Asymptotics 2025/2026
Problem Sheet 3
Question 1: Integration by Parts

Solution with XYZ Methodology

Problem 1. *Use integration by parts to obtain the first two terms in the asymptotic expansion of*

$$I(X) = \int_1^\infty e^{-X(t^2+1)}\mathrm{d}t$$

as $X \to \infty$.

Solution. **Overview and Strategy**

What do we see? *We have an integral of the form $\int_a^\infty (function) \cdot e^{-X(something)}\mathrm{d}t$ where X is a large parameter.*

Why is this significant? *This is a* Laplace-type integral *(Section 4.2 of lecture notes). As $X \to \infty$, the exponential factor $e^{-X(t^2+1)}$ decays extremely rapidly, causing the integral to be dominated by behavior near the* lower limit *of integration where the exponent is smallest.*

What method do we use? *The lecture notes (Section 4.2.1, "Integration by parts of Laplace integrals") provide the systematic approach: we must first transform the integral into standard form, then apply repeated integration by parts.*

Step 1: Transformation to Standard Form

Step 1.1: Identifying the Structure

What do we have? *The exponent is $-X(t^2+1)$, which we write as $-X\phi(t)$ where*

$$\phi(t) = t^2 + 1.$$

Why this decomposition? *The standard form for Laplace integrals (Equation 163 in notes) is*

$$I(x) = \int_a^b f(t)e^{-xt}\mathrm{d}t.$$

Our integral has $e^{-X\phi(t)}$ instead of e^{-Xt}, so we must use a change of variables to achieve the standard form.

Step 1.2: The Substitution

What substitution do we make? *Following Section 4.2.1, when $\phi'(t) \neq 0$ in the integration interval, we set*

$$\tau = \phi(t) = t^2 + 1.$$

Why this choice? *This substitution transforms $e^{-X\phi(t)}$ into $e^{-X\tau}$, giving us the standard Laplace form. Let's verify $\phi'(t) \neq 0$:*

$$\phi'(t) = 2t.$$

Since $t \in [1, \infty)$, we have $\phi'(t) = 2t > 0$ throughout the domain. ***Therefore the substitution is valid.***

Step 1.3: Computing the Differential

What do we need? *We need to express* $\mathrm{d}t$ *in terms of* $\mathrm{d}\tau$*.*

How do we find it? *From $\tau = t^2 + 1$, we differentiate:*

$$\mathrm{d}\tau = 2t\mathrm{d}t \quad \Rightarrow \quad \mathrm{d}t = \frac{\mathrm{d}\tau}{2t}.$$

What about t as a function of τ? *Since $\tau = t^2 + 1$, we have*

$$t^2 = \tau - 1 \quad \Rightarrow \quad t = \sqrt{\tau - 1},$$

where we take the positive root because $t \geq 1$ in our integration domain.

Step 1.4: Transforming the Limits

What are the new limits?

- *When $t = 1$: $\tau = 1^2 + 1 = 2$.*
- *When $t \to \infty$: $\tau = t^2 + 1 \to \infty$.*

Why is this important? *The transformation preserves the infinite upper limit, and we now know the integral starts at $\tau = 2$.*

Step 1.5: The Transformed Integral

Putting it all together:

$$\begin{aligned} I(X) &= \int_1^\infty e^{-X(t^2+1)}\mathrm{d}t \\ &= \int_2^\infty e^{-X\tau} \cdot \frac{\mathrm{d}\tau}{2t} \\ &= \int_2^\infty \frac{1}{2\sqrt{\tau - 1}} e^{-X\tau}\mathrm{d}\tau. \end{aligned}$$

What have we achieved? *We now have the standard form*

$$I(X) = \int_2^\infty f(\tau)e^{-X\tau}\mathrm{d}\tau$$

where

$$f(\tau) = \frac{1}{2\sqrt{\tau - 1}} = \frac{1}{2}(\tau - 1)^{-1/2}.$$

Why is this form useful? *This is precisely the setup for the integration by parts formula (Equation 165 in notes).*

Step 2: First Integration by Parts

Step 2.1: Setting Up Integration by Parts

What is our formula? *From calculus, $\int u\mathrm{d}v = uv - \int v\mathrm{d}u$.*

How do we choose u and $\mathrm{d}v$? *Following the lecture notes method (Section 4.2.1):*

- *Let* $\mathrm{d}v = e^{-X\tau}\mathrm{d}\tau$*, so* $v = -\frac{1}{X}e^{-X\tau}$*.*
- *Let* $u = f(\tau) = \frac{1}{2\sqrt{\tau-1}}$*, so* $\mathrm{d}u = f'(\tau)\mathrm{d}\tau$*.*

Why this assignment? *We want to "peel off" the exponential and generate a boundary term that will give us the leading asymptotic behavior.*

Step 2.2: Applying the Formula

Computing the parts:

$$
\begin{aligned}
I(X) &= \int_2^\infty f(\tau)e^{-X\tau}\mathrm{d}\tau \\
&= \left[f(\tau)\cdot\left(-\frac{e^{-X\tau}}{X}\right)\right]_2^\infty - \int_2^\infty \left(-\frac{e^{-X\tau}}{X}\right) f'(\tau)\mathrm{d}\tau \\
&= -\frac{1}{X}\left[f(\tau)e^{-X\tau}\right]_2^\infty + \frac{1}{X}\int_2^\infty f'(\tau)e^{-X\tau}\mathrm{d}\tau.
\end{aligned}
$$

Step 2.3: Evaluating the Boundary Term

What happens at $\tau \to \infty$***?***

We need to evaluate $\lim_{\tau\to\infty} f(\tau)e^{-X\tau}$*.*

Analyzing the behavior:

