SEMT30006
Methods of Applied Mathematics
Part 1 - ODEs

Mike Jeffrey
University of Bristol
mike.jeffrey@bristol.ac.uk

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Course Structure and Instructions

How the course works

Each week you should:

- work through the notes on Blackboard, and watch the videos for more help/context
- bring any questions to the in-class session on Tuesday, where we'll go over some examples
- then take a look at the exercise sheet
- come to the problem class on Thursday, work through the exercises and get any help you need.

Notes	· The notes for the course are released week-by-week on Blackboard · Work through them and make your own notes · Make a note of anything you don't understand to discuss in class
Videos	· Short (10-20 minutes) videos discussing the notes and exercises · These will give context for the written notes or highlight key ideas · They also contain some worked examples
Exercise Sheets	· Homework problems will be released week-by-week on Blackboard · Try to at least make an attempt before each week's Thursday problem sessio · Solutions will be released after the problem class.
In-class	· Tuesday lecture – 1 hour of worked examples or further explanations · Thursday problem class – 2 hours to work through exercises and get help

Assessment

- *Exam* (100%) – the assessment for the unit is one exam at the end of the semester, before xmas.

Textbooks

We won't follow any textbook directly, but you'll find useful resources in:

- Strogatz – *Nonlinear dynamics and chaos*
- Glendinning – *Stability, Instability and Chaos*
- Thompson & Stewart – *Nonlinear dynamics and chaos*
- Other interesting books on the topic include:
    - Seydel – *Practical bifurcation and stability analysis*
    - Stewart – *Does God play dice?*
- And . . . try out simulations using computational methods!

Plan of Topics

Week	Topic
1	Introduction to ODEs, flows, and population models
2	Stability, linearisation, and equilibria
3	Bifurcations
4	Bifurcations cont'd, limit cycles and periodic orbits
5	Maps, period doubling, and chaos
6	Reading week
7-12	PDEs

[Side Notes:] When you see one of these boxes ...

... they contain extra information or bits of theory that are important to know, but which you don't need to remember every detail of.

The aim of this course is to teach you **methods**, so while you need to know the important theorems that make all this work, you won't be asked to quote them in exam questions.

Contents

[Side Notes:] Revision

First a bit of notation:

- $x \in \mathbb{R}$ denotes that x is a real variable, and
  $x \in \mathbb{R}^n$ denotes that x is a real n-dimensional vector.

Taylor series:

- The expansion of a function $f(x)$ at a point $x = c$ is

$$\begin{aligned} f(x) &= \sum_{r=0}^{\infty} \frac{(x-c)^r}{r!} f^{(r)}(c) \\ &= f(c) + (x-c)f'(c) + \tfrac{1}{2}(x-c)^2 f''(c) + ... \end{aligned}$$

  for $x \in \mathbb{R}$, where $f^{(r)}(c) = \frac{d^r}{dx^r} f(x)\big|_{x=c}$, and

$$\begin{aligned} \begin{pmatrix} f(x,y) \\ g(x,y) \end{pmatrix} &= \sum_{r,s=0}^{\infty} \frac{(x-c)^r (y-d)^s}{r!s!} \cdot \begin{pmatrix} f^{(r,s)}(c,d) \\ g^{(r,s)}(c,d) \end{pmatrix} \\ &= \begin{pmatrix} f(c,d) + (x-c)f^{(1,0)}(c,d) + (y-d)f^{(0,1)}(c,d) + ... \\ g(c,d) + (x-c)g^{(1,0)}(c,d) + (y-d)g^{(0,1)}(c,d) + ... \end{pmatrix} \end{aligned}$$

  for $(x,y) \in \mathbb{R}^2$, with $f^{(r,s)}(c,d) = \frac{\partial^r}{\partial x^r}\frac{\partial^s}{\partial y^s} f(x,y)|_{(c,d)}$, etc. and so on if you have 3, 4, or more variables.

- We can use these to approximate the functions near $x = 0$ by taking just the first few terms. Obviously to use these requires the necessary derivatives to exist.

Differentiability:

- If all derivatives exist up to and including $f^{(r)}$ we say f is r-times differentiable (sometimes called *continuity class* $\mathcal{C}^r$).
- If all derivatives exist (the function is of class $\mathcal{C}^\infty$) we say it is **smooth**.
- If the function is smooth *and* equal to its Taylor series it is **analytic**.
- A function might be differentiable only on some subset or 'domain' of its space, not necessarily for all x.

Linear algebra:

- An $n \times n$ matrix $\underline{\underline{A}}$ has eigenvalues γ_r and associated eigenvectors $\mathbf{v}_r$, defined as satisfying

$$\underline{\underline{A}}.\mathbf{v}_r = \gamma_r \mathbf{v}_r \qquad \text{for} \quad r = 1, ..., n \tag{0.1}$$

  This says that when multiplying certain vectors ($\mathbf{v}_r$), the matrix $\underline{\underline{A}}$ behaves as if it was a scalar (γ_r). We'll see how important this is for understanding dynamics of systems in this course.

- We can find the eigenvalues by solving the characteristic polynomial

$$0 = (\underline{\underline{A}} - \gamma_r)\mathbf{v}_r \quad \Rightarrow \quad 0 = |\underline{\underline{A}} - \gamma_r| \tag{0.2}$$

  and then if we substitute back into (0.1) we can find the eigenvectors $\mathbf{v}_r$.

1 ODEs and Dynamical Systems

This part of the course is about the **qualitative** approach to studying dynamical systems.

- 'Qualitative' means we are interested in finding geometrical descriptions of a system's behaviour, in describing its general features rather than solving it exactly.
- Very few systems can be solved exactly. Even computational solutions can be difficult to interpret, and sometimes misleading. A qualitative understanding of a system is vital before rushing into computer simulations. And it can tell you everything you need to understand without solving its equations at all.

A dynamical system is any system of equations that tells you how something evolves in time.

- An equation $\dot{x} = f(x)$ defines an *ordinary differential equation* (ODE) for the dependent variable x, in terms of the independent variable t, where $\dot{x}$ is shorthand for $\frac{dx}{dt}$.
- An equation $x_{n+1} = f(x_n)$ defines a *difference equation* or *map* in the discrete variable x_n.
- Both of these are types of dynamical system. Their solutions are functions $x(t)$ (for the ODE) or sequences $x_0, x_1, x_2, ...$ (for the map) that tell us how a system behaves over time, from an initial condition $x(0)$ or x_0.
- We can have sets of such equations defining an n-dimensional system, e.g. a set of ODEs in continuous variables $x, y, ...$

  $$(\dot{x}, \dot{y}, ...) = \{f(x, y, ...), g(x, y, ...), ...\}$$

  or a set of equations in discrete variables $x_n, y_n, ...$

  $$(x_{n+1}, y_{n+1}, ...) = \{f(x_n, y_n, ...), g(x_n, y_n, ...), ...\}$$

- There are other kinds of dynamical systems such as partial differential equations, cellular automata, delay differential equations, integral equations, renewal equations, stochastic differential equations, hybrid systems, piecewise smooth dynamical systems, ... A lot of what we will study here provide ideas that can be extended to these.

2 An example of population growth

Take a population that has N individuals at time t, evolving as

$$\dot{N} \;=\; B \;-\; D \;+\; M \tag{2.1}$$

where B = births, D = deaths, M = migrations, per unit time.

A simple model is to say these changes are proportional to the number of individuals, so define a birth rate β such that $B = \beta N$, and death rate δ such that $D = \delta N$. These are defined per individual per unit time, with $\beta > 0$ and $\delta > 0$. For now say $M = 0$ (no migration, i.e. closed borders).

Then we have

$$\dot{N} \;=\; (\beta - \delta)N \;. \tag{2.2}$$

- This is easy to solve, e.g. by separation of variables

$$\frac{dN}{N} = (\beta - \delta)dt \quad \Rightarrow \quad \ln \tfrac{N(t)}{N_0} = (\beta - \delta)t \quad \Rightarrow \quad N(t) = N_0 e^{(\beta-\delta)t} \tag{2.3}$$

  with initial condition $N(0) = N_0$.

- If $N_0 = 0$ then $N(t) = 0$ for all times t. We call this an *equilibrium*, i.e. a state where $\frac{dN}{dt} = 0$ so the system feels no impulse to change.

- If $\beta > \delta$ the population $N(t)$ grows exponentially away from $N = 0$, without bound. This makes sense as births outweigh deaths. We say the equilibrium $N = 0$ is *unstable* or *repelling*.

- If $\beta < \delta$ the population $N(t)$ shrinks *asymptotically* (in infinite time) towards $N = 0$. This makes sense as deaths overwhelm births. We say the equilibrium $N = 0$ is *stable* or *attracting*.

- We could re-define the crucial parameter as $\alpha = \beta - \delta$. Then there is only one parameter in the system, $\alpha =$ the difference between the birth and death rates. The actual values of β and δ don't matter, only their difference α (so $\beta = 2$, $\delta = 1$, behaves the same as $\beta = 10$, $\delta = 9$, as in both cases $\alpha = 1$).

A more realistic model is to say that the death rate increases if the population gets too large (typical in a contained environment), so it becomes $\delta = \gamma N$ for a constant γ, then

$$\dot{N} = (\beta - \gamma N)N \tag{2.4}$$

with $\beta, \gamma > 0$.

In this case to minimize the number of parameters we can re-scale (re-scaling is like non-dimensionalization):

- Let $N = x/\gamma$ for a scaled population x, then

  $$\dot{x} = (\beta - x)x \tag{2.5}$$

  which has only one parameter. Now the constant γ just acts like a scale for measuring the population, a system of units for N if you like.

- You can see immediately that this now has two *equilibria*, as there are two solutions to $\dot{x} = 0$. There is still an equilibrium at $x = 0$, and now a new one at $x = \beta$.

- We can still solve the nonlinear system (2.5) by separation of variables

  $$dt = \tfrac{dx}{(\beta - x)x} = \left(\tfrac{1}{x} + \tfrac{1}{\beta - x}\right)\tfrac{dx}{\beta} \quad \Rightarrow \quad t = \tfrac{1}{\beta}\ln\tfrac{(\beta - x_0)x(t)}{(\beta - x(t))x_0} \tag{2.6}$$
  $$\Rightarrow \quad x(t) = \frac{\beta x_0 e^{\beta t}}{\beta - x_0 + x_0 e^{\beta t}}$$

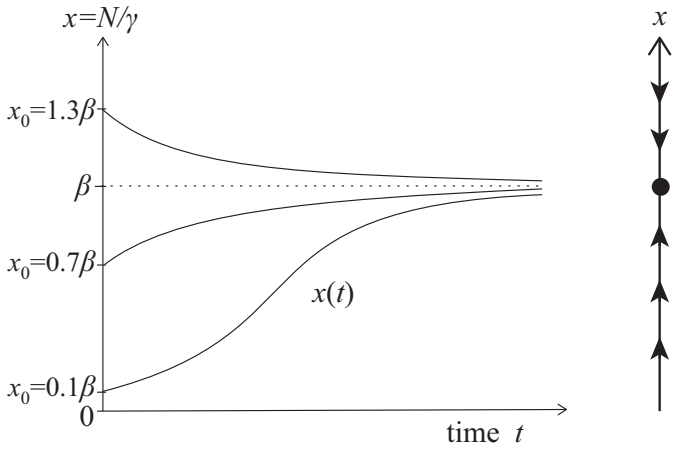
- The long time behaviour is now bounded as

  $$x(t) \to \frac{\beta x_0 e^{\beta t}}{x_0 e^{\beta t}} \to \beta \qquad \text{as} \quad t \to \infty\ . \tag{2.7}$$

- The nonlinear term stops the population exploding to infinity and instead cuts it off at $x = \beta$ (or $N = \beta/\gamma$). So
  - the equilibrium $x = 0$ is unstable,
  - the equilibrium $x = \beta$ is stable.

- Note this all assumed $\beta, \gamma > 0$. What would happen for $\beta < 0$ or $\gamma < 0$?

- Here's what these solutions look like, graphing $x(t)$ for different x_0 values...



  Note that changing the parameters β and γ would just change the scale on the vertical axis.

- On the right we've done away with the time axis, and just represented the flow of time by arrows on the x-axis. This is called the **phase portrait** of the system. It will be much more useful than the graph when we study systems with multiple variables $x, y, z, ...$

These are about the last systems we'll be able to solve exactly . . .

- From hereon we'll need something smarter — more qualitative — to study how things behave.

- We'll keep using the population model to illustrate more general and powerful ways to find the behaviour of systems, especially when we cannot solve them like we did above.

A small but extremely important thing we did above was to reduce the number of constants, which works like *non-dimensionalization*.

[Side Notes:] Rescaling / Non-dimensionalization

Given a system $\dot{X} = F(X; a, b, ...)$ in terms of a variable x and parameters $a, b, ...$, try to define new scaled quantities to reduce the number of parameters in the equation.

- You can scale any of the variables and/or parameters, say $x = AX$ and let some $\alpha, \beta, ...$ be new combinations of the old parameters $a, b, ...$, to give some $\dot{x} = f(x; \alpha, \beta, ...)$
- The object is for the number of parameters $\alpha, \beta, ...$ to be less than the number of the original parameters $a, b, ...$
- This can be an incredibly powerful tool, and is *vitally important* to do in mathematical modeling. The behaviour in the population models above only really depended on one parameter. The other just behaved like a scaling or 'set of units' for the system.
- Sometimes we start off with a physical or biological problem with many rate constants, material coefficients, and so on, which can be reduced to just one or two parameters that define the system's behaviour.
- When you've studied the system in the scaled quantities, *after* you've understood its behaviour, then you just work out what that all looks like back in the original unscaled variables and parameters.

3 ODEs and their flows

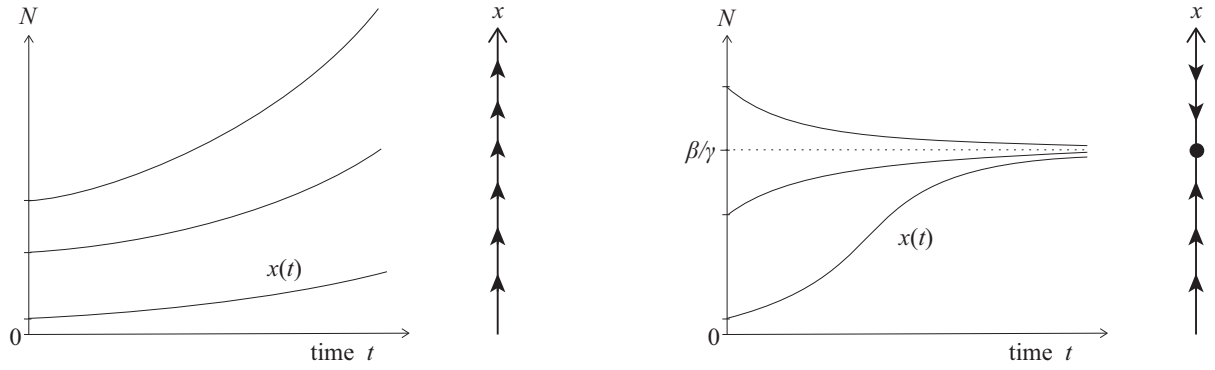
A lot of what we do will be **qualitative** dynamics, forming a conceptual sketch of a system.

