MTL-101 Mathematics: Practice Sheet 4
Semester 1: 2016-2017
Linear Transformations

1. Verify if the following $T : V_F \to W_F$ are linear transformations (LT)? If yes, determine their kernel and the range.

   (a) $V = \mathbb{R}^3,\ W = \mathbb{R}^2,\ F = \mathbb{R};\ T(x_1, x_2, x_3) = (x_1, x_2)$
   (b) $V = \mathbb{R}^2,\ W = \mathbb{R}^3,\ F = \mathbb{R};\ T(x_1, x_2) = (x_1, x_2, x_3)$
   (c) $V = \mathbb{R}^3,\ W = \mathbb{R}^2,\ F = \mathbb{R};\ T(x_1, x_2, x_3) = (x_2 - x_1, x_3 - x_2)$
   (d) $V = \mathbb{R}^3,\ W = \mathbb{R}^3,\ F = \mathbb{R};\ T(x_1, x_2, x_3) = (2x_1 - x_2 - x_3, 2x_2 - x_1 - x_3, 2x_3 - x_1 - x_2)$

2. Check if the following $T : P_2 \to P_2$ are LT?
   (i) $T(a_0 + a_1x + a_2x^2) = a_0 + a_1(x + 1) + a_2(x + 1)^2$
   (ii) $T(a_0 + a_1x + a_2x^2) = (a_0 + 1) + (a_1 + 1)x + (a_2 + 1)x^2$.

3. Find nullity of T when (i) $T : P_4 \to P_3$ has rank 1, (ii) $T : M_{22} \to M_{22}$ has rank 3.

4. For a positive integer $n > 1$, let $T : M_{nn}(\mathbb{R}) \to \mathbb{R}$ be a LT defined by $T(A) = \text{trace}\,(A)$. Determine the nullity of T.

5. Let $D : P_3 \to P_2$ be the differentiation transformation $D(p) = p^{'}$. Describe kernel of D.

6. Consider a basis $\{v_1 = (-2, 1), v_2 = (1, 3)\}$ of $\mathbb{R}^2$ and let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be a LT such that $T(v_1) = (-1, 2, 0)$ and $T(v_2) = (0, -3, 5)$. Find an expression for $T(x_1, x_2)$ and use it to find $T(2, -3)$.

7. Consider a basis $\{v_1 = (1, 1, 1), ((1, 1, 0), (1, 0, 0)\}$ of $\mathbb{R}^3$. Let T be a LT from $\mathbb{R}^3$ to $\mathbb{R}^3$ such that $T(v_1) = (2, -4, 1),\ T(v_2) = (3, 0, 1),\ T(v_3) = (-1, 5, 1)$. Find a formula for $T(x_1, x_2, x_3)$. Use it to find $T(2, 4, -1)$.

8. Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be given by $T(x_1, x_2, x_3) = (x_1 + x_2, x_1 - x_2)$. Write the matrix of T with respect to the ordered bases B_1 and B_2, where
   (i) $B_1 = \{(1, 1, 1), (0, 1, 0), (1, 0, 1)\},\ B_2 = \{(1, 1), (0, 1)\}$,
   (ii) $B_1 = \{(3, 4, 5), (1, 0, 1), (1, 2, 3)\},\ B_2 = \{(2, 3), (1, 2)\}$.

9. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be given by $T(x_1, x_2, x_3) = (x_1 + x_3, x_1 - x_2, x_1 + x_2 + x_3)$. Write the matrix M_1 of T with respect to the ordered basis $B_1 = \{(1, 2, 3), (3, 4, 5), (1, 0, 1)\}$. Write the matrix M_2 of T with respect to the ordered basis $B_2 = \{(1, 1, 0), (0, 1, 1), (1, 0, 1)\}$. Find an invertible matrix P, if it exists, such that $M_1 = P^{-1}M_2P$.

10. Let $T : P_2 \to P_2$ be a LT defined by $T(p(x)) = p(3x - 5)$. Find matrix of T with respect to basis $B_1 = \{1, x, x^2\}$. Use this matrix to compute $T(1 + 2x + 3x^2)$.

11. Let $A = \begin{pmatrix} 1 & 3 & -1 \\ 2 & 0 & 5 \\ 6 & -2 & 4 \end{pmatrix}$ be a matrix of $T : P_2 \to P_2$ with respect to the basis $B = \{v_1 = 3x + 3x^2, -1,\ v_2 = 3x + 2x^2,\ v_3 = 3 + 7x + 2x^2\}$. Find $T(v_i),\ i = 1, 2, 3$. Find a formula for $T(p(x)),\ p(z) \in P_2$. Use this formula to find $T(1 + x^2)$.

12. Write all eigenvalues and the corresponding eigenvectors of $T : \mathbb{R}^3 \to \mathbb{R}^3$ when
    (i) $T(x_1, x_2, x_3) = (2x_1, x_1 + x_3, x_1 - x_2)$, (ii) $T(x_1, x_2, x_3) = (3x_1 - 2x_3, x_2 - x_1 + x_3, 2x_1 - 3x_2)$.

13. Find the eigenvalues and eigenvectors of the following matrices:
    $$\begin{pmatrix} 1 & 1 & -2 \\ -1 & 2 & 1 \\ 0 & 1 & -1 \end{pmatrix} \quad \begin{pmatrix} 1 & 2 & 2 \\ 0 & 2 & 1 \\ -1 & 2 & 2 \end{pmatrix} \quad \begin{pmatrix} 2-i & 0 & i \\ 0 & 1+i & 0 \\ i & 0 & 1-i \end{pmatrix}$$

14. Prove that (i) the eigenvalues of a skew Hermitian matrix are purely imaginary or zero.
    (ii) Eigenvalues of a real symmetric matrix for different eigenvalues are orthogonal.
    (iii) The eigenvalues of a unitary matrix have the absolute value 1.