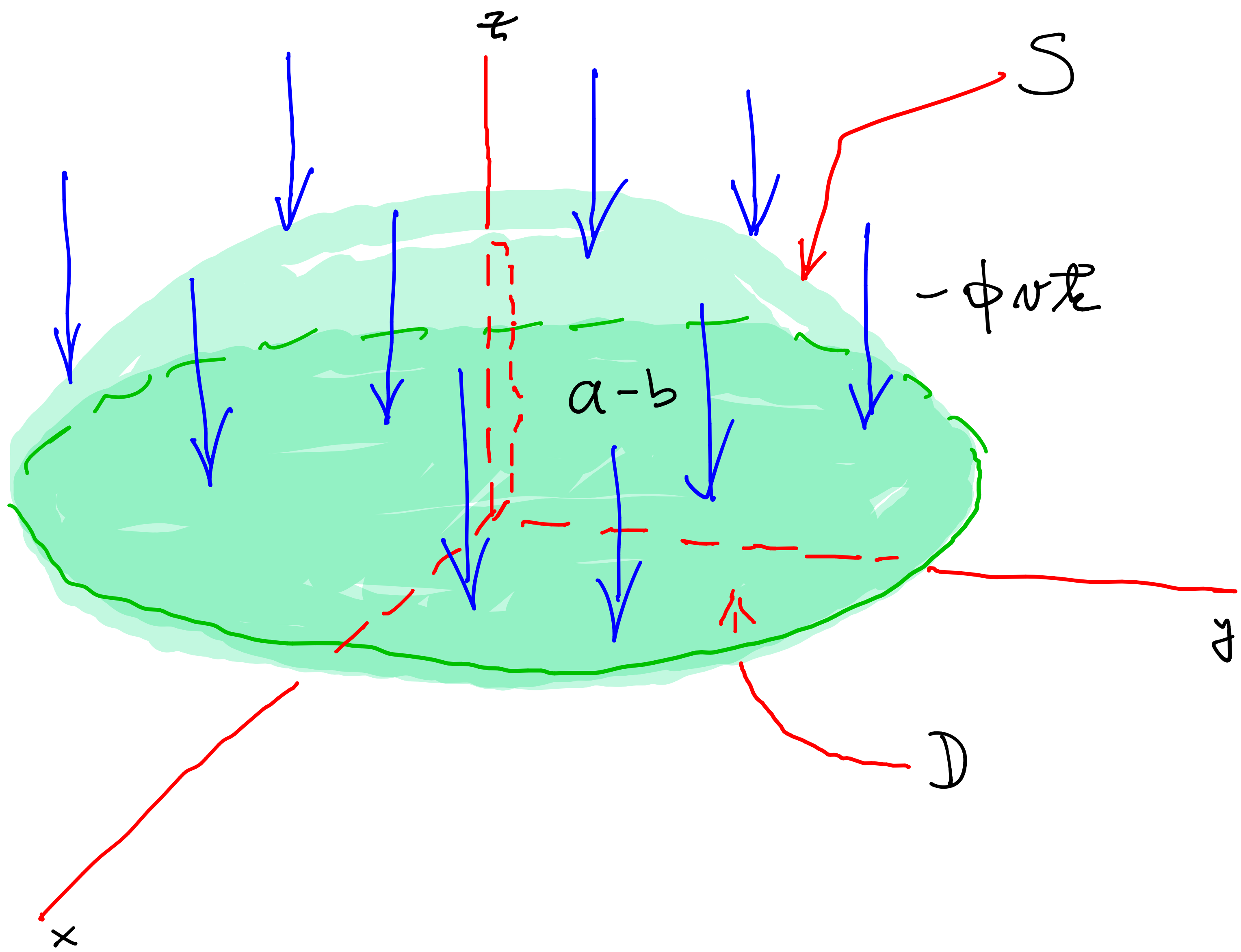
15.) $S: x^2+y^2+(z+b)^2=a^2 \quad 0<b<a \qquad z>0$

