Project 2

December 4th, 2021

- **This work is individual or group restricted**. It is however allowed to exchange opinions with colleagues and clarify doubts with lecturers.
- If the work is carried out in a group (maximum of 3 elements), the group can indicate the **contribution of each element** to the completion of 100 % of the report.
- The answers should be delivered as a report in **digital format**, using a text editor (and preferably submitted in pdf version). The report language should either be English or Portuguese.
- **Justify properly your answers**. If appropriate provide all intermediate calculations. In your answer insert all the elements that you consider relevant like graphics, code and the results from code or any other element that you consider relevant. The implemented codes must be submitted in a compressed file, within a directory for each problem, for a correct evaluation of the work developed.
- The expected duration is **1 week**; If the work is delivered after this deadline and during the following 1 or 2 days, there will be a penalty.
- The report must be **submitted** for evaluation through the online course (Moodle UPorto).
- **The project report requires an enclosed signed Personal Ethics Statement.**

Master in Economics, 2021-2022
Faculdade Economia Universidade Porto

1 Portfolio Optimization

Consider the portfolio consisting on n different assets. Each asset i has a mean rate of return f_i. The problem is to find what fraction x_i to invest in each asset i in order to minimize the risk

$$\frac{1}{2}x^T Hx,$$

where H is the covariance matrix and $x = [x_1, x_2, \cdots, x_n]^T$, subject to:

- the expected return should not be smaller than the minimal rate of portfolio return r desired by the investor,

$$\sum_{i=1}^{n} f_i x_i \geq r_i$$

- the sum of the investment fractions x_i, $i = 1, \cdots, n$, should add up 100% of the portfolio,

$$\sum_{i=1}^{n} x_i = 1$$

- and the fraction of each asset i in the portfolio should lie between zero and one (i.e. no short-selling),

$$0 \leq x_i \leq 1, \quad , i = 1, \cdots, n.$$

Retrieve the `Covariance` and `MeanReturns` data from `AssetData.mat`: just perform `load('AssetData.mat')`.

Challenge

Perform the portfolio optimization model exposed for a desired return $r = 0.002$ (0.2 percent), considering the following topics:

1. Solve the problem:
   a) Using the appropriate techniques, justify.
   b) What is the best portfolio? Show your answer making use of a bar plot.
2. Redo the same problem but now taking a sharper precision: `TolFun =` 10^{-10}. Compare the results with the default solution
3. Consider that assets 1 to 75 are related to technology, assets from 76 to 150 to automotive, and assets 151 to the last one to pharmaceutical industries. Add the following group constraints in the model: require that, at least, 30% of the investor's money has to be invested in technological assets, 30% in automotive assets, and 30% in pharmaceutical assets.
   a) Show the results using a bar plot, preferably considering different color for the three different types of assets.

b) Show that your solution satisfies the 30% minimum value for the investment in each asset group.

4. Consider other possible extensions on the model. Explore them numerically.

2 Dynamic choice

Searching for work, investing in human capital, installing or removing physical capital, are all decisions that should be considered in a dynamic way. In a static context, a consumer acts to equalize the cost-adjusted marginal utility of each unit of consumption. As well in a dynamic setting, when determining behavior over a number of periods, a decision maker should equalize its discounted marginal utility at each point in time.

A household maximizes its discounted utility over T periods: $U = \sum_{i=1}^{T} \beta^{t-1} u(c_t)$, where β is the household's discount factor. Given an initial amount of capital k_1, at period 1, the household's consumes c_1, leaving a maximum consumption of $k_1 - c_1$ for period 2. Generally speaking, we get the *flow equation* $k_{t+1} = k_t - c_t$, $t = 1, ..., T$.

The problem is formulated as

$$\max_{\{c_t\}_1^T} \sum_{i=1}^{T} \beta^{t-1} u(c_t)$$

subject to:

$$\begin{cases} \sum_{t=1}^{T} c_t + k_{T+1} & = k_1 \\ c_t & > 0 \\ k_t & > 0 \end{cases}$$

A code to perform such a dynamic choice on a finite horizon is available in `Dyn_choice_fh.m`, where a log utility function $u(c_t) = \ln(c_t)$ is considered, together with parameters $T = 10$, a discount factor of $\beta = 0.9$, and an initial endowment of $k_1 = 100$.

To understand the code, the concept of value function is introduced: the value function summarizes the value to the household of a given amount of capital, assuming that this capital will be used optimally in future periods. Starting with the final period, we know that the value of any capital which remains beyond the final period is zero $V_{T+1}(k_T + 1) = 0$. So a backward induction can be followed: when making its decision in period T, the household solves $V_T(k_T) = max_{c_T \in (0,k_T]} u(c_T) + \beta V_{T+1}(k_{T+1})$ where $k_{T+1} = k_T - c_T$, knowing $V_{T+1}(k_T + 1) = 0$. This process is repeated until $t = 1$: $V_t(k_t) = max_{c_t \in (0,k_t]} u(c_t) + \beta V_{t+1}(k_{t+1})$, $t = 1, ..., T$.

