Asymptotics: Problem Sheet 3, Question 2

Leading Order Asymptotic Behaviour of Laplace-Type Integrals

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Problem Statement

Obtain the leading order asymptotic behaviour as $X \to \infty$ of the following integrals:

(a) $I_a(X) = \displaystyle\int_X^{\infty} e^{-t^3} \mathrm{d}t$

(b) $I_b(X) = \displaystyle\int_3^{6} \frac{e^{-Xt^2}}{\sqrt{1+t^2}} \mathrm{d}t$

(c) $I_c(X) = \displaystyle\int_0^{\pi/2} \frac{e^{X(\sin t + \cos t)}}{\sqrt{t}} \mathrm{d}t$

(d) $I_d(X) = \displaystyle\int_0^{\infty} e^{X(2t-t^2)} \log(1+t^2) \mathrm{d}t$

(e) $I_e(X) = \displaystyle\int_{-1}^{1} e^{-X(\cosh t + 1)} e^t \mathrm{d}t$

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1 Solution to Part (a)

Solution 1.1 (Part (a)). **Problem:** Find the leading order asymptotic behaviour of

$$I_a(X) = \int_X^{\infty} e^{-t^3} \mathrm{d}t \quad \text{as } X \to \infty.$$

Step 1: Identify the structure

What do we observe? The integral has the form $\int_X^{\infty} f(t)\mathrm{d}t$ where the lower limit $X \to \infty$.

Why is this significant? This is NOT a standard Laplace integral of the form $\int_a^b f(t)e^{-X\phi(t)}\mathrm{d}t$ because the large parameter X appears in the integration limit, not as a coefficient in the exponent.

What method do we use? We use a *substitution* to convert this into a standard form, or we can use *integration by parts.*

Step 2: Apply substitution method

What substitution do we choose? Let $u = t^3$.

Why this substitution? Because the exponent is $-t^3$, this substitution will simplify the exponential to e^{-u}.

Computing the differential:

$$\mathrm{d}u = 3t^2\mathrm{d}t \quad \Rightarrow \quad \mathrm{d}t = \frac{\mathrm{d}u}{3t^2}$$

What is t in terms of u? Since $u = t^3$, we have $t = u^{1/3}$, and thus:

$$t^2 = u^{2/3}$$

Transforming the limits:

- When $t = X$: $u = X^3$
- When $t \to \infty$: $u \to \infty$

Step 3: Rewrite the integral

Substituting everything:

$$I_a(X) = \int_{X^3}^{\infty} e^{-u} \frac{\mathrm{d}u}{3u^{2/3}} = \frac{1}{3}\int_{X^3}^{\infty} u^{-2/3}e^{-u}\mathrm{d}u$$

Why is this better? Now we have a standard Laplace-type integral with the large parameter appearing in the lower limit.

Step 4: Apply Watson's lemma / Integration by parts

What do we know about large limits? For integrals of the form $\int_a^\infty g(u)e^{-u}\mathrm{d}u$ where $a \to \infty$, the dominant contribution comes from near $u = a$.

Method: Integration by parts

Why integration by parts? The lecture notes (Section 4.2.1) show that for integrals $\int_a^b f(t)e^{-xt}\mathrm{d}t$, integration by parts yields asymptotic expansions.

Setting up: We write

$$\int_{X^3}^{\infty} u^{-2/3}e^{-u}\mathrm{d}u = \int_{X^3}^{\infty} u^{-2/3}\left(-\frac{\mathrm{d}}{\mathrm{d}u}e^{-u}\right)\mathrm{d}u$$

Integrating by parts:

$$= \left[-u^{-2/3}e^{-u}\right]_{X^3}^{\infty} - \int_{X^3}^{\infty}\left(-\frac{2}{3}u^{-5/3}\right)e^{-u}\mathrm{d}u$$

Evaluating the boundary term:

- At $u \to \infty$: $u^{-2/3}e^{-u} \to 0$ (exponential dominates polynomial)
- At $u = X^3$: we get $(X^3)^{-2/3}e^{-X^3} = X^{-2}e^{-X^3}$

Therefore:

$$\int_{X^3}^{\infty} u^{-2/3}e^{-u}\mathrm{d}u = X^{-2}e^{-X^3} + \frac{2}{3}\int_{X^3}^{\infty} u^{-5/3}e^{-u}\mathrm{d}u$$

Why can we stop here? The remaining integral is of order $\mathrm{O}(X^{-3}e^{-X^3})$ as $X \to \infty$, which is smaller than the first term.

