Advanced Calculus, MATH-4600

Exam #2

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1. Let D be the region contained between the paraboloid $z = x^2 + y^2 - 1$ on the bottom and the paraboloid $z = 1 - x^2 - y^2$ on the top. Evauate the integral $\iiint_D \sqrt{x^2+y^2}^3 \, dxdydz$.

2. Compute the area of the surface S expressed as $z = xy$, with (x, y) residing in the disk $x^2 + y^2 \leq 3$.

3. Evaluate the integral $\iint_S \nabla \times \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F}(x, y, z) = (-y, x, \cos xyz)$ and S is the part of the paraboloid $z = x^2 + y^2$ that lies below the plane $z = 1$. The surface S is oriented so that its normal $\mathbf{n}$ points up on the paraboloid.

4. Compute the integral $\int_C \mathbf{F} \cdot d\mathbf{r}$, where C is the curve parametrized by the expression $\mathbf{r}(t) = \big(\cos(\pi t^2), \sin(\pi t^2), \pi t^2\big)$ with $0 \leq t \leq 2$, and $\mathbf{F}(x, y, z) = (3x^2y^2z + yz, 2x^3yz + xz, x^3y^2 + xy)$.

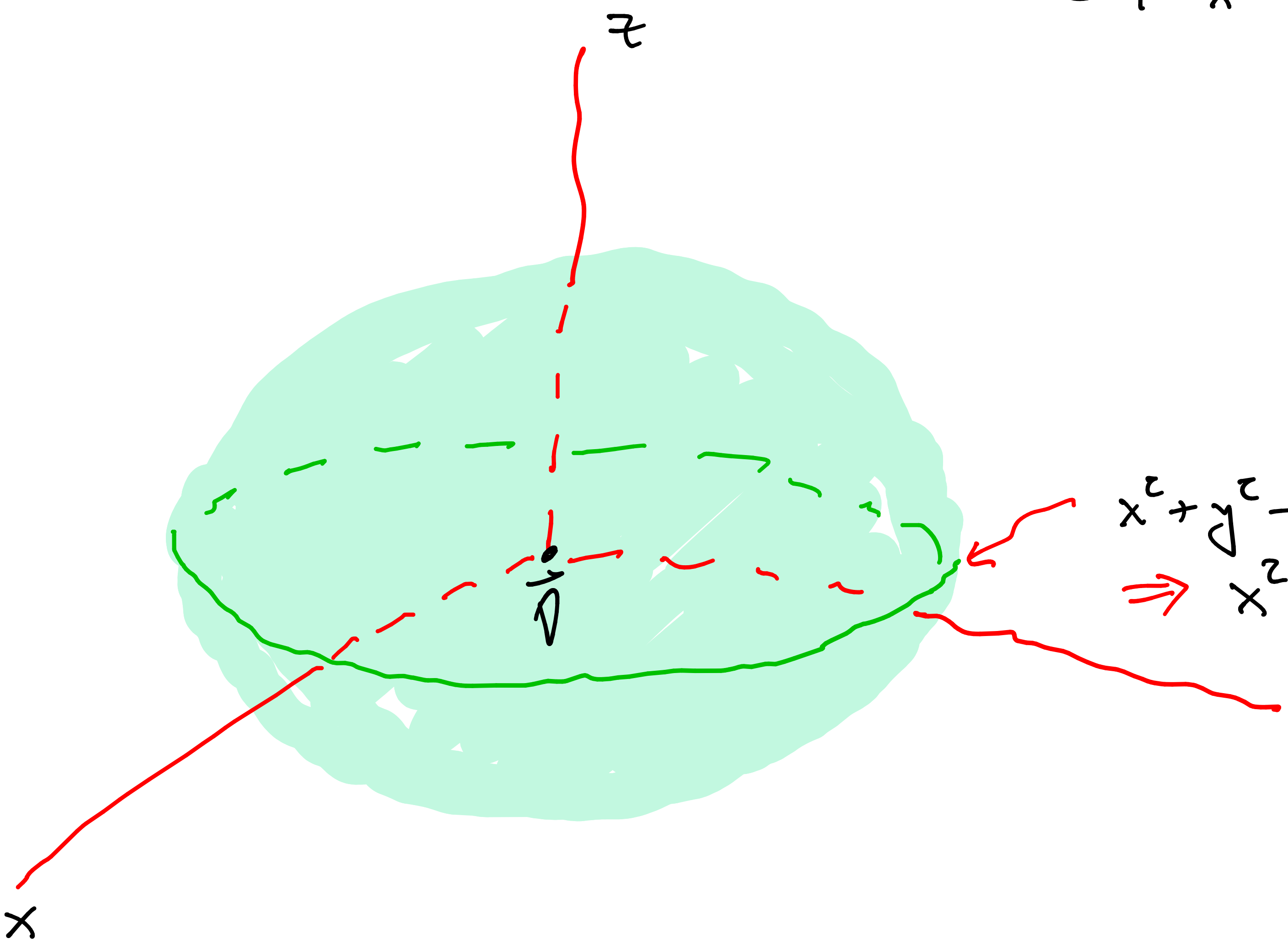
5. Find the extremal curve $y(x)$ of the functional $\int_0^1 \left[(y')^2 + 12xy \right] dx$, which satisfies the boundary conditions $y(0) = 0$ and $y(1) = 1$.

