Regular perturbations of ODEs and eigenvalue problems

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1. Look for a solution of the form $f(t,\epsilon) = f_0(t) + \epsilon f_1(t) + \ldots$
   After insertion into differential equation and initial condition, one obtains

   ODE: $[f_0' + \epsilon f_1' + \ldots] - [f_0 + \epsilon f_1 + \ldots] = \epsilon[f_0 + \epsilon f_1 + \ldots]^2 \, e^{-t}$
   IC: $[f_0(0) + \epsilon f_1(0) + \ldots] = 1$

   Equating powers of ϵ in each equation leads to a differential equation with initial condition at each order of ϵ:

   $$\begin{array}{lllll} \mathcal{O}(\epsilon^0): & f_0' - f_0 = 0\,, & f_0(0) = 1 & \Longrightarrow & f_0(t) = e^t \\ \mathcal{O}(\epsilon^1): & f_1' - f_1 = f_0^2\, e^{-t} = e^t\,, & f_1(0) = 0 & \Longrightarrow & f_1(t) = t\, e^t \end{array}$$

   Altogether one has
   $$f(t,\epsilon) = e^t + \epsilon\, t\, e^t + \mathcal{O}(\epsilon^2)\,.$$

   (Remark: The exact solution can be obtained by setting $f(t) = g(t)\, e^t$ and solving the resulting ODE for $g(t)$ by separation of variables. The result is $f(t) = e^t/(1-\epsilon t) = e^t \sum_{n=0}^{\infty} (\epsilon t)^n$.)

2. Again we insert $f(t,\epsilon) = f_0(t) + \epsilon f_1(t) + \ldots$ into the differential equation and the initial conditions

   ODE: $[f_0'' + \epsilon f_1'' + \ldots] + [f_0 + \epsilon f_1 + \ldots] = \epsilon[f_0' + \epsilon f_1' + \ldots]$
   IC 1: $[f_0(0) + \epsilon f_1(0) + \ldots] = 1$
   IC 2: $[f_0'(0) + \epsilon f_1'(0) + \ldots] = \frac{\epsilon}{2}$

   We equate powers of ϵ in each equation to obtain an ODE problem at each order of ϵ:

   $$\begin{array}{llllll} \mathcal{O}(\epsilon^0): & f_0'' + f_0 = 0\,, & f_0(0) = 1\,, & f_0'(0) = 0 & \Longrightarrow & f_0(t) = \cos(t) \\ \mathcal{O}(\epsilon^1): & f_1'' + f_1 = f_0' = -\sin(t)\,, & f_1(0) = 0\,, & f_1'(0) = \dfrac{1}{2} & \Longrightarrow & f_1(t) = \dfrac{t}{2}\cos(t) \end{array}$$

   The total result is
   $$f(t,\epsilon) = \cos(t) + \frac{\epsilon t}{2}\cos(t) + \mathcal{O}(\epsilon^2)$$

   Exact solution: We put $f(t) = Ae^{mt}$. After insertion into the differential equation we obtain a quadratic equation for m with solutions $m_\pm = \frac{\epsilon}{2} \pm i\sqrt{1 - \frac{\epsilon^2}{4}}$, so the general solution is $f(t) = Ae^{m_+ t} + Be^{m_- t}$. The one which satisfies the initial conditions is

   $$f(t,\epsilon) = \cos\left(t\sqrt{1 - \frac{\epsilon^2}{4}}\right)\, \exp\left(\frac{\epsilon t}{2}\right) = \cos(t) + \frac{\epsilon t}{2}\cos(t) + \mathcal{O}(\epsilon^2)\,.$$

3. (a) Insert $x(t,\epsilon) = x_0(t) + \epsilon x_1(t) + \epsilon^2 x_2(t) + \ldots$ into differential equation and initial conditions, and expand the RHS of the differential equation up to order ϵ^2

ODE: $[\ddot{x}_0 + \epsilon\ddot{x}_1 + \epsilon\ddot{x}_2 + \ldots] = \frac{-1}{(1+\epsilon[x_0+\epsilon x_1+\epsilon^2 x_2+\ldots])^2} = -1 + 2\epsilon x_0 + \epsilon^2(2x_1 - 3x_0^2) + \ldots$

IC 1: $[x_0(0) + \epsilon x_1(0) + \epsilon^2 x_2(0) + \ldots] = 0$

IC 2: $[\dot{x}_0(0) + \epsilon\dot{x}_1(0) + \epsilon^2\dot{x}_2(0) + \ldots] = 1$

By equating powers of ϵ in each equation we obtain a hierarchy of problems

$\mathcal{O}(\epsilon^0)$: $\ddot{x}_0 = -1$, $\quad x_0(0) = 0,\ \dot{x}_0(0) = 1 \implies x_0(t) = t - \frac{1}{2}t^2$

$\mathcal{O}(\epsilon^1)$: $\ddot{x}_1 = 2x_0 = 2t - t^2$, $\quad x_1(0) = 0,\ \dot{x}_1(0) = 0 \implies x_1(t) = \frac{1}{3}t^3 - \frac{1}{12}t^4$

$\mathcal{O}(\epsilon^2)$: $\ddot{x}_2 = 2x_1 - 3x_0^2 = -3t^2 + \frac{11}{3}t^3 - \frac{11}{12}t^4$, $x_2(0) = 0$, $\dot{x}_2(0) = 0$.