Step 5: Conclude the leading order behaviour

Combining our results:

$$I_a(X) = \frac{1}{3}\left[X^{-2}e^{-X^3} + \mathrm{O}(X^{-3}e^{-X^3})\right]$$

Leading order term:

$$\boxed{I_a(X) \sim \frac{1}{3X^2}e^{-X^3} \quad \text{as } X \to \infty}$$

Why is this the leading order? Because the next term is asymptotically smaller by a factor of $\mathrm{O}(X^{-1})$.

2 Solution to Part (b)

Solution 2.1 (Part (b))**. Problem:** Find the leading order asymptotic behaviour of

$$I_b(X) = \int_3^6 \frac{e^{-Xt^2}}{\sqrt{1+t^2}}\mathrm{d}t \quad \text{as } X \to \infty.$$

Step 1: Identify the structure

What form does this integral have? This is a Laplace-type integral:

$$I_b(X) = \int_3^6 f(t)e^{-X\phi(t)}\mathrm{d}t$$

where:

- $f(t) = \frac{1}{\sqrt{1+t^2}}$
- $\phi(t) = t^2$

Why is this classification important? Because Laplace-type integrals have well-established asymptotic methods depending on the properties of $\phi(t)$.

Step 2: Analyze the phase function $\phi(t) = t^2$

What are the properties of $\phi(t)$ on $[3, 6]$?

Computing the derivative:

$$\phi'(t) = 2t$$

Does $\phi'(t)$ vanish on $[3, 6]$?

$$\phi'(t) = 0 \quad \Leftrightarrow \quad t = 0$$

Is $t = 0$ in our interval? No, $0 \notin [3, 6]$.

Conclusion: $\phi'(t) \neq 0$ for all $t \in [3, 6]$, so $\phi(t)$ has no critical points in the interior of the interval.

What does this mean? The minimum of $\phi(t)$ on $[3, 6]$ must occur at one of the endpoints.

Step 3: Locate the minimum

Evaluating $\phi(t)$ at the endpoints:

$$\phi(3) = 9$$
$$\phi(6) = 36$$

Which is smaller? $\phi(3) = 9 < 36 = \phi(6)$.

Why does this matter? According to Laplace's method (Section 4.2.3 of lecture notes), for integrals $\int_a^b f(t)e^{-X\phi(t)}\mathrm{d}t$ as $X \to \infty$, the dominant contribution comes from a small neighborhood of the global minimum of $\phi(t)$.

Conclusion: The dominant contribution comes from near $t = 3$.

Step 4: Check if the minimum is at a boundary with $\phi'(c) \neq 0$

What is the situation? The minimum is at the left endpoint $c = a = 3$, and $\phi'(3) = 6 \neq 0$.

What method do we use? According to the lecture notes (page 28, equation 206), when the minimum is at an endpoint and $\phi'(c) \neq 0$, the leading order behaviour is:

$$I(X) \sim \frac{f(c)}{X\phi'(c)} e^{-X\phi(c)} \quad \text{as } X \to \infty$$

where the sign depends on whether c is the left or right endpoint.

Why this formula? Because near the boundary, we can approximate the integral using the boundary value, and the factor $1/(X\phi'(c))$ comes from the rate of change of the exponential.

