Asymptotics Problem 7.2: Complete Pedagogical Solution

WKB Method via Double Transformation

Problem 1. For the equation $\varepsilon^2 y''(x) + q(x)y(x) = 0$, look for transformations of both the dependent and independent variables, $z = \varphi(x)$, $\nu(z) = \psi(x)y(x)$ so that with suitable choice for the functions $\varphi(x)$ and $\psi(x)$, the ODE becomes $\varepsilon^2 \nu''(z) + \nu(z) = 0$ to the leading order as $\varepsilon \to 0$. Hence deduce the leading order solution and show it is equivalent to the WKB solution. (You may assume that $q(x)$ is a positive function.)

Solution: Step-by-Step Atomic Breakdown

Step 1: Understanding the Problem and Overall Strategy

Strategy: *We are performing a* double transformation*:*

- ***Independent variable transformation:*** $z = \varphi(x)$ *(change of coordinate)*
- ***Dependent variable transformation:*** $\nu(z) = \psi(x)y(x)$ *(amplitude rescaling)*

The goal is to transform the ODE with variable coefficient $q(x)$ *into the* simplest possible form *with constant coefficients:* $\varepsilon^2\nu'' + \nu = 0$.
Justification: *Why do we want this particular form? Because* $\varepsilon^2\nu'' + \nu = 0$ *is exactly solvable:*

$$\nu(z) = A\cos\left(\frac{z}{\varepsilon}\right) + B\sin\left(\frac{z}{\varepsilon}\right),$$

which exhibits the rapid oscillations characteristic of WKB solutions. By transforming our original equation into this form, we can:

1. *Solve it exactly in the transformed variables*
2. *Transform back to obtain the WKB approximation*
3. *Understand geometrically what the WKB method is doing*

This approach appears in Lecture Notes §7.2.1, equations (479)-(511), where it's used to derive WKB from the averaging method.

Step 2: Setting Up the Transformation Framework

What we have:

$$\begin{aligned} \text{Original equation:} \quad & \varepsilon^2 y''(x) + q(x)y(x) = 0 \\ \text{New independent variable:} \quad & z = \varphi(x) \\ \text{New dependent variable:} \quad & \nu(z) = \psi(x)y(x) \\ \text{Target equation:} \quad & \varepsilon^2\nu''(z) + \nu(z) = 0 \end{aligned}$$

Technique: *To substitute into the ODE, we need to express* y *and its derivatives in terms of* ν *and its derivatives. This requires careful application of the chain rule for both transformations.*

Step 3: Relating y to ν

From $\nu(z) = \psi(x)y(x)$, we can express y in terms of ν:

$$y(x) = \frac{\nu(z)}{\psi(x)} = \frac{\nu(\varphi(x))}{\psi(x)}.$$

Justification: *This inverse relationship is crucial: we're expressing the old unknown $y(x)$ in terms of the new unknown $\nu(z)$. Note that ν is a function of z, which itself depends on x through $z = \varphi(x)$.*

Step 4: Computing the First Derivative $y'(x)$

What we need: $\frac{dy}{dx}$ using $y = \frac{\nu(z)}{\psi(x)}$ and $z = \varphi(x)$.
Technique: *Use the quotient rule combined with the chain rule. We have:*

$$y(x) = \frac{\nu(z(x))}{\psi(x)}.$$

Applying the quotient rule:

$$\frac{dy}{dx} = \frac{\frac{d\nu}{dx} \cdot \psi(x) - \nu(z) \cdot \frac{d\psi}{dx}}{\psi(x)^2}.$$

But ν depends on x through z, so by the chain rule:

$$\frac{d\nu}{dx} = \frac{d\nu}{dz} \cdot \frac{dz}{dx} = \nu'(z) \cdot \varphi'(x).$$

Therefore:

$$y'(x) = \frac{\nu'(z)\varphi'(x) \cdot \psi(x) - \nu(z) \cdot \psi'(x)}{\psi(x)^2}.$$

