7.) $C: \ x = t + \frac{a^2}{t}, \quad y = t - \frac{a^2}{t}, \quad z = 2a \ln \frac{t}{a}$

$$\dot{x} = 1 - \frac{a^2}{t^2}, \quad \dot{y} = 1 + \frac{a^2}{t^2}, \quad \dot{z} = \frac{2a}{t}$$

$$\dot{s}^2 = 1 - 2\frac{a^2}{t^2} + \frac{a^4}{t^4} + 1 + 2\frac{a^2}{t^2} + \frac{a^4}{t^4} + 4\frac{a^2}{t^2} =$$

$$= 2 + 4\frac{a^2}{t^2} + 2\frac{a^4}{t^4} = 2\left(1 + \frac{a^2}{t^2}\right)^2$$

$$\dot{s} = \sqrt{2}\left(1 + \frac{a^2}{t^2}\right)$$

$$s = \sqrt{2}\int_a^b \left(1 + \frac{a^2}{t^2}\right)dt = \sqrt{2}\left(t - \frac{a^2}{t}\right)\Big|_a^b = \sqrt{2}\left(b - \frac{a^2}{b}\right)$$

8.) (i) IF $\vec{F} = \nabla f$ FOR f SATISFYING $\Delta f = 0$:

$$\nabla \cdot \vec{F} = \nabla \cdot \nabla f = \Delta f = 0$$

$$\nabla \times \vec{F} = \nabla \times \nabla f = 0$$

(ii) $$f_{mn}(x,y,z) = \sinh\left(\sqrt{m^2+n^2}\,x\right) \sin my \,\sin nz$$

ASIDE: $\sinh t = \frac{e^t - e^{-t}}{2}$, $\sinh' t = \frac{e^t + e^{-t}}{2} = \cosh t$

$$\sinh'' t = \frac{e^t - e^{-t}}{2} = \sinh t$$

$$\frac{\partial}{\partial x} f_{mn} = \sqrt{m^2+n^2}\cosh\left[\sqrt{m^2+n^2}\,x\right) \sin my \,\sin nz$$

$$\frac{\partial^2 f_{mn}}{\partial x^2} = (m^2+n^2)\sinh\left(\sqrt{m^2+n^2}\,x\right) \sin my \,\sin nz =$$

$$= (m^2+n^2) f_{mn}$$

$$\frac{\partial^2 f_{mn}}{\partial y^2} = -m^2 f_{mn}, \quad \frac{\partial^2 f_{mn}}{\partial z^2} = -n^2 f_{mn}$$

$$\Longrightarrow \nabla^2 f_{mn} = 0$$

$$\vec{F}_{mn} = \Big(\sqrt{m^2+n^2}\cosh\big(\sqrt{m^2+n^2}\,x\big)\sin my\,\sin nz,$$

$$m\sinh\big(\sqrt{m^2+n^2}\,x\big)\cos my\,\sin nz,$$

$$n\sinh\big(\sqrt{m^2+n^2}\,x\big)\sin my\,\cos nz\Big)$$

EXTRA CREDIT: $f_{mn}(x,y,z)$ IS NONZERO ON THE FACE $x=\pi$, $0 \le y,z \le \pi$ OF THE SQUARE.

ITS VALUES THERE ARE

$$f_{mn}(\pi,y,z) = \sinh\big(\sqrt{m^2+n^2}\,\pi\big)\sin my\,\sin nz$$