Asymptotics Problem 8.4: Complete Pedagogical Solution

Boundary Layer with Non-Standard Width

Problem 1. Find a first-order uniform expansion as $\varepsilon \to 0$ for $y(x)$ satisfying

$$\varepsilon y'' + x^2 y' - x^3 y = 0, \quad y(0) = \alpha, \quad y(1) = \beta.$$

Solution: Step-by-Step Atomic Breakdown

Step 1: Problem Classification and Initial Analysis

Strategy: *This is a singularly perturbed second-order linear ODE of the general form*

$$\varepsilon y'' + p(x)y' + q(x)y = 0$$

with $p(x) = x^2$ and $q(x) = -x^3$. Our systematic approach follows the workflow from Lecture Notes §6.2.3:

1. Identify candidate locations for boundary layers

2. Determine the boundary layer width $\delta(\varepsilon)$ via dominant balance

3. Compute outer and inner solutions

4. Match solutions using Prandtl's matching criterion

5. Construct the composite solution

Justification: *Why is this a boundary layer problem? The coefficient ε multiplying the highest derivative y'' is small. As $\varepsilon \to 0$, the order of the ODE effectively reduces from 2 to 1, but we have* two *boundary conditions. This is the hallmark of a singular perturbation problem where rapid variation (a boundary layer) must occur somewhere to accommodate both boundary conditions.*

Step 2: Identifying the Boundary Layer Location

What we examine: The sign and zeros of the coefficient $p(x) = x^2$.
Key Concept: *From the general theory of boundary layers (Lecture Notes §6.2.1, equations (340)–(354)), for the ODE $\varepsilon y'' + p(x)y' + q(x)y = 0$:*

- *If $p(x) > 0$ on $[0, 1]$: boundary layer at $x = 0$*
- *If $p(x) < 0$ on $[0, 1]$: boundary layer at $x = 1$*
- *If $p(x_0) = 0$ for some $x_0 \in [0, 1]$: special treatment required*

The physical intuition: the sign of $p(x)$ determines whether information "propagates" from left to right or right to left.
Analysis of $p(x) = x^2$:

- For $x > 0$: $p(x) = x^2 > 0$ ✓

- At $x = 0$: $p(0) = 0$ (vanishes at boundary!)
- At $x = 1$: $p(1) = 1 > 0$ ✓

Justification: *Since $p(x) = x^2 > 0$ for all $x > 0$, there* cannot *be a boundary layer in the interior or at $x = 1$. The only possible location is $x = 0$.*

However, this case is special because $p(0) = 0$—the coefficient of y' vanishes exactly at the boundary point where we expect the boundary layer! According to Lecture Notes §6.2.2 (equation (356) and surrounding discussion): "For a boundary point x_0 with $p(x_0) = 0$, we can have boundary layers of different width than $\sim \varepsilon$."

This is precisely our situation. We must determine the actual boundary layer width through careful dominant balance analysis.

$$\boxed{\textbf{Conclusion: } \text{Boundary layer at } x = 0 \text{ with non-standard width } \delta \neq O(\varepsilon).}$$

Step 3: Computing the Outer Solution

What we do: In the outer region (away from $x = 0$), the solution varies slowly, so we neglect the $\varepsilon y''$ term.

Technique: *Setting $\varepsilon = 0$ in the original ODE gives the leading-order outer equation:*

$$x^2 y_0' - x^3 y_0 = 0.$$

Solving the outer equation:

Divide by x^2 (valid for $x \neq 0$, which is the outer region):

$$y_0' - x y_0 = 0.$$

Technique: *This is a first-order linear homogeneous ODE. Separate variables:*

$$\frac{y_0'}{y_0} = x \quad \Longrightarrow \quad \frac{dy_0}{y_0} = x\,dx.$$

Integrate both sides:

$$\ln|y_0| = \frac{x^2}{2} + C'.$$

Exponentiate:

$$y_0(x) = C \exp\left(\frac{x^2}{2}\right),$$

where C is an arbitrary constant.

