Topological Stability and Geometric Symmetries in Neural Manifolds via High-Order State Space Augmentation

Enzo Boulin
Sant'Anna School of Advanced Studies

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Abstract

Neural coding theory suggests that information is represented in the collective activity of neuronal populations, forming trajectories within an N-dimensional state space. These trajectories are constrained to low-dimensional structures termed neural manifolds. This study investigates the geometry of these manifolds in the nonhuman primate sensorimotor cortex during reaching tasks. While standard unsupervised techniques (PCA, Autoencoders) yield suboptimal separability for behavioral decoding (37–54%), we propose a supervised framework utilizing Linear Discriminant Analysis (LDA) coupled with a quadratic state-space expansion ($O(N^2)$). This approach introduces necessary non-linearities while maintaining the stability of a linear classifier, increasing decoding accuracy to 89%. Furthermore, we identify robust cross-subject topological stability. By analyzing geometric symmetries, we demonstrate that neural manifolds are conserved across subjects (left vs. right arm reaching) up to a specific reflection transformation, suggesting the existence of universal latent structures in motor control.

1 Introduction

The instantaneous state of a neural population is represented by a vector $\mathbf{x}_t \in \mathbb{R}^N$, where each dimension corresponds to the firing rate of one of N recorded neurons. As the network evolves over time $t \in T$, the vector $\mathbf{x}_t$ traces a trajectory. The aggregate of these trajectories defines the neural manifold $\mathcal{M}$:

$$\mathcal{M} = \{\mathbf{x}_t : t \in T\} \subset \mathbb{R}^N \tag{1}$$

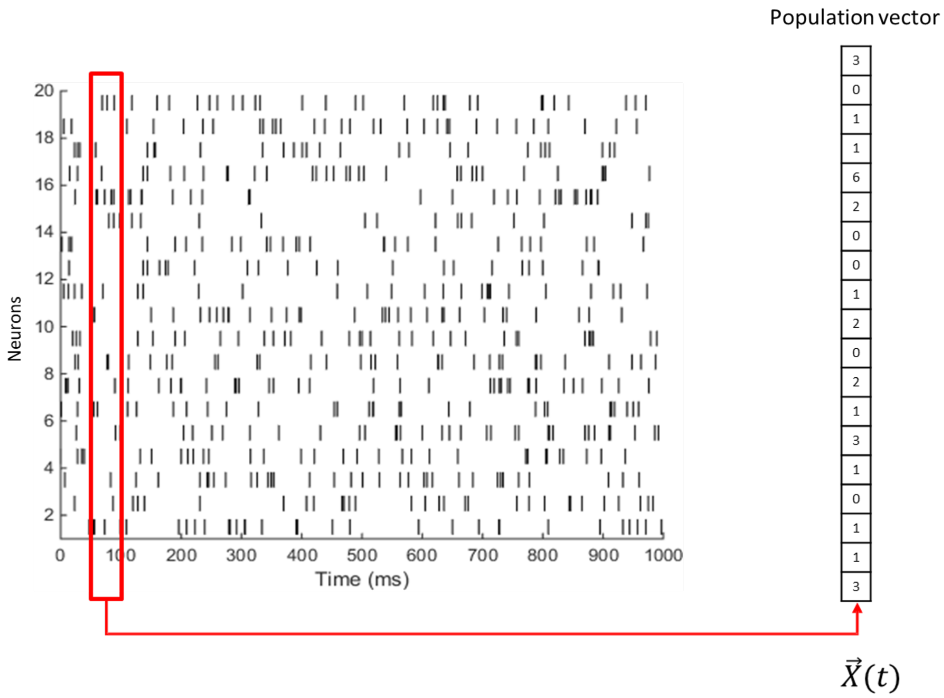

Figure 1: Conceptual framework: From multi-unit raster plots to population vectors in N-dimensional state space.

