MAT MEK 4270 Mandatory Assignment 1

1.2.3

$$\frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u$$

$$\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

$$u(t,x,y) = e^{i(k_x x + k_y y - \omega t)}$$

$$u_t = \frac{\partial}{\partial t}\left(e^{i(k_x x + k_y y - \omega t)}\right) = -i\omega u$$

$$u_{tt} = \frac{d}{dt}\left(-i\omega u\right) = (-i\omega)^2 u = -\omega^2 u$$

$$\frac{\partial u}{\partial x} = i k_x u \qquad \frac{\partial^2 u}{\partial x^2} = (i k_x)^2 u = -k_x^2 u$$

$$\frac{\partial u}{\partial y} = i k_y u \qquad \frac{\partial^2 u}{\partial y^2} = (i k_y)^2 u = -k_y^2 u$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -k_x^2 u - k_y^2 u = -(k_x^2 + k_y^2) u$$

$$\frac{\partial^2 u}{\partial t^2} = -\omega^2 u$$

$$-\omega^2 \cancel{u} = c^2\left[-(k_x^2 + k_y^2)\cancel{u}\right]$$

$$\boxed{\omega = \pm c\sqrt{k_x^2 + k_y^2}}$$

1.2.4

$$u_{i,j}^{n+1} = e^{-i\bar{\omega}\Delta t} u_{i,j}^{n} \qquad u_{i,j}^{n-1} = e^{+i\bar{\omega}\Delta t} u_{i,j}^{n}$$

LHS

$$\left(e^{-i\bar{\omega}\Delta t} - 2 + e^{i\bar{\omega}\Delta t}\right) u_{i,j}^{n} = 2\left(\cos(\bar{\omega}\Delta t) - 1\right) u_{i,j}^{n}$$

$$k_x = k_y = k$$

$$u_{i+1,j}^{n} = e^{ikh} u_{i,j}^{n}, \quad u_{i-1,j}^{n} = e^{-ikh} u_{i,j}^{n}, \quad u_{i,j+1}^{n} = e^{ikh} u_{i,j}^{n}$$

$$\circ \quad u_{i,j-1}^{n} = e^{-ikh} u_{i,j}^{n}$$

RHS $\quad C^2\left(4\cos(kh) - 4\right) u_{i,j}^{n} = 4C^2\left(\cos(kh) - 1\right) u_{i,j}^{n}$

$$2\left(\cos(\bar{\omega}\Delta t) - 1\right) \cancel{u_{i,j}^{n}} = 4C^2\left(\cos(kh) - 1\right) \cancel{u_{i,j}^{n}}$$

$$1 - \cos\theta = 2\sin^2(\theta/2)$$

$$1 - \cos(\bar{\omega}\Delta t) = 2C^2\left(1 - \cos(kh)\right)$$

$$\hookrightarrow 2\sin^2\left(\frac{\bar{\omega}\Delta t}{2}\right) = 4C^2\sin^2\left(\frac{kh}{2}\right)$$

$$\hookrightarrow \sin^2\left(\frac{\bar{\omega}\Delta t}{2}\right) = 2C^2\sin^2\left(\frac{kh}{2}\right)$$

$$\sin\left(\frac{\bar{\omega}\Delta t}{2}\right) = \sqrt{2}\,C\sin\left(\frac{kh}{2}\right)$$

$$\bar{\omega} = \frac{2}{\Delta t}\arcsin\left(\sqrt{2}\,C\sin\frac{kh}{2}\right) \quad \text{w/ } C = \frac{1}{\sqrt{2}}$$

$$\bar{\omega} = \frac{2}{\Delta t}\arcsin\left(\sin\frac{kh}{2}\right) \quad \text{w/ } \arcsin(\theta) = \theta$$

$$\boxed{\bar{\omega} = \frac{\cancel{2}}{\Delta t}\cdot\frac{kh}{\cancel{2}} = \frac{kh}{\Delta t}}$$

$$\omega = c\sqrt{k_x^2 + k_y^2} = ck\sqrt{2} \quad \text{for } k_x = k_y = k$$

$$C = \frac{c\Delta t}{h} \to C = \frac{1}{\sqrt{2}} \to c = \frac{h}{\Delta t}C = \frac{h}{\Delta t}\frac{1}{\sqrt{2}} \to \omega = ck\sqrt{2} = \frac{h}{\Delta t}\frac{1}{\sqrt{2}}k\sqrt{2}$$

$$\boxed{\omega = \frac{kh}{\Delta t}}$$

so $\omega = \bar{\omega}$ for $C = \frac{1}{\sqrt{2}}$