6. Find all the extremal curves $y(x)$ of the functional $\int_0^\pi (y')^2 \, dx$, which satisfies the boundary conditions $y(0) = y(\pi) = 0$, provided that $y(x)$ satisfies the constraint $\int_0^\pi y^2 \, dx = 1$. From among these curves, is there one that gives a maximum or a minimum? If yes, which one?

HINT: Just using Euler's equation directly may be easier than using the conserved quantity.

EXAM #2

1.) $I = \iiint\limits_D f\, dV \quad f = \sqrt{x^2+y^2}^{\,3} \quad D = \{(x,y,z),\ x^2+y^2-1 \le z \le$
$\le 1-x^2-y^2\}$



$x^2+y^2-1 = 1-x^2-y^2$
$\Rightarrow x^2+y^2 = 1$

(i) EASIEST: $I = 2\pi \int_0^1 r\,dr \int_{r^2-1}^{1-r^2} r^3\, dz =$

$= 2\pi \int_0^1 r^4 dr\ z\Big|_{r^2-1}^{1-r^2} = 4\pi \int_0^1 r^4(1-r^2)\, dr =$

$= 4\pi \left(\frac{r^5}{5} - \frac{r^7}{7}\right)\Big|_0^1 = 4\pi\left(\frac{1}{5}-\frac{1}{7}\right) =$

$= \frac{8\pi}{35}$

(ii) $-1+x^2+y^2 \le z \le 1-x^2-y^2 \Rightarrow |z| \le 1-r^2 \Rightarrow r^2 \le 1-|z|$

$$I = 2\pi \int_{-1}^{1} dz \int_0^{\sqrt{1-|z|}} r^4\, dr = \text{(BY SYMMETRY)}$$

$$4\pi \int_0^1 dz \int_0^{\sqrt{1-z}} r^4\, dr = \frac{4\pi}{5} \int_0^1 \sqrt{1-z}^{\,5}\, dz$$