That means not having an exact solution like we've plotted in the graphs above, but forming a picture of the geometry of solutions using things like equilibria, as in the phase portraits above.

This can actually be *more* powerful than having exact solutions.

Take the population models:

- Look at the arrows I've drawn on the graphs from the population models, showing the direction of travel according to the ODE, i.e. the vector $\dot{N}$ or $\dot{x}$. In higher dimensions we can't draw the graph (left of each picture below), but we can still understand things in terms of these vectors, drawn just in the space of x (right of each picture below), which are the **phase portraits** of the system.



- The phase portraits are often not only easier to sketch, and actually more useful, than the graphs of solutions. Still, as we get to more than two dimensions even these will be difficult to sketch, but fortunately we'll learn concepts to understand them that will work just as well in higher dimensions.

[Side Notes:] Flows, orbits and phase portraits

Consider an ODE

$$\dot{x} = f(x) \tag{3.1}$$

for $x \in D$ and $f \in R$, from the **domain** $D \subset \mathbb{R}^n$ to the **range** $R \subset \mathbb{R}^n$, that is $f : D \mapsto R$. If we are given an initial condition $x(0) = x_0$ we call this an **initial value problem**.

- The solution $x(t)$ to (3.1) traces out a **trajectory** through D as t changes.
- Often the solution we get depends crucially on the initial condition x_0, so it can help to write a solution of the initial value problem as

$$x(t) = \Phi_t(x_0) \quad s.t. \quad \frac{d}{dt}\Phi_t(x_0) = f\left(\Phi_t(x_0)\right) \quad \& \quad \Phi_0(x_0) = x_0 \; . \tag{3.2}$$

- The function $\Phi_t(x_0)$ is called the **flow operator** of the ODE.
- A complete trajectory $\{\Phi_t(x_0) \; : \; t \in [0, T]\}$ is called an **orbit** of the ODE through the point x_0.
- The collection of all orbits is called the **flow field** (or simply the **flow**).
- Its depiction in the **state space** or **phase space** of x is called the **phase portrait**.
- The system (3.1) is **autonomous** (time-independent).
  If instead time appears on the righthand side, say $\dot{x} = f(x, t)$, then the system is **non-autonomous**.
- An **equilibrium** is a point x_* where the system is stationary, i.e. where

$$f(x_*) = 0 \; . \tag{3.3}$$

All this is just the same if instead of $x \in \mathbb{R}$, we have a multivariable system with a vector $\mathbf{x} = (x, y, z, ...) \in \mathbb{R}^n$. Let's start with $\mathbf{x} = (x, y) \in \mathbb{R}^2$ . . .

To use these methods we have to work with **first order differential equations**.

- Typically we can turn a one-dimensional n^{th} order ODE, into an n-dimensional first order ODE, just by associating each derivative with a spatial coordinate, so . . .
- the ODE $\dot{x} = f(x)$ is a first order ODE.
- the second order ODE $\ddot{x} + b(x)\dot{x} + a(x) = 0$ becomes a first order ODE by letting $y = \dot{x}$, giving

$$\dot{x} = y \ , \qquad \dot{y} = -b(x)y - a(x) \ .$$

- the third order ODE $\dddot{x} + c(x)\ddot{x} + b(x)\dot{x} + a(x) = 0$ becomes a first order ODE by letting $y = \dot{x}$ and $z = \ddot{x}$, giving

$$\dot{x} = y \ , \qquad \dot{y} = z \ , \qquad \dot{z} = -c(x)z - b(x)y - a(x) \ .$$

- and so on. These are quite easy to understand, as y is then the speed, z the acceleration, etc. and in a first order system we include these to form the system's **state space**.
- Particularly with high order (large dimensional) systems, we sometimes prefer indexed variables, so for the last example $\dddot{x} + c(x)\ddot{x} + b(x)\dot{x} + a(x) = 0$ we might instead let $x_1 = x$, $x_2 = \dot{x}$, and $x_3 = \ddot{x}$, giving

$$\dot{x}_1 = x_2 \ , \qquad \dot{x}_2 = x_3 \ , \qquad \dot{x}_3 = -c(x_1)x_3 - b(x_1)x_2 - a(x_1) \ .$$

  This **index form** (called by some the **state space** form) is particularly useful for computer simulations.
- The **state space** is the space occupied by the variables $(x, y, z, ...)$ or $(x_1, x_2, x_3, ...)$ of the n-dimensional first order ODE, typically $\mathbb{R}^n$ or some subset of it (e.g. the population model's state space is $\mathbb{R}_+$ (the positive part of $\mathbb{R}$), the predator-prey model's state space is $\mathbb{R}^2_+$).

4 Two populations

An important model with two populations is the Lotka-Volterra predator-prey model. This considers a number of prey X, and number of predators Y.

$$\dot{X} = \alpha X - \beta XY \ , \qquad \dot{Y} = \delta XY - \gamma Y \ , \tag{4.1}$$

- The X equation says the prey population grows exponentially, minus the rate at which it is preyed on by Y.
- The Y equation says the predator population grows proportional to its rate of feeding on X, minus its natural death rate.

These equations are based on a number of assumptions (see box below).

We cannot solve these equations exactly. Instead, the way we study systems like this then typically starts the same way.

- First thing: try to reduce the number of parameters.

  Let $x = \delta X$ and $y = \beta Y$, giving

  $$\dot{x} = (\alpha - y)x \ , \qquad \dot{y} = (x - \gamma)y \ , \tag{4.2}$$

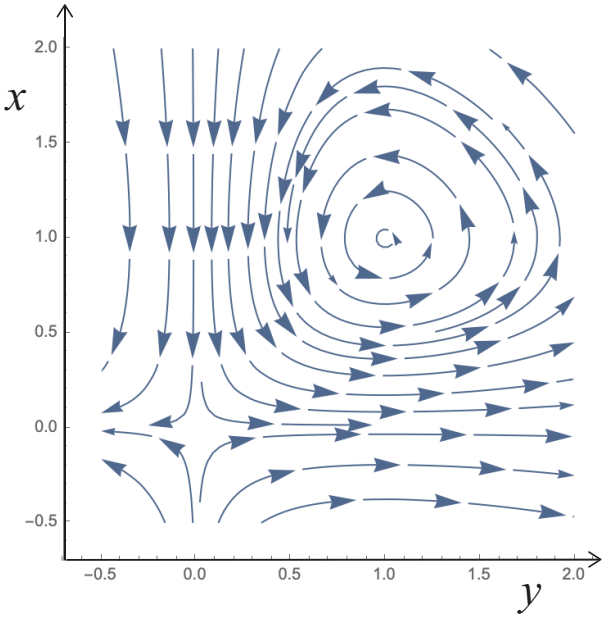
  so now we just have the two parameters α, γ (with δ and β just being the 'units' of x and y).

- Second thing: find any equilibria. There are two places that $\dot{x} = 0$ and $\dot{y} = 0$, and these are the 'trivial' state at $(x, y) = (0, 0)$, and a second state at $(x, y) = (\gamma, \alpha)$.
  At these points (and *only* at these points) the system won't change, elsewhere it will usually be traveling towards or away from these equilibria.

- Third thing: sketch. . .

Plot the **vector field** $(\dot{x}, \dot{y})$ defined by the ODE, by drawing an arrow at each point (x, y) representing the length and direction of $(\dot{x}, \dot{y})$:

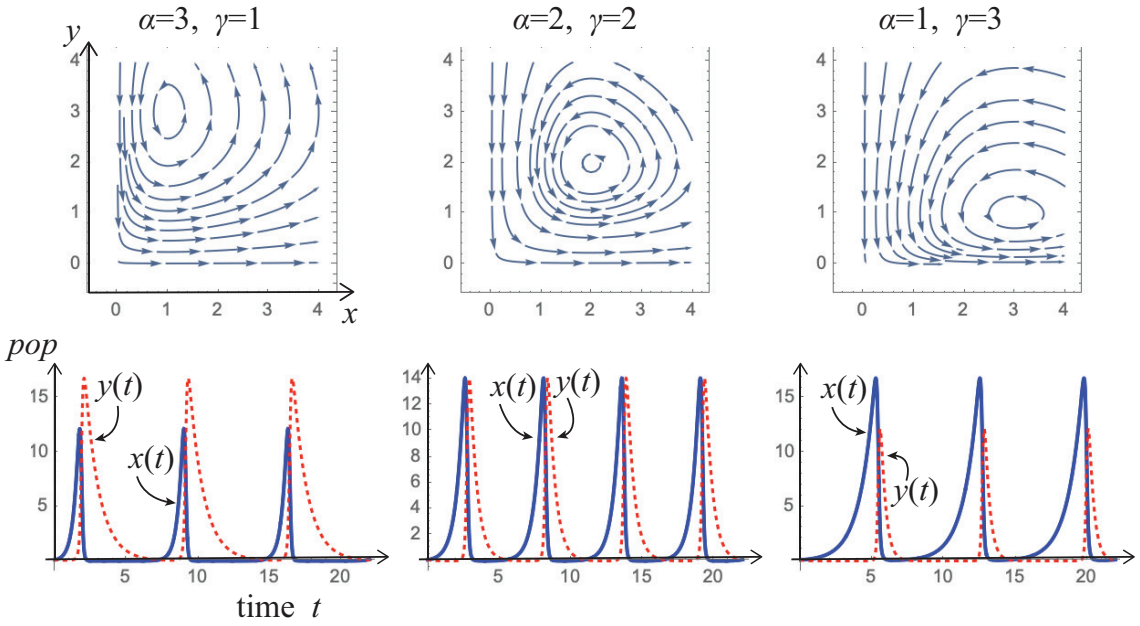
This shows the **phase portrait** of the system in (x, y) space.



- This is plotted for $\alpha = \gamma > 0$. Try out other values, and even try with α and/or γ negative.

- Actually I've cheated a bit in this picture, and not just plotted vectors at each point, but curves with vectors on them. These show bits of solutions of $(x(t), y(t))$, or their *flow*. Some packages like Mathematica can do this (using StreamPlot), in Maple or Matlab you might have to make do with a vector field plot (VectorField in Maple or Quiver in Matlab).

- Below are a few different positive values, along with the graphs of the solutions $x(t)$ and $y(t)$ (found numerically).



[Further Reading Only:] Assumptions of the Lotka-Volterra model

It is always important to understand the simplifying assumptions that we make about a system in order to write down equations modeling it. That's was allows us to understand the limitations of a model and improve it, to compare different models, or compare our model to reality.
For a predator-prey system the equations above assume:

1. The prey population finds ample food at all times.
2. The food supply of the predator population depends entirely on the size of the prey population.
3. The rate of change of population is proportional to its size.
4. During the process, the environment does not change in favour of one species, and genetic adaptation is inconsequential.
5. Predators have limitless appetite.

Other examples, see e.g. SIR model and others on Ex.Sht.

We can form the sketch for the predator-prey model without using a computer. Taking the equations

$$\dot{x} = (\alpha - y)x \ , \qquad \dot{y} = (x - \gamma)y \ ,$$

you have the same two main bits of information:

- nullclines: where $\dot{x} = 0$ or $\dot{y} = 0$, so the vector field $(\dot{x}, \dot{y})$ is vertical on $\dot{x} = 0$ and horizontal on $\dot{y} = 0$, and either side of those curves is up/down/left/right as given by their signs.

- equilibria: where $\dot{x} = \dot{y} = 0$, hence where the nullclines cross, the vector field is zero there and is attracted to and/or repelled from them.

- That's all you need to plot the phase portraits:

  Put points at the equilibria: solutions here cannot move, and other solutions will usually move around, towards, or away from these.

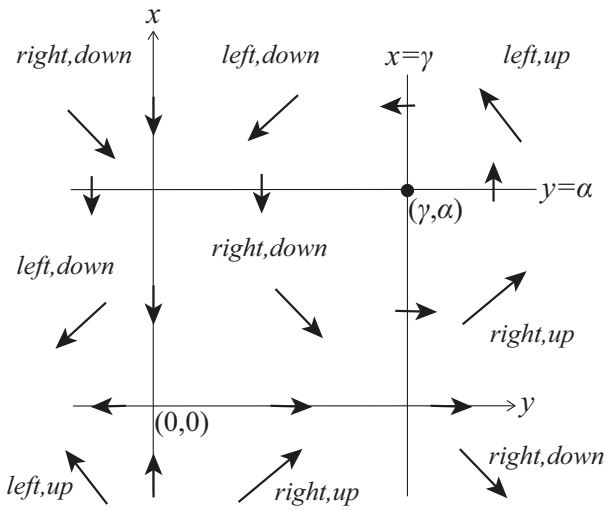
  Use the nullclines to infer whether the vector field $(\dot{x}, \dot{y})$ is pointing up/down and left/right in different regions, don't worry about the actual size of $\dot{x}$ and $\dot{y}$. Here we have 9 different regions:

	$x < 0$	$0 < x < \gamma$	$\gamma < x$
$\alpha < y$	$\dot{y} < 0 < \dot{x}$	$\dot{x}, \dot{y} < 0$	$\dot{x} < 0 < \dot{y}$
$0 < y < \alpha$	$\dot{x}, \dot{y} < 0$	$\dot{y} < 0 < \dot{x}$	$0 < \dot{x}, \dot{y}$
$y < 0$	$\dot{x} < 0 < \dot{y}$	$0 < \dot{x}, \dot{y}$	$\dot{y} < 0 < \dot{x}$

i.e.

	$x < 0$	$0 < x < \gamma$	$\gamma < x$
$\alpha < y$	right,down	left,down	left,up
$0 < y < \alpha$	left,down	right,down	right,up
$y < 0$	left,up	right,up	right,down

and on the nullclines themselves the arrows are either horizontal or vertical.



- Compare this to the flow we plotted earlier. With a bit of practice you can produce accurate phase portraits from a sketch like this. You will need to know how to tell if solutions will go towards or away from an equilibrium, or just circle it as they do in the Lotka-Volterra model. We'll learn how to do this in '*Linear stability*' below.

Want an easy million dollars?

Before you worry about what solutions look like, you need to know whether there are any at all. If they do exist, you then need to know whether there is just one solution, or a whole bunch of possible solutions.

This basic detail is one of the most important parts of dynamical systems theory.

Even for a simple viscous fluid for instance, described by the Navier-Stokes equations, we don't know generally whether solutions exist, and there is a $\$1 million$ pound prize (from the Clay Institute) if you can find a general solution, or even just show there is more than one.

