kreyszig-10-1

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0.1 Erwin Kresyzig 10.1 Problems

0.1.1 2-11 LINE INTEGRAL. WORK.

Calculate

$$\int_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r}$$

for the given data. If $\mathbf{F}$ is a force, this gives the work done by the force in the displacement along C. Show the details.

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[ ]:
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We repeat for clarity the definition of the "line" integral $\mathbf{F} \cdot d\mathbf{r}$ along the curve C from first principles.

The integral along C is defined as:

$$\int_C \mathbf{F} \cdot d\mathbf{r}$$

where $\mathbf{F}$ and $\mathbf{r}$ are vector functions to be expanded in vector components as:

$\mathbf{F} = [F_1, F_2, F_3] = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}$,

$\mathbf{r}(t) = [x(t), y(t), z(t)] = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$,

and the differential \$ d r = $[dx, dy, dz]$.\$

$\mathbf{F} = [F_1, F_2, F_3]$ is a vector function in the 3-dimensional space to be evaluated along the path C defined by $\mathbf{r}(t)$, i. e., $\mathbf{F}$ is \$ $([F\_1, F\_2, F\_3])\,(\mathbf{r}(t))$.\$

The differential $d\mathbf{r}$ may be calculated by chain rule:

$$d\mathbf{r} = \frac{d\mathbf{r}(t)}{dt}dt = \mathbf{r}'(t)dt.$$

The components are,

$$\frac{d}{dt}\mathbf{r}(t) = \frac{d}{dt}\left[x(t), y(t), z(t)\right] = \left[\frac{d}{dt}x(t), \frac{d}{dt}y(t), \frac{d}{dt}z(t)\right] = [x'(t), y'(t), z'(t)]$$

or more concisely:

$$\mathbf{d}r = [dx, dy, dz] = [x'(t), y'(t), z'(t)]\, dt.$$

Thus the dot product in the integrand is, along C,

$$\begin{aligned}\mathbf{F} \cdot d\mathbf{r} &= [F_1, F_2, F_3] \cdot [dx, dy, dz] \\ &= ([F_1, F_2, F_3])\, (\mathbf{r}(t)) \cdot [x'(t), y'(t), z'(t)]\, dt \\ &= ((F_1(\mathbf{r}(t)) \cdot x'(t) + (F_2(\mathbf{r}(t))) \cdot y'(t) + (F_3(\mathbf{r}(t))) \cdot z'(t))\, dt\end{aligned}$$

Thus the line integral in in the 3-dimensional space is reduced to an integral in one variable along a closed interval in the real line, say, $[a, b]$.

[ ]:

2. $F = [y^2, -x^2]$, $C : y = 4x^2$ from $(0, 0)$ to $(1, 4)$

Given $\mathbf{F} = [y^2, -x^2]$ and $C : y = 4x^2$ from $(0, 0)$ to $(1, 4)$, we have to compute the line integral:

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C (y^2\, dx - x^2\, dy).$$

Here $ F = y^2 i - x^2 j.$

So $F_1 = y^2$ and $F_2 = -x^2$.

Parameterize C using $x = t$, so $y = 4t^2$, and t goes from 0 to 1.

For differentials, $dx = dt$ and $dy = 8tdt$.

Substituting into the integrand: $ F = y^2 i - x^2 j$ and expanding dot product:

$$\begin{aligned}\mathbf{F} \cdot d\mathbf{r} &= F_1 dx + F_2 dy = y^2 dx - x^2 dy \\ &= (4t^2)^2\, dt - t^2(8t\, dt) \\ &= 16t^4\, dt - 8t^3\, dt \\ &= (16t^4 - 8t^3)\, dt\end{aligned}$$

Integrating:

$$\begin{aligned}\int_0^1 (16t^4 - 8t^3)\, dt &= \left[\frac{16t^5}{5} - \frac{8t^4}{4}\right]_0^1 \\ &= \frac{16}{5} - \frac{8}{4} \\ &= \frac{16}{5} - 2 = \frac{6}{5}.\end{aligned}$$

[ ]:

3 F as in Prob. 2, C from $(0,0)$ straight to $(1,4)$. Compare.

Using same $\mathbf{F} = [y^2, -x^2]$ from previous problem. Now C is the straight line from $(0,0)$ to $(1,4)$, i.e., $y = 4x$.

