kreyszig-9.5

March 19, 2025

1 Erwin Kreyzig - Problem Set 9.5 Parametric Curves

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1.1 1-10 What curves are represented by the following? Sketch them.

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1.2 1.

Parametric Representation: $[3 + 2\cos t, 2\sin t, 0]$

$x = 3 + 2\cos t,\ y = 2\sin t,\ z = 0.$

Let $u = \cos t,\ v = \sin t$, so $u^2 + v^2 = 1$.

Then $u = \frac{x-3}{2},\ v = \frac{y}{2}$.

Thus,

$$\left(\frac{x-3}{2}\right)^2 + \left(\frac{y}{2}\right)^2 = u^2 + v^2 = 1.$$

Curve: Circle with center $(3, 0)$ in the xy-plane $(z = 0)$.

Equation:

$$\left(\frac{x-3}{2}\right)^2 + \left(\frac{y}{2}\right)^2 = 1$$

.

[1]:

```
import numpy as np
import matplotlib.pyplot as plt

# Create parameter t for parametric equations
t = np.linspace(0, 2*np.pi, 100)

# Parametric equations for the ellipse
# x = h + a*cos(t), y = k + b*sin(t)
# where (h,k) is center, a is horizontal radius, b is vertical radius
# circle special case when a = b = r, radius r
x = 3 + 2 * np.cos(t)  # h=3, a=2
y = 0 + 2 * np.sin(t)  # k=0, b=2
```

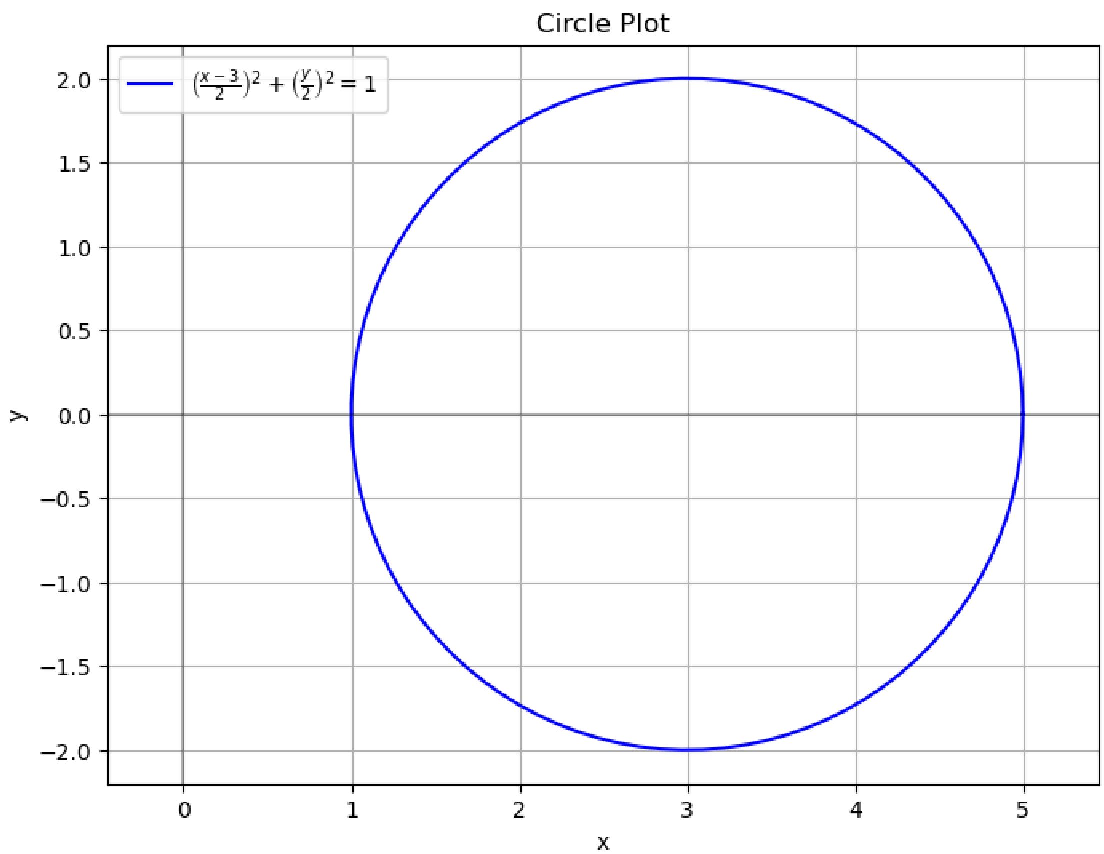
```
# Create the plot
plt.figure(figsize=(8, 6))
plt.plot(x, y, 'b-', label=r'$\left(\frac{x-3}{2}\right)^2 +␣
↪\left(\frac{y}{2}\right)^2 = 1$')

# Add grid and axes
plt.grid(True)
plt.axhline(y=0, color='k', linestyle='-', alpha=0.3)
plt.axvline(x=0, color='k', linestyle='-', alpha=0.3)

# Set equal aspect ratio to show true ellipse shape
plt.axis('equal')

# Add labels and title
plt.xlabel('x')
plt.ylabel('y')
plt.title('Circle Plot')
plt.legend()

# Show the plot
plt.show()
```



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1.2.1 2.

Parametric Equation: $[a+t, b+3t, c-5t]$

$x = a+t$, $y = b+3t$, $z = c-5t$.

Let $t = 0$. Then, $x = a, y = b, z = c$. Thus the curve passes through (a, b, c).

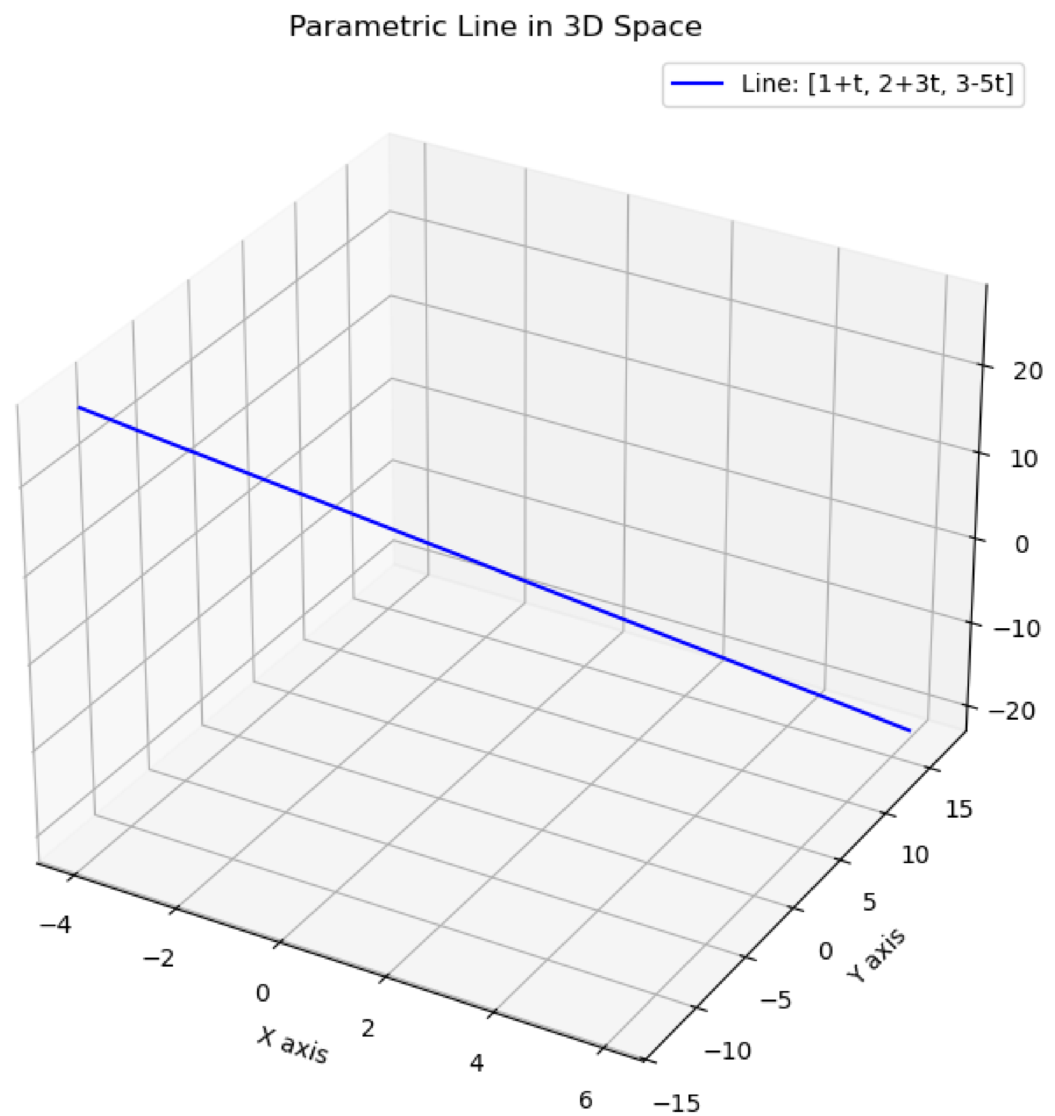
If $t \neq 0$, the displacement $(t, 3t, -5t)$ is added to the point (a, b, c).. Thus, it is a line in the direction $(1, 3, -5)$ through (a, b, c).

To get non-parametric equation relating x, y, z without involving t, solving for t, we get $t = x - a$. Substituting back in, $y = b + 3(x - a)$, $z = c - 5(x - a)$.

