Question 3: Irregular Singular Points and Asymptotic Solutions Complete Analysis with Controlling Factor Method

Asymptotics Course — Sheet 4

Problem Statement

Consider the differential equation:

$$\frac{d^2y}{dx^2} - \left(1 + \frac{1}{x}\right) y = 0$$

(a) Show that it has an irregular singular point at $x = \infty$.
(b) Compute the two linearly independent solutions at leading order as $x \to \infty$.

1 Part (a): Verify Irregular Singular Point at $x = \infty$

1.1 Step 1: Strategy for Analyzing Point at Infinity

- **STAGE X (What we need):** To analyze whether $x = \infty$ is a singular point, we must transform it to a finite point and apply the classification from Section 3.1.
- **STAGE Y (Why this approach):** The standard classification (regular vs. irregular singular point) is defined for finite points. We use the transformation $x = 1/t$ to map $x = \infty$ to $t = 0$.
- **STAGE Z (What this means):** If $t = 0$ is an irregular singular point after transformation, then $x = \infty$ is an irregular singular point.

1.2 Step 2: Transform the ODE

Change of Variable

Set:

$$x = \frac{1}{t} \quad \Rightarrow \quad t = \frac{1}{x}$$

As $x \to \infty$, we have $t \to 0$.

Transform Derivatives (ESSENTIAL)

Using the chain rule:

$$\frac{d}{dx} = \frac{dt}{dx} \cdot \frac{d}{dt} = -t^2 \frac{d}{dt}$$

For the second derivative:

$$\begin{aligned}
\frac{d^2}{dx^2} &= \frac{d}{dx}\left(\frac{d}{dx}\right) = \frac{d}{dx}\left(-t^2\frac{d}{dt}\right) \\
&= -\frac{d(t^2)}{dx}\cdot\frac{d}{dt} - t^2\frac{d}{dx}\left(\frac{d}{dt}\right) \\
&= -(-2t^3)\frac{d}{dt} - t^2\left(-t^2\frac{d^2}{dt^2}\right) \\
&= 2t^3\frac{d}{dt} + t^4\frac{d^2}{dt^2}
\end{aligned}$$

Therefore:

$$\frac{d^2y}{dx^2} = t^4\frac{d^2y}{dt^2} + 2t^3\frac{dy}{dt}$$

Transform the Coefficient

$$1 + \frac{1}{x} = 1 + t$$

1.3 Step 3: Write the Transformed ODE

Substituting into the original equation:

$$t^4\frac{d^2y}{dt^2} + 2t^3\frac{dy}{dt} - (1+t)y = 0$$

Divide through by t^4 to obtain standard form:

$$\frac{d^2y}{dt^2} + \frac{2}{t}\frac{dy}{dt} - \frac{1+t}{t^4}y = 0$$

1.4 Step 4: Apply Classification Criteria

Standard Form Comparison (Section 3.1)

The general second-order linear ODE is:

$$\frac{d^2y}{dt^2} + p(t)\frac{dy}{dt} + q(t)y = 0$$

In our transformed equation:

$$p(t) = \frac{2}{t}, \qquad q(t) = -\frac{1+t}{t^4}$$

Classification Criteria for Regular Singular Point

From Section 3.1 of lecture notes:

A point t_0 is a **regular singular point** if:

1. $(t - t_0)p(t)$ is analytic at t_0
2. $(t - t_0)^2 q(t)$ is analytic at t_0

For our case with $t_0 = 0$:
Check Condition 1:

$$t \cdot p(t) = t \cdot \frac{2}{t} = 2$$

This is analytic at $t = 0$ (it's a constant). ✓
Check Condition 2:

$$t^2 \cdot q(t) = t^2 \cdot \left(-\frac{1+t}{t^4}\right) = -\frac{1+t}{t^2} = -\frac{1}{t^2} - \frac{1}{t}$$

This is **NOT analytic** at $t = 0$ because it has terms $-1/t^2$ and $-1/t$ that diverge as $t \to 0$. ×

1.5 Step 5: Conclusion

- **STAGE X (What we found):** Condition 1 is satisfied, but Condition 2 fails.
- **STAGE Y (Why this matters):** Since $(t-0)^2 q(t)$ is not analytic at $t = 0$, the point $t = 0$ is NOT a regular singular point. Therefore, it must be an irregular singular point.
- **STAGE Z (Final conclusion):** Since $t = 0$ (corresponding to $x = \infty$ in the original coordinates) is an irregular singular point, we have proven:

