13 Bifurcations

Look back at the nonlinear population model (2.5). We said β was positive, giving:

- an unstable equilibrium at $x = 0$,
- a stable equilibrium at $x = \beta$, which the population grows towards,

but what happens if β becomes negative? You should see that then there is:

- a stable equilibrium at $x = 0$, which the population shrinks towards,
- an unstable equilibrium at $x = 0$.

That is, if β becomes negative then the stable equilibrium at $x = \beta > 0$ collapses to zero, and moves to negative values $x = \beta < 0$, and in doing so it becomes unstable. The unstable equilibrium $x = 0$ becomes stable in the process, so for $\beta < 0$ the population shrinks to zero, i.e. becomes extinct.

The population has undergone a **bifurcation** in which the two equilibria switch ordering, and switch stability. In other bifurcations equilibria can appear or disappear, change stability in different ways, or the oscillatory behaviour around them can change.

- The idea of a bifurcation is: as we vary a parameter of the model smoothly, the model should change smoothly. But there are places where this doesn't happen. Where small changes don't lead to an equivalent system.
- These are **bifurcation points**: places where a system may turn into one of two non-equivalent systems if we change a parameter slightly.
- In dynamics:
  - **Local bifurcations** occur when an eigenvalue of an equilibrium passes through the imaginary axis. They result only in local changes around the equilibrium.
  - **Global bifurcations** occur when two invariant objects meet (e.g. a periodic orbit and an equilibrium). They result in changes that cannot be entirely described locally (near any one point).

To look at some simple examples, first consider the roots of a function $f(x)$, i.e. the points where $x = 0$. (You'll need to sketch these simple functions to follow this argument; you'll want them to refer back to for the next part too).

- The equation $f(x) = ax + b$ always has one root, $x = -b/a$. Nothing about this can change qualitatively.
- The equation $f(x) = x^2 + ax + b$ is more interesting.
  - First let $a = b = 0$, then $f = x^2 = 0$ has two roots at $x = 0$.
  - Now vary b slightly, so $f = x^2 + b$. Now we get different scenarios depending on the sign of b. If $b < 0$ there are two real roots $x = \pm\sqrt{-b}$, but if $b > 0$ there are no real roots. A bifurcation between the two happens at $b = 0$.
  - Now instead vary a slightly, so $f = x^2 + ax$. We get different scenarios depending on the sign of a. If $a > 0$ then one of the roots moves to $x = -a < 0$, if $a < 0$ then it moves to $x = -a > 0$, while the other stays at $x = 0$ throughout. That's a minor change, but more important is the slope of f. If $a > 0$ then the slope of f is positive at the root $x = 0$, if $a < 0$ then the slope is negative at $x = 0$ (with the opposite sign slopes at the root $x = -a$). A bifurcation happens between the situations for $a > 0$ and $a < 0$.
  - For the full picture we have to vary both a and b. The two roots lie at $x_* = (-a \pm \sqrt{a^2 - 4b})/2$, so there is a *bifurcation curve* in (a, b) space given by $a^2 - 4b = 0$, giving two real roots in $a^2 > 4b$ and no real roots in $a^2 < 4b$. In the parameter region $a^2 > 4b$, whichever root is furthest to the right (most positive x) has a positive slope of f, and they switch slopes when they collide at $a = 0$.
- The first case above is an example of a **fold** bifurcation, where two special points (the roots) can 'fold' together to annihilate each other (or the quadratic curve, viewed from the side, looks 'folded' at its trough point where the roots collide).
- The second case above is an example of a **transcritical** bifurcation, where two special points pass through each other and exchange properties.

As we increase the order of our polynomial, more roots can appear, so more things can happen. For example:

- Take the cubic $f = x^3 + ax^2 + bx + c = 0$ and look at its roots. (Doing things algebraically is difficult for this case, but why not look up how to solve a cubic?!)

