MTL 100 (Calculus): Tutorial Sheet 1(a)

Taylors Theorem

1. Show that

   (a) $e^{ax}\cos bx = 1 + ax + \frac{a^2-b^2}{2!}x^2 + \frac{a(a^2-3b^2)}{3!}x^3 + \cdots$

   (b) $\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$

   (c) $\sin(x) = 1 - \frac{(x-\frac{\pi}{2})^2}{2!} + \frac{(x-\frac{\pi}{2})^4}{4!} - \cdots$

   (d) $\tan(x) = x + \frac{x^3}{3} + \frac{2}{15}x^5 + \cdots$

   (e) $\sin^{-1} x = x + \frac{x^3}{6} + \cdots$

2. (a) If $f(x) = x^8 - 2x^7 + 5x^6 - x + 3$, find three terms of the expansion by Taylor's formula about the point 2 and show that $f(2.02)$ and $f(1.97)$ are approximately 343.4 and 289.9.

   (b) Using $\log(1+x) \approx x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4}$, find $\log(1.5)$ and estimate the error.

3. Using Taylor's formula, compute the following:

   (a) $\lim_{x\to 0} \frac{(\ln(1+x))^2 - \sin^2 x}{1-e^{-x^2}}$

   (b) $\lim_{x\to 0} \frac{\sin x - \cos x}{x^2 \sin x}$

4. Let n be the degree of the approximating polynomial.

   (a) Write down the Maclaurin's formula for

   (i) $f(x) = \sqrt{1+x}$, $n = 2$; (ii) $f(x) = \cos^2(x)$, $n = 4$

   (iii) $f(x) = \sin(x^3)$, $n = 4$; (iv) $f(x) = \frac{1}{1+x+x^2+x^3}$, $n = 3$

   (b) Write down the Taylor's formula for $f(x) = \sqrt{x}$ about 1, for $n = 3$.

5. Estimate the error in the following:

   (a) $\ln(\cos x) \approx -\frac{x^2}{2} - \frac{x^4}{12}$

   (b) $\ln(x + \sqrt{(1-x^2)}) \approx x - x^2 + \frac{5x^3}{6}$

6. Prove that if $x > 0$ then

   $\ln(\frac{x+1}{x}) = 2((2x+1)^{-1} + \frac{(2x+1)^{-3}}{3} + \frac{(2x+1)^{-5}}{5} + \cdots))$

   Given that $\ln 2$=0.69315, show without using log tables that ln(17)=2.8332.

7. In Maclaurin's expansion, show that if $f^{(n+1)}(x) \neq 0$ is continuous and θ_n is a number between 0 and 1 that appears in $R_n(x) = \frac{x^n}{n!}f^{(n)}(\theta_n x)$, then $\theta_n \to \frac{1}{n+1}$ as $x \to 0$.