Solution 5.2(e)

Problem: Find the asymptotic behavior of

$$I(X) = \int_{-1}^{\infty} \sqrt{1+t}\,\cos(Xt^2)\,e^{X(t-t^3/3)}\,dt$$

as $X \to \infty$.

Solution:
Step 1: Recast as the real part of a complex integral.
We write the integral in complex form by expressing $\cos(Xt^2) = \mathrm{Re}[e^{iXt^2}]$:

$$I(X) = \mathrm{Re}\int_{-1}^{\infty} \sqrt{1+z}\,e^{X(z+iz^2-z^3/3)}\,dz.$$

Define the phase function:

$$\Phi(z) = z + iz^2 - \frac{z^3}{3}.$$

Thus:

$$I(X) = \mathrm{Re}\int_{-1}^{\infty} \sqrt{1+z}\,e^{X\Phi(z)}\,dz.$$

Step 2: Find the saddle points.
The derivative of the phase function is:

$$\Phi'(z) = 1 + 2iz - z^2 = -(z^2 - 2iz - 1) = -(z-i)^2.$$

Setting $\Phi'(z) = 0$:

$$-(z-i)^2 = 0 \implies z = i.$$

This is a saddle point of **order two** (a double root), which means $\Phi'(i) = 0$ and $\Phi''(i) = 0$, but $\Phi'''(i) \neq 0$.
We verify:

$$\Phi''(z) = 2i - 2z \implies \Phi''(i) = 2i - 2i = 0.$$
$$\Phi'''(z) = -2 \implies \Phi'''(i) = -2 \neq 0.$$

Step 3: Evaluate $\Phi(z)$ at the saddle point.
At $z = i$:

$$\Phi(i) = i + i(i)^2 - \frac{(i)^3}{3} = i + i(-1) - \frac{-i}{3} = i - i + \frac{i}{3} = \frac{i}{3}.$$

Step 4: Determine the steepest descent paths through $z = i$.
The real and imaginary parts of $\Phi(z)$ for $z = x + iy$ are:

$$\Phi(x+iy) = \left(-\frac{x^3}{3} + xy^2 - 2xy + x\right) + i\left(\frac{(y-1)^3}{3} - x^2(y-1) + \frac{1}{3}\right).$$

More directly, expanding around the saddle point:

$$\Phi(z) = \Phi(i) + \frac{\Phi'''(i)}{6}(z-i)^3 + O((z-i)^4) = \frac{i}{3} - \frac{1}{3}(z-i)^3 + O((z-i)^4).$$

The steepest descent/ascent contours through $z = i$ are determined by $\mathrm{Im}[\Phi(z)] = \mathrm{Im}[\Phi(i)] = \frac{1}{3}$.
For a saddle point of order two (where the first non-vanishing derivative is the third), the steepest contours consist of **three lines** meeting at equal angles of 120 at the saddle point. These alternate between steepest descent and steepest ascent paths.
From the condition $\mathrm{Im}[\Phi(z)] = \frac{1}{3}$, the steepest contours are:

- $y = 1$ (the horizontal line through $z = i$)
- $y = \sqrt{3}x + 1$ (line at angle 60 from horizontal)
- $y = -\sqrt{3}x + 1$ (line at angle -60 from horizontal, i.e., 120)

Examining the real part $\text{Re}[\Phi(z)]$ along these curves determines which are descent and which are ascent:

- Along $y = 1$, $x > 0$: This is a steepest **descent** curve.
- Along $y = \sqrt{3}x + 1$, $x < 0$: This is also a steepest **descent** curve.
- The remaining directions are steepest ascent curves.

Step 5: Deform the integration contour.

The original contour is the real axis from $z = -1$ to $z = +\infty$. We need to deform this contour to pass through the saddle point at $z = i$ along steepest descent paths.

