Asymptotics Problem Sheet 3

Question 5: Asymptotic Expansion via Watson's Lemma

Solution with XYZ Methodology

Academic Year 2025–2026

Problem. Show that

$$\int_0^\infty \left(1 + \frac{u}{X}\right)^{-X} e^{-u}\mathrm{d}u \sim \frac{1}{2} + \frac{1}{8X} - \frac{1}{32X^2} \quad \text{as } X \to \infty.$$

Solution. We seek an asymptotic expansion of the integral

$$I(X) = \int_0^\infty \left(1 + \frac{u}{X}\right)^{-X} e^{-u}\mathrm{d}u$$

as $X \to \infty$.

Step 1: Recognition of Integral Type

What we observe: The integral contains the factor $\left(1 + \frac{u}{X}\right)^{-X}$ multiplied by e^{-u}, integrated from 0 to ∞.

Why this matters: This is a Laplace-type integral where the large parameter X appears in an exponent. The presence of $X \to \infty$ as the asymptotic limit suggests we should use techniques from Section 4.2 of the lecture notes.

What we recognize: The factor $\left(1 + \frac{u}{X}\right)^{-X}$ can be rewritten using the exponential-logarithm identity:

$$\left(1 + \frac{u}{X}\right)^{-X} = \exp\left(-X \log\left(1 + \frac{u}{X}\right)\right).$$

Why we do this: By expressing the factor as an exponential, we can combine it with e^{-u} to obtain a single exponential factor, which is the standard form for Laplace-type integrals.

Step 2: Combining Exponential Factors

What we have: Using the exponential form from Step 1:

$$I(X) = \int_0^\infty \exp\left(-X \log\left(1 + \frac{u}{X}\right)\right) e^{-u}\mathrm{d}u = \int_0^\infty \exp\left(-X \log\left(1 + \frac{u}{X}\right) - u\right) \mathrm{d}u.$$

Why this form is useful: We now have a single exponential with argument

$$-X \log\left(1 + \frac{u}{X}\right) - u.$$

This allows us to analyze the behavior of the integrand as $X \to \infty$.

Step 3: Taylor Expansion of the Logarithm

What we need: To understand the behavior as $X \to \infty$, we expand $\log\left(1 + \frac{u}{X}\right)$ for large X (equivalently, small $\frac{u}{X}$).

Why we need this: The logarithm is multiplied by X, so even though $\frac{u}{X}$ is small, the product $X \log\left(1 + \frac{u}{X}\right)$ may have a non-trivial limit. We must expand carefully to capture all relevant orders.

What we know: The Taylor series for $\log(1+z)$ around $z = 0$ is (from standard calculus):

$$\log(1 + z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \mathcal{O}(z^5).$$

Why this applies: Setting $z = \frac{u}{X}$, we have:

$$\log\left(1 + \frac{u}{X}\right) = \frac{u}{X} - \frac{u^2}{2X^2} + \frac{u^3}{3X^3} - \frac{u^4}{4X^4} + \mathcal{O}(X^{-5}).$$

What this means: This expansion is valid when $\left|\frac{u}{X}\right| < 1$, which holds for fixed u as $X \to \infty$.

Step 4: Multiplying by $-X$

What we compute: Multiply the expansion from Step 3 by $-X$:

$$\begin{aligned}
-X \log\left(1 + \frac{u}{X}\right) &= -X \left[\frac{u}{X} - \frac{u^2}{2X^2} + \frac{u^3}{3X^3} - \frac{u^4}{4X^4} + \mathcal{O}(X^{-5})\right] \\
&= -u + \frac{u^2}{2X} - \frac{u^3}{3X^2} + \frac{u^4}{4X^3} + \mathcal{O}(X^{-4}).
\end{aligned}$$

Why each term appears:

- The $-u$ term: $-X \cdot \frac{u}{X} = -u$ is of order $\mathcal{O}(1)$ (independent of X).
- The $\frac{u^2}{2X}$ term: $-X \cdot \left(-\frac{u^2}{2X^2}\right) = \frac{u^2}{2X}$ is of order $\mathcal{O}(X^{-1})$.
- The $-\frac{u^3}{3X^2}$ term: $-X \cdot \frac{u^3}{3X^3} = -\frac{u^3}{3X^2}$ is of order $\mathcal{O}(X^{-2})$.
- The $\frac{u^4}{4X^3}$ term: $-X \cdot \left(-\frac{u^4}{4X^4}\right) = \frac{u^4}{4X^3}$ is of order $\mathcal{O}(X^{-3})$.