Answer to Part (a):
The differential equation $y'' - (1 + 1/x)y = 0$ has an **irregular singular point at** $x = \infty$ because:

1. The transformation $x = 1/t$ maps $x = \infty$ to $t = 0$
2. The transformed ODE has $t^2 q(t) = -(1+t)/t^2$, which is not analytic at $t = 0$
3. Therefore $t = 0$ is an irregular singular point, making $x = \infty$ an irregular singular point

2 Part (b): Compute Leading Order Solutions as $x \to \infty$

2.1 Step 1: Strategy — Controlling Factor Method

- **STAGE X (What we have):** An ODE with irregular singular point at $x = \infty$, which means standard Frobenius series fails.
- **STAGE Y (Why controlling factor):** For irregular singular points, we use the **controlling factor method** (Section 3.2) which seeks solutions of the form $y = e^{S(x)}$.
- **STAGE Z (What this means):** We'll find $S(x)$ by successive approximation, starting with dominant balance.

2.2 Step 2: Set Up the Controlling Factor Ansatz

Ansatz (Section 3.2.1)

Set:

$$y(x) = e^{S(x)}$$

Then:

$$y'(x) = S'(x)e^{S(x)} = S'(x)y(x)$$
$$y''(x) = \left[S''(x) + \left(S'(x)\right)^2\right] e^{S(x)} = \left[S''(x) + \left(S'(x)\right)^2\right] y(x)$$

Substitute into ODE

The original ODE is:

$$y'' - \left(1 + \frac{1}{x}\right) y = 0$$

Substituting:

$$\left[S'' + (S')^2\right] y - \left(1 + \frac{1}{x}\right) y = 0$$

Since $y = e^{S(x)} \neq 0$, divide by y:

$$S'' + (S')^2 - 1 - \frac{1}{x} = 0$$

Rearrange:

$$S'' + (S')^2 = 1 + \frac{1}{x}$$

2.3 Step 3: Leading Order — Dominant Balance

Standard Assumption (Equation 121)

From Section 3.2.1: Near irregular singular points, we typically assume:

$$S'' = o\left((S')^2\right) \quad \text{as } x \to \infty$$

Apply Dominant Balance

If $S'' \ll (S')^2$, then to leading order:

$$(S')^2 \approx 1 + \frac{1}{x} \quad \text{as } x \to \infty$$

For large x, $1/x \to 0$, so:

$$(S')^2 \approx 1 \quad \Rightarrow \quad S' \approx \pm 1$$

Integrating:

$$S_0(x) = \pm x + \text{const}$$

We can absorb the constant into the overall multiplicative constant of the solution, so:

$$\boxed{S_0(x) = \pm x}$$

- **STAGE X (What we found):** Two leading order solutions: $S_0(x) = x$ and $S_0(x) = -x$.
- **STAGE Y (Verify assumption):** With $S_0' = \pm 1$, we have $S_0'' = 0$, so indeed $S_0'' = o((S_0')^2)$
- **STAGE Z (Next step):** Find the next correction to determine the complete leading order behavior including prefactors.

2.4 Step 4: Next Order Correction

Refinement (Section 3.2.3)

Set:

$$S(x) = \pm x + C(x)$$

where $C(x) = o(x)$ as $x \to \infty$ (i.e., $C(x)$ grows slower than linearly).

Compute Derivatives

$$\begin{aligned} S'(x) &= \pm 1 + C'(x) \\ S''(x) &= C''(x) \end{aligned}$$

Substitute Back into ODE Equation

$$C'' + (\pm 1 + C')^2 = 1 + \frac{1}{x}$$

Expand:

$$C'' + 1 \pm 2C' + (C')^2 = 1 + \frac{1}{x}$$

Simplify:

$$C'' \pm 2C' + (C')^2 = \frac{1}{x}$$

Apply Dominant Balance for $C(x)$

Since $C(x) = o(x)$, we have:

- $C'(x) = o(1)$ as $x \to \infty$
- $C''(x) = o(1/x)$ as $x \to \infty$
- $(C')^2 = o(1)$ as $x \to \infty$

