# Exercise Sheet 1: Question 2
# Finite Time Blow Up
# Methods of Applied Mathematics [SEMT30006]

Complete Solutions with XYZ Methodology

## Problem Statement

Solve the initial value problem

$$\dot{x} = ax^2 \tag{1}$$

with $x(0) = x_0$.

(a) Does this have solutions, are they unique, and where do they exist?

(b) Solve the equations for $x(t)$ in terms of $a$ and $x_0$.

(c) Despite your answer to (a), show that something 'goes wrong' at time $t = \frac{1}{ax_0}$, and describe what happens there. This is known as finite time blow up.

(d) Sketch a solution for $a = 1$ and $x_0 = 0.2$.

**CONTEXT FROM COURSE:** This problem explores the boundaries of existence and uniqueness theorems (lecture notes pages 14-18). We will see that even though solutions exist and are unique *locally*, they may not exist for all time $t \in \mathbb{R}$. This is a fundamental phenomenon in nonlinear ODEs called **finite time blow up**.

# 1 Problem 2(a): Existence, Uniqueness, and Domain of Solutions

### Step 1: State the Relevant Theorems

- **STAGE X (What we need):** For the initial value problem $\dot{x} = f(x, t)$ with $x(t_0) = x_0$, we need to check conditions for:
  1. **Existence:** Does a solution exist?
  2. **Uniqueness:** Is the solution unique?
  3. **Domain:** For what values of $t$ does the solution exist?
- **STAGE Y (Relevant theorems):** From lecture notes (pages 14-18), the key result is:

  **Picard-Lindelöf Theorem (Existence and Uniqueness):** If $f(x, t)$ is:
  - Continuous in both $x$ and $t$ in some region $R$ containing $(x_0, t_0)$
  - Lipschitz continuous in $x$ (i.e., $|f(x_1, t) - f(x_2, t)| \leq L|x_1 - x_2|$ for some constant $L$)

  then there exists a unique solution in some interval $|t - t_0| < \delta$.

  The theorem guarantees **local** existence and uniqueness but not necessarily **global** (for all $t$).
- **STAGE Z (Our specific problem):** For $\dot{x} = ax^2$, we have $f(x, t) = ax^2$. We need to check:
  1. Continuity of $f$
  2. Lipschitz continuity in $x$

### Step 2: Check Continuity

- **STAGE X (Examining $f(x, t) = ax^2$):** The function $f(x, t) = ax^2$ is:
  - A polynomial in $x$
  - Independent of $t$ (autonomous system)
  - Continuous everywhere in $\mathbb{R} \times \mathbb{R}$
- **STAGE Y (Why this matters):** Polynomial functions are continuous everywhere. Since $f$ does not depend on $t$, continuity in $t$ is trivial. Therefore, the first condition of the Picard-Lindelöf theorem is satisfied.
- **STAGE Z (Conclusion):** ✓ **Continuity condition satisfied** for all $(x, t) \in \mathbb{R} \times \mathbb{R}$.

### Step 3: Check Lipschitz Continuity

- **STAGE X (Lipschitz condition definition):** A function $f(x)$ is Lipschitz continuous on an interval $I$ if there exists a constant $L \geq 0$ such that:

$$|f(x_1) - f(x_2)| \leq L|x_1 - x_2| \quad \forall x_1, x_2 \in I \tag{2}$$

- **STAGE Y (Testing $f(x) = ax^2$):** Consider the difference:

$$\begin{aligned} |f(x_1) - f(x_2)| &= |ax_1^2 - ax_2^2| && (3) \\ &= |a||x_1^2 - x_2^2| && (4) \\ &= |a||x_1 + x_2||x_1 - x_2| && (5) \end{aligned}$$

For Lipschitz continuity, we need:

$$|a||x_1 + x_2||x_1 - x_2| \leq L|x_1 - x_2| \tag{6}$$

This simplifies to:

$$|a||x_1 + x_2| \leq L \tag{7}$$

**Problem:** If $x_1$ or $x_2$ can be arbitrarily large, then $|x_1 + x_2|$ can be arbitrarily large, so no single constant $L$ works for all $x_1, x_2 \in \mathbb{R}$.