- $f(\tau) = \frac{1}{2\sqrt{\tau-1}} \sim \frac{1}{2\sqrt{\tau}}$ *as* $\tau \to \infty$*, which grows like* $\tau^{-1/2}$*.*
- $e^{-X\tau}$ *decays exponentially.*

Why does it vanish? *Exponential decay always dominates polynomial growth, so*

$$\lim_{\tau\to\infty} \frac{1}{2\sqrt{\tau-1}}e^{-X\tau} = 0.$$

What about at $\tau = 2$***?***

$$f(2) = \frac{1}{2\sqrt{2-1}} = \frac{1}{2\sqrt{1}} = \frac{1}{2}.$$

Therefore:

$$-\frac{1}{X}\left[f(\tau)e^{-X\tau}\right]_2^\infty = -\frac{1}{X}(0 - f(2)e^{-2X}) = \frac{1}{X}\cdot\frac{1}{2}\cdot e^{-2X} = \frac{e^{-2X}}{2X}.$$

Step 2.4: Intermediate Result

What have we found?

$$I(X) = \frac{e^{-2X}}{2X} + \frac{1}{X}\int_2^\infty f'(\tau)e^{-X\tau}\mathrm{d}\tau.$$

What is the significance?

- *The first term* $\frac{e^{-2X}}{2X}$ *is our* leading order term.
- *The integral contains* $f'(\tau)$*, which will yield the next correction.*
- *The factor* $\frac{1}{X}$ *in front means this integral contributes at lower order.*

Step 3: Second Integration by Parts

Step 3.1: Computing $f'(\tau)$

What is the derivative of f?

$$f(\tau) = \frac{1}{2}(\tau - 1)^{-1/2}.$$

Using the power rule:

$$f'(\tau) = \frac{1}{2} \cdot \left(-\frac{1}{2}\right)(\tau - 1)^{-3/2} \cdot 1 = -\frac{1}{4}(\tau - 1)^{-3/2}.$$

At $\tau = 2$:

$$f'(2) = -\frac{1}{4}(2 - 1)^{-3/2} = -\frac{1}{4}.$$

Step 3.2: Applying Integration by Parts Again

What do we do? *We apply the same technique to $\int_2^\infty f'(\tau)e^{-X\tau}\mathrm{d}\tau$:*

$$\begin{aligned}
\int_2^\infty f'(\tau)e^{-X\tau}\mathrm{d}\tau &= \left[f'(\tau) \cdot \left(-\frac{e^{-X\tau}}{X}\right)\right]_2^\infty + \frac{1}{X}\int_2^\infty f''(\tau)e^{-X\tau}\mathrm{d}\tau \\
&= -\frac{1}{X}\left[f'(\tau)e^{-X\tau}\right]_2^\infty + \frac{1}{X}\int_2^\infty f''(\tau)e^{-X\tau}\mathrm{d}\tau.
\end{aligned}$$

Step 3.3: Evaluating the Boundary Term

At $\tau \to \infty$: *Again, exponential decay dominates, so the term vanishes.*

At $\tau = 2$:

$$-\frac{1}{X}\left[f'(\tau)e^{-X\tau}\right]_2^\infty = -\frac{1}{X}(0 - f'(2)e^{-2X}) = \frac{1}{X} \cdot \left(-\frac{1}{4}\right) \cdot e^{-2X} = -\frac{e^{-2X}}{4X}.$$

Step 3.4: Substituting Back

Combining results:

$$\begin{aligned}
I(X) &= \frac{e^{-2X}}{2X} + \frac{1}{X}\left[-\frac{e^{-2X}}{4X} + \frac{1}{X}\int_2^\infty f''(\tau)e^{-X\tau}\mathrm{d}\tau\right] \\
&= \frac{e^{-2X}}{2X} - \frac{e^{-2X}}{4X^2} + \frac{1}{X^2}\int_2^\infty f''(\tau)e^{-X\tau}\mathrm{d}\tau.
\end{aligned}$$

Step 4: Asymptotic Interpretation

Step 4.1: Understanding the Remainder

What about the remaining integral? *The lecture notes (proof of Equation 167) show that*

$$\int_2^\infty f''(\tau)e^{-X\tau}\mathrm{d}\tau = o\left(\frac{e^{-2X}}{X^2}\right) \quad \text{as } X \to \infty.$$

Why? *The integral can be bounded, and the exponential decay at $\tau = 2$ determines its order.*

Step 4.2: The Asymptotic Expansion

Conclusion: *The first two terms in the asymptotic expansion are:*

$$\boxed{I(X) \sim \frac{e^{-2X}}{2X} - \frac{e^{-2X}}{4X^2} \text{ as } X \to \infty}$$

Alternative form: *Factoring out e^{-2X}/X:*

$$\boxed{I(X) \sim \frac{e^{-2X}}{X}\left(\frac{1}{2} - \frac{1}{4X}\right) \text{ as } X \to \infty}$$

Step 4.3: Interpretation

What does this tell us?

- *The factor e^{-2X} comes from the exponential evaluated at $\tau = 2$ (equivalently $t = 1$), the lower limit where the exponent $t^2 + 1$ is smallest.*
- *The powers of X in the denominator come from repeated differentiation and integration by parts.*
- *Each successive term is smaller by a factor of order $1/X$.*
- *The expansion is asymptotic but not necessarily convergent for any fixed X.*

Verification and Consistency

Does our answer make sense?

- *As $X \to \infty$, the integral $I(X) \to 0$ because the exponential kills the integrand.*
- *The leading behavior is $\sim e^{-2X}/X$, which goes to zero.*
- *The next term provides a correction of order $1/X$ smaller.*
- *The form matches the general result (Equation 164) from the lecture notes.*