Homework Problems

1. Find the tangent plane to the surface $z = f(x, y) = \sqrt{x^2 + y^2}$ at the point $P = (x, y) = (1, 2)$. Also find the gradient of $f(x, y)$ at the point P, and the directional derivative of $f(x, y)$ in the direction of the vector $(3, -4)$ at P. Draw a figure! Can you find the analogous objects at the point $Q = (0, 0)$? Again, draw a figure and explain!

2. For the function $z = f(x, y)$, use the chain rule to express the formula

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y}$$

in terms of the polar coordinates $x = r\cos\theta$, $y = r\sin\theta$.

3. Find the tangent plane and the normal line to the surface defined implicitly by the equation $2x^2 - y^2 - z^2 = 0$ at the point $(2, 2, -2)$. Attempt to find the corresponding two objects at the point $(0, 0, 0)$. What happens and why? Draw a picture and provide an explanation.

4. Find the Taylor expansion around the origin of the function $f(x, y, z) = \cos\left(xz - y^2\right)$ up to including terms of $\mathcal{O}\left[\left(x^2 + y^2 + z^2\right)^4\right]$. What order are the first neglected terms?

5. Determine how the location and type of the extrema of the function

$$f(x, y, z) = \frac{1}{2}x^2 + xy - 2\alpha xz + y^2 - \alpha z^2$$

depend on the parameter α.

6. Find the maximum of the function $f(x, y, z) = x^2y^2z^2$ on the sphere $x^2 + y^2 + z^2 = c^2$. Conclude the inequality

$$\left(x^2y^2z^2\right)^{\frac{1}{3}} \le \frac{x^2 + y^2 + z^2}{3},$$

which states that the geometric mean of three nonnegative numbers x^2, y^2, z^2 is never greater than their arithmetic mean.

7. Consider the curve C parametrized by the expression

$$\mathbf{r}(t) = \left(t + \frac{a^2}{t}, t - \frac{a^2}{t}, 2a\ln\frac{t}{a}\right),$$

where $a > 0$ is a parameter. Calculate the length of the curve C over the parameter interval $a \le t \le b$, with $b > a$.

8. (i) Show that a vector field of the form $\mathbf{F}(x, y, z) = \nabla f(x, y, z)$, where the potential $f(x, y, z)$ satisfies Laplace's equation

$$\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0,$$

is both incompressible and irrotational.
(ii) Show that any function of the form

$$f_{m,n}(x, y, z) = \sinh\left(\sqrt{m^2 + n^2}x\right) \sin my \sin nz$$

satisfies Laplace's equation on the square $0 < x, y, z < \pi$. Compute the corresponding incompressible and irrotational vector field $\mathbf{F}_{m,n}(x, y, z) = \nabla f_{m,n}(x, y, z)$.
Extra Credit (2 pts): On which face of the square does the function $f_{m,n}(x, y, z)$ not vanish, and what are its values there?
Clarification: $\sinh t = \dfrac{e^t - e^{-t}}{2}$

9. Compute the integral $\displaystyle\iint_D \frac{\sin xy}{x}\, dA$, where A is the figure bounded by the straight lines $y = 0$, $x = \pi$, and $x = 2\pi$, as well as the curve $y = \dfrac{\pi}{x}$.

10. Compute the volume of the solid region D bounded by the planes $x = 1$, $y = 0$, $z = 0$, and $x = y$, and the paraboloid $z = \alpha x^2 + \beta y^2$, where α and β are positive constants.

11. The points (x, y, z) in a bowl-shaped region D can be described by the inequalities $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - z^2 \leq 1$ and $0 \leq z \leq 3$. Modify the cylindrical coordinates in an appropriate way and compute the volume of D.

12. Let D be the region whose points (x, y, z) satisfy the inequalities $z > 0$, $z^2 \geq (x^2 + y^2)/3$, $z^2 \leq 3(x^2 + y^2)$, and $x^2 + y^2 + z^2 \leq a^2$. Compute the integral $\displaystyle\iiint_D z\, dV$.

13. Let C be the intersection curve between the plane $x - 2y = 1$ and the hyperboloid $9(x - 1)^2 - 27y^2 + (z - 2)^2 = 1$, traversed so that z increases along C when $y > 0$ and decreases when $y < 0$. (Draw a sketch!) Find an appropriate parametrization of C and compute the integral $\displaystyle\int_C (2 - z)\, dx + (2 - z)\, dy + (x + y - 1)\, dz$.

14. Compute the area of the portion of the saddle-like surface $z = \dfrac{b}{2}(x^2 - y^2)$ that lies inside the cylinder $x^2 + y^2 \leq a^2$. (Draw a sketch!) What is the leading-order term in this area as either $a \to 0$ or $b \to 0$?

15. Rain is falling straight down at the rate ϕv, where ϕ is its intensity per unit area, and v its velocity. Assume that your umbrella is the part of the circle $S:\ x^2+y^2+(z+b)^2=a^2,\ 0<b<a$, that lies above the xy-plane. Compute the volume flow rate that your umbrella deflects. Show also that, as far as this rate is concerned, it might as well be a disk. What is the radius of this disk?

16. Compute the surface integral $I=\iint_S (\nabla\times\mathbf{F})\cdot d\mathbf{S}$, where S is the portion of the ellipsoid $3x^2+3y^2+z^2=28$ that lies above the plane $z=1$, $\mathbf{F}=(yz^2,4xz,x^2y)$, and $d\mathbf{S}=\mathbf{n}\,dS$ with $\mathbf{n}$ being the unit normal to S with a positive $\mathbf{k}$ component.

17. Compute the integral $\int_C \mathbf{F}\cdot d\mathbf{r}$ along the curve parametrized by $\mathbf{r}(t)=(t\cos\pi t^2, t^2\sin\pi t^2, t^3)$, $0\le t\le 1$, where $\mathbf{F}=\Big(yz[2x(y+z)+yz], xz[2y(x+z)+xz], xy[2z(x+y)+xy]\Big)$.

18. Find and sketch the curve $y(x)$, which passes through the points $(0,1)$ and $(1,0)$, and along which the functional $J[y(x)]=\int_0^1 \left[(y')^2+y^2\right]\,dx$ is extremal.

19. What two-dimensional geometric figure D with area $A=\dfrac{1}{2}\oint_{\partial D} y\,dx-x\,dy$ has the shortest perimeter?

HINT: Parametrize ∂D by $\mathbf{r}(t)=(x(t),y(t))$, $\alpha<t<\beta$, $\mathbf{r}(\alpha)=\mathbf{r}(\beta)$. The perimeter length of D then equals $\int_\alpha^\beta |\dot{\mathbf{r}}(t)|\,dt$. Show that each of the two Euler's equations represents a total t-derivative, and consequently integrate it once. Manipulate the resulting two equations so that you find the usual implicit equation of a circle.

20. (i) For Newton's equation $\ddot{x}+x^2-x=0$, find the potential energy $U(x)$, the kinetic energy, as well as the Lagrangian. What are the equilibrium points for this equation?

(ii) Multiply Newton's equation in part (i) by $\dot{x}$ and integrate to obtain the total energy, E. Sketch $U(x)$, and use it and E to sketch the trajectories of the equation in the x-$\dot{x}$-plane.

(iii) <u>Extra credit, 2 points:</u> What is the value of the energy along the separatrix loop?