[3]:

```python
# Import required libraries
import numpy as np
import matplotlib.pyplot as plt
from mpl_toolkits.mplot3d import Axes3D

# Define constants
a = 1  # You can change these values
b = 2
c = 3

# Create parameter t
t = np.linspace(-5, 5, 100)  # Creates 100 points from -5 to 5

# Calculate x, y, z coordinates
x = a + t
y = b + 3*t
z = c - 5*t

# Create the 3D plot
fig = plt.figure(figsize=(10, 8))
ax = fig.add_subplot(111, projection='3d')

# Plot the line
ax.plot(x, y, z, 'b-', label=f'Line: [{a}+t, {b}+3t, {c}-5t]')

# Set labels
ax.set_xlabel('X axis')
ax.set_ylabel('Y axis')
ax.set_zlabel('Z axis')
ax.set_title('Parametric Line in 3D Space')
```

```python
# Add legend
ax.legend()

# Add grid
ax.grid(True)

# Show plot
plt.show()
```



[ ]:

1.2.2 3.

Parametric Representation: $[0, t, t^3]$

We want to eliminate t.

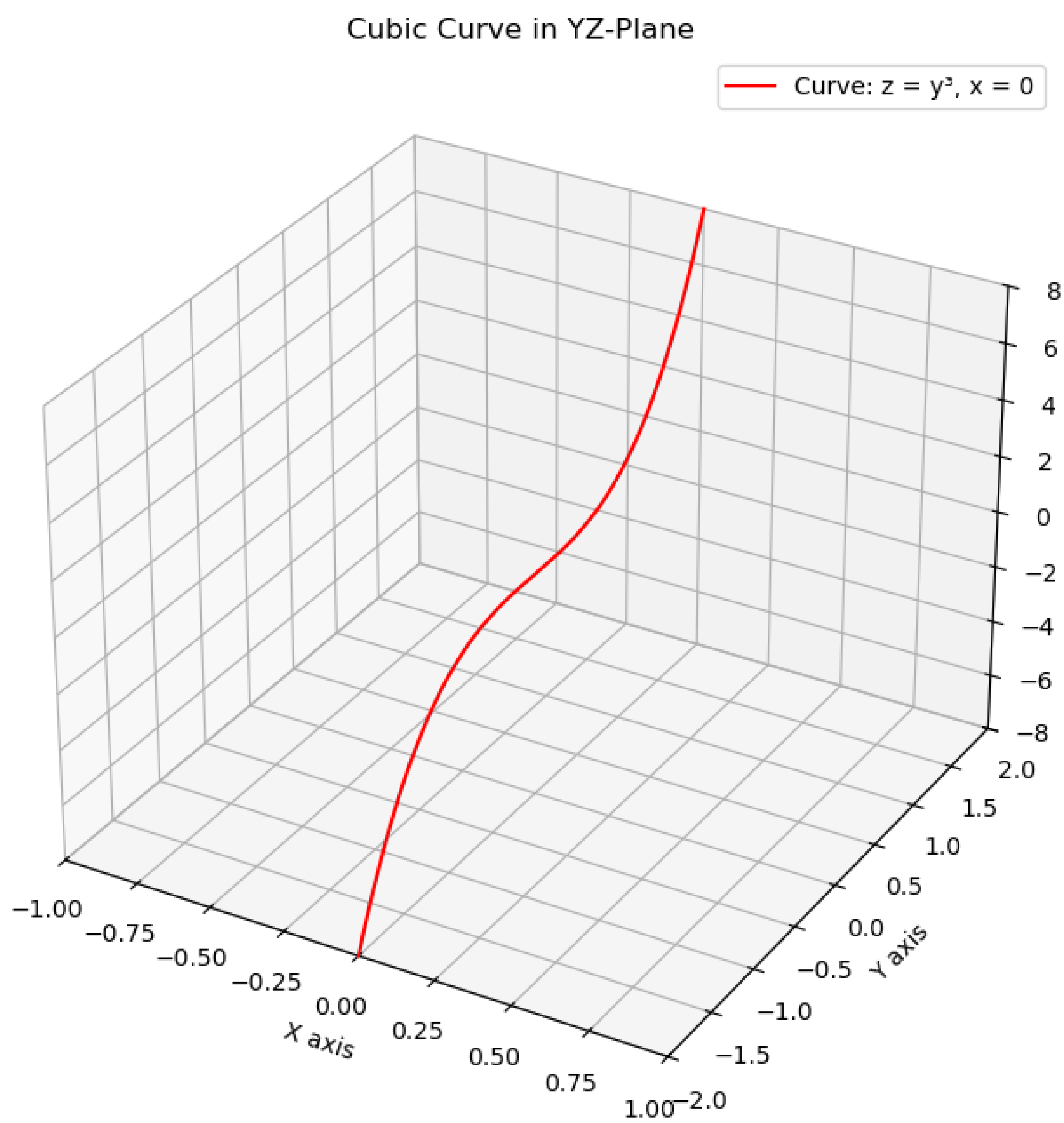
Using $y = t$, we get $z = t^3$.

Substitute $t = y$ back in, so $z = y^3$.

Curve: Cubic curve in the yz-plane ($x = 0$) given by quation $z = y^3$.

[5]:

```
# Import required libraries
import numpy as np
import matplotlib.pyplot as plt
from mpl_toolkits.mplot3d import Axes3D

# Create y values
y = np.linspace(-2, 2, 100)  # Creates 100 points from -2 to 2

# Calculate x and z coordinates
x = np.zeros_like(y)  # x = 0 for all points
z = y**3  # z = y³

# Create the 3D plot
fig = plt.figure(figsize=(10, 8))
ax = fig.add_subplot(111, projection='3d')

# Plot the curve
ax.plot(x, y, z, 'r-', label='Curve: z = y³, x = 0')

# Set labels
ax.set_xlabel('X axis')
ax.set_ylabel('Y axis')
ax.set_zlabel('Z axis')
ax.set_title('Cubic Curve in YZ-Plane')

# Set axis limits to better visualize the curve
ax.set_xlim(-1, 1)  # Small range since x is always 0
ax.set_ylim(-2, 2)
ax.set_zlim(-8, 8)  # Larger range since z = y³

# Add legend
ax.legend()

# Add grid
ax.grid(True)

# Show plot
plt.show()
```

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1.2.3 4.

Parametric Representation: $[-2, 2 + 5\cos t, -1 + 5\sin t]$

We have $x = -2, y = 2 + 5\cos t$ and $z = -1 + 5\sin t$.

Let $u = \cos t$, $v = \sin t$, so $u^2 + v^2 = 1$.

But solving for $u = \cos t$ and $v = \sin t$ in terms of y and z gives \$ u = y - 2$_{5\$andv=\frac{z+1}{5}}$.

Thus, we have equations without t:

$$\left(\frac{y-2}{5}\right)^2 + \left(\frac{z+1}{5}\right)^2 = 1.$$

Or:

$$(y-2)^2 + (z+1)^2 = 5^2,$$

while $x = -2$.

This is a circle in the yz-plane centered at $(-2, 2, -1)$, radius 5, translated to $x = -2$.

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1.2.4 5.

Paremetric Representation:$[2 + 4\cos t, 1 + \sin t, 0]$

$x = 2 + 4\cos t$, $y = 1 + \sin t$, $z = 0$.

Let $u = \cos t$, $v = \sin t$, so $u^2 + v^2 = 1$.

Then $\frac{x-2}{4} = u$, $\frac{y-1}{1} = v$.

Thus,

$$\left(\frac{x-2}{4}\right)^2 + \left(\frac{y-1}{1}\right)^2 = 1$$

.

Curve: Ellipse in the xy-plane ($z = 0$) centered at $(2, 1, 0)$, semi-major axis 4 (x), semi-minor axis 1 (y).

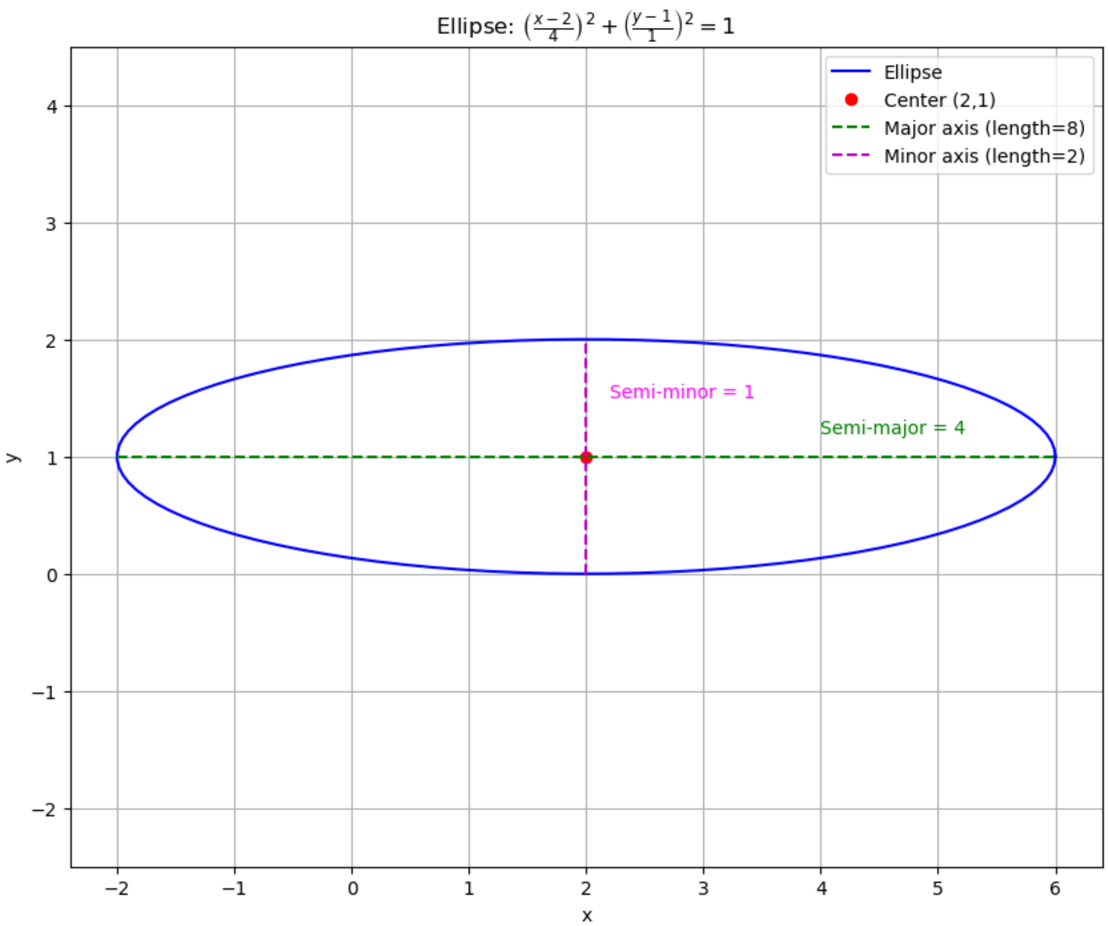
An ellipse has two axes. The bigger axis is called major axis and smaller one minor axis. Instead of radius for circle, we have two values, the lengths of semi-major axis and semi-minor axis.