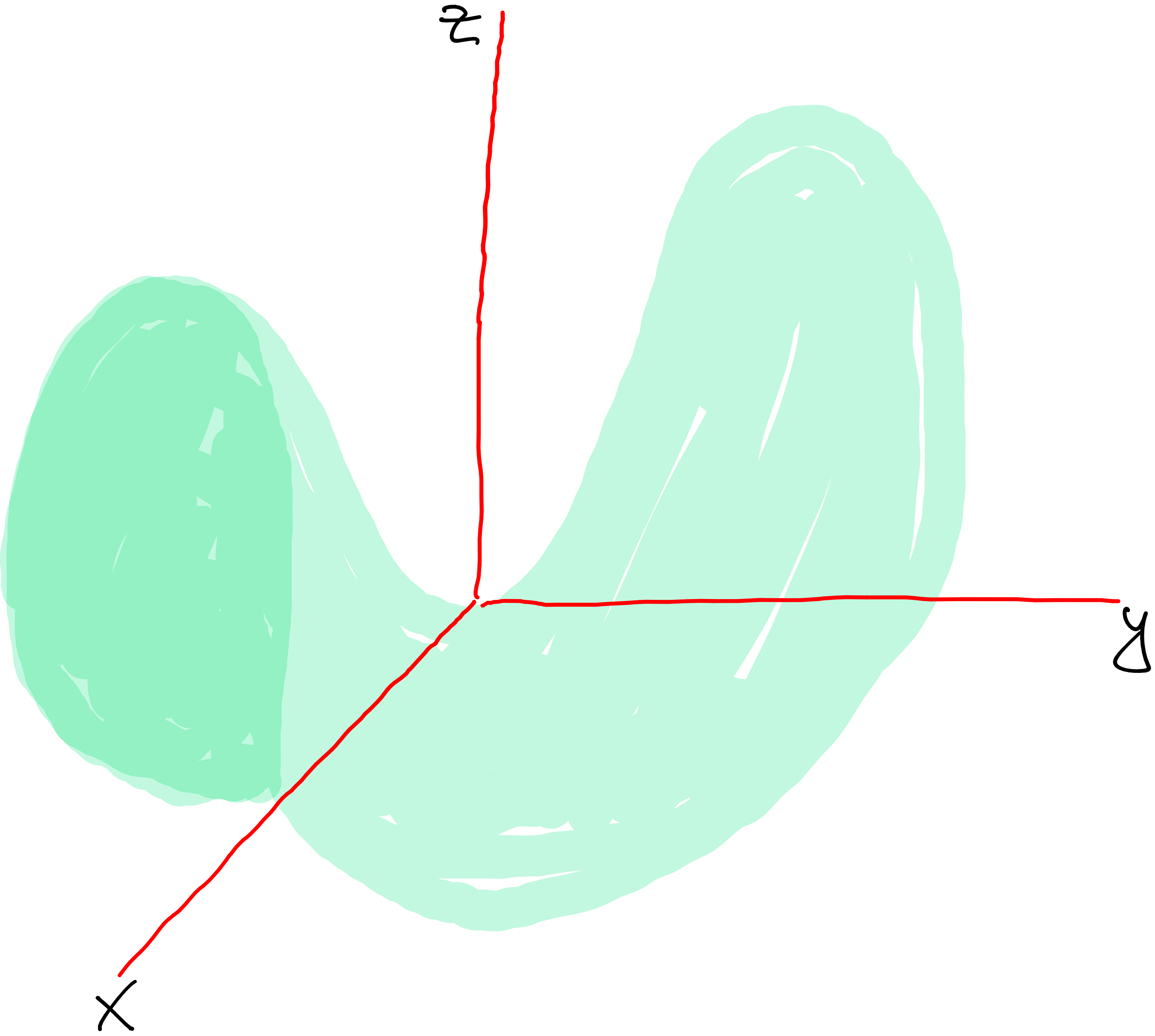
LET $1-z = u^2 \quad dz = -2u\, du$

$$z=0 \longrightarrow u=1$$
$$z=1 \longrightarrow u=0$$

$$\Rightarrow I = \frac{4\pi}{5} \int_1^0 u^5 (-2u)\, du = \frac{8\pi}{5} \int_0^1 u^6\, du =$$

$$= \frac{8\pi}{5} u^7 \Big|_0^1 = \frac{8\pi}{35}$$

2.) $S: z = xy, \quad x^2+y^2 \le 3$



$$A(S) = \iint_{x^2+y^2\le 3} \sqrt{z_x^2+z_y^2+1}\,dx\,dy = \iint_{x^2+y^2\le 3} \sqrt{y^2+x^2+1}\,dx\,dy =$$

