Advanced Calculus, MATH-4600

Exam #1

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1. Let $u = e^s \cos t$ and $v = e^s \sin t$, and let $s = x^2 - y^2$ and $t = 2xy$. Use the chain rule to calculate the expression $\dfrac{\partial u}{\partial x} - \dfrac{\partial v}{\partial y}$.

2. Find the tangent plane and the normal line to the surface defined implicitly by the equation $x^2y^2z^2 - 2xyz = 48$ at the point $(2, 2, 2)$. Attempt also to find the tangent plane and the normal line to this surface at the point $(-2, -2, -2)$. What happens and why?

3. Find and classify all the extrema of the function $f(x, y) = 3x^2 - 2x^3 + y^2$.

4. Find the maximum volume V_{max} of a cylindrical cup whose surface area S is 3π? The volume of such a cup is $V = \pi r^2 h$, and the surface area is $S = \pi r^2 + 2\pi rh$, where r is the radius of its base and h is its height.

5. Consider the curve C parametrized by the expression $\mathbf{r}(t) = \left(2c \ln \dfrac{t}{c}, t - \dfrac{c^2}{t}, t + \dfrac{c^2}{t}\right)$, where $c > 0$ is a parameter. Calculate the length of the curve C along the parameter interval $a < t < b$, $a \geq c$.

6. Compute the volume of the solid region D bounded by the planes $x = 0$, $y = 1$, $z = 0$, and $x = y$, and the cubic paraboloid $z = \alpha x^3 + \beta y^3$, where α and β are positive constants.

EXAM 1

1.) $u = e^s \cos t \quad , \quad v = e^s \sin t$

$s = x^2 - y^2 \qquad t = 2xy$

$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial s}\frac{\partial s}{\partial x} + \frac{\partial u}{\partial t}\frac{\partial t}{\partial x} \qquad \frac{\partial v}{\partial y} = \frac{\partial v}{\partial s}\frac{\partial s}{\partial y} + \frac{\partial v}{\partial t}\frac{\partial t}{\partial y}$

$u_s = e^s \cos t \qquad u_t = -e^s \cos t$

$v_s = e^s \sin t \qquad v_t = e^s \cos t$

$s_x = 2x \;, \quad s_y = -2y \quad t_x = 2y \quad t_y = 2x$

$u_x = e^s \cos t \, 2x - e^s \sin t \, 2y$

$v_y = e^s \sin t \, (-2y) + e^s \cos t \, 2x$

$\Rightarrow u_x - v_y = 0$

2.) $F(x,y,z) = x^2y^2z^2 - 2xyz = 48$

$$\nabla F = (2xy^2z^2 - 2yz,\ 2x^2yz - 2xz,\ 2x^2y^2z - 2xy)$$

$$\nabla F(2,2,2) = (64-8,\ 64-8,\ 64-8) =$$

$$= 56\,(1,1,1)$$

TANGENT PLANE: $(1,1,1)\cdot(x-2, y-2, z-2) =$

$= x+y+z-6 = 0$

NORMAL LINE: $(x,y,z) = (2,2,2) + t\,(1,1,1) =$

$$= \lambda\,(1,1,1).$$

$$F(-2,-2,-2) = 64 + 8 = 72 \neq 48$$

$(-2,-2,-2)$ IS NOT ON THE SURFACE.

3.) $f(x,y) = 3x^2 - 2x^3 + y^2$

$f_x = 6x - 6x^2 = 6x(1-x) = 0 \Rightarrow x = 0, 1$

$f_y = 2y = 0 \Rightarrow y = 0$

CANDIDATES FOR EXTREMA: $(0,0)$, $(1,0)$

$f_{xx} = 6 - 12x \qquad f_{xy} = 0 \qquad f_{yy} = 2$

$(0,0)$: $f_{xx} = 6 > 0$, $f_{yy} = 2 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = 12 > 0$

