Department of Mathematics
MTL 100: Calculus
2014-15: Semester I
Minor 2

10 October 2014

Please begin the answer to each question on a new page, and give adequate explanation for full credit.

1. Let f be a real-valued function, which is uniformly continuous on (a, b). Show that there exists a real-valued function $\overline{f}$, continuous on $[a, b]$, such that $\overline{f}(x) = f(x)$ for each $x \in (a, b)$. [4]

2. (a) Suppose f is a real-valued function, continuous on $[a, b]$ and differentiable on (a, b), with $f(a) = f(b) = 0$. Prove that for every real number α, there is some $c \in (a, b)$ such that $f'(c) = \alpha f(c)$.
   [HINT: Apply Rolle's Theorem to $g(x)f(x)$ for some suitable $g(x)$ depending on α.] [3]

   (b) Determine the Taylor's expansion for $f(x) = \sqrt{x}$ about $x = 1$, giving an explicit formula for the nth term, its radius of convergence, and show that the remainder $R_n(x)$ tends to 0 as $n \to \infty$ for each $x \in [1, 2]$. [3]

3. (a) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by

$$f(x,y) = \begin{cases} xy\left(\dfrac{x^2 - y^2}{x^2 + y^2}\right) & \text{if } (x,y) \neq (0,0); \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

   Discuss the differentiability of f at $(0, 0)$. [3]

   (b) If $f(x,y) = e^{xy}$, where $x = \log(u^2 + v^2)^{1/2}$ and $y = \tan^{-1}\left(\frac{u}{v}\right)$, find $\frac{\partial f}{\partial u}$ and $\frac{\partial f}{\partial v}$ using `Chain Rule`. [3]

4. Show that the closed cylinder (with lids) with the greatest surface area that can be inscribed in a sphere of radius a has the altitude $h = a\sqrt{2 - \frac{2}{\sqrt{5}}}$ and the radius of the base $r = \frac{a}{2}\sqrt{2 + \frac{2}{\sqrt{5}}}$. [4]