Asymptotics Problem 7.1: Complete Pedagogical Solution

WKB Method — Elimination of First Derivative

Problem 1. Consider the ordinary differential equation

$$u''(x) + p(x)u'(x) + q(x)u(x) = 0.$$

Set $u(x) = f(x)y(x)$ and find a function $f(x)$ such that the resulting ODE for $y(x)$ has no term in $y'(x)$.

Solution: Step-by-Step Atomic Breakdown

Step 1: Understanding the Problem

Strategy: *We are given a second-order linear ODE with non-constant coefficients. Our goal is to perform a transformation that simplifies the equation by eliminating the first derivative term. This is a standard technique in asymptotic analysis and is particularly important for:*

- *Connecting boundary layer problems to WKB-type equations (as noted in Lecture Notes §6.3.1, equation (377))*
- *Simplifying ODEs before applying the WKB approximation*
- *Converting equations into a form amenable to asymptotic analysis*

Justification: *Why do we want to eliminate the first derivative term? From the lecture notes §6.3, we learned that the WKB method works most naturally on equations of the form* $\varepsilon^2 y'' + q(x)y = 0$*. If we encounter an equation with a* y' *term, we need to transform it away first. This transformation reveals the underlying oscillatory or exponential structure of solutions.*

Step 2: Setting Up the Transformation

What we do: We substitute $u(x) = f(x)y(x)$ into the original ODE.
Justification: *This substitution is called a* dependent variable transformation*. We're expressing the unknown function* $u(x)$ *as a product of:*

- *A known (to-be-determined) function* $f(x)$ *that will absorb the unwanted* y' *term*
- *A new unknown function* $y(x)$ *that will satisfy a simpler ODE*

This is analogous to an integrating factor in first-order ODEs, but here we seek to change the structure of a second-order equation.

Step 3: Computing the Required Derivatives

What we need: To substitute $u = fy$ into the ODE, we need u' and u''.

Step 3a: First Derivative

Technique: *Apply the product rule: If $u = f \cdot y$, then*

$$u' = \frac{d}{dx}(f \cdot y) = f'y + fy'.$$

This is straightforward differentiation. We obtain:

$$u'(x) = f'(x)y(x) + f(x)y'(x).$$

Step 3b: Second Derivative

Technique: *We differentiate $u' = f'y + fy'$ using the product rule on each term:*

$$\begin{aligned} u'' &= \frac{d}{dx}(f'y + fy') \\ &= \frac{d}{dx}(f'y) + \frac{d}{dx}(fy') \\ &= (f''y + f'y') + (f'y' + fy'') \\ &= f''y + 2f'y' + fy''. \end{aligned}$$

Therefore:

$$u''(x) = f''(x)y(x) + 2f'(x)y'(x) + f(x)y''(x).$$

Justification: *Notice the structure: The second derivative u'' produces three terms involving y, y', and y''. The term $2f'y'$ is particularly important — this is where we'll have leverage to eliminate the first derivative from the transformed equation.*

Step 4: Substituting into the Original ODE

What we do: Substitute u, u', and u'' into $u'' + pu' + qu = 0$.

$$\begin{aligned} u'' + p(x)u' + q(x)u = 0 \\ \left[f''y + 2f'y' + fy''\right] + p(x)\left[f'y + fy'\right] + q(x)\left[fy\right] = 0. \end{aligned}$$

Expanding the middle term:

$$f''y + 2f'y' + fy'' + pf'y + pfy' + qfy = 0.$$

Step 5: Collecting Terms by Derivative Order

Technique: *Group terms according to which derivative of y they contain: y'', y', or y. This reveals the structure of the transformed ODE.*
Grouping by y and its derivatives:

$$\begin{aligned} \text{Coefficient of } y'' : &\quad f \\ \text{Coefficient of } y' : &\quad 2f' + pf \\ \text{Coefficient of } y : &\quad f'' + pf' + qf \end{aligned}$$

Therefore, the equation becomes:

$$f \cdot y'' + (2f' + pf) \cdot y' + (f'' + pf' + qf) \cdot y = 0.$$

Step 6: Converting to Standard Form

What we do: Divide through by $f(x)$ (assuming $f(x) \neq 0$) to obtain:

$$y'' + \left(\frac{2f'}{f} + p\right) y' + \left(\frac{f''}{f} + p\frac{f'}{f} + q\right) y = 0.$$

Justification: *Dividing by f gives us the standard form of a second-order ODE with the coefficient of y'' equal to 1. This makes it easier to see what condition on f will eliminate the y' term.*

Step 7: Imposing the Condition to Eliminate y'

The Key Step: We want the coefficient of y' to vanish.
Strategy: *Set the coefficient of y' equal to zero:*

$$\frac{2f'}{f} + p = 0.$$

This is a first-order ODE for $f(x)$ that we can solve.
Justification: *Why does this work? If the coefficient of y' is zero, then the transformed ODE contains only y'' and y terms, which is exactly what we want. This is a necessary and sufficient condition for eliminating the first derivative.*