$\Rightarrow$ MINIMUM

$(1,0)$: $f_{xx} = -6 < 0$, $f_{yy} = 2 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = -12 < 0$

$\Rightarrow$ SADDLE

4.) $V = \pi r^2 h \qquad S = \pi r^2 + 2\pi r h = 3\pi$

$$F = r^2 h - \lambda(r^2 + 2rh - 3)$$

$$F_r = 2rh - 2\lambda r - 2\lambda h \Rightarrow rh - \lambda(r+h) = 0$$

$$F_\theta = r^2 - 2\lambda r = r(r - 2\lambda) = 0$$

(i) $r = 0 \Rightarrow S = 0 \Rightarrow$ NOT A SOLUTION

(ii) $\lambda = \frac{r}{2} = \frac{rh}{r+h} \Rightarrow r(r+h) = 2rh$

SINCE $r \neq 0 \Rightarrow r + h = 2h \Rightarrow r = h = 1$

$$V_{MAX} = \pi$$

5.) $\vec{r} = \left(2c \ln \frac{t}{c},\ t - \frac{c^2}{t},\ t + \frac{c^2}{t}\right)$

$$\dot{\vec{r}} = \left(\frac{2c}{t},\ 1 + \frac{c^2}{t^2},\ 1 - \frac{c^2}{t^2}\right)$$

$$\dot{s}^2 = \frac{4c^2}{t^2} + 1 + \cancel{\frac{2c^2}{t^2}} + \frac{c^4}{t^4} + 1 - \cancel{\frac{2c^2}{t^2}} + \frac{c^4}{t^4} =$$

$$= 2\left(1 + \frac{2c^2}{t^2} + \frac{c^4}{t^4}\right) = 2\left(1 + \frac{c^2}{t^2}\right)^2$$

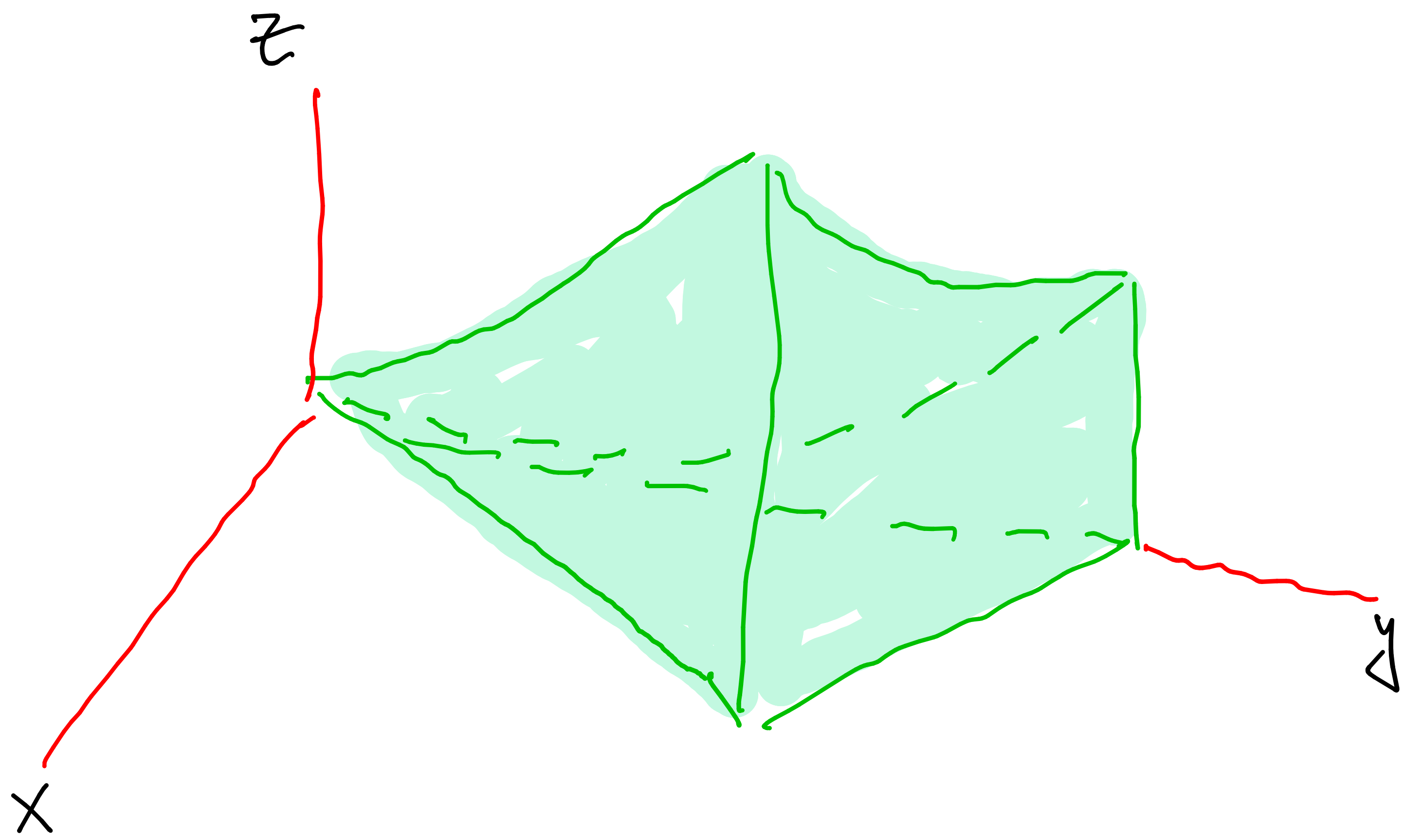
$$\dot{s} = \sqrt{2}\left(1 + \frac{c^2}{t^2}\right)$$

