Problem 7, Question 5: WKB Approximation for Eigenvalue Problem

Pedagogical Breakdown

Question Statement

Use the WKB approximation to estimate the large eigenvalues, λ, of the eigenvalue problem

$$y'' + \frac{\lambda^2}{x^2}y = 0, \quad y(1) = 0, \quad y(e) = 0. \tag{1}$$

Find also the exact solutions and the exact eigenvalues. (Try $y(x) = x^\alpha$.) Consider the two sets of eigenvalues:

(i) Are the discrepancies between them consistent with the approximation made? If so, explain briefly why.

(ii) Will more terms of the WKB approximation give a better result? If your answer is yes, determine the form of the next term in the approximation to $y(x)$ and show how this gives a better result for the eigenvalues.

Part A: Identifying the Structure and Setting Up WKB

Step 1: Recognize the Form of the ODE

What are we doing? We begin by identifying that the given eigenvalue problem is of a form amenable to WKB analysis.

Why? The WKB method, as developed in Section 6.3 of the lecture notes, applies to equations of the form $\varepsilon^2 y'' + q(x)y = 0$. We must first recognize how our equation fits this pattern, and what plays the role of the small parameter ε.

Rewriting the equation: The given ODE is

$$y'' + \frac{\lambda^2}{x^2}y = 0. \tag{2}$$

Key observation: We are told to consider *large* eigenvalues λ. This suggests we should think of $1/\lambda$ as a small parameter. However, the standard WKB form has the small parameter multiplying y'', not inside $q(x)$.

Identifying the WKB parameter: Let us define

$$\varepsilon := \frac{1}{\lambda} \tag{3}$$

so that $\lambda = 1/\varepsilon$ and $\lambda \to \infty$ corresponds to $\varepsilon \to 0$.

Does this give WKB form? Rewriting:

$$y'' + \frac{1}{\varepsilon^2 x^2}y = 0 \tag{4}$$

Multiplying through by ε^2:

$$\varepsilon^2 y'' + \frac{1}{x^2} y = 0 \tag{5}$$

Identification: This is precisely the WKB form $\varepsilon^2 y'' + q(x)y = 0$ with

$$q(x) = \frac{1}{x^2}. \tag{6}$$

Domain and positivity: Since $x \in [1, e]$, we have $x > 0$, hence $q(x) > 0$ throughout the domain. This means we are in the *oscillatory regime* where WKB solutions involve trigonometric functions (Section 6.3.3, page 69).

Step 2: Verify That WKB is Appropriate for Large λ

What are we doing? Before proceeding with the WKB calculation, we verify that the method is valid for our problem.

Why? The WKB approximation is an asymptotic method valid when $\varepsilon \to 0$. We must check that the conditions stated in Section 6.3.3 (page 69) are satisfied.

The WKB validity criterion: From the lecture notes, the WKB approximation is good when

$$\frac{1}{\varepsilon} S_0(x) \gg S_1(x) \gg \varepsilon S_2(x) \quad \text{and} \quad \varepsilon S_2(x) \ll 1 \tag{7}$$

in the interval considered.

Order of magnitude estimates: For $q(x) = 1/x^2$ on $[1, e]$:

$$S_0 \sim \int \frac{1}{x} \mathrm{d}x \sim \log x \sim O(1) \tag{8}$$

$$S_1 \sim \log(x^{-1/2}) \sim O(1) \tag{9}$$

$$S_2 \sim O(1) \tag{10}$$

Thus:

$$\frac{1}{\varepsilon} S_0 \sim \frac{1}{\varepsilon} \gg 1 \gg \varepsilon \quad \text{for } \varepsilon \to 0 \tag{11}$$

Conclusion: The WKB approximation is valid for large λ (small ε) on the interval $[1, e]$.

Note on turning points: Since $q(x) = 1/x^2 > 0$ for all $x \in [1, e]$, there are no turning points in the domain. We are entirely in the oscillatory regime.

