Exercise Sheet 3: Bifurcations
Question 2 - Complete Solution

Methods of Applied Mathematics

Problem Statement

Consider the system:

$$\dot{x} = a + 2x + x^2$$

Tasks:

(a) Find the equilibria of the system and their stability

(b) Conjecture the bifurcation that occurs in the system

(c) Evaluate the bifurcation and genericity conditions to prove your conjecture

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1 Step 1: Find Equilibria

Set up equilibrium condition

For equilibria, we require $\dot{x} = 0$:

$$a + 2x + x^2 = 0$$

Rearranging as standard quadratic:

$$x^2 + 2x + a = 0$$

Apply quadratic formula

$$x = \frac{-2 \pm \sqrt{4 - 4a}}{2} = \frac{-2 \pm 2\sqrt{1 - a}}{2} = -1 \pm \sqrt{1 - a}$$

Analyze discriminant

The discriminant is $\Delta = 4(1 - a)$. The number of real equilibria depends on its sign:

$$\begin{aligned}
a < 1 : \quad & \Delta > 0 \quad \Rightarrow \quad \text{Two distinct real equilibria:} \\
& \boxed{x_+^* = -1 + \sqrt{1 - a}} \quad \text{and} \quad \boxed{x_-^* = -1 - \sqrt{1 - a}} \\
a = 1 : \quad & \Delta = 0 \quad \Rightarrow \quad \text{One repeated equilibrium:} \\
& \boxed{x^* = -1} \\
a > 1 : \quad & \Delta < 0 \quad \Rightarrow \quad \text{No real equilibria}
\end{aligned}$$

XYZ Analysis of Equilibrium Structure

- **STAGE X (What we found):** The number of equilibria changes as parameter a varies: two for $a < 1$, one for $a = 1$, zero for $a > 1$.
- **STAGE Y (Why this happens):** The equation $x^2 + 2x + a = 0$ represents the intersection of parabola $y = x^2 + 2x$ (opening upward, vertex at $x = -1, y = -1$) with horizontal line $y = -a$:
  - For $a < 1$: Line $y = -a$ is above $y = -1$ (vertex), intersecting parabola twice
  - For $a = 1$: Line $y = -1$ touches parabola at vertex (tangent)
  - For $a > 1$: Line $y = -a$ is below vertex, missing parabola entirely

  As a increases through 1, two equilibria approach each other along the x-axis, collide at $x = -1$, then disappear into the complex plane.
- **STAGE Z (What this means):** Equilibria are created/destroyed at $a = 1$, suggesting a fold (saddle-node) bifurcation. Unlike transcritical (where equilibria pass through each other), here they annihilate.

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2 Step 2: Determine Stability

Compute derivative

For $f(x) = a + 2x + x^2$:

$$f'(x) = 2 + 2x$$

Evaluate at each equilibrium

At $x_+^* = -1 + \sqrt{1-a}$:

$$\begin{aligned} f'(x_+^*) &= 2 + 2(-1 + \sqrt{1-a}) \\ &= 2 - 2 + 2\sqrt{1-a} \\ &= 2\sqrt{1-a} \end{aligned}$$

For $a < 1$: $\sqrt{1-a} > 0$, so $f'(x_+^*) = 2\sqrt{1-a} > 0$

$$\boxed{\text{UNSTABLE}}$$

At $x_-^* = -1 - \sqrt{1-a}$:

$$\begin{aligned} f'(x_-^*) &= 2 + 2(-1 - \sqrt{1-a}) \\ &= 2 - 2 - 2\sqrt{1-a} \\ &= -2\sqrt{1-a} \end{aligned}$$

For $a < 1$: $\sqrt{1-a} > 0$, so $f'(x_-^*) = -2\sqrt{1-a} < 0$

$$\boxed{\text{STABLE}}$$

At $a = 1$ (critical point $x^* = -1$):

$$f'(-1) = 2 + 2(-1) = 0 \quad \Rightarrow \quad \boxed{\text{NEUTRAL}}$$

Stability summary

Parameter	Equilibrium	Stability
2*$a < 1$	$x_+^* = -1 + \sqrt{1-a}$	Unstable
	$x_-^* = -1 - \sqrt{1-a}$	Stable
$a = 1$	$x^* = -1$	Neutral
$a > 1$	None	—

