counted as a simplex if the intersection $\bigcap_Z p(y)$ is non-empty.*

(b) *The simplicial complex $V(p)$ is the simplicial scheme with the vertex set X, and a subset Z of X is counted as a simplex if Z is contained in some $p(y)$.*

Geometric realization

The construction goes as follows. First, define $|K|$ as a subset of $[0,1]^S$ consisting of functions $t : S \to [0,1]$ satisfying the two conditions: □

$$\{s \in S : t_s > 0\} \in K$$
$$\sum_{s \in S} t_s = 1$$

Now think of the set of elements of $[0,1]^S$ with finite support as the direct limit of $[0,1]^A$ where A ranges over finite subsets of S, and give that direct limit the induced topology. Now give $|K|$ the subspace topology. *It is always Hausdorff.* We will identify an abstract simplicial complex with its geometric realization.

1.2 Simplical sets

Let $\mathbf{\Delta}$ be the category of finite ordinal numbers, with order-preserving maps between them. More precisely, the objects for Δ consist of elements $\mathbf{n}, n \geq 0$, where $\mathbf{n}$ is a string of relations

$$0 \to 1 \to 2 \to \cdots \to n$$

(in other words $\mathbf{n}$ is a totally ordered set with $n+1$ elements). A morphism $\theta : \mathbf{m} \to \mathbf{n}$ is an order-preserving set function, or alternatively a functor. We usually commit the abuse of saying that $\mathbf{\Delta}$ is the ordinal number category.

A simplicial set is a contravariant functor $X : \Delta^{op} \to$ Sets, where Sets is the category of sets.

Remark 1. *The standard covariant functor:* $\mathbf{n} \mapsto |\Delta^n|$ *from* Δ *to* ***Top***. *The singular set $S(T)$ is the simplicial set given by*

$$\mathbf{n} \mapsto \hom\left(|\Delta^n|, T\right).$$

This is the object that gives the singular homology of the space T.

The standard n-simplex, simplicial Δ^n in the simplicial set category $\mathbf{S}$ *is defined by*

$$\Delta^n = \hom_\Delta(, \mathbf{n}).$$

In other words, Δ^n is the contravariant functor on Δ which is represented by n.

A map $f : X \to Y$ of simplicial sets (or, more simply, a simplicial map) is a natural transformation of contravariant set-valued functors defined on $\mathbf{\Delta}$. We shall use $\mathbf{S}$ to denote the resulting category of simplicial sets and simplicial maps.

From a simplicial set Y, one may construct a simplicial abelian group $\mathbb{Z}Y$ (ie. a contravariant functor $\mathbf{\Delta}^{op} \to \mathbf{Ab}$), with $\mathbb{Z}Y_n$ set equal to the free abelian group on Y_n. The simplicial abelian group $\mathbb{Z}Y$ has associated to it a chain complex, called its Moore complex and also written $\mathbb{Z}Y$, with

$$\mathbb{Z}Y_0 \overset{\partial}{\leftarrow} \mathbb{Z}Y_1 \overset{\partial}{\leftarrow} \mathbb{Z}Y_2 \leftarrow \dots \quad \text{and}$$
$$\partial = \sum_{i=0}^{n} (-1)^i d_i$$

in degree n. Recall that the integral singular homology groups $H_*(X;\mathbb{Z})$ of the space X are defined to be the homology groups of the chain complex $\mathbb{Z}SX$. The homology groups $H_n(Y, A)$ of a simplicial set Y with coefficients in an abelian group A are defined to be the homology groups $H_n(\mathbb{Z}Y \otimes A)$ of the chain complex $\mathbb{Z}Y \otimes A$.

Classifying space

Suppose that $\mathcal{C}$ is a (small) category. The classifying space (or nerve) $B\mathcal{C}$ of $\mathcal{C}$ is the simplicial set with

$$B\mathcal{C}_n = \hom_{\text{cat}}\,(\mathbf{n}, \mathcal{C}),$$

n-simplex is a string

$$a_0 \xrightarrow{\alpha_1} a_1 \xrightarrow{\alpha_2} \dots \xrightarrow{\alpha_n} a_n$$

of composeable arrows of length n in $\mathcal{C}$.

If G is a group, then G can be identified with a category (or groupoid) with one object $*$ and one morphism $g : * \to *$ for each element g of G, and so the classifying space BG of G is defined. Moreover $|BG|$ is an Eilenberg-Mac Lane space of the form $K(G, 1)$, as the notation suggests; this is now the standard construction.

Geometric realization

The simplex category: $\Delta \downarrow X$ of a simplicial set X. The objects of $\Delta \downarrow X$ are the maps $\sigma : \Delta^n \to X$, or simplices of X. An arrow of $\Delta \downarrow X$ is a commutative diagram of simplicial maps .....