- Immediately you can see it has up to three roots. Plot it, change a, b, c, to move the graph around. The cubic can change shape in ways the quadratic cannot, so for some values it has turning points, for others it doesn't.

- It can have one or three roots. See if you can work out what values of a, b, c, the following things happen at. . .

- Two roots can collide and annihilate in a *fold bifurcation* like the quadratic we looked at above.

- Two roots can pass through each other like the *transcritical bifurcation*.

- Set $a = c = 0$ and notice there is always a root at $x = 0$ then, and as b changes sign two roots either side of $x = 0$ come in and collide and annihilate, while the slope at $x = 0$ changes; this is a cousin of the transcritical and we call it a **pitchfork** bifurcation.

- At $a = b = 0$ all three roots collide, and depending how we change a, b, c, away from zero we might get one or three roots; this point is called a **cusp bifurcation**.

- If you investigate the bifurcation curves in a, b, c, space you'll find out why these are called *cusps* and *pitchforks*.

And we can go on. Try a quartic, or a quintic, and so on. What about a sin curve, or an exponential curve? What about having a function of more than one variable $x, y, z, ...$? We get more and more elaborate bifurcations.

Fortunately in this endless zoo of things that can happen, there are a set of 'standard animals' which we call the **normal forms** of the bifurcations. For the bifurcations of roots in the functions above:

- the normal form of the fold is $f = x^2 - b$
- the normal form of the transcritical is $f = x^2 - bx$
- the normal form of the cusp is $f = x^3 - bx + c$
- the normal form of the pitchfork is $f = x^3 - bx$

Importantly, if one of these bifurcations happens at a point $x = 0$, then we can change variables to make the function f equal to these normal forms plus higher order terms (so plus things involving x^3 or higher for the fold or transcritical, plus things involving x^4 or higher for the cusp or pitchfork, and so on).

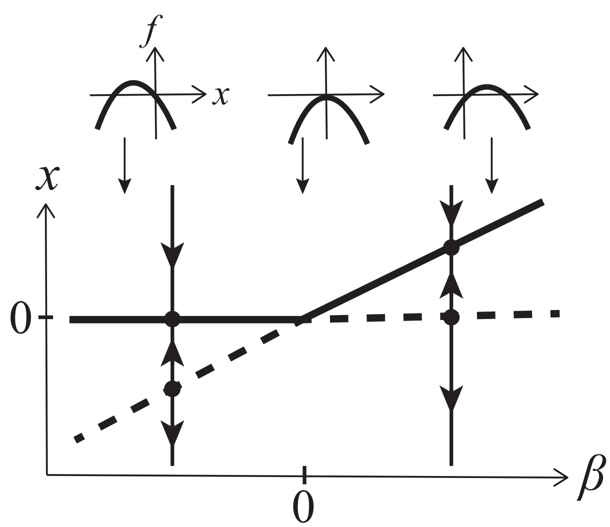
Bifurcations get interesting when we apply these functions to something where these roots have physical relevance, e.g. to dynamical systems.

Transcritical bifurcation

Let's go back to the 1d population model

$$\dot{x} = f(x) = (\beta - x)x \tag{13.1}$$

- There is an equilibrium at $x = 0$. The stability there is given by $f'(0) = \beta - 2x = \beta$, so for $\beta < 0$ it is stable and for $\beta > 0$ it is unstable.
- There is also an equilibrium at $x = \beta$. The stability there is given by $f'(\beta) = \beta - 2\beta = -\beta$, so for $\beta < 0$ it is unstable and for $\beta > 0$ it is stable.
- We see that at $\beta = 0$ the equilibria coincide.
- As β changes sign the equilibria swap ordering along the real x-line, and they swap stability.
- This is a **transcritical bifurcation**.
- We make a *bifurcation diagram* of the event by plotting the x-position of the equilibria against the *bifurcation parameter* β:



The figure shows the bifurcation diagram in x, β, space. Also on this, I've drawn two examples of the phase portraits in x (the vertical lines with arrows and equilibria on). I've also sketched (at the top) the graphs of $f(x)$ at $\beta < 0$, at $\beta = 0$, and at $\beta > 0$.