Part B: Applying the WKB Method

Step 3: Recall the WKB Solution in the Oscillatory Regime

What are we doing? We now write down the general form of the WKB solution for $q(x) > 0$.

Why? Before imposing boundary conditions to find eigenvalues, we need the general solution. From Section 6.3.2, equations (382) on page 69, the WKB solution for $q(x) > 0$ is:

The WKB solution:

$$y(x) = \frac{A}{[q(x)]^{1/4}} \exp\left(\frac{i}{\varepsilon} \int^x \sqrt{q(s)}\, \mathrm{d}s\right) + \frac{B}{[q(x)]^{1/4}} \exp\left(-\frac{i}{\varepsilon} \int^x \sqrt{q(s)}\, \mathrm{d}s\right) \tag{12}$$

where A and B are constants to be determined by boundary conditions.

Alternative form: Using Euler's formula, this can be rewritten as:

$$y(x) = \frac{C}{[q(x)]^{1/4}} \cos\left(\frac{1}{\varepsilon} \int^x \sqrt{q(s)}\, \mathrm{d}s + \phi\right) \tag{13}$$

where C and ϕ are constants related to A and B.

Step 4: Evaluate the WKB Phase Integral

What are we doing? We compute the phase integral $\int^x \sqrt{q(s)}\,\mathrm{d}s$ explicitly for $q(x) = 1/x^2$.

Why? This integral appears in the exponential/trigonometric arguments of the WKB solution. To apply boundary conditions and find eigenvalues, we need its explicit form.

Computation: For $q(x) = 1/x^2$:

$$\sqrt{q(x)} = \frac{1}{x} \tag{14}$$

Setting the integration limits: We choose the lower limit as $x = 1$ (the left boundary) for convenience:

$$\int_1^x \sqrt{q(s)}\,\mathrm{d}s = \int_1^x \frac{1}{s}\,\mathrm{d}s = [\log s]_1^x = \log x - \log 1 = \log x \tag{15}$$

The amplitude factor: The amplitude prefactor is:

$$[q(x)]^{-1/4} = \left(\frac{1}{x^2}\right)^{-1/4} = x^{1/2} \tag{16}$$

Step 5: Write the Explicit WKB Solution

What are we doing? We substitute our computed integrals into the general WKB form.

Why? This gives us the concrete form of the solution that we can apply boundary conditions to.

The WKB solution becomes:

$$y(x) = \sqrt{x}\left[A\exp\left(\frac{i\log x}{\varepsilon}\right) + B\exp\left(-\frac{i\log x}{\varepsilon}\right)\right] \tag{17}$$

Simplifying the exponentials: Using the property $e^{i\log x} = e^{\log x^i} = x^i$:

$$\exp\left(\frac{i\log x}{\varepsilon}\right) = x^{i/\varepsilon} \tag{18}$$

Thus:

$$y(x) = \sqrt{x}\left[Ax^{i/\varepsilon} + Bx^{-i/\varepsilon}\right] \tag{19}$$

Alternative trigonometric form: Using $x^{i/\varepsilon} = e^{i\log x/\varepsilon} = \cos(\log x/\varepsilon) + i\sin(\log x/\varepsilon)$, we can write:

$$y(x) = \sqrt{x}\left[C\cos\left(\frac{\log x}{\varepsilon}\right) + D\sin\left(\frac{\log x}{\varepsilon}\right)\right] \tag{20}$$

where C and D are real constants if we want a real solution.

Step 6: Apply the First Boundary Condition

What are we doing? We impose the boundary condition $y(1) = 0$.

Why? Eigenvalue problems require the solution to satisfy both boundary conditions. Each condition constrains the constants in the general solution.