Applying the boundary condition at $x = 1$:

Justification: *Since the boundary layer is at $x = 0$, the outer solution is valid at $x = 1$. We can therefore apply the boundary condition $y(1) = \beta$ directly to the outer solution:*

$$y_0(1) = C \exp\left(\frac{1}{2}\right) = \beta \quad \Longrightarrow \quad C = \beta e^{-1/2} = \frac{\beta}{\sqrt{e}}.$$

Final outer solution:

$$\boxed{y_0(x) = \beta \exp\left(\frac{x^2 - 1}{2}\right)}$$

Reflection: *Notice that $y_0(0) = \beta e^{-1/2} = \beta/\sqrt{e} \neq \alpha$ in general. This confirms that the outer solution cannot satisfy both boundary conditions—there must be a boundary layer at $x = 0$ to transition from $y(0) = \alpha$ to the outer solution.*

Step 4: Setting Up the Inner Region

What we do: Introduce stretched (inner) coordinates to resolve the rapid variation near $x = 0$.
Technique: *Define the inner variable X by:*

$$x = \delta X, \quad \textit{where } \delta = \delta(\varepsilon) \to 0 \textit{ as } \varepsilon \to 0.$$

Define the inner solution $Y(X) = y(x) = y(\delta X)$.
Justification: *The scaling $x = \delta X$ "zooms in" on the boundary layer region near $x = 0$. When $x = O(\delta)$, we have $X = O(1)$, so the inner variable X is order one within the boundary layer. The function $\delta(\varepsilon)$ represents the boundary layer* width*—the region where rapid variation occurs.*
Transforming derivatives:
Using the chain rule:

$$\frac{dy}{dx} = \frac{dY}{dX} \cdot \frac{dX}{dx} = \frac{1}{\delta} Y'(X),$$
$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{1}{\delta}Y'\right) = \frac{1}{\delta} \cdot \frac{dY'}{dX} \cdot \frac{dX}{dx} = \frac{1}{\delta^2} Y''(X).$$

Substituting into the original ODE:
The original equation is:

$$\varepsilon y'' + x^2 y' - x^3 y = 0.$$

With $x = \delta X$ and $y = Y$:

$$\varepsilon \cdot \frac{1}{\delta^2} Y'' + (\delta X)^2 \cdot \frac{1}{\delta} Y' - (\delta X)^3 Y = 0$$
$$\frac{\varepsilon}{\delta^2} Y'' + \frac{\delta^2 X^2}{\delta} Y' - \delta^3 X^3 Y = 0$$
$$\frac{\varepsilon}{\delta^2} Y'' + \delta X^2 Y' - \delta^3 X^3 Y = 0.$$

$$\boxed{\text{Inner equation: } \frac{\varepsilon}{\delta^2} Y'' + \delta X^2 Y' - \delta^3 X^3 Y = 0}$$

Step 5: Dominant Balance Analysis

Strategy: *To find the boundary layer width δ, we require that the leading terms in the inner equation balance each other as $\varepsilon \to 0$. This is the* dominant balance *principle (Lecture Notes §2.2.2 and §6.2.2).*
Examining the three terms:
The inner equation has three terms with coefficients:

1. Y'' term: coefficient ε/δ^2
2. Y' term: coefficient δ
3. Y term: coefficient δ^3

Justification: *We seek $\delta \to 0$ as $\varepsilon \to 0$. Compare the magnitudes:*

- *The Y term has coefficient δ^3.*
- *The Y' term has coefficient δ.*
- *Since $\delta \to 0$, we have $\delta^3 \ll \delta$, so the Y term is* smaller *than the Y' term.*

Therefore, the third term ($-\delta^3 X^3 Y$) is subdominant and can be neglected at leading order.
Balancing the remaining two terms:
Technique: *For a distinguished limit (where both remaining terms contribute at the same order), we require:*

$$\frac{\varepsilon}{\delta^2} \sim \delta.$$

Solving for δ:

$$\varepsilon \sim \delta^3 \quad \Longrightarrow \quad \boxed{\delta = \varepsilon^{1/3}}$$