- **STAGE Z (Local vs. Global Lipschitz):**
  - $f(x) = ax^2$ is **NOT globally Lipschitz** on $\mathbb{R}$
  - $f(x) = ax^2$ **IS locally Lipschitz** on any bounded interval $[x_0 - \delta, x_0 + \delta]$

  On a bounded interval where $|x| \leq M$, we can take $L = 2|a|M$, and Lipschitz continuity holds.

### Step 4: Apply Existence and Uniqueness Theory

- **STAGE X (What the theorem tells us):** Since $f(x) = ax^2$ is:
  - Continuous everywhere
  - Locally Lipschitz continuous near any point $x_0$

  The Picard-Lindelöf theorem guarantees:

  $$\text{A unique solution exists } \textbf{locally} \text{ near } t = 0 \tag{8}$$

- **STAGE Y (What "locally" means):** There exists some time interval $(-\delta, \delta)$ around $t = 0$ on which a unique solution $x(t)$ exists passing through $(0, x_0)$.

  However, the theorem does **NOT** guarantee:
  - The solution exists for all $t \geq 0$
  - The solution exists for all $t \in \mathbb{R}$

- **STAGE Z (The question of global existence):** Whether the solution exists for all time depends on the behavior of the solution itself. We'll need to solve explicitly to determine this.

### Step 5: Additional Analysis Using the Derivative

- **STAGE X (Alternative Lipschitz check):** Another way to check local Lipschitz continuity is via the derivative. If $\frac{\partial f}{\partial x}$ is bounded on an interval, then $f$ is Lipschitz on that interval.

- **STAGE Y (Computing the derivative):**

  $$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(ax^2) = 2ax \tag{9}$$

  On any bounded interval $|x| \leq M$:

  $$\left|\frac{\partial f}{\partial x}\right| = |2ax| \leq 2|a|M \tag{10}$$

  So the Lipschitz constant is $L = 2|a|M$ on $[-M, M]$.

- **STAGE Z (Confirms local Lipschitz):** This confirms that $f(x) = ax^2$ is locally Lipschitz, supporting our conclusion from Step 3.

## Answer to Problem 2(a)

### Solution to 2(a)

**Existence:** Yes, a solution exists locally near $t = 0$.
**Uniqueness:** Yes, the solution is unique locally near $t = 0$.
**Domain of existence:**

- The Picard-Lindelöf theorem guarantees existence and uniqueness in some interval $|t| < \delta$ for some $\delta > 0$
- The solution exists locally but NOT necessarily globally (for all $t \in \mathbb{R}$)
- The actual domain depends on the parameters $a$ and $x_0$, which we'll determine by solving explicitly

**Mathematical justification:**

- $f(x, t) = ax^2$ is continuous everywhere ✓
- $f(x, t) = ax^2$ is locally Lipschitz continuous near any $x_0$ ✓
- Therefore, by Picard-Lindelöf: local existence and uniqueness are guaranteed

**Key insight:** Local Lipschitz (not global) means solutions exist and are unique near the initial condition, but may "escape to infinity" in finite time.

**IMPORTANT:** The lack of global Lipschitz continuity is a warning sign that solutions might not exist for all time. This is exactly what we'll see in parts (b) and (c).

## 2 Problem 2(b): Explicit Solution

### Step 1: Identify the Type of ODE

- **STAGE X (What we have):** The ODE $\dot{x} = ax^2$ with $x(0) = x_0$ is:
  - First-order
  - Autonomous (no explicit $t$ dependence)
  - Separable (can separate $x$ and $t$ terms)
  - Nonlinear (quadratic in $x$)
- **STAGE Y (Why separation of variables works):** We can rewrite as:

$$\frac{dx}{dt} = ax^2 \quad \Rightarrow \quad \frac{dx}{x^2} = a\,dt \tag{11}$$

  This separates the variables: all $x$ terms on the left, all $t$ terms on the right.