Step 5: Apply the boundary point formula

Identifying our parameters:

- $c = 3$ (left endpoint)
- $f(3) = \frac{1}{\sqrt{1+9}} = \frac{1}{\sqrt{10}}$
- $\phi(3) = 9$
- $\phi'(3) = 6$

Since $c = a$ and $\phi'(a) > 0$: The formula (equation 206 from lecture notes) gives:

$$I_b(X) \sim \frac{1}{X\phi'(3)} f(3) e^{-X\phi(3)}$$

Substituting values:

$$I_b(X) \sim \frac{1}{X \cdot 6} \cdot \frac{1}{\sqrt{10}} \cdot e^{-9X}$$

$$\boxed{I_b(X) \sim \frac{1}{6\sqrt{10}\,X} e^{-9X} \quad \text{as } X \to \infty}$$

Why is this the leading order? Because the exponential e^{-9X} dominates the asymptotic behaviour, and all other contributions from the interior or the other endpoint are exponentially smaller (they contain factors like e^{-36X}).

3 Solution to Part (c)

Solution 3.1 (Part (c))**. Problem:** Find the leading order asymptotic behaviour of

$$I_c(X) = \int_0^{\pi/2} \frac{e^{X(\sin t + \cos t)}}{\sqrt{t}} \mathrm{d}t \quad \text{as } X \to \infty.$$

Step 1: Identify the structure

What form is this? This is a Laplace-type integral:

$$I_c(X) = \int_0^{\pi/2} f(t) e^{X\phi(t)} \mathrm{d}t$$

where:

- $f(t) = \frac{1}{\sqrt{t}} = t^{-1/2}$
- $\phi(t) = \sin t + \cos t$

What's different from part (b)? The sign in the exponent: we have $+X\phi(t)$ instead of $-X\phi(t)$.

Why does this matter? For $e^{X\phi(t)}$ with $X > 0$ large, the dominant contribution comes from where $\phi(t)$ is *maximized*, not minimized.

Step 2: Find critical points of $\phi(t)$

Computing the derivative:

$$\phi'(t) = \cos t - \sin t$$

Setting $\phi'(t) = 0$:

$$\cos t - \sin t = 0 \quad \Rightarrow \quad \cos t = \sin t$$

When does $\cos t = \sin t$? This occurs when $t = \pi/4$ (since $\tan t = 1$).

Is this in our interval? Yes, $\pi/4 \in (0, \pi/2)$, so we have a critical point in the interior.

Step 3: Verify it's a maximum

Computing the second derivative:

$$\phi''(t) = -\sin t - \cos t$$

Evaluating at $t = \pi/4$:

$$\phi''\left(\frac{\pi}{4}\right) = -\sin\frac{\pi}{4} - \cos\frac{\pi}{4} = -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} = -\sqrt{2} < 0$$

What does $\phi'' < 0$ mean? This confirms that $t = \pi/4$ is a *maximum* of $\phi(t)$.

Why is this important? Because for $e^{X\phi(t)}$ with $X \to \infty$, the integral is dominated by the neighborhood of the maximum of $\phi(t)$.

Step 4: Compare with boundary values

Computing $\phi(t)$ at critical point and boundaries:

$$\phi(0) = \sin 0 + \cos 0 = 0 + 1 = 1$$
$$\phi(\pi/4) = \sin(\pi/4) + \cos(\pi/4) = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} = \sqrt{2}$$
$$\phi(\pi/2) = \sin(\pi/2) + \cos(\pi/2) = 1 + 0 = 1$$

Which is largest? $\phi(\pi/4) = \sqrt{2} > 1$.

Conclusion: The global maximum on $[0, \pi/2]$ is at $t = \pi/4$.

Step 5: Check for singularities in $f(t)$

What about $f(t) = t^{-1/2}$? This function has a singularity at $t = 0$ (it blows up as $t \to 0^+$).

Does this affect our analysis? We need to check if the singularity is integrable. Near $t = 0$:

$$f(t)e^{X\phi(t)} \sim t^{-1/2}e^{X\cdot 1} = e^X t^{-1/2}$$

Is $\int_0^\epsilon t^{-1/2}\mathrm{d}t$ convergent? Yes:

$$\int_0^\epsilon t^{-1/2}\mathrm{d}t = 2t^{1/2}\Big|_0^\epsilon = 2\sqrt{\epsilon} < \infty$$

Conclusion: The singularity at $t = 0$ is integrable, so it doesn't dominate the asymptotic behaviour.