While the raw high-dimensional geometry reflects physical connectivity, the functional relationships governing behavior often reside on a lower-dimensional embedding. Understanding these dynamics requires dimensionality reduction to project $\mathcal{M}$ into a latent space where behavioral intent becomes discriminative and interpretable.

2 Methodology

2.1 Data Acquisition

Neural activity was recorded from nonhuman primates performing a self-paced 2D reaching task toward targets on an 8×8 grid. Data was captured from the Primary Motor Cortex (M1) and Somatosensory Cortex (S1) using multi-channel microelectrode arrays (96–192 channels).

2.2 Dimensionality Reduction Benchmarking

We evaluated the efficacy of several projection techniques using a linear SVM for classifying reaching directions into four cardinal quadrants.

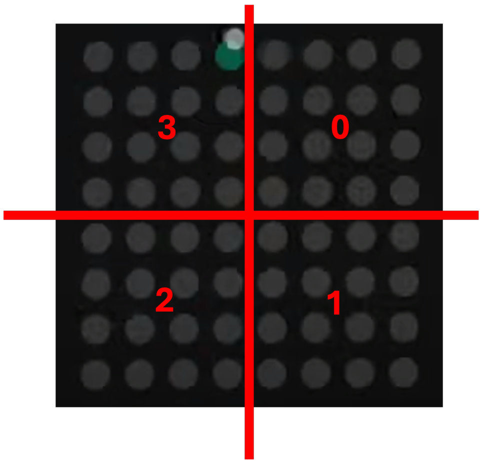

Figure 2: Experimental setup for primate sensorimotor recording during 2D reaching.

- **PCA:** Principal Component Analysis achieved only 37% accuracy, suggesting that maximum variance axes do not necessarily align with task-relevant features.
- **Autoencoders:** Non-linear compression via deep bottleneck architectures improved performance to 54%, yet remained limited by the lack of explicit class-supervision during feature extraction.

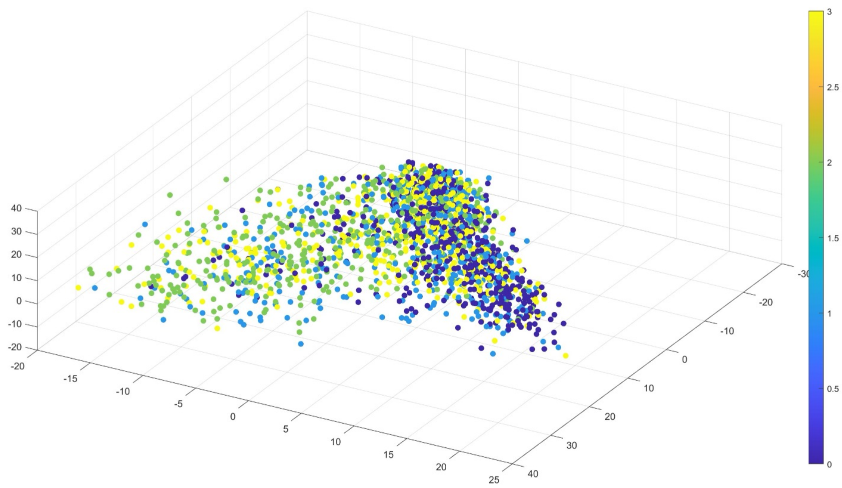

Figure 3: Visualization of the latent space via Principal Component Analysis (PCA).

2.3 Linear Discriminant Analysis (LDA)

To maximize class separability, we employed Linear Discriminant Analysis. LDA seeks a projection vector $\vec{w}$ that maximizes the ratio of between-class variance to within-class variance.