5 Existence and uniqueness

In an ODE

$$\dot{x} = f(x,t) \tag{5.1}$$

if f is differentiable on some region, then there exist unique solutions to the ODE there. For example:

- The linear population model:

  $$\dot{N} = f(N) = (\beta - \delta)N$$

  Here $f(N)$ is continuous and differentiable with respect to N. The solution through any initial point $N(0) = N_0$ clearly exists, since it is

  $$N(t) = N_0 e^{(\beta-\delta)t}$$

  and this is clearly unique for any N_0.

- The non-linear population model:

  $$\dot{N} = f(N) = (\beta - \gamma N)N$$

  Here $f(N)$ is continuous and differentiable with respect to N. The solution through any initial point $N(0) = N_0$ clearly exists, since it is (using the result we found before for $x = \gamma N$)

  $$N(t) = \frac{\beta N_0 e^{\beta t}}{\beta - \gamma N_0 + \gamma N_0 e^{\beta t}}$$

  This is also unique for any N_0, but does not exist for all N because it cannot pass the equilibrium at $N_* = \beta/\gamma$. So:

  - if $N_0 > \beta/\gamma$ then the population is restricted to $N(t) > \beta/\gamma$ for all t,
  - if $N_0 < \beta/\gamma$ then the population is restricted to $N(t) < \beta/\gamma$ for all t.

For more than two dimensions (or more than first order ODEs) things can be more complicated, but roughly speaking the same rule applies: existence and uniqueness can be expected to hold locally where the equations are continuous and differentiable. For example:

- The predator-prey model:

$$\dot{x} = f(x, y) = (\alpha - y)x \ , \qquad \dot{y} = g(x, y) = (x - \gamma)y \ ,$$

  We cannot write down explicit solutions to these, but they are continuous and differentiable in x and y, so given any initial point with (scaled) prey number x_0 and predator number y_0, we can expect solutions to exist and be unique, at least for a limited time and a limited region around (x_0, y_0).

[Side Notes:] Formal statements and sketch proofs

- **Theorem 1 [Existence]** Suppose that $f(x,t)$ is a function that is uniformly Lipschitz continuous in a region

$$R = \{(x,t) \ : \ |x - x_0| < \delta \ , \ |t - t_0| < \varepsilon\} \ . \tag{5.2}$$

  Then there exists an interval $|t - t_0| < E \leq \varepsilon$ where the solution $x(t)$ to (5.1) is defined.

- **Theorem 2 [Uniqueness]** Suppose that $f(x,t)$ and $\frac{\partial}{\partial x} f(x,t)$ are continuous on R with respect to x and t, then there exists an interval $|t - t_0| < \hat{E} \leq E$ where the solution $x(t)$ to (5.1) is unique.

Caveats:

- These are easily generalised to higher-order ODEs

$$\dot{x} = f(x,y,z,...,t) \ , \quad \dot{y} = g(x,y,z,...,t) \ , ...$$

- These only apply on the intervals described.

- For an initial value problem (an ODE plus initial data) these only provide a local statement: near t_0 and x_0. They existence and uniqueness for all space and time, but do not guarantee it — the region of existence may be smaller, e.g. due to finite-time blow up (see Exercise Sheets).

- The criteria are sufficient but not necessary (we can still have existence of unique solutions even if the conditions on f are violated).

- We can estimate $\hat{E} \leq \min\{\varepsilon, \delta/D\}$ where $|f(x,...,t)| \leq D$ on R.

- The concept of *Lipschitz continuity* is weaker than being differentiable but stronger than just being continuous, essentially it says the change in value of f is constrained between any two points $x = a$ and $x = b$, meaning that
$$|f(b) - f(a)| \leq L|b - a|$$
where L is called the *Lipschitz constant*, and 'uniformly' means L does not depend on t . . . the most full version of this Theorem also allows f to depend on and be continuous in t, i.e. a non-autonomous system.

Sketch proof of Existence Theorem:

- Let $\phi(t)$ be a function with continuous derivative on R, then ϕ satisfies the initial value problem
$$\phi'(t) = f(\phi(t), t) \qquad \& \qquad \phi(t_0) = x_0 \tag{5.3}$$
if and only if it satisfies the integral equation
$$\phi(t) = x_0 + \int_{t_0}^{t} f(\phi(s), s)ds \; . \tag{5.4}$$

- To prove the "if": given (5.3), integrate the lefthand side to give $\int_{t_0}^{t} \phi'(s)ds = \phi(t) - \phi(t_0)$, the initial condition gives $\int_{t_0}^{t} \phi'(s)ds = \phi(t) - x_0$, equate this to the integral of the righthand side and we get (5.4).

- To prove the "only if", assuming (5.4), the initial condition follows by substitution, and by the fundamental theorem of calculus we can differentiate, giving (5.3).

So solutions exist, but are they unique?

Sketch proof of Uniqueness Theorem:

- Assume that two functions $\phi(t)$ and $\psi(t)$ satisfy the ODE. Can they be *different* functions?
- Say $|f(\phi(t),t) - f(\psi(t),t)| \le K\,|\phi(t) - \psi(t)|$,

$$\text{then} \qquad |\phi(t) - \psi(t)| \le \left| \int_{t_0}^{t} (f(\phi(s),s) - f(\psi(s),s))\, ds \right|$$
$$\le \int_{t_0}^{t} |f(\phi(s),s) - f(\psi(s),s)|\, ds$$
$$\le K \int_{t_0}^{t} |\phi(s) - \psi(s)|\, ds \;:=\; KU(t) \qquad (5.5)$$

  defining $U(t)$ where clearly $U(t) \ge 0$.

- Eq.(5.5) reads $U' - KU \le 0$. We can multiply by $e^{-K(t-t_0)} \ge 0$ to get $(U' - KU)e^{-K(t-t_0)} \le 0$ or $\left(Ue^{-K(t-t_0)}\right)' \le 0$, whose integral is $U(t)e^{-K(t-t_0)} - U(t_0)e^{-K(t-t_0)} \le 0$, but $U(t_0) = 0$ (since $\phi(t_0) = \psi(t_0) = x_0$), implying $U(t)e^{-K(t-t_0)} \le 0$ implying $U(t) \le 0$.
- Hence $U(t) \ge 0$ & $U(t) \le 0 \;\;\Rightarrow\;\; U(t) = 0 \;\;\Rightarrow\;\; \phi(t) = \psi(t)$.

These important statements are sometimes referred to as the Picard-Lindelöf theorem, and date back to Émile Picard, Ernst Lindelöf, Rudolf Lipschitz, and Augustin-Louis Cauchy. The precise statement can be weakened to not require $\partial f/\partial x$ to be continuous, but only f to be 'uniformly Lipschitz continuous in x' and continuous in t, which just requires that the change in f is smaller than some Lipschitz constant L as x varies (so $\partial f/\partial x$ might not exist).

6 Linear stability

A lot of what we do is **local dynamics**, which means looking at behaviour only near certain points.

- For the simple population model $\dot{N} = \alpha N$ (with $\alpha = \beta - \delta$) we had solutions

$$N(t) = N_0 e^{\alpha t} \tag{6.1}$$

  For $\alpha < 0$ we saw that this tends towards $N(t) \to 0$ as $t \to \infty$.

- For the cut-off population model $\dot{x} = (\beta - x)x$ (with $N = x/\gamma$) we had solutions

$$x(t) = \frac{\beta x_0 e^{\beta t}}{\beta - x_0 + x_0 e^{\beta t}} \tag{6.2}$$

- Taking (6.2), let's ask what happens when t is large and negative ($t \to -\infty$). Then the term e^t is small. We can expand in a Taylor series (in small $w = e^t$ if you like), giving

$$\begin{aligned} x(t) &\approx \tfrac{\beta x_0}{\beta - x_0} e^{\beta t} \left\{ 1 - \tfrac{x_0 e^{\beta t}}{\beta - x_0} + \tfrac{x_0^2 e^{2\beta t}}{(\beta - x_0)^2} - \ldots \right\} \\ &\approx \rho e^{\beta t} \qquad \text{with } \rho = \tfrac{\beta x_0}{\beta - x_0} \end{aligned} \tag{6.3}$$

  which looks rather like the solution (6.1) just with different constants.

  In fact if the initial condition x_0 is small, then

$$\rho = \tfrac{\beta x_0}{\beta - x_0} = x_0 \{ 1 + \tfrac{x_0}{\beta} + \tfrac{x_0^2}{\beta^2} + \ldots \} \approx x_0$$

  so

$$x(t) \approx x_0 e^{\beta t} \qquad \text{or} \qquad N(t) \approx N_0 e^{\beta t} \tag{6.4}$$

  which behaves exactly like (6.1) when the population is small, just with the expansion rate α replaced by β.

- The function (6.4) is a solution to the **local** problem

$$\dot{x} = \beta x \tag{6.5}$$

namely the local approximation $\dot{x} = (\beta - x)x \approx \beta x$ near $x = 0$.

- We call $\dot{x} = \beta x$ the **linearization** of $\dot{x} = (\beta - x)x$ about $x = 0$.

It tells us what happens near $x = 0$ when approximated to linear order in x.

- Let's go back to (6.2) and ask what happens when t is large and positive ($t \to +\infty$). Then the term $e^{\beta t}$ is large. But we can re-write

$$x(t) = \frac{\beta x_0}{(\beta - x_0)e^{-\beta t} + x_0} \tag{6.6}$$

then the term $e^{-\beta t}$ is small. Now we can expand in a Taylor series (in small $w = e^{-\beta t}$ if you like), so this looks like

$$\begin{aligned} x(t) &\approx \beta \left\{ 1 - \tfrac{(\beta - x_0)e^{-\beta t}}{x_0} + \tfrac{(\beta - x_0)^2 e^{-2\beta t}}{x_0^2} - ... \right\} \\ &\approx \beta + \tfrac{x_0 - \beta}{x_0} \beta e^{-\beta t} \\ \Rightarrow \qquad \hat{x}(t) &\approx \hat{\rho} e^{-\beta t} \end{aligned} \tag{6.7}$$

where $\hat{x}(t) = x(t) - \beta$ and $\hat{\rho} = \frac{x_0 - \beta}{x_0}\beta$.

So again we have the same kind of exponential behaviour near $\hat{x} = 0$, which means near $x = \beta$ or $N = \beta/\gamma$.

- The function $\hat{x}(t) = \rho e^{-\beta t}$ is a solution to the **local** problem

$$\dot{\hat{x}} = -\beta \hat{x} \tag{6.8}$$

namely the local approximation $\dot{\hat{x}} = (\beta - x)x \approx -\beta \hat{x}$ near $\hat{x} = 0$ (equivalently the local approximation $\dot{x} = (\beta - x)x \approx \beta - x$ near $x = \beta$).

- We call $\dot{x} = \beta - x$ the **linearization** of $\dot{x} = (\beta - x)x$ about $x = \beta$. It tells us what happens near $x = \beta$ when approximated to linear order in x.

- So we are starting to see that we will *always* find this kind of exponential behaviour around an equilibrium, i.e. any point x_* where $f(x_*) = 0$.
- As we look into the distant future ($t \to +\infty$) or past ($t \to -\infty$), solutions tend towards equilibria exponentially, or else they fly off (diverge) to infinity. We say *solutions asymptote towards/away from the equilibria.*

Let's take the two-dimensional system and try the same.

The predator-prey model (4.1) had two equilibria (solutions of $\dot{x} = 0$ and $\dot{y} = 0$), at $(x_{*1}, y_{*1}) = (0, 0)$ and $(x_{*2}, y_{*2}) = (\gamma, \alpha)$.

- Let's approximate about these two equilibria:
  - Linearize about $(x_{*1}, y_{*1}) = (0, 0)$:

$$\begin{aligned} \dot{x} &= (\alpha - y)x \approx \alpha x + ... \\ \dot{y} &= (x - \gamma)y \approx -\gamma y + ... \end{aligned} \tag{6.9}$$

    These look a bit like the linearizations of the 1d population model, an exponential growth in x away from an equilibrium at $x = 0$, and an exponential contraction in y towards an equilibrium at $y = 0$.
  - Linearize about $(x_{*2}, y_{*2}) = (\gamma, \alpha)$:

$$\begin{aligned} \dot{x} &= (\alpha - y)x \approx (\alpha - y)\gamma + ... \\ \dot{y} &= (x - \gamma)y \approx (x - \gamma)\alpha + ... \end{aligned} \tag{6.10}$$

    Again these look a bit like the linearizations of the 1d population model but the x and y terms are mixed up. You might recognize this as a second order ODE that gives oscillations, since differentiating gives $\ddot{x} \approx -\gamma\alpha x + \gamma^2\alpha$ and $\ddot{y} \approx -\alpha\gamma y + \alpha^2\gamma$.

General solution of the local system

- Because these local systems are linear we can solve them exactly (see Exercise Sheet).
- If a system has an equilibrium at $\mathbf{x} = 0$, nearby it will look in vector form like $\dot{\mathbf{x}} = \underline{\underline{A}}\mathbf{x}$, whose solution is $\mathbf{x}(t) = e^{\underline{\underline{A}}t}\mathbf{x}_0$.
- More generally around an equilibrium at $\mathbf{x} = \mathbf{x}_*$ the system will look like

$$\dot{\mathbf{x}} = \underline{\underline{A}}(\mathbf{x} - \mathbf{x}_*) \qquad \text{with solution} \qquad \mathbf{x}(t) = \mathbf{x}_* + e^{\underline{\underline{A}}t}(\mathbf{x}_0 - \mathbf{x}_*) \ .$$

  - Superficially that just looks the same as the 1d systems above.
  - In a solution to a scalar problem $x(t) = x_0 e^{\alpha t}$, the exponent gives us the rate α of repulsion from ($\alpha > 0$) or attraction to ($\alpha < 0$) an equilibrium.
  - Now we have a 'rate' matrix $\underline{\underline{A}}$. Just like the rate α this tells us whether solutions are repelled to or attracted from the equilibrium, but also whether they circulate around it.
- We must remember these are only local approximations near the equilibria. Let's go back to our predator-prey example.

- For the predator-prey system, re-writing $\mathbf{x} = \binom{x}{y}$,
  - the linearization about $(x_{*1}, y_{*1}) = (0, 0)$ is:

$$\dot{\mathbf{x}} \approx \underline{\underline{A}}.\mathbf{x} \qquad \text{where} \qquad \underline{\underline{A}} = \begin{pmatrix} \alpha & 0 \\ 0 & -\gamma \end{pmatrix} \tag{6.11}$$

    whose solution is

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} \approx e^{\underline{\underline{A}}t}\mathbf{x}_0 = \begin{pmatrix} e^{\alpha t}x_0 \\ e^{-\gamma t}y_0 \end{pmatrix} \tag{6.12}$$

    That's easy enough, the x and y variables just grow or shrink with t, depending whether α and γ or positive or negative.

