Topological Stability and Geometric Symmetries in Neural Manifolds via High-Order State Space Augmentation

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Abstract

Neural coding postulates that information is encoded in the collective activity of neuronal populations, forming trajectories in an N-dimensional state space. These trajectories define a low-dimensional topological structure known as the neural manifold. This study investigates the geometry of these manifolds in nonhuman primate sensorimotor cortex during reaching tasks. While standard dimensionality reduction techniques (PCA, Autoencoders) yield suboptimal separability for behavioral decoding (37-54%), we propose a supervised learning approach utilizing Linear Discriminant Analysis (LDA) coupled with a quadratic state-space expansion ($O(N^2)$). This method introduces non-linearity into the linear classifier, increasing decoding accuracy from 65% (raw state space) to 89%. Furthermore, we identify robust cross-subject topological stability. By analyzing geometric symmetries, we demonstrate that neural manifolds are conserved across subjects (left vs. right arm reaching) up to a specific linear transformation ($x \to -x, z \to -z$), implying universal latent structures in motor control.

1 Introduction

The instantaneous state of a neural network can be summarized by a point X_t in an N-dimensional state space, where N represents the number of recorded neurons. The coordinates of this point constitute the population vector. Over time, the evolution of X_t traces a trajectory, and the aggregate of these trajectories forms the neural manifold $\mathcal{M}$:

$$\mathcal{M} = \{X_t; t \in T\}$$

While the geometry of the raw data reflects the connectivity between neuronal units, it does not necessarily provide an interpretable geometry of the functional manifold. To understand the functional relationships, dimensionality reduction is required to project the N-dimensional activity into a latent space where behavioral dynamics are observable.

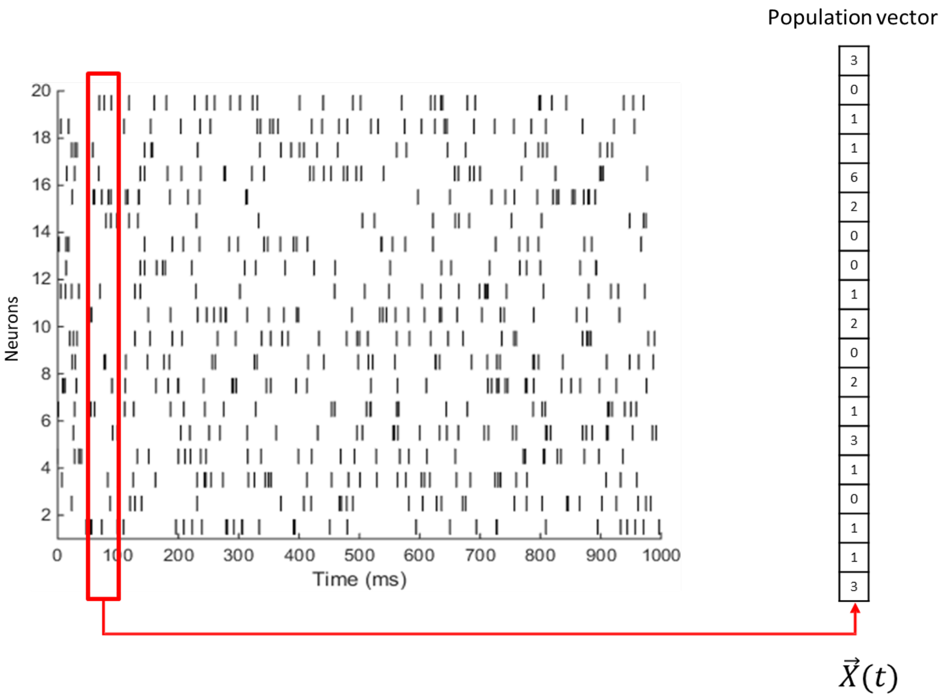

Figure 1: From Raster plot to population vector in N-dimensional state space

2 Methodology

2.1 Data Acquisition

Data was acquired from nonhuman primates performing a self-paced reaching task. Targets were arranged in an 8×8 grid. Neural activity was recorded from the Motor Cortex (M1) and Somatosensory Cortex (S1) using multi-channel arrays (96 to 192 channels).

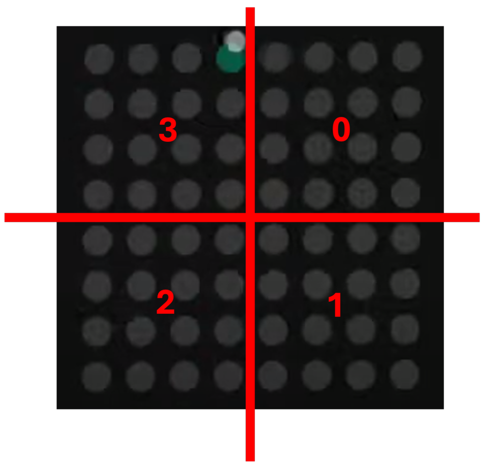

Figure 2: Primate sensorimotor recording during a 2D reaching task

2.2 Dimensionality Reduction Benchmarking

We evaluated several unsupervised and supervised reduction techniques to project the manifold into a low-dimensional space for classification. The objective was to classify the reaching direction into four distinct quadrants/classes.