XYZ Analysis of Stability

- **STAGE X (What we found):** For $a < 1$, the right equilibrium (x_+^*, closer to zero) is unstable, the left equilibrium (x_-^*, more negative) is stable. At $a = 1$, both collapse to a neutral equilibrium.
- **STAGE Y (Why these stabilities):** The derivative $f'(x) = 2 + 2x$ is a linear function of x:
  - $f'(x) > 0$ for $x > -1$ (right of critical point) $\Rightarrow$ unstable
  - $f'(x) < 0$ for $x < -1$ (left of critical point) $\Rightarrow$ stable
  - $f'(x) = 0$ at $x = -1$ (the critical point) $\Rightarrow$ neutral

  Since $x_+^* = -1 + \sqrt{1-a} > -1$ (always to the right), it's unstable. Since $x_-^* = -1 - \sqrt{1-a} < -1$ (always to the left), it's stable. As $a \to 1^-$, both approach $x = -1$ from opposite sides, maintaining their stabilities until they meet.
- **STAGE Z (What this means):** A stable and unstable equilibrium collide at $a = 1$. This is the characteristic signature of a fold/saddle-node bifurcation: one equilibrium attracts, one repels, they meet and mutually annihilate. No stability exchange occurs (contrast with transcritical).

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3 Step 3: Conjecture Bifurcation Type

Observed features

From our analysis:

1. Two equilibria exist for $a < 1$
2. Equilibria collide at $a = 1$, $x = -1$
3. No equilibria exist for $a > 1$
4. One equilibrium is stable, one unstable before collision
5. At collision point, derivative is zero: $f'(-1) = 0$

Conjecture

$$\boxed{\text{FOLD BIFURCATION (also called SADDLE-NODE BIFURCATION)}}$$

occurs at $(a, x) = (1, -1)$.

XYZ Analysis of Bifurcation Conjecture

- **STAGE X (What we observe):** Equilibria are created/destroyed (not exchanged or multiplied). Number changes from 2 to 0 across the bifurcation.
- **STAGE Y (Why fold specifically):** The distinguishing features match fold bifurcation:
  - **Creation/annihilation**: Equilibria appear from nowhere (for decreasing a) or disappear into complex plane (for increasing a)
  - **Pairing**: A stable-unstable pair collides (in higher dimensions, typically a saddle and node, hence "saddle-node")
  - **Tangency**: At bifurcation, $f' = 0$ and the equilibrium is tangent to the a-axis in (a, x) space
  - **Generic**: No special symmetry required (unlike pitchfork) and no pinned equilibrium (unlike transcritical)

  This is *not* transcritical because equilibria don't pass through each other - they vanish. It's *not* pitchfork because only two equilibria are involved (not one splitting into three) and there's no symmetry $f(x) = f(-x)$.
- **STAGE Z (What this means):** The fold is the most common codimension-1 bifurcation in generic systems. It represents a threshold phenomenon: for $a > 1$, the system has no steady states and diverges; for $a < 1$, steady behavior is possible. The critical value $a = 1$ is the "tipping point."

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4 Step 4: Verify Bifurcation Conditions

Fold bifurcation requirements

From lecture notes (Section 13, pages 46-47), a fold bifurcation at (a^*, x^*) requires:

Bifurcation conditions:

(B1) $f(x^*, a^*) = 0 \quad$ (equilibrium exists)

(B2) $\left.\frac{\partial f}{\partial x}\right|_{(x^*,a^*)} = 0 \quad$ (zero eigenvalue)

Genericity conditions:

(G1) $\left.\frac{\partial^2 f}{\partial x^2}\right|_{(x^*,a^*)} \neq 0 \quad$ (second derivative nonzero)

(G2) $\left.\frac{\partial f}{\partial a}\right|_{(x^*,a^*)} \neq 0 \quad$ (positive speed of f in a)

Check conditions at $(a, x) = (1, -1)$

For $f(x, a) = a + 2x + x^2$:

(B1) Equilibrium condition:

$$f(-1, 1) = 1 + 2(-1) + (-1)^2 = 1 - 2 + 1 = 0 \quad \checkmark$$

(B2) Zero eigenvalue:

$$\frac{\partial f}{\partial x} = 2 + 2x \quad \Rightarrow \quad \left.\frac{\partial f}{\partial x}\right|_{x=-1} = 2 + 2(-1) = 0 \quad \checkmark$$

