Asymptotics 2025/2026 Sheet 1
Problem 2: Complete Solution

Problem Statement

Find two terms in the expansion for each root of

$$\epsilon^2 x^3 + x^2 + 2x + \epsilon = 0, \quad \epsilon \ll 1. \tag{1}$$

1 Initial Analysis and Strategy

1.1 Why We Begin With The Unperturbed Problem

What we observe: The equation contains a small parameter ϵ multiplying certain terms.

Why this matters: According to Section 2 of the lecture notes, when we have an equation depending on a small parameter ϵ, we seek to understand how solutions behave as $\epsilon \to 0$. The fundamental strategy is to first solve the simpler problem where $\epsilon = 0$, then understand how solutions "perturb" away from this base case.

What we do: Set $\epsilon = 0$ to obtain the unperturbed equation.

Why we do this: The unperturbed equation gives us candidate locations where roots might exist for small ϵ. These are our "base points" around which we'll construct expansions.

1.2 The Unperturbed Equation

Setting $\epsilon = 0$ in equation (1):

$$x^2 + 2x = 0. \tag{2}$$

What we have: A quadratic equation.

Why we factor: Factoring reveals the structure of solutions clearly.

$$x(x+2) = 0. \tag{3}$$

What this tells us: The unperturbed equation has exactly two solutions:

$$x_0 = 0 \quad \text{and} \quad x_0 = -2. \tag{4}$$

1.3 Critical Observation: Order Reduction

What we notice: The original equation (1) is cubic (degree 3), but the unperturbed equation (2) is quadratic (degree 2).

Why this is significant: This is the hallmark of a **singular perturbation problem**. From Section 2.2 of the notes:

> *Singular perturbation problem: The perturbed and unperturbed problem differ in an essential way: Not all solutions of the perturbed problem can be expressed as an expansion around the unperturbed solution(s).*

What this means concretely:

- The perturbed equation has 3 roots (it's cubic)
- The unperturbed equation has 2 roots
- Therefore, at least one root of the perturbed equation must behave in a qualitatively different way as $\epsilon \to 0$

What we expect:

- Some roots will be "regular" - approaching the unperturbed roots smoothly
- At least one root will be "singular" - either going to infinity, or requiring non-standard expansions

1.4 Strategy: The Two-Pronged Approach

Our plan:

1. **Regular solutions:** Try standard power series expansions $x = x_0 + x_1\epsilon + x_2\epsilon^2 + \cdots$ around each unperturbed root $x_0 \in \{0, -2\}$
2. **Singular solution:** If we don't get all three roots from step 1, perform dominant balance analysis to find the "missing" root

2 Finding Regular Solutions

2.1 Expansion Method Framework

The general approach (from Section 2.1.1):
For each unperturbed solution x_0, we make the ansatz:

$$x(\epsilon) = x_0 + x_1\epsilon + x_2\epsilon^2 + x_3\epsilon^3 + \cdots \tag{5}$$

Why this form: We assume the solution depends smoothly on ϵ, so a Taylor series in ϵ is natural. The coefficients $x_1, x_2, x_3, \ldots$ are constants to be determined.

The key principle: After substitution, we collect terms by powers of ϵ. For the equation to hold for all small ϵ, each power of ϵ must vanish independently.

Why powers must vanish independently: If

$$a_0 + a_1\epsilon + a_2\epsilon^2 + \cdots = 0$$

for all small ϵ, then setting $\epsilon = 0$ gives $a_0 = 0$. Dividing by ϵ and taking $\epsilon \to 0$ gives $a_1 = 0$, and so forth.

2.2 Attempt 1: Regular Solution Near $x_0 = 0$

Our ansatz:

$$x = 0 + x_1\epsilon + x_2\epsilon^2 + x_3\epsilon^3 + \cdots = x_1\epsilon + x_2\epsilon^2 + x_3\epsilon^3 + \cdots \tag{6}$$

Why we start here: $x_0 = 0$ is simpler algebraically (fewer terms to track).

2.2.1 Substituting Into The Original Equation

We substitute (6) into (1):

$$\epsilon^2(x_1\epsilon + x_2\epsilon^2 + \cdots)^3 + (x_1\epsilon + x_2\epsilon^2 + \cdots)^2 + 2(x_1\epsilon + x_2\epsilon^2 + \cdots) + \epsilon = 0. \tag{7}$$

What we need to do: Expand each term and collect by powers of ϵ.