Parameterize: $x = t$, $y = 4t$, t from 0 to 1.

Differentials: $dx = dt$, $dy = 4\,dt$.

Substituting into integrand:

$$\begin{aligned} \mathbf{F} \cdot d\mathbf{r} &= (4t)^2\,dt - t^2(4\,dt) \\ &= 16t^2\,dt - 4t^2\,dt \\ &= 12t^2\,dt \end{aligned}$$

Integrating:

$$\int_0^1 12t^2\,dt = \left[12\frac{t^3}{3}\right]_0^1 = 12 \cdot \frac{1}{3} = 4.$$

Comparison: The work done differs from problem 2 for same function along the parabolic path, where it was $\frac{6}{5} = 1.2$. Here, it is 4. Thus integral of $\mathbf{F}$ is not path-independent.

[ ]:

4. $F = [xy, x^2y^2]$, C from $(2,0)$ straight to $(0,2)$

C is the straight line from $(2,0)$ to $(0,2)$, i.e., $y = -x + 2$.

Parameterize: $x = 2 - t$, $y = t$, t from 0 to 2.

Differentials: $dx = -dt$, $dy = dt$.

Substitute:

$$\begin{aligned} \mathbf{F} \cdot d\mathbf{r} &= (2-t)t(-dt) + (2-t)^2t^2(dt) \\ &= (t^4 - 4t^3 + 5t^2 - 2t)\,dt \end{aligned}$$

Integrate:

$$\begin{aligned} \int_0^2 (t^4 - 4t^3 + 5t^2 - 2t)\,dt &= \left[\frac{t^5}{5} - \frac{4t^4}{4} + \frac{5t^3}{3} - \frac{2t^2}{2}\right]_0^2 \\ &= \frac{32}{5} - 16 + \frac{40}{3} - 4 \\ &= -\frac{4}{15}. \end{aligned}$$

[ ]:

5 F as in Prob. 4, C the quarter-circle from $(2,0)$ to $(0,2)$ with center $(0,0)$

Using $\mathbf{F} = [xy, x^2y^2]$, C is the quarter-circle from $(2,0)$ to $(0,2)$, center $(0,0)$, radius 2.

Parameterize: $\mathbf{r} = [2\cos t, 2\sin t]$, t from 0 to $\frac{\pi}{2}$.

Differentials: $dx = -2\sin t\,dt$, $dy = 2\cos t\,dt$.

Thus

$$\mathbf{F} = [4\cos t\sin t, 16\cos^2 t\sin^2 t].$$

$$\mathbf{F}\cdot d\mathbf{r} = 8\cos t\sin^2 t(-1 + 4\cos^2 t)\,dt$$

Substitute $u = \sin t$, $du = \cos t\,dt$. Thus $du = \cos t dt$ and $u^2 = \sin^2 t$ and $\cos^2 t = 1 - u^2$.

Grouping terms in $8\cos t\sin^2 t(-1 + 4\cos^2 t)\,dt$, we get:

$$8\cos t\sin^2 t((-1 + 4\cos^2 t))\,dt = 3 - 4u^2u^2(3 - 4u^2)\,du.$$

For the substitution, when $t = 0, \pi/2$ resp., corresponding $u = 0, 1$ resp. as $t = 0 \implies u = \sin 0 = 0$ and $t = \pi/2 \implies u = \sin\pi/2 = 1$.