Observe that θ is induced by a unique ordinal number map $\theta : \mathbf{m} \to \mathbf{n}$.

Lemma 1. *There is an isomorphism*

$$X \cong \varinjlim_{\Delta^n \longrightarrow X} \Delta^n.$$

in $\Delta \downarrow X$

The realization $|X|$ of a simplicial set X is defined by the colimit

$$|X| = \xrightarrow{\lim} |\Delta^n|.$$
$$\Delta^n \to X$$
$$\text{in } \Delta \downarrow X$$

in the category of topological spaces. The construction $X \mapsto |X|$ is seen to be functorial in simplicial sets X, by using the fact that any simplicial map $f : X \to Y$ induces a functor $f_* : \Delta \downarrow X \to \Delta \downarrow Y$ by composition with f.

Proposition 1. *The realization functor is left adjoint to the singular functor in the sense that there is an isomorphism*

$$\hom_{Top}(|X|, Y) \cong \hom_{\mathbf{S}}(X, SY)$$

which is natural in simplicial sets X and topological spaces Y. In particular, since $\mathbf{S}$ *has all colimits and the realization functor, // preserves them.*

Proposition 2. $|X|$ *is a CW-complex for each simplicial set X. In particular it is a compactly generated Hausdorff space.*

1.3 CW-complexes

They can be defined in an inductive way:

1. Start with a discrete set X^0, whose points are regarded as 0 -cells.
2. Inductively, form the $\boldsymbol{n}$-skeleton X^n from X^{n-1} by attaching n-cells e^n_α via maps $\varphi_\alpha : S^{n-1} \to X^{n-1}$. This means that X^n is the quotient space of the disjoint union $X^{n-1} \amalg_\alpha D^n_\alpha$ of X^{n-1} with a collection of n-disks D^n_α under the identifications $x \sim \varphi_\alpha(x)$ for $x \in \partial D^n_\alpha$. Thus as a set, $X^n = X^{n-1} \amalg_\alpha e^n_\alpha$ where each e^n_α is an open n-disk.
3. One can either stop this inductive process at a finite stage, setting $X = X^n$ for some $n < \infty$, or one can continue indefinitely, setting $X = \cup_n X^n$. In the latter case X is given the weak topology: A set $A \subset X$ is open (or closed) iff $A \cap X^n$ is open (or closed) in X^n for each n.

Example 2. *1. A 1-dimensional cell complex $X = X^1$ is what is called a graph in algebraic topology. It consists of vertices (the 0 -cells) to which edges (the 1-cells) are attached. The two ends of an edge can be attached to the same vertex.*

2. *The sphere S^n has the structure of a cell complex with just two cells, e^0 and e^n, the n-cell being attached by the constant map $S^{n-1} \to e^0$. This is equivalent to regarding S^n as the quotient space $D^n/\partial D^n$.*

3. ***Real projective n-space*** $\mathbb{R}\mathrm{P}^n$***.*** *It is equivalent to the quotient space of a hemisphere D^n with antipodal points of ∂D^n identified. Since ∂D^n with antipodal points identified is just $\mathbb{R}\mathbb{P}\mathrm{P}^{n-1}$, we see that $\mathbb{R}\mathrm{P}^n$ is obtained from $\mathbb{R}\mathrm{P}^{n-1}$ by attaching an n-cell, with the quotient projection $S^{n-1} \to \mathbb{R}P^{n-1}$ as the attaching map. It follows by induction on n that $\mathbb{R}\mathbb{P}^n$ has a cell complex structure $e^0 \cup e^1 \cup \cdots \cup e^n$ with one cell e^i in each dimension $i \leq n$.*
   The infinite union $\mathbb{R}P^\infty = U_n \mathbb{R}P^n$ becomes a cell complex with one cell in each dimension. We can view $\mathbb{R}P^\infty$ as the space of lines through the origin in $\mathbb{R}^\infty = \bigcup_n \mathbb{R}^n$.

