

kreyszig-10-1

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0.1 Erwin Kresyzig 10.1 Problems

0.1.1 2-11 LINE INTEGRAL. WORK.

Calculate

$$\int_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r}$$

for the given data. If \mathbf{F} is a force, this gives the work done by the force in the displacement along C . Show the details.

[]:

We repeat for clarity the definition of the “line” integral $\mathbf{F} \cdot d\mathbf{r}$ along the curve C from first principles.

The integral along C is defined as:

$$\int_C \mathbf{F} \cdot d\mathbf{r}$$

where \mathbf{F} and \mathbf{r} are vector functions to be expanded in vector components as:

$$\mathbf{F} = [F_1, F_2, F_3] = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k},$$

$$\mathbf{r}(t) = [x(t), y(t), z(t)] = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k},$$

and the differential $d\mathbf{r} = [dx, dy, dz]$.

$\mathbf{F} = [F_1, F_2, F_3]$ is a vector function in the 3-dimensional space to be evaluated along the path C defined by $\mathbf{r}(t)$, i. e., \mathbf{F} is $([F_1, F_2, F_3])(\mathbf{r}(t))$.

The differential $d\mathbf{r}$ may be calculated by chain rule:

$$d\mathbf{r} = \frac{d\mathbf{r}(t)}{dt} dt = \mathbf{r}'(t) dt.$$

The components are,

$$\frac{d}{dt}\mathbf{r}(t) = \frac{d}{dt}[x(t), y(t), z(t)] = \left[\frac{d}{dt}x(t), \frac{d}{dt}y(t), \frac{d}{dt}z(t) \right] = [x'(t), y'(t), z'(t)]$$

or more concisely:

$$d\mathbf{r} = [dx, dy, dz] = [x'(t), y'(t), z'(t)] dt.$$

Thus the dot product in the integrand is, along C ,

$$\begin{aligned}\mathbf{F} \cdot d\mathbf{r} &= [F_1, F_2, F_3] \cdot [dx, dy, dz] \\ &= ([F_1, F_2, F_3])(\mathbf{r}(t)) \cdot [x'(t), y'(t), z'(t)] dt \\ &= ((F_1(\mathbf{r}(t)) \cdot x'(t) + (F_2(\mathbf{r}(t))) \cdot y'(t) + (F_3(\mathbf{r}(t))) \cdot z'(t)) dt\end{aligned}$$

Thus the line integral in the 3-dimensional space is reduced to an integral in one variable along a closed interval in the real line, say, $[a, b]$.

[]:

$$2. \quad F = [y^2, -x^2], \quad C : y = 4x^2 \text{ from } (0, 0) \text{ to } (1, 4)$$

Given $\mathbf{F} = [y^2, -x^2]$ and $C : y = 4x^2$ from $(0, 0)$ to $(1, 4)$, we have to compute the line integral:

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C (y^2 dx - x^2 dy).$$

Here $\mathbf{F} = y^2 \mathbf{i} - x^2 \mathbf{j}$.

Parameterize C using $x = t$, $y = 4t^2$, and t goes from 0 to 1.

For differentials, $dx = dt$ and $dy = 8t dt$.

Substituting into the integrand:

$$\mathbf{F} = y^2 \mathbf{i} - x^2 \mathbf{j}$$

$$\mathbf{F} \cdot d\mathbf{r} = (4t^2)^2 dt - t^2(8t dt) = 16t^4 dt - 8t^3 dt = (16t^4 - 8t^3) dt.$$

Integrating:

$$\int_0^1 (16t^4 - 8t^3) dt = \left[\frac{16t^5}{5} - \frac{8t^4}{4} \right]_0^1 = 16 \cdot \frac{1}{5} - 8 \cdot \frac{1}{4} = \frac{16}{5} - 2 = \frac{6}{5}.$$

[]:

3 F as in Prob. 2, C from $(0, 0)$ straight to $(1, 4)$. Compare.

Using same $\mathbf{F} = [y^2, -x^2]$ from previous problem. Now C is the straight line from $(0, 0)$ to $(1, 4)$, i.e., $y = 4x$.

