Methods of Applied Mathematics - Part 1 [SEMT30006]

Exercise Sheet 3 – Bifurcations

1. Consider the system
$$\dot{x} = (5 - x)(1 - ax)$$
with $a > 0$. Find the equilibria of the system and their stability. Identify the bifurcation that occurs at $a = 1/5$ and sketch the bifurcation diagram.

2. Consider the system
$$\dot{x} = a + 2x + x^2$$
   (a) Find the equilibria of the system and their stability.
   (b) Conjecture the bifurcation that occurs in the system.
   (c) Evaluate the bifurcation and genericity conditions for your conjectured bifurcation (the B's and G's from our definitions of each bifurcation in lectures) to prove that it indeed occurs.

3. Consider the dynamical system
$$\dot{x} = y - 3x$$
$$\dot{y} = \alpha x - x^2$$
for $-9/4 < \alpha < 9/4$.
   (a) Compute and classify the stability/type of any equilibria.
   (b) What bifurcation happens in the system at $\alpha = 0$?
   (c) Draw a bifurcation diagram with α on the horizontal axis, and x on the vertical. What would the diagram look like if you drew α against y?

4. Consider the dynamical system
$$\dot{x} = \alpha x - x^3$$
$$\dot{y} = -y$$
   (a) Compute and classify the stability/type of any equilibria.
   (b) What bifurcation happens in the system at $\alpha = 0$?
   (c) Draw a bifurcation diagram with α on the horizontal axis, and x on the vertical. What would the diagram look like if you drew α against y?

5. Show that a Hopf bifurcation happens in the system $\frac{dx}{dt} = 1 + x^2y - \mu x - x$, $\frac{dy}{dt} = \mu x - x^2y$ as μ varies.

6. Consider the Brusselator system (a chemical reaction equation)

$$\dot{x} = a - bx + px^2y - qx$$
$$\dot{y} = bx - px^2y$$

Let $a = q = p = 1$ and consider what happens as b varies. In this problem b is a reaction rate so it is positive.

(a) Find any equilibria.

(b) Find their stability.

(c) Conjecture the bifurcation that occurs in the system, stating where (in x, y, and b it happens), and sketch the bifurcation diagram.

Bifurcations are a big part of this course, so here are some extra questions for practicing on later. . .

7. Determine what bifurcation happens as μ changes in the systems:

   (a) $\frac{dx}{dt} = \mu x - x^3$

   (b) $\frac{dx}{dt} = -\mu x + (1+\mu)x^2 - x^3$

   (c) $\frac{dx}{dt} = \tanh(x) - \mu x$

   (d) $\frac{d^2x}{dt^2} + \frac{dx}{dt} + \mu x + x^3 = 0$

   (e) $\frac{dx}{dt} = \mu y - x$, $\frac{dy}{dt} = -\frac{1}{3}y^3 + y^2 - y + x$

8. Determine what bifurcation happens as μ changes in the system $\frac{dx}{dt} = x - x^3 + \mu$

9. In lectures we wrote the normal form of the Hopf bifurcation in complex variables as

$$\dot{z} = (\rho + i\omega)z + \ell_1 z|z|^2 \tag{1}$$

   By letting $z = x + iy$ show that this is just a neat way of writing the two-dimensional ODE

$$\begin{aligned} \dot{x} &= \rho x - \omega y + \ell_1 x(x^2 + y^2) \\ \dot{y} &= \omega x + \rho y + \ell_1 y(x^2 + y^2) \end{aligned}$$

10. Consider the system

$$\begin{aligned} \dot{x} &= \rho x - \omega y + \alpha(x^2 + y^2)x \\ \dot{y} &= \omega x + \rho y + \alpha(x^2 + y^2)y \end{aligned}$$

   for $\omega > 0$ and $\alpha > 0$, with ρ allowed to vary.

   (a) Find any equilibria of the system and their stability. [Hint: there's an obvious equilibrium, and if its gets to hard to find any others, have a try but don't waste too much time ... you'll find a better way to look at this in (c)).

   (b) Determine what bifurcation occurs at the origin when $\rho = 0$.

   (c) Expressing the system in polar coordinates $(x, y) = (r\cos\phi, r\sin\phi)$, derive dynamical equations for $\dot{r}$ and $\dot{\phi}$. Describe what these tell you about the dynamics.

   (d) From the polar form, show that there exists a limit cycle in the system (corresponding to an equilibrium of the radial $\dot{r}$ system). Hint: show that there is a bifurcation in the $\dot{r}$ system when $\rho = 0$, identify it, and interpret what happens in the system].

   (e) Derive the Poincaré map from the section $\phi = 0$ back to itself, that is, a map of any point r_n on $\phi = 0$ to a point r_{n+1} when ϕ returns to $\phi = 2\pi$. Argue that this verifies the existence of a limit cycle.