Reflection: *This is a* ***non-standard boundary layer width!*** *In the typical case where $p(x_0) \neq 0$ at the boundary, we get $\delta = \varepsilon$. Here, because $p(0) = 0$, we obtain $\delta = \varepsilon^{1/3}$, which is* larger *than ε (since $\varepsilon^{1/3} \gg \varepsilon$ for small ε).*

This confirms the statement from Lecture Notes §6.2.2: when the coefficient of y' vanishes at the boundary, "we can have boundary layers of different width than $\sim \varepsilon$, here $\sim \sqrt{\varepsilon}$" (in some cases) or $\sim \varepsilon^{1/3}$ (in our case).
Key Concept: *Verification of dominant balance: With $\delta = \varepsilon^{1/3}$:*

- $\varepsilon/\delta^2 = \varepsilon/\varepsilon^{2/3} = \varepsilon^{1/3} = \delta$ ✓ *(these balance)*
- $\delta^3 = \varepsilon \ll \delta = \varepsilon^{1/3}$ ✓ *(third term is subdominant)*

The analysis is self-consistent.

Step 6: Deriving the Leading-Order Inner Equation

What we do: With $\delta = \varepsilon^{1/3}$, write the inner equation and identify the leading-order problem.
Substituting $\delta = \varepsilon^{1/3}$:

$$\varepsilon^{1/3} Y'' + \varepsilon^{1/3} X^2 Y' - \varepsilon X^3 Y = 0.$$

Divide through by $\varepsilon^{1/3}$:

$$Y'' + X^2 Y' - \varepsilon^{2/3} X^3 Y = 0.$$

Technique: *As $\varepsilon \to 0$, the term $\varepsilon^{2/3} X^3 Y \to 0$. At leading order:*

$$\boxed{Y_0'' + X^2 Y_0' = 0}$$

This is the leading-order inner equation.
Justification: *Why can we neglect $\varepsilon^{2/3} X^3 Y$? In the inner region, $X = O(1)$, so $X^3 = O(1)$. The coefficient $\varepsilon^{2/3} \to 0$, making this term small compared to the $O(1)$ terms Y'' and $X^2 Y'$.*

Step 7: Solving the Inner Equation

What we solve: $Y_0'' + X^2 Y_0' = 0$.

Step 7a: First Integration

Technique: *This is a second-order ODE that can be reduced to first order by the substitution $P = Y_0'$:*

$$P' + X^2 P = 0.$$

This is a separable first-order linear ODE.
Separate variables:

$$\frac{dP}{P} = -X^2\, dX.$$

Integrate:

$$\ln |P| = -\frac{X^3}{3} + C_1.$$

Exponentiate:

$$P = Y_0' = A \exp\left(-\frac{X^3}{3}\right),$$

where A is an arbitrary constant.

Step 7b: Second Integration

Technique: *Integrate Y_0' to find Y_0:*

$$Y_0(X) = A \int_0^X \exp\left(-\frac{s^3}{3}\right) ds + B,$$

where B is another arbitrary constant, and we choose the lower limit of integration as 0 for convenience.
Justification: *The choice of lower limit 0 is convenient because:*

1. *At $X = 0$, the integral vanishes, giving $Y_0(0) = B$ directly.*
2. *This form makes applying the boundary condition at $x = 0$ (i.e., $X = 0$) straightforward.*

General inner solution:

$$\boxed{Y_0(X) = A \int_0^X \exp\left(-\frac{s^3}{3}\right) ds + B}$$

Step 8: Applying the Boundary Condition at $x = 0$

What we apply: $y(0) = \alpha$, which in inner variables is $Y_0(0) = \alpha$.
From the inner solution:

$$Y_0(0) = A \int_0^0 \exp\left(-\frac{s^3}{3}\right) ds + B = 0 + B = B.$$

Therefore:

$$\boxed{B = \alpha}$$

Inner solution with boundary condition:

$$Y_0(X) = A \int_0^X \exp\left(-\frac{s^3}{3}\right) ds + \alpha$$

Reflection: *We have one remaining unknown constant A, which will be determined by matching the inner and outer solutions in the overlap (intermediate) region.*