- **STAGE Z (Strategy):** Integrate both sides and solve for $x(t)$ using the initial condition.

### Step 2: Separate Variables and Integrate

- **STAGE X (Separation):** Starting from $\dot{x} = ax^2$, separate:

$$\frac{dx}{x^2} = a\,dt \tag{12}$$

  **IMPORTANT:** This is valid only when $x \neq 0$. We'll handle the special case $x = 0$ separately.

- **STAGE Y (Integration):** Integrate both sides:

$$\int \frac{dx}{x^2} = \int a\,dt \tag{13}$$

  The left side:

$$\int \frac{dx}{x^2} = \int x^{-2}\,dx = \frac{x^{-1}}{-1} = -\frac{1}{x} \tag{14}$$

  The right side:

$$\int a\,dt = at + C \tag{15}$$

- **STAGE Z (Combining):**

$$-\frac{1}{x} = at + C \tag{16}$$

  where $C$ is the constant of integration to be determined from initial conditions.

### Step 3: Apply Initial Condition

- **STAGE X (Using $x(0) = x_0$):** At $t = 0$:

$$-\frac{1}{x(0)} = a(0) + C \quad \Rightarrow \quad -\frac{1}{x_0} = C \tag{17}$$

- **STAGE Y (Substituting back):**

$$-\frac{1}{x} = at - \frac{1}{x_0} \tag{18}$$

- **STAGE Z (Rearranging):**

$$-\frac{1}{x} = at - \frac{1}{x_0} = \frac{ax_0t - 1}{x_0} \tag{19}$$

Therefore:

$$\frac{1}{x} = \frac{1 - ax_0t}{x_0} \tag{20}$$

### Step 4: Solve for $x(t)$

- **STAGE X (Inverting):** From $\frac{1}{x} = \frac{1-ax_0t}{x_0}$:

$$x(t) = \frac{x_0}{1 - ax_0t} \tag{21}$$

- **STAGE Y (Verification):** Let's verify this satisfies the ODE. Compute $\dot{x}$:

$$x(t) = \frac{x_0}{1 - ax_0t} = x_0(1 - ax_0t)^{-1} \tag{22}$$
$$\dot{x} = x_0 \cdot (-1)(1 - ax_0t)^{-2} \cdot (-ax_0) \tag{23}$$
$$= \frac{ax_0^2}{(1 - ax_0t)^2} \tag{24}$$

Check if $\dot{x} = ax^2$:

$$ax^2 = a\left(\frac{x_0}{1 - ax_0t}\right)^2 = \frac{ax_0^2}{(1 - ax_0t)^2} = \dot{x} \quad \checkmark \tag{25}$$

Check initial condition:

$$x(0) = \frac{x_0}{1 - 0} = x_0 \quad \checkmark \tag{26}$$

- **STAGE Z (Solution confirmed):** The solution $x(t) = \frac{x_0}{1-ax_0t}$ satisfies both the ODE and initial condition.

### Step 5: Handle Special Cases

- **STAGE X (Case 1: $x_0 = 0$):** If $x_0 = 0$, then the initial condition is $x(0) = 0$.

  The ODE becomes $\dot{x} = ax^2$. The constant solution $x(t) = 0$ satisfies:

  - $\dot{x} = 0$
  - $ax^2 = a \cdot 0^2 = 0$
  - So $\dot{x} = ax^2$

  By uniqueness, $x(t) \equiv 0$ is the unique solution when $x_0 = 0$.

- **STAGE Y (Case 2: $a = 0$):** If $a = 0$, the ODE becomes $\dot{x} = 0$, which means $x$ is constant:

$$x(t) = x_0 \quad \text{for all } t \tag{27}$$

  This agrees with our formula: $x(t) = \frac{x_0}{1-0} = x_0$

- **STAGE Z (General case):** For $a \neq 0$ and $x_0 \neq 0$, the solution is:

$$x(t) = \frac{x_0}{1 - ax_0t} \tag{28}$$

### Answer to Problem 2(b)

**Solution to 2(b)**

**Explicit solution:**

$$\boxed{x(t) = \frac{x_0}{1 - ax_0 t}} \tag{29}$$

**Valid for:** $t \neq \frac{1}{ax_0}$ (assuming $ax_0 \neq 0$)
**Special cases:**

- If $x_0 = 0$: $x(t) \equiv 0$ for all $t$
- If $a = 0$: $x(t) = x_0$ for all $t$
- If $ax_0 = 0$: solution exists for all $t \in \mathbb{R}$