[Side Notes:] Transcritical bifurcation normal form

Let

$$\dot{x} = f(x, \beta) \tag{13.2}$$

with $x, \beta \in \mathbb{R}$.
Then a transcritical bifurcation occurs at $x = x_*$ when $\beta = \beta_*$ if the following conditions hold:

(B1) $f(x_*, \beta) = 0$ "equilibrium at $x = x_*$",

(B2) $\det(\frac{\partial f}{\partial x}) = 0$ at $x = x_*$, $\beta = \beta_*$, "zero eigenvalue",

(B3) $\frac{\partial f}{\partial \beta} = 0$ at $x = x_*$, $\beta = \beta_*$, "zero speed of f w.r.t. β",

(G1) $\frac{\partial^2 f}{\partial x^2} \neq 0$ at $x = x_*$, $\beta = \beta_*$, "second order derivative nonzero",

(G2) $\frac{\partial}{\partial \beta}\frac{\partial f}{\partial x} \neq 0$ at $x = x_*$, $\beta = \beta_*$, "positive speed of $\frac{\partial f}{\partial x}$ in β",

then (13.2) has the topological normal form

$$\dot{y} = \beta y - y^2 \tag{13.3}$$

in a neighbourhood of (x_*, β_*).

- We call the (B..) conditions **bifurcation conditions**.
  They tell us quantities that must be zero for the bifurcation to occur.
- We call the (G..) conditions **genericity conditions**
  (also sometimes called *non-degeneracy conditions*).
  They tell us quantities that must not be zero for the bifurcation to have its typical form. Usually if any of these genericity conditions fail (i.e. are zero) then a more complicated bifurcation is happening (because they are degenerate in some way, e.g. multiple bifurcations are happening at once).

The transcritical bifurcation is actually a special situation because of condition (B3), which more typically won't hold.

Fold bifurcation

Let's go back to the 1d population model, but consider some population control that gives a constant growth rate β rather than βx,

$$\dot{x} = f(x) = \beta - x^2 \tag{13.4}$$

- Now we see this has two equilibria at $x = \pm\sqrt{\beta}$ for $\beta > 0$.

  Their stabilities are given by $f'(\pm\sqrt{\beta}) = \mp 2\sqrt{\beta}$, so one is stable and the other unstable.

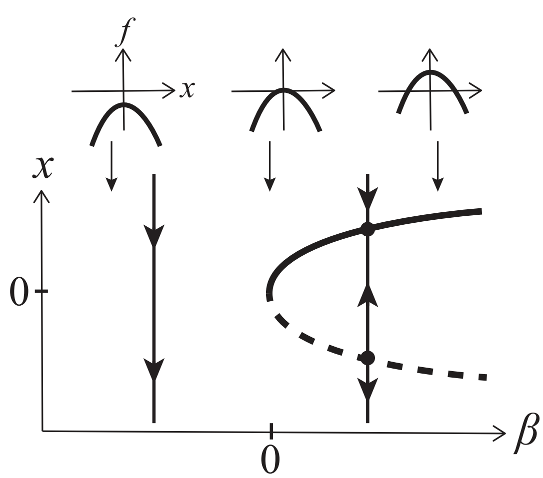
- When $\beta = 0$ the two equilibria collide at $x = 0$.

- For $\beta < 0$ there are no equilibria in the system at all (then $\dot{x} = f(x) < 0$ for all x, so the population always shrinks and dies out).

- Okay, the equilibrium at $x = -\sqrt{\beta}$ isn't physically realistic in this situation, since you can't have a negative population! But it is a dynamically meaningful point that helps us understand the system.

- At $\beta = 0$ a **fold** bifurcation has occurred.