Step 8: Solving for $f(x)$

What we need to solve:

$$\frac{2f'}{f} + p(x) = 0 \quad \Longrightarrow \quad \frac{2f'}{f} = -p(x).$$

Technique: *This is a separable first-order ODE. We can write:*

$$\frac{f'}{f} = -\frac{p(x)}{2}.$$

Recall that $\frac{d}{dx} \ln|f| = \frac{f'}{f}$, so:

$$\frac{d}{dx} \ln|f| = -\frac{p(x)}{2}.$$

Integrating both sides with respect to x:

$$\ln|f(x)| = -\frac{1}{2}\int p(x)\, dx + C,$$

where C is a constant of integration.
Exponentiating both sides:

$$|f(x)| = \exp\left(-\frac{1}{2}\int p(x)\, dx + C\right) = e^C \cdot \exp\left(-\frac{1}{2}\int p(x)\, dx\right).$$

Since e^C is just a positive constant, and we can absorb it (we only need one particular function f, not the general solution with arbitrary constants), we write:

$$f(x) = \exp\left(-\frac{1}{2}\int p(x)\, dx\right).$$

Justification: *We drop the absolute value and the arbitrary constant because:*

- *We only need* one *function $f(x)$ that eliminates the y' term, not a family of solutions*
- *The constant would factor out of the entire ODE and can be absorbed into $y(x)$*
- *The exponential is always positive, so $|f| = f$*

Step 9: Verifying the Result

Check: Let's verify that with $f(x) = \exp\left(-\frac{1}{2}\int p(x)\,dx\right)$, we have $2f'/f + p = 0$.
Technique: *Differentiate f using the chain rule:*

$$f'(x) = \exp\left(-\frac{1}{2}\int p(x)\,dx\right) \cdot \left(-\frac{1}{2}p(x)\right) = -\frac{p(x)}{2} \cdot f(x).$$

Therefore:

$$\frac{f'}{f} = -\frac{p(x)}{2} \implies \frac{2f'}{f} = -p(x) \implies \frac{2f'}{f} + p(x) = 0. \quad \checkmark$$

Step 10: The Transformed ODE for $y(x)$

With the y' term eliminated, the ODE for $y(x)$ becomes:

$$y''(x) + \left(\frac{f''}{f} + p\frac{f'}{f} + q\right) y(x) = 0.$$

Technique: *Let's simplify the coefficient of y. We know:*

- $\frac{f'}{f} = -\frac{p}{2}$
- $\frac{f''}{f}$ *can be computed by differentiating f'/f*

Computing f''/f: Differentiate $f' = -\frac{p}{2}f$:

$$\begin{aligned} f'' &= -\frac{p'}{2}f - \frac{p}{2}f' \\ &= -\frac{p'}{2}f - \frac{p}{2}\left(-\frac{p}{2}f\right) \\ &= -\frac{p'}{2}f + \frac{p^2}{4}f. \end{aligned}$$

Therefore:

$$\frac{f''}{f} = -\frac{p'}{2} + \frac{p^2}{4}.$$

The coefficient of y becomes:

$$\begin{aligned} \frac{f''}{f} + p\frac{f'}{f} + q &= \left(-\frac{p'}{2} + \frac{p^2}{4}\right) + p\left(-\frac{p}{2}\right) + q \\ &= -\frac{p'}{2} + \frac{p^2}{4} - \frac{p^2}{2} + q \\ &= -\frac{p'}{2} - \frac{p^2}{4} + q. \end{aligned}$$

Final Answer

The function $f(x)$ that eliminates the first derivative term is:

$$\boxed{f(x) = \exp\left(-\frac{1}{2}\int p(x)\,dx\right)}$$

The transformed ODE for $y(x)$ is:

$$\boxed{y''(x) + \left(q(x) - \frac{p'(x)}{2} - \frac{p(x)^2}{4}\right) y(x) = 0}$$

Connection to the Lecture Notes

Justification: *This result appears in the lecture notes §6.3.1, equation (377). There, the transformation* $y(x) = \exp\left(-\frac{1}{2\varepsilon}\int p(s)\,ds\right) z(x)$ *is used to eliminate the first derivative from:*

$$\varepsilon y'' + p(x)y' + q(x)y = 0,$$

yielding:

$$\varepsilon^2 z'' + \left(-\frac{p^2}{4} - \frac{p'}{2} + q\right) z = 0.$$

This confirms that our transformation is correct and shows its importance in connecting boundary layer problems to WKB-type equations.

Physical Interpretation

Justification: *The transformation* $u = fy$ *with* $f = \exp\left(-\frac{1}{2}\int p\,dx\right)$ *has a physical meaning:*

- *The function* $f(x)$ *acts as an amplitude modulation factor*
- *It absorbs the damping or growth encoded in the* $p(x)u'$ *term*
- *What remains in* $y(x)$ *is the purely oscillatory or exponential behavior determined by the effective potential* $q - p'/2 - p^2/4$
- *This is why the transformation is essential before applying WKB: it separates amplitude effects from phase effects*