5.) $f(x,y,z) = \frac{1}{2}x^2 + xy - 2\alpha xz + y^2 - \alpha z^2$

$$f_x = x + y - 2\alpha z = 0$$
$$f_y = x + 2y \qquad = 0$$
$$f_z = -2\alpha x \quad - 2\alpha z = 0$$

IF $\alpha \neq 0 \Rightarrow z = -x = 2y = \frac{1}{2\alpha}(x+y) =$

$$= \frac{1}{2\alpha}\left(-z + \frac{1}{2}z\right) = -\frac{1}{4\alpha}z$$

IF $\alpha \neq 0, \alpha = -\frac{1}{4} \Rightarrow x = y = z = 0$

IF $\alpha = 0 \Rightarrow x = y = 0$, z IS ARBITRARY: z - AXIS

IF $\alpha = -\frac{1}{4} \Rightarrow (x,y,z) = t(2,-1,-1)$, t IS ARBITRARY

LOOK AT $(0,0,0)$

HESSIAN: $f_{xx} = 1 \qquad f_{yy} = 2 \qquad f_{zz} = -2\alpha$

$f_{xy} = 1 \qquad f_{xz} = -2\alpha \qquad f_{yz} = 0$

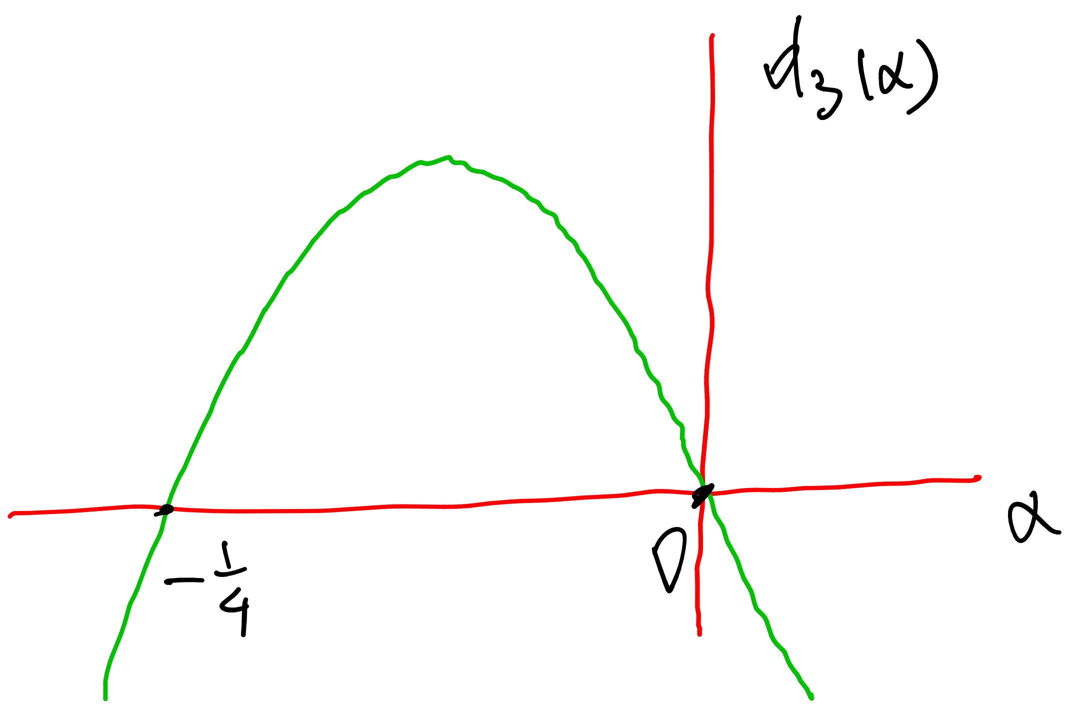
$$D^2 f = \begin{bmatrix} 1 & 1 & -2\alpha \\ 1 & 2 & 0 \\ -2\alpha & 0 & -2\alpha \end{bmatrix}$$

SYLVESTER'S CRITERION

$$d_1 = 1 \qquad d_2 = \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} = 1$$

$$d_3 = \begin{vmatrix} 1 & 1 & -2\alpha \\ 1 & 2 & 0 \\ -2\alpha & 0 & -2\alpha \end{vmatrix} = \begin{vmatrix} 2 & 0 \\ 0 & -2\alpha \end{vmatrix} - \begin{vmatrix} 1 & 0 \\ -2\alpha & -2\alpha \end{vmatrix} - 2\alpha \begin{vmatrix} 1 & 2 \\ -2\alpha & 0 \end{vmatrix} =$$

$$= -4\alpha + 2\alpha - 8\alpha^2 = -2\alpha(1+4\alpha) = -8\alpha\left(\alpha + \frac{1}{4}\right)$$



$$-\frac{1}{4} < \alpha < 0 \Longrightarrow d_1 > 0,\ d_2 > 0,\ d_3 > 0$$

$\Longrightarrow$ $(0,0,0)$ IS A MINIMUM

$\alpha < -\frac{1}{4}$ OR $\alpha > 0 \rightarrow d_1, d_2 > 0, \; d_3 < 0$

$\Rightarrow (0,0,0)$ IS A SADDLE

$\alpha = 0 \Rightarrow$ A "TROUGH" ALONG THE LINE OF DEGENERATE EXTREMA $(0,0,z)$

$\alpha = -\frac{1}{4} \Rightarrow$ A "TROUGH" ALONG THE LINE OF DEGENERATE EXTREMA $(2t, -t, -t)$

6.) $f(x,y,z) = x^2 y^2 z^2 =$ MAX

$g(x,y,z) = x^2+y^2+z^2-c^2 = 0$

$\Rightarrow$ FIND EXTREMA OF $F = f + \lambda g$

$$F_x = 2xy^2z^2 + 2\lambda x = 2x(y^2z^2+\lambda) = 0$$

$$F_y = 2x^2yz^2 + 2\lambda y = 2y(x^2z^2+\lambda) = 0$$

$$F_z = 2x^2y^2z + 2\lambda z = 2z(x^2y^2+\lambda) = 0$$

(i) $x=y=z=0$ DOES NOT SATISFY $g=0$.

(ii) $x=0$, $\lambda=0$, y, z ARBITRARY WITH

$y^2+z^2=c^2$

PLUS THE TWO PERMUTATIONS W/ $y=0$, $z=0$

ON ALL THESE $f=0$

(iii) $y^2z^2 = x^2z^2 = x^2y^2 = -\lambda \Rightarrow \frac{1}{x^2} = \frac{1}{y^2} = \frac{1}{z^2}$

$x = \pm y = \pm z$

CONSTRAINT: $3x^2 = c^2 \Rightarrow x^2 = \frac{c^2}{3}$

LIKEWISE: $y^2 = \frac{c^2}{3}$, $z^2 = \frac{c^2}{3}$

$\Rightarrow f = \frac{c^6}{27} > 0$, MAX

MAX: $(x, y, z) = \frac{c}{\sqrt{3}}(\pm 1, \pm 1, \pm 1)$

AWAY FROM MAX: $x^2 y^2 z^2 \leq \frac{c^6}{27} = \frac{(x^2+y^2+z^2)^3}{27}$

$\Rightarrow (x^2 y^2 z^2)^{1/3} \leq \frac{x^2+y^2+z^2}{3}$