Integrating with u from 0 to 1:

$$\begin{aligned}\int_0^1 8(3u^2 - 4u^4)\,du &= 8\left[3\frac{u^3}{3} - 4\frac{u^5}{5}\right]_0^1 \\ &= 8\left(1 - \frac{4}{5}\right) \\ &= \frac{8}{5}.\end{aligned}$$

[ ]:

6 $F = [x - y, y - z, z - x]$, $C : \mathbf{r} = [2\cos t, t, 2\sin t]$ from $(2,0,0)$ to $(2, 2\pi, 0)$

$C : \mathbf{r} = [2\cos t, t, 2\sin t]$, t from 0 to 2π.

Differentials: $dx = -2\sin t\,dt$, $dy = dt$, $dz = 2\cos t\,dt$.

Thus

$$\mathbf{F} = [2\cos t - t, t - 2\sin t, 2\sin t - 2\cos t]$$

and

$$
\begin{aligned}
\mathbf{F}\cdot d\mathbf{r} &= \left((2\cos t - t)(-2\sin t) + (t - 2\sin t) + (2\sin t - 2\cos t)(2\cos t)\right)\,dt \\
&= \left(t(2\sin t + 1) - 2\sin t - 4\cos^2 t\right)\,dt.
\end{aligned}
$$

Thus to compute the line integral:

$$
\begin{aligned}
\int_C \mathbf{F}\cdot d\mathbf{r} &= \int_0^{2\pi} \left(t(2\sin t + 1) - 2\sin t - 4\cos^2 t\right)\,dt \\
&= \int_0^{2\pi} (2t\sin t + t - 2\sin t - 4\cos^2 t)\,dt \\
&= 2\int_0^{2\pi} t\sin t\,dt + \int_0^{2\pi} t\,dt - 2\int_0^{2\pi} \sin t\,dt - 4\int_0^{2\pi} \cos^2 t\,dt \\
&= 2[-t\cos t + \sin t]_0^{2\pi} + \left[\frac{t^2}{2}\right]_0^{2\pi} - 2[-\cos t]_0^{2\pi} - 4\left[\frac{t}{2} + \frac{\sin 2t}{4}\right]_0^{2\pi} \\
&= 2(-2\pi\cdot 1 + 0 - 0) + \left(\frac{(2\pi)^2}{2} - 0\right) - 2(-1+1) - 4\left(\frac{2\pi}{2} + 0 - 0\right) \\
&= 2(-2\pi) + 2\pi^2 - 0 - 4\pi \\
&= -4\pi + 2\pi^2 - 4\pi \\
&= 2\pi^2 - 8\pi
\end{aligned}
$$

[ ]:

7 $F = [x^2, y^2, z^2]$, $C : \mathbf{r} = [\cos t, \sin t, e^t]$ from $(1,0,1)$ to $(1,0,e^{2\pi})$. Sketch C.

Given $\mathbf{F} = [x^2, y^2, z^2]$, $C : \mathbf{r} = [\cos t, \sin t, e^t]$, t from 0 to 2π.

Differentials: $dx = -\sin t\,dt$, $dy = \cos t\,dt$, $dz = e^t\,dt$.

$\mathbf{F} = [\cos^2 t, \sin^2 t, e^{2t}]$.

$\mathbf{F}\cdot d\mathbf{r} = \sin t\cos t(\sin t - \cos t)\,dt + e^{3t}\,dt$.

To integrate the first term: The integrand and integral are periodic functions of period 2π, thus, evaluating, the definite integral between 0 and 2π is 0.