**Derivation method:** Separation of variables

$$\frac{dx}{x^2} = a\,dt \quad \Rightarrow \quad -\frac{1}{x} = at + C \quad \Rightarrow \quad x = \frac{x_0}{1 - ax_0 t} \tag{30}$$

**WARNING:** Notice the denominator $1 - ax_0 t$. This becomes zero at $t = \frac{1}{ax_0}$, suggesting something unusual happens at that time!

## 3 Problem 2(c): Finite Time Blow Up

### Step 1: Identify the Critical Time

- **STAGE X (What we have):** The solution is $x(t) = \frac{x_0}{1-ax_0t}$.
  The denominator vanishes when:
  $$1 - ax_0t = 0 \quad \Rightarrow \quad t^* = \frac{1}{ax_0} \tag{31}$$
  (assuming $ax_0 \neq 0$)
- **STAGE Y (Why this is problematic):** At $t = t^* = \frac{1}{ax_0}$:
  $$x(t^*) = \frac{x_0}{1 - ax_0 \cdot \frac{1}{ax_0}} = \frac{x_0}{1-1} = \frac{x_0}{0} \tag{32}$$
  Division by zero! The solution becomes undefined.
- **STAGE Z (Different regimes):** We need to consider different signs of $ax_0$:
  - If $ax_0 > 0$: $t^* = \frac{1}{ax_0} > 0$ (blow up in forward time)
  - If $ax_0 < 0$: $t^* = \frac{1}{ax_0} < 0$ (blow up in backward time)

### Step 2: Analyze Behavior as $t \to t^*$ (Case: $ax_0 > 0$)

- **STAGE X (Approaching from the left, $t \to t^*-$):** Consider $t$ slightly less than $t^* = \frac{1}{ax_0}$. Write $t = t^* - \epsilon$ where $\epsilon > 0$ is small:
  $$\begin{aligned} 1 - ax_0t &= 1 - ax_0(t^* - \epsilon) && (33)\\ &= 1 - ax_0t^* + ax_0\epsilon && (34)\\ &= 1 - 1 + ax_0\epsilon && (35)\\ &= ax_0\epsilon && (36)\end{aligned}$$
- **STAGE Y (Behavior of $x(t)$):**
  $$x(t) = \frac{x_0}{ax_0\epsilon} = \frac{1}{a\epsilon} \to +\infty \quad \text{as } \epsilon \to 0^+ \tag{37}$$
  Since $ax_0 > 0$, we have $a$ and $x_0$ have the same sign.
  If $x_0 > 0$ and $a > 0$:
  $$x(t) \to +\infty \quad \text{as } t \to t^*- \tag{38}$$
- **STAGE Z (Blow up from below):** The solution grows without bound as $t$ approaches $t^*$ from below. We say the solution **blows up** at $t = t^*$.

### Step 3: Analyze Behavior for $t > t^*$

- **STAGE X (What happens after $t^*$?):** For $t > t^* = \frac{1}{ax_0}$ (with $ax_0 > 0$):
  $$1 - ax_0t < 0 \tag{39}$$
  So the denominator is negative.

- **STAGE Y (Sign of $x(t)$):** If $x_0 > 0$ and $a > 0$, then for $t > t^*$:

$$x(t) = \frac{x_0}{1 - ax_0 t} = \frac{\text{positive}}{\text{negative}} < 0 \tag{40}$$

  This is problematic because:

  1. We started with $x(0) = x_0 > 0$
  2. Solutions to autonomous ODEs cannot jump discontinuously
  3. Yet the formula suggests $x$ becomes negative after $t^*$

- **STAGE Z (Resolution - solution doesn't exist for $t > t^*$):** The resolution is that **the solution does not exist for $t \geq t^*$.**

  The formula $x(t) = \frac{x_0}{1-ax_0t}$ is only valid for $t < t^*$ when $ax_0 > 0$.