Let C be the number of classes, μ_i the mean of class i, and μ the global mean. The between-class scatter matrix Σ_b is defined as:

$$\Sigma_b = \frac{1}{C}\sum_{i=1}^{C}(\mu_i - \mu)(\mu_i - \mu)^T \tag{2}$$

To account for the internal spread of the data, we utilize the global covariance matrix Σ. Under the LDA assumption that all classes share the same covariance structure, Σ is estimated as the weighted average of the individual class covariance matrices:

$$\Sigma = \frac{1}{n - C} \sum_{i=1}^{C} \sum_{x \in C_i} (x - \mu_i)(x - \mu_i)^T \tag{3}$$

The optimization objective (Rayleigh quotient) is to maximize the separation score S:

$$S = \frac{\sigma^2_{\text{between}}}{\sigma^2_{\text{within}}} = \frac{\vec{w}^T \Sigma_b \vec{w}}{\vec{w}^T \Sigma \vec{w}} \tag{4}$$

Using LDA on the raw manifold improved classification accuracy to 70%, surpassing the 65% baseline of an SVM trained on the full N-dimensional space.

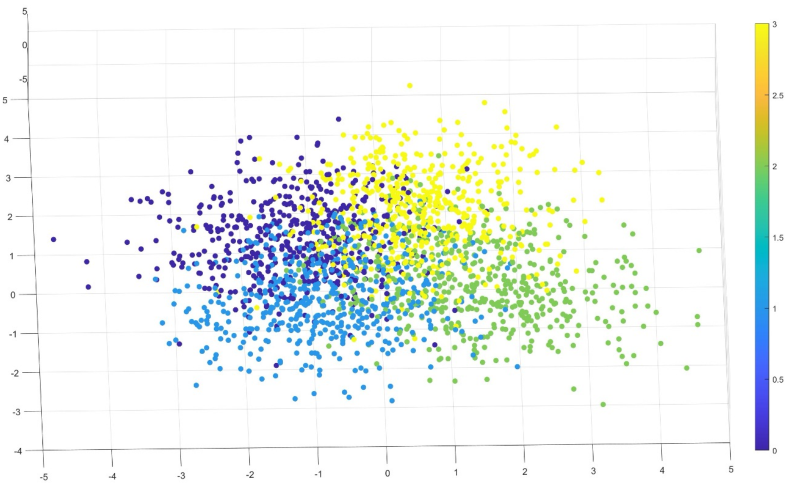

Figure 4: Unsupervised latent space visualization via Linear Discriminant Analysis

2.4 High-Order State Space Augmentation

To capture non-linear decision boundaries while preserving the robustness of linear discriminants, we employed an explicit feature mapping $\Phi(\mathbf{x})$. We performed an Order-2 polynomial expansion, mapping the original state space to a higher-dimensional space of dimension $M = N + \frac{N(N+1)}{2}$:

$$\Phi(\mathbf{x}) = [x_1, \ldots, x_N, x_1^2, x_1 x_2, \ldots, x_N^2]^T \tag{5}$$

This expansion allows the linear LDA hyperplane in the augmented space to function as a quadratic boundary in the original manifold. While the computational complexity scales as $O(N^2)$, this method provides a deterministic approach to resolving overlapping class densities.

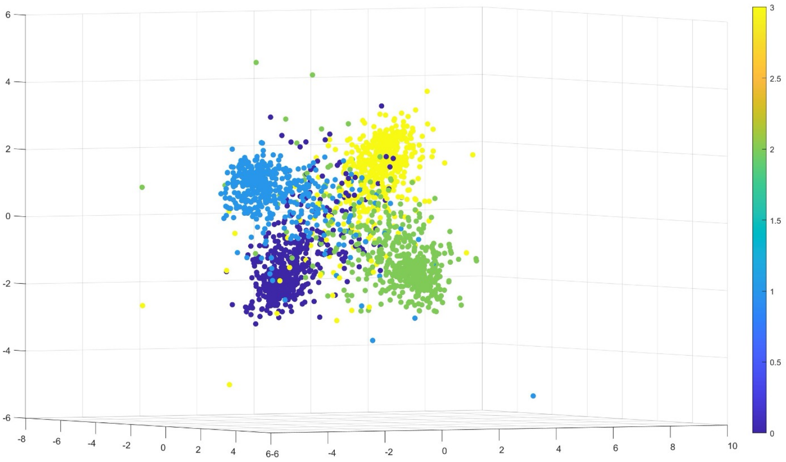

Figure 5: Discriminative latent space following quadratic state-space expansion and LDA projection.