Step 9: Asymptotic Matching

Strategy: *We use **Prandtl's matching rule** (Lecture Notes §6.1.2): the outer limit of the inner solution must equal the inner limit of the outer solution. This is valid here because both limits approach constants, making the matching straightforward.*

Step 9a: Inner Limit of the Outer Solution

Technique: *The "inner limit" means taking $x \to 0^+$ in the outer solution:*

$$\lim_{x \to 0^+} y_0(x) = \lim_{x \to 0^+} \beta \exp\left(\frac{x^2 - 1}{2}\right) = \beta \exp\left(\frac{0 - 1}{2}\right) = \beta e^{-1/2} = \frac{\beta}{\sqrt{e}}.$$

Step 9b: Outer Limit of the Inner Solution

Technique: *The "outer limit" means taking $X \to +\infty$ in the inner solution (since $X = x/\varepsilon^{1/3} \to \infty$ as we leave the boundary layer):*

$$\lim_{X\to\infty} Y_0(X) = \lim_{X\to\infty}\left[A\int_0^X \exp\left(-\frac{s^3}{3}\right) ds + \alpha\right] = A\cdot I + \alpha,$$

where we define the integral:

$$I = \int_0^\infty \exp\left(-\frac{s^3}{3}\right) ds.$$

Step 9c: Evaluating the Integral I

Technique: *This is a generalized Gaussian integral. Using the substitution $u = s^3/3$, so $s = (3u)^{1/3}$ and $ds = (3u)^{-2/3}du$:*

$$I = \int_0^\infty e^{-u}\cdot(3u)^{-2/3}\, du = 3^{-2/3}\int_0^\infty u^{-2/3}e^{-u}\, du = 3^{-2/3}\Gamma\left(\frac{1}{3}\right),$$

where $\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}dt$ is the gamma function (Lecture Notes §2.6.1).
Justification: *We used the gamma function identity $\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}dt$ with $z = 1/3$:*

$$\int_0^\infty u^{1/3-1}e^{-u}\, du = \int_0^\infty u^{-2/3}e^{-u}\, du = \Gamma\left(\frac{1}{3}\right).$$

The numerical value is $\Gamma(1/3) \approx 2.679$.
Result:

$$I = \int_0^\infty \exp\left(-\frac{s^3}{3}\right) ds = 3^{-2/3}\Gamma\left(\frac{1}{3}\right)$$

Step 9d: Applying Prandtl's Matching Condition

Technique: *Equate the inner limit of outer and outer limit of inner:*

$$\frac{\beta}{\sqrt{e}} = \alpha + A\cdot I.$$

Solve for A:

$$A = \frac{\beta e^{-1/2} - \alpha}{I} = \frac{\beta/\sqrt{e} - \alpha}{I}.$$

Matched constant:

$$\boxed{A = \frac{\beta e^{-1/2} - \alpha}{I} = \frac{\beta e^{-1/2} - \alpha}{3^{-2/3}\Gamma(1/3)}}$$

Reflection: *The constant A encodes the "mismatch" between the boundary value α and the limiting value $\beta e^{-1/2}$ of the outer solution as $x \to 0$. If $\alpha = \beta e^{-1/2}$, then $A = 0$ and the inner solution reduces to the constant α—no boundary layer correction is needed!*

Step 10: Constructing the Composite Solution

Strategy: *The composite solution is constructed by adding the outer and inner solutions and subtracting their common limit (to avoid double-counting in the overlap region). This follows Lecture Notes §6.1.3 and equation (353):*

$$y_c(x) = y_{outer}(x) + Y_{inner}\left(\frac{x}{\delta}\right) - (\textit{common limit}).$$