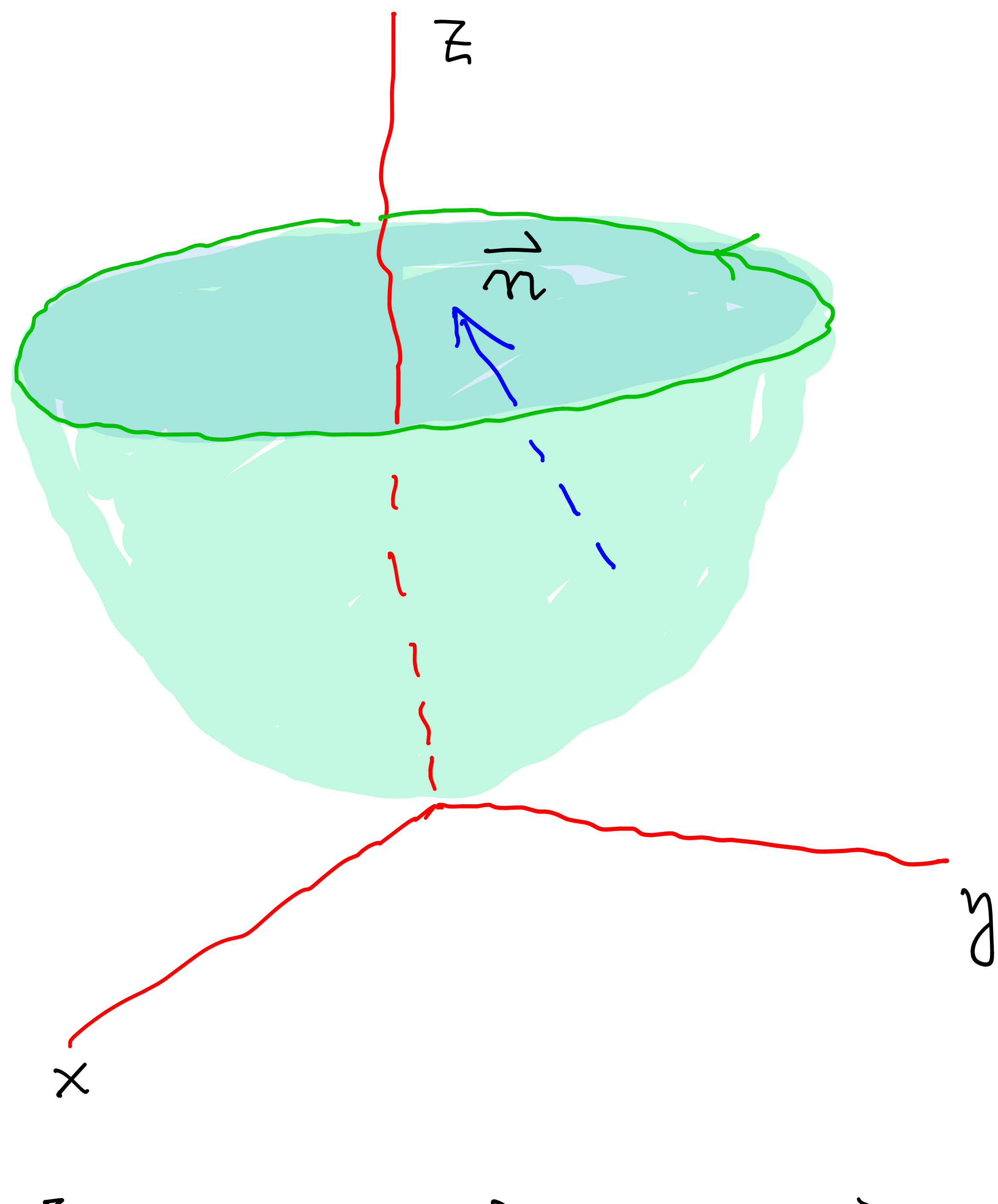
$$= 2\pi \int_0^{\sqrt{3}} \sqrt{r^2+1}\; r\,dr =$$

$$= \pi \, \frac{2}{3}\sqrt{r^2+1}^{\,3} \Big|_0^{\sqrt{3}} = \frac{2\pi}{3}\left(\sqrt{3+1}^{\,3} - 1\right) = \frac{14\pi}{3}$$