Simplifying:

$$y'(x) = \frac{\varphi'(x)}{\psi(x)}\nu'(z) - \frac{\psi'(x)}{\psi(x)^2}\nu(z).$$

Step 5: Computing the Second Derivative $y''(x)$

What we need: Differentiate $y'(x)$ with respect to x.
Technique: *We have two terms to differentiate:*

$$y'(x) = \frac{\varphi'(x)}{\psi(x)}\nu'(z) - \frac{\psi'(x)}{\psi(x)^2}\nu(z).$$

Each term requires the product rule and chain rule.

Step 5a: Differentiating the First Term

For the term $\frac{\varphi'}{\psi}\nu'$, apply the product rule:

$$\frac{d}{dx}\left[\frac{\varphi'(x)}{\psi(x)}\nu'(z)\right] = \frac{d}{dx}\left[\frac{\varphi'}{\psi}\right] \cdot \nu' + \frac{\varphi'}{\psi} \cdot \frac{d\nu'}{dx}.$$

For the first factor, use the quotient rule:

$$\frac{d}{dx}\left[\frac{\varphi'}{\psi}\right] = \frac{\varphi''\psi - \varphi'\psi'}{\psi^2}.$$

For the second factor, use the chain rule:

$$\frac{d\nu'}{dx} = \frac{d}{dx}\left[\frac{d\nu}{dz}\right] = \frac{d^2\nu}{dz^2} \cdot \frac{dz}{dx} = \nu''(z) \cdot \varphi'(x).$$

Therefore:

$$\frac{d}{dx}\left[\frac{\varphi'}{\psi}\nu'\right] = \frac{\varphi''\psi - \varphi'\psi'}{\psi^2}\nu' + \frac{\varphi'}{\psi}\nu''\varphi' = \frac{\varphi''\psi - \varphi'\psi'}{\psi^2}\nu' + \frac{(\varphi')^2}{\psi}\nu''.$$

Step 5b: Differentiating the Second Term

For the term $-\frac{\psi'}{\psi^2}\nu$, apply the product rule:

$$\frac{d}{dx}\left[-\frac{\psi'}{\psi^2}\nu\right] = -\frac{d}{dx}\left[\frac{\psi'}{\psi^2}\right]\cdot\nu - \frac{\psi'}{\psi^2}\cdot\frac{d\nu}{dx}.$$

For the first factor, use the quotient rule:

$$\frac{d}{dx}\left[\frac{\psi'}{\psi^2}\right] = \frac{\psi''\psi^2 - \psi'\cdot 2\psi\psi'}{\psi^4} = \frac{\psi''\psi - 2(\psi')^2}{\psi^3}.$$

For the second factor:

$$\frac{d\nu}{dx} = \nu'(z)\varphi'(x).$$

Therefore:

$$\frac{d}{dx}\left[-\frac{\psi'}{\psi^2}\nu\right] = -\frac{\psi''\psi - 2(\psi')^2}{\psi^3}\nu - \frac{\psi'}{\psi^2}\nu'\varphi'.$$

Step 5c: Combining Both Terms

Adding the results from Steps 5a and 5b:

$$y''(x) = \frac{\varphi''\psi - \varphi'\psi'}{\psi^2}\nu' + \frac{(\varphi')^2}{\psi}\nu'' - \frac{\psi''\psi - 2(\psi')^2}{\psi^3}\nu - \frac{\psi'\varphi'}{\psi^2}\nu'.$$

Collecting terms by derivative order of ν:

$$y''(x) = \frac{(\varphi')^2}{\psi}\nu'' + \left[\frac{\varphi''\psi - \varphi'\psi'}{\psi^2} - \frac{\psi'\varphi'}{\psi^2}\right]\nu' - \frac{\psi''\psi - 2(\psi')^2}{\psi^3}\nu.$$