3 Results

3.1 Classification Performance

The polynomial expansion significantly reorganized the manifold topology for decoding. The results are summarized in Table 1.

Table 1: Comparison of Decoding Accuracy across Models

Method	Type	Accuracy (%)
PCA	Unsupervised	37%
Autoencoder	Unsupervised	54%
Raw LDA	Supervised	70%
Augmented LDA (Order 2)	**Supervised**	**89%**

3.2 Topological Stability and Symmetry

We tested the hypothesis that neural manifolds represent a fundamental language of motor control across individuals.

1. **Intra-subject Stability:** Manifold structures were highly conserved across different experimental sessions within the same subject.

2. **Cross-subject Symmetry:** Initial projections between two primates "Loco" (right-handed reach) and "Indy" (left-handed reach) appeared disparate. However, we discovered a topological equivalence via a parity transformation:

$$\begin{pmatrix} x' \\ z' \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ z \end{pmatrix} \tag{6}$$

Applying this reflection aligned the manifolds, demonstrating that the latent dynamics are identical modulo the biological symmetry of the hemispheric recordings.

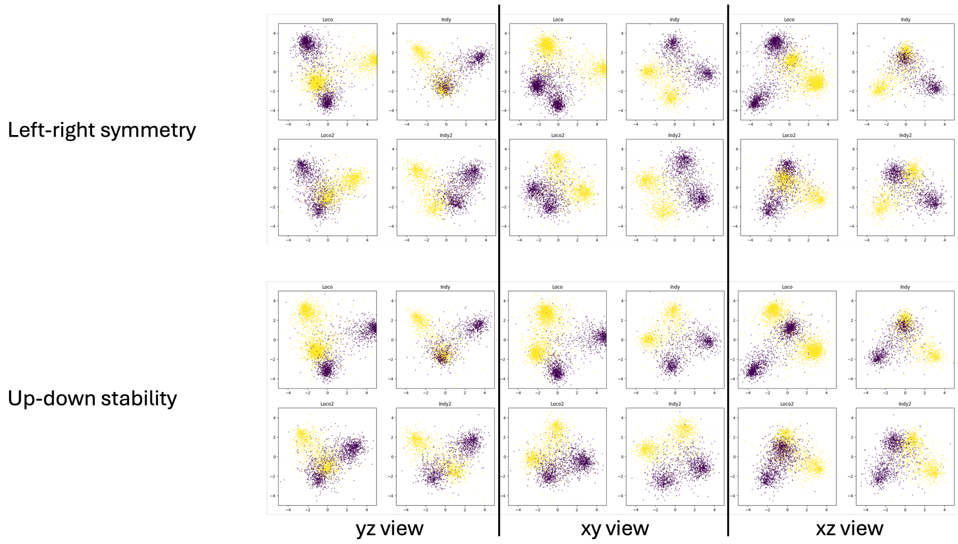

Figure 6: Invariance and geometric symmetry of the manifold

4 Discussion and Conclusion

Our findings suggest that high-dimensional neural activity is not stochastic but is constrained to specific, low-dimensional manifolds. By employing high-order state space augmentation, we achieved an 89% decoding accuracy, indicating that task-relevant information is embedded in the second-order interactions of the population activity.

The discovery of cross-subject symmetry has profound implications for Brain-Computer Interfaces (BCI). It points toward a "universal manifold" for motor control. Future BCI systems could potentially be pre-trained on large-scale datasets and "calibrated" to new users through simple geometric transformations rather than extensive supervised training.

References

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