3.) $I = \iint_S (\nabla \times \vec{F}) \cdot dS$

$$S = \{(x,y,z) \mid z = x^2 + y^2, \quad z \le 1\}$$

$$\vec{F} = (-y, x, \cos(xyz))$$



(i) $\partial S = \{x^2 + y^2 = 1, \ z = 1\}$ B/C $\vec{n}$ POINTS UPWARD

THE CIRCLE IS TRAVERSED COUNTERCLOCKWISE:

$$x = \cos\theta, \quad y = \sin\theta, \quad z = 1$$

$$dx = -\sin\theta, \quad dy = \cos\theta, \quad dz = 0$$

BY STOKES' THEOREM

$$I = \oint_{\partial S} \vec{F} \cdot d\vec{r} =$$

$$= \oint_{\partial S} -y\,dx + x\,dy + \cos(xyz)\,dz =$$

$$= \int_0^{2\pi} (\sin^2\theta + \cos^2\theta)\,d\theta = 2\pi$$

(ii) DIRECTLY

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y & x & \cos(xyz) \end{vmatrix} =$$

$$= (-xz\sin xyz,\ yz\sin xyz,\ 2)$$

$$\vec{N} = (-2x, -2y, 1) \qquad \text{POINTS UPWARDS}$$

$$I = \iint_S (\nabla \times \vec{F}) \cdot \vec{N}\, dS$$

$$= \iint_{x^2+y^2 \le 1} \left(-2x^2 z \sin(xyz) - 2y^2 z \sin(xyz) + 2\right) dx\,dy$$

$$= \iint_{x^2+y^2 \le 1} \{\underbrace{-2(x^2+y^2)^2 \sin[xy(x^2+y^2)]}_{\text{ODD F'N OF } x,y} + 2)\, dx\,dy =$$

$$= 2\iint_{x^2+y^2 \le 1} dx\,dy = 2\pi$$

4.) $I = \int_C \vec{F} \cdot d\vec{r}$

$$C = \{ (\cos \pi t^2, \sin \pi t^2, \pi t^2) \mid 0 \le t \le 2 \}$$

$$\vec{F} = (3x^2y^2z + yz, 2x^3yz + xz, x^3y^2 + xy)$$

(i) GUESS THAT $\vec{F}$ IS CONSERVATIVE

$$\vec{F} = \nabla f$$

$\Rightarrow$ $3x^2y^2z + yz = \frac{\partial f}{\partial x}$, $f = x^3y^2z + xyz + \phi(y,z)$

$$\frac{\partial f}{\partial y} = 2x^3yz + xz + \frac{\partial \phi}{\partial y} = 2x^3yz + xz \Rightarrow \frac{\partial \phi}{\partial y} = 0$$

$\Rightarrow$ $f = x^3y^2z + xyz + \phi(z)$

$$\frac{\partial f}{\partial z} = x^3y^2 + xy + \phi'(z) = x^3y^2 + xy \Rightarrow \frac{\partial \phi}{\partial z} = 0$$

$\Rightarrow$ $f = x^3y^2z + xyz$

$$\Rightarrow I = f(\vec{r}(4)) - f(\vec{r}(0)) =$$

$$= f(1,0,16\pi) - f(1,0,0) = 0 - 0 = 0$$

(ii) DIRECTLY:

$$\dot{\vec{r}} = \left(-2\pi t \sin(\pi t^2),\ 2\pi t \cos(\pi t^2),\ 2\pi t\right) =$$

$$= 2\pi t\left(-\sin \pi t^2,\ \cos \pi t^2,\ 1\right) =$$

$$= 2\pi t(-y, x, 1)$$

$$I = \int_C 2\pi t\left(3x^2y^2z + yz,\ 2x^3yz + xz,\ x^3y^2 + xy\right)\cdot(-y, x, 1)\, dt =$$

$$= \int_0^2 2\pi t\left(-3x^2y^3z - y^2z + 2x^4yz + x^2z + x^3y^2 + xy\right) dt$$

LET $t^2 = u$, $2t\,dt = du$

$$\Rightarrow I = \int_0^4 \pi\left[-3\pi u \cos^2\pi u \sin^3\pi u - \pi u \sin^2\pi u + 2\pi u \cos^4\pi u \sin\pi u + \pi u \cos^2\pi u + \cos^3\pi u \sin^2\pi u + \cos\pi u \sin\pi u\right) du$$

$$I=\pi\int_0^4\Big\{\pi x\Big[-3\left(\cos^2\pi x-\cos^4\pi x\right)\sin\pi x+$$