Parameterize: $x = t$, $y = 4t$, t from 0 to 1.

Differentials: $dx = dt$, $dy = 4 dt$.

Substituting into integrand:

$$\mathbf{F} \cdot d\mathbf{r} = (4t)^2 dt - t^2(4 dt) = 16t^2 dt - 4t^2 dt = 12t^2 dt.$$

Integrating:

$$\int_0^1 12t^2 dt = \left[12 \frac{t^3}{3} \right]_0^1 = 12 \cdot \frac{1}{3} = 4.$$

Comparison: The work done differs from problem 2 for same function along the parabolic path, where it was $\frac{6}{5} = 1.2$. Here, it is 4. Thus integral of \mathbf{F} is not path-independent.

4. $F = [xy, x^2y^2]$, C from $(2, 0)$ straight to $(0, 2)$

C is the straight line from $(2, 0)$ to $(0, 2)$, i.e., $y = -x + 2$.

Parameterize: $x = 2 - t$, $y = t$, t from 0 to 2.

Differentials: $dx = -dt$, $dy = dt$.

Substitute:

$$\mathbf{F} \cdot d\mathbf{r} = (2 - t)t(-dt) + (2 - t)^2t^2(dt) = (t^4 - 4t^3 + 5t^2 - 2t) dt.$$

Integrate:

$$\int_0^2 (t^4 - 4t^3 + 5t^2 - 2t) dt = \left[\frac{t^5}{5} - 4 \frac{t^4}{4} + 5 \frac{t^3}{3} - 2 \frac{t^2}{2} \right]_0^2 = \frac{32}{5} - 16 + \frac{40}{3} - 4 = -\frac{4}{15}.$$

[]:

5 F as in Prob. 4, C the quarter-circle from $(2, 0)$ to $(0, 2)$ with center $(0, 0)$

Using $\mathbf{F} = [xy, x^2y^2]$, C is the quarter-circle from $(2, 0)$ to $(0, 2)$, center $(0, 0)$, radius 2.

Parameterize: $\mathbf{r} = [2 \cos t, 2 \sin t]$, t from 0 to $\frac{\pi}{2}$.

Differentials: $dx = -2 \sin t \, dt$, $dy = 2 \cos t \, dt$.

$\mathbf{F} = [4 \cos t \sin t, 16 \cos^2 t \sin^2 t]$.

$\mathbf{F} \cdot d\mathbf{r} = 8 \cos t \sin^2 t (3 - 4 \sin^2 t) \, dt$.

Substitute $u = \sin t$, $du = \cos t \, dt$, u from 0 to 1:

$$\int_0^1 8(3u^2 - 4u^4) \, du = 8 \left(1 - \frac{4}{5}\right) = \frac{8}{5}.$$

[]:

6 $F = [x - y, y - z, z - x]$, $C : \mathbf{r} = [2 \cos t, t, 2 \sin t]$ from $(2, 0, 0)$ to $(2, 2\pi, 0)$

$C : \mathbf{r} = [2 \cos t, 2 \sin t, 0]$, t from 0 to 2π (a closed loop).

Differentials: $dx = -2 \sin t \, dt$, $dy = 2 \cos t \, dt$, $dz = 0$.

$\mathbf{F} = [2 \cos t - 2 \sin t, 2 \sin t, -2 \cos t]$.

$\mathbf{F} \cdot d\mathbf{r} = 4 \sin^2 t \, dt$.

Integrate using $\sin^2 t = \frac{1 - \cos 2t}{2}$:

$$\int_0^{2\pi} 4 \sin^2 t \, dt = 2 \left[t - \frac{\sin 2t}{2} \right]_0^{2\pi} = 4\pi.$$

[]:

7 $F = [x^2, y^2, z^2]$, $C : \mathbf{r} = [\cos t, \sin t, e^t]$ from $(1, 0, 1)$ to $(1, 0, e^{2\pi})$. Sketch C .