Applying $y(1) = 0$: Using the trigonometric form:

$$y(1) = \sqrt{1}\left[C\cos\left(\frac{\log 1}{\varepsilon}\right) + D\sin\left(\frac{\log 1}{\varepsilon}\right)\right] = 0 \tag{21}$$

Simplification: Since $\log 1 = 0$:

$$C\cos(0) + D\sin(0) = C\cdot 1 + D\cdot 0 = C = 0 \tag{22}$$

Conclusion: The first boundary condition forces $C = 0$, so:

$$y(x) = D\sqrt{x}\sin\left(\frac{\log x}{\varepsilon}\right) \tag{23}$$

Step 7: Apply the Second Boundary Condition to Find Eigenvalues

What are we doing? We now impose $y(e) = 0$ to determine the allowed values of ε (and hence λ).

Why? The second boundary condition, combined with the requirement of non-trivial solutions ($D \neq 0$), yields the eigenvalue equation.

Applying $y(e) = 0$:

$$y(e) = D\sqrt{e}\sin\left(\frac{\log e}{\varepsilon}\right) = 0 \tag{24}$$

Non-triviality: For a non-trivial solution, we need $D \neq 0$ and $\sqrt{e} \neq 0$. Therefore:

$$\sin\left(\frac{\log e}{\varepsilon}\right) = 0 \tag{25}$$

General solution of sine equation: The sine function vanishes when its argument is an integer multiple of π:

$$\frac{\log e}{\varepsilon} = n\pi, \quad n = 1, 2, 3, \ldots \tag{26}$$

(We exclude $n = 0$ as it would give $y \equiv 0$, and negative n give the same eigenvalues as positive n.)

Recalling $\log e = 1$:

$$\frac{1}{\varepsilon} = n\pi \tag{27}$$

Substituting $\varepsilon = 1/\lambda$:

$$\lambda = n\pi \tag{28}$$

Step 8: State the WKB Eigenvalue Prediction

What have we found? The leading-order WKB approximation predicts the eigenvalues are:

$$\boxed{\lambda_n^{\mathrm{WKB}} = n\pi, \quad n = 1, 2, 3, \ldots} \tag{29}$$

The corresponding eigenfunctions (WKB):

$$y_n^{\mathrm{WKB}}(x) = \sqrt{x}\sin(n\pi\log x) \tag{30}$$

Verification of boundary conditions:

- At $x = 1$: $y_n(1) = \sqrt{1}\sin(n\pi\cdot 0) = \sin(0) = 0$ ✓
- At $x = e$: $y_n(e) = \sqrt{e}\sin(n\pi\cdot 1) = \sqrt{e}\sin(n\pi) = 0$ ✓

Part C: Finding the Exact Solution

Step 9: Use the Suggested Ansatz $y(x) = x^\alpha$

What are we doing? Following the hint in the problem, we try a power-law solution $y(x) = x^\alpha$ for some constant α to be determined.

Why this ansatz? The original ODE has the form $y'' + (\lambda^2/x^2)y = 0$. Since the coefficient λ^2/x^2 is a power of x, and differentiation of power functions yields power functions, a power-law ansatz is natural.

This is called an Euler equation or Cauchy-Euler equation: Equations of the form

$$x^2y'' + pxy' + qy = 0 \tag{31}$$

are known to have power-law solutions. Our equation is of this type.

Step 10: Compute Derivatives of the Ansatz

What are we doing? We compute the derivatives of $y(x) = x^\alpha$ to substitute into the ODE.

Why? To determine α, we must substitute our ansatz into the differential equation and find which values of α satisfy it.

Computation:

$$y(x) = x^\alpha \tag{32}$$

$$y'(x) = \alpha x^{\alpha-1} \tag{33}$$

$$y''(x) = \alpha(\alpha-1)x^{\alpha-2} \tag{34}$$

Step 11: Substitute into the ODE and Derive the Characteristic Equation

What are we doing? We substitute $y = x^\alpha$ and its derivatives into the ODE $y''+(\lambda^2/x^2)y = 0$.

Why? This will give us an algebraic equation for α, known as the characteristic equation or indicial equation.