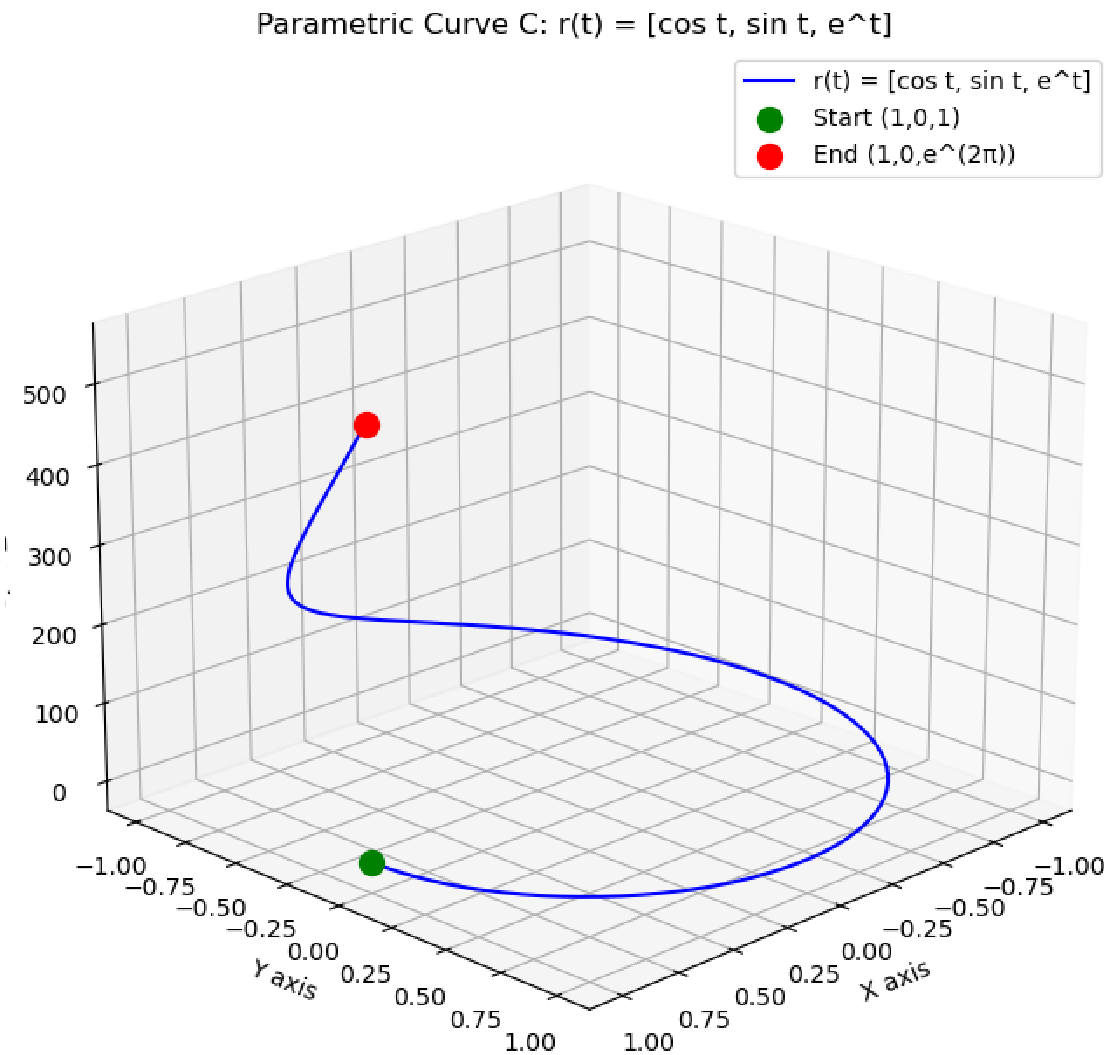
Another way to see is to integrate $\sin^2 t\cos t$ and $\sin t\cos^2 t$ separately via appropriate substitutions, and observe that the upper and lower limits are same (= 0.)

To integrate second term:

$$
\int_0^{2\pi} e^{3t}\,dt = \left[\frac{e^{3t}}{3}\right]_0^1 = \frac{e^{6\pi} - 1}{3}.
$$

Sketch: The curve is a spiral on the cylinder $x^2 + y^2 = 1$, with z increasing from 1 to $e^{2\pi}$ at the speed of exponential function.