Given $\mathbf{F} = [x^2, y^2, z^2]$, $C : \mathbf{r} = [\cos t, \sin t, e^t]$, t from 0 to 2π .

Differentials: $dx = -\sin t \, dt$, $dy = \cos t \, dt$, $dz = e^t \, dt$.

$\mathbf{F} = [\cos^2 t, \sin^2 t, e^{2t}]$. $\mathbf{F} \cdot d\mathbf{r} = \sin t \cos t (\sin t - \cos t) \, dt + e^{3t} \, dt$.

Integrate: First term is 0 (closed loop in $u = \sin t$). Second term:

$$\int_0^{2\pi} e^{3t} \, dt = \frac{e^{6\pi} - 1}{3}.$$

Sketch: The curve is a spiral on the cylinder $x^2 + y^2 = 1$, with z increasing from 1 to $e^{2\pi}$ at the speed of exponential function.

```

[3]: import numpy as np
import matplotlib.pyplot as plt
from mpl_toolkits.mplot3d import Axes3D

# Create parameter t from 0 to 2
t = np.linspace(0, 2*np.pi, 100)

# Calculate x, y, z coordinates
x = np.cos(t)
y = np.sin(t)
z = np.exp(t)

# Create 3D plot
fig = plt.figure(figsize=(10, 8))
ax = fig.add_subplot(111, projection='3d')

# Plot the curve
ax.plot(x, y, z, 'b-', label='r(t) = [cos t, sin t, e^t]')

# Plot start and end points
ax.scatter([1], [0], [1], color='green', s=100, label='Start (1,0,1)')
ax.scatter([1], [0], [np.exp(2*np.pi)], color='red', s=100, label='End ↴
↵(1,0,e^(2))')

# Set labels
ax.set_xlabel('X axis')
ax.set_ylabel('Y axis')
ax.set_zlabel('Z axis')
ax.set_title('Parametric Curve C: r(t) = [cos t, sin t, e^t]')

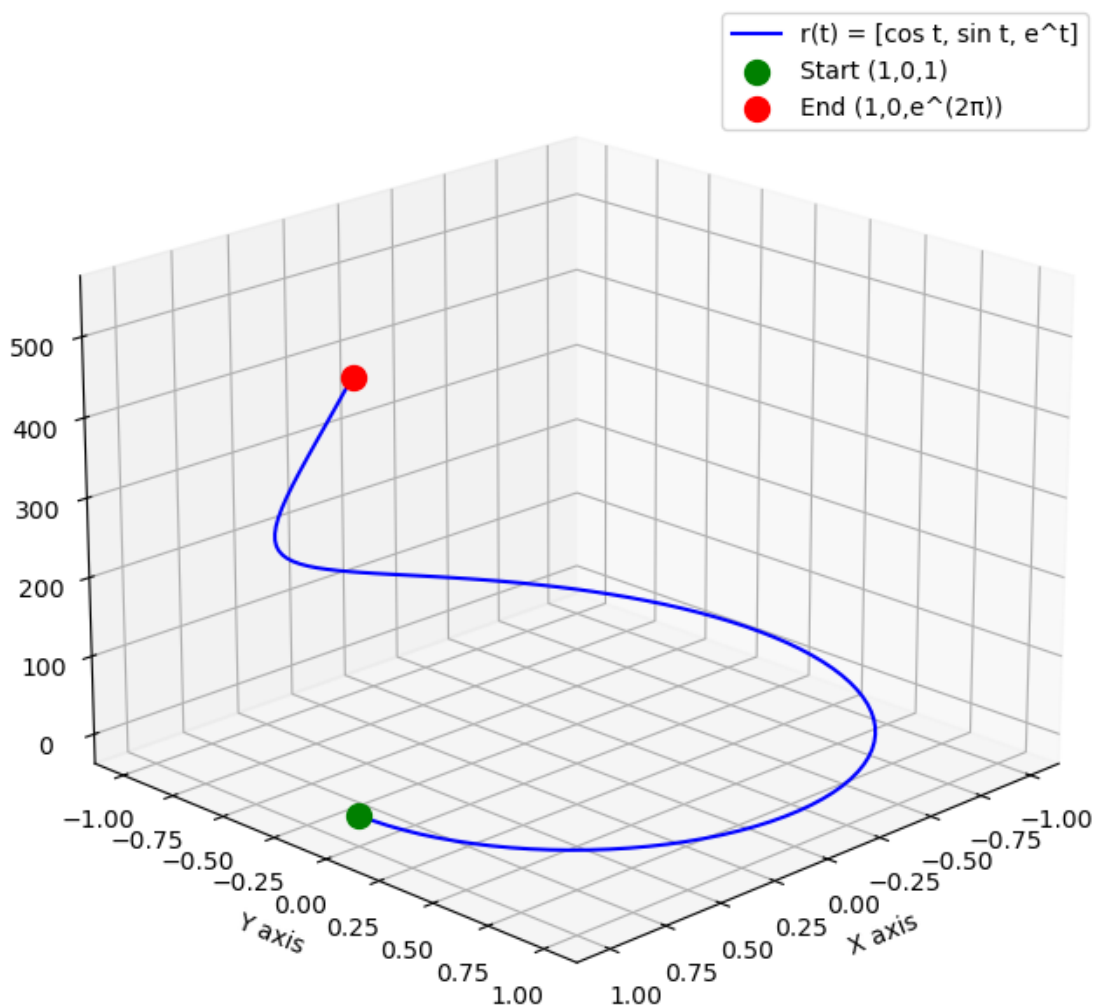
# Add legend
ax.legend()

# Adjust the view angle
ax.view_init(elev=20, azimuth=45)

plt.show()