Substitution:

$$\alpha(\alpha-1)x^{\alpha-2} + \frac{\lambda^2}{x^2}x^\alpha = 0 \tag{35}$$

Simplification:

$$\alpha(\alpha-1)x^{\alpha-2} + \lambda^2 x^{\alpha-2} = 0 \tag{36}$$

Factoring out $x^{\alpha-2}$:

$$x^{\alpha-2}\left[\alpha(\alpha-1) + \lambda^2\right] = 0 \tag{37}$$

Key observation: Since $x^{\alpha-2} \neq 0$ for $x \in [1, e]$, we must have:

$$\alpha(\alpha-1) + \lambda^2 = 0 \tag{38}$$

This is the **characteristic equation**.

Step 12: Solve the Characteristic Equation for α

What are we doing? We solve the quadratic equation $\alpha(\alpha-1) + \lambda^2 = 0$ for α.

Why? This determines the exponents in the power-law solutions, giving us the two linearly independent solutions needed for a second-order ODE.

Rearranging:

$$\alpha^2 - \alpha + \lambda^2 = 0 \tag{39}$$

Using the quadratic formula:

$$\alpha = \frac{1 \pm \sqrt{1-4\lambda^2}}{2} \tag{40}$$

For large λ: When $\lambda^2 > 1/4$, we have $1 - 4\lambda^2 < 0$, so:

$$\alpha = \frac{1 \pm \sqrt{-(4\lambda^2-1)}}{2} = \frac{1 \pm i\sqrt{4\lambda^2-1}}{2} \tag{41}$$

Defining μ: Let us define

$$\mu := \sqrt{\lambda^2 - \frac{1}{4}} \tag{42}$$

Then $\sqrt{4\lambda^2-1} = 2\mu$, and:

$$\alpha = \frac{1 \pm 2i\mu}{2} = \frac{1}{2} \pm i\mu \tag{43}$$

The two roots:

$$\alpha_1 = \frac{1}{2} + i\mu, \quad \alpha_2 = \frac{1}{2} - i\mu \tag{44}$$

Critical observation: Note that $\mu = \sqrt{\lambda^2 - 1/4} \neq \lambda$. This distinction is crucial for finding the exact eigenvalues.

Step 13: Write the General Exact Solution

What are we doing? We construct the general solution as a linear combination of the two linearly independent power-law solutions.

Why? A second-order linear ODE has a two-dimensional solution space. Any solution can be written as a linear combination of two independent solutions.

The general solution:

$$y(x) = Ax^{\alpha_1} + Bx^{\alpha_2} = Ax^{\frac{1}{2}+i\mu} + Bx^{\frac{1}{2}-i\mu} \tag{45}$$

Factoring out common power:

$$y(x) = \sqrt{x}\left(Ax^{i\mu} + Bx^{-i\mu}\right) \tag{46}$$

Converting to trigonometric form: Using $x^{i\mu} = e^{i\mu \log x}$ and Euler's formula:

$$x^{i\mu} = \cos(\mu \log x) + i \sin(\mu \log x) \tag{47}$$

$$x^{-i\mu} = \cos(\mu \log x) - i \sin(\mu \log x) \tag{48}$$

For real solutions:

$$y(x) = \sqrt{x}\left[C \cos(\mu \log x) + D \sin(\mu \log x)\right] \tag{49}$$

where C and D are real constants and $\mu = \sqrt{\lambda^2 - 1/4}$.

Comparison with WKB: The WKB solution has the form $\sqrt{x}\sin(\lambda \log x)$, while the exact solution has $\sqrt{x}\sin(\mu \log x)$ where $\mu = \sqrt{\lambda^2 - 1/4}$. These are *not* the same!

Step 14: Apply Boundary Conditions to Find Exact Eigenvalues

What are we doing? We impose the boundary conditions $y(1) = 0$ and $y(e) = 0$ on the exact solution.

Why? This determines the allowed values of λ exactly.

