Asymptotics 2025/2026 Sheet 6[1]

Regular perturbations of ODEs and eigenvalue problems

Please hand in questions 1 and 3 on Thursday 13 November 2025 at 12pm

1. Obtain a two-term expansion when $\epsilon \ll 1$ for the solution of
$$\frac{df}{dt} - f = \epsilon f^2 e^{-t}, \qquad f(0) = 1.$$

2. Obtain a two-term expansion when $\epsilon \ll 1$ for the solution of
$$\frac{d^2 f}{dt^2} + f = \epsilon \frac{df}{dt}, \qquad f(0) = 1, \quad \frac{df}{dt}(0) = \frac{\epsilon}{2}.$$
Compare this expansion with that obtained by expanding the exact solution.

3. (a) The distance x of a projectile from the Earth at time t is governed by the following equation and initial conditions
$$\ddot{x} = \frac{-1}{(1+\epsilon x)^2}, \qquad x(0) = 0, \quad \dot{x}(0) = 1.$$
Here $\epsilon = V^2/(gR)$ where V is the initial speed upward, g is the gravitational acceleration and R is the Earth's radius (distance has been non-dimensionalised by V^2/g and time by V/g). Find the time taken for the projectile to reach its maximum height correct to order ϵ. (Optional: to order ϵ^2.) (Remark: two expansions in ϵ have to be carried out, first for $x(t, \epsilon)$ and then for the time to reach the maximum height $t_m(\epsilon)$.)

   (b) If, instead of incorporating variations in g, we include air resistance effects, the problem to be solved is
$$\ddot{x} + \beta \dot{x} = -1, \qquad x(0) = 0, \quad \dot{x}(0) = 1$$
where $0 < \beta \ll 1$ is the ratio of air resistance at speed V to gravitational force. What now is the time for the projectile to reach its maximum height? (just find the leading correction.)

4. Consider the eigenvalue problem
$$y'' + \lambda(1+\epsilon x)y = 0, \qquad x \in [0, \pi], \qquad y(0) = y(\pi) = 0$$
where $0 < \epsilon \ll 1$. Determine the first order correction to the unperturbed eigenvalues given by $\epsilon = 0$.

5. Estimate the eigenvalues and eigenvectors of the matrix

$$\begin{bmatrix} 1 & 1-\epsilon \\ \epsilon - 1 & 1 \end{bmatrix}$$

for $\epsilon \ll 1$. Compare your results with the exact solution.