Components:

$$\text{Outer solution:} \quad y_0(x) = \beta \exp\left(\frac{x^2-1}{2}\right)$$

$$\text{Inner solution:} \quad Y_0(X) = \alpha + A\int_0^X \exp\left(-\frac{s^3}{3}\right) ds$$

$$\text{Common limit:} \quad \frac{\beta}{\sqrt{e}} = \beta e^{-1/2}$$

Technique: *The common limit is what both solutions approach in the intermediate (matching) region:*

- *As $x \to 0^+$: $y_0(x) \to \beta e^{-1/2}$*
- *As $X \to \infty$: $Y_0(X) \to \alpha + AI = \beta e^{-1/2}$ (by matching)*

Composite solution formula:

$$y_c(x) = y_0(x) + Y_0\left(\frac{x}{\varepsilon^{1/3}}\right) - \frac{\beta}{\sqrt{e}}$$

Substituting:

$$y_c(x) = \beta \exp\left(\frac{x^2-1}{2}\right) + \alpha + A\int_0^{x/\varepsilon^{1/3}} \exp\left(-\frac{s^3}{3}\right) ds - \frac{\beta}{\sqrt{e}}$$

Simplifying:
Technique: *Using $A = (\beta e^{-1/2} - \alpha)/I$ and the fact that $\int_0^\infty e^{-s^3/3} ds = I$:*

$$A\int_0^X e^{-s^3/3} ds = \frac{\beta e^{-1/2} - \alpha}{I}\int_0^X e^{-s^3/3} ds = \left(\frac{\beta}{\sqrt{e}} - \alpha\right)\frac{\int_0^X e^{-s^3/3} ds}{\int_0^\infty e^{-s^3/3} ds}.$$

Therefore, the composite solution can be written as:

$$y_c(x) = \beta \exp\left(\frac{x^2-1}{2}\right) + \left(\frac{\beta}{\sqrt{e}} - \alpha\right)\left[\frac{\int_0^{x\varepsilon^{-1/3}} e^{-s^3/3} ds}{\int_0^\infty e^{-s^3/3} ds} - 1\right]$$

Step 11: Final Answer in Standard Form

First-Order Uniform Expansion:

$$y_c(x) = \beta \exp\left(\frac{x^2-1}{2}\right) + \left(\frac{\beta}{\sqrt{e}} - \alpha\right)\left[\frac{\displaystyle\int_0^{x\varepsilon^{-1/3}} e^{-s^3/3}\, ds}{\displaystyle\int_0^\infty e^{-s^3/3}\, ds} - 1\right]$$

Equivalently, with explicit integral notation:

$$y_c(x) = \beta \exp\left(\frac{x^2-1}{2}\right) + \left(\frac{\beta}{\sqrt{e}} - \alpha\right)\left[\frac{\displaystyle\int_0^{x/\varepsilon^{1/3}} \exp\left(-\frac{s^3}{3}\right) ds}{\displaystyle\int_0^\infty \exp\left(-\frac{s^3}{3}\right) ds} - 1\right]$$

where $\displaystyle\int_0^\infty \exp\left(-\frac{s^3}{3}\right) ds = 3^{-2/3}\Gamma\left(\frac{1}{3}\right)$.

Step 12: Verification of the Solution

Verification 1: Boundary Condition at $x = 0$

Technique: *As $x \to 0^+$:*

- $\beta \exp\left(\frac{x^2-1}{2}\right) \to \beta e^{-1/2}$
- $x\varepsilon^{-1/3} \to 0$*, so* $\int_0^{x\varepsilon^{-1/3}} e^{-s^3/3} ds \to 0$

Therefore:

$$y_c(0) = \frac{\beta}{\sqrt{e}} + \left(\frac{\beta}{\sqrt{e}} - \alpha\right)(0-1) = \frac{\beta}{\sqrt{e}} - \frac{\beta}{\sqrt{e}} + \alpha = \alpha \quad \checkmark$$

Verification 2: Boundary Condition at $x = 1$

Technique: *At $x = 1$, the inner variable $X = 1/\varepsilon^{1/3} \to \infty$ as $\varepsilon \to 0$:*