$$+2\cos^4\pi x\sin\pi x+\cos 2\pi x\Big]+$$

$$\left(\sin^2\pi x-\sin^4\pi x\right)\cos\pi x+\sin\pi x\cos\pi x\Big\}dx=$$

$$=\pi\Big[\pi x\Big(\frac{\cos^3\pi x}{\pi}-\overbrace{\frac{3}{5\pi}\cos^5\pi x-\frac{2}{5\pi}\cos^5\pi x}^{=-\frac{1}{\pi}\cos^5\pi x}+$$

$$+\frac{1}{2\pi}\sin 2\pi x\Big)\Big|_0^4-\int_0^4\left(\cos^3\pi x-\cos^5\pi x+\frac{1}{2}\sin 2\pi x\right)dx$$

$$+\left(\frac{\sin^3\pi x}{3\pi}-\frac{\sin^5\pi x}{5\pi}+\frac{\sin^2\pi x}{2\pi}\right)\Big|_0^4\Big]=$$

$$=\pi\Big[4\pi(1-1)-0-\int_0^4\left[1-\sin^2\pi x-\left(1-\sin^2\pi x\right)^2\right]\cos\pi x\,dx$$

$$+\underbrace{\frac{1}{4\pi}\cos 2\pi x\Big|_0^4}_{0}+0$$

$$=-\int_0^4\left(\sin^2\pi x-\sin^4\pi x\right)\cos\pi x\,dx=$$

$$=\left[-\frac{\sin^3\pi x}{3\pi}+\frac{\sin^5\pi x}{5\pi}\right]_0^4=0$$

5.) $J[y(x)] = \int_0^1 [(y')^2 + 12xy]\,dx$, $y(0)=0$, $y(1)=1$

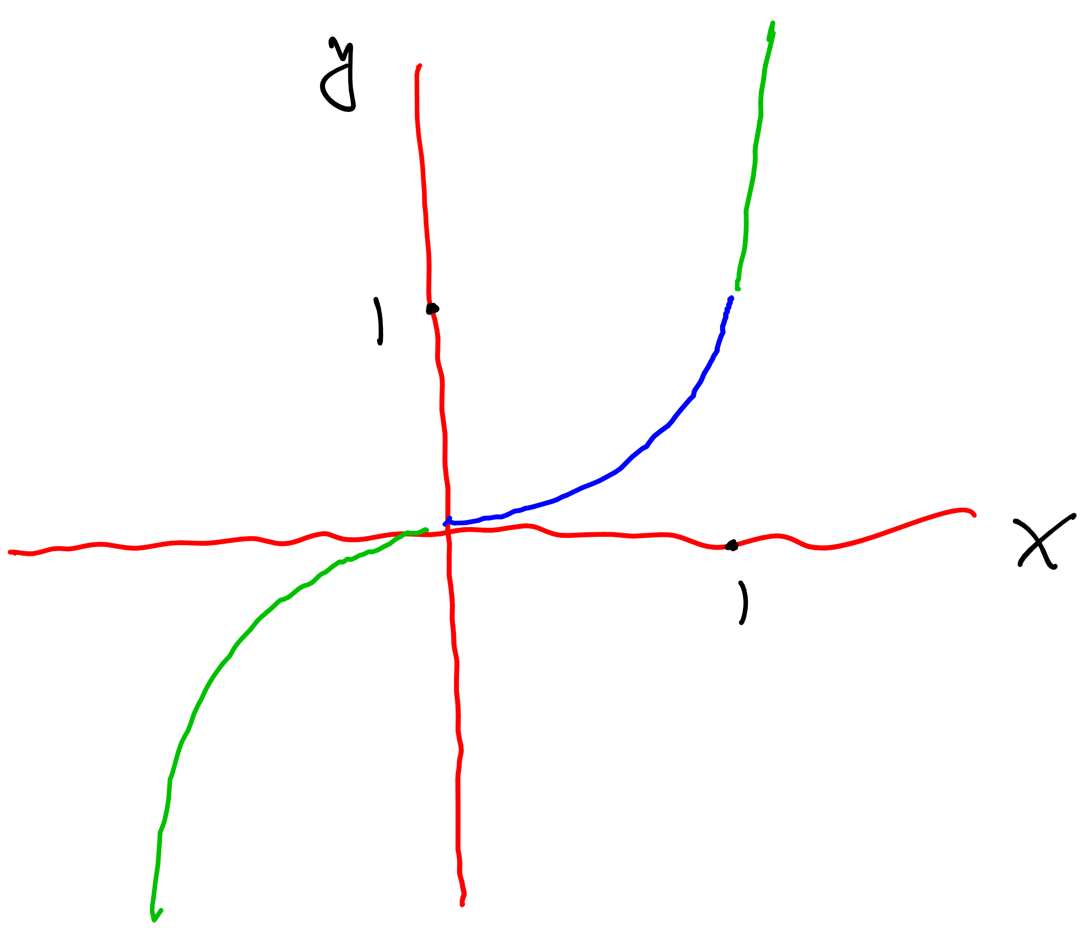
EULER'S EQ'N: $F = (y')^2 + 12xy$

$$\frac{d}{dx}F_{y'} - F_y = \frac{d}{dx}(2y') - 12x = 2(y'' - 6x) = 0$$

$\Rightarrow$ $y'' = 6x$, $y' = 3x^2 + C$, $y = x^3 + Cx + D$

$$\left.\begin{aligned} y(0) &= D = 0 \\ y(1) &= 1 + C = 1 \end{aligned}\right\} \quad C = D = 0$$

$\Rightarrow$ $y = x^3$



6.) $J = \int_0^\pi (y')^2 dx \qquad K = \int_0^\pi y^2 dx = 1$

$\Rightarrow$ USE LAGRANGE MULTIPLIERS

FIND EXTREMA OF $\quad J - \lambda K = \int_0^\pi [(y')^2 - \lambda y^2] dx$