Initial benchmarking using Principal Component Analysis (PCA) yielded poor separability, with a linear Support Vector Machine (SVM) achieving only 37% accuracy on the latent space. Similarly, Deep Learning approaches using Auto-Encoders (Encoder-Decoder architecture) achieved only 54% accuracy, suggesting that standard non-linear compression captures variance that is not necessarily discriminative for the specific behavioral task.

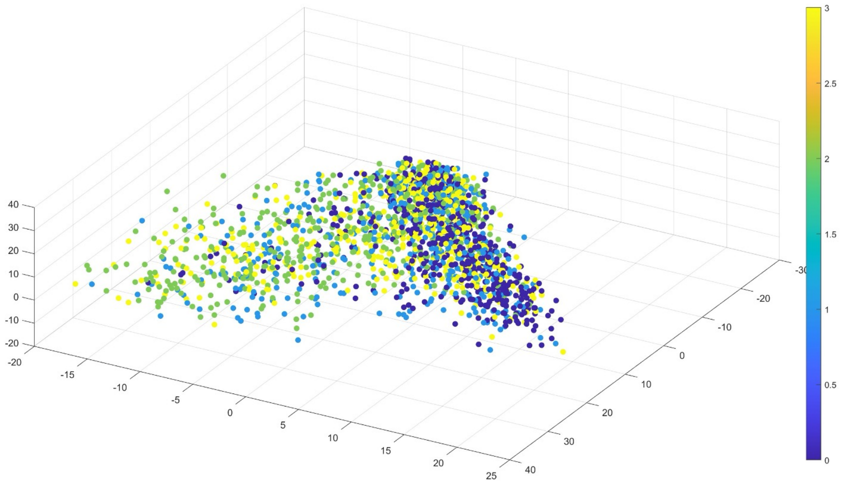

Figure 3: Unsupervised latent space visualization via Principal Component Analysis

2.3 Linear Discriminant Analysis (LDA)

To maximize class separability, we employed Linear Discriminant Analysis (LDA). LDA seeks a projection vector $\vec{w}$ that maximizes the ratio of between-class variance to within-class variance.

Let C be the number of classes, μ_i the mean of class i, and μ the global mean. The between-class scatter matrix Σ_b is defined as:

$$\Sigma_b = \frac{1}{C} \sum_{i=1}^{C} (\mu_i - \mu)(\mu_i - \mu)^T \tag{1}$$

The optimization objective (Rayleigh quotient) is to maximize the separation score S:

$$S = \frac{\sigma^2_{\text{between}}}{\sigma^2_{\text{within}}} = \frac{\vec{w}^T \Sigma_b \vec{w}}{\vec{w}^T \Sigma \vec{w}} \tag{2}$$

Using LDA on the raw manifold improved classification accuracy to 70%, surpassing the 65% baseline of an SVM trained on the full N-dimensional space.

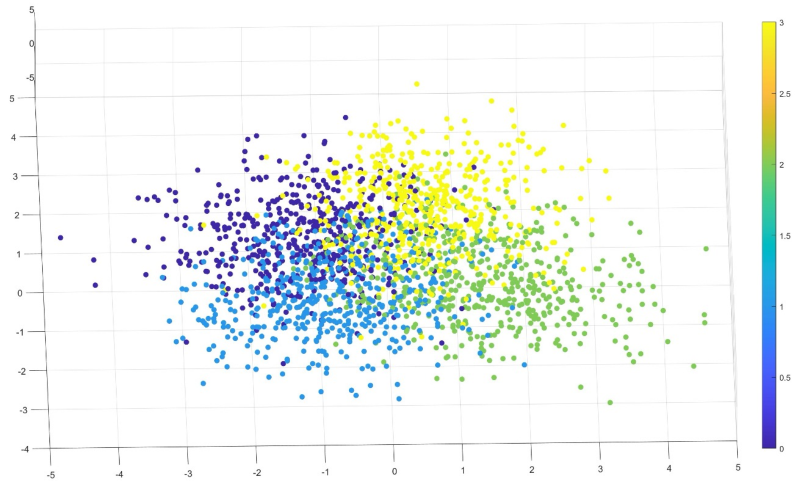

Figure 4: Unsupervised latent space visualization via Linear Discriminant Analysis

2.4 State Space Augmentation (Order 2 Expansion)

To resolve non-linear boundaries (e.g., separating concentric data) while retaining the stability of linear classifiers, we utilized the kernel trick principle via explicit feature mapping.

We augmented the state space by performing an Order 2 polynomial expansion. For a state vector with axes $x_1, \ldots, x_N$, we introduced $(N+1)N/2$ additional dimensions corresponding to interaction and quadratic terms:

$$\Phi(X) = [x_1, \ldots, x_N, x_1^2, x_1 x_2, \ldots, x_N^2]$$

We then applied LDA on this high-dimensional augmented manifold. While computationally more expensive ($O(N^2)$), this method allows a linear hyperplane in the augmented space to act as a quadratic decision boundary in the original space.