$\vec{F}(x,y,z) = -\phi v \vec{k}$



EXPLICIT WAY: $z = f(x,y) = \sqrt{a^2-x^2-y^2} - b \geq 0$

$z=0 \Rightarrow x^2+y^2 = a^2-b^2$

$z_x = -\dfrac{x}{z+b} \qquad\qquad z_y = -\dfrac{y}{z+b}$

$$\vec{N} = \left(\frac{x}{z+b}, \frac{y}{z+b}, 1\right) \qquad D: x^2+y^2 \le a^2-b^2$$

FLOW RATE $= \iint_S \vec{F}\cdot d\vec{S} = \iint_D (-\phi v \hat{k})\cdot \vec{N}\, dxdy =$

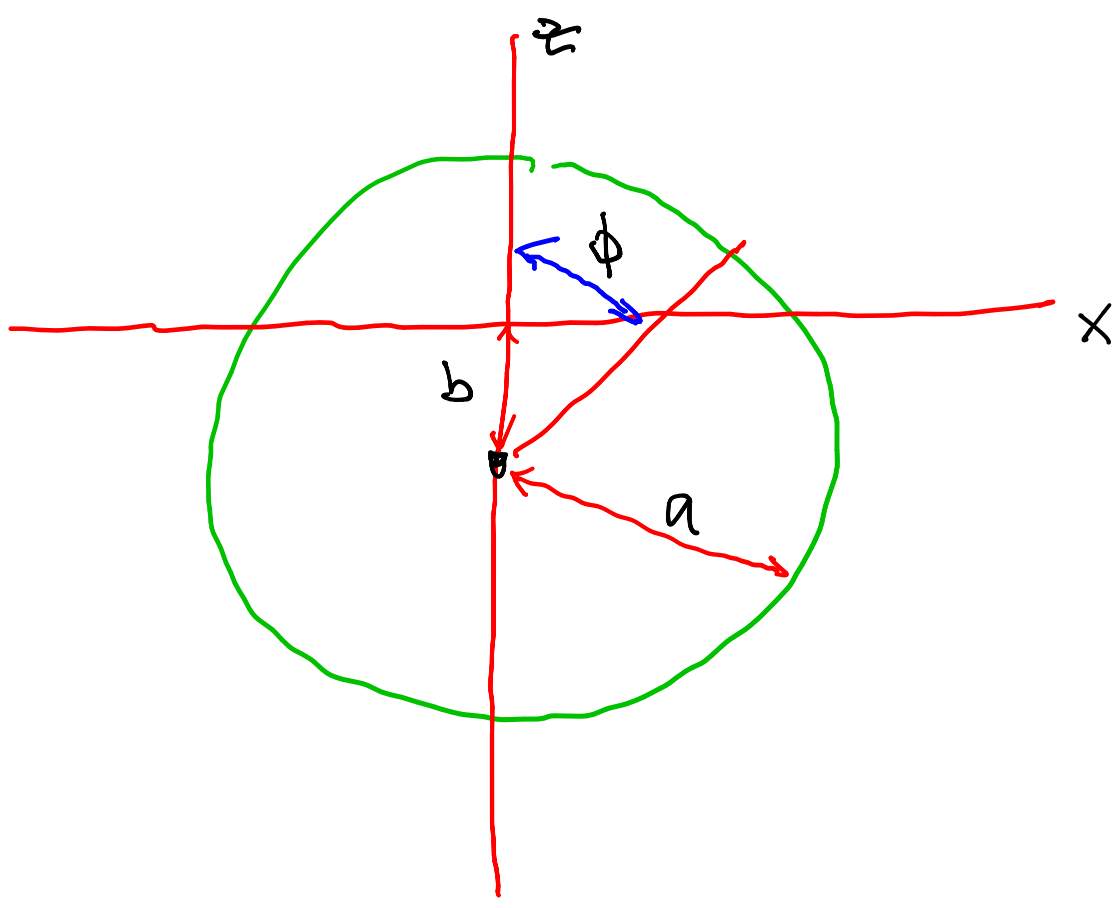
$$= -\phi v \underbrace{\pi(a^2-b^2)}_{\text{AREA OF CIRCLE}} = -\phi v \pi R^2$$