First boundary condition, $y(1) = 0$:

$$y(1) = \sqrt{1}\left[C \cos(\mu \log 1) + D \sin(\mu \log 1)\right] = C \cos(0) + D \sin(0) = C = 0 \tag{50}$$

So $C = 0$ and:

$$y(x) = D\sqrt{x}\sin(\mu \log x) \tag{51}$$

Second boundary condition, $y(e) = 0$:

$$y(e) = D\sqrt{e}\sin(\mu \log e) = D\sqrt{e}\sin(\mu) = 0 \tag{52}$$

For non-trivial solutions ($D \neq 0$):

$$\sin(\mu) = 0 \tag{53}$$

Solution:

$$\mu = n\pi, \quad n = 1, 2, 3, \ldots \tag{54}$$

Step 15: Convert from μ to λ

What are we doing? We use the relation $\mu = \sqrt{\lambda^2 - 1/4}$ to find the exact eigenvalues λ_n.

Why? The boundary condition determines $\mu_n = n\pi$, but the problem asks for λ_n.

From $\mu_n = n\pi$:

$$\sqrt{\lambda_n^2 - \frac{1}{4}} = n\pi \tag{55}$$

Squaring both sides:

$$\lambda_n^2 - \frac{1}{4} = n^2\pi^2 \tag{56}$$

Solving for λ_n:

$$\lambda_n^2 = n^2\pi^2 + \frac{1}{4} \tag{57}$$

$$\boxed{\lambda_n^{\text{exact}} = \sqrt{n^2\pi^2 + \frac{1}{4}}, \quad n = 1, 2, 3, \ldots} \tag{58}$$

Step 16: Compare WKB and Exact Eigenvalues

What have we found? Let us compare the two sets of eigenvalues:

$$\lambda_n^{\text{WKB}} = n\pi \tag{59}$$

$$\lambda_n^{\text{exact}} = \sqrt{n^2\pi^2 + \frac{1}{4}} \tag{60}$$

Key observation: The WKB and exact eigenvalues are *not* identical! There is a discrepancy.

Expanding the exact eigenvalue for large n:

$$\lambda_n^{\text{exact}} = \sqrt{n^2\pi^2 + \frac{1}{4}} = n\pi\sqrt{1 + \frac{1}{4n^2\pi^2}} \tag{61}$$

$$= n\pi\left(1 + \frac{1}{8n^2\pi^2} - \frac{1}{128n^4\pi^4} + O(n^{-6})\right) \tag{62}$$

$$= n\pi + \frac{1}{8n\pi} + O(n^{-3}) \tag{63}$$

The discrepancy:

$$\boxed{\Delta\lambda_n = \lambda_n^{\text{exact}} - \lambda_n^{\text{WKB}} = \frac{1}{8n\pi} + O(n^{-3})} \tag{64}$$

This is an $O(1/\lambda)$ correction, which is consistent with the WKB approximation being accurate to leading order but not exact.

Part D: Analysis of the Results

Step 17: Answer Question (i) – Are the Discrepancies Consistent?

What are we being asked? Part (i) asks: "Are the discrepancies between them consistent with the approximation made? If so, explain briefly why."

Answer: Yes, the discrepancies are consistent with the WKB approximation.

Explanation:

1. **The WKB method is an asymptotic approximation for large λ.** The leading-order WKB solution captures the behavior as $\lambda \to \infty$ but includes errors of relative order $O(1/\lambda)$.

2. **The discrepancy $\Delta\lambda_n = 1/(8n\pi) + O(n^{-3})$ is indeed $O(1/\lambda)$.** Since $\lambda_n \sim n\pi$, we have:

$$\frac{\Delta\lambda_n}{\lambda_n} \sim \frac{1/(8n\pi)}{n\pi} = \frac{1}{8n^2\pi^2} = O(\lambda^{-2}) \tag{65}$$

   This is a small relative error that vanishes as $\lambda \to \infty$, exactly as expected for a leading-order asymptotic approximation.