[2]:
```
import numpy as np
import matplotlib.pyplot as plt
from mpl_toolkits.mplot3d import Axes3D

# Create parameter t from 0 to 2
t = np.linspace(0, 2*np.pi, 100)

# Calculate x, y, z coordinates
x = np.cos(t)
y = np.sin(t)
z = np.exp(t)

# Create 3D plot
fig = plt.figure(figsize=(10, 8))
ax = fig.add_subplot(111, projection='3d')

# Plot the curve
ax.plot(x, y, z, 'b-', label='r(t) = [cos t, sin t, e^t]')

# Plot start and end points
ax.scatter([1], [0], [1], color='green', s=100, label='Start (1,0,1)')
ax.scatter([1], [0], [np.exp(2*np.pi)], color='red', s=100, label='End␣
↪(1,0,e^(2 ))')

# Set labels
ax.set_xlabel('X axis')
ax.set_ylabel('Y axis')
ax.set_zlabel('Z axis')
ax.set_title('Parametric Curve C: r(t) = [cos t, sin t, e^t]')

# Add legend
ax.legend()

# Adjust the view angle
ax.view_init(elev=20, azim=45)

plt.show()
```



[ ]:

8 Let $F = [e^x, \cosh y, \sinh z]$, $C : \mathbf{r} = [t, t^2, t^3]$ from $(0,0,0)$ to $\left(\frac{1}{2}, \frac{1}{4}, \frac{1}{8}\right)$. Sketch C.

Let $\mathbf{F} = [e^x, \cosh y, \sinh z]$, $C : \mathbf{r}(t) = [t, t^2, t^3]$ from $(0,0,0)$ to $\left(\frac{1}{2}, \frac{1}{4}, \frac{1}{8}\right)$, so t ranges from 0 to $\frac{1}{2}$.

Differentials: $dx = dt$, $dy = 2t\,dt$, $dz = 3t^2\,dt$.

Then, $d\mathbf{r} = [1, 2t, 3t^2]\,dt$, and along C, $\mathbf{F} = [e^t, \cosh(t^2), \sinh(t^3)]$, so:

$$
\begin{aligned}
\int_C \mathbf{F} \cdot d\mathbf{r} &= \int_0^{\frac{1}{2}} [e^t + 2t\cosh(t^2) + 3t^2\sinh(t^3)]\, dt \\
&= \int_0^{\frac{1}{2}} e^t\, dt + \int_0^{\frac{1}{4}} \cosh u\, du + \int_0^{\frac{1}{8}} \sinh v\, dv \\
&= [e^t]_0^{\frac{1}{2}} + [\sinh u]_0^{\frac{1}{4}} + [\cosh v]_0^{\frac{1}{8}} \\
&= (e^{\frac{1}{2}} - 1) + \sinh\frac{1}{4} + (\cosh\frac{1}{8} - 1) \\
&= e^{\frac{1}{2}} + \sinh\frac{1}{4} + \cosh\frac{1}{8} - 2.
\end{aligned}
$$