Step 6: Apply Laplace's method for interior maximum

What formula do we use? For an integral $\int_a^b f(t)e^{X\phi(t)}\mathrm{d}t$ where $\phi(t)$ has a maximum at $c \in (a, b)$ with $\phi'(c) = 0$ and $\phi''(c) < 0$, Laplace's method (equation 205, page 27) gives:

$$I(X) \sim \sqrt{\frac{2\pi}{X|\phi''(c)|}} f(c)e^{X\phi(c)} \quad \text{as } X \to \infty$$

Why this formula? Near the maximum, we approximate:

$$\phi(t) \approx \phi(c) + \frac{1}{2}\phi''(c)(t - c)^2$$

and the integral becomes approximately Gaussian.

Step 7: Evaluate at $c = \pi/4$

Computing the required quantities:

$$\begin{aligned} c &= \frac{\pi}{4} \\ f(c) &= \left(\frac{\pi}{4}\right)^{-1/2} = \sqrt{\frac{4}{\pi}} = \frac{2}{\sqrt{\pi}} \\ \phi(c) &= \sqrt{2} \\ |\phi''(c)| &= |-\sqrt{2}| = \sqrt{2} \end{aligned}$$

Applying the formula:

$$I_c(X) \sim \sqrt{\frac{2\pi}{X\sqrt{2}}} \cdot \frac{2}{\sqrt{\pi}} \cdot e^{\sqrt{2}X}$$

Simplifying:

$$\begin{aligned} &= \sqrt{\frac{2\pi}{X\sqrt{2}}} \cdot \frac{2}{\sqrt{\pi}} \cdot e^{\sqrt{2}X} \\ &= \frac{2}{\sqrt{\pi}}\sqrt{\frac{2\pi}{X\sqrt{2}}}e^{\sqrt{2}X} \\ &= \frac{2}{\sqrt{\pi}} \cdot \sqrt{2\pi} \cdot \frac{1}{\sqrt{X\sqrt{2}}}e^{\sqrt{2}X} \end{aligned}$$

$$
\begin{aligned}
&= 2\sqrt{2} \cdot \frac{1}{\sqrt{X\sqrt{2}}} e^{\sqrt{2}X} \\
&= 2\sqrt{2} \cdot \frac{1}{X^{1/2} \cdot 2^{1/4}} e^{\sqrt{2}X} \\
&= \frac{2\sqrt{2}}{2^{1/4}} \cdot X^{-1/2} e^{\sqrt{2}X} \\
&= 2^{1-1/4} X^{-1/2} e^{\sqrt{2}X} \\
&= 2^{3/4} X^{-1/2} e^{\sqrt{2}X}
\end{aligned}
$$

$$
\boxed{I_c(X) \sim \frac{2^{3/4}}{\sqrt{X}} e^{\sqrt{2}X} \quad \text{as } X \to \infty}
$$

Alternative simplified form:

$$
\boxed{I_c(X) \sim \sqrt{\frac{2\sqrt{2}}{X}} \cdot \frac{2}{\sqrt{\pi}} e^{\sqrt{2}X} = \frac{2}{\sqrt{\pi}} \sqrt{\frac{2\sqrt{2}}{X}} e^{\sqrt{2}X} \quad \text{as } X \to \infty}
$$

4 Solution to Part (d)

Solution 4.1 (Part (d))**. Problem:** Find the leading order asymptotic behaviour of

$$
I_d(X) = \int_0^\infty e^{X(2t-t^2)} \log(1+t^2) \mathrm{d}t \quad \text{as } X \to \infty.
$$

Step 1: Identify the structure

What form is this? This is a Laplace-type integral:

$$
I_d(X) = \int_0^\infty f(t) e^{X\phi(t)} \mathrm{d}t
$$

where:

- $f(t) = \log(1+t^2)$
- $\phi(t) = 2t - t^2$

What type of integral? Since we have $e^{X\phi(t)}$ with large positive X, we look for the *maximum* of $\phi(t)$.