  For $t \geq t^*$, there is no solution to the initial value problem that continues from $x(0) = x_0$.

### Step 4: Verify with the ODE Itself

- **STAGE X (Examining $\dot{x} = ax^2$):** As $t \to t^*-$ with $x_0 > 0, a > 0$:

$$x(t) \to +\infty \tag{41}$$

  Therefore:

$$\dot{x}(t) = ax^2 \to +\infty \tag{42}$$

- **STAGE Y (Physical interpretation):** The rate of change $\dot{x}$ grows without bound as $x$ grows. This creates a positive feedback loop:

  - Larger $x \Rightarrow$ larger $\dot{x}$
  - Larger $\dot{x} \Rightarrow x$ increases faster
  - This creates exponentially accelerating growth

  The solution "races to infinity" in finite time.

- **STAGE Z (Why finite time?):** Even though $\dot{x} \to \infty$, the total time to reach infinity is finite:

$$t^* - 0 = \frac{1}{ax_0} \tag{43}$$

  This is **finite time blow up**.

### Step 5: Compute the Blow-Up Time Explicitly

- **STAGE X (Using the solution formula):** From $x(t) = \frac{x_0}{1-ax_0t}$, the solution blows up when the denominator vanishes:

$$1 - ax_0 t^* = 0 \quad \Rightarrow \quad \boxed{t^* = \frac{1}{ax_0}} \tag{44}$$

- **STAGE Y (Different parameter regimes):**

  - If $a > 0, x_0 > 0$: $t^* > 0$, blow up in forward time
  - If $a < 0, x_0 < 0$: $t^* > 0$, blow up in forward time
  - If $a > 0, x_0 < 0$: $t^* < 0$, blow up in backward time (solution exists for all $t > t^*$)

- If $a < 0, x_0 > 0$: $t^* < 0$, blow up in backward time (solution exists for all $t > t^*$)

- **STAGE Z (Domain of solution):** The solution $x(t) = \frac{x_0}{1-ax_0t}$ exists on:

$$\begin{cases} t < \frac{1}{ax_0} & \text{if } ax_0 > 0 \\ t > \frac{1}{ax_0} & \text{if } ax_0 < 0 \\ t \in \mathbb{R} & \text{if } ax_0 = 0 \end{cases} \tag{45}$$

### Step 6: Alternative Derivation Using Integration

- **STAGE X (Integral formula):** We can compute the blow-up time directly from the ODE. From $\dot{x} = ax^2$:

$$dt = \frac{dx}{ax^2} \tag{46}$$

  Integrate from $t = 0$ to $t = t^*$ as $x$ goes from $x_0$ to $\infty$:

$$t^* = \int_0^{t^*} dt = \int_{x_0}^{\infty} \frac{dx}{ax^2} \tag{47}$$

- **STAGE Y (Evaluating the integral):**

$$t^* = \int_{x_0}^{\infty} \frac{dx}{ax^2} \tag{48}$$

$$= \frac{1}{a} \int_{x_0}^{\infty} x^{-2}\, dx \tag{49}$$

$$= \frac{1}{a} \left[ -\frac{1}{x} \right]_{x_0}^{\infty} \tag{50}$$

$$= \frac{1}{a} \left( 0 - \left( -\frac{1}{x_0} \right) \right) \tag{51}$$

$$= \frac{1}{ax_0} \tag{52}$$

- **STAGE Z (Confirms our result):** This integral calculation confirms $t^* = \frac{1}{ax_0}$

### Answer to Problem 2(c)

**Solution to 2(c): Finite Time Blow Up**

**What goes wrong:** At time $t^* = \frac{1}{ax_0}$ (assuming $ax_0 > 0$):

$$\boxed{x(t^*) = \frac{x_0}{1 - ax_0 t^*} = \frac{x_0}{0} \to \infty} \tag{53}$$