3. **The source of the discrepancy:** The characteristic equation gives $\mu = \sqrt{\lambda^2 - 1/4}$, not $\mu = \lambda$. The WKB method effectively approximates $\mu \approx \lambda$ for large λ, which introduces an error of order $O(1/\lambda)$.

4. **Physical interpretation:** The $1/4$ term in $\lambda^2 - 1/4$ arises from the amplitude modulation factor $q(x)^{-1/4} = x^{1/2}$ in the WKB solution. This amplitude factor contributes to the effective phase of the exact solution, but the leading-order WKB eigenvalue condition only accounts for the explicit phase integral $\int \sqrt{q}\,\mathrm{d}x$.

Conclusion for part (i): The discrepancy $\Delta\lambda_n = O(1/n)$ is fully consistent with the leading-order WKB approximation, which is accurate to $O(1)$ in λ but not beyond.

Step 18: Answer Question (ii) – Will More WKB Terms Help?

What are we being asked? Part (ii) asks whether including higher-order WKB corrections will give a better result for the eigenvalues.

Answer: Yes, including the next WKB correction gives a better result. Let us demonstrate this explicitly.

Step 19: The Higher-Order WKB Expansion

What are we doing? We include the next term in the WKB expansion to improve the eigenvalue estimate.

Why? The leading-order WKB gave eigenvalues accurate to $O(1)$. The next correction should capture the $O(1/\lambda)$ term.

Recall from Section 6.3.2: The WKB expansion for $p = S'$ is:

$$p(x, \varepsilon) = \frac{1}{\varepsilon} p_0(x) + p_1(x) + \varepsilon p_2(x) + \cdots \tag{66}$$

where:

$$p_0 = \pm i\sqrt{q} = \pm \frac{i}{x} \tag{67}$$

$$p_1 = -\frac{q'}{4q} = -\frac{-2x^{-3}}{4x^{-2}} = \frac{1}{2x} \tag{68}$$

The next term p_2: From the lecture notes (page 68), the recursion relation gives:

$$p_2 = -\frac{p_1' + p_1^2}{2p_0} \tag{69}$$

Computing p_1' and p_1^2:

$$p_1 = \frac{1}{2x} \tag{70}$$

$$p_1' = -\frac{1}{2x^2} \tag{71}$$

$$p_1^2 = \frac{1}{4x^2} \tag{72}$$

Thus:

$$p_1' + p_1^2 = -\frac{1}{2x^2} + \frac{1}{4x^2} = -\frac{1}{4x^2} \tag{73}$$

And:

$$p_2 = -\frac{-1/(4x^2)}{2 \cdot (\pm i/x)} = \frac{1/(4x^2)}{\pm 2i/x} = \mp \frac{i}{8x} \tag{74}$$

Step 20: The Improved WKB Phase

What are we doing? We integrate to find the corrected phase function.

Why? The phase $S(x) = \int p \, \mathrm{d}x$ determines the oscillatory behavior of the solution.

The total phase:

$$S(x) = \frac{1}{\varepsilon} \int p_0 \, \mathrm{d}x + \int p_1 \, \mathrm{d}x + \varepsilon \int p_2 \, \mathrm{d}x + O(\varepsilon^2) \tag{75}$$

Computing each integral from $x = 1$:

$$\int_1^x p_0 \, \mathrm{d}s = \pm i \int_1^x \frac{1}{s} \, \mathrm{d}s = \pm i \log x \tag{76}$$

$$\int_1^x p_1 \, \mathrm{d}s = \int_1^x \frac{1}{2s} \, \mathrm{d}s = \frac{1}{2} \log x \tag{77}$$

$$\int_1^x p_2 \, \mathrm{d}s = \mp i \int_1^x \frac{1}{8s} \, \mathrm{d}s = \mp \frac{i \log x}{8} \tag{78}$$

The corrected WKB solution (taking the sine combination):

$$y(x) = D\sqrt{x} \sin\left(\frac{\log x}{\varepsilon} - \frac{\varepsilon \log x}{8} \right) \tag{79}$$

Recalling $\varepsilon = 1/\lambda$:

$$y(x) = D\sqrt{x} \sin\left(\lambda \log x - \frac{\log x}{8\lambda} \right) \tag{80}$$

Step 21: Apply Boundary Conditions with the Correction

What are we doing? We apply the boundary conditions to the corrected WKB solution.

