PYL100: Electromagnetic Waves and Quantum Mechanics (II Semester, 2016-17)

<u>Exercise Sheet No.7</u>

Part 2: QM basics -- Operators, Commutator, Eigenfunctions & Eigenvalues, Postulates

1. The wave function at t = 0 is given by: $\psi(x,0)=\frac{1}{\sqrt{2}}\phi_1+\frac{1}{\sqrt{2}}\phi_2$, where ϕ_1 and ϕ_2 are normalized eigenstates of the Hamiltonian with energy eigenvalues $E_1, E_2\ (E_2 > E_1)$ respectively. Calculate the shortest time after which $\psi(x,0)$ will become orthogonal to $\psi(x,t)$? [**Ans.** $\frac{\pi\hbar}{E_2-E_1}$]

2. The wave function of a spinless particle moving under a one-dimensional potential $\psi(x)=Ae^{-\alpha^2x^2}\ (-\infty < x < \infty)$, where A is Normalization constant corresponding to the energy eigenvalue $E_0=\frac{\alpha^2\hbar^2}{m}$. What will be the form of the one-dimensional potential? [**Ans.** $2E_0\alpha^2x^2$]

3. The wave function corresponding to a particle is given by: $\psi(x)=\frac{1}{\sqrt{a}}\exp\left(-\frac{|x|}{a}\right)$
   (a) Calculate the probability of finding the particle in the region –a < x <a.
   (b) Find the value of 'b' such that probability of finding the particle between -b < x < b is 0.5.
   [**Ans.** (a) $1-\frac{1}{e^2}$, (b) $\frac{a\ln 2}{2}$]

4. Consider two states Ψ_1 and Ψ_2, defined as follows:
$$\Psi_1 = 2\,\phi_1 - 3\,\phi_2,$$
$$\Psi_2 = \phi_1 + \alpha\,\phi_2,$$
   Where ϕ_1 and ϕ_2 are the orthonormal eigenfunctions in a 1D infinite potential. For what value of 'α' is Ψ_1 orthogonal to Ψ_2?

5. Assume $\Psi(\boldsymbol{r})=\frac{1}{r}\,e^{ikr}$, where $r=\sqrt{x^2+y^2+z^2}$. Calculate J(**r**,t) and interpret the result physically.

6. Find out the probability current density $\mathrm{J}(x,t)$ for a plane wave given by: $\Psi(x,t)=Be^{i(kx-\omega t)}$.

7. Derive the Continuity equation for probability current density *J(x,t)* for a one-dimensional motion of a flux of electrons, considering their wave function as $\Psi(\mathrm{x},\mathrm{t})$.

8. Show that: $\mathbf{J}=Re\left(\Psi^*\frac{\hbar}{im}\nabla\Psi\right)$. Hence, show that $\mathbf{J}=Re\left(\Psi^*\frac{\hat{\boldsymbol{p}}}{m}\nabla\Psi\right)=Re\left(\Psi^*\,\hat{\boldsymbol{v}}\,\nabla\Psi\right)$ where $\hat{\boldsymbol{v}}=\frac{\hat{\boldsymbol{p}}}{m}$ is the velocity operator.

9. Show that the Energy Eigenfunctions are orthonormal.

10. Which of the following is an eigenfunction the kinetic energy operator?
    (a) $Ae^{-x/a}$ (b) $Ae^{x/a}$ [**Ans.** None of them]

11. If a wave function $\psi(x)$ satisfies the following relation $\hat{x}\,\psi(x) = -\left(\frac{ia^2}{\hbar}\right)\hat{p}_x\,\psi(x)$ ('a' is real positive quantity), then what will be the form of $\psi(x)$? [**Ans.** $Ae^{-\frac{x^2}{2a^2}}$]

12. Calculate the value of the commutator brackets:
    (a) $\left[\hat{x}^2, \hat{p}_x{}^2\right]$
    (b) $\left[\hat{x}, \left[\hat{x}, \hat{H}\right]\right]$
    (c) $\left[x, \frac{\partial^2}{\partial x^2}\right]$ [Ans. (a) $2i\hbar\left(\hat{p}_x\hat{x} + \hat{x}\hat{p}_x\right)$, (b) $-\frac{\hbar^2}{m}$, (c) $-2\frac{\partial}{\partial x}$]

13. Discuss the hermiticity of the following operators:
    (i) $\hat{B} = \left(\hat{A}^\dagger + \hat{A}\right)$,
    (ii) $\hat{C} = i\left(\hat{A}^\dagger - \hat{A}\right)$

14. Find the Hermitian conjugate of: $f(\hat{A}) = \left(1 + \mathrm{i}\hat{A} + 3\hat{A}^2\right)\left(1 - 2\mathrm{i}\hat{A} - 9\hat{A}^2\right)$

15. Discuss the Hermiticity of the operators: (a) $\frac{d}{dx}$ and (b) $e^{i\,d/dx}$

16. Show that (a) $\left[\hat{L}_x, \hat{L}_y\right] = i\hbar\,\hat{L}_z$
    (b) $\left[\hat{L}_y, \hat{L}_z\right] = i\hbar\,\hat{L}_x$
    (c) $\left[\hat{L}^2, \hat{L}_z\right] = 0$
    (d) $\left[\hat{L}^2, \hat{\boldsymbol{L}}\right] = 0$

17. Consider a Gaussian wave packet for a free particle with the normalized wave function:

$$\Psi(x,0) = \left(\frac{2}{\pi\,a^2}\right)^{1/4} e^{-\frac{x^2}{a^2}}\; e^{i\,k_0 x}$$

    i) Verify that the wave function is normalized.
    ii) Evaluate the expectation value for position: $\langle x\rangle$.
    iii) Evaluate the expectation value for momentum: $\langle k\rangle$
    iv) Given $\langle x^2\rangle = \frac{a^2}{4}$ and $\langle k^2\rangle = k_0{}^2 + \frac{1}{a^2}$. Show that the product of the uncertainties is minimum and equal to: $\Delta x\,\Delta k = \frac{1}{2}$.