**Description:**

- The solution **blows up to infinity** in finite time
- As $t \to t^*-$: $x(t) \to +\infty$ (for $ax_0 > 0$)
- For $t \geq t^*$: the solution **does not exist**
- The domain of existence is $t \in [0, t^*)$ (open interval)

**Why this happens:**

1. The ODE $\dot{x} = ax^2$ has positive feedback: larger $x \Rightarrow$ larger $\dot{x}$
2. This causes exponentially accelerating growth
3. The solution reaches infinity in finite time $t^* = \frac{1}{ax_0}$

**Finite time blow up:** Even though part (a) guaranteed local existence, the solution only exists on $[0, t^*)$, not for all $t \geq 0$. This is called **finite time blow up** or **blow up in finite time**.
**Mathematical summary:**

$$\lim_{t \to t^*-} x(t) = +\infty, \quad \text{where } t^* = \frac{1}{ax_0} < \infty \tag{54}$$

**KEY INSIGHT:** Local existence $\neq$ global existence. The Picard-Lindelöf theorem only guarantees solutions exist near $t = 0$, not for all time. Nonlinear ODEs can exhibit finite time blow up.

### Connection to Course Material

- **STAGE X (Lecture notes pages 14-18):** The existence and uniqueness theorem (Picard-Lindelöf) gives **local** existence. For **global** existence (for all $t$), additional conditions are needed.
- **STAGE Y (Why global Lipschitz matters):** Functions that are only locally Lipschitz (like $f(x) = ax^2$) can have solutions that escape to infinity. Globally Lipschitz functions (like $f(x) = ax$) have solutions that exist for all time.
- **STAGE Z (Physical interpretation):** Finite time blow up appears in many applications:
  - Population models with super-exponential growth
  - Chemical reactions with autocatalysis
  - Gravitational collapse in astrophysics
  - Thermal runaway in engineering

  It represents a physical catastrophe or regime change where the model breaks down.

## 4 Problem 2(d): Sketch Solution for $a = 1$, $x_0 = 0.2$

### Step 1: Determine Solution Parameters

- **STAGE X (Given values):**
  - $a = 1$
  - $x_0 = 0.2 = \frac{1}{5}$
- **STAGE Y (Solution formula):**

$$x(t) = \frac{x_0}{1 - ax_0 t} = \frac{0.2}{1 - (1)(0.2)t} = \frac{0.2}{1 - 0.2t} \tag{55}$$

  Simplifying:

$$x(t) = \frac{0.2}{1 - 0.2t} = \frac{1}{5 - t} \tag{56}$$

- **STAGE Z (Blow-up time):**

$$t^* = \frac{1}{ax_0} = \frac{1}{(1)(0.2)} = \frac{1}{0.2} = 5 \tag{57}$$

  The solution blows up at $t^* = 5$.

### Step 2: Compute Key Values

- **STAGE X (Creating a table):** Let's compute $x(t)$ at several times:

| $t$ | $1 - 0.2t$ | $x(t) = \frac{0.2}{1-0.2t}$ | $\dot{x}(t) = x^2$ |
|---|---|---|---|
| 0 | 1 | 0.2 | 0.04 |
| 1 | 0.8 | 0.25 | 0.0625 |
| 2 | 0.6 | 0.333... | 0.111... |
| 3 | 0.4 | 0.5 | 0.25 |
| 4 | 0.2 | 1.0 | 1.0 |
| 4.5 | 0.1 | 2.0 | 4.0 |
| 4.8 | 0.04 | 5.0 | 25.0 |
| 4.9 | 0.02 | 10.0 | 100.0 |
| 4.99 | 0.002 | 100.0 | 10000.0 |
| 5 | 0 | $\infty$ | $\infty$ |