Why? This will give us improved eigenvalue estimates.

The boundary condition $y(1) = 0$: At $x = 1$, $\log 1 = 0$, so $y(1) = 0$ is automatically satisfied. ✓

The boundary condition $y(e) = 0$:

$$y(e) = D\sqrt{e} \sin\left(\lambda - \frac{1}{8\lambda} \right) = 0 \tag{81}$$

For non-trivial solutions:

$$\lambda - \frac{1}{8\lambda} = n\pi, \quad n = 1, 2, 3, \ldots \tag{82}$$

Solving for λ: Multiplying by λ:

$$\lambda^2 - n\pi\lambda - \frac{1}{8} = 0 \tag{83}$$

Wait, this should be:

$$\lambda^2 - \frac{1}{8} = n\pi\lambda \tag{84}$$

Actually, let me redo this. From $\lambda - \frac{1}{8\lambda} = n\pi$:

$$\lambda^2 - \frac{1}{8} = n\pi\lambda \tag{85}$$

$$\lambda^2 - n\pi\lambda - \frac{1}{8} = 0 \tag{86}$$

Hmm, this doesn't match what we want. Let me reconsider the sign of the correction.

Re-examining the correction: The phase correction from p_2 should be reconsidered. Looking at the structure more carefully, the higher-order WKB correction to the eigenvalue condition can be written as:

$$\lambda - \frac{1}{8\lambda} = n\pi \tag{87}$$

For large λ: We can solve this perturbatively. Write $\lambda = n\pi + \delta$ where $\delta \ll n\pi$:

$$n\pi + \delta - \frac{1}{8(n\pi + \delta)} = n\pi \tag{88}$$

$$\delta = \frac{1}{8(n\pi + \delta)} \approx \frac{1}{8n\pi} \tag{89}$$

The improved WKB eigenvalue:

$$\lambda_n^{\text{WKB (improved)}} = n\pi + \frac{1}{8n\pi} + O(n^{-3}) \tag{90}$$

Step 22: Compare with the Exact Result

What are we doing? We compare the improved WKB eigenvalue with the exact eigenvalue.

Recall the exact eigenvalue expansion:

$$\lambda_n^{\text{exact}} = n\pi + \frac{1}{8n\pi} + O(n^{-3}) \tag{91}$$

The improved WKB eigenvalue:

$$\lambda_n^{\text{WKB (improved)}} = n\pi + \frac{1}{8n\pi} + O(n^{-3}) \tag{92}$$

These agree to order $O(1/n)$!

$$\boxed{\lambda_n^{\text{WKB (improved)}} = \lambda_n^{\text{exact}} + O(n^{-3})} \tag{93}$$

Step 23: The Form of the Next Term in the Approximation to $y(x)$

What are we doing? We state explicitly the form of the higher-order WKB correction to the eigenfunction.

The improved WKB solution:

$$y(x) = \sqrt{x} \sin\left(\lambda \log x - \frac{\log x}{8\lambda}\right) \tag{94}$$

Alternative form: This can be written as:

$$y(x) = \sqrt{x} \sin\left[\left(\lambda - \frac{1}{8\lambda}\right) \log x\right] \tag{95}$$

Comparison with the exact solution: The exact solution is:

$$y(x) = \sqrt{x} \sin(\mu \log x) = \sqrt{x} \sin\left(\sqrt{\lambda^2 - \frac{1}{4}} \log x\right) \tag{96}$$

For large λ:

$$\mu = \sqrt{\lambda^2 - \frac{1}{4}} = \lambda\sqrt{1 - \frac{1}{4\lambda^2}} \approx \lambda - \frac{1}{8\lambda} + O(\lambda^{-3}) \tag{97}$$

So the improved WKB solution matches the exact solution to next order in $1/\lambda$.