The deformation proceeds as follows:

1. Start at $z = -1$ and follow the steepest descent path determined by $\text{Im}[\Phi(z)] = \text{Im}[\Phi(-1)] = 1$ into the third quadrant until it approaches the asymptote $y = \sqrt{3}x + 1$.
2. This asymptote is itself a steepest descent path of the saddle point $z = i$, so we follow it to reach $z = i$.
3. From $z = i$, we follow the steepest descent path along $y = 1$, $x > 0$ (i.e., the line $\text{Im}(z) = 1$) to infinity.

The contributions from paths far from the saddle point are subdominant. The leading asymptotic contribution comes from the vicinity of the saddle point $z = i$.

Step 6: Evaluate the contribution from the saddle point.

For an order-two saddle point (where $\Phi'''(z_0) \neq 0$ is the first non-vanishing derivative), the local behavior is:

$$\Phi(z) \approx \Phi(i) + \frac{\Phi'''(i)}{6}(z - i)^3 = \frac{i}{3} - \frac{1}{3}(z - i)^3.$$

Contribution from the path $y = 1$, $x > 0$:

Along this path, set $z = i + s$ where $s \in [0, \infty)$ is real. Then:

$$\Phi(z) \approx \frac{i}{3} - \frac{s^3}{3}.$$

The amplitude function at the saddle is:

$$\sqrt{1 + i} = (1 + i)^{1/2} = 2^{1/4} e^{i\pi/8}.$$

The integral contribution from this path is:

$$I_1(X) \sim \text{Re}\left[\sqrt{1 + i} \int_0^\infty e^{X(i/3 - s^3/3)}\, ds\right] = \text{Re}\left[2^{1/4} e^{i\pi/8} e^{iX/3} \int_0^\infty e^{-Xs^3/3}\, ds\right].$$

Using the substitution $t = Xs^3/3$, so $s = (3t/X)^{1/3}$ and $ds = \frac{1}{3}(3/X)^{1/3} t^{-2/3}\, dt$:

$$\int_0^\infty e^{-Xs^3/3}\, ds = \frac{1}{3}\left(\frac{3}{X}\right)^{1/3} \int_0^\infty t^{-2/3} e^{-t}\, dt = \frac{3^{1/3}}{3X^{1/3}} \Gamma\left(\frac{1}{3}\right) = \frac{\Gamma(1/3)}{3^{2/3} X^{1/3}}.$$

Thus:

$$I_1(X) \sim \mathrm{Re}\left[\frac{2^{1/4}\Gamma(1/3)}{3^{2/3}X^{1/3}} e^{i(X/3+\pi/8)}\right].$$

Contribution from the path $y = \sqrt{3}x + 1$, $x < 0$:
Along this path, set $z = i + e^{-2\pi i/3}s$ where $s \in [0, \infty)$ is real (note the direction is into the third quadrant). Then:

$$(z - i)^3 = e^{-2\pi i}s^3 = s^3.$$

So:

$$\Phi(z) \approx \frac{i}{3} - \frac{s^3}{3}.$$

The integral contribution is:

$$I_2(X) \sim \mathrm{Re}\left[\sqrt{1+i}\int_\infty^0 e^{X(i/3-s^3/3)}e^{-2\pi i/3}\,ds\right] = -\mathrm{Re}\left[2^{1/4}e^{i\pi/8}e^{-2\pi i/3}e^{iX/3}\int_0^\infty e^{-Xs^3/3}\,ds\right].$$

Thus:

$$I_2(X) \sim -\mathrm{Re}\left[\frac{2^{1/4}\Gamma(1/3)}{3^{2/3}X^{1/3}} e^{i(X/3+\pi/8-2\pi/3)}\right].$$

Step 7: Combine the contributions.
The total asymptotic contribution is:

$$I(X) = I_1(X) + I_2(X) \sim \frac{2^{1/4}\Gamma(1/3)}{3^{2/3}X^{1/3}}\mathrm{Re}\left[e^{i(X/3+\pi/8)} - e^{i(X/3+\pi/8-2\pi/3)}\right].$$