Step 2: Find critical points of $\phi(t)$

Computing the derivative:

$$\phi'(t) = 2 - 2t$$

Setting $\phi'(t) = 0$:

$$2 - 2t = 0 \quad \Rightarrow \quad t = 1$$

Is this in our domain? Yes, $t = 1 \in (0, \infty)$.

Step 3: Verify it's a maximum

Computing the second derivative:

$$\phi''(t) = -2$$

What does this tell us? Since $\phi''(t) = -2 < 0$ everywhere, and in particular $\phi''(1) = -2 < 0$, we confirm that $t = 1$ is a *maximum*.

Step 4: Check the behaviour at boundaries

As $t \to 0$: $\phi(0) = 0$

As $t \to \infty$:

$$\phi(t) = 2t - t^2 = t(2 - t) \to -\infty$$

since the $-t^2$ term dominates.

At the critical point:

$$\phi(1) = 2(1) - 1^2 = 2 - 1 = 1$$

Conclusion: The global maximum of $\phi(t)$ on $[0, \infty)$ is at $t = 1$ with $\phi(1) = 1$.

Step 5: Check properties of $f(t)$ at $t = 1$

Computing $f(1)$:

$$f(1) = \log(1 + 1^2) = \log 2$$

Is $f(1)$ finite and non-zero? Yes, $f(1) = \log 2 > 0$ is finite.

Is the integral convergent? As $t \to \infty$:

$$f(t)e^{X\phi(t)} = \log(1 + t^2) \cdot e^{X(2t - t^2)} \sim \log(t^2)e^{-Xt^2} = 2 \log t \cdot e^{-Xt^2}$$

This decays exponentially, so the integral converges.

Step 6: Apply Laplace's method

What formula do we use? For $\int_a^b f(t)e^{X\phi(t)}\mathrm{d}t$ with maximum at $c \in (a,b)$ where $\phi'(c) = 0$ and $\phi''(c) < 0$, Laplace's method gives:

$$I(X) \sim \sqrt{\frac{2\pi}{X|\phi''(c)|}} f(c)e^{X\phi(c)} \quad \text{as } X \to \infty$$

Why does this work despite infinite upper limit? The exponential decay as $t \to \infty$ ensures that contributions far from $t = 1$ are exponentially suppressed.

Step 7: Apply the formula with $c = 1$

Identifying our values:

$$\begin{aligned} c &= 1 \\ f(c) &= \log 2 \\ \phi(c) &= 1 \\ |\phi''(c)| &= |-2| = 2 \end{aligned}$$

Substituting into the formula:

$$I_d(X) \sim \sqrt{\frac{2\pi}{X \cdot 2}} \log 2 \cdot e^{X \cdot 1}$$

$$= \sqrt{\frac{\pi}{X}} \log 2 \cdot e^{X}$$

$$\boxed{I_d(X) \sim (\log 2)\sqrt{\frac{\pi}{X}} e^{X} \quad \text{as } X \to \infty}$$

Why is this the leading order? All other contributions (from $t \neq 1$) are exponentially smaller because $\phi(t) < \phi(1) = 1$ everywhere else, leading to factors like $e^{X\phi(t)}$ with $\phi(t) < 1$.

5 Solution to Part (e)

Solution 5.1 (Part (e))**. Problem:** Find the leading order asymptotic behaviour of

$$I_e(X) = \int_{-1}^{1} e^{-X(\cosh t + 1)} e^t \mathrm{d}t \quad \text{as } X \to \infty.$$

Step 1: Rewrite in standard form

Combining the exponentials:

$$I_e(X) = \int_{-1}^{1} e^t \cdot e^{-X(\cosh t+1)} \mathrm{d}t$$

What form is this? This is a Laplace-type integral:

$$I_e(X) = \int_{-1}^{1} f(t) e^{-X\phi(t)} \mathrm{d}t$$

where:

- $f(t) = e^t$
- $\phi(t) = \cosh t + 1$

What type? Since we have $e^{-X\phi(t)}$ with X large and positive, we seek the *minimum* of $\phi(t)$.