Step 24: Final Answer to Part (ii)

Answer to part (ii):

Yes, including more terms of the WKB approximation gives a better result for the eigenvalues.
The form of the next term in the approximation to $y(x)$:
The leading-order WKB solution is:

$$y^{(0)}(x) = \sqrt{x} \sin(\lambda \log x) \tag{98}$$

Including the next-order correction, the improved solution is:

$$y^{(1)}(x) = \sqrt{x} \sin\left(\lambda \log x - \frac{\log x}{8\lambda}\right) \tag{99}$$

How this gives a better result for the eigenvalues:

1. The leading-order WKB eigenvalue condition $\sin(\lambda) = 0$ gives $\lambda_n^{(0)} = n\pi$.
2. The improved WKB eigenvalue condition $\sin\left(\lambda - \frac{1}{8\lambda}\right) = 0$ gives:

$$\lambda - \frac{1}{8\lambda} = n\pi \tag{100}$$

   Solving perturbatively: $\lambda_n^{(1)} = n\pi + \frac{1}{8n\pi} + O(n^{-3})$.

3. The exact eigenvalues are:

$$\lambda_n^{\text{exact}} = \sqrt{n^2\pi^2 + \frac{1}{4}} = n\pi + \frac{1}{8n\pi} + O(n^{-3}) \tag{101}$$

4. **Comparison:**

$$|\lambda_n^{(0)} - \lambda_n^{\text{exact}}| = \frac{1}{8n\pi} + O(n^{-3}) \tag{102}$$

$$|\lambda_n^{(1)} - \lambda_n^{\text{exact}}| = O(n^{-3}) \tag{103}$$

   The improved WKB approximation reduces the error from $O(n^{-1})$ to $O(n^{-3})$.

Summary and Conclusion

Complete answer to Question 5:

1. **Leading-order WKB eigenvalues:** $\lambda_n^{\text{WKB}} = n\pi$, $n = 1, 2, 3, \ldots$

2. **Exact eigenvalues:** $\lambda_n^{\text{exact}} = \sqrt{n^2\pi^2 + \frac{1}{4}}$, $n = 1, 2, 3, \ldots$

3. **Discrepancy:** $\Delta\lambda_n = \lambda_n^{\text{exact}} - \lambda_n^{\text{WKB}} = \dfrac{1}{8n\pi} + O(n^{-3})$

4. **Part (i):** Yes, the discrepancies are consistent with the WKB approximation. The WKB method is an asymptotic approximation for large λ, and the $O(1/n)$ discrepancy is precisely the expected error for a leading-order asymptotic result.

5. **Part (ii):** Yes, including more WKB terms gives a better result. The next term in the WKB approximation to $y(x)$ is:

$$y(x) = \sqrt{x} \sin\left(\lambda \log x - \frac{\log x}{8\lambda}\right) \tag{104}$$

   This leads to the improved eigenvalue condition $\lambda - \frac{1}{8\lambda} = n\pi$, which gives eigenvalues accurate to $O(n^{-3})$ instead of $O(n^{-1})$.

Key lessons:

- WKB is an asymptotic method that typically gives approximate, not exact, results.
- For this problem, the WKB eigenvalues differ from exact eigenvalues by $O(1/\lambda)$.
- The discrepancy arises because the exact characteristic exponent is $\mu = \sqrt{\lambda^2 - 1/4}$, not $\mu = \lambda$.
- Higher-order WKB corrections systematically improve the approximation.
- The next-order correction captures the 1/4 term in $\lambda^2 - 1/4$, reducing the error from $O(1/\lambda)$ to $O(1/\lambda^3)$.