  - the linearization about $(x_{*2}, y_{*2}) = (\gamma, \alpha)$ is:

$$\dot{\mathbf{x}} \approx \underline{\underline{A}}.(\mathbf{x} - \mathbf{x}_{*2}) \qquad \text{where} \qquad \underline{\underline{A}} = \begin{pmatrix} 0 & -\gamma \\ \alpha & 0 \end{pmatrix}, \quad \mathbf{x}_{*2} = \begin{pmatrix} \gamma \\ \alpha \end{pmatrix}, \tag{6.13}$$

    whose solution is (writing $\mathbf{x}(t) - \mathbf{x}_*$ on the lefthand side)

$$\begin{aligned} \begin{pmatrix} x(t) - \gamma \\ y(t) - \alpha \end{pmatrix} &\approx \mathbf{x}_* + e^{\underline{\underline{A}}t}(\mathbf{x}_0 - \mathbf{x}_*) \\ &= \begin{pmatrix} \cos(t\sqrt{\alpha\gamma}) & -\frac{\gamma}{\sqrt{\alpha\gamma}}\sin(t\sqrt{\alpha\gamma}) \\ \frac{\alpha}{\sqrt{\alpha\gamma}}\sin(t\sqrt{\alpha\gamma}) & \cos(t\sqrt{\alpha\gamma}) \end{pmatrix} \begin{pmatrix} x_0 - \gamma \\ y_0 - \alpha \end{pmatrix} \end{aligned} \tag{6.14}$$

    (you can find that using ODE methods you've learned, or directly expanding the matrix exponential [see Exercise Sheet]).

- So you can easily see from this what solutions do, yes?
- No? Me neither. We're going to need a slightly smarter way to understand matrices. The Exercise Sheet tries to explore this a bit. (You can get a rough idea that for $\alpha\gamma > 0$ the solution around (x_{*2}, y_{*2}) is going to oscillate, because the cos and sin functions are going to oscillate as t changes, but for $\alpha\gamma < 0$ these become cosh and sinh functions which are like exponentials, giving growth or decay similar to the solution around (x_{*1}, y_{*1}). But we need to do better . . .)

7 Stability & Eigendecomposition

Here's where the eigenvectors and eigenvalues of a matrix come in.

- On the whole the behaviour created by a 'rate' matrix like we saw above is complicated. But along certain directions in space it is very simple . . . the matrix's eigenvectors.
- If we take an initial condition $\mathbf{x}_0$ that happens to lie along the direction $\mathbf{v}$ from an equilibrium, where $\mathbf{v}$ is an eigenvector of a matrix $\underline{\underline{A}}$ with an eigenvalue λ, then

$$\underline{\underline{A}}.\mathbf{v} = \lambda\mathbf{v} \tag{7.1}$$

  then it follows that

$$e^{\underline{\underline{A}}t}.\mathbf{v} = e^{\lambda t}\mathbf{v} \tag{7.2}$$

[Side Notes:] Proof

$$\begin{aligned} \underline{\underline{A}}.\mathbf{v} = \lambda\mathbf{v} \quad &\Rightarrow \quad \underline{\underline{A}}^2.\mathbf{v} = \lambda\underline{\underline{A}}\mathbf{v} = \lambda^2\mathbf{v} \quad \Rightarrow \quad \ldots \\ &\Rightarrow \quad \underline{\underline{A}}^n.\mathbf{v} = \lambda^n\mathbf{v} \end{aligned} \tag{7.3}$$

so

$$e^{\underline{\underline{A}}t}.\mathbf{v} \;=\; \sum_{n=0}^{\infty} \tfrac{1}{n!}t^n\underline{\underline{A}}^n.\mathbf{v} \;=\; \sum_{n=0}^{\infty} \tfrac{1}{n!}t^n\lambda^n\mathbf{v} \;=\; e^{\lambda t}\mathbf{v} \tag{7.4}$$

In the predator-prey model near (x_{*1}, y_{*1}) the 'rate' matrix was $\underline{\underline{A}} = \begin{pmatrix} \alpha & 0 \\ 0 & -\gamma \end{pmatrix}$. This has :

- eigenvalue α along eigenvector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$,
  so if $\mathbf{x}_0 - \mathbf{x}_* = \begin{pmatrix} x_0 \\ 0 \end{pmatrix} = x_0 \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ then

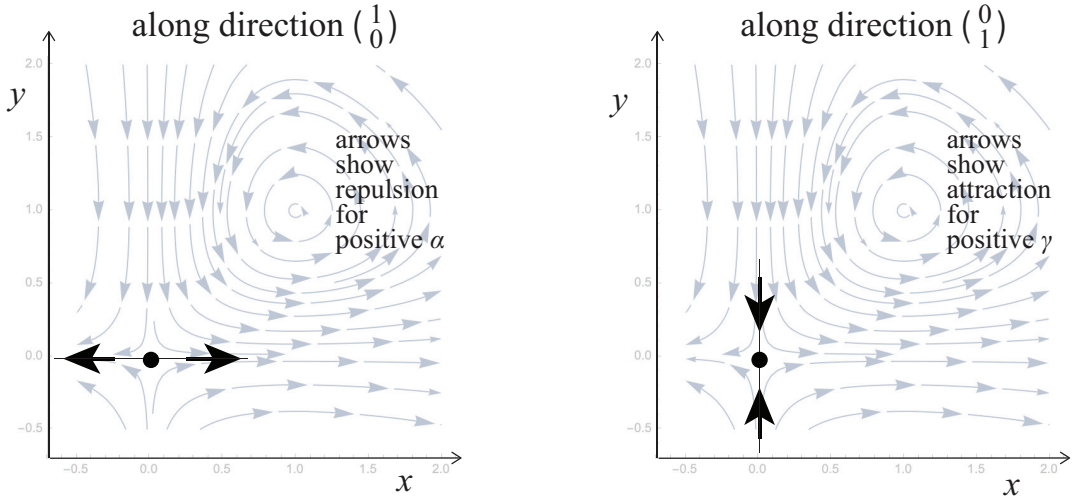
$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} \approx e^{\underline{\underline{A}}t}\mathbf{x}_0 = x_0 e^{\alpha t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{7.5}$$

  i.e. simple 1d attraction/repulsion along direction $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ with rate α.

- eigenvalue $-\gamma$ along eigenvector $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$,
  so if $\mathbf{x}_0 - \mathbf{x}_* = \begin{pmatrix} 0 \\ y_0 \end{pmatrix} = y_0 \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ then

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} \approx e^{\underline{\underline{A}}t}\mathbf{x}_0 = y_0 e^{-\gamma t} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{7.6}$$

  i.e. simple 1d attraction/repulsion along direction $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ with rate $-\gamma$.



- That's too easy, now look at the other equilibrium. . .

Near (x_{*2}, y_{*2}) the 'rate' matrix was $\underline{\underline{A}} = \begin{pmatrix} 0 & -\gamma \\ \alpha & 0 \end{pmatrix}$. This has :

- eigenvalues $\pm\sqrt{-\alpha\gamma}$ along eigenvectors $\begin{pmatrix} \pm\sqrt{-\gamma} \\ \sqrt{\alpha} \end{pmatrix}$,
  so if $\mathbf{x}_0 - \mathbf{x}_* = r\begin{pmatrix} \pm\sqrt{-\gamma} \\ \sqrt{\alpha} \end{pmatrix}$ then

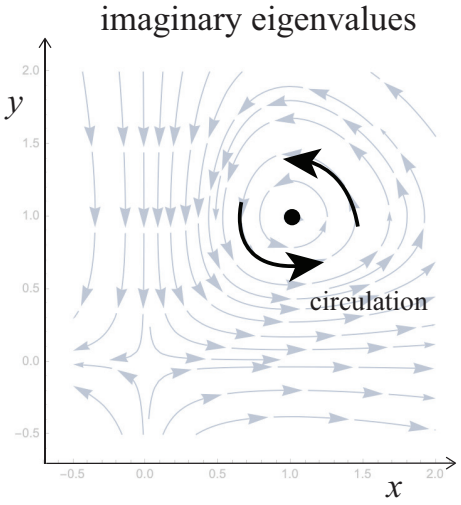
$$\begin{pmatrix} x(t) - \gamma \\ y(t) - \alpha \end{pmatrix} \approx e^{\underline{\underline{A}}t}\mathbf{x}_0 = re^{\pm\sqrt{-\alpha\gamma}t}\begin{pmatrix} \pm\sqrt{-\gamma} \\ \sqrt{\alpha} \end{pmatrix} \tag{7.7}$$

  - If $\gamma\alpha < 0$ this gives simple 1d attraction along direction $\begin{pmatrix} -\sqrt{-\gamma} \\ \sqrt{\alpha} \end{pmatrix}$ and repulsion along direction $\begin{pmatrix} +\sqrt{-\gamma} \\ \sqrt{\alpha} \end{pmatrix}$.
    (Note I didn't say r had to be real).
  - If $\gamma\alpha > 0$ this can't give a real solution, as $\sqrt{-\alpha\gamma} = i\sqrt{\alpha\gamma}$ so this looks like the complex valued

$$\begin{pmatrix} x(t) - \gamma \\ y(t) - \alpha \end{pmatrix} \approx e^{\underline{\underline{A}}t}\mathbf{x}_0 = re^{\pm i\sqrt{\alpha\gamma}t}\begin{pmatrix} \pm\sqrt{-\gamma} \\ \sqrt{\alpha} \end{pmatrix} \tag{7.8}$$

    but it is easy to enough to play around and turn this into something real-valued [see sidenote below].

    More importantly note what this means. You know
    $e^{\pm i\sqrt{\alpha\gamma}t} = \cos(\sqrt{\alpha\gamma}t) + i\sin(\sqrt{\alpha\gamma}t)$
    so solutions with this term oscillate with frequency $\sqrt{\gamma\alpha}$, and in a manner shaped by the eigenvectors.



[Side Notes:] To find the real-plane oscillations

Consider an initial condition that is a linear combination of the two eigenvectors

$$\mathbf{x}_0 - \mathbf{x}_* = r\begin{pmatrix} +\sqrt{-\gamma} \\ \sqrt{\alpha} \end{pmatrix} + r^*\begin{pmatrix} -\sqrt{-\gamma} \\ \sqrt{\alpha} \end{pmatrix}$$

for $r \in \mathbb{C}$ and $\alpha, \gamma > 0$. Despite appearances this, and the solutions $\mathbf{x}(t) = ((x(t), y(t))$ that flow out of it, must be real.

With a little work we can re-arrange these into something more useful, in particular to see they are indeed real. Write

$$\begin{aligned}
\begin{pmatrix} x(t) - x_* \\ y(t) - y_* \end{pmatrix} &\approx e^{\underline{\underline{A}}t}(\mathbf{x}_0 - \mathbf{x}_*) \qquad (7.9) \\
&= re^{+i\sqrt{\alpha\gamma}t}\begin{pmatrix} +\sqrt{-\gamma} \\ \sqrt{\alpha} \end{pmatrix} + r^*e^{-i\sqrt{\alpha\gamma}t}\begin{pmatrix} -\sqrt{-\gamma} \\ \sqrt{\alpha} \end{pmatrix} \\
&= r[C + iS]\begin{pmatrix} +\sqrt{-\gamma} \\ \sqrt{\alpha} \end{pmatrix} + r^*[C - iS]\begin{pmatrix} -\sqrt{-\gamma} \\ \sqrt{\alpha} \end{pmatrix} \\
&\quad \text{using shorthand } C = \cos\left(\sqrt{\alpha\gamma}t\right),\ S = \sin\left(\sqrt{\alpha\gamma}t\right) \\
&= \begin{pmatrix} (r - r^*)\sqrt{-\gamma} \\ (r + r^*)\sqrt{\alpha} \end{pmatrix}\cos\left(\sqrt{\alpha\gamma}t\right) + i\begin{pmatrix} (r + r^*)\sqrt{-\gamma} \\ (r - r^*)\sqrt{\alpha} \end{pmatrix}\sin\left(\sqrt{\alpha\gamma}t\right) \\
&\quad \text{let } r = (a + ib)/2 \\
&= \begin{pmatrix} \sqrt{\gamma} & 0 \\ 0 & \sqrt{\alpha} \end{pmatrix}\left\{\begin{pmatrix} -b \\ a \end{pmatrix}\cos\left(\sqrt{\alpha\gamma}t\right) - \begin{pmatrix} a \\ b \end{pmatrix}\sin\left(\sqrt{\alpha\gamma}t\right)\right\}
\end{aligned}$$

So we see that a solution oscillates as cos and sin along two orthogonal directions, creating (uneven) circulation around the equilibrium. The factors $\sqrt{\gamma}$ and $\sqrt{\alpha}$ just skew the whole picture by a constant.

[Side Notes:] Eigendecomposition

You can write any ODE or its solutions in terms of the eigenvectors associated with a given equilibrium.

- A non-singular $n \times n$ matrix $\underline{\underline{A}}$ has n linearly independent eigenvectors $\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_n$. We can therefore express any vector as a linear combination of them, say

$$\mathbf{x} - \mathbf{x}_* = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + ... + c_n\mathbf{v}_n \tag{7.10}$$

  for some coefficients $c_1, ..., c_n$.

- Take the linearization of the ODE

$$\begin{aligned} \dot{\mathbf{x}} = \underline{\underline{A}}.(\mathbf{x} - \mathbf{x}_*) &= \underline{\underline{A}}.(c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + ... + c_n\mathbf{v}_n) \\ &= c_1\lambda_1\mathbf{v}_1 + c_2\lambda_2\mathbf{v}_2 + ... + c_n\lambda_n\mathbf{v}_n \end{aligned} \tag{7.11}$$

  giving us equations of motion along these different eigenvectors.

- Similarly for the solutions of the ODE

$$\begin{aligned} \mathbf{x}(t) = \mathbf{x}_* + e^{\underline{\underline{A}}t}.(\mathbf{x}_0 - \mathbf{x}_*) &= \mathbf{x}_* + e^{\underline{\underline{A}}t}.(c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + ... + c_n\mathbf{v}_n) \\ &= \mathbf{x}_* + c_1e^{\lambda_1 t}\mathbf{v}_1 + c_2e^{\lambda_2 t}\mathbf{v}_2 + ... + c_ne^{\lambda_n t}\mathbf{v}_n \end{aligned} \tag{7.12}$$

  giving us rates of attraction to / repulsion from $\mathbf{x}_*$ along the eigenvector directions (and rotation if the λ_rs are complex).

8 Saddle, Node, Focus, or Center?

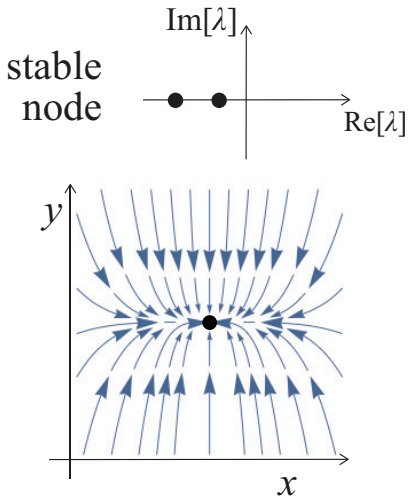
You've seen now pretty much everything that can happen around an equilibrium: as t increases you have either attraction ($x \sim e^{-t}$) or repulsion ($x \sim e^{+t}$), with or without circulation ($x \sim e^{it}$).