$$R = \sqrt{a^2-b^2}$$

PARAMETRIC WAY: SPHERICAL COORDINATES

$$\vec{r} = (a\sin\phi\cos\theta, a\sin\phi\sin\theta, a\cos\phi - b)$$

$$0 \le \phi \le \arccos\frac{b}{a}, \quad 0 \le \theta \le 2\pi$$



$$\vec{r}_\phi = a(\cos\phi\cos\theta,\ \cos\phi\sin\theta,\ -\sin\phi)$$

$$\vec{r}_\theta = a(-\sin\phi\sin\theta,\ \sin\phi\cos\theta,\ 0)$$

$$\vec{N} = \vec{r}_\phi \times \vec{r}_\theta = a^2\sin\phi \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ \cos\phi\cos\theta & \cos\phi\sin\theta & -\sin\phi \\ -\sin\theta & \cos\theta & 0 \end{vmatrix} =$$

$$= a^2\sin\phi(\sin\phi\cos\theta,\ \sin\phi\sin\theta,\ \cos\phi) =$$

$$= a\sin\phi(\vec{r} + b\vec{k})$$

FLOW RATE:

$$\iint_S \vec{F}\cdot d\vec{S} = \int_0^{2\pi} d\theta \int_0^{\arccos\frac{b}{a}} (-\phi v\vec{k})\cdot\vec{N}\, d\phi =$$

$$= 2\pi \int_0^{\arccos\frac{b}{a}} (-\phi v\vec{k})\cdot(\vec{r} + b\vec{k})\, a\sin\phi =$$

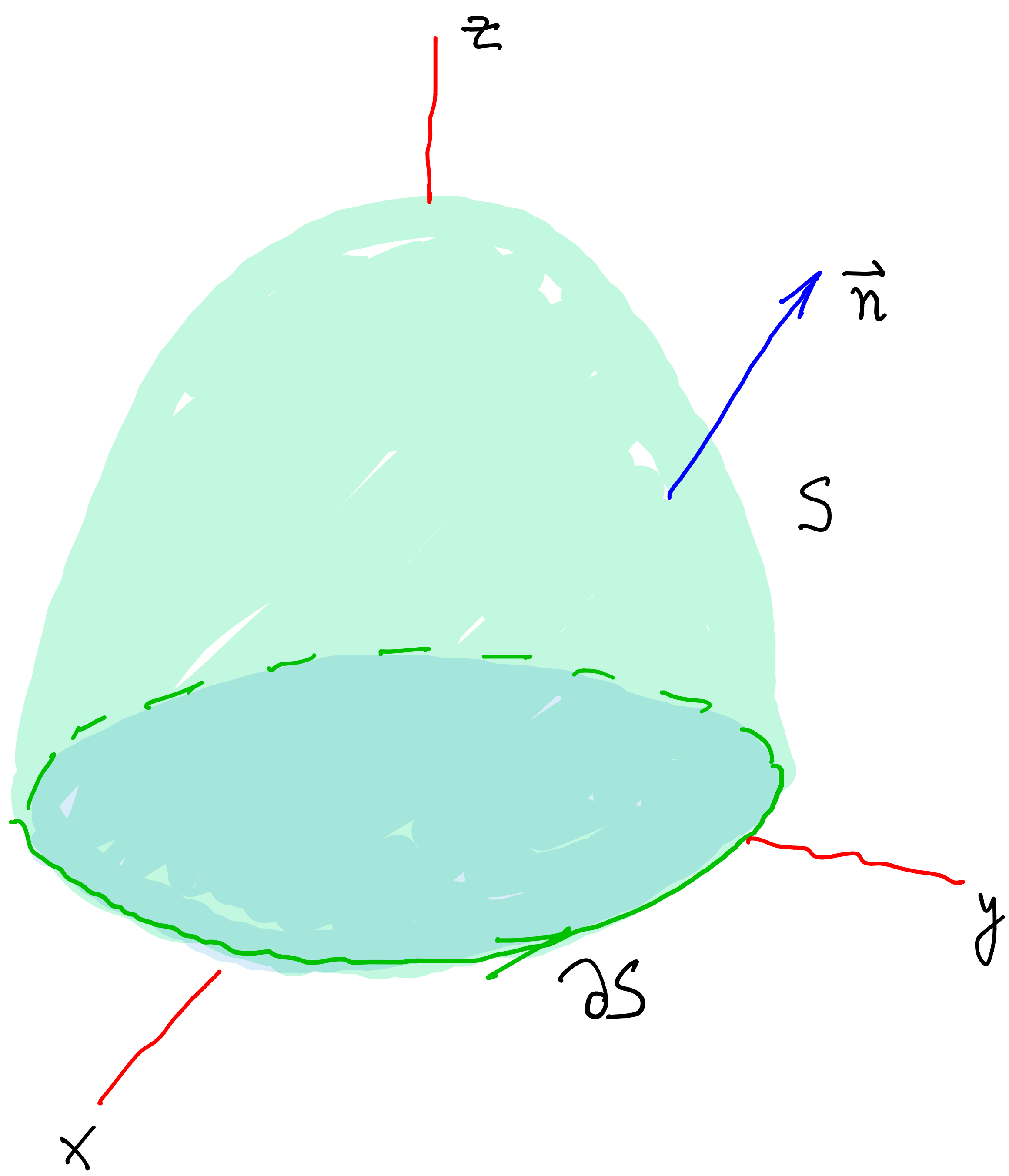
$$= 2\pi a(-\phi v) \int_0^{\arccos\frac{b}{a}} a\cos\phi\sin\phi\, d\phi =$$