Therefore, to leading order in the equation for C:

$$\pm 2C' \approx \frac{1}{x}$$

This gives:

$$C'(x) \approx \pm \frac{1}{2x}$$

Integrating:

$$C(x) = \pm \frac{1}{2} \log x + \text{const}$$

Again, absorbing the constant:

$$\boxed{C(x) = \pm \frac{1}{2} \log x}$$

2.5 Step 5: Combine Leading and Next Order

Complete Expression for $S(x)$

$$S(x) = \pm x \pm \frac{1}{2} \log x + o(\log x) \quad \text{as } x \to \infty$$

Note: The two $\pm$ signs are independent. For the two linearly independent solutions, we take:

$$S_+(x) = x + \frac{1}{2} \log x \quad \text{and} \quad S_-(x) = -x - \frac{1}{2} \log x$$

Solutions

$$y_+(x) = e^{S_+(x)} = e^{x+\frac{1}{2}\log x} = e^x \cdot e^{\log x^{1/2}} = e^x\sqrt{x}$$
$$y_-(x) = e^{S_-(x)} = e^{-x-\frac{1}{2}\log x} = e^{-x} \cdot e^{-\log x^{1/2}} = \frac{e^{-x}}{\sqrt{x}}$$

2.6 Step 6: State Final Answer

Answer to Part (b):
The two linearly independent solutions at leading order as $x \to \infty$ are:

$$y_1(x) \sim \sqrt{x}\,e^x \quad \text{as } x \to \infty$$

$$y_2(x) \sim \frac{e^{-x}}{\sqrt{x}} \quad \text{as } x \to \infty$$

Or equivalently:

$$y_1(x) \sim Ax^{1/2}e^x, \qquad y_2(x) \sim Bx^{-1/2}e^{-x} \quad \text{as } x \to \infty$$

where A and B are arbitrary constants.

2.7 Step 7: Verification

Check Linear Independence

The Wronskian:

$$W = y_1y_2' - y_1'y_2$$

For $y_1 \sim \sqrt{x}e^x$ and $y_2 \sim \frac{e^{-x}}{\sqrt{x}}$:

$$W \sim \left(\sqrt{x}e^x\right) \cdot \left(-\frac{e^{-x}}{\sqrt{x}} - \frac{e^{-x}}{2x^{3/2}}\right) - \left(\sqrt{x}e^x + \frac{e^x}{2\sqrt{x}}\right) \cdot \frac{e^{-x}}{\sqrt{x}}$$
$$\sim -1 - \frac{1}{2x} - 1 - \frac{1}{2x} = -2 + O(1/x) \neq 0 \quad \checkmark$$

The solutions are linearly independent.

Verify Dominant Balance Consistency

For $y_1 \sim \sqrt{x}e^x$:

$$y_1' \sim \sqrt{x}e^x + \frac{e^x}{2\sqrt{x}} \sim \sqrt{x}e^x \quad \text{(leading order)}$$
$$y_1'' \sim \sqrt{x}e^x \quad \text{(leading order)}$$

Substitute into ODE:

$$y_1'' - (1+1/x)y_1 \sim \sqrt{x}e^x - (1+1/x)\sqrt{x}e^x \sim \sqrt{x}e^x(1-1-1/x)$$

The leading terms cancel, leaving $O(x^{-1/2}e^x) = o(\sqrt{x}e^x)$ ✓

3 Verification Checklist

✓ **Part (a) — Transformation:** $x = 1/t$ correctly transforms derivatives

✓ **Part (a) — Classification:** Checked both conditions for regular singular point

✓ **Part (a) — Conclusion:** $t^2 q(t)$ not analytic $\Rightarrow$ irregular singular point

✓ **Part (b) — Method:** Used controlling factor ansatz $y = e^{S(x)}$

✓ **Part (b) — Leading order:** Found $S_0' \approx \pm 1$ via dominant balance

✓ **Part (b) — Next order:** Found $C(x) \approx \pm\frac{1}{2} \log x$

✓ **Part (b) — Solutions:** Combined to get $y_1 \sim \sqrt{x} e^x$ and $y_2 \sim e^{-x}/\sqrt{x}$

✓ **Verification:** Checked linear independence via Wronskian

✓ **Consistency:** Verified dominant balance assumption holds

This solution follows the methodology of Section 3.1 (classification of singular points) and Section 3.2 (controlling factor method for irregular singular points) from the lecture notes.