Let $\theta = X/3 + \pi/8$. Then:

$$e^{i\theta} - e^{i(\theta-2\pi/3)} = e^{i\theta}\left(1 - e^{-2\pi i/3}\right).$$

Now:

$$1 - e^{-2\pi i/3} = 1 - \left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right) = \frac{3}{2} + \frac{\sqrt{3}}{2}i = \sqrt{3}\left(\frac{\sqrt{3}}{2} + \frac{1}{2}i\right) = \sqrt{3}\,e^{i\pi/6}.$$

Therefore:

$$e^{i\theta} - e^{i(\theta-2\pi/3)} = \sqrt{3}\,e^{i(\theta+\pi/6)} = \sqrt{3}\,e^{i(X/3+\pi/8+\pi/6)}.$$

Simplifying the phase:

$$\frac{\pi}{8} + \frac{\pi}{6} = \frac{3\pi + 4\pi}{24} = \frac{7\pi}{24}.$$

So:

$$I(X) \sim \frac{2^{1/4}\Gamma(1/3)}{3^{2/3}X^{1/3}} \cdot \sqrt{3} \cdot \mathrm{Re}\left[e^{i(X/3+7\pi/24)}\right] = \frac{2^{1/4}\sqrt{3}\,\Gamma(1/3)}{3^{2/3}X^{1/3}}\cos\left(\frac{X}{3} + \frac{7\pi}{24}\right).$$

Step 8: Simplify the coefficient.

$$\frac{\sqrt{3}}{3^{2/3}} = \frac{3^{1/2}}{3^{2/3}} = 3^{1/2-2/3} = 3^{-1/6} = \frac{1}{3^{1/6}}.$$

Therefore, the coefficient becomes:

$$\frac{2^{1/4}\Gamma(1/3)}{3^{1/6}X^{1/3}}.$$

Step 9: Final result.

$$\boxed{I(X) \sim \frac{2^{1/4}\Gamma(1/3)}{3^{1/6}X^{1/3}} \cos\left(\frac{X}{3} + \frac{7\pi}{24}\right) \quad \text{as } X \to \infty.}$$

Remarks:

1. **Order of the saddle point:** The saddle point at $z = i$ is of order two because $\Phi'(i) = \Phi''(i) = 0$ but $\Phi'''(i) = -2 \neq 0$. This is sometimes called a "monkey saddle" or a degenerate saddle point.

2. **Asymptotic order:** For a standard (order-one) saddle point, the asymptotic expansion gives $O(X^{-1/2})$ behavior. For an order-two saddle point, the expansion instead gives $O(X^{-1/3})$ behavior, which decays more slowly. This is reflected in the appearance of $X^{1/3}$ in the denominator and $\Gamma(1/3)$ from the integral $\int_0^\infty t^{-2/3}e^{-t}\,dt$.

3. **Phase of the oscillation:** The argument of the cosine is $X/3 + 7\pi/24$, not X as one might naively expect. The factor of $1/3$ comes from the value $\text{Im}[\Phi(i)] = 1/3$, and the constant phase shift $7\pi/24$ arises from combining the phase of $\sqrt{1+i} = 2^{1/4}e^{i\pi/8}$ with the geometric factor $e^{i\pi/6}$ from the steepest descent analysis.

4. **Why the naive approach fails:** Treating this as a standard Laplace integral with a maximum at $t = 1$ (or any other point on the real axis) would give incorrect results because the $\cos(Xt^2)$ term creates rapid oscillations that fundamentally change the character of the integral. The proper treatment requires the method of steepest descent in the complex plane.

5. **Numerical value:** For reference, $\Gamma(1/3) \approx 2.6789$, $2^{1/4} \approx 1.1892$, and $3^{1/6} \approx 1.2009$, so the coefficient $\frac{2^{1/4}\Gamma(1/3)}{3^{1/6}} \approx 2.653$.