PYL100: Electromagnetic Waves and Quantum Mechanics (I Semester, 2016-17)

Exercise Sheet No.8

QM applications: One-dimensional problems involving the Schrödinger Equation

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Infinite Potential Well

1. Calculate the Zero point energy for a particle in an infinite potential well for the following cases:
   (a) A 100 g ball confined on a 5 m long line,
   (b) An oxygen atom confined to a 2×10^{-10} m lattice.
   (c) An electron confined to a 10^{-10} m atom.
   [**Ans.**: (a) 1.25×10^{-49} eV, (b) 3×10^{-4} eV, (c) 30 eV.]

2. A proton is confined in an infinite square well of width 10 fm. (The nuclear potential that binds protons and neutrons in the nucleus of an atom is often approximated by an infinite square well potential.) (a) Calculate the energy and wavelength of the photon emitted when the proton undergoes a transition from the first excited state (n = 2) to the ground state (n = 1). (b) In what region of the electromagnetic spectrum does this wavelength belong?
   [**Ans.**: 6.16 MeV, 202 fm → Gamma ray region of the electromagnetic spectrum]

3. The normalized wave-functions and the corresponding energies of various states for a particle of mass 'm' in an infinite-square-well of dimension 'a' are given as,
   $$\psi_n(x) = \sqrt{\frac{2}{a}} \sin\left\{\frac{n\pi x}{a}\right\} \quad \text{and} \quad E_n = \frac{n^2\pi^2\hbar^2}{2ma^2}, \quad n = 1, 2, \ldots$$
   If the initial wave function is given to be $\Psi(x,0) = A \sin\left(\frac{3\pi}{2a}x\right)\cos\left(\frac{\pi}{2a}x\right)$, then
   (a) Identify the *states* involved in the above wave function
   (b) Write down the wave function at a later time *t*, i.e. $\Psi(x,t)$
   (c) Determine the normalization constant *A*.

Finite Potential Well

4. Show that the ground state energy of a particle of mass 'm' in a symmetric finite square well potential of width 'a' is less than the ground state energy of the same particle in an infinite square potential well of same width.

5. A particle of mass 'm' and energy 'E' $(E < V_0)$ moves in the following potential:
   $$V(x) = \begin{cases} \infty & \text{for } x \le 0 \\ 0 & \text{for } 0 < x < L \\ V_0 & \text{for } x \ge L \end{cases}$$

(a) Write down the general solutions for the wave functions in the regions $0 < x < L$ and $x \geq L$.
(b) Using appropriate boundary conditions, show that,

$$\cot\left(\frac{\sqrt{2mE}}{\hbar}L\right) = -\sqrt{\frac{V_0 - E}{E}}$$

6. A particle of mass 'm' is in a 1-D **finite** potential well of the form given below:

$$V(x) = \begin{cases} 0 & for\,|x| < \frac{a}{2} \\ V_0 & for\,|x| \geq \frac{a}{2} \end{cases}$$

Given the depth of the potential well, V_0 lies in the range: $\frac{\pi^2\hbar^2}{2ma^2} < V_0 < \frac{2\pi^2\hbar^2}{ma^2}$.

i) Using graphical method, determine how many bound states are possible.
ii) Draw the form of the appropriate eigenfunctions corresponding to these bound states.

7. There are only **three** bound states for a particle of mass 'm' in a 1-D finite potential well of the form given below:

$$V(x) = \begin{cases} 0 & for\,|x| < \frac{a}{2} \\ V_0 & for\,|x| \geq \frac{a}{2} \end{cases}$$

Find the range of the depth of the potential well. [**Ans.** $\frac{2\pi^2\hbar^2}{ma^2} < V_0 < \frac{9\pi^2\hbar^2}{2ma^2}$]

8. A particle of mass 'm' is confined in finite square well potential extending from $x = -a/2$ to $x = a/2$. It is in its **ground state** stationary state. The value of the ground state eigen function at $x = 0$ is $\Psi = B$. And at $x = a/4$ it is $\Psi = \frac{\sqrt{3}}{2}B$, at a particular instant.
(a) Calculate the ground state energy of the particle.
(b) Calculate the depth of the potential well.

[**Ans.**(a) $\frac{2\pi^2\hbar^2}{9mL^2}$, (b) $\frac{8\pi^2\hbar^2}{9mL^2}$]

Potential Step

9. For the Potential Step problem the wave function in the region $x > 0$ is of the form $e^{-\gamma x}$. Calculate $\langle x\rangle$, $\langle x^2\rangle$ and show that $\Delta x \sim 1/\gamma$. [**Hint:** $\langle x\rangle = \frac{\int_0^\infty x\, e^{-2\kappa x}\, dx}{\int_0^\infty e^{-2\kappa x}\, dx}$], etc.]

10. Particles of energy 9 eV are sent towards a potential step 8 eV high. What percentage of particles will reflect back? [**Ans**.: 25 %]

Barrier Potential

11. Consider a barrier potential, defined as

$$
\begin{aligned}
V &= 0 && \text{for} \quad x < 0, \\
&= V_0 && \text{for} \quad 0 < x < a, \\
&= 0 && \text{for} \quad x > 0.
\end{aligned}
$$

Show that for a bound state problem $(E < V_0)$, when the barrier is quite high as compared to the energy $(V_0 \gg E)$ or the barrier width 'a' is large, such that $\gamma a \gg 1$, the Transmission coefficient or Transmission probability is given by: $T = \frac{16\, E(V_0 - E)}{{V_0}^2}\, e^{-2\gamma a} \approx e^{-2\gamma a}$.

12. Electrons of energy 1 eV are incident on a barrier of height 10 eV and width 0.5 nm. Find the transmission probability *T*. [**Ans.**: 3×10^{-7}]

13. Particles of energy E are incident on a rectangular potential barrier of height V_0 and width a. Consider the case $E > V_0$. Show that for a given V_0, there is perfect transmission and the particles do not reflect for certain widths of the barrier. Find the values of these widths. Sketch the transmission probability T as a function of the width of the barrier.