The eigenvalues and eigenvectors tell us everything we need about this:

- the eigenvalues tell us the rates of attraction/repulsion/circulation
- the eigenvectors tell us the directions along which these rates apply
- (along other directions the behaviour is just a mixture of these).

We get three types of behaviour:

- A **node** has two real eigenvalues λ_1, λ_2, with the same signs:
  - $\lambda_1, \lambda_2 > 0$ means an unstable node (repeller)
  - $\lambda_1, \lambda_2 < 0$ means a stable node (attractor)
  - Order them as $|\lambda_1| < |\lambda_2|$, then λ_2 gives the **strong** eigendirection and λ_1 gives the **weak** eigendirection (telling us the fastest and slowest directions of motion).
  - Note $\det \underline{\underline{A}} = \lambda_1 \lambda_2 > 0$.
- A **saddle** has two real eigenvalues λ_1, λ_2, with different signs:
  - There is attraction along one direction and repulsion along the other.
  - Order them as $\lambda_1 < 0 < \lambda_2$, then λ_2 gives the **unstable** (repelling) eigendirection and λ_1 gives the **stable** (attracting) eigendirection.
  - Note $\det \underline{\underline{A}} = \lambda_1 \lambda_2 < 0$.
- A **focus** has two complex eigenvalues $\lambda_1, \lambda_2 \in \mathbb{C}$:
  - they are complex conjugates $\lambda_1 = \lambda_2^*$
  - the real part $\mathsf{Re}[\lambda_1] = \mathsf{Re}[\lambda_2]$ tells us the rate of attraction/repulsion just like the real eigenvalues of a node (so $\mathsf{Re}[\lambda_1] > 0$ mean unstable, $\mathsf{Re}[\lambda_1] < 0$ means stable).
  - the imaginary part $\mathsf{Im}[\lambda_1] = -\mathsf{Im}[\lambda_2]$ gives the frequency of circulation.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node
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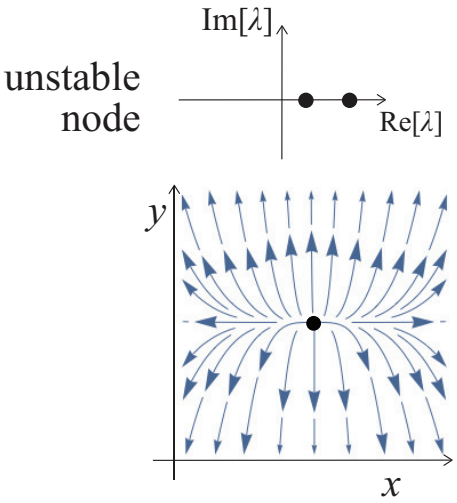
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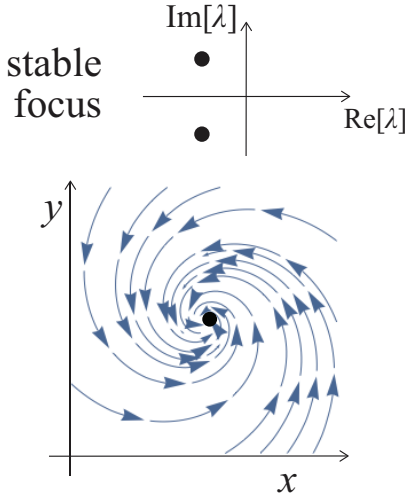
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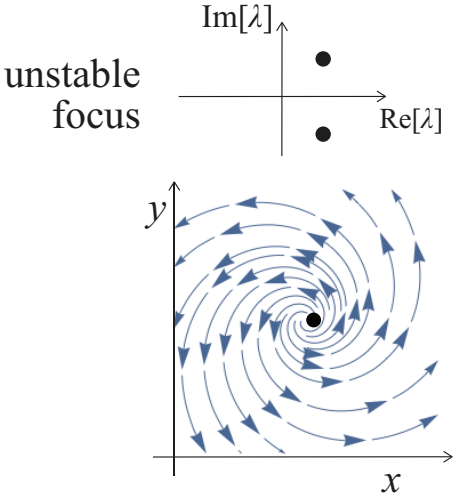
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y
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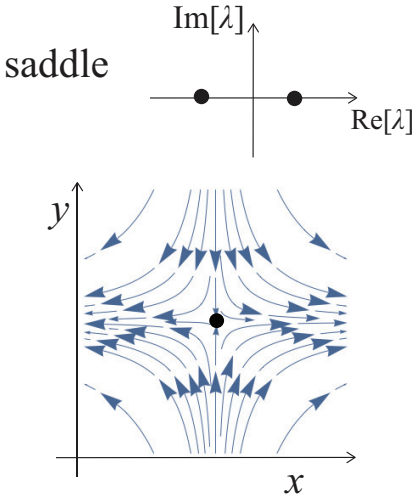
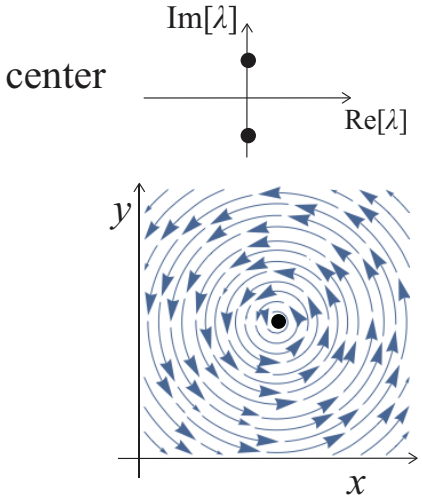

saddle
Im[λ]
Re[λ]
y
x


center
Im[λ]
Re[λ]
y
x

- A **centre** is the special case of a focus with no attraction/repulsion:
  - $\lambda_1, \lambda_2 \in \mathbb{I}$ so $\mathsf{Re}[\lambda_1] = \mathsf{Re}[\lambda_2] = 0$

- The predator-prey model equilibrium (x_{*1}, y_{*1}) classifies as:

	$\gamma < 0$	$\gamma > 0$
$\alpha > 0$	unstable node	saddle
$\alpha < 0$	saddle	stable node

  while the equilibrium (x_{*2}, y_{*2}) classifies as:

	$\gamma < 0$	$\gamma > 0$
$\alpha > 0$	saddle	center
$\alpha < 0$	center	saddle

- In higher dimensions things are just an extension of these:
  - in n dimensions, there will be n eigenvalues. Some might be real, others complex, some positive and others negative, so an equilibrium can be stable and unstable in different directions, and can be like a node, focus, saddle, or centre along different directions. But always:
  - if there are complex eigenvalues they always occur in conjugate pairs,
  - a node has at least two real eigenvalues with the same sign,
  - a saddle has at least two real eigenvalues with different signs,
  - a focus has at least two eigenvalues that are a complex conjugate pair.
  - An equilibrium is typically a composite of these, e.g. a saddle-focus has at least two eigenvalues that are a complex conjugate pair, but not all real parts or real eigenvalues have the same sign.

9 Local stability

Linear systems are actually the only ones we can solve in general, and this is certainly true when we go to multiple dimensions (i.e. multiple variables, like the two population model above).

- But the **linear theory** above would be rather useless if it was only good for solving linear systems.
- As we have seen, linearization provides a local approximation to equilibria more generally, so in a system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$, if $\mathbf{x}_*$ is any point where $\mathbf{f}(\mathbf{x}_*) = 0$, then locally the system looks like

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) \approx \underline{\underline{A}}.(\mathbf{x} - \mathbf{x}_*)$$

  which has solutions

$$\mathbf{x}(t) \approx \mathbf{x}_* + e^{\underline{\underline{A}}t}.(\mathbf{x}_0 - \mathbf{x}_*)$$

- We rarely use this solution, we just infer the behaviour from the eigenvalues and eigenvectors of $\underline{\underline{A}}$.
- The matrix $\underline{\underline{A}}$ is found from the derivatives of the ODE at the equilibrium:

$$\text{in 1d} \quad \text{for} \quad \dot{x} = f(x) \quad : \quad A \equiv \lambda = \left.\frac{df(x)}{dx}\right|_{x=x_*} \tag{9.1}$$

$$\text{in } n\text{d} \quad \text{for} \quad \dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) \quad : \quad \underline{\underline{A}} = \left.\frac{d\mathbf{f}(\mathbf{x})}{d\mathbf{x}}\right|_{\mathbf{x}=\mathbf{x}_*}$$

$$\& \quad ||\underline{\underline{A}} - \lambda \underline{\underline{I}}|| = 0 \tag{9.2}$$

  where generally $\underline{\underline{A}}$ is the Jacobian matrix

$$\frac{d\mathbf{f}}{d\mathbf{x}} = \frac{\partial(f, g, ...)}{\partial(x, y, ...)} = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \cdots \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix} \tag{9.3}$$

  with $\mathbf{x} = (x, y, ...)$, $\mathbf{f} = (f, g, ...)$.

[Side Notes:] Linearization

Look back at the revision notes:

- Approximating a 1d system about a point x_* we have

$$\dot{x} = f(x) = f(x_*) + f'(x_*)(x - x_*) + ... \tag{9.4}$$

  and $f(x_*) = 0$ at an equilibrium, so removing that and dropping higher orders we have

$$\dot{x} = f(x) = A(x - x_*) \tag{9.5}$$

  called the *linearization* of the system, where $A = f'(x_*)$ where $f'(x_*) = \frac{d}{dx} f(x)\big|_{x=x_*}$.

- Similarly for an nd system about a point $\mathbf{x}_* = (x_*, y_*, ...)$ we have

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) = \mathbf{f}(\mathbf{x}_*) + \underline{\underline{A}}.(\mathbf{x} - \mathbf{x}_*) + ... \tag{9.6}$$

  and $\mathbf{f}(\mathbf{x}_*) = 0$ at an equilibrium, so again removing that and dropping higher orders we have

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) = \underline{\underline{A}}.(\mathbf{x} - \mathbf{x}_*) \tag{9.7}$$

  called the *linearization* of the system, where $\underline{\underline{A}}$ is the Jacobian of $\mathbf{f}$ at $\mathbf{x}_*$.

But actually we can say something stronger. The linearization isn't just an approximation, *close enough* to the equilibrium it captures the local behaviour *exactly* (the way the derivative of a graph captures its gradient) . . .

10 Stable and unstable manifolds

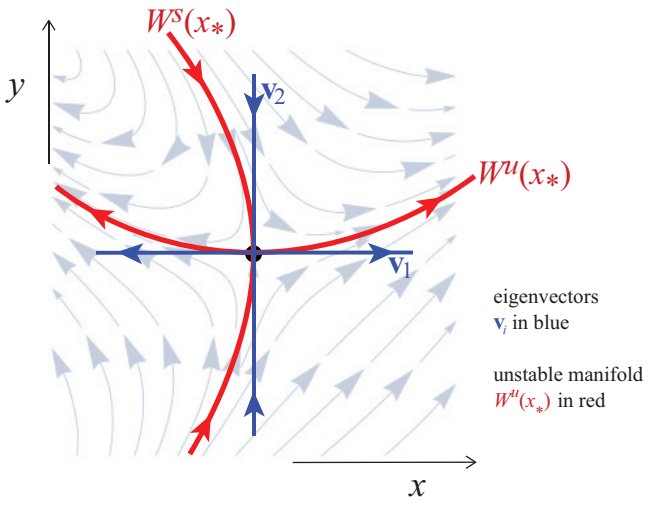
The linearization, and the eigenvectors, only tell us the *local* picture.
What happens as you look further away from an equilibrium?

- As we follow the eigenvector direction out away from the equilibrium, we follow an orbit that begins to bend (typically, if there are nonlinear terms in the ODE).

- E.g. take a simple saddle $\dot{x} = x$, $\dot{y} = -y$, and add some nonlinear terms, say

$$\begin{aligned} \dot{x} &= x + y^2 \ , \\ \dot{y} &= -y + x^2 \end{aligned}$$

  The figure shows the eigenvectors pointing along the axes from the saddle at the origin. There is an orbit that runs along each eigenvector at the saddle itself, but further away becomes curved.



- Let x_* be a saddle point of the vector field $\dot{x} = f(x)$. The set of all points "ending up at x_*" under the flow of Φ_t of the vector field

$$W^s(x_*) = \{x \in \mathbb{R}^n \mid \Phi_t(x) \to x_* \ \text{ as } \ t \to +\infty\} \ , \tag{10.1}$$

  is called the *stable manifold* of x_*.

- Similarly, the set of all points "coming from x_*"

$$W^u(x_*) = \{x \in \mathbb{R}^n \mid \Phi_t(x) \to x_* \ \text{ as } \ t \to -\infty\} \ , \tag{10.2}$$

  is called the *unstable manifold* of x_*.

- The dimensions of the stable manifolds, plus the dimensions of the unstable manifolds, of an equilibrium, must add up to the total dimension of the system.

Invariant manifolds are hard to find in general but ...

[Side Notes:] **Theorem**

Let x_* be a saddle point of $\dot{x} = f(x)$. Then x_* is also a saddle point of the linearised system

$$\dot{x} = A(x - x_*).$$

Let $E^s(x_*)$ and $E^u(x_*)$ denote the stable and unstable eigenspaces of the linearised system (the directions the eigenvectors point along). In a small enough neighbourhood U of x_*, there exists a local piece of $W^s(x_*)$, that is a smooth manifold that is a graph of some function $h : E^s(x_*) \to W^s(x_*)$. Furthermore, $W^s(x_*)$ is tangent to $E^s(x_*)$ at x_*. The same is true for the local unstable manifold $W^u(x_*)$ which is defined in the same way.

So this says the stable and unstable manifolds, whatever shape they have, are tangent to the eigenvectors at an equilibrium, with the appropriate stability (a stable manifold if the eigenvalue along that direction has negative real part, an unstable manifold if the eigenvalue along that direction has positive real part).

Basin of attraction

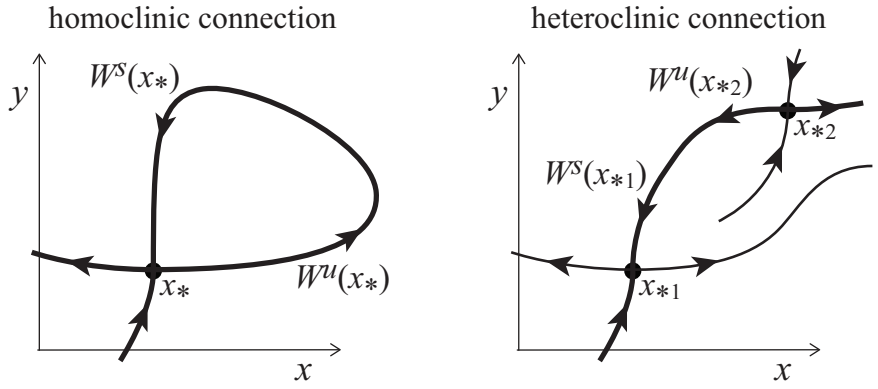
In a nonlinear system there might be many equilibria, some stable and some unstable.