2.2.2 Expanding The Cubic Term

The cubic term:

$$\epsilon^2(x_1\epsilon + x_2\epsilon^2 + \cdots)^3 = \epsilon^2 \cdot \epsilon^3(x_1 + x_2\epsilon + \cdots)^3 \quad (8)$$
$$= \epsilon^5(x_1^3 + 3x_1^2x_2\epsilon + \cdots) \quad (9)$$

Why we can ignore this: The leading power is ϵ^5. Since we're finding the first two terms (coefficients of ϵ and ϵ^2), terms starting at ϵ^5 won't affect our calculation.

What we write:

$$\epsilon^2 x^3 = O(\epsilon^5). \quad (10)$$

2.2.3 Expanding The Quadratic Term

The quadratic term:

$$(x_1\epsilon + x_2\epsilon^2 + x_3\epsilon^3 + \cdots)^2 = x_1^2\epsilon^2 + 2x_1x_2\epsilon^3 + (x_2^2 + 2x_1x_3)\epsilon^4 + \cdots \quad (11)$$

Why each term appears:

- $x_1^2\epsilon^2$: from $(x_1\epsilon)^2$
- $2x_1x_2\epsilon^3$: from $2(x_1\epsilon)(x_2\epsilon^2)$
- Higher order terms involve more products

What we retain: For two-term accuracy, we need up to ϵ^2:

$$x^2 = x_1^2\epsilon^2 + O(\epsilon^3). \quad (12)$$

2.2.4 The Linear Term

The linear term:

$$2x = 2x_1\epsilon + 2x_2\epsilon^2 + 2x_3\epsilon^3 + \cdots \quad (13)$$

Why this is straightforward: Linear terms don't mix - each power of ϵ stays separate.

2.2.5 Collecting All Terms

The full equation becomes:

$$O(\epsilon^5) + x_1^2\epsilon^2 + 2x_1\epsilon + 2x_2\epsilon^2 + \epsilon + O(\epsilon^3) = 0. \quad (14)$$

Grouping by powers of ϵ:

$$(2x_1 + 1)\epsilon + (x_1^2 + 2x_2)\epsilon^2 + O(\epsilon^3) = 0. \quad (15)$$

2.2.6 Solving Order By Order

At $O(\epsilon)$: The coefficient of ϵ must vanish

$$2x_1 + 1 = 0. \quad (16)$$

Why this must be zero: For the equation to hold for all small $\epsilon > 0$, we can't have a non-zero coefficient of ϵ (it would dominate as $\epsilon \to 0$).

Solving:

$$x_1 = -\frac{1}{2}. \quad (17)$$

What this means: The leading correction to $x = 0$ is negative, pushing the root toward negative values.

At $O(\epsilon^2)$: The coefficient of ϵ^2 must vanish

$$x_1^2 + 2x_2 = 0. \tag{18}$$

Why we can solve this now: We know $x_1 = -\frac{1}{2}$ from the previous order, so we can substitute:

$$\left(-\frac{1}{2}\right)^2 + 2x_2 = 0. \tag{19}$$

$$\frac{1}{4} + 2x_2 = 0. \tag{20}$$

$$x_2 = -\frac{1}{8}. \tag{21}$$

What this tells us: The $O(\epsilon^2)$ correction is also negative, continuing the trend.

2.2.7 First Root: The Result

Our first root:

$$\boxed{x_1(\epsilon) = -\frac{1}{2}\epsilon - \frac{1}{8}\epsilon^2 + O(\epsilon^3)}. \tag{22}$$

Physical interpretation: Starting from $x = 0$, the perturbation pushes the root in the negative direction, with corrections of order ϵ and ϵ^2.

2.3 Attempt 2: Regular Solution Near $x_0 = -2$

Our ansatz:

$$x = -2 + x_1\epsilon + x_2\epsilon^2 + x_3\epsilon^3 + \cdots \tag{23}$$

Why we try this: The second unperturbed root is $x_0 = -2$, and we expect a regular expansion around it.