4. ***Complex projective space*** $\mathbb{C}P^n$***.*** *It is equivalent to the quotient of the unit sphere $S^{2n+1} \subset \mathbb{C}^{n+1}$ with $v \sim \lambda v$ for $|\lambda| = 1$.*
   It is also possible to obtain $\mathbb{C}\mathbb{P}^n$ as a quotient space of the disk D^{2n} under the identifications $v \sim \lambda v$ for $v \in \partial D^{2n}$, in the following way. The vectors in $S^{2n+1} \subset \mathbb{C}^{n+1}$ with last coordinate real and nonnegative are precisely the vectors of the form $\left(w, \sqrt{1-|w|^2}\right) \in \mathbb{C}^n \times \mathbb{C}$ with $|w| \leq 1$. Such vectors form the graph of the function $w \mapsto \sqrt{1-|w|^2}$. This is a disk D^{2n}_+ bounded by the sphere $S^{2n-1} \subset S^{2n+1}$ consisting of vectors $(w, 0) \in \mathbb{C}^n \times \mathbb{C}$ with $|w| = 1$. Each vector in S^{2n+1} is equivalent under the identifications $v \sim \lambda v$ to a vector in D^{2n}_+, and the latter vector is unique if its last coordinate is nonzero. If the last coordinate is zero, we have just the identifications $v \sim \lambda v$ for $v \in S^{2n-1}$.
   It follows that $\mathbb{P}^n$ is obtained from $\mathbb{C}\mathrm{P}^{n-1}$ by attaching a cell e^{2n} via the quotient map $S^{2n-1} \to \mathbb{C}\mathbb{P}^{n-1}$. So by induction on n we obtain a cell structure $\mathbb{C}\mathbb{P}^n = e^0 \cup e^2 \cup \cdots \cup e^{2n}$ with cells only in even dimensions. Similarly, $\mathbb{C}\mathbb{P}^\infty$ has a cell structure with one cell in each even dimension.

Each cell e^n_α in a cell complex X has a **characteristic map** $\Phi_\alpha : D^n_\alpha \to X$ which extends the attaching map φ_α and is a homeomorphism from the interior of D^n_α onto e^n_α. Namely, we can take Φ_α to be the composition $D^n_\alpha \hookrightarrow X^{n-1} \coprod_\alpha D^n_\alpha \to X^n \hookrightarrow X$ where the middle map is the quotient map defining X^n.

Chapter 2

Homotopy theory

Let I^n be the n-dimensional unit cube, the product of n copies of the interval $[0,1]$. The boundary ∂I^n of I^n is the subspace consisting of points with at least one coordinate equal to 0 or 1 . For a space X with basepoint $x_0 \in X$, define $\pi_n(X, x_0)$ to be the set of homotopy classes of maps $f : (I^n, \partial I^n) \to (X, x_0)$, where homotopies f_t are required to satisfy $f_t(\partial I^n) = x_0$ for all t. The definition extends to the case $n = 0$ by taking I^0 to be a point and ∂I^0 to be empty, so $\pi_0(X, x_0)$ is just the set of path-components of X.

When $n \geq 2$, a sum operation in $\pi_n(X, x_0)$, generalizing the composition operation in π_1, is defined by

$$(f+g)(s_1, s_2, \cdots, s_n) = \begin{cases} f(2s_1, s_2, \cdots, s_n), & s_1 \in [0, 1/2] \\ g(2s_1 - 1, s_2, \cdots, s_n), & s_1 \in [1/2, 1] \end{cases}$$

It is evident that this sum is well-defined on homotopy classes. Since only the first coordinate is involved in the sum operation, the same arguments as for π_1 show that $\pi_n(X, x_0)$ is a group, with identity element the constant map sending I^n to x_0 and with inverses given by $-f(s_1, s_2, \cdots, s_n) = f(1 - s_1, s_2, \cdots, s_n)$.

Proposition 3. *If $n \geq 2$, then $\pi_n(X, x_0)$ is abelian.*

Part II

K-theory

Chapter 3

K-theory constructions

3.1 Volodin's K-theory

Let G be a group and $\{G_i\}_{i\varepsilon I}$ a family of subgroups. Define $V(G, \{G_i\})$, or just $V(G)$ to be the simplicial complex, whose vertices are the elements of G, where $g_0, \ldots, g_p$ $(g_i \neq g_j)$ form a p-simplex if for some G_i all the elements $g_j g_k^{-1}$ lie in G_i. If H is another group with a family of subgroups $\{H_j\}$ and $\phi : G \to H$ is a homomorphism sending each G_i into some H_j, then ϕ induces a simplicial map $V(\phi) : V(G) \to V(H)$.

In many situations it is more convenient to use simplicial sets instead of simplicial complexes: Denote by $W(G, \{G_i\})$ the geometric realization of the simplicial set whose p-simplices are the sequences $(g_0, \ldots, g_p)$ of elements of G (not necessarily distinct) such that for some G_i al1 $g_j g_k^{-1}$ lie in G_i, the r-th face (resp. degeneracy) of this simplex being obtained by omitting g_r (resp., repeating g_r). Associating with any p-simplex $(g_0, \ldots, g_p)$ the linear singular simplex of the space $V(G)$ which sends the i-th vertex of the standard simplex to g_j, we obtain a map of simplicial sets from $W(G)$ to the simplicial set of singular simplices of $V(G)$ and hence a cellular map (linear on any simplex) from $W(G)$ to $V(G)$. This map is a homotopy equivalence ....