- We make a *bifurcation diagram* of the event by plotting the x-position of the equilibria against the *bifurcation parameter* β:



The figure shows the bifurcation diagram in x, β, space. Also on this, I've drawn two examples of the phase portraits in x (the vertical lines with arrows and equilibria on). I've also sketched (at the top) the graphs of $f(x)$ at $\beta < 0$, at $\beta = 0$, and at $\beta > 0$.

[Side Notes:] Fold normal form

Let

$$\dot{x} = f(x, \beta) \tag{13.5}$$

with $x, \beta \in \mathbb{R}$. Then a fold bifurcation occurs at $x = x_*$ when $\beta = \beta_*$ if the following conditions hold:

(B1) $f(x_*, \beta) = 0$ "equilibrium at $x = x_*$",

(B2) $\det(\frac{\partial f}{\partial x}) = 0$ at $x = x_*$, $\beta = \beta_*$, "zero eigenvalue",

(G1) $\frac{\partial^2 f}{\partial x^2} \neq 0$ at $x = x_*$, $\beta = \beta_*$, "second order derivative nonzero",

(G2) $\frac{\partial f}{\partial \beta} \neq 0$ at $x = x_*$, $\beta = \beta_*$, "positive speed of f in β",

then the system has the topological normal form

$$\dot{y} = \beta \pm y^2 \tag{13.6}$$

in a neighbourhood of (x_*, β_*).

If more than one dimension the two equilibria involved are usually a saddle and a node, which collide and annihilate at some $\beta = 0$. For this reason the fold bifurcation in dynamics is often known as a **saddlenode bifurcation**.

A note on multiple dimensions:

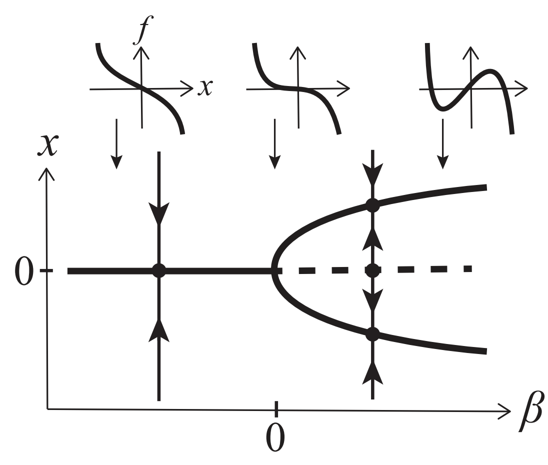
- If we have a system with multiple dimensions, e.g. $(\dot{x}, \dot{y}, ...) = (f, g, ...)$, we can still find bifurcations like the fold (and the others we'll see below), but you can see that the derivative conditions like $\frac{\partial f}{\partial x}$ will get a bit more involved.
- One way to study them is to find coordinates so that the bifurcation can be analysed just in the $\dot{x}$ equation, and we can ignore the others. Technically this means finding the **centre manifold** of the bifurcation, and taking the coordinate x along it. We won't go into that here.
- We'll do things more methodologically: find the equilibria of the system, observe what happens to them as you change a parameter, and deduce what bifurcation occurs. For example, if two equilibria collide and annihilate, with their eigenvalues becoming zero, the bifurcation is a fold.

Pitchfork bifurcation

Let's see what would happen if the 1d population model had a cubic decay rate instead, say

$$\dot{x} = f(x) = \beta x - x^3 \tag{13.7}$$

- There is an equilibrium at $x = 0$. The stability there is given by $f'(0) = \beta - 3x^2 = \beta$, so for $\beta < 0$ it is stable and for $\beta > 0$ it is unstable.
- There are two equilibrium at $x = \pm\sqrt{\beta}$, so these exist only for $\beta > 0$. Their stability is given by $f'(\pm\sqrt{\beta}) = \beta - 3x^2 = \beta - 3(\pm\sqrt{\beta})^2 = \beta - 3\beta = -2\beta$, so these are stable.
- We see that at $\beta = 0$ the equilibria coincide.
- As β changes sign we go from having just one stable equilibrium for $\beta < 0$, to this becoming unstable for $\beta > 0$ and two stable equilibria being created.
- This is a **pitchfork bifurcation**.
- We make a *bifurcation diagram* of the event by plotting the x-position of the equilibria against the *bifurcation parameter* β:



