Asymptotics 2025/2026 – Problem Sheet 5
Solution to Question 2(a)

Method of Steepest Descent

Problem Statement

Use the method of steepest descent to find the leading asymptotic behaviour as $X \to \infty$ of:

$$I(X) = \int_{-1}^{\infty} e^{X(t+\mathrm{i}t-t^2/2)} \mathrm{d}t \tag{1}$$

Solution

Step 1: Identify the Complex Function

The integral has the form

$$I(X) = \int_{-1}^{\infty} e^{X\phi(t)} \mathrm{d}t \tag{2}$$

where the complex function $\phi(t)$ is given by:

$$\phi(t) = t + \mathrm{i}t - \frac{t^2}{2} = t(1+\mathrm{i}) - \frac{t^2}{2} \tag{3}$$

This is a steepest descent problem with $\phi(t) \in \mathbb{C}$ and integration parameter $X \to \infty$.

Step 2: Find Critical Points (Saddle Points)

Following the methodology from Section 4.4.2 of the lecture notes, we locate saddle points by finding where $\phi'(t) = 0$:

$$\phi'(t) = 1 + \mathrm{i} - t = 0 \tag{4}$$

Thus, the unique saddle point is:

$$t_0 = 1 + \mathrm{i} \tag{5}$$

The second derivative at this point is:

$$\phi''(t) = -1 \quad \text{for all } t \tag{6}$$

In particular, $\phi''(t_0) = -1 = e^{\mathrm{i}\pi}$, so $|\phi''(t_0)| = 1$ and $\alpha = \pi$.

Step 3: Evaluate ϕ at the Saddle Point

We compute:

$$\begin{aligned}
\phi(t_0) &= (1+\mathrm{i})(1+\mathrm{i}) - \frac{(1+\mathrm{i})^2}{2} && (7)\\
&= (1+\mathrm{i})^2 - \frac{(1+\mathrm{i})^2}{2} && (8)\\
&= \frac{(1+\mathrm{i})^2}{2} && (9)\\
&= \frac{1+2\mathrm{i}-1}{2} && (10)\\
&= \mathrm{i} && (11)
\end{aligned}$$

Step 4: Determine Steepest Descent Contours

To understand the geometry of the steepest descent paths, we decompose $\phi(t)$ into real and imaginary parts. Setting $t = x + \mathrm{i}y$:

$$\begin{aligned}
\phi(x+\mathrm{i}y) &= (x+\mathrm{i}y)(1+\mathrm{i}) - \frac{(x+\mathrm{i}y)^2}{2} && (12)\\
&= x + \mathrm{i}x + \mathrm{i}y - y - \frac{x^2 - y^2 + 2\mathrm{i}xy}{2} && (13)\\
&= \left(x - y - \frac{x^2-y^2}{2}\right) + \mathrm{i}\,(x + y - xy) && (14)
\end{aligned}$$

Therefore:

$$u(x,y) = x - y - \frac{x^2-y^2}{2}, \qquad v(x,y) = x + y - xy \tag{15}$$

At the saddle point $(x_0, y_0) = (1,1)$:

$$v(1,1) = 1 + 1 - 1 = 1 \tag{16}$$

The constant phase contour through the saddle point satisfies $v(x,y) = 1$.

Local analysis near the saddle point:

Near $t_0 = 1 + \mathrm{i}$, set $t = t_0 + se^{\mathrm{i}\theta}$ for small $s \geq 0$. Then:

$$\phi(t) \approx \phi(t_0) + \frac{1}{2}\phi''(t_0)(t-t_0)^2 = \mathrm{i} - \frac{1}{2}s^2 e^{2\mathrm{i}\theta} \tag{17}$$

The real part is:

$$\mathrm{Re}[\phi(t)] \approx -\frac{s^2}{2}\cos(2\theta) \tag{18}$$

Using the general formula from the lecture notes (page 72), for $\phi''(t_0) = ae^{\mathrm{i}\alpha}$ with $a = 1$ and $\alpha = \pi$, the directions of steepest descent are:

$$\theta_{\text{descent}} = -\frac{\alpha}{2} + (2p+1)\frac{\pi}{2}, \quad p = 0, 1 \tag{19}$$

For $p = 0$: $\theta = -\pi/2 + \pi/2 = 0$

For $p = 1$: $\theta = -\pi/2 + 3\pi/2 = \pi$

Thus, the steepest descent directions are along $\theta = 0$ and $\theta = \pi$, i.e., along the horizontal direction (parallel to the real axis) passing through $t_0 = 1 + \mathrm{i}$.

