Methods of Applied Mathematics - Part 1
Exercise Sheet 3: Bifurcations - Question 1

1 Question 1: $\dot{x} = (5 - x)(1 - ax)$ with $a > 0$

Solution 1. Goal: Find equilibria, their stability, identify the bifurcation at $a = 1/5$, and sketch the bifurcation diagram

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Finding Equilibria

Step 1: Set $\dot{x} = 0$

$$(5 - x)(1 - ax) = 0$$

Step 2: Solve
Two equilibria:

$$\boxed{x_1^* = 5} \quad \text{and} \quad \boxed{x_2^* = \frac{1}{a}}$$

Note: $x_1^* = 5$ is independent of a (fixed), while $x_2^* = 1/a$ moves as a varies. At $a = 1/5$, both equal 5.

Stability Analysis

Step 1: Compute $f'(x)$ where $f(x) = (5 - x)(1 - ax)$
Expand: $f(x) = 5 - 5ax - x + ax^2$
Derivative:

$$f'(x) = -5a - 1 + 2ax$$

Step 2: Evaluate at $x_1^* = 5$

$$f'(5) = -5a - 1 + 2a(5) = -5a - 1 + 10a = -1 + 5a$$

Stability:

- $a < 1/5$: $f'(5) < 0 \Rightarrow$ Stable
- $a = 1/5$: $f'(5) = 0 \Rightarrow$ Neutral
- $a > 1/5$: $f'(5) > 0 \Rightarrow$ Unstable

Step 3: Evaluate at $x_2^* = 1/a$

$$f'(1/a) = -5a - 1 + 2a(1/a) = -5a - 1 + 2 = 1 - 5a$$

Stability:

- $a < 1/5$: $f'(1/a) > 0 \Rightarrow$ Unstable
- $a = 1/5$: $f'(1/a) = 0 \Rightarrow$ Neutral
- $a > 1/5$: $f'(1/a) < 0 \Rightarrow$ Stable

Step 4: Summary

Parameter	$x^* = 5$	$x^* = 1/a$
$a < 1/5$	Stable	Unstable
$a = 1/5$	Both at $x = 5$, neutral	
$a > 1/5$	Unstable	Stable

The equilibria exchange stability as they pass through each other at $a = 1/5$.

Bifurcation Identification

Step 1: Check Bifurcation Conditions at $a = 1/5$, $x = 5$
(B1) Equilibrium: $f(5) = 0$ ✓
(B2) Zero eigenvalue: $f'(5) = -1 + 5(1/5) = 0$ ✓
(B3) $\partial f/\partial a = -(5 - x) \cdot x|_{x=5} = 0$ ✓
Step 2: Check Genericity Conditions
(G1) $f''(x) = 2a|_{a=1/5} = 2/5 \neq 0$ ✓
(G2) $\partial f'/\partial a = -5 + 2x|_{x=5} = 5 \neq 0$ ✓
Conclusion:

$$\boxed{\text{Transcritical bifurcation at } a = 1/5}$$

One equilibrium is pinned at $x = 5$, the other passes through it, and they exchange stability.

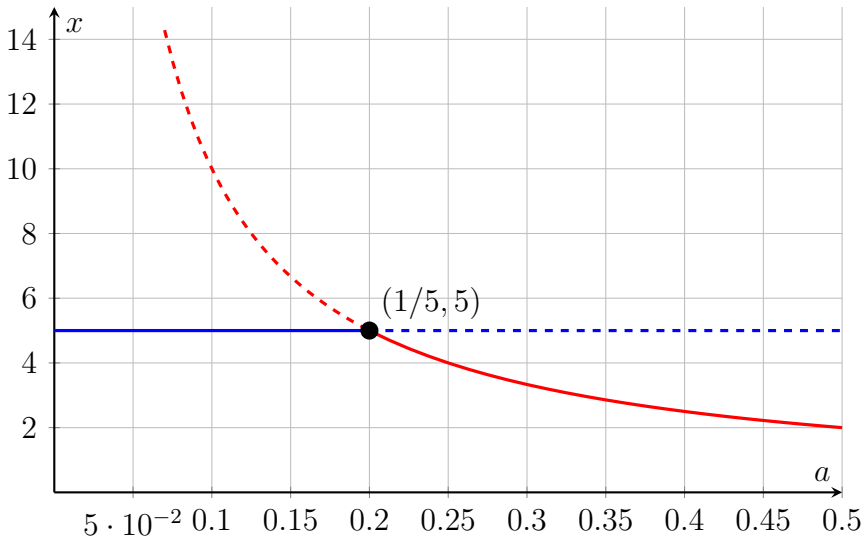
Bifurcation Diagram

Step 1: Identify branches

- Branch 1: $x = 5$ (horizontal line)
  - Stable for $a < 1/5$ (solid)
  - Unstable for $a > 1/5$ (dashed)
- Branch 2: $x = 1/a$ (hyperbola)
  - Unstable for $a < 1/5$ (dashed)
  - Stable for $a > 1/5$ (solid)

Bifurcation point: $(a, x) = (1/5, 5)$

Step 2: Sketch



Solid = stable, dashed = unstable

Summary

- Equilibria: $x^* = 5$ and $x^* = 1/a$
- At $a = 1/5$: transcritical bifurcation where equilibria collide and exchange stability
- For $a < 1/5$: $x = 5$ stable, $x = 1/a$ unstable
- For $a > 1/5$: $x = 5$ unstable, $x = 1/a$ stable