(G1) Second derivative nonzero:

$$\frac{\partial^2 f}{\partial x^2} = 2 \quad \Rightarrow \quad \left.\frac{\partial^2 f}{\partial x^2}\right|_{(1,-1)} = 2 \neq 0 \quad \checkmark$$

(G2) Parameter derivative nonzero:

$$\frac{\partial f}{\partial a} = 1 \quad \Rightarrow \quad \left.\frac{\partial f}{\partial a}\right|_{(1,-1)} = 1 \neq 0 \quad \checkmark$$

Conclusion

All four conditions satisfied:

$$\boxed{\text{FOLD BIFURCATION CONFIRMED at } (a, x) = (1, -1)}$$

XYZ Analysis of Condition Verification

- **STAGE X (What we verified):** All bifurcation conditions (B1-B2) and genericity conditions (G1-G2) hold at the critical point.
- **STAGE Y (Why each condition matters):**
  - **(B1)** $f = 0$: Fundamental requirement - we need an equilibrium at the bifurcation point
  - **(B2)** $f' = 0$: The linearization has zero eigenvalue, meaning we can't determine stability from linear terms alone. This is the algebraic signature of equilibria colliding - both have the same derivative value
  - **(G1)** $f'' \neq 0$: Ensures the nonlinearity is "strong enough" to create the fold. If $f'' = 0$ too, we'd have a higher-order (degenerate) bifurcation like a cusp. Here $f'' = 2 > 0$ means the parabola opens upward
  - **(G2)** $\partial f/\partial a \neq 0$: Ensures equilibrium positions actually change with parameter a. If this were zero, varying a wouldn't move the equilibria, and no bifurcation would occur. Here $\partial f/\partial a = 1 > 0$ means increasing a shifts f upward, pushing equilibria together
- **STAGE Z (What this rigor provides):** These conditions aren't just a checklist - they guarantee the system near $(1, -1)$ can be transformed to the normal form $\dot{y} = \mu + y^2$ (or $\mu - y^2$ depending on sign of f''). This universality means our local analysis captures the essential behavior shared by all fold bifurcations.

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5 Step 5: Normal Form Connection

Standard fold normal form

The canonical form is:

$$\dot{y} = \mu \pm y^2$$

where μ is the bifurcation parameter near zero.

Transform to normal form

Let's verify our system near $(a, x) = (1, -1)$ matches this structure.

Define shifted coordinates:

$$\begin{aligned} \mu &= a - 1 \quad \text{(bifurcation parameter, zero at critical point)} \\ y &= x + 1 \quad \text{(spatial coordinate, zero at critical equilibrium)} \end{aligned}$$

Then $a = \mu + 1$ and $x = y - 1$. Substitute into $\dot{x} = a + 2x + x^2$:

$$\begin{aligned} \dot{y} = \dot{x} &= a + 2x + x^2 \\ &= (\mu + 1) + 2(y - 1) + (y - 1)^2 \\ &= \mu + 1 + 2y - 2 + y^2 - 2y + 1 \\ &= \mu + y^2 \end{aligned}$$

Therefore:

$$\boxed{\dot{y} = \mu + y^2}$$

This is precisely the fold bifurcation normal form with the "+" sign (since $f'' = 2 > 0$).

XYZ Analysis of Normal Form

- **STAGE X (What we derived):** Through coordinate shift $(a, x) \to (\mu, y)$ centered at the bifurcation point, our equation reduces exactly to the standard form $\dot{y} = \mu + y^2$ with no higher-order terms needed.
- **STAGE Y (Why this works):** The Taylor expansion of $f(x, a)$ around $(1, -1)$ is:

  $$f(x, a) = \underbrace{f(-1, 1)}_{=0} + \underbrace{f_x(-1, 1)}_{=0}(x + 1) + \underbrace{f_a(-1, 1)}_{=1}(a - 1) + \frac{1}{2}\underbrace{f_{xx}(-1, 1)}_{=2}(x + 1)^2 + \cdots$$

  The linear term in x vanishes (condition B2), leaving only the parameter term and quadratic term - exactly the normal form structure. Higher-order terms (y^3, μy, etc.) are genuinely absent because our f is only quadratic in x and linear in a.
- **STAGE Z (What this means):** The normal form isn't just an approximation for our system - it's exact. This makes our analysis particularly clean. For general systems, we'd have $\dot{y} = \mu + y^2 + O(|y|^3, |y||\mu|)$, but the qualitative behavior (equilibria colliding along a square-root curve) would be the same.