The figure shows the bifurcation diagram in x, β, space. Also on this, I've drawn two examples of the phase portraits in x (the vertical lines with arrows and equilibria on). I've also sketched (at the top) the graphs of $f(x)$ at $\beta < 0$, at $\beta = 0$, and at $\beta > 0$.

- Again, in the population model strictly only the equilibria in $x \geq 0$ are physically meaningful.
- Try the same analysis with $f(x) = \beta x + x^3$, you should find something similar but with the opposite stability of the equilibria.

[Side Notes:] **Pitchfork normal form**

Let

$$\dot{x} = f(x, \beta) \tag{13.8}$$

with reflectional symmetry $f(-x, \beta) = -f(x, \beta)$, with $x, \beta \in \mathbb{R}$.
Then a pitchfork bifurcation occurs at $x = x_*$ when $\beta = \beta_*$ if the following conditions hold:

(B1) $f(x_*, \beta) = 0$ "equilibrium at $x = x_*$",

(B2) $\det(\frac{\partial f}{\partial x}) = 0$ at $x = x_*$, $\beta = \beta_*$, "zero eigenvalue",

(B3) $\frac{\partial^2}{\partial x^2} f(x_*, \beta_*) = 0$ at $x = x_*$, $\beta = \beta_*$, "second order derivative zero"',

(B4) $\frac{\partial}{\partial \beta} f(x_*, \beta_*) = 0$ at $x = x_*$, $\beta = \beta_*$, "parameter derivative zero"',

(G1) $\frac{\partial^3}{\partial x^3} f(x_*, \beta_*) \neq 0$ at $x = x_*$, $\beta = \beta_*$, "third order derivative nonzero"',

(G2) $\frac{\partial}{\partial \beta} \frac{\partial}{\partial x} f(x_*, \beta_*) \neq 0$ at $x = x_*$, $\beta = \beta_*$, "positive speed in β",

then the system has the topological normal form

$$\dot{y} = \beta y \pm y^3, \tag{13.9}$$

in a neighbourhood of (x_*, β_*).

The pitchfork is a special situation (like the transcritical we looked at before), because it only happens in systems with reflectional symmetry.

Cusp bifurcation

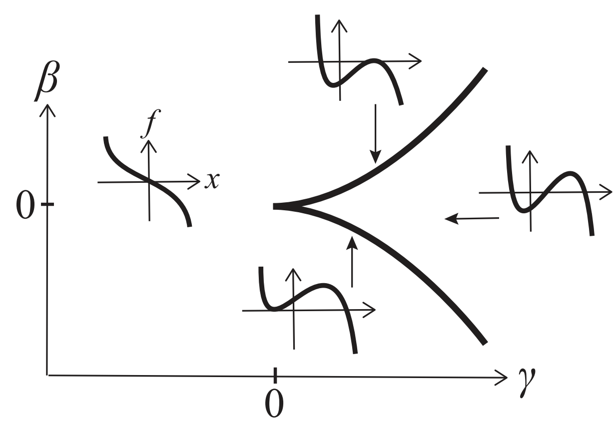
Let's go back to the 1d population model one last time. What if it had more parameters? Say a constant growth rate β plus a linear growth and a cubic decay,