2.3.1 Substituting Into The Original Equation

We substitute (20) into (1):

$$\epsilon^2(-2 + x_1\epsilon + x_2\epsilon^2 + \cdots)^3 + (-2 + x_1\epsilon + x_2\epsilon^2 + \cdots)^2 \\ + 2(-2 + x_1\epsilon + x_2\epsilon^2 + \cdots) + \epsilon = 0. \tag{24}$$

2.3.2 Expanding The Cubic Term

Using the binomial expansion:

$$(-2 + x_1\epsilon + x_2\epsilon^2 + \cdots)^3 = (-2)^3 + 3(-2)^2(x_1\epsilon) + \cdots \tag{25}$$

$$= -8 + 12x_1\epsilon + O(\epsilon^2). \tag{26}$$

Why we stop here: For two-term accuracy, we only need up to $O(\epsilon^2)$ in the final equation. The cubic term is multiplied by ϵ^2, so:

$$\epsilon^2 \cdot (-8 + 12x_1\epsilon + \cdots) = -8\epsilon^2 + 12x_1\epsilon^3 + \cdots \tag{27}$$

What we retain:

$$\epsilon^2 x^3 = -8\epsilon^2 + O(\epsilon^3). \tag{28}$$

2.3.3 Expanding The Quadratic Term

Using binomial expansion:

$$(-2 + x_1\epsilon + x_2\epsilon^2 + \cdots)^2 = (-2)^2 + 2(-2)(x_1\epsilon) + 2(-2)(x_2\epsilon^2) + (x_1\epsilon)^2 + \cdots \tag{29}$$

$$= 4 - 4x_1\epsilon - 4x_2\epsilon^2 + x_1^2\epsilon^2 + O(\epsilon^3) \tag{30}$$

$$= 4 - 4x_1\epsilon + (x_1^2 - 4x_2)\epsilon^2 + O(\epsilon^3). \tag{31}$$

Why each term:

- $4 = (-2)^2$: the zeroth order term
- $-4x_1\epsilon = 2(-2)(x_1\epsilon)$: cross term from binomial
- $-4x_2\epsilon^2 = 2(-2)(x_2\epsilon^2)$: cross term with x_2
- $x_1^2\epsilon^2 = (x_1\epsilon)^2$: square of the first correction

2.3.4 The Linear Term

Straightforward expansion:

$$2x = 2(-2 + x_1\epsilon + x_2\epsilon^2 + \cdots) = -4 + 2x_1\epsilon + 2x_2\epsilon^2 + \cdots \tag{32}$$

2.3.5 Collecting All Terms

The full equation:

$$-8\epsilon^2 + [4 - 4x_1\epsilon + (x_1^2 - 4x_2)\epsilon^2] + [-4 + 2x_1\epsilon + 2x_2\epsilon^2] + \epsilon + O(\epsilon^3) = 0. \tag{33}$$

Why we group by powers: This is the standard method to extract coefficients systematically.

Collecting constant terms:

$$4 - 4 = 0. \quad \checkmark \tag{34}$$

Why this must work: We chose $x_0 = -2$ because it's a root of the unperturbed equation, so the $O(1)$ terms must cancel.

Collecting $O(\epsilon)$ terms:

$$-4x_1 + 2x_1 + 1 = 0. \tag{35}$$

$$-2x_1 + 1 = 0. \tag{36}$$

$$x_1 = \frac{1}{2}. \tag{37}$$

What this means: The perturbation pushes the root at $x = -2$ in the positive direction (toward zero).

Collecting $O(\epsilon^2)$ terms:

$$-8 + x_1^2 - 4x_2 + 2x_2 = 0. \tag{38}$$

$$-8 + \left(\frac{1}{2}\right)^2 - 2x_2 = 0. \tag{39}$$

$$-8 + \frac{1}{4} - 2x_2 = 0. \tag{40}$$

$$-2x_2 = 8 - \frac{1}{4} = \frac{31}{4}. \tag{41}$$

$$x_2 = -\frac{31}{8}. \tag{42}$$

What this tells us: The $O(\epsilon^2)$ correction is negative, partially offsetting the positive $O(\epsilon)$ correction.

2.3.6 Second Root: The Result

Our second root:

$$\boxed{x_2(\epsilon) = -2 + \frac{1}{2}\epsilon - \frac{31}{8}\epsilon^2 + O(\epsilon^3)}. \tag{43}$$

3 Finding The Singular Solution

3.1 Why We Need To Continue

What we have found so far: Two roots.

What we need: Three roots (the equation is cubic).

Conclusion: There must be a third root that doesn't fit the regular expansion pattern around $x_0 = 0$ or $x_0 = -2$.

3.2 Dominant Balance Analysis

The method (from Section 2.2.2):

When regular expansions fail, we use dominant balance to determine which terms in the equation balance each other as $\epsilon \to 0$, revealing the scaling of the "missing" root.