```

Parametric Curve C: $\mathbf{r}(t) = [\cos t, \sin t, e^t]$



[]:

8 Let $F = [e^x, \cosh y, \sinh z]$, $C : \mathbf{r} = [t, t^2, t^3]$ from $(0, 0, 0)$ to $(\frac{1}{2}, \frac{1}{4}, \frac{1}{8})$. Sketch C .

For \mathbf{r} from $(0, 0, 0)$ to $(\frac{1}{2}, \frac{1}{4}, \frac{1}{8})$, t is from 0 to $\frac{1}{2}$.

Differentials: $dx = dt$, $dy = 2t dt$, $dz = 3t^2 dt$. (Or, $d\mathbf{r} = [1, 2t, 3t^2]dt$.)

$\mathbf{F} \cdot d\mathbf{r} = e^t + 2t \cosh(t^2) + 3t^2 \sinh(t^3) dt$.

Integrate:

$$e^{\frac{1}{2}} - 1 + \int_0^{\frac{1}{4}} \cosh u \, du + \int_0^{\frac{1}{8}} \sinh v \, dv = e^{\frac{1}{2}} + \sinh \frac{1}{4} + \cosh \frac{1}{8} - 2.$$

Sketch: The curve follows $y = x^2$, $z = x^3$, a winding 3D curve from $(0, 0, 0)$ to $(\frac{1}{2}, \frac{1}{4}, \frac{1}{8})$.

```
[1]: # Import required libraries
import numpy as np
import matplotlib.pyplot as plt
from mpl_toolkits.mplot3d import Axes3D

# Define the parameter t
t = np.linspace(0, 0.5, 100) # 100 points from t=0 to t=0.5