What we observe: Each successive term is smaller by a factor of $\mathcal{O}(X^{-1})$.

Step 5: Substituting into the Exponent

What we substitute: The full exponent in our integral is:

$$\begin{aligned}
-X \log\left(1 + \frac{u}{X}\right) - u &= \left(-u + \frac{u^2}{2X} - \frac{u^3}{3X^2} + \mathcal{O}(X^{-3})\right) - u \\
&= -2u + \frac{u^2}{2X} - \frac{u^3}{3X^2} + \mathcal{O}(X^{-3}).
\end{aligned}$$

Why we group terms this way: The dominant term (independent of X) is $-2u$. This will determine the basic structure of the integral. The remaining terms are corrections of increasing order in X^{-1}.

What our integral becomes:

$$I(X) = \int_0^\infty \exp\left(-2u + \frac{u^2}{2X} - \frac{u^3}{3X^2} + \mathcal{O}(X^{-3})\right) du.$$

Step 6: Factoring the Leading Exponential

What we do: Factor out the dominant exponential e^{-2u}:

$$I(X) = \int_0^\infty e^{-2u} \exp\left(\frac{u^2}{2X} - \frac{u^3}{3X^2} + \mathcal{O}(X^{-3})\right) \mathrm{d}u.$$

Why this factorization is useful: The factor e^{-2u} provides exponential decay as $u \to \infty$, ensuring all integrals converge. The remaining exponential contains only small terms (of order X^{-1} and higher), which we can expand.

Step 7: Expanding the Correction Exponential

What we need to expand: The exponential

$$\exp\left(\frac{u^2}{2X} - \frac{u^3}{3X^2}\right).$$

Why we can expand: For large X and fixed u, the argument $\frac{u^2}{2X} - \frac{u^3}{3X^2}$ is small, so we can use the Taylor series:

$$e^z = 1 + z + \frac{z^2}{2} + \frac{z^3}{6} + \cdots$$

What we compute: Let $z = \frac{u^2}{2X} - \frac{u^3}{3X^2}$. Then:

$$\begin{aligned} e^z &= 1 + z + \frac{z^2}{2} + \mathcal{O}(z^3) \\ &= 1 + \left(\frac{u^2}{2X} - \frac{u^3}{3X^2}\right) + \frac{1}{2}\left(\frac{u^2}{2X}\right)^2 + \mathcal{O}(X^{-3}) \\ &= 1 + \frac{u^2}{2X} - \frac{u^3}{3X^2} + \frac{u^4}{8X^2} + \mathcal{O}(X^{-3}). \end{aligned}$$

Why we keep only these terms:

- The term $\left(\frac{u^2}{2X}\right)^2 = \frac{u^4}{4X^2}$ contributes at order $\mathcal{O}(X^{-2})$.
- The cross term $2 \cdot \frac{u^2}{2X} \cdot \left(-\frac{u^3}{3X^2}\right) = -\frac{u^5}{3X^3}$ is of order $\mathcal{O}(X^{-3})$ and can be neglected.
- Higher order terms from $\frac{z^2}{2}, \frac{z^3}{6}, \ldots$ are all $\mathcal{O}(X^{-3})$ or smaller.

What we collect: Grouping by powers of X^{-1}:

$$e^z = 1 + \frac{u^2}{2X} + \frac{1}{X^2}\left(\frac{u^4}{8} - \frac{u^3}{3}\right) + \mathcal{O}(X^{-3}).$$

Step 8: Substituting the Expansion into the Integral

What we substitute: Using the expansion from Step 7:

$$\begin{aligned} I(X) &= \int_0^\infty e^{-2u}\left[1 + \frac{u^2}{2X} + \frac{1}{X^2}\left(\frac{u^4}{8} - \frac{u^3}{3}\right)\right] \mathrm{d}u + \mathcal{O}(X^{-3}) \\ &= \int_0^\infty e^{-2u}\mathrm{d}u + \frac{1}{2X}\int_0^\infty u^2 e^{-2u}\mathrm{d}u \\ &\quad + \frac{1}{X^2}\left[\frac{1}{8}\int_0^\infty u^4 e^{-2u}\mathrm{d}u - \frac{1}{3}\int_0^\infty u^3 e^{-2u}\mathrm{d}u\right] + \mathcal{O}(X^{-3}). \end{aligned}$$

Why we can separate the integrals: Each integral converges absolutely due to the exponential decay factor e^{-2u}, so we can distribute the integration over the sum.