- $\beta \exp\left(\frac{1-1}{2}\right) = \beta$
- $\int_0^{\varepsilon^{-1/3}} e^{-s^3/3} ds \to \int_0^\infty e^{-s^3/3} ds = I$

Therefore:

$$y_c(1) = \beta + \left(\frac{\beta}{\sqrt{e}} - \alpha\right)\left(\frac{I}{I} - 1\right) = \beta + \left(\frac{\beta}{\sqrt{e}} - \alpha\right)(0) = \beta \quad \checkmark$$

Verification 3: Behavior in Outer Region

Technique: *For fixed $x > 0$ as $\varepsilon \to 0$:*

$$x\varepsilon^{-1/3} \to \infty, \quad \text{so} \quad \frac{\int_0^{x\varepsilon^{-1/3}} e^{-s^3/3} ds}{I} \to 1.$$

Therefore:

$$y_c(x) \to \beta \exp\left(\frac{x^2-1}{2}\right) + \left(\frac{\beta}{\sqrt{e}} - \alpha\right)(1-1) = \beta \exp\left(\frac{x^2-1}{2}\right) = y_0(x) \quad \checkmark$$

Step 13: Physical Interpretation and Key Insights

Reflection: *This problem illustrates several important asymptotic concepts:*

1. ***Non-standard boundary layer width:*** *When the coefficient of y' vanishes at the boundary ($p(0) = x^2|_{x=0} = 0$), the boundary layer width is determined by dominant balance as $\delta = \varepsilon^{1/3}$, not $\delta = \varepsilon$. This is because near $x = 0$, the y' term "turns on" gradually (like x^2), requiring a wider region for the solution to transition.*

2. ***The integral $\int_0^\infty e^{-s^3/3} ds$:*** *This generalized Gaussian integral appears because the inner equation $Y'' + X^2 Y' = 0$ has solutions involving $e^{-X^3/3}$. The exponent $X^3/3$ comes from integrating the coefficient X^2.*

3. ***Matching via Prandtl's rule:*** *Both the inner and outer solutions approach constants as they enter the intermediate region, making Prandtl's matching (comparing limits) straightforward. Van Dyke matching would give the same result here.*

4. ***Composite solution structure:*** *The additive form $y_c = y_{outer} + Y_{inner} - (\text{common limit})$ ensures that:*

   - *In the inner region: $y_{outer} \approx$ common limit, so $y_c \approx Y_{inner}$*
   - *In the outer region: $Y_{inner} \approx$ common limit, so $y_c \approx y_{outer}$*

   This is the essence of matched asymptotic expansions from Lecture Notes §6.1.3.

Summary Table

Quantity	**Value**
Boundary layer location	$x = 0$
Boundary layer width	$\delta = \varepsilon^{1/3}$
Inner variable	$X = x/\varepsilon^{1/3}$
Outer solution	$y_0(x) = \beta \exp\left(\dfrac{x^2 - 1}{2}\right)$
Inner equation	$Y_0'' + X^2 Y_0' = 0$
Inner solution	$Y_0(X) = \alpha + A \int_0^X e^{-s^3/3}\, ds$
Matching constant	$A = \dfrac{\beta e^{-1/2} - \alpha}{3^{-2/3}\Gamma(1/3)}$
Common limit	$\beta e^{-1/2} = \dfrac{\beta}{\sqrt{e}}$

Connection to Lecture Material

Key Concept: *This problem is a direct application of the boundary layer workflow from Lecture Notes §6.2.3, with the special feature that $p(x_0) = 0$ at the boundary (case discussed in §6.2.2, equation (356)). The key steps parallel Example 1 on page 61–62 of the lecture notes, which treats $\varepsilon y'' + x^2 y' - y = 0$—a closely related problem.*

The dominant balance analysis follows §2.2.2, the matching procedure follows §6.1.2 (Prandtl's rule), and the composite solution construction follows §6.1.3 (equation (353)).