$$\dot{x} = f(x) = \beta + \gamma x - x^3 \tag{13.10}$$

- This can have one or three equilibria depending on the values of β and γ. (It isn't easy to write out x values of these, but we usually don't need to).
- If there is only one equilibrium, say x_*, then it must have derivative $f'(x) < 0$ (sketch $f(x)$ to convince yourself of this, or try to prove it . . . essentially it is because the $-x^3$ term demands f is decreasing). So this equilibrium is stable.
- This system can exhibit folds, which would create two more equilibria. Folds require $f'(x) = \gamma - 3x^2 = 0$, so at $x = \pm\sqrt{\gamma/3}$, and if we combine this with the equilibrium condition $f(x) = 0$ we have

  $$0 = \beta + \gamma(\pm\sqrt{\gamma/3}) - (\pm\sqrt{\gamma/3})^3 \quad = \quad \beta \pm \tfrac{2}{3^{3/2}}\gamma^{3/2}$$

  i.e. folds happen when the two parameters satisfy $(\beta/2)^2 = (\gamma/3)^3$.
- Since folds create two equilibria, of different stabilities, there then exist two stable and one unstable equilibria.
- These folds lie along curves in the (β, γ) parameter plane, that meet at $\beta = \gamma$ in a cusp shape:
- At $\beta = \gamma = 0$ the three equilibria all coincide.
- This is a **cusp bifurcation**.
- The full *bifurcation diagram* now requires plotting the x-position of the equilibria against the *bifurcation parameters* β and γ:



- If we change the decay $-x^3$ to a growth $+x^3$ we will obtain a similar bifurcation but with the opposite stabilities of equilibria.

[Side Notes:] Cusp normal form

Let

$$\dot{x} = f(x, \beta, \gamma) \tag{13.11}$$

with $x, \beta, \gamma \in \mathbb{R}$.
Then a cusp bifurcation occurs at $x = x_*$ when $\beta = \beta_*$, $\gamma = \gamma_*$, if the following conditions hold:

(B1) $f(x_*, \beta, \gamma) = 0$ "equilibrium at x_*",

(B2) $\det(\frac{\partial f}{\partial x}) = 0$ at $x = x_*$, $\beta = \beta_*$, $\gamma = \gamma_*$, "zero eigenvalue",

(B3) $\frac{\partial^2 f}{\partial x^2} = 0$ at $x = x_*$, $\beta = \beta_*$, $\gamma = \gamma_*$, "second order derivative vanishes"',

(G1) $\frac{\partial^3 f}{\partial x^3} \neq 0$ at $x = x_*$, $\beta = \beta_*$, $\gamma = \gamma_*$, "third order derivative nonzero",

(G2) $\frac{\partial f}{\partial \beta}\frac{\partial^2 f}{\partial x \partial \gamma} - \frac{\partial f}{\partial \gamma}\frac{\partial^2 f}{\partial x \partial \beta} \neq 0$ at $x = x_*$, $\beta = \beta_*$, $\gamma = \gamma_*$, "positive speed in β",

then the system has the topological normal form

$$\dot{y} = \beta + \gamma y \pm y^3 \tag{13.12}$$

in a neighbourhood of (x_*, β_*, γ_*).

Codimension

When defining bifurcations two things are paramount:

- what is the class of systems you are studying?
  (E.g. are they symmetric $f(x) = -f(x)$, reflective $-f(x) = f(-x)$, reversible $f(x,t) = -f(x,-t)$, or conservative $f(x) = \nabla\phi(x)$, . . . ?)

- how many parameters must be varied for the bifurcation to happen?

This last one is called the **codimension** of the bifurcation.

- Each bifurcation has a number of conditions, the (B..) conditions in the bifurcation definitions above, that must be satisfied for them to happen.

- The first condition (B1) is just that there is an equilibrium. When we solve the other conditions, we then find parameter values that satisfy them. Each condition fixes one parameter (or removes one parameter freedom, e.g. fixes one parameter in relation to another).

These conditions are the (B..) conditions in the bifurcation definitions above, excluding the equilibrium condition. So:

- A fold has codimension one, and can occur in a system $\dot{x} = f(x)$ without any special conditions (a *generic* system).