$$= -2\pi a\phi v\left(-\frac{a}{2}\cos^2\phi\right)\Big|_0^{\arccos\frac{b}{a}} =$$

$$= -\pi a^2\phi v\left(-\frac{b^2}{a^2} + 1\right) = -\phi v\pi(a^2 - b^2) = -\phi v\pi R^2$$

16.) $I = \iint_S (\nabla \times \vec{F}) \cdot d\vec{S}$ $\quad S: 3x^2 + 3y^2 + z^2 = 28, \; z > 1$

$d\vec{S} = \vec{n}\, dS, \quad \vec{n} \cdot \vec{k} > 0, \quad \vec{F} = (yz^2, 4xz, x^2yz)$



USE STOKES' THEOREM:

$$I = \int_{\partial S} \vec{F} \cdot d\vec{r}$$

$$\partial S : 3x^2 + 3y^2 + z^2 = 28, \; z = 1$$

$\Rightarrow$ $3x^2+3y^2=27$, $\quad x^2+y^2=9 \quad$ ON ∂S

∂S IS TRAVERSED COUNTERCLOCKWISE B/C $\vec{n}\cdot\hat{z}>0$

$\Rightarrow$ CHOOSE PARAMETRIZATION

$$x=3\cos\theta,\quad y=3\sin\theta,\quad z=1,\quad 0\le\theta\le 2\pi$$

$$\vec{F}=(y,\,4x,\,x^2y)=$$

$$=(3\sin\theta,\,12\cos\theta,\,27\cos^2\theta\sin\theta)$$

$$d\vec{r}=(3\cos\theta,\,3\sin\theta,\,1)'\,d\theta=(-3\sin\theta,\,3\cos\theta,\,0)\,d\theta$$

$$\Rightarrow I=\int_0^{2\pi}(-9\sin^2\theta+36\cos^2\theta)\,d\theta=\pi(-9+36)=$$

$$=27\pi$$

(SHOWED IN CLASS: $\displaystyle\int_0^{N\frac{\pi}{2}}\cos^2\theta\,d\theta=\int_0^{N\frac{\pi}{2}}\sin^2\theta\,d\theta=\frac{N\pi}{4}$)