Parameterization of steepest descent path:

The steepest descent path through t_0 is the horizontal line at height $y = 1$:

$$t = x + \mathrm{i}, \quad x \in (-\infty, \infty) \tag{20}$$

Along this path:

$$\phi(x + \mathrm{i}) = (x + \mathrm{i})(1 + \mathrm{i}) - \frac{(x + \mathrm{i})^2}{2} \tag{21}$$

$$= x - 1 + \mathrm{i}(x + 1) - \frac{x^2 - 1 + 2\mathrm{i}x}{2} \tag{22}$$

$$= x - 1 - \frac{x^2 - 1}{2} + \mathrm{i}\,(x + 1 - x) \tag{23}$$

$$= -\frac{x^2}{2} + x - \frac{1}{2} + \mathrm{i} \tag{24}$$

$$= -\frac{(x - 1)^2}{2} + \mathrm{i} \tag{25}$$

The real part is:

$$u(x, 1) = -\frac{(x - 1)^2}{2} \tag{26}$$

This has a maximum at $x = 1$ (the saddle point) and decreases to $-\infty$ as $|x| \to \infty$, confirming exponential decay along the steepest descent path.

Step 5: Deform the Integration Contour

The original integration path runs from -1 to ∞ along the real axis (at $y = 0$). The saddle point $t_0 = 1 + \mathrm{i}$ lies off this path.

By Cauchy's integral theorem (Section 4.4, page 68), since $e^{X\phi(t)}$ is entire (analytic everywhere in $\mathbb{C}$), we can deform the contour without changing the integral value, provided boundary contributions vanish.

We deform the contour to consist of:

(i) A vertical segment from -1 to $-1 + \mathrm{i}$

(ii) The horizontal steepest descent path from $-1 + \mathrm{i}$ to $+\infty + \mathrm{i}$ (passing through the saddle point $t_0 = 1 + \mathrm{i}$)

Justification for neglecting the vertical segment:
Along the vertical segment $t = -1 + \mathrm{i}s$ for $s \in [0, 1]$:

$$\mathrm{Re}[\phi(-1 + \mathrm{i}s)] = -1 - s - \frac{1 - s^2}{2} \tag{27}$$

$$= -\frac{3}{2} - s + \frac{s^2}{2} \tag{28}$$

$$\leq -\frac{3}{2} + \frac{1}{2} = -1 \tag{29}$$

Therefore:

$$\left| \int_0^1 e^{X\phi(-1+\mathrm{i}s)} \mathrm{i}\mathrm{d}s \right| \leq \int_0^1 e^{-X} \mathrm{d}s = e^{-X} \tag{30}$$

As $X \to \infty$, this contribution is exponentially small compared to the saddle point contribution and can be neglected: $e^{-X} = o(e^0) = o(1)$.

Step 6: Evaluate the Integral Along the Steepest Descent Path

Along the horizontal line $y = 1$, we have $t = x + \mathrm{i}$ with $x \in (-\infty, \infty)$ and $\mathrm{d}t = \mathrm{d}x$. Using our result from Step 4:

$$\phi(x + \mathrm{i}) = \mathrm{i} - \frac{(x - 1)^2}{2} \tag{31}$$

Therefore:

$$I(X) \sim \int_{-\infty}^{\infty} e^{X[\mathrm{i}-(x-1)^2/2]}\mathrm{d}x \tag{32}$$

$$= e^{\mathrm{i}X} \int_{-\infty}^{\infty} e^{-X(x-1)^2/2}\mathrm{d}x \tag{33}$$

Change of variables:
Let $u = x - 1$, so $\mathrm{d}u = \mathrm{d}x$:

$$I(X) = e^{\mathrm{i}X} \int_{-\infty}^{\infty} e^{-Xu^2/2}\mathrm{d}u \tag{34}$$

Evaluate the Gaussian integral:
Using the standard result $\int_{-\infty}^{\infty} e^{-au^2}\mathrm{d}u = \sqrt{\pi/a}$ for $a > 0$:

$$\int_{-\infty}^{\infty} e^{-Xu^2/2}\mathrm{d}u = \sqrt{\frac{2\pi}{X}} \tag{35}$$

Step 7: Final Result

Combining the results, we obtain the leading asymptotic behaviour:

$$\boxed{I(X) \sim \sqrt{\frac{2\pi}{X}}\, e^{\mathrm{i}X} \quad \text{as } X \to \infty} \tag{36}$$

This can also be written as:

$$I(X) \sim \sqrt{\frac{2\pi}{X}}\, [\cos(X) + \mathrm{i}\sin(X)] \quad \text{as } X \to \infty \tag{37}$$

Verification

The result is consistent with the general steepest descent formula from the lecture notes. For an integral of the form $\int f(z)e^{\lambda\phi(z)}\mathrm{d}z$ with a saddle point at z_0 where $\phi'(z_0) = 0$ and $\phi''(z_0) \neq 0$, the leading asymptotic contribution is:

$$I(\lambda) \sim \sqrt{\frac{2\pi}{\lambda|\phi''(z_0)|}}\, f(z_0)\, e^{\lambda\phi(z_0)}\, e^{\mathrm{i}\beta} \tag{38}$$

where β accounts for the phase factor depending on the direction of the steepest descent path.

In our case:

- $\lambda = X$
- $f(t) = 1$, so $f(t_0) = 1$
- $\phi(t_0) = \mathrm{i}$
- $|\phi''(t_0)| = |-1| = 1$
- The steepest descent path is horizontal, contributing no additional phase

This yields:

$$I(X) \sim \sqrt{\frac{2\pi}{X}}\, e^{\mathrm{i}X} \tag{39}$$

confirming our result. □