- A transcritical has codimension one, but only happens if a system $\dot{x} = f(x)$ is fixed such that $(B3)$ holds, e.g. when the bifurcation involves an immovable equilibrium at $x = 0$.

- A pitchfork has codimension one, but only happens if a system is symmetric. E.g. if we take our cusp example above, but fix $\beta = 0$, we get a symmetric system; then varying β gives a pitchfork.

- A cusp has codimension two, and can occur in a system $\dot{x} = f(x)$ without any special conditions (a *generic* system).

- There is an endless list of higher codimension bifurcations that involve varying two or more parameters, in one dimension or in higher dimensions. They are all extensions of these basis ideas, and involve equilibria (dis)appearing and/or swapping stability in collisions.

- There is just one more, we should look at, because it does something rather different. It requires at least two dimensions . . .

14 Hopf bifurcation

(This is sometimes called an *Andronov-Hopf* bifurcation after the two mathematicians most associated with its discovery).

What happens if we have a system with an equilibrium that changes stability, but doesn't encounter any other equilibria in doing so? Take a simple equilibrium at the origin

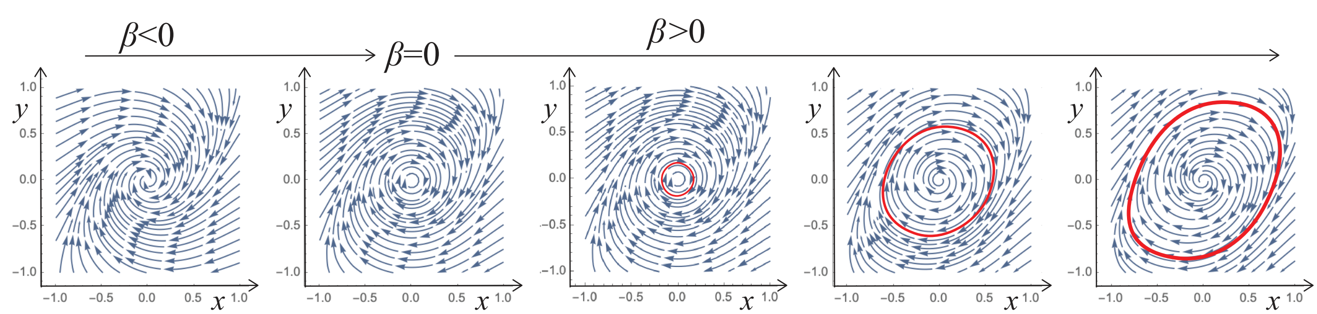
$$\dot{x} = f_1(x) = \beta x + y \ , \qquad \dot{y} = f_2(x) = \beta y - x \ , \tag{14.1}$$

- You should easily be able to see now that this equilibrium has eigenvalues $\beta \pm i$, so it is a focus with stability changing with the sign of β.
- So when $\beta = 0$ the equilibrium is null-stable, and actually the entire flow consists of closed circles around the origin — this is a centre, which is actually a very special situation in itself. If there were any nonlinear terms, typically we wouldn't get a centre, they'd cause slow attraction to or repulsion from the equilibrium.

So consider adding a nonlinear term that gives an attraction growing with the radius from the origin, say

$$\dot{x} = f_1(x) = \beta x + y - x(x^2 + y^2) \ , \qquad \dot{y} = f_2(x) = \beta y - x \ , \tag{14.2}$$

- Now watch what happens as we change the sign of β.