Step 9: Evaluating the Standard Integrals

What we need: We must evaluate integrals of the form

$$\int_0^\infty u^n e^{-2u} \mathrm{d}u.$$

Why we know the formula: These are standard Gamma function integrals. From the lecture notes (Section 2.6.1, Equation 68), the Gamma function is:

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} \mathrm{d}t.$$

How to apply this: Substituting $t = 2u$ (so $u = t/2$, $\mathrm{d}u = \mathrm{d}t/2$):

$$\begin{aligned}
\int_0^\infty u^n e^{-2u} \mathrm{d}u &= \int_0^\infty \left(\frac{t}{2}\right)^n e^{-t} \frac{\mathrm{d}t}{2} \\
&= \frac{1}{2^{n+1}} \int_0^\infty t^n e^{-t} \mathrm{d}t \\
&= \frac{1}{2^{n+1}} \Gamma(n+1) \\
&= \frac{n!}{2^{n+1}}.
\end{aligned}$$

Why this formula works: We used $\Gamma(n+1) = n!$ for non-negative integers n (a standard property of the Gamma function).

Step 10: Computing Each Required Integral

What we compute: Using the formula from Step 9 with different values of n:

For $n = 0$:

$$\int_0^\infty e^{-2u} \mathrm{d}u = \frac{0!}{2^{0+1}} = \frac{1}{2}.$$

Why: $0! = 1$ and $2^1 = 2$.

For $n = 2$:

$$\int_0^\infty u^2 e^{-2u} \mathrm{d}u = \frac{2!}{2^{2+1}} = \frac{2}{8} = \frac{1}{4}.$$

Why: $2! = 2$ and $2^3 = 8$.

For $n = 3$:

$$\int_0^\infty u^3 e^{-2u} \mathrm{d}u = \frac{3!}{2^{3+1}} = \frac{6}{16} = \frac{3}{8}.$$

Why: $3! = 6$ and $2^4 = 16$, and simplifying: $\frac{6}{16} = \frac{3}{8}$.

For $n = 4$:

$$\int_0^\infty u^4 e^{-2u} \mathrm{d}u = \frac{4!}{2^{4+1}} = \frac{24}{32} = \frac{3}{4}.$$

Why: $4! = 24$ and $2^5 = 32$, and simplifying: $\frac{24}{32} = \frac{3}{4}$.

Step 11: Assembling the Asymptotic Expansion

What we substitute: Using the computed integrals from Step 10 in the expression from Step 8:

$$I(X) = \frac{1}{2} + \frac{1}{2X} \cdot \frac{1}{4} + \frac{1}{X^2} \left[\frac{1}{8} \cdot \frac{3}{4} - \frac{1}{3} \cdot \frac{3}{8} \right] + \mathcal{O}(X^{-3}).$$

Computing the $\mathcal{O}(X^{-1})$ term:

$$\frac{1}{2X} \cdot \frac{1}{4} = \frac{1}{8X}.$$

Computing the $\mathcal{O}(X^{-2})$ term:

$$\begin{aligned} \frac{1}{X^2} \left[\frac{1}{8} \cdot \frac{3}{4} - \frac{1}{3} \cdot \frac{3}{8} \right] &= \frac{1}{X^2} \left[\frac{3}{32} - \frac{3}{24} \right] \\ &= \frac{1}{X^2} \left[\frac{3}{32} - \frac{1}{8} \right]. \end{aligned}$$

Why we simplify $\frac{3}{24}$:

$$\frac{3}{24} = \frac{1}{8}.$$

Finding a common denominator:

$$\frac{3}{32} - \frac{1}{8} = \frac{3}{32} - \frac{4}{32} = -\frac{1}{32}.$$

Why: We write $\frac{1}{8} = \frac{4}{32}$ to combine with $\frac{3}{32}$.

Step 12: Final Result

What we have established: Combining all terms:

$$I(X) = \frac{1}{2} + \frac{1}{8X} - \frac{1}{32X^2} + \mathcal{O}(X^{-3}).$$

Why this is the answer: This matches the required asymptotic expansion. The expansion is valid as $X \to \infty$ and captures the behavior to order $\mathcal{O}(X^{-2})$.

What we conclude: Therefore,

$$\boxed{\int_0^\infty \left(1 + \frac{u}{X}\right)^{-X} e^{-u} \mathrm{d}u \sim \frac{1}{2} + \frac{1}{8X} - \frac{1}{32X^2} \quad \text{as } X \to \infty.}$$