An ellipse in general co-ordinates in two dimensions may be transformed to the above form equation by translation and rotation of the cartesian co-ordinates.

[11]:

```python
import numpy as np
import matplotlib.pyplot as plt

# Create parameter t for parametric form
t = np.linspace(0, 2*np.pi, 100)

# Parametric equations for ellipse
x = 2 + 4 * np.cos(t)  # center x=2, semi-major axis=4
y = 1 + 1 * np.sin(t)  # center y=1, semi-minor axis=1

# Create the plot
plt.figure(figsize=(10, 8))
plt.plot(x, y, 'b-', label='Ellipse')
```

```python
# Plot center point
plt.plot(2, 1, 'ro', label='Center (2,1)')

# Major axis (x-direction, length 8 = 2*4)
plt.plot([2-4, 2+4], [1, 1], 'g--', label='Major axis (length=8)')
# Minor axis (y-direction, length 2 = 2*1)
plt.plot([2, 2], [1-1, 1+1], 'm--', label='Minor axis (length=2)')

# Add annotations for axes lengths
plt.text(2+2, 1.2, 'Semi-major = 4', color='green')
plt.text(2.2, 1.5, 'Semi-minor = 1', color='magenta')

# Set equal aspect ratio
plt.axis('equal')

# Add grid and labels
plt.grid(True)
plt.xlabel('x')
plt.ylabel('y')
plt.title('Ellipse: $\\left(\\frac{x-2}{4}\\right)^2 + \\left(\\frac{y-1}{1}\\right)^2 = 1$')
plt.legend()

# Show the plot
plt.show()
```



[ ]:

1.2.5 6.

Parametric Representation: $[a+3\cos\pi t, b-2\sin\pi t, 0]$ - **Curve**: Ellipse in the xy-plane ($z=0$). - **Derivation**: $x = a + 3\cos\pi t$, $y = b - 2\sin\pi t$. Let $u = \cos\pi t$, $v = \sin\pi t$, so $u^2 + v^2 = 1$. Then $\frac{x-a}{3} = u$, $\frac{y-b}{-2} = v$. Thus,

$$\left(\frac{x-a}{3}\right)^2 + \left(\frac{y-b}{2}\right)^2 = 1$$

. - **Equation**:

$$\left(\frac{x-a}{3}\right)^2 + \left(\frac{y-b}{2}\right)^2 = 1$$

. - **Sketch**: Ellipse centered at $(a, b, 0)$, semi-major axis 3 (x), semi-minor axis 2 (y).

[ ]:

1.2.6 7.

Parametric Representation: $[4\cos t, 4\sin t, 3t]$

$x = 4\cos t,\ y = 4\sin t,\ z = 3t.$

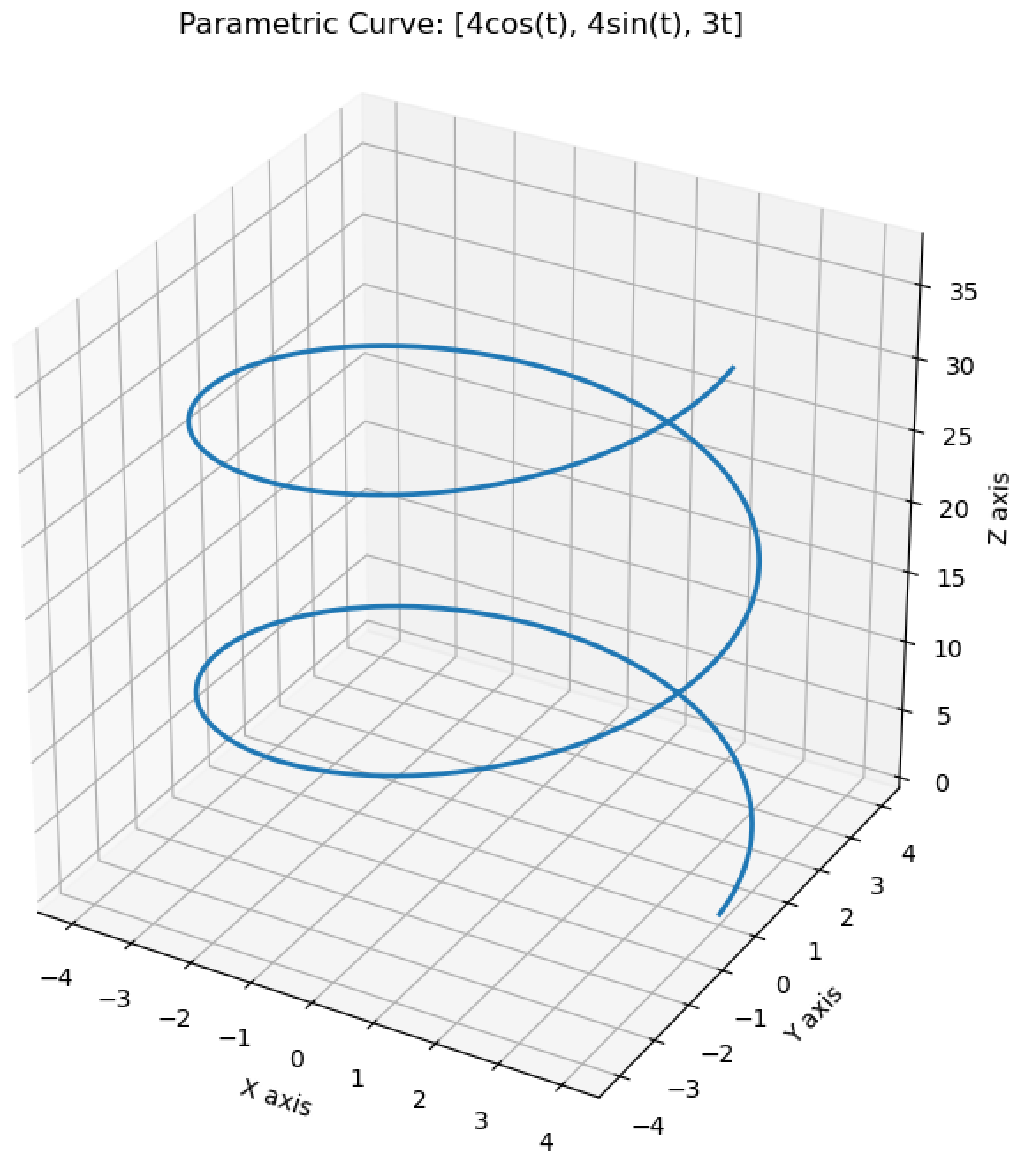
Then $x^2 + y^2 = (4\cos t)^2 + (4\sin t)^2 = 16(\cos^2 t + \sin^2 t) = 16 = 4^2$.

So, $x^2 + y^2 = 16$, with $z = 3t$.

Curve: Circular helix in 3D, with x, y co ordinates along a circle with center at origin and radius 4. The helix rises along z co-ordinate at the rate of three units per t..

Even though the equation relating x, y can be obtained as that of a circle, the the parametric form for the helix cannot be really removed as it's required for the z-coordinate.

[1]:
```python
import numpy as np
import matplotlib.pyplot as plt
from mpl_toolkits.mplot3d import Axes3D

# Create parameter t
t = np.linspace(0, 4*np.pi, 1000)  # Adjust range as needed

# Calculate x, y, z coordinates
x = 4 * np.cos(t)
y = 4 * np.sin(t)
z = 3 * t

# Create the 3D plot
fig = plt.figure(figsize=(10, 8))
ax = fig.add_subplot(111, projection='3d')

# Plot the parametric curve
ax.plot(x, y, z, lw=2)

# Add labels and title
ax.set_xlabel('X axis')
ax.set_ylabel('Y axis')
ax.set_zlabel('Z axis')
ax.set_title('Parametric Curve: [4cos(t), 4sin(t), 3t]')

# Optional: Set equal aspect ratio
ax.set_box_aspect([1,1,1])

# Show the plot
plt.show()
```

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1.2.7 8.

Parametric Representation: $[\cosh t, \sinh t, 2]$

Instead of the circular trigonometric functions sin, cos, the parametrization is by the hyperbolic trigonometric functions sinh and cosh.

We have $x = \cosh t$, $y = \sinh t, z = 2$.