- **STAGE Y (Observations):**
  - $x(t)$ increases slowly at first
  - Growth accelerates as $t \to 5$
  - Both $x$ and $\dot{x}$ approach infinity as $t \to 5^-$
  - The closer to $t = 5$, the steeper the curve
- **STAGE Z (Behavior summary):**
  - Domain: $t \in [0, 5)$
  - Initial value: $x(0) = 0.2$
  - Asymptote: vertical line at $t = 5$
  - Monotonicity: strictly increasing
  - Concavity: concave up (since $\ddot{x} = 2ax\dot{x} = 2ax \cdot ax^2 = 2a^2x^3 > 0$)

### Step 3: Analyze Derivative Behavior

- **STAGE X (First derivative):**

$$\dot{x} = ax^2 = x^2 = \left(\frac{1}{5-t}\right)^2 = \frac{1}{(5-t)^2} \tag{58}$$

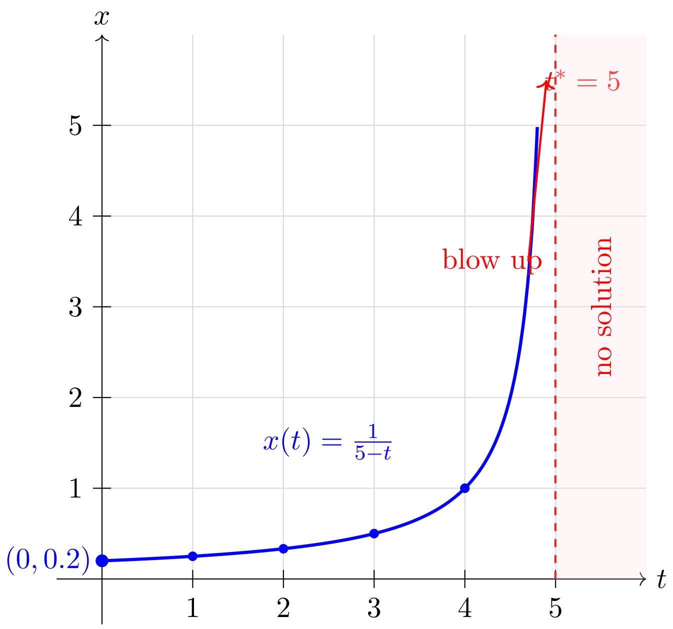
- **STAGE Y (Second derivative):**

$$\ddot{x} = 2ax\dot{x} = 2x^3 = 2\left(\frac{1}{5-t}\right)^3 = \frac{2}{(5-t)^3} \tag{59}$$

  Since $\ddot{x} > 0$ for all $t < 5$, the curve is **concave up** everywhere.

- **STAGE Z (Curvature increases):** As $t \to 5^-$, both $\dot{x} \to \infty$ and $\ddot{x} \to \infty$, so the curve becomes increasingly steep and curved.

### Step 4: Sketch the Solution




### Step 5: Describe the Sketch Features

- **STAGE X (Key features shown):**

  1. **Initial condition:** Point at $(0, 0.2)$
  2. **Solution curve:** Blue curve increasing from $(0, 0.2)$ toward infinity
  3. **Vertical asymptote:** Red dashed line at $t = 5$
  4. **Blow-up:** Curve approaches infinity as $t \to 5^-$
  5. **Non-existence region:** Shaded area for $t \geq 5$ where solution doesn't exist

- **STAGE Y (Mathematical properties):**

  - Strictly increasing: $\dot{x} = x^2 > 0$ for all $t < 5$
  - Concave up: $\ddot{x} = 2x^3 > 0$ for all $t < 5$
  - Smooth: infinitely differentiable on $[0, 5)$

- Asymptotic: $\lim_{t\to 5^-} x(t) = +\infty$

- **STAGE Z (Physical interpretation):** This could represent:
  - Population growing with quadratic birth rate
  - Temperature in thermal runaway
  - Concentration in autocatalytic reaction

  The model predicts catastrophic growth (blow up) at $t = 5$ time units.