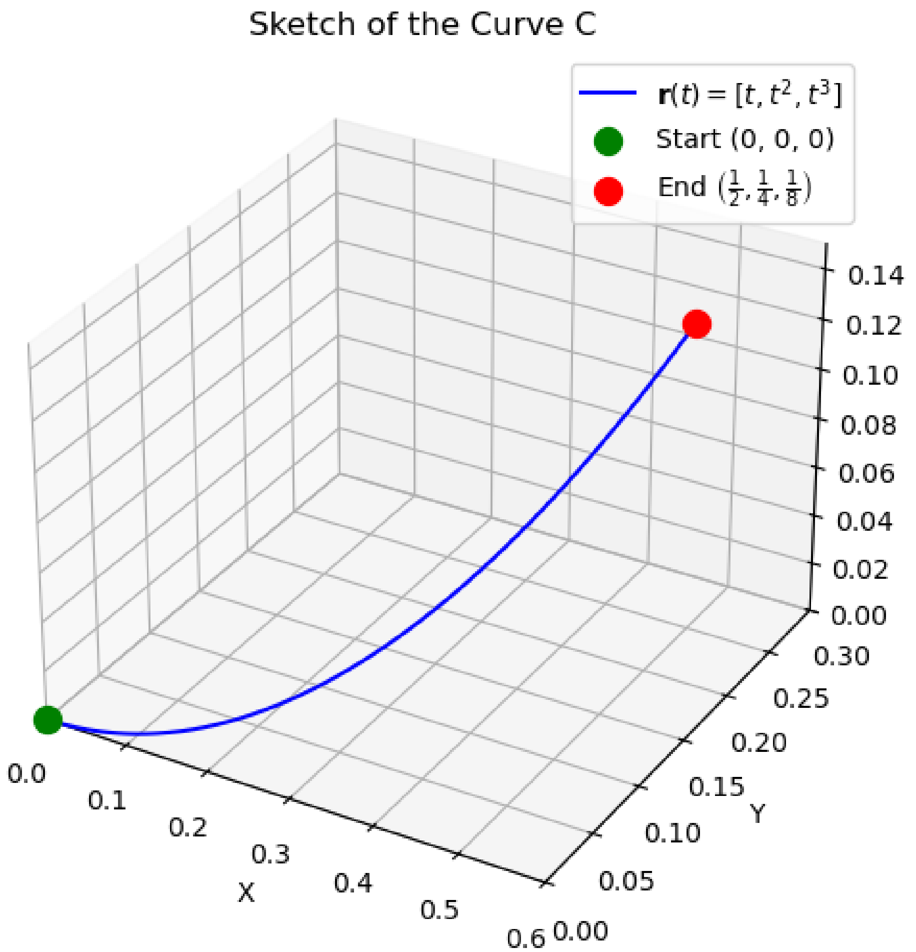
(Substitutions: $u = t^2$, $v = t^3$.)

Sketch: The curve follows $y = x^2$, $z = x^3$, a winding 3D curve from $(0, 0, 0)$ to $\left(\frac{1}{2}, \frac{1}{4}, \frac{1}{8}\right)$. The projections to xy and xz axes give $y = x^2$ and $z = x^3$.

[3]:

```python
# Import required libraries
import numpy as np
import matplotlib.pyplot as plt
from mpl_toolkits.mplot3d import Axes3D

# Define the parameter t
t = np.linspace(0, 0.5, 100)  # 100 points from t=0 to t=0.5

# Parametric equations
x = t
y = t**2
z = t**3

# Create a 3D plot
fig = plt.figure(figsize=(8, 6))
ax = fig.add_subplot(111, projection='3d')

# Plot the curve
ax.plot(x, y, z, label=r'$\mathbf{r}(t) = [t, t^2, t^3]$', color='blue')

# Mark the start and end points
ax.scatter([0], [0], [0], color='green', s=100, label='Start (0, 0, 0)')
ax.scatter([0.5], [0.25], [0.125], color='red', s=100, label=r'End␣
↪$\left(\frac{1}{2}, \frac{1}{4}, \frac{1}{8}\right)$')

# Set labels and title
ax.set_xlabel('X')
ax.set_ylabel('Y')
ax.set_zlabel('Z')
ax.set_title('Sketch of the Curve C')
ax.legend()
```

```python
# Set axis limits for better visualization
ax.set_xlim(0, 0.6)
ax.set_ylim(0, 0.3)
ax.set_zlim(0, 0.15)

# Show the plot
plt.show()
```



[ ]:

9 Let $F = [x + y, y + z, z + x]$, $C : \mathbf{r} = [2t, 5t, t]$ from $t = 0$ to 1. Also from $t = -1$ to 1.

Given $\mathbf{F} = [x + y, y + z, z + x]$, $C : \mathbf{r} = [2t, 5t, t]$.

Differentials: $dx = 2\,dt$, $dy = 5\,dt$, $dz = dt$.

$\mathbf{F} = [7t, 6t, 3t]$.

$d\mathbf{r} = [2, 5, 1]dt$.

$\mathbf{F} \cdot d\mathbf{r} = 47t\, dt$.