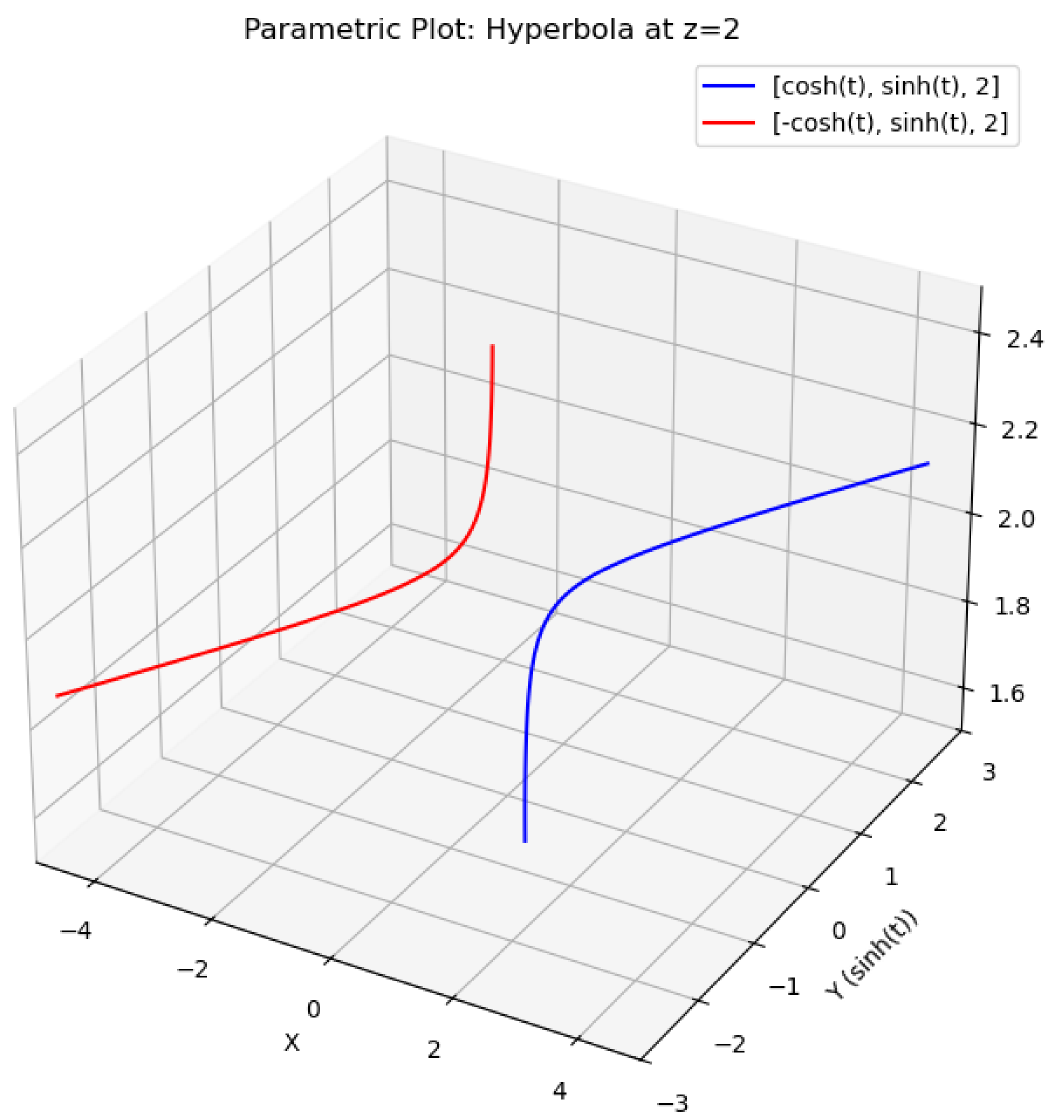
Using $\cosh^2 t - \sinh^2 t = 1$, we get $x^2 - y^2 = 1$.

Curve: Hyperbola in the xy-plane with offset $z = 2$ with equation: $x^2 - y^2 = 1$, opens along

x-axis, centered at $(0, 0, 2)$.

When we have the hyperbolic trigonometric functions in the parametric representation, we get a hyperbola as the equation. When we had the circular trigonometric functions in parametric representation, we had got the circle.

[4]:
```python
import numpy as np
import matplotlib.pyplot as plt
from mpl_toolkits.mplot3d import Axes3D

# Create parameter t range
t = np.linspace(-2, 2, 100)

# Calculate coordinates for first branch (positive x)
x1 = np.cosh(t)
y1 = np.sinh(t)
z1 = np.full_like(t, 2)

# Calculate coordinates for second branch (negative x)
x2 = -np.cosh(t)
y2 = np.sinh(t)
z2 = np.full_like(t, 2)

# Create the 3D plot
fig = plt.figure(figsize=(10, 8))
ax = fig.add_subplot(111, projection='3d')

# Plot both branches
ax.plot(x1, y1, z1, 'b-', label='[cosh(t), sinh(t), 2]')
ax.plot(x2, y2, z2, 'r-', label='[-cosh(t), sinh(t), 2]')

# Set labels
ax.set_xlabel('X')
ax.set_ylabel('Y (sinh(t))')
ax.set_zlabel('Z')
ax.set_title('Parametric Plot: Hyperbola at z=2')

# Add a legend
ax.legend()

# Set reasonable axis limits to see both branches clearly
ax.set_xlim(-5, 5)
ax.set_ylim(-3, 3)
ax.set_zlim(1.5, 2.5)

# Show the plot
plt.show()
```

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[ ]:
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1.2.8 9.

Parametric Representation: $[\cos t, \sin 2t, 0]$

We have $x = \cos t$, $y = \sin 2t$, $z = 0$.

Since the z-component is zero, the curve lies in the xy-plane, with:

Parametric Behavior:
- $x = \cos t$: Period 2π. Value between ± 1. - $y = \sin 2t$: Period π.

Such parametrization using *cos* and *sin* functions with different speed/frequency for x and y give rise to a type of figure-8 shape called Lissajous curve.

Use the trigonometric formula $\sin 2t = 2 \sin t \cos t.$,

But $\cos t = x$, and $y = \sin t = \pm\sqrt{1-\cos^2 t} = \pm\sqrt{1-x^2}$

The frequency ratio $1:2$ suggests a Lissajous curve, where x and y are sinusoidal functions with integer-related frequencies.

Such curves are explained here: https://mathworld.wolfram.com/LissajousCurve.html

In the following page, this curve is traced in the first example, with horizontal figure-8 or bow tie shape: https://jwilson.coe.uga.edu/emt668/emat6680.2003.fall/shiver/assignment10/assignment10.htm

Eliminating the Parameter

- $x = \cos t$, so $\cos t = x$, $\sin t = \pm\sqrt{1-x^2}$.
- $y = \sin 2t = 2 \sin t \cos t = 2(\pm\sqrt{1-x^2})x$.

Thus:

$$y = \pm 2x\sqrt{1-x^2}$$

Square both sides:

$$y^2 = 4x^2(1-x^2)$$

Shape $1-x^2$ is negative if $x > 1$ or $x < -1$. Thus y^2, which has to be a positive quanity, is defined only if $-1 \leq x \leq 1$.

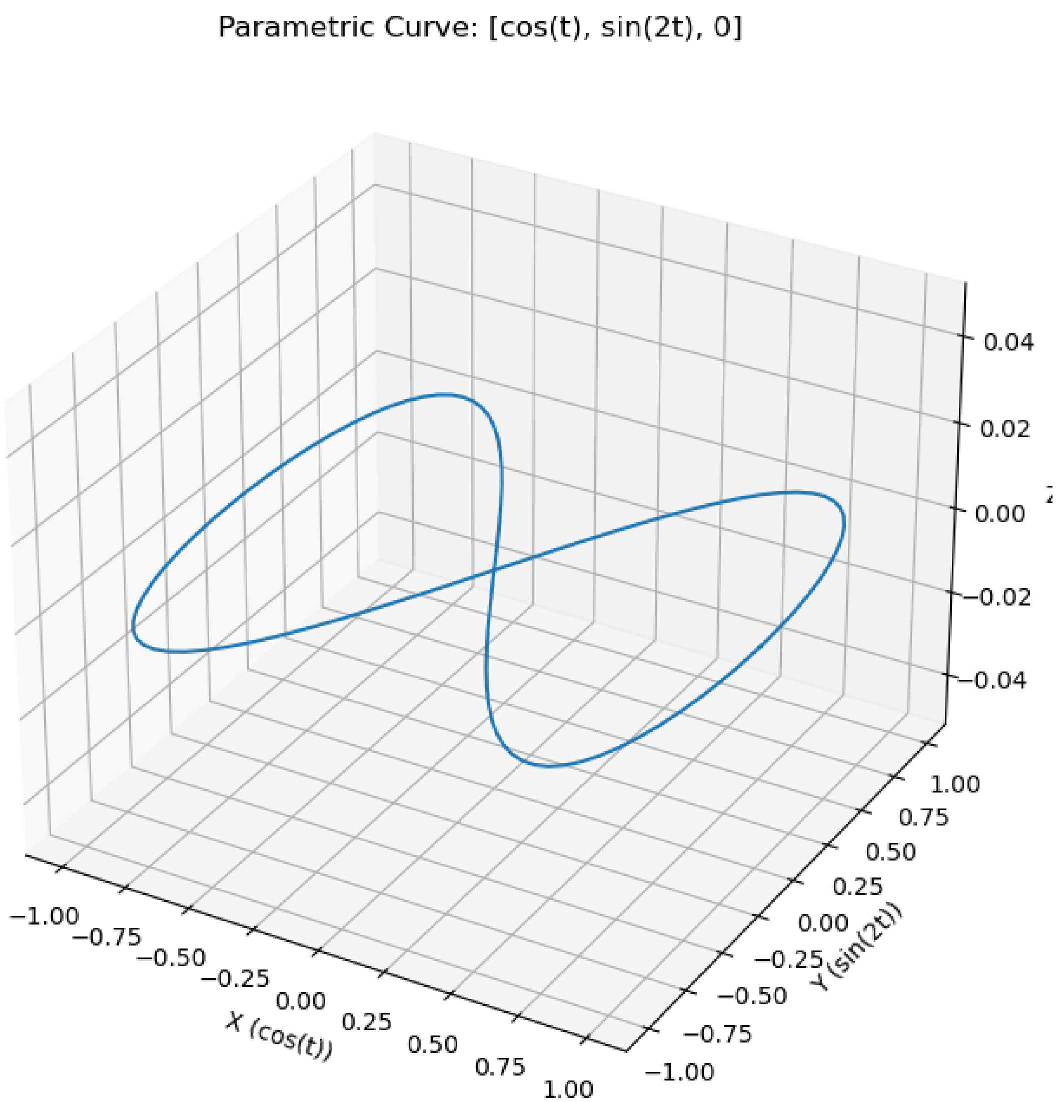
If y is a solution, then so is $-y$. The figure is symmetric about $x-$ axis. Likewise it is symmetric about $y-$ axis.