EULER'S EQ'N: $\frac{d}{dx} y' + \lambda y = y'' + \lambda y = 0$

$\lambda = \mu^2 > 0 \Rightarrow \quad y = e^{rx}, \quad r^2 + \mu^2 = 0, \quad r = \pm i$

$$y = a \cos \mu t + b \sin \mu t$$

$$y(0) = a = 0$$

$y(\pi) = b \sin \mu\pi = 0 \Rightarrow \mu = n$ INTEGER

$\Rightarrow \quad y_n(x) = b_n \sin nx, \qquad \lambda_n = n^2$

$$\int_0^\pi b_n^2 \sin^2 nx \, dx = \frac{\pi}{2} b_n^2 = 1 \Rightarrow b_n = \sqrt{\frac{2}{\pi}}$$

$$\Rightarrow \quad y_n(x) = \sqrt{\frac{2}{\pi}} \sin nx$$

$\lambda \le 0 \Rightarrow$ NO NEW EIGENVALUES/EIGENFUNCTIONS

$$\int_0^\pi (y_n')^2 dx = \int_0^\pi \frac{2}{\pi} n^2 \cos^2 nx \, dx = n^2$$

$\Rightarrow \quad n = 1 \quad$ MINIMUM

IF YOU INSIST ON USING THE CONSERVED QUANTITY: $H = y' 2y' - (y')^2 + \lambda y^2 = (y')^2 + \lambda y^2$

$$= a^2 \Rightarrow \text{CONST}$$

$$y' = \sqrt{a^2 - \lambda y^2}$$

$$\lambda = \mu^2 > 0: \quad y' = \sqrt{a^2 - \mu^2 y^2} = a\sqrt{1 - \left(\frac{\mu}{a}\right)^2 y^2}$$

$$\frac{dy}{a\sqrt{1-\left(\frac{\mu}{a}\right)^2 y^2}} = dx$$

$$\frac{d\left(\frac{\mu y}{a}\right)}{\sqrt{1-\left(\frac{\mu y}{a}\right)^2}} = \mu\, dx$$

$$\arcsin\left(\frac{\mu y}{a}\right) = \mu(x + x_0)$$

$$y = \frac{a}{\mu} \sin\left(\mu(x + x_0)\right)$$

$$y(0) = 0 \Rightarrow \mu x_0 = 0 \Rightarrow x_0 = 0$$

$$y(\pi) = 0 \Rightarrow \mu\pi = n\pi, \quad \mu = n$$

$$\Rightarrow \quad y_n(x) = \frac{a_n}{n} \sin nx = b_n \sin nx$$