# Methods of Applied Mathematics - Part 1 [SEMT30006]

## Exercise Sheet 4 – Maps

1. Derive the discrete population model $N_{n+1} = N_n\left(1 + \beta - \gamma N_n\right)$ from the solution of the nonlinear ODE population model.

2. Derive a discrete map for the predator-prey system, in a similar way we did for the 1d population model.

3. In lectures we looked at the logistic map

$$x_{n+1} = r x_n (1 - x_n)$$

which has fixed points at

$$x_{*1} = 0 \qquad \& \qquad x_{*2} = (r-1)/r \tag{2}$$

Derive the linearization of the map about each of these fixed points, and hence show that $x_{*1}$ is unstable and $x_{*2}$ is stable for $r > 1$.

4. Newton's Method is a common numerical tool for finding the roots of functions (i.e. $x$ such that $f(x) = 0$). When applying the method to find the roots of a function $f$ we choose an initial value of $x_0$ and then repeatedly apply the mapping

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

where $f' \equiv df/dx$, until the method converges on a root. In fact we are just iterating a map until it reaches a stable fixed point.

   (a) Show that all fixed points of the map are points where $f(x)$ is zero.

   (b) Show that certain roots cannot be found with this method, using the concept of local stability to derive the condition that must be met by a fixed point for it to be reachable by Newton's Method.

5. Solve the map

$$
\begin{aligned}
x_{n+1} &= 2x_n - y_n \\
y_{n+1} &= 2y_n - x_n
\end{aligned}
$$

with initial condition $x_0 = 1$, $y_0 = 0$. You can do this in two different ways:

(a) Simply iterate the map repeatedly (as we did when we solved the population map at the start of the course), until you reach $x_0$ and $y_0$ on the righthand side.

(b) Try to use the same kind of decomposition we used for ODEs, trying a solution

$$
\begin{pmatrix} x_n \\ y_n \end{pmatrix} = \alpha_1 \mathbf{c}_1 {\lambda_1}^n + \alpha_2 \mathbf{c}_2 \lambda_2^n
$$

where $\mathbf{c}_1$, $\mathbf{c}_2$, are the eigenvectors of the 'Jacobian' of the righthand side

$$
\frac{\partial(x_{n+1}, y_{n+1})}{\partial(x_n, y_n)}
$$

and $\lambda_1$, $\lambda_2$, are the eigenvalues of the system's equilibrium, and the terms $\alpha_1$, $\alpha_2$, are constants that need to be found (using the initial condition). Note the decomposition is a bit different to ODEs, as the $\lambda_i$'s appear as powers rather than as $e^{\lambda_i t}$ terms.