The three steps:

1. Assume which terms balance
2. Solve for the implied scaling
3. Check consistency

3.2.1 The Original Equation Structure

Our equation:

$$\epsilon^2 x^3 + x^2 + 2x + \epsilon = 0. \tag{44}$$

Four terms, four possible sizes:

- $\epsilon^2 x^3$: depends on both ϵ and x
- x^2: depends only on x
- $2x$: depends only on x
- ϵ: depends only on ϵ

3.2.2 Scenario 1: Large Negative x

Hypothesis: The missing root satisfies $|x| \gg 1$ as $\epsilon \to 0$.

Why we consider this: If x is large, different terms might dominate.

Size estimates for $x \to -\infty$:

- $\epsilon^2 x^3 \sim -\epsilon^2 |x|^3$ (large, negative)
- $x^2 \sim |x|^2$ (large, positive)
- $2x \sim -2|x|$ (large, negative)
- ϵ (small, positive)

Which terms could balance?

Balance Attempt 1: $\epsilon^2 x^3 \sim x^2$ **The balance equation:**

$$\epsilon^2 x^3 \sim x^2 \implies \epsilon^2 x \sim 1 \implies x \sim \frac{1}{\epsilon^2}. \tag{45}$$

Check other terms:

- $2x \sim \frac{2}{\epsilon^2}$: This is $O(1/\epsilon^2)$, same size as x^2
- ϵ: This is $O(\epsilon)$, much smaller

Problem: The terms x^2 and $2x$ are both $O(1/\epsilon^2)$, so they would both be important. But they have the same sign structure for large x, so we need to check more carefully.

For $x = c/\epsilon^2$ with $c > 0$:

- $\epsilon^2 x^3 = \epsilon^2 \cdot \frac{c^3}{\epsilon^6} = \frac{c^3}{\epsilon^4}$
- $x^2 = \frac{c^2}{\epsilon^4}$
- $2x = \frac{2c}{\epsilon^2}$

Observation: The first two terms are $O(1/\epsilon^4)$ while $2x$ is only $O(1/\epsilon^2)$.
But wait: Let's check if they can balance:

$$\frac{c^3}{\epsilon^4} + \frac{c^2}{\epsilon^4} \sim 0 \implies c^3 + c^2 = 0 \implies c^2(c+1) = 0. \tag{46}$$

This gives $c = 0$ (not large) or $c = -1$ (contradicts $c > 0$).
Conclusion: This balance doesn't work cleanly.

Balance Attempt 2: $\epsilon^2 x^3 \sim 2x$ **The balance equation:**

$$\epsilon^2 x^3 \sim 2x \implies \epsilon^2 x^2 \sim 2 \implies x^2 \sim \frac{2}{\epsilon^2} \implies x \sim \pm\frac{\sqrt{2}}{\epsilon}. \tag{47}$$

Check other terms with $x \sim -\sqrt{2}/\epsilon$ (choosing negative):

- $\epsilon^2 x^3 \sim \epsilon^2 \cdot \frac{-2\sqrt{2}}{\epsilon^3} = \frac{-2\sqrt{2}}{\epsilon}$
- $2x \sim \frac{-2\sqrt{2}}{\epsilon}$: Same order! ✓
- $x^2 \sim \frac{2}{\epsilon^2}$: Larger by factor $1/\epsilon$
- ϵ: Much smaller

Problem: The x^2 term is larger than our balanced terms, so it can't be neglected.
Conclusion: This balance doesn't work either.

Balance Attempt 3: $x^2 \sim 2x$ **The balance equation:**

$$x^2 \sim 2x \implies x \sim 2. \tag{48}$$

Why this is wrong: $x = O(1)$ doesn't give us a singular solution - we'd just get the regular solutions we already found.

Balance Attempt 4: $x^2 \sim \epsilon$ **The balance equation:**

$$x^2 \sim \epsilon \implies x \sim \pm\sqrt{\epsilon}. \tag{49}$$

Check other terms with $x \sim \sqrt{\epsilon}$:

- $\epsilon^2 x^3 \sim \epsilon^2 \cdot \epsilon^{3/2} = \epsilon^{7/2}$: Much smaller
- $x^2 \sim \epsilon$: Balanced ✓
- $2x \sim 2\sqrt{\epsilon}$: Larger by factor $1/\sqrt{\epsilon}$
- ϵ: Balanced ✓

Problem: The $2x$ term dominates, so this balance fails.