Simplifying the coefficient of ν':

$$\frac{\varphi''\psi - \varphi'\psi' - \psi'\varphi'}{\psi^2} = \frac{\varphi''\psi - 2\varphi'\psi'}{\psi^2}.$$

Therefore:

$$y''(x) = \frac{(\varphi')^2}{\psi}\nu'' + \frac{\varphi''\psi - 2\varphi'\psi'}{\psi^2}\nu' - \frac{\psi''\psi - 2(\psi')^2}{\psi^3}\nu.$$

Step 6: Substituting into the Original ODE

What we do: Substitute y and y'' into $\varepsilon^2 y'' + qy = 0$.

$$\varepsilon^2 y'' + q(x)y = 0$$
$$\varepsilon^2\left[\frac{(\varphi')^2}{\psi}\nu'' + \frac{\varphi''\psi - 2\varphi'\psi'}{\psi^2}\nu' - \frac{\psi''\psi - 2(\psi')^2}{\psi^3}\nu\right] + q(x)\frac{\nu}{\psi} = 0.$$

Multiply through by ψ to clear denominators:

$$\varepsilon^2(\varphi')^2\nu'' + \varepsilon^2\frac{\varphi''\psi - 2\varphi'\psi'}{\psi}\nu' - \varepsilon^2\frac{\psi''\psi - 2(\psi')^2}{\psi^2}\nu + q(x)\nu = 0.$$

Step 7: Imposing Conditions for the Target Form

Our goal: We want $\varepsilon^2\nu''(z) + \nu(z) = 0$, which means:

- Coefficient of ν'': should be ε^2
- Coefficient of ν': should be 0
- Coefficient of ν: should be 1

Strategy: *We have three conditions and two unknown functions (φ and ψ). We'll impose the conditions strategically:*

1. *Make the coefficient of ν'' equal to ε^2*
2. *Make the coefficient of ν' equal to 0*
3. *Check what remains for the coefficient of ν*

Step 7a: Condition from the ν'' Term

The coefficient of ν'' is $\varepsilon^2(\varphi')^2$. We require:

$$\varepsilon^2(\varphi')^2 = \varepsilon^2 \quad \Longrightarrow \quad (\varphi')^2 = 1 \quad \Longrightarrow \quad \varphi'(x) = \pm 1.$$

Justification: *We choose the positive sign (the negative would just reverse the direction of z):*

$$\varphi'(x) = 1 \quad \Longrightarrow \quad \varphi(x) = x + \mathit{const.}$$

Without loss of generality, we can set the constant to zero, so $\varphi(x) = x$, which means $z = x$.
Reflection: *Wait! If $\varphi(x) = x$, then we're not really changing the independent variable at all. This seems strange at first, but it makes sense: the essence of the WKB transformation is in the* dependent variable *transformation $\psi(x)$, not the independent variable. However, let's not assume this yet and continue more generally.*

Step 7b: General Case - Condition from the ν' Term

Let's continue with general $\varphi'(x)$ satisfying $(\varphi')^2 = 1$. The coefficient of ν' is:

$$\varepsilon^2 \frac{\varphi''\psi - 2\varphi'\psi'}{\psi}.$$

We require this to vanish:

$$\varphi''\psi - 2\varphi'\psi' = 0 \quad \Longrightarrow \quad \varphi''\psi = 2\varphi'\psi'.$$

Rearranging:

$$\frac{\varphi''}{\varphi'} = 2\frac{\psi'}{\psi}.$$

Technique: *Recognize that $\frac{\varphi''}{\varphi'} = \frac{d}{dx}\ln|\varphi'|$ and $\frac{\psi'}{\psi} = \frac{d}{dx}\ln|\psi|$. Therefore:*

$$\frac{d}{dx}\ln|\varphi'| = 2\frac{d}{dx}\ln|\psi| = \frac{d}{dx}\ln|\psi^2|.$$

Integrating:

$$\ln|\varphi'| = \ln|\psi^2| + C \quad \Longrightarrow \quad |\varphi'| = K\psi^2,$$

where $K = e^C$ is a positive constant.