The value of the right side expression approaches zero at the origin. It also approaches 0 at $x = \pm 1$..

Plotting the curve, we see it's a horizontal figure eight curve (bowtie).$.

[5]:
```
import numpy as np
import matplotlib.pyplot as plt
from mpl_toolkits.mplot3d import Axes3D

# Create parameter t
t = np.linspace(0, 2*np.pi, 100)

# Calculate x, y, z coordinates
x = np.cos(t)
y = np.sin(2*t)
z = np.zeros_like(t)  # z = 0 for all points

# Create the 3D plot
fig = plt.figure(figsize=(10, 8))
ax = fig.add_subplot(111, projection='3d')
```

```python
# Plot the curve
ax.plot(x, y, z)

# Set labels
ax.set_xlabel('X (cos(t))')
ax.set_ylabel('Y (sin(2t))')
ax.set_zlabel('Z')
ax.set_title('Parametric Curve: [cos(t), sin(2t), 0]')

# Add grid
ax.grid(True)

# Show the plot
plt.show()
```



[ ]:

1.2.9 10.

Parametric Representation: $[t, 2, 1/t]$

y is costant, with value 2. Therefore the curve is in the plane $y = 2$, parallel to the xz plane.

$x = t$, $z = 1/t$. Substitute $t = x$, so $z = 1/x$ (for $x \neq 0$).

Thus it's a hyperbola in the $z = 1/x$ in xz-plane, with $y = 2$ where x and z are inversely related, and the two axes x and z are asymptotes.

[ ]:

1.3 11–20 FIND A PARAMETRIC REPRESENTATION

[ ]:

11 Circle in the plane $z = 1$ with center $(3, 2)$ and passing through the origin.

A circle is parameterized using trigonometric functions cos and sin .

Since the circle lies in the plane $z = 1$, the z-coordinate is constant.

The center is $(3, 2, 1)$, and the circle passes through the point $(0, 0, 1)$ in the plane $z = 1$ corresponding to origin $0, 0$ in xy plane.

The equation for the circle in xy plane with center (a, b) with radius r is $(x - x_c)^2 + (y - y_c)^2 = r^2)$.

Here $(x_c, y_c) = (3, 2)$ and r is the distance between $(3, 2)$ and $(0, 0)$, which is $\sqrt{3^2 + 2^2} = \sqrt{13}$.

The co-ordinate equation is $(x - x_c)^2 + (z - 1)^2 = r^2$, where r is the radius, along with $z = 1$. Thus the equation is $(y - 2)^2 + (z - 1)^2 = 13$, along with $z = 1$.

The parametric representation for the same is set up as $(x - x_c) = r \cos t$ and $(y - y_c) = r \sin t$, where $t = \theta$ represents the angle made by the point (x, y) on the circle. Thus the equation is:

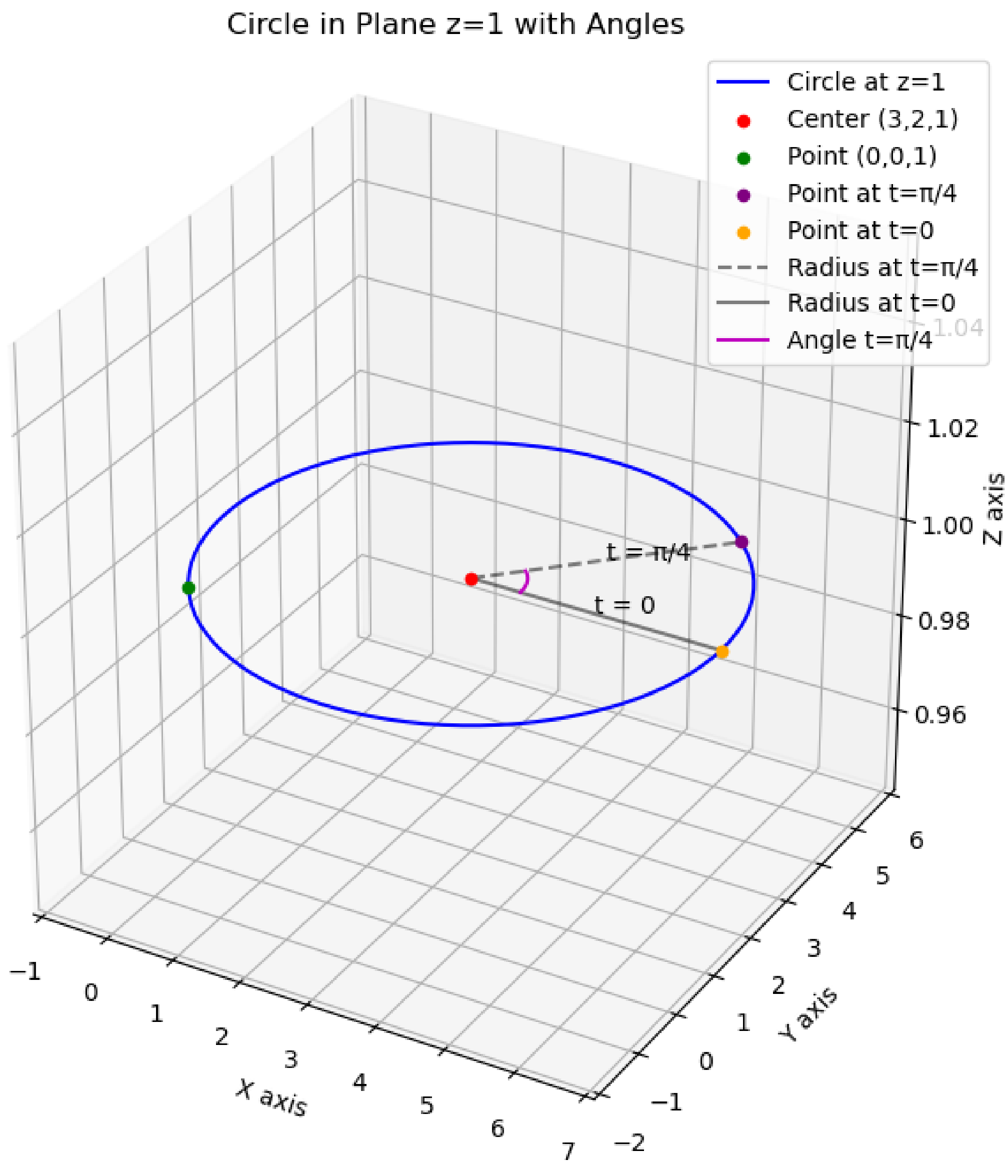
$$x = 3 + \sqrt{13} \cos t, \quad y = 2 + \sqrt{13} \sin t, \quad z = 1,$$

where $t \in [0, 2\pi)$.

[17]:
```
import numpy as np
import matplotlib.pyplot as plt
from mpl_toolkits.mplot3d import Axes3D

# Create parameter t for the circle
t = np.linspace(0, 2*np.pi, 100)  # 100 points from 0 to 2

# Calculate radius
```

```python
r = np.sqrt(13)

# Parametric equations for the circle
x = 3 + r * np.cos(t)
y = 2 + r * np.sin(t)
z = np.ones_like(t)  # z = 1 for all points

# Specific point at t = /4
t_point = np.pi/4
x_point = 3 + r * np.cos(t_point)
y_point = 2 + r * np.sin(t_point)
z_point = 1

# Point at t = 0
x_zero = 3 + r * np.cos(0)
y_zero = 2 + r * np.sin(0)
z_zero = 1

# Points for angle arc (smaller radius near center)
arc_radius = r * 0.2  # Small radius for angle visualization
t_arc = np.linspace(0, t_point, 20)  # Points from 0 to /4
x_arc = 3 + arc_radius * np.cos(t_arc)
y_arc = 2 + arc_radius * np.sin(t_arc)
z_arc = np.ones_like(t_arc)

# Create 3D plot
fig = plt.figure(figsize=(10, 8))
ax = fig.add_subplot(111, projection='3d')

# Plot the circle
ax.plot(x, y, z, 'b-', label='Circle at z=1')

# Plot the center point
ax.scatter([3], [2], [1], color='red', label='Center (3,2,1)')

# Plot the origin point
ax.scatter([0], [0], [1], color='green', label='Point (0,0,1)')

# Plot the specific point at t = /4
ax.scatter([x_point], [y_point], [z_point], color='purple', label='Point at t= /
 ↪4')

# Plot the point at t = 0
ax.scatter([x_zero], [y_zero], [z_zero], color='orange', label='Point at t=0')

# Add line from center to t = /4 point
```

```
ax.plot([3, x_point], [2, y_point], [1, 1], 'k--', alpha=0.5, label='Radius at t= /4')

# Add line from center to t = 0 point
ax.plot([3, x_zero], [2, y_zero], [1, 1], 'k-', alpha=0.5, label='Radius at t=0')

# Plot the angle arc
ax.plot(x_arc, y_arc, z_arc, 'm-', label='Angle t= /4')

# Annotate the angles
ax.text(x_point/2 + 1.5, y_point/2 + 1, 1, 't =  /4', fontsize=10)
ax.text(x_zero/2 + 1.5, y_zero/2 + 1, 1, 't = 0', fontsize=10)

# Set labels and title
ax.set_xlabel('X axis')
ax.set_ylabel('Y axis')
ax.set_zlabel('Z axis')
ax.set_title('Circle in Plane z=1 with Angles')

# Set equal aspect ratio
ax.set_box_aspect([1,1,1])

# Add legend
ax.legend()

# Add grid
ax.grid(True)

# Show plot
plt.show()
```



[ ]:

12 Circle in the yz-plane with center $(4, 0)$ and passing through $(0, 3)$. Sketch it.

In the yz-plane, $x = 0$. The circle is parameterized using y and z coordinates with trigonometric functions, centered at $(0, 4, 0)$ (adjusted for the yz-plane context).