Balance Attempt 5: $2x \sim \epsilon$ **The balance equation:**

$$2x \sim \epsilon \implies x \sim \frac{\epsilon}{2}. \tag{50}$$

Check other terms:

- $\epsilon^2 x^3 \sim \epsilon^2 \cdot \epsilon^3 = \epsilon^5$: Much smaller
- $x^2 \sim \epsilon^2$: Smaller by factor ϵ
- $2x \sim \epsilon$: Balanced ✓
- ϵ: Balanced ✓

Balancing terms:

$$x^2 + 2x + \epsilon \approx 0 \implies 2x + \epsilon \approx 0 \implies x \approx -\frac{\epsilon}{2}. \tag{51}$$

This is consistent! The subdominant term $x^2 \sim \epsilon^2$ is indeed much smaller.

3.2.3 Constructing The Singular Solution

Our ansatz (motivated by dominant balance):

$$x = -\frac{\epsilon}{2} + x_1\epsilon^2 + x_2\epsilon^3 + \cdots \tag{52}$$

Why this form:

- The leading term is $-\epsilon/2$ from dominant balance
- We don't include an $O(1)$ term because that would change the balance
- The next correction should be $O(\epsilon^2)$ (the size of the neglected x^2 term)

3.2.4 Substituting The Ansatz

Into equation (1):

$$\epsilon^2 \left(-\frac{\epsilon}{2} + x_1\epsilon^2\right)^3 + \left(-\frac{\epsilon}{2} + x_1\epsilon^2\right)^2 + 2\left(-\frac{\epsilon}{2} + x_1\epsilon^2\right) + \epsilon = 0. \tag{53}$$

Cubic term:

$$\epsilon^2 \cdot \left(-\frac{\epsilon}{2}\right)^3 + \cdots = \epsilon^2 \cdot \frac{-\epsilon^3}{8} + \cdots = -\frac{\epsilon^5}{8} + O(\epsilon^6). \tag{54}$$

Why we can ignore: This is $O(\epsilon^5)$, far smaller than ϵ^2.
Quadratic term:

$$\left(-\frac{\epsilon}{2} + x_1\epsilon^2\right)^2 = \frac{\epsilon^2}{4} - 2 \cdot \frac{\epsilon}{2} \cdot x_1\epsilon^2 + x_1^2\epsilon^4 \tag{55}$$

$$= \frac{\epsilon^2}{4} - x_1\epsilon^3 + O(\epsilon^4). \tag{56}$$

Linear term:

$$2\left(-\frac{\epsilon}{2} + x_1\epsilon^2\right) = -\epsilon + 2x_1\epsilon^2. \tag{57}$$

Collecting everything:

$$\frac{\epsilon^2}{4} + 2x_1\epsilon^2 - \epsilon + \epsilon + O(\epsilon^3) = 0. \tag{58}$$

Simplifying:

$$\left(\frac{1}{4} + 2x_1\right)\epsilon^2 + O(\epsilon^3) = 0. \tag{59}$$

At $O(\epsilon^2)$:

$$\frac{1}{4} + 2x_1 = 0 \implies x_1 = -\frac{1}{8}. \tag{60}$$

3.2.5 Third Root: The Result

Our third root:

$$\boxed{x_3(\epsilon) = -\frac{1}{2}\epsilon - \frac{1}{8}\epsilon^2 + O(\epsilon^3)}. \tag{61}$$

Wait - this is identical to $x_1(\epsilon)$!

3.3 Resolution: The Roots Are Distinct But Close

What happened: Both the regular expansion near $x_0 = 0$ and the singular balance $2x \sim \epsilon$ give the same two-term expansion!

Why this makes sense: For $x = O(\epsilon)$, the distinction between "starting from 0" and "being determined by balance" becomes subtle. The root is actually evolving continuously from $x = 0$ as ϵ increases from 0.

The actual situation: We have only found **two distinct roots** to two-term accuracy:

1. $x_1(\epsilon) = -\frac{1}{2}\epsilon - \frac{1}{8}\epsilon^2 + O(\epsilon^3)$
2. $x_2(\epsilon) = -2 + \frac{1}{2}\epsilon - \frac{31}{8}\epsilon^2 + O(\epsilon^3)$

Where is the third root?

3.4 Finding The True Third Root

Reconsider the structure: The equation can be rewritten as:

$$\epsilon^2 x^3 = -x^2 - 2x - \epsilon = -(x^2 + 2x + \epsilon). \tag{62}$$

For very large $|x|$: If $x \to -\infty$, then $\epsilon^2 x^3 \to -\infty$ (large negative), while the RHS $\to +\infty$ (large positive). So there must be a balance.

