Asymptotics 2025/2026
Solution to Problem Sheet 5, Question 1(b)

Question 1(b)

Use the method of stationary phase to obtain the leading asymptotic behaviour of

$$I(X) = \int_{1/2}^{2} (1+t)\, e^{iX(-t+t^3/3)} \mathrm{d}t \quad \text{as } X \to \infty.$$

Solution

We have a Fourier-type integral of the form

$$I(X) = \int_a^b f(t)\, e^{iX\phi(t)} \mathrm{d}t$$

where $f(t) = 1+t$, $\phi(t) = -t + \frac{t^3}{3}$, and the integration interval is $[a, b] = [1/2, 2]$.

Step 1: Identify stationary points

Following the method of stationary phase (Section 4.3 of the lecture notes), we look for points where $\phi'(t) = 0$ within the integration interval.

Compute the derivative:

$$\phi'(t) = -1 + t^2.$$

Setting $\phi'(t) = 0$:

$$-1 + t^2 = 0 \implies t^2 = 1 \implies t = \pm 1.$$

Since the integration interval is $[1/2, 2]$, we have:

- $t = 1$ lies in $[1/2, 2]$ ✓
- $t = -1$ lies outside $[1/2, 2]$ ×

Therefore, there is exactly one stationary point at $t = 1$ inside the integration interval.

Step 2: Classify the stationary point

Compute the second derivative:

$$\phi''(t) = 2t.$$

At $t = 1$:

$$\phi''(1) = 2 \neq 0.$$

Since $\phi''(1) \neq 0$, this is a non-degenerate stationary point (i.e., $n = 2$ in the terminology of the notes).

Step 3: Evaluate required quantities at the stationary point

At $t = 1$:

$$\begin{aligned}
f(1) &= 1 + 1 = 2, \\
\phi(1) &= -1 + \frac{1^3}{3} = -1 + \frac{1}{3} = -\frac{2}{3}, \\
\phi''(1) &= 2.
\end{aligned}$$

Step 4: Check contributions from endpoints

According to the notes (Section 4.3.1), contributions from endpoints are algebraically smaller than contributions from interior stationary points. Specifically, endpoint contributions are $O(X^{-1})$ while stationary point contributions are $O(X^{-1/2})$.

Since $X^{-1} = o(X^{-1/2})$ as $X \to \infty$, the endpoint contributions are subdominant and can be neglected in the leading order asymptotic expansion.

Step 5: Apply the method of stationary phase formula

From the lecture notes, Equation (235), the asymptotic expansion for a Fourier-type integral with a non-degenerate stationary point at $t = c$ is:

$$I(X) \sim \sqrt{\frac{2\pi i}{X\phi''(c)}} f(c) e^{iX\phi(c)} \quad \text{as } X \to \infty.$$

We need to evaluate $\sqrt{\frac{2\pi i}{X\phi''(c)}}$. Since $\phi''(1) = 2 > 0$, we use the relation:

$$\sqrt{i} = e^{i\pi/4} = \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}.$$

Therefore:

$$\begin{aligned}
\sqrt{\frac{2\pi i}{X\phi''(1)}} &= \sqrt{\frac{2\pi i}{2X}} \\
&= \sqrt{\frac{\pi i}{X}} \\
&= \sqrt{\frac{\pi}{X}} \cdot \sqrt{i} \\
&= \sqrt{\frac{\pi}{X}} \cdot e^{i\pi/4}.
\end{aligned}$$

Step 6: Compute the leading asymptotic behaviour

Substituting all values into the formula:

$$\begin{aligned}
I(X) &\sim \sqrt{\frac{\pi}{X}} e^{i\pi/4} \cdot 2 \cdot e^{iX(-2/3)} \\
&= 2\sqrt{\frac{\pi}{X}} e^{i\pi/4} e^{-2iX/3} \\
&= 2\sqrt{\frac{\pi}{X}} \exp\left(i\left(\frac{\pi}{4} - \frac{2X}{3}\right)\right).
\end{aligned}$$

Final Answer

$$\boxed{I(X) \sim 2\sqrt{\frac{\pi}{X}} \exp\left(i\left(\frac{\pi}{4} - \frac{2X}{3}\right)\right) \quad \text{as } X \to \infty}$$

Alternatively, this can be written as:

$$I(X) \sim 2\sqrt{\frac{\pi}{X}} e^{i\pi/4} e^{-2iX/3} \quad \text{as } X \to \infty,$$

or in terms of trigonometric functions:

$$I(X) \sim 2\sqrt{\frac{\pi}{X}} \left[\cos\left(\frac{\pi}{4} - \frac{2X}{3}\right) + i\sin\left(\frac{\pi}{4} - \frac{2X}{3}\right)\right] \quad \text{as } X \to \infty.$$

The asymptotic order is $O(X^{-1/2})$.