Challenge

The main topic of this challenge is to understand how dynamic choice on a finite horizon works. Possible avenues to explore might include, but are not limited to, the following suggestions:

- Execute the `Dyn_choice_fh.m` file.
  - Comment on the results.
  - Improve the `Dyn_choice_fh.m` file to deliver the results in a table with 3 columns: `t`, `capital` and `consumption`,

  - You have experimented with optimal consumption patterns based on a given discount rate. How does the optimal consumption path vary with β? Write your own script to produce a graph to illustrate optimal consumption over a range of values of β.
  - Make shocks on the initial capital. Are the results expected? Comment.

- Imagine now that the household is both a producer and a consumer: consumption decisions in one period affect future production. Assume that the household extracts c_t from their capital stock at the beginning of period t leaving $k_t - c_t$ as an input to the production function, f. The new *flow equation* takes the form: $k_{t+1} = f(k_t - c_t, \theta) = \theta(k_t - c_t)^\alpha$, where θ is a time-invariant technology parameter.
  - Change the flow equation in `Dyn_choice_fh.m` with this one. Hint: only two changes are expected (formulas which define consumption (`c` and `consumption`)).
  - Execute the code and comment on the results.
  - Compare the solutions with the two different flow equations.

- You may want to extend further your analysis for this problem.

3 Ramsey-Cass-Koopmans model

The well-known neoclassical economic Solow-Swan model explains the long-run economic growth by looking at capital accumulation, labor or population growth, and increases in productivity (or technological progress). It is expressed via an initial value differential problem with a single nonlinear equation that models the evolution of the per capita stock of capital: $\dot{k}(t) = sf(k(t)) - \delta k(t)$, where $k(t)$ stands for the per capita capital and $f(k(t))$ is a production function. The differential equation illustrates that a total change of an individual's capital accumulation at year t is the value of the goods retained (the individual product times the saving rate s) deducted by the depreciation of previous capital holdings ($\delta k(t)$).

Several extensions have been made to this model, in particular by endogenizing the savings rate giving rise to the Ramsey-Cass-Koopmans (RCK) model:

$$\begin{aligned} \dot{k}(t) &= \quad f(k(t)) - (n+\delta)k(t) - c(t) \\ \dot{c}(t) &= \quad c(t)(f'(k(t)) - (\rho+\delta+n))\sigma \end{aligned}$$

where $k(t)$ is the per capita capital, $c(t)$ the per capita consumption, δ is the depreciation rate, n the population change rate, ρ the rate of time preference, σ is the intertemporal elasticity of substitution. This is a boundary value problem, where an initial capital is set along with a value for the consumption in the infinite.

Challenge

In this challenge, the RCK model will be tackled as an initial value problem, for very particular values of $k(0)$ and $c(0)$. In the following take the production function $f(k) = \frac{5k}{1+3k}$ and the parameters: $\delta = 0.1$, $n = 0.01$, $\rho = 0.02$, $\sigma = 1.1$.

1. Solve the problem for the initial condition $[k_0, c_0] = [0.05, 0.044518277]$ in the interval $[0, 20]$.
   a) What is the equilibrium?
   b) Draw the transition dynamics to the steady state.
   c) Comment on the result.
2. Solve the model for the following initial conditions and time period
   a) $[k_0, c_0] = [2.55, 1.509891]$ in the interval $[0, 35]$.
   b) $[k_0, c_0] = [1, 0.5]$ in the interval $[0, 50]$.
   c) $[k_0, c_0] = [0.05, 0.1]$ in the interval $[0, 0.5]$.
   d) Discuss the previous results. Are all solutions equilibrium points? Do the results have economic meaning?
3. Consider the problem with initial condition $[k_0, c_0] = [0.05, 0.044518277]$ in the interval $[0, 20]$. To analyze the stability of the model in the neighborhood of the

equilibrium point (k^*, c^*), the Jacobian matrix $J(k^*, c^*)$ of the linearization process is relevant. Discuss the stability by analyzing this matrix for our problem $J(k^*, c^*) = \begin{bmatrix} \rho & -1 \\ \sigma \times c^* \times f''(k^*) & 0 \end{bmatrix}$.

4. Consider again the problem with initial condition $[k_0, c_0] = [0.05, 0.044518277]$ in the interval $[0, 20]$.
   a) Simulate a shock on the depreciation rate: δ doubles. Discuss the result.
   b) Simulate a shock on the rate of time preference: ρ doubles. Discuss the result.

5. Make other simulations and explain them.

4 Free component

For this question, the student (or group) is urged to make a short report on a topic of her/his/their choice.

Challenge

The material taught in the course is expected to be explored. Typically this component can consist on solving an economic model using numerical methods studied in the course, along with simulation and interpretation of results.

The model can be an extension of those studied during the course or others.

This part corresponds to 25% of the total grade of P2.

5 Pine's model

Consider the Pine's model (**Pine_model.m**) attached to the project.

Challenge

Run the model and comment its results in a single sentence.