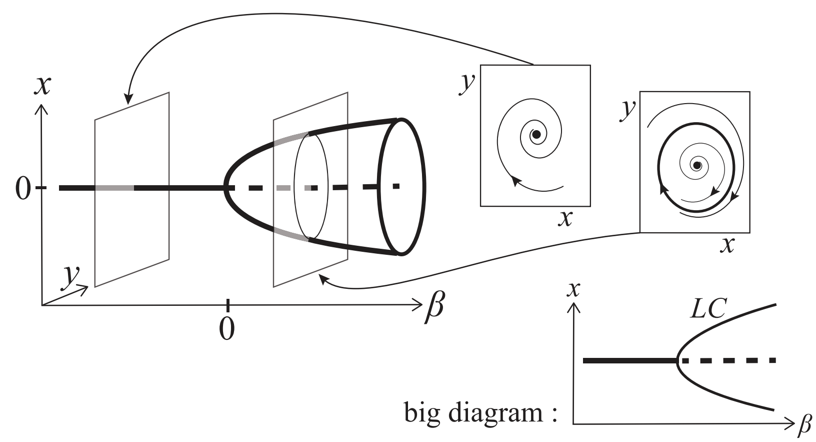


- The stability at the origin is determined by the local terms, so that is still stable for $\beta < 0$, unstable for $\beta > 0$, and null-stable for $\beta = 0$.

- But now, the nonlinear term gives an over-riding attraction towards the origin from far away — we call this a *global* attraction to the origin.

- So when $\beta > 0$, the origin is unstable so the flow spirals outward, but far away the flow spirals inward. Somewhere in the middle they must balance out, where they form a close orbits called a **limit cycle**.

- This limit cycle grows with a radius proportional to roughly $\sqrt{\beta}$, giving a bifurcation diagram something like:



The main picture shows a branch representing the equilibrium, unmoving as β increases until β changes sign, then the equilibrium changes stability, and throws out a limit cycle whose radius grows with $\sqrt{\beta}$. The cross-sections show the phase portraits. The little picture on the bottom-right is a bit simpler, it shows what we usually draw on an actual bifurcation diagram, just the branches of the equilibrium, and thin curves denoting the limit cycle (sometimes labelled LC).

- By our argument, this limit cycle must be stable (i.e. attracting). We call this a **supercritical Hopf bifurcation**.

- Change the nonlinear term from $-x(x^2+y^2)$ to $+x(x^2+y^2)$, and instead the limit cycle appears when $\beta < 0$ and is unstable. We call this a **subcritical Hopf bifurcation**.

- The size of the nonlinear terms that decide whether the bifurcation is supercritical or subcritical is called the *first Lyapunov quantity*.

[Side Notes:] Hopf bifurcation

Let

$$\dot{x} = f(x, \beta) \tag{14.3}$$

with $x, \beta, \gamma \in \mathbb{R}$.
Then a Hopf bifurcation occurs at $x = x_*$ when $\beta = \beta_*$, at an equilibrium with eigenvalues

$$\lambda_{\pm} = \rho(\beta) \pm i\omega(\beta) \ , \tag{14.4}$$

if the following conditions hold:

(B1) $f(x_*, \beta) = 0$ "equilibrium at x_*",

(B2) $\rho(\beta_*) = 0$ at $x = x_*$, $\beta = \beta_*$, "imaginary eigenvalues",

(G1) $\ell_1 \neq 0$ at $x = x_*$, $\beta = \beta_*$, where ℓ_1 is the "first Lyapunov quantity"

(G2) $\omega(\beta_*) \neq 0$ at $x = x_*$, $\beta = \beta_*$, "positive speed in β",

then the system has the topological normal form $z \in \mathbb{C}$

$$\dot{z} = (\rho + i\omega)z + \ell_1 z|z|^2 \tag{14.5}$$

giving a

- supercritical Hopf bifurcation if $\ell_1 < 0$ and
- subcritical Hopf bifurcation if $\ell_1 > 0$.

A little strangely we've decided to write this normal form in complex variables. If we let $z = x+iy$ we can see that this is just a neat way of writing a two-dimensional ODE

$$\begin{aligned}\dot{x} &= \rho x - \omega y + \ell_1 x(x^2 + y^2)\\ \dot{y} &= \omega x + \rho y + \ell_1 y(x^2 + y^2)\end{aligned}$$

[see Ex.Sht].