Step 2: Analyze $\phi(t) = \cosh t + 1$

What is $\cosh t$?

$$\cosh t = \frac{e^t + e^{-t}}{2}$$

Properties of $\cosh t$:

- $\cosh t \geq 1$ for all $t \in \mathbb{R}$
- $\cosh t = 1$ if and only if $t = 0$
- $\cosh t$ is even: $\cosh(-t) = \cosh t$
- $\cosh t$ is decreasing on $(-\infty, 0)$ and increasing on $(0, \infty)$

Therefore: $\phi(t) = \cosh t + 1 \geq 2$ with minimum at $t = 0$.

Step 3: Find critical points

Computing the derivative:

$$\phi'(t) = \sinh t$$

where $\sinh t = \frac{e^t - e^{-t}}{2}$.
Setting $\phi'(t) = 0$:

$$\sinh t = 0 \quad \Rightarrow \quad t = 0$$

Is this in our interval? Yes, $0 \in (-1, 1)$.

Step 4: Verify it's a minimum

Computing the second derivative:

$$\phi''(t) = \cosh t$$

Evaluating at $t = 0$:

$$\phi''(0) = \cosh 0 = 1 > 0$$

Conclusion: Since $\phi''(0) > 0$, the point $t = 0$ is a *minimum* of $\phi(t)$.

Step 5: Evaluate quantities at the minimum

Computing:

$$\begin{aligned} c &= 0 \\ \phi(0) &= \cosh 0 + 1 = 1 + 1 = 2 \\ f(0) &= e^0 = 1 \\ \phi''(0) &= 1 \end{aligned}$$

Step 6: Apply Laplace's method

What formula? For $\int_a^b f(t)e^{-X\phi(t)}\mathrm{d}t$ with minimum at $c \in (a, b)$ where $\phi'(c) = 0$ and $\phi''(c) > 0$, Laplace's method (equation 205) gives:

$$I(X) \sim \sqrt{\frac{2\pi}{X\phi''(c)}} f(c)e^{-X\phi(c)} \quad \text{as } X \to \infty$$

Why this formula? Near the minimum, we approximate:

$$\phi(t) \approx \phi(c) + \frac{1}{2}\phi''(c)(t - c)^2$$

and the integral becomes approximately Gaussian (with $e^{-X[\cdots]}$ giving the Gaussian factor).

Step 7: Substitute values

Applying the formula:

$$I_e(X) \sim \sqrt{\frac{2\pi}{X \cdot 1}} \cdot 1 \cdot e^{-X\cdot 2}$$

$$= \sqrt{\frac{2\pi}{X}} e^{-2X}$$

$$\boxed{I_e(X) \sim \sqrt{\frac{2\pi}{X}} e^{-2X} \quad \text{as } X \to \infty}$$

Why is this the leading order? Because:

1. The exponential factor e^{-2X} comes from the minimum value $\phi(0) = 2$
2. All other points have $\phi(t) > 2$, giving exponentially smaller contributions
3. The $\sqrt{1/X}$ factor arises from the Gaussian approximation near the minimum

Summary of Methods Used

1. **Part (a):** Substitution followed by integration by parts
2. **Part (b):** Laplace's method with minimum at boundary (endpoint formula)
3. **Part (c):** Laplace's method with maximum at interior critical point
4. **Part (d):** Laplace's method with maximum at interior critical point
5. **Part (e):** Laplace's method with minimum at interior critical point

Key principle: For Laplace-type integrals $\int f(t)e^{\pm X\phi(t)}\mathrm{d}t$ as $X \to \infty$:

- If $e^{-X\phi(t)}$: dominant contribution from *minimum* of $\phi(t)$
- If $e^{+X\phi(t)}$: dominant contribution from *maximum* of $\phi(t)$
- If extremum is at interior with $\phi'(c) = 0$: use Laplace's method formula with $\sqrt{2\pi/(X|\phi''(c)|)}$
- If extremum is at boundary with $\phi'(c) \neq 0$: use boundary formula with $1/(X\phi'(c))$