- Around each stable equilibrium there will be a region of orbits that are all attracted into that equilibrium. This is called its **basin of attraction**.
- The size and shape of the basin of attraction is typically determined by the positions of the stable and unstable manifolds from the various equilibria. These can be hard to find exactly.

Connections

The stable and unstable manifolds of two equilibria can connect to each other.

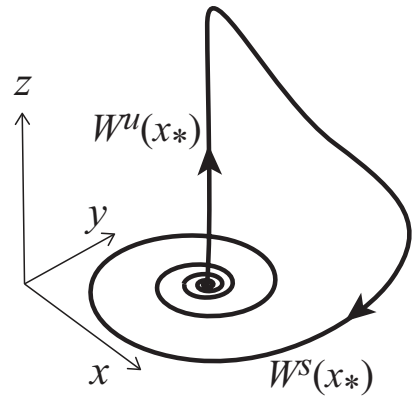
- A **homoclinic connection** occurs when the unstable manifold of an equilibrium connects back to the stable manifold of the same equilibrium.
- A **heteroclinic connection** occurs when the unstable manifold of one equilibrium connects to the stable manifold of another equilibrium.



- Note in each case the dimension of the stable and unstable manifolds do not have to be the same.
- E.g. The 'Shilnikov bifurcation' is a homoclinic connection where a one-dimensional unstable manifold connects back to the two-dimensional stable manifold of a saddle-focus (two complex eigenvalues with negative real part, and a positive real eigenvalue), for example in the Rössler system

$$\begin{aligned}\dot{x} &= -y - z \\ \dot{y} &= x + ay \\ \dot{z} &= bx - cz + xz\end{aligned}$$

with $a = 0.38$, $b = 0.3$, $c = 4.820$.



11 Similar systems

We have seen from the cut-off population model, and the predatory-prey model, that:

- the linearization gives a local approximation around the equilibrium, or put another way,
- the system looks *similar to* the linearization near the equilibrium.

This idea of two systems looking *similar* is extremely powerful.
We call it **topological equivalence**.

- Essentially two systems are topologically equivalent if we can find a homeomorphism between them — a map that continuously (and invertibly) warps the orbits of one system into those of another.

[Side Notes:] Topological equivalence

Suppose we have two vector fields

$$\dot{x} = f(x) \qquad \text{and} \qquad \dot{y} = g(y) \tag{11.1}$$

where $x \in U$ and $y \in V$, on domains $U, V \subset \mathbb{R}^n$. Then these are *topologically equivalent* if we can find a continuous and invertible map (a *homeomorphism*) $h : U \to V$ that maps orbits of one system to the other, respecting the direction of time.

- An important concept in topological equivalence is hyperbolicity. An equilibrium is **hyperbolic** if none of its eigenvalues lie on the imaginary axis.

[Side Notes:] Theorem: on Topological Equivalence for linear flows

Consider the two linear vector fields

$$\dot{x} = Ax \qquad \text{and} \qquad \dot{x} = Bx \tag{11.2}$$

where $x \in \mathbb{R}^n$, and A and B are $n \times n$ matrices. Let $n_{\pm}(A)$ denote the number of eigenvalues of a matrix A that have a $\pm$ve real part.
Then these systems are topologically equivalent if and only if

$$n_+(A) = n_+(B) \neq 0 \qquad \text{and} \qquad n_-(A) = n_-(B) \neq 0 \, .$$

- Now that isn't much use if we can't extend it beyond linear systems. So we need to be able to describe topological equivalence, between nonlinear systems, or between a nonlinear system and its linearization. The next theorem does this.

[Side Notes:] Theorem of Hartman & Grobman

If the system

$$\dot{x} = f(x) \tag{11.3}$$

has a hyperbolic equilibrium at $x = x_*$, then there exists a neighbourhood U of x_* such that the system on U is topologically equivalent to the linearised system

$$\dot{y} = Ay \tag{11.4}$$

on an (arbitrary) neighbourhood V of the origin, where A is the Jacobian of the first system at x_*, and $y = x - x_*$.

12 Stability and genericity

- If a small change in a system (e.g. a small change in its parameter values) results in a topologically equivalent system, we say it is **structurally stable**.
- We say a system is **generic** if it occurs '*typically*', i.e. has a nonzero probability of occurring in a given system (or its occurrence is 'not measure zero').
- Genericity therefore depends on the *class* of systems we are talking about — the defining conditions. E.g. is the system in $\mathbb{R}$ or $\mathbb{R}^n$ or only a limited domain $x > 0$, is the system conservative, reversible, does it have a symmetry, and so on? If so, we can talk about genericity within a given class of system.
- Usually a generic system is structurally stable, otherwise any small change will result in a different system, so the system has a zero-measure chance of occurring.
- If a major structural (qualtitative/topological) change does take place we'll call it a *bifurcation*.

13 Bifurcations

Look back at the nonlinear population model (2.5). We said β was positive, giving:

- an unstable equilibrium at $x = 0$,
- a stable equilibrium at $x = \beta$, which the population grows towards,

but what happens if β becomes negative? You should see that then there is:

- a stable equilibrium at $x = 0$, which the population shrinks towards,
- an unstable equilibrium at $x = 0$.

That is, if β becomes negative then the stable equilibrium at $x = \beta > 0$ collapses to zero, and moves to negative values $x = \beta < 0$, and in doing so it becomes unstable. The unstable equilibrium $x = 0$ becomes stable in the process, so for $\beta < 0$ the population shrinks to zero, i.e. becomes extinct.

The population has undergone a **bifurcation** in which the two equilibria switch ordering, and switch stability. In other bifurcations equilibria can appear or disappear, change stability in different ways, or the oscillatory behaviour around them can change.

- The idea of a bifurcation is: as we vary a parameter of the model smoothly, the model should change smoothly. But there are places where this doesn't happen. Where small changes don't lead to an equivalent system.
- These are **bifurcation points**: places where a system may turn into one of two non-equivalent systems if we change a parameter slightly.
- In dynamics:
  - **Local bifurcations** occur when an eigenvalue of an equilibrium passes through the imaginary axis. They result only in local changes around the equilibrium.
  - **Global bifurcations** occur when two invariant objects meet (e.g. a periodic orbit and an equilibrium). They result in changes that cannot be entirely described locally (near any one point).

To look at some simple examples, first consider the roots of a function $f(x)$, i.e. the points where $x = 0$. (You'll need to sketch these simple functions to follow this argument; you'll want them to refer back to for the next part too).

- The equation $f(x) = ax + b$ always has one root, $x = -b/a$. Nothing about this can change qualitatively.
- The equation $f(x) = x^2 + ax + b$ is more interesting.
  - First let $a = b = 0$, then $f = x^2 = 0$ has two roots at $x = 0$.
  - Now vary b slightly, so $f = x^2 + b$. Now we get different scenarios depending on the sign of b. If $b < 0$ there are two real roots $x = \pm\sqrt{-b}$, but if $b > 0$ there are no real roots. A bifurcation between the two happens at $b = 0$.
  - Now instead vary a slightly, so $f = x^2 + ax$. We get different scenarios depending on the sign of a. If $a > 0$ then one of the roots moves to $x = -a < 0$, if $a < 0$ then it moves to $x = -a > 0$, while the other stays at $x = 0$ throughout. That's a minor change, but more important is the slope of f. If $a > 0$ then the slope of f is positive at the root $x = 0$, if $a < 0$ then the slope is negative at $x = 0$ (with the opposite sign slopes at the root $x = -a$). A bifurcation happens between the situations for $a > 0$ and $a < 0$.
  - For the full picture we have to vary both a and b. The two roots lie at $x_* = (-a \pm \sqrt{a^2 - 4b})/2$, so there is a *bifurcation curve* in (a, b) space given by $a^2 - 4b = 0$, giving two real roots in $a^2 > 4b$ and no real roots in $a^2 < 4b$. In the parameter region $a^2 > 4b$, whichever root is furthest to the right (most positive x) has a positive slope of f, and they switch slopes when they collide at $a = 0$.
- The first case above is an example of a **fold** bifurcation, where two special points (the roots) can 'fold' together to annihilate each other (or the quadratic curve, viewed from the side, looks 'folded' at its trough point where the roots collide).
- The second case above is an example of a **transcritical** bifurcation, where two special points pass through each other and exchange properties.

As we increase the order of our polynomial, more roots can appear, so more things can happen. For example:

- Take the cubic $f = x^3 + ax^2 + bx + c = 0$ and look at its roots. (Doing things algebraically is difficult for this case, but why not look up how to solve a cubic?!)
- Immediately you can see it has up to three roots. Plot it, change a, b, c, to move the graph around. The cubic can change shape in ways the quadratic cannot, so for some values it has turning points, for others it doesn't.
- It can have one or three roots. See if you can work out what values of a, b, c, the following things happen at. . .
- Two roots can collide and annihilate in a *fold bifurcation* like the quadratic we looked at above.
- Two roots can pass through each other like the *transcritical bifurcation*.
- Set $a = c = 0$ and notice there is always a root at $x = 0$ then, and as b changes sign two roots either side of $x = 0$ come in and collide and annihilate, while the slope at $x = 0$ changes; this is a cousin of the transcritical and we call it a **pitchfork** bifurcation.
- At $a = b = 0$ all three roots collide, and depending how we change a, b, c, away from zero we might get one or three roots; this point is called a **cusp bifurcation**.
- If you investigate the bifurcation curves in a, b, c, space you'll find out why these are called *cusps* and *pitchforks*.

And we can go on. Try a quartic, or a quintic, and so on. What about a sin curve, or an exponential curve? What about having a function of more than one variable $x, y, z, ...$? We get more and more elaborate bifurcations.

Fortunately in this endless zoo of things that can happen, there are a set of 'standard animals' which we call the **normal forms** of the bifurcations. For the bifurcations of roots in the functions above:

- the normal form of the fold is $f = x^2 - b$
- the normal form of the transcritical is $f = x^2 - bx$
- the normal form of the cusp is $f = x^3 - bx + c$
- the normal form of the pitchfork is $f = x^3 - bx$

Importantly, if one of these bifurcations happens at a point $x = 0$, then we can change variables to make the function f equal to these normal forms plus higher order terms (so plus things involving x^3 or higher for the fold or transcritical, plus things involving x^4 or higher for the cusp or pitchfork, and so on).

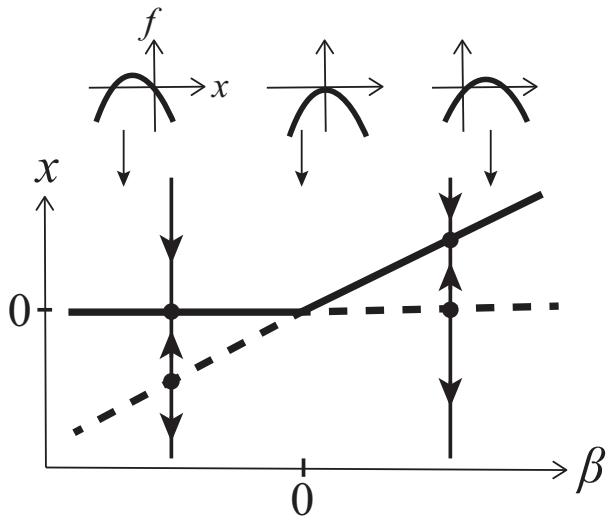
Bifurcations get interesting when we apply these functions to something where these roots have physical relevance, e.g. to dynamical systems.

Transcritical bifurcation

Let's go back to the 1d population model

$$\dot{x} = f(x) = (\beta - x)x \tag{13.1}$$

- There is an equilibrium at $x = 0$. The stability there is given by $f'(0) = \beta - 2x = \beta$, so for $\beta < 0$ it is stable and for $\beta > 0$ it is unstable.
- There is also an equilibrium at $x = \beta$. The stability there is given by $f'(\beta) = \beta - 2\beta = -\beta$, so for $\beta < 0$ it is unstable and for $\beta > 0$ it is stable.
- We see that at $\beta = 0$ the equilibria coincide.
- As β changes sign the equilibria swap ordering along the real x-line, and they swap stability.
- This is a **transcritical bifurcation**.
- We make a *bifurcation diagram* of the event by plotting the x-position of the equilibria against the *bifurcation parameter* β:



The figure shows the bifurcation diagram in x, β, space. Also on this, I've drawn two examples of the phase portraits in x (the vertical lines with arrows and equilibria on). I've also sketched (at the top) the graphs of $f(x)$ at $\beta < 0$, at $\beta = 0$, and at $\beta > 0$.

[Side Notes:] Transcritical bifurcation normal form

Let

$$\dot{x} = f(x, \beta) \tag{13.2}$$

with $x, \beta \in \mathbb{R}$.
Then a transcritical bifurcation occurs at $x = x_*$ when $\beta = \beta_*$ if the following conditions hold:

(B1) $f(x_*, \beta) = 0$ "equilibrium at $x = x_*$",

(B2) $\det(\frac{\partial f}{\partial x}) = 0$ at $x = x_*$, $\beta = \beta_*$, "zero eigenvalue",

(B3) $\frac{\partial f}{\partial \beta} = 0$ at $x = x_*$, $\beta = \beta_*$, "zero speed of f w.r.t. β",

(G1) $\frac{\partial^2 f}{\partial x^2} \neq 0$ at $x = x_*$, $\beta = \beta_*$, "second order derivative nonzero",

(G2) $\frac{\partial}{\partial \beta}\frac{\partial f}{\partial x} \neq 0$ at $x = x_*$, $\beta = \beta_*$, "positive speed of $\frac{\partial f}{\partial x}$ in β",

then (13.2) has the topological normal form

$$\dot{y} = \beta y - y^2 \tag{13.3}$$

in a neighbourhood of (x_*, β_*).

- We call the (B..) conditions **bifurcation conditions**.
  They tell us quantities that must be zero for the bifurcation to occur.
- We call the (G..) conditions **genericity conditions**
  (also sometimes called *non-degeneracy conditions*).
  They tell us quantities that must not be zero for the bifurcation to have its typical form. Usually if any of these genericity conditions fail (i.e. are zero) then a more complicated bifurcation is happening (because they are degenerate in some way, e.g. multiple bifurcations are happening at once).

The transcritical bifurcation is actually a special situation because of condition (B3), which more typically won't hold.

Fold bifurcation

Let's go back to the 1d population model, but consider some population control that gives a constant growth rate β rather than βx,

$$\dot{x} = f(x) = \beta - x^2 \tag{13.4}$$

- Now we see this has two equilibria at $x = \pm\sqrt{\beta}$ for $\beta > 0$.

  Their stabilities are given by $f'(\pm\sqrt{\beta}) = \mp 2\sqrt{\beta}$, so one is stable and the other unstable.