Integrate from $t = 0$ to 1:

$$\int_0^1 47t\, dt = \frac{47}{2}.$$

From $t = -1$ to 1:

$$\int_{-1}^1 47t\, dt = 0.$$

[ ]:

10 Let $F = [x, -z, 2y]$ from $(0, 0, 0)$ straight to $(1, 1, 0)$, then to $(1, 1, 1)$, back to $(0, 0, 0)$.

C is a closed triangular loop: $(0, 0, 0)$ to $(1, 1, 0)$, to $(1, 1, 1)$, to $(0, 0, 0)$.

Side 1:

$\mathbf{r}(t) = [t, t, 0]$, t from 0 to 1, $d\mathbf{r} = [1, 1, 0]\, dt$, $\mathbf{F} = [t, 0, 2t]$, so $\mathbf{F} \cdot d\mathbf{r} = t\, dt$. Then,

$$\int_0^1 t\, dt = \frac{1}{2}.$$

Side 2:

$\mathbf{r}(t) = [1, 1, t]$, t from 0 to 1, $d\mathbf{r} = [0, 0, 1]\, dt$, $\mathbf{F} = [1, -t, 2]$, so $\mathbf{F} \cdot d\mathbf{r} = 2\, dt$. Then,

$$\int_0^1 2\, dt = 2.$$

Side 3:

$\mathbf{r}(t) = [1 - t, 1 - t, 1 - t]$, t from 0 to 1, $d\mathbf{r} = [-1, -1, -1]\, dt$, $\mathbf{F} = [1 - t, -(1 - t), 2(1 - t)]$, so:

$$\begin{aligned} \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} &= (1 - t)(-1) - (1 - t)(-1) + 2(1 - t)(-1) \\ &= 2(t - 1), \end{aligned}$$

and

$$\int_0^1 (2t - 2)\, dt = [t^2 - 2t]_0^1 = -1.$$

[ ]:

11 Let $F = [e^{-x}, e^{-y}, e^{-z}]$, $C : \mathbf{r} = [t, t^2, t]$ from $(0, 0, 0)$ to $(2, 4, 2)$. Sketch C.

For the given endpoints, t is from 0 to 2.

Differentials: $dx = dt$, $dy = 2t\,dt$, $dz = dt$.

Integrand:

$\mathbf{F} \cdot d\mathbf{r} = 2e^{-t} + 2te^{-t^2}\,dt$.

To integrate:

$$2\int_0^2 e^{-t}\,dt + \int_0^2 2te^{-t^2}\,dt = 2(1 - e^{-2}) + (1 - e^{-4}) = 3 - 2e^{-2} - e^{-4}.$$

Substituting $u = -t^2$ in second integral; thus $du = -2tdt$, corresponding range of u from 0 to -4 etc.

Sketch: The curve follows $y = x^2$, $z = x$, a parabolic path in 3D from $(0, 0, 0)$ to $(2, 4, 2)$.

[4]:
```python
# Import required libraries
import numpy as np
import matplotlib.pyplot as plt
from mpl_toolkits.mplot3d import Axes3D

# Define the parameter t
t = np.linspace(0, 2, 100)  # 100 points from t=0 to t=2

# Parametric equations
x = t
y = t**2
z = t

# Create a 3D plot
fig = plt.figure(figsize=(8, 6))
ax = fig.add_subplot(111, projection='3d')

# Plot the curve
ax.plot(x, y, z, label=r'$\mathbf{r}(t) = [t, t^2, t]$', color='blue')

# Mark the start and end points
ax.scatter([0], [0], [0], color='green', s=100, label='Start (0, 0, 0)')
ax.scatter([2], [4], [2], color='red', s=100, label='End (2, 4, 2)')

# Set labels and title
ax.set_xlabel('X')
ax.set_ylabel('Y')
ax.set_zlabel('Z')
ax.set_title('Sketch of the Curve C')
```

```python
ax.legend()

# Set axis limits for better visualization
ax.set_xlim(0, 2.5)
ax.set_ylim(0, 4.5)
ax.set_zlim(0, 2.5)

# Show the plot
plt.show()
```