Combined with $(\varphi')^2 = 1$, we have $|\varphi'| = 1$, so:

$$\psi^2 = \frac{1}{K}.$$

Choosing $K = 1$ (absorbing the constant into ν), we get:

$$\psi(x) = 1.$$

Reflection: *This is surprising! The dependent variable transformation is trivial: $\psi(x) = 1$ means $\nu(z) = y(x)$. Combined with $\varphi(x) = x$, we seem to have done nothing. But this can't be right - let's reconsider our approach.*

Step 8: Reconsidering the Transformation Strategy

Strategy: *The issue is that we've been too restrictive. Let's instead require that the coefficient of ν'' be ε^2 times* something that could vary with x*, but which we'll handle at leading order as $\varepsilon \to 0$. The key insight is that we want:*

$$\varepsilon^2(\varphi')^2 = \varepsilon^2 \quad \textit{to leading order.}$$

But actually, let's try a different approach: set $(\varphi')^2 = q(x)$ instead!
Justification: *Why this choice? Because if $(\varphi')^2 = q(x)$, then:*

$$\varphi'(x) = \sqrt{q(x)},$$

and the coefficient of ν'' becomes $\varepsilon^2 q(x)$. If we then divide the entire equation by $q(x)$, we can potentially get the desired form. Let's try this!

Step 9: New Approach - Setting $\varphi'(x) = \sqrt{q(x)}$

Choice: Let $\varphi'(x) = \sqrt{q(x)}$, so:

$$z = \varphi(x) = \int^x \sqrt{q(s)}\, ds.$$

Justification: *This is the natural choice because $\sqrt{q(x)}$ is the local "frequency" of oscillation in the WKB solution. By integrating it, we're creating a new coordinate z that measures accumulated phase. This connects directly to the WKB solution formula (Lecture Notes §6.3.2, equation (382)):*

$$y_\pm(x) \sim \frac{A_\pm}{|q(x)|^{1/4}} \exp\left(\pm\frac{i}{\varepsilon}\int^x \sqrt{q(s)}\, ds\right).$$

With this choice, the coefficient of ν'' becomes:

$$\varepsilon^2(\varphi')^2 = \varepsilon^2 q(x).$$

Step 10: Finding $\psi(x)$ with the New φ

Now we need to find $\psi(x)$ such that the ν' term vanishes. The coefficient of ν' is:

$$\varepsilon^2 \frac{\varphi''\psi - 2\varphi'\psi'}{\psi}.$$

Setting this to zero:

$$\varphi''\psi - 2\varphi'\psi' = 0 \quad \implies \quad \frac{\varphi''}{\varphi'} = 2\frac{\psi'}{\psi}.$$

With $\varphi' = \sqrt{q}$:

$$\varphi'' = \frac{d}{dx}\sqrt{q} = \frac{q'}{2\sqrt{q}} = \frac{q'}{2\varphi'}.$$

Therefore:

$$\frac{\varphi''}{\varphi'} = \frac{q'}{2q}.$$

The condition becomes:

$$\frac{q'}{2q} = 2\frac{\psi'}{\psi} \quad \Longrightarrow \quad \frac{\psi'}{\psi} = \frac{q'}{4q}.$$