Suppose that R is a ring, n a natural number and σ a partial ordering of $\{1, \ldots, n\}$. Define $T_n^\sigma(R)$ to be the subgroup of $GL_n(R)$ consisting of the α with $\alpha_{ij} = 1$ and $\alpha_{ij} = 0$ if $i\&j$. Subgroups of this form will be called triangular subgroups of $GL_n(R)$. The space $V(GL_n(R), \{T_n^\sigma(R)\})$ will be denoted by $V_n(R)$. Since any partial ordering may be extended to a linear ordering, it suffices to consider linear orderings when defining $V_n(R)$. The natural embedding $GL_n \hookrightarrow GL_{n+1}(R)$ defines an embedding $V_n(R) \longleftrightarrow V_{n+1}(R)$ and we'l1 define $V_\infty(R)$ as $\lim_{\substack{\rightarrow \\ n}} V_n(R)$.

Finally for $i \geq 1$, put

$$k_{i,n}(R) = \pi_{i-1}(V_n(R))$$

and $k_i(R) = k_{i,\infty}(R) = \lim_{\rightarrow} k_{i,n}(R)$ (compare [26], [27]). Evidently $K_{1,n}(R) = GL_n(R)/E_n(R)$ and $K_{i,n}(R)$ is a group if $i \geq 2$, and this group is abelian if

$i \geq 3$. Moreover the $K_i(R)$ are abelian groups for all $i \geq 1$ (see [26], [27]). The connected component of $V_n(R)$ passing through T_n equals $V\left(E_n(R), \{T_n^\sigma(R)\}\right)$. It is easy to show that the universal covering space of $V_n\left(E_n(R), \{T_n^\sigma(R)\}\right)$ equals $V\left(St(R), \{T_n^\sigma(R)\}\right)$, where T_n^σ is identified with the subgroup of $St_n(R)$ generated by the $x_{ij}(a)$ with a $\varepsilon R, i \overset{\sigma}{<} j (n \geq 3)$. Hence

Lemma 2. $K_{2,n}(R) = \ker\left(St_n(R) + E_n(R)\right)$, *and* $K_{i,n}(R) = \pi_{i-1}\left(V\left(St_n(R)\right)\right) = \pi_{i-1}\left(W\left(St_n(R)\right)\right) \quad$ *if* $\quad i \geq 3 \quad (n \geq 3)$.

Let's define $\overline{St}_n(R)$ to be the inverse image of $GL_n(R)$ under the projection $St(R) \to E(R)$. There is a canonical homomorphism $St_n(R) \to \overline{st}_n(R)$ and stability for K_1, k_2 ([10], [20], [22]) shows that this homomorphism is surjective if $n \geq s.r.R + 1$ and bijective if $n \geq s.r.R + 2$. The spaces $W\left(St_n(R)\right)$ and $W\left(\overline{St}_n(R)\right)$ will play an essential role in the sequel. We'll denote them by $W_n(R), \bar{W}_n(R)$, resp. (So $W_n(R) = \bar{W}_n(R)$ if $n \geq$ s.r. $R + 2$.)

3.2 Milnor's K-theory

3.3 Whitehead's K-theory

3.4 Quillen's K-theory

Chapter 4

Homological stability

4.1 Motivation

The symmetric group Σ_n is the group of bijections of the finite set $\underline{n} = \{1, \ldots, n\}$, under composition. The classifying space BG of a discrete group G, such as Σ_n, is the connected space determined uniquely up to weak homotopy equivalence by the property

$$\pi_*(BG) = \begin{cases} G & \text{if } * = 1, \\ 0 & \text{otherwise} \end{cases}$$

It can be constructed by extracting from G the groupoid $*//G$ given by: - a single object $*$, - morphisms given by $* \xrightarrow{g} *$ for $g \in G$, and - composition given by multiplication.

We then take its nerve to obtain a simplicial set, and take the geometric realisation to get a topological space $|N(*//G)|$; this is a model for BG. Exercise 1.3.1 proves it indeed has the desired property.

Proposition 4. *$H_*(B\Sigma_n; \mathbb{Z})$ is the same as computing the group homology of Σ_n with coefficients in $\mathbb{Z}$.*

Let us compute these groups and the homology of their classifying spaces for the first few values of n.

Example 3.

1. *For $n = 0, 1$, the group Σ_n is trivial so its classifying space is weakly contractible and hence has trivial homology.*

2. *Example 1.1.4. For $n = 2, \Sigma_2$ is isomorphic to the cyclic abelian group $\mathbb{Z}/2$. Then $B\mathbb{Z}/2$, as constructed above, is homotopy equivalent to $\mathbb{R}P^\infty$. We conclude that*

$$H_*(B\mathbb{Z}/2; \mathbb{Z}) = H_*(\mathbb{R}P^\infty; \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } * = 0 \\ \mathbb{Z}/2 & \text{if } * > 0 \text{ is odd,} \\ 0 & \text{if } * > 0 \text{ is even.} \end{cases}$$