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6 Step 6: Bifurcation Diagram

Equilibrium curves in (a, x) space

From $x = -1 \pm \sqrt{1 - a}$, the equilibria trace out curves:

Upper branch: $x_+(a) = -1 + \sqrt{1 - a}$ (unstable, dashed)

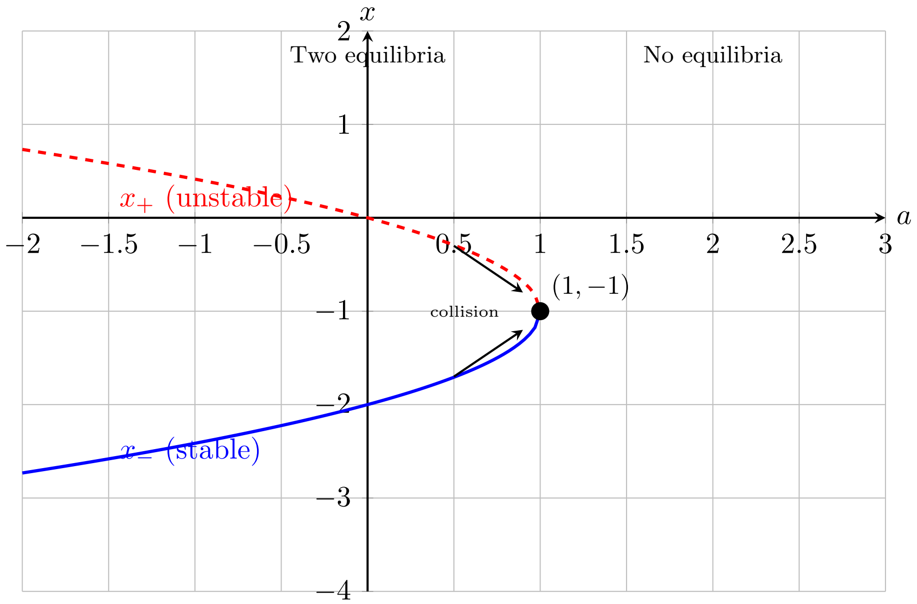
- Exists for $a \leq 1$
- At $a = 1$: $x_+ = -1$
- As $a \to -\infty$: $x_+ \to +\infty$

- As $a \to 1^-$: $x_+ \to -1$ (approaches bifurcation from right)

Lower branch: $x_-(a) = -1 - \sqrt{1-a}$ (stable, solid)

- Exists for $a \leq 1$
- At $a = 1$: $x_- = -1$
- As $a \to -\infty$: $x_- \to -\infty$
- As $a \to 1^-$: $x_- \to -1$ (approaches bifurcation from left)

Bifurcation diagram



XYZ Analysis of Bifurcation Diagram

- **STAGE X (What the diagram shows):** Two parabolic branches meeting at $(1, -1)$ like a "fold" in the (a, x) plane. The branches exist only for $a \leq 1$ and terminate at the bifurcation point. For $a > 1$, no branches exist - no real equilibria.
- **STAGE Y (Why this shape):** The equilibrium equation $x^2 + 2x + a = 0$ can be rearranged as:

  $$a = -x^2 - 2x = -(x+1)^2 + 1$$

  This is a parabola in (a, x) space, opening leftward, with vertex at $(a, x) = (1, -1)$. Each horizontal line $a =$ const intersects this parabola:

  - Twice if $a < 1$ (line to left of vertex) $\Rightarrow$ two equilibria
  - Once if $a = 1$ (line through vertex) $\Rightarrow$ one equilibrium
  - Not at all if $a > 1$ (line to right of vertex) $\Rightarrow$ no equilibria

  The square-root dependence $x \propto \pm\sqrt{1-a}$ near $a = 1$ is universal to fold bifurcations - equilibria approach the critical point along two branches that are tangent to each other (both have infinite slope dx/da at $a = 1$).
- **STAGE Z (What this means dynamically):** As a increases:

  - For $a \ll 1$: Two well-separated equilibria; stable one far left attracts trajectories

– As $a \to 1^-$: Equilibria approach; stable attractor moves right toward $x = -1$
– At $a = 1$: Equilibria merge; system at critical point
– For $a > 1$: No equilibria exist; system has no steady state, $\dot{x} = a + 2x + x^2 > 0$ for all x when $a > 1$ and x near -1, so solutions diverge to $+\infty$

This represents a catastrophic transition: beyond $a = 1$, stable behavior is impossible.