Technique: *Integrate:* $\ln|\psi| = \frac{1}{4}\ln|q| + C$*, so:*

$$\psi(x) = Kq(x)^{1/4},$$

where K *is a constant. Choosing* $K = 1$*:*

$$\psi(x) = q(x)^{1/4}.$$

Step 11: Verifying the Coefficient of ν

With $\varphi' = \sqrt{q}$ and $\psi = q^{1/4}$, let's check the coefficient of ν in our transformed equation. The original coefficient (after multiplying by ψ) was:

$$-\varepsilon^2\frac{\psi''\psi - 2(\psi')^2}{\psi^2} + q(x).$$

Technique: *We need to compute* ψ' *and* ψ''*:*

$$\begin{aligned}
\psi &= q^{1/4} \\
\psi' &= \frac{1}{4}q^{-3/4}q' = \frac{q'}{4q^{3/4}} \\
\psi'' &= \frac{d}{dx}\left[\frac{q'}{4q^{3/4}}\right] = \frac{1}{4}\left[\frac{q''q^{3/4} - q' \cdot \frac{3}{4}q^{-1/4}q'}{q^{3/2}}\right] \\
&= \frac{1}{4q^{3/4}}\left[q'' - \frac{3(q')^2}{4q}\right].
\end{aligned}$$

Computing $\psi''\psi$:

$$\begin{aligned}
\psi''\psi &= \frac{1}{4q^{3/4}}\left[q'' - \frac{3(q')^2}{4q}\right] \cdot q^{1/4} \\
&= \frac{1}{4q^{1/2}}\left[q'' - \frac{3(q')^2}{4q}\right].
\end{aligned}$$

Computing $(\psi')^2$:

$$(\psi')^2 = \frac{(q')^2}{16q^{3/2}}.$$

Therefore:

$$\begin{aligned}
\psi''\psi - 2(\psi')^2 &= \frac{1}{4q^{1/2}}\left[q'' - \frac{3(q')^2}{4q}\right] - \frac{(q')^2}{8q^{3/2}} \\
&= \frac{1}{4q^{1/2}}q'' - \frac{3(q')^2}{16q^{3/2}} - \frac{2(q')^2}{16q^{3/2}} \\
&= \frac{1}{4q^{1/2}}q'' - \frac{5(q')^2}{16q^{3/2}}.
\end{aligned}$$

And:

$$\frac{\psi''\psi - 2(\psi')^2}{\psi^2} = \frac{\frac{1}{4q^{1/2}}q'' - \frac{5(q')^2}{16q^{3/2}}}{q^{1/2}} = \frac{1}{4q}q'' - \frac{5(q')^2}{16q^2}.$$

The coefficient of ν is:

$$-\varepsilon^2\left[\frac{1}{4q}q'' - \frac{5(q')^2}{16q^2}\right] + q = q\left[1 - \varepsilon^2\left(\frac{q''}{4q^2} - \frac{5(q')^2}{16q^3}\right)\right].$$

Step 12: Taking the Leading Order as $\varepsilon \to 0$

Justification: *As $\varepsilon \to 0$, the term multiplied by ε^2 becomes negligible. To leading order:*

$$\textit{Coefficient of } \nu \to q(x) \quad \textit{as } \varepsilon \to 0.$$

So our transformed equation is (to leading order):

$$\varepsilon^2 q(x)\nu''(z) + q(x)\nu(z) = 0.$$

Dividing by $q(x)$:

$$\varepsilon^2\nu''(z) + \nu(z) = 0.$$

Success! With the transformations:

$$z = \varphi(x) = \int^x \sqrt{q(s)}\,ds$$
$$\nu(z) = \psi(x)y(x) = q(x)^{1/4}y(x)$$

the ODE $\varepsilon^2 y'' + q(x)y = 0$ becomes $\varepsilon^2\nu'' + \nu = 0$ to leading order as $\varepsilon \to 0$.