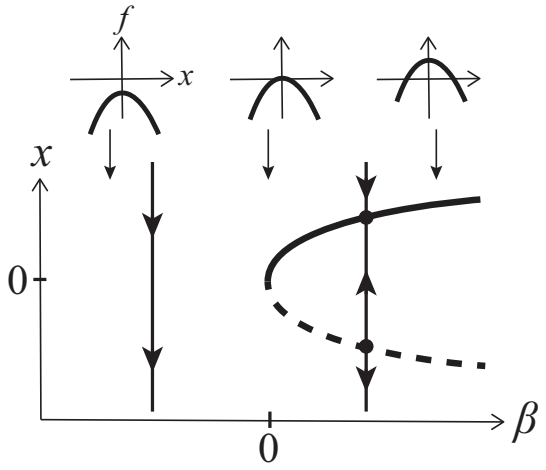
- When $\beta = 0$ the two equilibria collide at $x = 0$.

- For $\beta < 0$ there are no equilibria in the system at all (then $\dot{x} = f(x) < 0$ for all x, so the population always shrinks and dies out).

- Okay, the equilibrium at $x = -\sqrt{\beta}$ isn't physically realistic in this situation, since you can't have a negative population! But it is a dynamically meaningful point that helps us understand the system.

- At $\beta = 0$ a **fold** bifurcation has occurred.

- We make a *bifurcation diagram* of the event by plotting the x-position of the equilibria against the *bifurcation parameter* β:



The figure shows the bifurcation diagram in x, β, space. Also on this, I've drawn two examples of the phase portraits in x (the vertical lines with arrows and equilibria on). I've also sketched (at the top) the graphs of $f(x)$ at $\beta < 0$, at $\beta = 0$, and at $\beta > 0$.

[Side Notes:] **Fold normal form**

Let

$$\dot{x} = f(x, \beta) \tag{13.5}$$

with $x, \beta \in \mathbb{R}$. Then a fold bifurcation occurs at $x = x_*$ when $\beta = \beta_*$ if the following conditions hold:

(B1) $f(x_*, \beta) = 0$ "equilibrium at $x = x_*$",

(B2) $\det(\frac{\partial f}{\partial x}) = 0$ at $x = x_*$, $\beta = \beta_*$, "zero eigenvalue",

(G1) $\frac{\partial^2 f}{\partial x^2} \neq 0$ at $x = x_*$, $\beta = \beta_*$, "second order derivative nonzero",

(G2) $\frac{\partial f}{\partial \beta} \neq 0$ at $x = x_*$, $\beta = \beta_*$, "positive speed of f in β",

then the system has the topological normal form

$$\dot{y} = \beta \pm y^2 \tag{13.6}$$

in a neighbourhood of (x_*, β_*).

If more than one dimension the two equilibria involved are usually a saddle and a node, which collide and annihilate at some $\beta = 0$. For this reason the fold bifurcation in dynamics is often known as a **saddlenode bifurcation**.

A note on multiple dimensions:

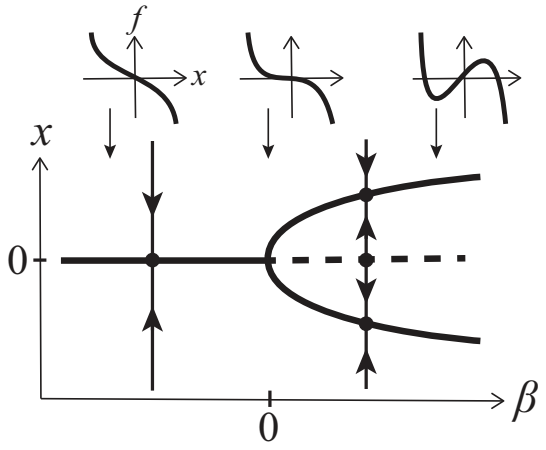
- If we have a system with multiple dimensions, e.g. $(\dot{x}, \dot{y}, ...) = (f, g, ...)$, we can still find bifurcations like the fold (and the others we'll see below), but you can see that the derivative conditions like $\frac{\partial f}{\partial x}$ will get a bit more involved.
- One way to study them is to find coordinates so that the bifurcation can be analysed just in the $\dot{x}$ equation, and we can ignore the others. Technically this means finding the **centre manifold** of the bifurcation, and taking the coordinate x along it. We won't go into that here.
- We'll do things more methodologically: find the equilibria of the system, observe what happens to them as you change a parameter, and deduce what bifurcation occurs. For example, if two equilibria collide and annihilate, with their eigenvalues becoming zero, the bifurcation is a fold.

Pitchfork bifurcation

Let's see what would happen if the 1d population model had a cubic decay rate instead, say

$$\dot{x} = f(x) = \beta x - x^3 \tag{13.7}$$

- There is an equilibrium at $x = 0$. The stability there is given by $f'(0) = \beta - 3x^2 = \beta$, so for $\beta < 0$ it is stable and for $\beta > 0$ it is unstable.
- There are two equilibrium at $x = \pm\sqrt{\beta}$, so these exist only for $\beta > 0$. Their stability is given by $f'(\pm\sqrt{\beta}) = \beta - 3x^2 = \beta - 3(\pm\sqrt{\beta})^2 = \beta - 3\beta = -2\beta$, so these are stable.
- We see that at $\beta = 0$ the equilibria coincide.
- As β changes sign we go from having just one stable equilibrium for $\beta < 0$, to this becoming unstable for $\beta > 0$ and two stable equilibria being created.
- This is a **pitchfork bifurcation**.
- We make a *bifurcation diagram* of the event by plotting the x-position of the equilibria against the *bifurcation parameter* β:



The figure shows the bifurcation diagram in x, β, space. Also on this, I've drawn two examples of the phase portraits in x (the vertical lines with arrows and equilibria on). I've also sketched (at the top) the graphs of $f(x)$ at $\beta < 0$, at $\beta = 0$, and at $\beta > 0$.

- Again, in the population model strictly only the equilibria in $x \geq 0$ are physically meaningful.
- Try the same analysis with $f(x) = \beta x + x^3$, you should find something similar but with the opposite stability of the equilibria.

[Side Notes:] Pitchfork normal form

Let

$$\dot{x} = f(x, \beta) \tag{13.8}$$

with reflectional symmetry $f(-x, \beta) = -f(x, \beta)$, with $x, \beta \in \mathbb{R}$.
Then a pitchfork bifurcation occurs at $x = x_*$ when $\beta = \beta_*$ if the following conditions hold:

(B1) $f(x_*, \beta) = 0$ "equilibrium at $x = x_*$",

(B2) $\det(\frac{\partial f}{\partial x}) = 0$ at $x = x_*$, $\beta = \beta_*$, "zero eigenvalue",

(B3) $\frac{\partial^2}{\partial x^2} f(x_*, \beta_*) = 0$ at $x = x_*$, $\beta = \beta_*$, "second order derivative zero"',

(B4) $\frac{\partial}{\partial \beta} f(x_*, \beta_*) = 0$ at $x = x_*$, $\beta = \beta_*$, "parameter derivative zero"',

(G1) $\frac{\partial^3}{\partial x^3} f(x_*, \beta_*) \neq 0$ at $x = x_*$, $\beta = \beta_*$, "third order derivative nonzero"',

(G2) $\frac{\partial}{\partial \beta} \frac{\partial}{\partial x} f(x_*, \beta_*) \neq 0$ at $x = x_*$, $\beta = \beta_*$, "positive speed in β",

then the system has the topological normal form

$$\dot{y} = \beta y \pm y^3, \tag{13.9}$$

in a neighbourhood of (x_*, β_*).

The pitchfork is a special situation (like the transcritical we looked at before), because it only happens in systems with reflectional symmetry.

Cusp bifurcation

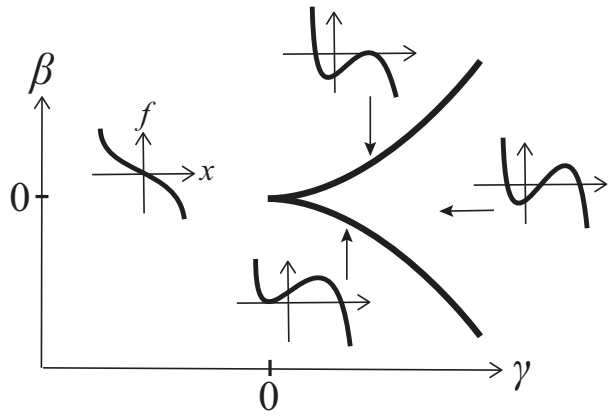
Let's go back to the 1d population model one last time. What if it had more parameters? Say a constant growth rate β plus a linear growth and a cubic decay,

$$\dot{x} = f(x) = \beta + \gamma x - x^3 \tag{13.10}$$

- This can have one or three equilibria depending on the values of β and γ. (It isn't easy to write out x values of these, but we usually don't need to).
- If there is only one equilibrium, say x_*, then it must have derivative $f'(x) < 0$ (sketch $f(x)$ to convince yourself of this, or try to prove it . . . essentially it is because the $-x^3$ term demands f is decreasing). So this equilibrium is stable.
- This system can exhibit folds, which would create two more equilibria. Folds require $f'(x) = \gamma - 3x^2 = 0$, so at $x = \pm\sqrt{\gamma/3}$, and if we combine this with the equilibrium condition $f(x) = 0$ we have

  $$0 = \beta + \gamma(\pm\sqrt{\gamma/3}) - (\pm\sqrt{\gamma/3})^3 \quad = \quad \beta \pm \tfrac{2}{3^{3/2}}\gamma^{3/2}$$

  i.e. folds happen when the two parameters satisfy $(\beta/2)^2 = (\gamma/3)^3$.
- Since folds create two equilibria, of different stabilities, there then exist two stable and one unstable equilibria.
- These folds lie along curves in the (β, γ) parameter plane, that meet at $\beta = \gamma$ in a cusp shape:
- At $\beta = \gamma = 0$ the three equilibria all coincide.
- This is a **cusp bifurcation**.
- The full *bifurcation diagram* now requires plotting the x-position of the equilibria against the *bifurcation parameters* β and γ:



- If we change the decay $-x^3$ to a growth $+x^3$ we will obtain a similar bifurcation but with the opposite stabilities of equilibria.

[Side Notes:] Cusp normal form

Let

$$\dot{x} = f(x, \beta, \gamma) \tag{13.11}$$

with $x, \beta, \gamma \in \mathbb{R}$.
Then a cusp bifurcation occurs at $x = x_*$ when $\beta = \beta_*$, $\gamma = \gamma_*$, if the following conditions hold:

(B1) $f(x_*, \beta, \gamma) = 0$ "equilibrium at x_*",

(B2) $\det(\frac{\partial f}{\partial x}) = 0$ at $x = x_*$, $\beta = \beta_*$, $\gamma = \gamma_*$, "zero eigenvalue",

(B3) $\frac{\partial^2 f}{\partial x^2} = 0$ at $x = x_*$, $\beta = \beta_*$, $\gamma = \gamma_*$, "second order derivative vanishes"',

(G1) $\frac{\partial^3 f}{\partial x^3} \neq 0$ at $x = x_*$, $\beta = \beta_*$, $\gamma = \gamma_*$, "third order derivative nonzero",

(G2) $\frac{\partial f}{\partial \beta}\frac{\partial^2 f}{\partial x \partial \gamma} - \frac{\partial f}{\partial \gamma}\frac{\partial^2 f}{\partial x \partial \beta} \neq 0$ at $x = x_*$, $\beta = \beta_*$, $\gamma = \gamma_*$, "positive speed in β",

then the system has the topological normal form

$$\dot{y} = \beta + \gamma y \pm y^3 \tag{13.12}$$

in a neighbourhood of (x_*, β_*, γ_*).

Codimension

When defining bifurcations two things are paramount:

- what is the class of systems you are studying?
  (E.g. are they symmetric $f(x) = -f(x)$, reflective $-f(x) = f(-x)$, reversible $f(x,t) = -f(x,-t)$, or conservative $f(x) = \nabla\phi(x)$, . . . ?)
- how many parameters must be varied for the bifurcation to happen?

This last one is called the **codimension** of the bifurcation.

- Each bifurcation has a number of conditions, the (B..) conditions in the bifurcation definitions above, that must be satisfied for them to happen.
- The first condition (B1) is just that there is an equilibrium. When we solve the other conditions, we then find parameter values that satisfy them. Each condition fixes one parameter (or removes one parameter freedom, e.g. fixes one parameter in relation to another).

These conditions are the (B..) conditions in the bifurcation definitions above, excluding the equilibrium condition. So:

- A fold has codimension one, and can occur in a system $\dot{x} = f(x)$ without any special conditions (a *generic* system).
- A transcritical has codimension one, but only happens if a system $\dot{x} = f(x)$ is fixed such that $(B3)$ holds, e.g. when the bifurcation involves an immovable equilibrium at $x = 0$.
- A pitchfork has codimension one, but only happens if a system is symmetric. E.g. if we take our cusp example above, but fix $\beta = 0$, we get a symmetric system; then varying β gives a pitchfork.
- A cusp has codimension two, and can occur in a system $\dot{x} = f(x)$ without any special conditions (a *generic* system).
- There is an endless list of higher codimension bifurcations that involve varying two or more parameters, in one dimension or in higher dimensions. They are all extensions of these basis ideas, and involve equilibria (dis)appearing and/or swapping stability in collisions.
- There is just one more, we should look at, because it does something rather different. It requires at least two dimensions . . .

14 Hopf bifurcation

(This is sometimes called an *Andronov-Hopf* bifurcation after the two mathematicians most associated with its discovery).

What happens if we have a system with an equilibrium that changes stability, but doesn't encounter any other equilibria in doing so? Take a simple equilibrium at the origin

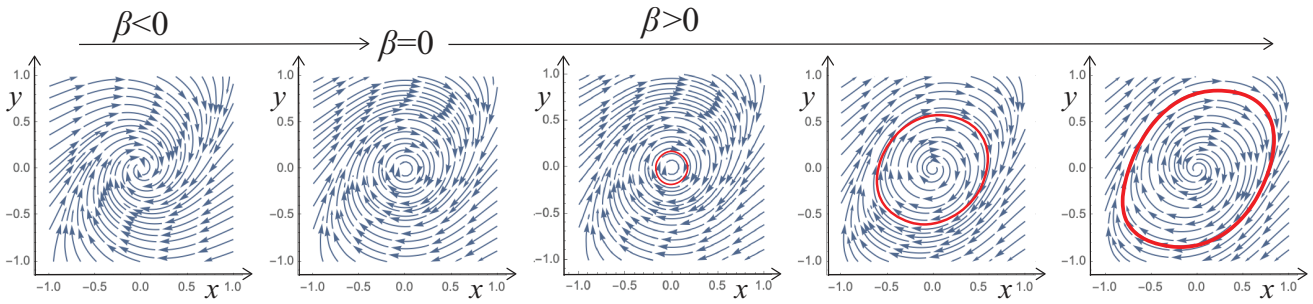
$$\dot{x} = f_1(x) = \beta x + y \ , \qquad \dot{y} = f_2(x) = \beta y - x \ , \tag{14.1}$$

- You should easily be able to see now that this equilibrium has eigenvalues $\beta \pm i$, so it is a focus with stability changing with the sign of β.
- So when $\beta = 0$ the equilibrium is null-stable, and actually the entire flow consists of closed circles around the origin — this is a centre, which is actually a very special situation in itself. If there were any nonlinear terms, typically we wouldn't get a centre, they'd cause slow attraction to or repulsion from the equilibrium.