The center is $(0, 4, 0)$ (since it's in the yz-plane, $x = 0$), and it passes through $(0, 3, 0)$.

The radius is the distance from $(0, 4, 0)$ to $(0, 3, 0)$. Thus $ r = |4 - 3| = 1. $

The parametric equations are $y = y_c + r\cos t$, $z = z_c + r\sin t$, with $x = 0$.

Parametric Representation:

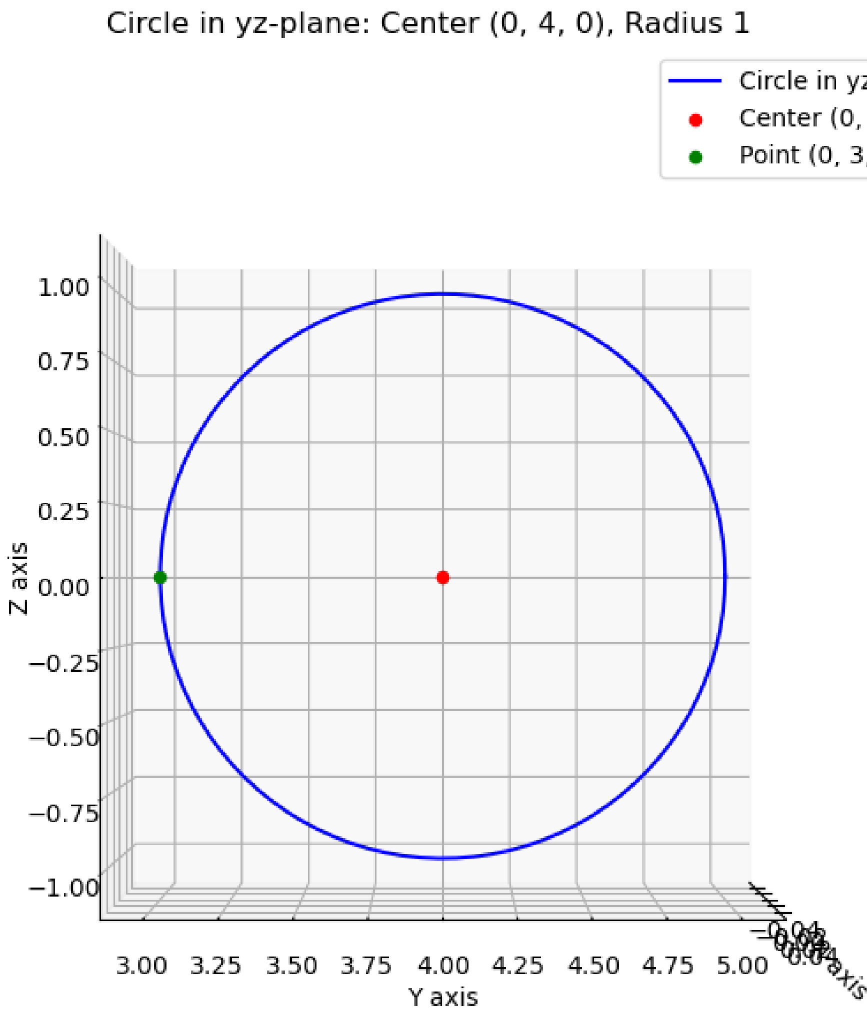
$$x = 0, \quad y = 4 + \cos t, \quad z = \sin t,$$

where $\theta \in [0, 2\pi)$.

Sketch a circle in the yz-plane centered at $(0, 4, 0)$ with radius 1, passing through $(0, 3, 0)$. It extends from $y = 3$ to $y = 5$ and z from -1 to 1.

[18]:
```python
import numpy as np
import matplotlib.pyplot as plt
from mpl_toolkits.mplot3d import Axes3D

# Create parameter t
t = np.linspace(0, 2*np.pi, 100)

# Parametric equations
x = np.zeros_like(t)   # x = 0 (yz-plane)
y = 4 + np.cos(t)      # y = 4 + cos(t)
z = np.sin(t)          # z = sin(t)

# Create 3D plot
fig = plt.figure(figsize=(8, 8))
ax = fig.add_subplot(111, projection='3d')

# Plot the circle
ax.plot(x, y, z, 'b-', label='Circle in yz-plane')

# Plot the center point
ax.scatter([0], [4], [0], color='red', label='Center (0, 4, 0)')

# Plot the point it passes through
ax.scatter([0], [3], [0], color='green', label='Point (0, 3, 0)')

# Set labels
ax.set_xlabel('X axis')
ax.set_ylabel('Y axis')
ax.set_zlabel('Z axis')
ax.set_title('Circle in yz-plane: Center (0, 4, 0), Radius 1')

# Set equal aspect ratio
ax.set_box_aspect([1,1,1])

# Add legend
ax.legend()

# Adjust view to emphasize yz-plane
ax.view_init(elev=0, azim=0)
```

```python
plt.show()
```



[ ]:

13 Straight line through $(2, 1, 3)$ in the direction of $\mathbf{i} + 2\mathbf{j}$.

A line in 3D space is parameterized using a point on the line and a direction vector. The direction vector $\mathbf{i} + 2\mathbf{j}$ corresponds to $(1, 2, 0)$.

The line passes through $(2, 1, 3)$ with direction vector $\mathbf{d} = (1, 2, 0)$.

The parametric form is $\mathbf{r}(t) = \mathbf{r_0} + t\mathbf{d}$, where $\mathbf{r_0} = (2, 1, 3)$.

Parametric Representation:

$$x = 2 + t, \quad y = 1 + 2t, \quad z = 3,$$

where $t \in \mathbb{R}$.

Note: The z-coordinate remains constant since the direction vector has no z-component.

[ ]:

14 Straight line through $(1, 1, 1)$ and $(4, 0, 2)$. Sketch it.

The parametric equation of a line through two points uses the direction vector as the difference of the points.

Direction vector $\mathbf{d} = (4 - 1, 0 - 1, 2 - 1) = (3, -1, 1)$.

The line passes through $(1, 1, 1)$, so $\mathbf{r}(t) = (1, 1, 1) + t(3, -1, 1)$.

Thus, Parametric Representation:

$$x = 1 + 3t, \quad y = 1 - t, \quad z = 1 + t,$$

where $t \in \mathbb{R}$.

The line starts at $(1, 1, 1)$ and moves toward $(4, 0, 2)$. As t increases, x and z increases and y decreases.

[ ]:

15 Straight line $y = 4x - 1, z = 5x$.

We have to represent x, y, z in terms of a parameter t.. Here already y and z can be represented in terms of x.. Therefore we can use x as the parameter and set $t = x$.. Then $y = 4t - 1$ and $z = 5t$. Thus Parametric representation is:

$$x = t, \quad y = 4t - 1, \quad z = 5t,$$

where $t \in \mathbb{R}$.

[ ]:

16 The intersection of the circular cylinder of radius 1 about the z-axis and the plane $z = y$.

A cylinder of radius 1 about the z-axis has the equation $x^2 + y^2 = 1$.

If x, y are represented in parametric form by a parameter t, then the intersection of above with $z = y$ has $z = y$ in the parametrization.

For parametric representation, use $x = \cos t$, $y = \sin t$, and $z = y = \sin t$.

The intersection is an ellipse in the plane $z = y$. It projects to a circle to both the xy-plane and xz-plane. Its semi minor axis has length 1 while semi major axis has length $\sqrt{2}$.

[ ]:

17 Circle $\frac{1}{2}x^2 + y^2 = 1, z = y$.

The first equation is:

$$x^2/2 + y^2 = 1$$

This is an ellipse in the xy-plane with: - Semi-major axis $a = \sqrt{2}$ along x-axis (since $x^2/2 = x^2/(\sqrt{2})^2$) - Semi-minor axis $b = 1$ along y-axis

To parametrize this in xy plane, we can use: - $x = \sqrt{2}\cos(t)$ - $y = \sin(t)$ where $t \in [0, 2\pi)$

Since $z = y$, we can write parametrization for the third co-ordinate also: - $z = \sin(t)$.

[ ]:

18 Helix $x^2 + y^2 = 25, z = 2\arctan(y/x)$.

A helix is parameterized using circular motion in the xy-plane with a linear or other z-component.

Here, $x^2 + y^2 = 25$ is a circle of radius 5.

This can be parametrized as $x = 5\cos t$, $y = 5\sin t$.

Thus $z = 2\arctan(y/x) = 2\arctan\left(\frac{\sin t}{\cos t}\right) = 2t$.

Thus, Parametric Representation is:

$$x = 5\cos t, \quad y = 5\sin t, \quad z = 2t,$$

where $t = \theta \in \mathbb{R}$ represents the angle (e.g., $[0, 2\pi)$ for one turn).

[ ]:

19 Hyperbola $4x^2 - 3y^2 = 4, z = -2$

A hyperbola in the xy-plane is parameterized using hyperbolic functions $\cosh t$ and $\sinh t$. In the given case, it is a hyperbola in the plane $z = -2$.

Just as we have the relation $\cos^2 t + \sin^2 t = 1$ for the circular trigonometric functions, we have $\cosh^2 t - \sinh^2 t = 1$ for the hyperbolic trigonometric functions. We use the latter to represent a hyperbola in parametric form.

The standard form of an ellipse centered at the origin is:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

where $a > b > 0$, a is the semi-major axis, and b is the semi-minor axis.

Similarly the standard form of a hyperbola centered at the origin (opening left and right) is:

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

where a and b are positive constants, and the asymptotes are $y = \pm\frac{b}{a}x$.

Rewrite given curve equation as $4x^2 - 3y^2 = 4$ as $\frac{x^2}{1} - \frac{y^2}{4/3} = 1$.