Step 13: Solving the Transformed Equation

The equation: $\varepsilon^2\nu''(z) + \nu(z) = 0$ is a constant-coefficient ODE.
Technique: *Try $\nu(z) = e^{\lambda z}$:*

$$\varepsilon^2\lambda^2 e^{\lambda z} + e^{\lambda z} = 0 \implies \lambda^2 = -\frac{1}{\varepsilon^2} \implies \lambda = \pm\frac{i}{\varepsilon}.$$

The general solution is:

$$\nu(z) = A\exp\left(\frac{iz}{\varepsilon}\right) + B\exp\left(-\frac{iz}{\varepsilon}\right),$$

or equivalently:

$$\nu(z) = C\cos\left(\frac{z}{\varepsilon}\right) + D\sin\left(\frac{z}{\varepsilon}\right).$$

Step 14: Transforming Back to $y(x)$

From $\nu(z) = \psi(x)y(x) = q(x)^{1/4}y(x)$, we have:

$$y(x) = \frac{\nu(z)}{q(x)^{1/4}} = \frac{\nu(\varphi(x))}{q(x)^{1/4}}.$$

Substituting $\nu(z) = A\exp(iz/\varepsilon) + B\exp(-iz/\varepsilon)$ and $z = \int^x \sqrt{q(s)}\,ds$:

$$y(x) = \frac{1}{q(x)^{1/4}}\left[A\exp\left(\frac{i}{\varepsilon}\int^x \sqrt{q(s)}\,ds\right) + B\exp\left(-\frac{i}{\varepsilon}\int^x \sqrt{q(s)}\,ds\right)\right].$$

Step 15: Comparing with the WKB Solution

Justification: *From Lecture Notes §6.3.2, equation (382), the WKB solution for* $\varepsilon^2 y'' + q(x)y = 0$ *with* $q(x) > 0$ *is:*

$$y_{\pm}(x) \sim \frac{A_{\pm}}{|q(x)|^{1/4}} \exp\left(\pm \frac{i}{\varepsilon} \int^x \sqrt{q(s)}\, ds + O(\varepsilon)\right) \quad \text{as } \varepsilon \to 0.$$

Since $q(x) > 0$*, we have* $|q(x)| = q(x)$*, so this becomes:*

$$y_{\pm}(x) \sim \frac{A_{\pm}}{q(x)^{1/4}} \exp\left(\pm \frac{i}{\varepsilon} \int^x \sqrt{q(s)}\, ds\right).$$

Our solution from the transformation method is:

$$y(x) = \frac{A}{q(x)^{1/4}} \exp\left(\frac{i}{\varepsilon} \int^x \sqrt{q(s)}\, ds\right) + \frac{B}{q(x)^{1/4}} \exp\left(-\frac{i}{\varepsilon} \int^x \sqrt{q(s)}\, ds\right).$$

Conclusion: The solution obtained by the double transformation method is *exactly* the leading order WKB solution! This proves that the WKB approximation can be understood as a systematic coordinate and amplitude transformation that converts the variable-coefficient ODE into a constant-coefficient one.

Final Summary and Physical Interpretation

Reflection: *What have we learned?*

1. ***The transformation*** $z = \int^x \sqrt{q(s)}\, ds$ *converts the spatial variable* x *into a "phase variable"* z *that measures accumulated oscillations. Where* $q(x)$ *is large, oscillations are rapid and* $dz/dx = \sqrt{q}$ *is large.*
2. ***The transformation*** $\nu = q^{1/4} y$ *rescales the amplitude. The factor* $q^{1/4}$ *appears because it's the unique power that eliminates the first derivative term after the coordinate transformation.*
3. ***Geometric insight:*** *In the* (z, ν) *coordinates, solutions are pure sinusoids with wavelength* $\sim \varepsilon$*. When transformed back to* (x, y) *coordinates, these become:*
   - *Amplitude-modulated by* $q(x)^{-1/4}$
   - *Phase-modulated by* $\int \sqrt{q(s)}\, ds$
4. ***Connection to §7.2.1:*** *This is exactly the approach used in Lecture Notes equations (479)-(511) to derive WKB from the averaging method. The key equations there are:*
   - *Equation (488):* $\dot{h}(t) = \sqrt{q(\varepsilon t)}$ *(same as our* φ'*)*
   - *Equation (510):* $R(T) = c_2/q(\tau)^{1/4}$ *(same as our* ψ^{-1}*)*
   - *Equation (511): The final WKB form*