# Parametric equations
x = t
y = t**2
z = t**3

# Create a 3D plot
fig = plt.figure(figsize=(8, 6))
ax = fig.add_subplot(111, projection='3d')

# Plot the curve
ax.plot(x, y, z, label=r'$\mathbf{r}(t) = [t, t^2, t^3]$', color='blue')

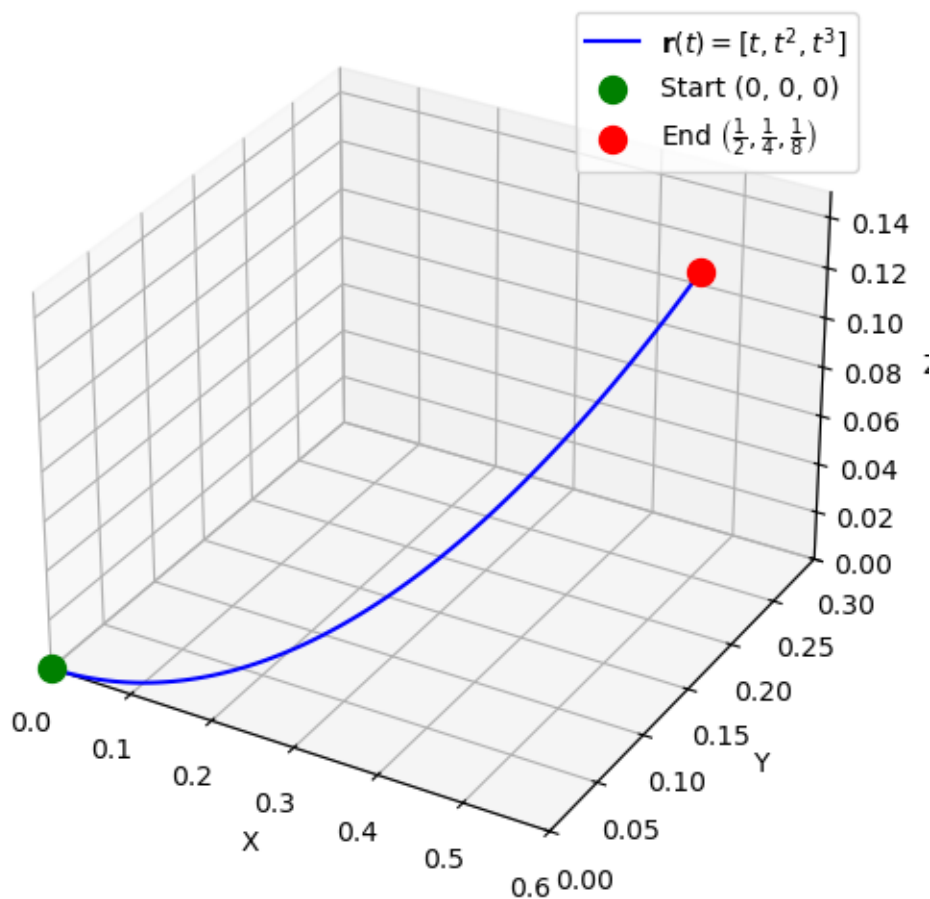
# Mark the start and end points
ax.scatter([0], [0], [0], color='green', s=100, label='Start (0, 0, 0)')
ax.scatter([0.5], [0.25], [0.125], color='red', s=100, label=r'End_
↳ $\left(\frac{1}{2}, \frac{1}{4}, \frac{1}{8}\right)$')

# Set labels and title
ax.set_xlabel('X')
ax.set_ylabel('Y')
ax.set_zlabel('Z')
ax.set_title('Sketch of the Curve C')
ax.legend()

# Set axis limits for better visualization
ax.set_xlim(0, 0.6)
ax.set_ylim(0, 0.3)
ax.set_zlim(0, 0.15)

# Show the plot
plt.show()
```

Sketch of the Curve C



[]:

9 Let $F = [x + y, y + z, z + x]$, $C : \mathbf{r} = [2t, 5t, t]$ from $t = 0$ to 1. Also from $t = -1$ to 1.

Given $\mathbf{F} = [x + y, y + z, z + x]$, $C : \mathbf{r} = [2t, 5t, t]$.