Use $x = \cosh t$, $y = \frac{2}{\sqrt{3}} \sinh t$, so:

$$4(\cosh t)^2 - 3\left(\frac{2}{\sqrt{3}} \sinh t\right)^2 = 4(\cosh^2 t - \sinh^2 t) = 4.$$

Also set set $z = -2$. Thus the parametric representation is:

$$x = \cosh t, \quad y = \frac{2}{\sqrt{3}} \sinh t, \quad z = -2,$$

where $t \in \mathbb{R}$.

[ ]:

20 Intersection of $2x - y + 3z = 2$ and $x + 2y - z = 3$

The intersection of two planes in 3D space is a straight line (unless the planes are parallel). To find a parametric representation, solve the system of equations to express two variables in terms of a third, then parameterize the line.

We can do in the manner of Gauss elimination for matrices or work directly with equations.

Derivation:
- The given planes are:

$$2x - y + 3z = 2 \quad (1)$$

$$x + 2y - z = 3 \quad (2)$$

- Solve the system by eliminating one variable. Multiply equation (2) by 3 to align the z terms:

$$3(x + 2y - z) = 3(3) \implies 3x + 6y - 3z = 9 \quad (3)$$

- Add equation (1) and equation (3) to eliminate z:

$$(2x - y + 3z) + (3x + 6y - 3z) = 2 + 9 \implies 5x + 5y = 11 \implies x + y = \frac{11}{5} \implies y = \frac{11}{5} - x.$$

- Substitute $y = \frac{11}{5} - x$ into equation (2) to solve for z:

$$x+2\left(\frac{11}{5} - x\right)-z = 3 \implies x+\frac{22}{5}-2x-z = 3 \implies -x+\frac{22}{5}-z = 3 \implies z = -x+\frac{22}{5}-3 = -x+\frac{22}{5}-\frac{15}{5} = -x+\frac{7}{5}.$$

- Now, let $x = t$ (as the parameter). Then:

$$y = \frac{11}{5} - t, \quad z = -t + \frac{7}{5}.$$

Parametric Representation:

$$x = t, \quad y = \frac{11}{5} - t, \quad z = \frac{7}{5} - t,$$

where $t \in \mathbb{R}$.

Verification:
- Substitute into equation (1): $2(t) - \left(\frac{11}{5} - t\right) + 3\left(\frac{7}{5} - t\right) = 2t - \frac{11}{5} + t + \frac{21}{5} - 3t = \frac{10}{5} = 2$, which holds.
- Substitute into equation (2): $t + 2\left(\frac{11}{5} - t\right) - \left(\frac{7}{5} - t\right) = t + \frac{22}{5} - 2t - \frac{7}{5} + t = \frac{15}{5} = 3$, which holds.

Note: The line passes through the point $\left(0, \frac{11}{5}, \frac{7}{5}\right)$ when $t = 0$, and its direction vector is $(1, -1, -1)$.

```
[ ]:
```

1.4 24-28 Tangent

Given a curve $C : \mathbf{r}(t)$, find a tangent vector $\mathbf{r}'(t)$, a unit tangent vector $\mathbf{u}'(t)$, and the tangent of C at P. Sketch curve and tangent.

```
[ ]:
```

24 \$ r(t) = [t, 1$_{\overline{2t\hat{}2,1]\$,\$P{:}(2,2,1)\$}}$

To calculate tangent vector first compute derivative: \$ r'(t) = [1, t, 0] \$

The point P corresponds to \$ t = 2 .\$ There the tangent vector is \$ r'(2) = [1, 2, 0] .\$

To calculate unit tangent vector, compute magnitude and divide the tangent vector by it:

Magnitude: \$ |r'(2)| = $\sqrt{1^2 + 2^2 + 0^2} = \sqrt{5}$\$

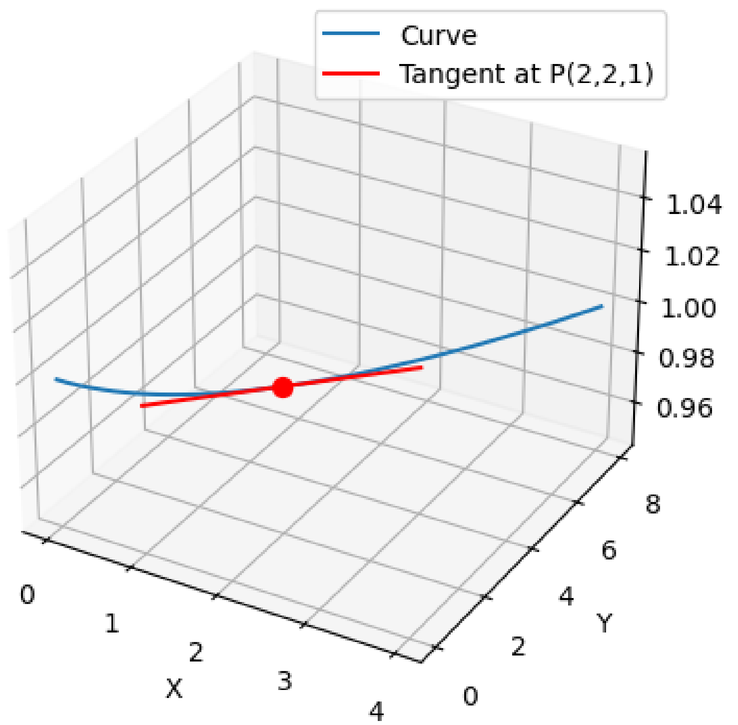
Denoting by $\mathbf{u}'$ the unit tangent vector, \$ u'(t) = $\left[\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}, 0\right]$ \$.

The tangent line at is the line through the point P in the direction of the tangent vector.

Given by parametric equations \$ x = 2 + s \$, \$ y = 2 + 2s \$, \$ z = 1 \$ for parameter s.

```
[1]: import numpy as np
import matplotlib.pyplot as plt
from mpl_toolkits.mplot3d import Axes3D

# Curve
t = np.linspace(0, 4, 100)
x = t
y = 0.5 * t**2
z = np.ones_like(t)

# Tangent at t=2
t0 = 2
s = np.linspace(-1, 1, 20)
x_tan = 2 + s
```

```
y_tan = 2 + 2 * s
z_tan = np.ones_like(s)

# Plot
fig = plt.figure()
ax = fig.add_subplot(111, projection='3d')
ax.plot(x, y, z, label='Curve')
ax.plot(x_tan, y_tan, z_tan, 'r', label='Tangent at P(2,2,1)')
ax.scatter([2], [2], [1], color='red', s=50)
ax.set_xlabel('X')
ax.set_ylabel('Y')
ax.set_zlabel('Z')
ax.legend()
plt.show()
```




```
[ ]:
```

25 $ **r(t)** $= [$**10** $\cos t, 1, 10\sin t]$\$, \$$P : (6, 1, 8)$\$

For tangent vector:

\$ **r'(t)** = [**-10** $\sin t, 0, 10\cos t]$\$

For t **corresponding to** P **above, we know that** $\sin t = 0.8, \cos t = 0.6$. **Thus, \$ r'(t)** = [**-8,**

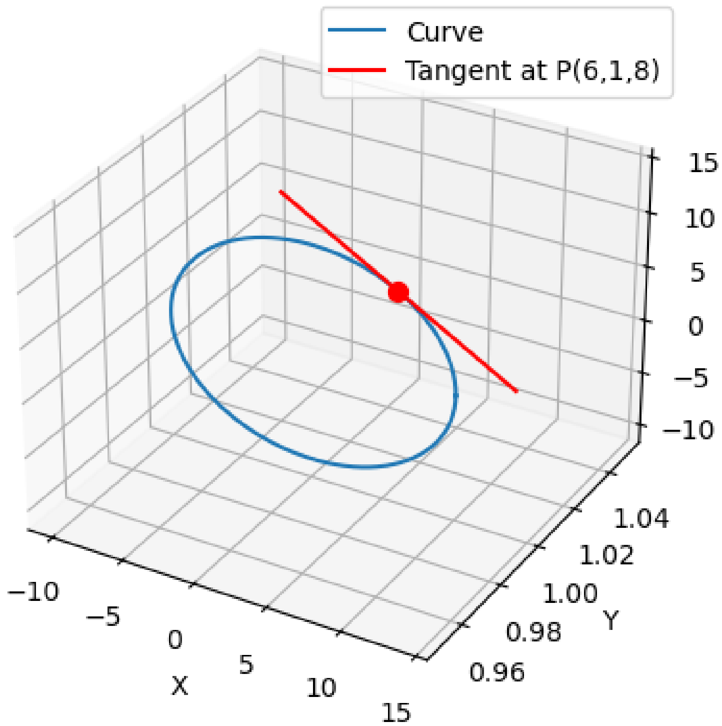
0, 6] $.

Magnitude: $|\mathbf{r}'(t)| = \sqrt{(-8)^2 + 0^2 + 6^2} = \sqrt{64+36} = \sqrt{100} = 10$

Unit tangent vector: $\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = (-0.8, 0, 0.6)$

Tangent line at $ P :$ x = 6 - 8s , y = 1 , z = 8 + 6s .$$

[2]:
```
import numpy as np
import matplotlib.pyplot as plt
from mpl_toolkits.mplot3d import Axes3D

# Curve
t = np.linspace(0, 2*np.pi, 100)
x = 10 * np.cos(t)
y = np.ones_like(t)
z = 10 * np.sin(t)

# Tangent at t=0.927
t0 = 0.927
s = np.linspace(-1, 1, 20)
x_tan = 6 - 8 * s
y_tan = np.ones_like(s)
z_tan = 8 + 6 * s

# Plot
fig = plt.figure()
ax = fig.add_subplot(111, projection='3d')
ax.plot(x, y, z, label='Curve')
ax.plot(x_tan, y_tan, z_tan, 'r', label='Tangent at P(6,1,8)')
ax.scatter([6], [1], [8], color='red', s=50)
ax.set_xlabel('X')
ax.set_ylabel('Y')
ax.set_zlabel('Z')
ax.legend()
plt.show()
```



[ ]:

```

```

26 $ **r(t)** $= [\cos t, \sin t, 9t]$\$, \$$P : (1, 0, 18\pi)$\$

Tangent vector:

\$ **r'(t)** $= [$-$\sin t, \cos t, 9]$\$

At \$ t = 2 \$, \$ r'(2) = [0, 1, 9] \$.