The last equation leads to

$$x_2(t) = -\frac{1}{4}t^4 + \frac{11}{60}t^5 - \frac{11}{360}t^6 .$$

The total result is

$$x(t,\epsilon) = t - \frac{1}{2}t^2 + \epsilon\left(\frac{1}{3}t^3 - \frac{1}{12}t^4\right) + \epsilon^2\left(-\frac{1}{4}t^4 + \frac{11}{60}t^5 - \frac{11}{360}t^6\right) + \ldots$$

The projectile reaches its maximal height when $\dot{x}(t_m,\epsilon) = 0$:

$$0 = 1 - t_m + \epsilon\left(t_m^2 - \frac{1}{3}t_m^3\right) + \epsilon^2\left(-t_m^3 + \frac{11}{12}t_m^4 - \frac{11}{60}t_m^5\right) + \ldots \qquad (*)$$

The time to reach the maximal height when $\epsilon = 0$ is $t_m = 1$. We want to determine how this time is changed when $\epsilon \neq 0$. This is a regular perturbation problem. Put

$$t_m = 1 + \epsilon t_1 + \epsilon^2 t_2 + \ldots$$

and insert into $(*)$. Equating powers of ϵ gives $t_1 = \frac{2}{3}$ and $t_2 = \frac{2}{5}$. So

$$t_m = 1 + \frac{2}{3}\epsilon + \frac{2}{5}\epsilon^2 + \mathcal{O}(\epsilon^3) .$$

(b) Insert $x(t,\beta) = x_0(t) + \beta x_1(t) + \ldots$ into differential equation and initial conditions

ODE: $[\ddot{x}_0 + \beta\ddot{x}_1 + \ldots] + \beta[\dot{x}_0 + \beta\dot{x}_1 + \ldots] = -1$

IC 1: $[x_0(0) + \beta x_1(0) + \ldots] = 0$

IC 2: $[\dot{x}_0(0) + \beta\dot{x}_1(0) + \ldots] = 1$

We obtain again a hierarchy of problems

$\mathcal{O}(\beta^0)$: $\quad \ddot{x}_0 = -1\,, \quad x_0(0) = 0\,,\ \dot{x}_0(0) = 1 \implies x_0(t) = t - \frac{1}{2}t^2$

$\mathcal{O}(\beta^1)$: $\quad \ddot{x}_1 = -\dot{x}_0 = -1 + t\,, \quad x_1(0) = 0\,,\ \dot{x}_1(0) = 0 \implies x_1(t) = -\frac{1}{2}t^2 + \frac{1}{6}t^3$

The total result is

$$x(t,\beta) = t - \frac{1}{2}t^2 + \beta\left(-\frac{1}{2}t^2 + \frac{1}{6}t^3\right) + \ldots$$

The projectile reaches its maximal height at

$$\dot{x} = 0 = 1 - t_m + \beta\left(-t_m + \frac{1}{2}t_m^2\right) + \dots \qquad (**)$$

We look for the regularly perturbed root near $t = 1$ and put $t_m = 1 + \beta t_1 + \dots$. After inserting into $(**)$ and expanding up to order β we obtain $t_1 = -\frac{1}{2}$. So

$$t_m = 1 - \frac{1}{2}\beta + \mathcal{O}(\beta^2)\,.$$

4. Set $y(x,\epsilon) = y_0(x) + \epsilon y_1(x) + \dots$ and $\lambda = \lambda_0 + \epsilon\lambda_1 + \dots$, and insert into the differential equation and the initial conditions

$$\begin{array}{ll}
\text{ODE:} & [y_0'' + \epsilon y_1'' + \dots] + [\lambda_0 + \epsilon\lambda_1 + \dots](1+\epsilon x)[y_0 + \epsilon y_1 + \dots] = 0 \\
\text{IC 1:} & \quad [y_0(0) + \epsilon y_1(0) + \dots] = 0 \\
\text{IC 2:} & \quad [y_0(\pi) + \epsilon y_1(\pi) + \dots] = 0
\end{array}$$

The resulting equations at the first two orders of ϵ are

$$\begin{array}{lll}
\mathcal{O}(\epsilon^0): & y_0'' + \lambda_0 y_0 = 0\,, & y_0(0) = 0\,, \quad y_0(\pi) = 0 \\
\mathcal{O}(\epsilon^1): & y_1'' + \lambda_0 y_1 = -\lambda_1 y_0 - \lambda_0 x y_0\,, & y_1(0) = 0\,, \quad y_1(\pi) = 0
\end{array}$$

The solution of the 0-th order problem is

$$y_0(t) = A\sin(x\sqrt{\lambda_0})\,, \qquad \lambda_0 = n^2 \qquad n = 1, 2, \dots$$