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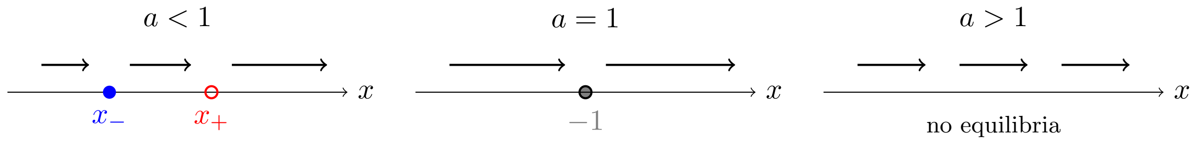
7 Step 7: Phase Portraits

Three scenarios



Notation: Filled circle = stable, hollow circle = unstable, half-filled = half-stable

XYZ Analysis of Phase Portrait Evolution

- **STAGE X (What we see):** Three qualitatively different flow patterns. Before bifurcation: two equilibria with flows toward stable one. At bifurcation: one half-stable equilibrium (attracts from left, repels to right). After bifurcation: no equilibria, all trajectories escape to infinity.

- **STAGE Y (Why these dynamics):** The sign of $\dot{x} = a + 2x + x^2$ determines flow direction:
  – **For** $a < 1$: Between equilibria $x_- < x < x_+$, we have $\dot{x} = a + 2x + x^2 < 0$ because the parabola $x^2 + 2x$ is below $-a$ in this interval. Flow is leftward toward x_-. Outside the equilibria, flow is rightward ($x < x_-$) or rightward ($x > x_+$).
  – **For** $a = 1$: At $x = -1$, $\dot{x} = 0$. For $x < -1$, $\dot{x} < 0$ (leftward), for $x > -1$, $\dot{x} > 0$ (rightward). The equilibrium is "half-stable" - stable from left, unstable to right.
  – **For** $a > 1$: No equilibria exist. For x near where they used to be (around $x = -1$), we have $\dot{x} = a + 2x + x^2 \approx 1 + 2x + x^2 = (x + 1)^2 > 0$ for $x \neq -1$. Since the parabola has no roots, it's always positive, so $\dot{x} > 0$ everywhere and all solutions diverge.

- **STAGE Z (What this means):** The fold bifurcation represents a "tipping point" beyond which stable behavior is lost:
  – **Before**: System has stable attractor at x_-; trajectories settle to steady state
  – **After**: System has no attractor; trajectories grow without bound

  In applications (e.g., climate models, population dynamics, electrical circuits), this corresponds to sudden catastrophic failure or "escape" from desired operating regime as a control parameter is varied.

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8 Summary

Part (a): Equilibria and Stability

$$\begin{aligned}
\text{For } a < 1: \quad & x_+^* = -1 + \sqrt{1-a} \quad \text{(unstable)} \\
& x_-^* = -1 - \sqrt{1-a} \quad \text{(stable)} \\
\text{For } a = 1: \quad & x^* = -1 \quad \text{(neutral/half-stable)} \\
\text{For } a > 1: \quad & \text{No real equilibria}
\end{aligned}$$

Part (b): Bifurcation Conjecture

$$\boxed{\text{Fold (Saddle-Node) Bifurcation at } (a, x) = (1, -1)}$$

Reasoning: Equilibria are created/destroyed (not exchanged), changing from 2 to 0 as parameter increases.

Part (c): Condition Verification

At $(a, x) = (1, -1)$:

$$\begin{aligned}
&\textbf{(B1)} \quad f(-1, 1) = 0 \quad \checkmark \\
&\textbf{(B2)} \quad f_x(-1, 1) = 0 \quad \checkmark \\
&\textbf{(G1)} \quad f_{xx}(-1, 1) = 2 \neq 0 \quad \checkmark \\
&\textbf{(G2)} \quad f_a(-1, 1) = 1 \neq 0 \quad \checkmark
\end{aligned}$$

Normal form: $\dot{y} = \mu + y^2$ where $\mu = a - 1$, $y = x + 1$

Physical interpretation: Fold bifurcation represents threshold phenomenon where stable steady state is destroyed beyond critical parameter value, leading to divergence.