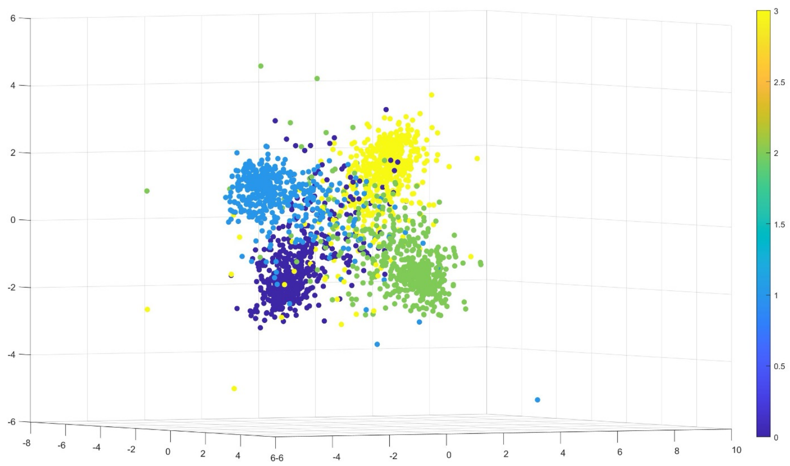

Figure 5: Discriminative manifold after quadratic state-space expansion and LDA

3 Results

3.1 Classification Performance

The polynomial expansion significantly improved the manifold topology for classification tasks. The accuracy of the linear classifier trained on the Order-2 augmented manifold reached **89%**, a substantial improvement over PCA (37%), Auto-Encoders (54%), and standard LDA (70%).

3.2 Topological Stability and Symmetry

We analyzed the manifold structure across different subjects (Monkey 1: "Loco", Monkey 2: "Indy") to test for cross-subject generalization.

1. **Intra-subject Stability:** The manifold geometry remained stable across different sessions for the same subject.

2. **Cross-subject Symmetry:** Initial comparisons showed misaligned manifolds. However, upon inspecting the experimental setup, we noted that one subject reached with the right arm (left hemisphere recording) and the other with the left arm (right hemisphere recording).

We identified that the manifolds were topologically equivalent modulo a geometric transformation reflecting the biological symmetry. By applying a reflection transformation to the latent axes:

$$z' \leftarrow -z, \quad x' \leftarrow -x$$

we achieved alignment between the subjects. This confirms that the low-dimensional neural dynamics governing motor control are conserved across subjects, despite differences in physical implementation.

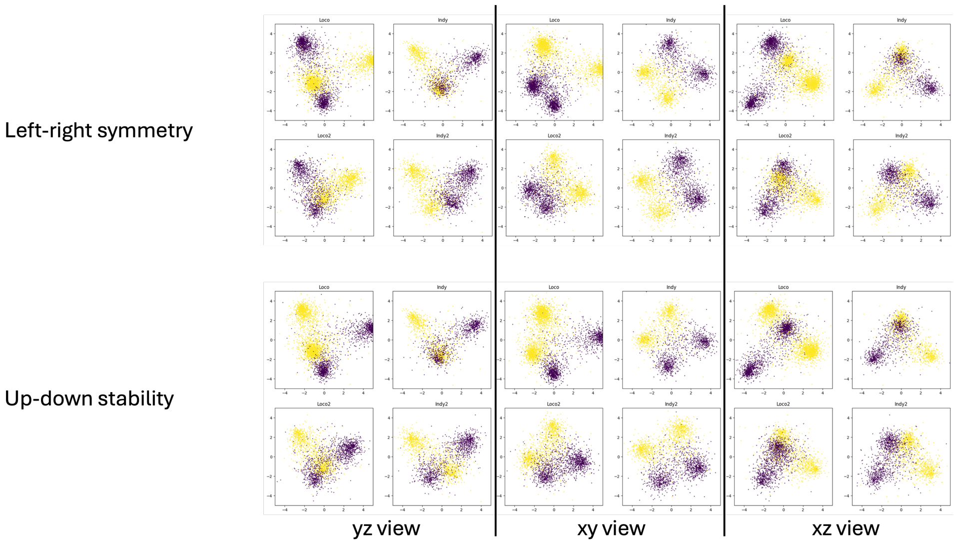

Figure 6: Invariance and geometric symmetry of the manifold

4 Discussion and Conclusion

This study demonstrates that high-dimensional neural activity collapses onto specific low-dimensional manifolds that are conserved across subjects. By employing polynomial state-space augmentation, we revealed that the functional geometry of these manifolds is highly discriminative (89% decoding accuracy).

The identification of these conserved structures and their geometric symmetries has significant implications for Brain-Computer Interfaces (BCI). It suggests that BCIs could be pre-trained on a generalized "template" manifold and adapted to new users via simple geometric transformations (rotation/reflection), reducing the need for extensive retraining.

References

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3. O'Doherty, Joseph E., et al. Nonhuman Primate Reaching with Multichannel Sensorimotor Cortex Electrophysiology.