$$L = \int_a^b \dot{s}\, dt = \sqrt{2}\int_a^b \left(1 + \frac{c^2}{t^2}\right) dt = \sqrt{2}\left(t - \frac{c^2}{t}\right)\Big|_a^b =$$

$$= \sqrt{2}\left(b - \frac{c^2}{b} - a + \frac{c^2}{a}\right) = \sqrt{2}\left(b - a + \frac{c^2(b-a)}{ab}\right)$$

$$= \sqrt{2}(b-a)\left(1 + \frac{c^2}{ab}\right)$$

6.)



$$V = \int_0^1 dx \int_x^1 dy \int_0^{\alpha x^3+\beta y^3} dz = \int_0^1 dx \int_x^1 dy\, (\alpha x^3+\beta y^3) =$$

$$= \int_0^1 dx \left(\alpha x^3 y + \beta \frac{y^4}{4}\right)\Big|_x^1 =$$

$$= \int_0^1 dx \left[\alpha x^3(1-x) + \beta\left(\frac{1}{4} - \frac{x^4}{4}\right)\right] =$$

$$= \left[\alpha\left(\frac{1}{4}-\frac{1}{5}\right) + \beta\left(\frac{1}{4}-\frac{1}{20}\right)\right] =$$

$$= \frac{\alpha}{20} + \frac{\beta}{5} = \frac{1}{5}\left(\frac{\alpha}{4}+\beta\right)$$

ALTERNATIVELY:

$$V = \int_0^1 dy \int_0^y dx \int_0^{\alpha x^3 + \beta y^3} dz = \int_0^1 dy \int_0^y dx \, (\alpha x^3 + \beta y^3) =$$

$$= \int_0^1 dy \left(\alpha \frac{x^4}{4} + \beta y^3 x \right) \Big|_0^y = \int_0^1 dy \left(\alpha \frac{y^4}{4} + \beta y^4 \right) =$$

$$= \left(\frac{\alpha}{4} + \beta \right) \int_0^1 dy \, y^4 = \frac{1}{5} \left(\frac{\alpha}{4} + \beta \right)$$