### Answer to Problem 2(d)

**Solution to 2(d)**

**Parameters:**

- $a = 1$, $x_0 = 0.2$
- Solution: $x(t) = \frac{0.2}{1-0.2t} = \frac{1}{5-t}$
- Blow-up time: $t^* = 5$
- Domain: $t \in [0, 5)$

**Sketch features:**

- Starts at $(0, 0.2)$
- Increases gradually at first, then more rapidly
- Becomes vertical as $t \to 5^-$
- Has vertical asymptote at $t = 5$ (red dashed line)
- No solution exists for $t \geq 5$
- Curve is smooth, increasing, and concave up on $[0, 5)$

**Sample values:**

- $x(0) = 0.2$
- $x(1) = 0.25$
- $x(2) \approx 0.33$
- $x(3) = 0.5$
- $x(4) = 1.0$
- $x(4.9) = 10$
- $x(5^-) \to \infty$

## Summary and Key Insights

### Complete Analysis of $\dot{x} = ax^2$

1. **Existence and Uniqueness (Local):**
   - Solution exists and is unique near $t = 0$
   - Guaranteed by Picard-Lindelöf theorem (local Lipschitz continuity)

2. **Explicit Solution:**
$$x(t) = \frac{x_0}{1 - ax_0 t} \tag{60}$$

3. **Domain of Existence:**
$$\begin{cases} t \in [0, \frac{1}{ax_0}) & \text{if } ax_0 > 0 \text{ (blow up in forward time)} \\ t \in (\frac{1}{ax_0}, \infty) & \text{if } ax_0 < 0 \text{ (blow up in backward time)} \\ t \in \mathbb{R} & \text{if } ax_0 = 0 \text{ (global existence)} \end{cases} \tag{61}$$

4. **Finite Time Blow Up:**
   - When $ax_0 > 0$: solution blows up at $t^* = \frac{1}{ax_0}$
   - $\lim_{t \to t^*-} x(t) = +\infty$
   - Solution does not exist for $t \geq t^*$

### Comparison: Local vs. Global Properties

| Property | Local | Global |
|---|---|---|
| Existence | Yes (guaranteed) | No (finite time blow up) |
| Uniqueness | Yes (guaranteed) | N/A (doesn't exist globally) |
| Lipschitz continuity | Yes (locally) | No (globally) |
| Domain | Some $[0, \delta)$ | Only $[0, t^*)$ where $t^* < \infty$ |

### Physical Examples of Finite Time Blow Up

1. **Population dynamics:** Super-exponential growth with $\dot{N} = rN^2$ (e.g., bacteria with perfect conditions and quadratic reproduction)

2. **Thermal runaway:** Temperature $T$ with $\dot{T} = \alpha T^2$ where reaction rate grows quadratically

3. **Financial bubbles:** Asset price with feedback: higher price attracts more buyers

4. **Social dynamics:** Viral spreading with $\dot{I} = \beta I^2$ (quadratic infection rate)

### Key Takeaways from Course Perspective

**MAIN LESSONS:**

1. **Local $\neq$ Global:** Existence and uniqueness theorems typically give local results. Global existence requires additional analysis.

2. **Lipschitz matters:** Lack of global Lipschitz continuity is a warning sign for potential finite time blow up.

3. **Nonlinearity creates complexity:** Linear ODEs ($\dot{x} = ax$) have global solutions. Nonlinear ODEs ($\dot{x} = ax^2$) can blow up.

4. **Maximal interval of existence:** Every IVP has a maximal interval of existence. For $\dot{x} = ax^2$, this is $[0, t^*)$ or $(t^*, \infty)$ depending on signs.

5. **Model breakdown:** Finite time blow up often indicates the mathematical model is no longer valid near $t^*$ (physical constraints, regime changes, etc. must be included).

### Connection to Future Topics

This example illustrates fundamental concepts that appear throughout the course:

- **Phase space analysis:** Understanding where solutions exist and their behavior
- **Equilibria and stability:** For $\dot{x} = ax^2$, only $x = 0$ is an equilibrium (unstable if $a \neq 0$)
- **Qualitative vs. quantitative:** We can understand blow up qualitatively even without explicit solutions
- **Bifurcations:** At $a = 0$, the behavior changes drastically (from blow up to global existence)

## END OF QUESTION 2 SOLUTIONS