New attempt: $\epsilon^2 x^3 + x^2 \sim 0$

This gives:

$$x^2(\epsilon^2 x + 1) \sim 0 \implies x \sim -\frac{1}{\epsilon^2}. \tag{63}$$

Check: For $x = -1/\epsilon^2$:

- $\epsilon^2 x^3 = \epsilon^2 \cdot \frac{-1}{\epsilon^6} = -\frac{1}{\epsilon^4}$
- $x^2 = \frac{1}{\epsilon^4}$: Same magnitude! ✓
- $2x = -\frac{2}{\epsilon^2}$: Smaller by factor ϵ^2
- ϵ: Much smaller

The balance works!

3.4.1 Expansion For The Large Root

Ansatz:

$$x = -\frac{1}{\epsilon^2} + \frac{x_1}{\epsilon} + x_0 + x_1'\epsilon + \cdots \tag{64}$$

Why we include $1/\epsilon$ term: The neglected $2x$ term is $O(1/\epsilon^2)$ when $x = O(1/\epsilon^2)$, suggesting corrections at intermediate powers.

Actually, let's be more systematic. Write:

$$x = -\frac{1}{\epsilon^2} + y, \tag{65}$$

where $y = o(1/\epsilon^2)$ as $\epsilon \to 0$.

Substituting:

$$\epsilon^2 \left(-\frac{1}{\epsilon^2} + y\right)^3 + \left(-\frac{1}{\epsilon^2} + y\right)^2 + 2\left(-\frac{1}{\epsilon^2} + y\right) + \epsilon = 0. \tag{66}$$

Expanding the cubic:

$$\left(-\frac{1}{\epsilon^2} + y\right)^3 = -\frac{1}{\epsilon^6} + 3 \cdot \frac{1}{\epsilon^4} \cdot y - 3 \cdot \frac{1}{\epsilon^2} \cdot y^2 + y^3. \tag{67}$$

So:

$$\epsilon^2 x^3 = -\frac{1}{\epsilon^4} + \frac{3y}{\epsilon^2} - 3y^2 + \epsilon^2 y^3. \tag{68}$$

The quadratic:

$$x^2 = \frac{1}{\epsilon^4} - \frac{2y}{\epsilon^2} + y^2. \tag{69}$$

The linear:

$$2x = -\frac{2}{\epsilon^2} + 2y. \tag{70}$$

Full equation:

$$\left[-\frac{1}{\epsilon^4}+\frac{3y}{\epsilon^2}\right]+\left[\frac{1}{\epsilon^4}-\frac{2y}{\epsilon^2}\right]+\left[-\frac{2}{\epsilon^2}+2y\right]+\epsilon+\text{higher order}=0. \tag{71}$$

At $O(1/\epsilon^4)$: $-1+1=0$ ✓
At $O(1/\epsilon^2)$:

$$3y-2y-2=0 \implies y=2. \tag{72}$$

So to leading order:

$$x=-\frac{1}{\epsilon^2}+2+O(\epsilon). \tag{73}$$

For the next term, substitute $y=2+z$ where $z=O(\epsilon)$:

$$\frac{y}{\epsilon^2}-2+2y+\epsilon-3y^2+O(\epsilon^2)=0. \tag{74}$$

This becomes intricate, but the key point is established.

3.4.2 Third Root: Final Form

$$\boxed{x_3(\epsilon)=-\frac{1}{\epsilon^2}+2+O(\epsilon)}. \tag{75}$$

4 Final Answer: All Three Roots

The complete solution to Problem 2:

$$\boxed{x_1(\epsilon)}=\boxed{-\frac{1}{2}\epsilon-\frac{1}{8}\epsilon^2+O(\epsilon^3)} \tag{76}$$

$$\boxed{x_2(\epsilon)}=\boxed{-2+\frac{1}{2}\epsilon-\frac{31}{8}\epsilon^2+O(\epsilon^3)} \tag{77}$$

$$\boxed{x_3(\epsilon)}=\boxed{-\frac{1}{\epsilon^2}+2+O(\epsilon)} \tag{78}$$

Summary:

- Two roots are $O(\epsilon)$ and $O(1)$ respectively - regular solutions
- One root is $O(1/\epsilon^2)$ - a singular solution that escapes to $-\infty$ as $\epsilon\to 0$