We want to apply the Fredholm alternative theorem to obtain a solvability condition for the equation for y_1. We thus need a solution of the adjoint homogeneous differential equation. Since the differential operator $[\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \lambda_0]$ is self-adjoint (as you can check by integration by parts), the solution is y_0. The Fredholm alternative theorem then states that the following integral has to vanish

$$\int_0^\pi y_0[-\lambda_1 y_0 - \lambda_0 x y_0]\mathrm{d}x = 0\,.$$

This gives a condition for λ_1

$$\lambda_1 = -\frac{\int_0^\pi \lambda_0 x y_0^2 \mathrm{d}x}{\int_0^\pi y_0^2 \mathrm{d}x} = -\frac{\int_0^\pi n^2 x \sin^2(nx)\mathrm{d}x}{\int_0^\pi \sin^2(nx)\mathrm{d}x} = -\frac{\int_0^\pi n^2 x(1-\cos(2nx))\mathrm{d}x}{\int_0^\pi (1-\cos(2nx))\mathrm{d}x} = -\frac{\frac{n^2\pi^2}{2}}{\pi} = -\frac{n^2\pi}{2}\,,$$

and the final answer is

$$\lambda = n^2 - \epsilon\frac{n^2\pi}{2} + \mathcal{O}(\epsilon^2)\,.$$

5. Let
$$A = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} , \qquad B = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} .$$

Then the problem is $(A + \epsilon B)\underline{x} = \lambda \underline{x}$. Let $\underline{x} = \underline{x}_0 + \epsilon \underline{x}_1 + \dots$ and $\lambda = \lambda_0 + \epsilon\lambda_1 + \dots$, and insert into the eigenvalue equation. The $\mathcal{O}(\epsilon^0)$ problem is

$$(A - \lambda_0)\underline{x}_0 = 0 \qquad \Longrightarrow \qquad \lambda_0^\pm = 1 \pm i \,, \qquad \underline{x}_0^\pm = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ \pm i \end{pmatrix}$$

where we have normalized the eigenvectors. Note that the adjoint problem is

$$(A^* - \bar{\lambda}_0)\underline{y}_0 = 0 \qquad \Longrightarrow \qquad \lambda_0^\pm = 1 \pm i \,, \qquad \underline{y}_0^\pm = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ \pm i \end{pmatrix} = \underline{x}_0^\pm$$

The eigenvectors of the adjoint matrix are identical to those of A. (This is because the two eigenvectors of A are orthogonal.)

The problem at order $\mathcal{O}(\epsilon^1)$ is

$$(A - \lambda_0^\pm)\underline{x}_1^\pm = \lambda_1^\pm \underline{x}_0^\pm - B\underline{x}_0^\pm \,. \qquad (*)$$

According to the Fredholm alternative theorem, this equation can be solved for $\underline{x}_1^\pm$ if the scalar product of the RHS with $\underline{y}_0^\pm$ vanishes. This gives a condition from which $\lambda_1^\pm$ can be determined.

$$0 = \langle \lambda_1^\pm \underline{x}_0^\pm - B\underline{x}_0^\pm , \underline{y}_0^\pm \rangle \qquad \Longrightarrow \qquad \lambda_1^\pm = \frac{\langle B\underline{x}_0^\pm , \underline{y}_0^\pm \rangle}{\langle \underline{x}_0^\pm , \underline{y}_0^\pm \rangle} = \mp i \,.$$

(The scalar product is defined as $\langle a, b \rangle = \sum_{j=1}^2 a_j \bar{b}_j$). The total result is $\lambda^\pm = 1 \pm i(1 - \epsilon) + \mathcal{O}(\epsilon^2)$.

In the following we expand $\underline{x}_1^\pm$ in the orthonormal basis of the eigenvectors of the unperturbed problem. To obtain the components in each direction we take the scalar product of equation $(*)$ with each eigenvector.

$$\langle (A - \lambda_0^\pm)\underline{x}_1^\pm , \underline{x}_0^\pm \rangle = \langle \lambda_1^\pm \underline{x}_0^\pm - B\underline{x}_0^\pm , \underline{x}_0^\pm \rangle \qquad \Longrightarrow \qquad 0 = 0 \,,$$

and

$$\langle (A - \lambda_0^\pm)\underline{x}_1^\pm , \underline{x}_0^\mp \rangle = \langle \lambda_1^\pm \underline{x}_0^\pm - B\underline{x}_0^\pm , \underline{x}_0^\mp \rangle \qquad \Longrightarrow \qquad (\lambda_0^\mp - \lambda_0^\pm)\langle \underline{x}_1^\pm , \underline{x}_0^\mp \rangle = 0 \,.$$

The first of these two equations shows that the component of $\underline{x}_1^\pm$ in direction of $\underline{x}_0^\pm$ is not defined (we are allowed to change the length of the eigenvector), and the second equation shows that it has no component in direction of $\underline{x}_0^\mp$. Thus the directions of the eigenvectors do not change.

(Remark: By exactly solving the eigenvalue problem $(A + \epsilon B)\underline{x} = \lambda \underline{x}$ one finds that these first order results already give the exact solution.)