Magnitude: \$ $|\mathbf{r'(2\ )}| = \sqrt{82}$\$

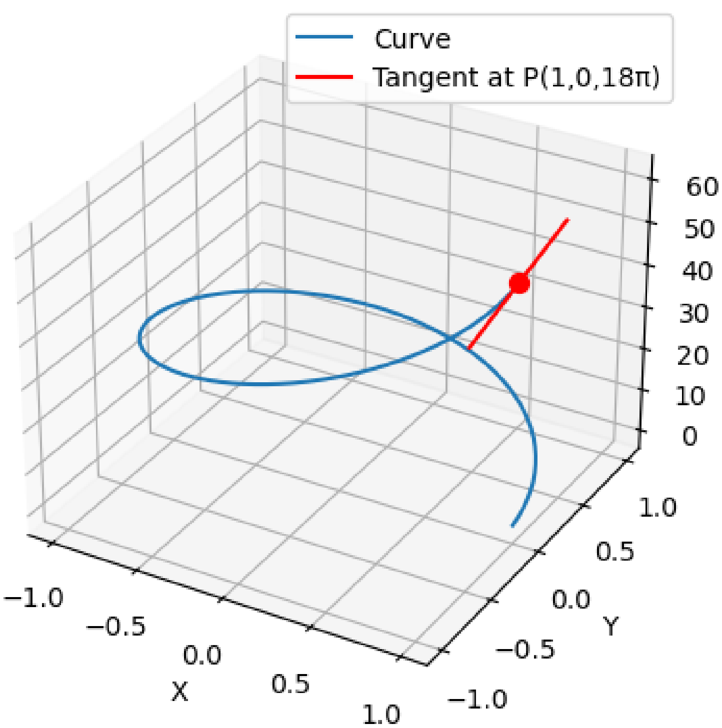
Unit tangent vector: \$ u'(t) $= \left[0, \frac{1}{\sqrt{82}}, \frac{9}{\sqrt{82}}\right]$ \$.

Tangent line at P**:**

$$x = 1, \quad y = s, \quad z = 18\pi + 9s.$$

[5]:
```
import numpy as np
import matplotlib.pyplot as plt
from mpl_toolkits.mplot3d import Axes3D

# Curve
t = np.linspace(0, 2*np.pi, 100)
x = np.cos(t)
```

```
y = np.sin(t)
z = 9 * t

# Tangent at t=2pi
t0 = 2 * np.pi
s = np.linspace(-0.5, 0.5, 20)
x_tan = np.ones_like(s)
y_tan = s
z_tan = 18 * np.pi + 9 * s

# Plot
fig = plt.figure()
ax = fig.add_subplot(111, projection='3d')
ax.plot(x, y, z, label='Curve')
ax.plot(x_tan, y_tan, z_tan, 'r', label='Tangent at P(1,0,18 )')
ax.scatter([1], [0], [18*np.pi], color='red', s=50)
ax.set_xlabel('X')
ax.set_ylabel('Y')
ax.set_zlabel('Z')
ax.legend()
plt.show()
```



[ ]:

```

```

27 **\$ r(t) = [t, 1**$_{\overline{t,0]\$,\$P:(2,\frac{1}{2},0)\$}}$

Tangent vector:

\$ r'(t) = [1, -1$_{\overline{t^2,0]\$}}$

At \$ t = 2 \$, \$ r'(2) = [1, -1$_{\overline{4,0]\$.}}$

Magnitude: \$ |r'(2)| = $\sqrt{17}_{\overline{4\$}}$

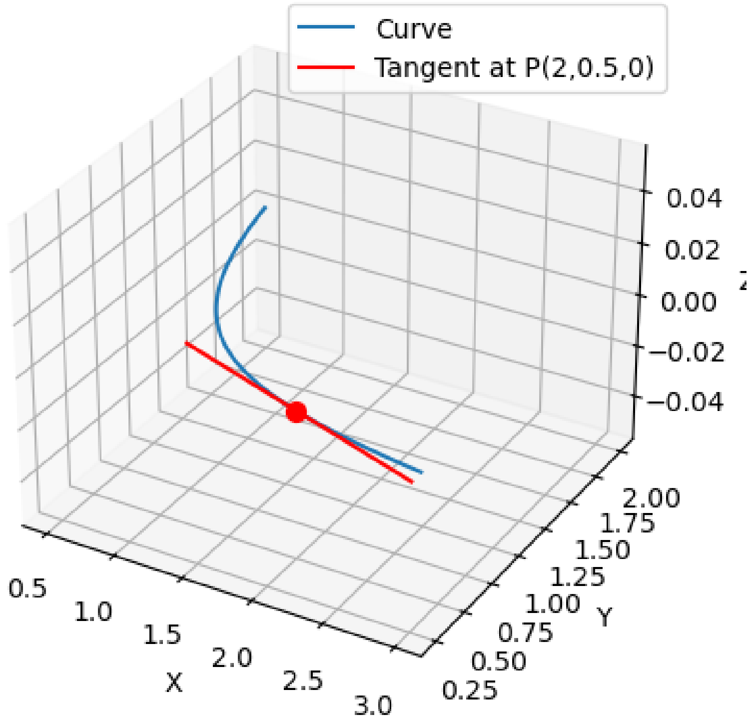
Unit tangent vector: \$ u'(t) = [4$_{\overline{\sqrt{17},-\frac{1}{\sqrt{17}},0]\$.}}$

Tangent line at \$ P \$:

$$x = 2 + s, \quad y = \frac{1}{2} - \frac{1}{4}s, \quad z = 0.$$

[6]:

```
import numpy as np
import matplotlib.pyplot as plt
from mpl_toolkits.mplot3d import Axes3D

# Curve
t = np.linspace(0.5, 3, 100)
x = t
y = 1 / t
z = np.zeros_like(t)

# Tangent at t=2
t0 = 2
s = np.linspace(-1, 1, 20)
x_tan = 2 + s
y_tan = 0.5 - 0.25 * s
z_tan = np.zeros_like(s)

# Plot
fig = plt.figure()
ax = fig.add_subplot(111, projection='3d')
ax.plot(x, y, z, label='Curve')
ax.plot(x_tan, y_tan, z_tan, 'r', label='Tangent at P(2,0.5,0)')
ax.scatter([2], [0.5], [0], color='red', s=50)
ax.set_xlabel('X')
ax.set_ylabel('Y')
ax.set_zlabel('Z')
ax.legend()
plt.show()
```



[ ]:

```

```

1.4.1 28

$r(t) = [t, t^2, t^3]$, $P: (1, 1, 1)$

Tangent vector:

$r'(t) = [1, 2t, 3t^2]$

At $t = 1$, $r'(1) = [1, 2, 3]$.

Magnitude: $|r'(1)| = \sqrt{14}$

Unit tangent vector: $u'(t) = \left[\frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}}\right]$.

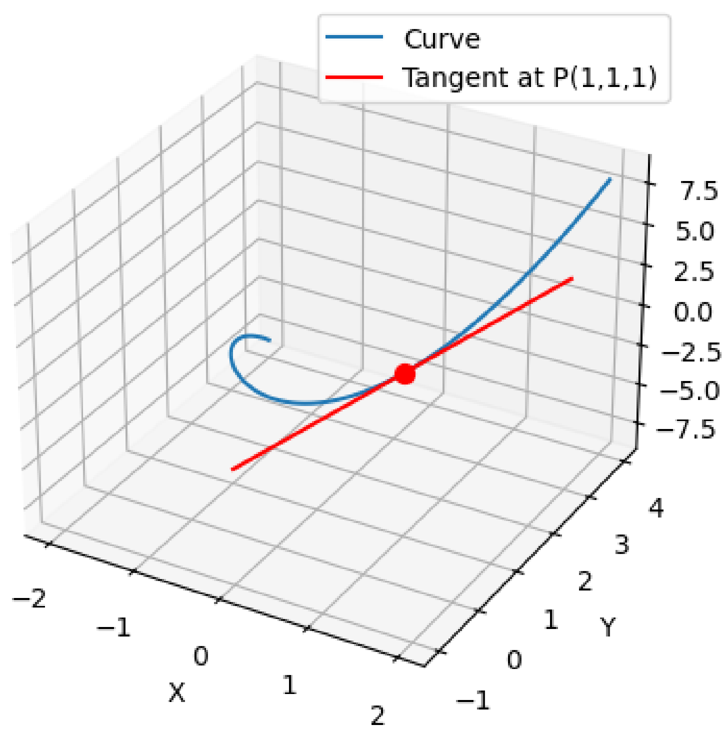
Tangent line at P:

$$x = 1 + s, \quad y = 1 + 2s, \quad z = 1 + 3s.$$

[7]:

```python
import numpy as np
import matplotlib.pyplot as plt
from mpl_toolkits.mplot3d import Axes3D

# Curve
```

```
t = np.linspace(-2, 2, 100)
x = t
y = t**2
z = t**3

# Tangent at t=1
t0 = 1
s = np.linspace(-1, 1, 20)
x_tan = 1 + s
y_tan = 1 + 2 * s
z_tan = 1 + 3 * s

# Plot
fig = plt.figure()
ax = fig.add_subplot(111, projection='3d')
ax.plot(x, y, z, label='Curve')
ax.plot(x_tan, y_tan, z_tan, 'r', label='Tangent at P(1,1,1)')
ax.scatter([1], [1], [1], color='red', s=50)
ax.set_xlabel('X')
ax.set_ylabel('Y')
ax.set_zlabel('Z')
ax.legend()
plt.show()
```



```
[ ]:
```