Differentials: $dx = 2 dt$, $dy = 5 dt$, $dz = dt$.

$\mathbf{F} = [7t, 6t, 3t]$.

$d\mathbf{r} = [2, 5, 1]dt$.

$\mathbf{F} \cdot d\mathbf{r} = 47t dt$.

Integrate from $t = 0$ to 1:

$$\int_0^1 47t dt = \frac{47}{2}.$$

From $t = -1$ to 1 :

$$\int_{-1}^1 47t \, dt = 0.$$

[]:

10 Let $F = [x, -z, 2y]$ from $(0, 0, 0)$ straight to $(1, 1, 0)$, then to $(1, 1, 1)$, back to $(0, 0, 0)$.

C is a closed triangular loop: $(0, 0, 0)$ to $(1, 1, 0)$, to $(1, 1, 1)$, to $(0, 0, 0)$.

Side 1:

$x = t, y = t, z = 0, t$ from 0 to 1 .

$$\mathbf{F} \cdot d\mathbf{r} = t \, dt$$

$$, \int_0^1 t \, dt = \frac{1}{2}.$$

Side 2:

$x = 1, y = 1, z = t, t$ from 0 to 1 .

$$\mathbf{F} \cdot d\mathbf{r} = -t \, dt$$

$$, \int_0^1 -t \, dt = -\frac{1}{2}.$$

Side 3:

$x = 1 - t, y = 1 - t, z = 1 - t, t$ from 0 to 1 .

$$\mathbf{F} \cdot d\mathbf{r} = (-4 + 4t) \, dt$$

$$, \int_0^1 (-4 + 4t) \, dt = -2.$$

$$\text{Total: } \frac{1}{2} - \frac{1}{2} - 2 = -2.$$

[]:

11 Let $F = [e^{-x}, e^{-y}, e^{-z}]$, $C : \mathbf{r} = [t, t^2, t]$ from $(0, 0, 0)$ to $(2, 4, 2)$. Sketch C .

For the given endpoints, t is from 0 to 2 .

Differentials: $dx = dt, dy = 2t \, dt, dz = dt$.

Integrand:

$$\mathbf{F} \cdot d\mathbf{r} = 2e^{-t} + 2te^{-t^2} \, dt.$$

To integrate:

$$2 \int_0^2 e^{-t} \, dt + 2 \int_0^2 te^{-t^2} \, dt = 2(1 - e^{-2}) + (1 - e^{-4}) = 3 - 2e^{-2} - e^{-4}.$$

Sketch: The curve follows $y = x^2, z = x$, a parabolic path in 3D from $(0, 0, 0)$ to $(2, 4, 2)$.

```
[2]: # Import required libraries
import numpy as np
import matplotlib.pyplot as plt
from mpl_toolkits.mplot3d import Axes3D

# Define the parameter t
t = np.linspace(0, 2, 100) # 100 points from t=0 to t=2

# Parametric equations
x = t
y = t**2
z = t

# Create a 3D plot
fig = plt.figure(figsize=(8, 6))
ax = fig.add_subplot(111, projection='3d')

# Plot the curve
ax.plot(x, y, z, label=r'$\mathbf{r}(t) = [t, t^2, t]$', color='blue')

# Mark the start and end points
ax.scatter([0], [0], [0], color='green', s=100, label='Start (0, 0, 0)')
ax.scatter([2], [4], [2], color='red', s=100, label='End (2, 4, 2)')

# Set labels and title
ax.set_xlabel('X')
ax.set_ylabel('Y')
ax.set_zlabel('Z')
ax.set_title('Sketch of the Curve C')
ax.legend()

# Set axis limits for better visualization
ax.set_xlim(0, 2.5)
ax.set_ylim(0, 4.5)
ax.set_zlim(0, 2.5)

# Show the plot
plt.show()
```

Sketch of the Curve C