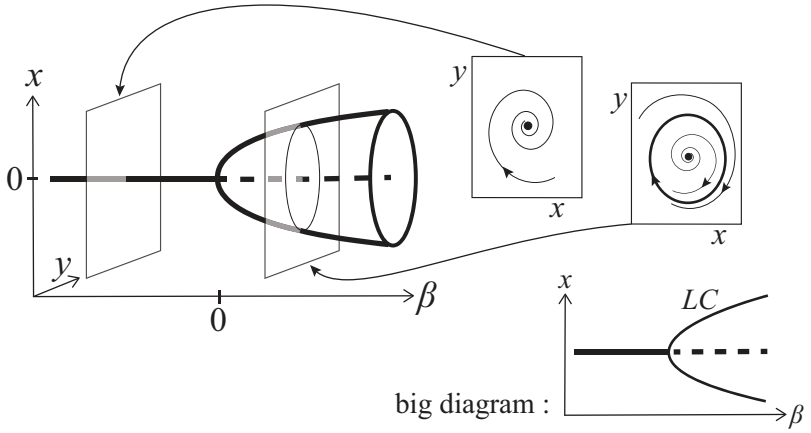
So consider adding a nonlinear term that gives an attraction growing with the radius from the origin, say

$$\dot{x} = f_1(x) = \beta x + y - x(x^2 + y^2) \ , \qquad \dot{y} = f_2(x) = \beta y - x \ , \tag{14.2}$$

- Now watch what happens as we change the sign of β.



- The stability at the origin is determined by the local terms, so that is still stable for $\beta < 0$, unstable for $\beta > 0$, and null-stable for $\beta = 0$.
- But now, the nonlinear term gives an over-riding attraction towards the origin from far away — we call this a *global* attraction to the origin.
- So when $\beta > 0$, the origin is unstable so the flow spirals outward, but far away the flow spirals inward. Somewhere in the middle they must balance out, where they form a close orbits called a **limit cycle**.
- This limit cycle grows with a radius proportional to roughly $\sqrt{\beta}$, giving a bifurcation diagram something like:



The main picture shows a branch representing the equilibrium, unmoving as β increases until β changes sign, then the equilibrium changes stability, and throws out a limit cycle whose radius grows with $\sqrt{\beta}$. The cross-sections show the phase portraits. The little picture on the bottom-right is a bit simpler, it shows what we usually draw on an actual bifurcation diagram, just the branches of the equilibrium, and thin curves denoting the limit cycle (sometimes labelled LC).

- By our argument, this limit cycle must be stable (i.e. attracting). We call this a **supercritical Hopf bifurcation**.
- Change the nonlinear term from $-x(x^2+y^2)$ to $+x(x^2+y^2)$, and instead the limit cycle appears when $\beta < 0$ and is unstable. We call this a **subcritical Hopf bifurcation**.
- The size of the nonlinear terms that decide whether the bifurcation is supercritical or subcritical is called the *first Lyapunov quantity*.

[Side Notes:] Hopf bifurcation

Let

$$\dot{x} = f(x, \beta) \tag{14.3}$$

with $x, \beta, \gamma \in \mathbb{R}$.
Then a Hopf bifurcation occurs at $x = x_*$ when $\beta = \beta_*$, at an equilibrium with eigenvalues

$$\lambda_\pm = \rho(\beta) \pm i\omega(\beta) \ , \tag{14.4}$$

if the following conditions hold:

(B1) $f(x_*, \beta) = 0$ "equilibrium at x_*",

(B2) $\rho(\beta_*) = 0$ at $x = x_*$, $\beta = \beta_*$, "imaginary eigenvalues",

(G1) $\ell_1 \neq 0$ at $x = x_*$, $\beta = \beta_*$, where ℓ_1 is the "first Lyapunov quantity"

(G2) $\omega(\beta_*) \neq 0$ at $x = x_*$, $\beta = \beta_*$, "positive speed in β",

then the system has the topological normal form $z \in \mathbb{C}$

$$\dot{z} = (\rho + i\omega)z + \ell_1 z|z|^2 \tag{14.5}$$

giving a

- supercritical Hopf bifurcation if $\ell_1 < 0$ and
- subcritical Hopf bifurcation if $\ell_1 > 0$.

A little strangely we've decided to write this normal form in complex variables. If we let $z = x+iy$ we can see that this is just a neat way of writing a two-dimensional ODE

$$\begin{aligned} \dot{x} &= \rho x - \omega y + \ell_1 x(x^2 + y^2) \\ \dot{y} &= \omega x + \rho y + \ell_1 y(x^2 + y^2) \end{aligned}$$

[see Ex.Sht].

15 Limit cycles and periodic orbits

Equilibria are points where a system is stationary, i.e. doesn't change.
There are also points to which a system might return again and again, forming closed orbits or cycles like the one we saw in the Hopf bifurcation.

- A **periodic orbit** is the orbit of a solution that satisfies

$$x(t) = x(t+T) \tag{15.1}$$

  and so returns to any point along its orbit after a time T, called its **period**.

- A **limit cycle** is a periodic orbit that is stable or unstable, i.e. attracts or repels surrounding orbits a bit like an equilibrium.

- We sometimes use the terms interchangeably, but be careful. A limit cycle is also a periodic orbit (because it is a cycle so it repeats), but a periodic orbit might not be a limit cycle (other orbits might not limit to/from it).

- E.g. In a center every orbit is a periodic orbit. Since this means no orbit is attracted to or repelled from any other, none of these are limit cycles.

- E.g. In the Hopf bifurcation we saw a limit cycle born from an equilibrium as it changed stability.

We usually cannot write the equations of a periodic orbit or cycle exactly. So we need a different way to study them, a simpler way.

But if an orbit always returns to the same point, then why bother studying the whole orbit, why not just study one point and the flow nearby? This is the idea of a **return map**, which we'll look at shortly below.

16 Maps from ODEs

All of the theory we did above was **local** — behaviour at or near equilibria.

- **Global** (or non-local) behaviour concerns orbits that travel large distances in a system. They might:
  - connect the unstable manifold of a saddle back to its stable manifold (a **homoclinic** connection),
  - connect the unstable manifold of one equilibrium to the stable manifold of another (a **heteroclinic** connection),
  - travel in closed repeating orbits called **periodic orbits**, on which

$$x(t+T) = x(T) \qquad \text{for period } T$$

  - become trapped in enclosed shapes that almost but never precisely repeat, giving **chaos**.

To study **global** behaviour in ODEs we often use maps.

- Poincaré maps are most commonly used to study periodic orbits and chaos (which we'll shortly get to).
- Stroboscopic maps are most commonly used in systems with an obvious imposed period, e.g. an electric circuit driven by an alternating current with a known frequency.

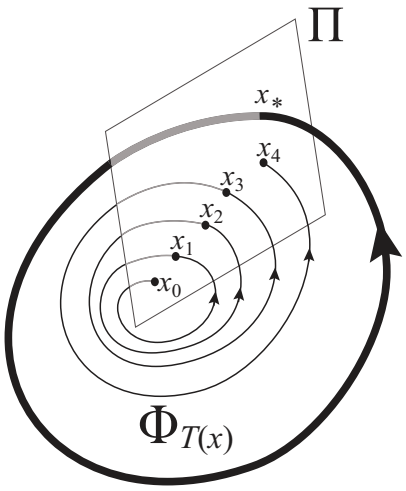
The rough idea is to take a cross-section Π through an ODEs phase portrait, then follow the orbits that pass through that section. There are two main types:

- Π is called a **Poincaré section** if the orbits return repeatedly through the same section Π. To find the map:
  - say an orbit crosses Π at a point x_0, then evolves through the ODE and returns to Π at a point $x_1 = \Phi_{T(x_0)}(x_0)$ after a *return time* $T(x_0)$ (this Φ is the integral of the solution from x_0 to x_1),
  - assume we can write this as a function f where $x_1 = f(x_0) = \Phi_{T(x_0)}(x_0)$,
  - more generally we'll want to look at multiple returns through points x_0, $x_1 = \Phi_{T(x_0)}(x_0)$, $x_2 = \Phi_{T(x_1)}(x_1)$, ..., so instead we write

    $$x_{n+1} = f(x_n)$$

    we call this the **Poincaré map** $f : \Pi \mapsto \Pi$.

  For example in a pendulum you might define a section $\theta = 0$ corresponding to "pendulum is vertical".



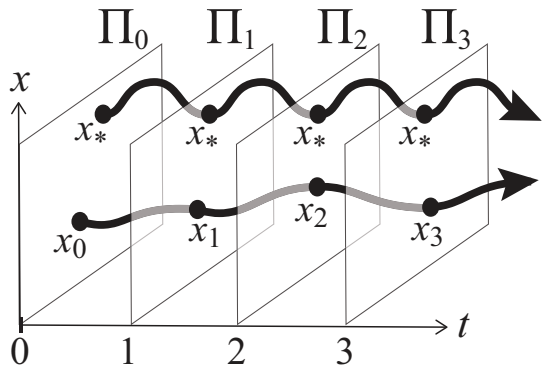
The picture shows a section Π, an orbit integrated through four returns, and also a periodic orbit that goes through x_*.

- Π is called a **stroboscopic section** if it is one of a set of sections $\Pi, \Pi_1, \Pi_2, ...$ taken at fixed time intervals. To find the map:
  - say an orbit crosses Π at a point x_0, then evolves through the ODE and crosses Π_1 at a point $x_1 = \Phi_T(x_0)$, then crosses Π_2 at a point $x_2 = \Phi_T(x_1)$, etc. . . . each after a *return time* T,
  - if we can write each map as

$$x_{n+1} = f(x_n)$$

    we call this the **stroboscopic map** $f : \Pi_n \mapsto \Pi_{n+1}$.

For example in the population model we took stroboscopic sections at time intervals $\Delta t = 1$.



The picture shows a section Π every period $T = 1$, an orbit integrated through three periods, and also a periodic orbit that goes through x_*.

17 Example: Maps for the population model

Let's derive the same population model, but in map form.
We'll do this two ways, first as a stroboscopic map, and then from first principles.

Stroboscopic map for the population model

Starting from the ODE, we had $\dot{N} = N(\beta - \gamma N)$. The derivative is defined as

$$\dot{N} = \frac{dN}{dt} = \lim_{\Delta t \to 0} \frac{N(t+\Delta t) - N(t)}{\Delta t} = N(t)\left(\beta - \gamma N(t)\right) \tag{17.1}$$

- So if Δt is small (but not zero) we can at least write

$$\begin{aligned} & \frac{N(t+\Delta t) - N(t)}{\Delta t} \approx N(t)\left(\beta - \gamma N(t)\right) \\ \Rightarrow \quad & N(t+\Delta t) - N(t) \approx N(t)\left(\beta - \gamma N(t)\right)\Delta t \\ \Rightarrow \quad & N(t+\Delta t) \approx N(t)\left(\beta - \gamma N(t)\right)\Delta t + N(t) \end{aligned} \tag{17.2}$$

  Now let $N_{n+1} = N(t+\Delta t)$ and $N_n = N(t)$, and we obtain a map from each point N_n to the next point N_{n+1} in the population's growth. If we just let the time-step be $\Delta t = 1$ we have

$$N_{n+1} \approx N_n\left(1 + \beta - \gamma N_n\right) \tag{17.3}$$

  sometimes called a '*time one map*'.

- You can also derive this from the solution $N(t)$ itself [see Ex.Sht.]

- Let's re-scale this discrete time system by letting $N = \frac{1+\beta}{\gamma}x$, and let $r = 1+\beta$, so (17.3) becomes

$$\tfrac{1+\beta}{\gamma}x_{n+1} = \tfrac{(1+\beta)^2}{\gamma}x_n(1-x_n) \quad \Rightarrow \quad x_{n+1} = rx_n(1-x_n) \tag{17.4}$$

  which is an important system called the **logistic map**.

- The ODE and the map are different ways of approximating — and hence modeling — the population. But be careful, their behaviours aren't the same (more on this soon).

Discrete time population model from first principles

We don't only get maps from ODEs, they are often derived as models in their own right, when we want a model in steps rather than in continuous time.

For example, we might only be interested in the population level measured once a day. So let's re-derive the population model as a map in discrete time steps.

- An ODE gives the rate of change of a quantity, $\dot{x} = \ldots$
- A *difference equation* gives the amount a quantity changes between two instants.
- Say a population N at some 'time' $n+1$ relates to an earlier 'time' n as

$$N_{n+1} - N_n \;=\; B \;-\; D \;+\; M \tag{17.5}$$

  and similiar to the ODE, let's take: births $B = \beta N_n$, deaths $D = \delta N_n$, migration $M = 0$, so

$$N_{n+1} \;=\; (\beta - \delta + 1)N_n \tag{17.6}$$

- A solution of this would tell us N_n for any initial value N_0.
- To find this try iterating:

$$\begin{aligned}
N_n &= (\beta - \delta + 1)N_{n-1} && \text{just the difference eq}^{\text{n}} \\
&= (\beta - \delta + 1)^2 N_{n-2} && \text{subbing in } N_{n-1} = (\beta - \delta + 1)N_{n-2} \\
&= (\beta - \delta + 1)^3 N_{n-3} && \text{subbing in } N_{n-2} = (\beta - \delta + 1)N_{n-3} \\
&\vdots \\
N_n &= (\beta - \delta + 1)^n N_0 && \text{... keeping going until you reach } N_0
\end{aligned} \tag{17.7}$$

- Like an ODE we can describe this with a flow operator Φ as

$$N_n = \Phi_n(N_0) \qquad \text{where} \qquad \Phi_n(N_0) = ((\beta - \delta + 1)^n N_0 \tag{17.8}$$

- It's pretty easy to see what happens to this solution. If $|\beta - \delta + 1| > 1$ population size grows exponentially, if $|\beta - \delta + 1| < 1$ it shrinks asymptotically to zero. Assuming $\beta - \delta + 1$ is positive, these conditions just become $\beta > \delta$ and $\beta < \delta$ as in the continuous time model.
- Note if $\beta - \delta + 1 < 0$ then N_n will flip between being positive and negative on each iteration. Clearly this isn't physically realistic for the population model.

As we did for the ODE, let's improve the population model by making the death rate proportional to the population, introducing a cut-off.

- Assume that the death rate increases with the population a $\delta = \gamma N$, so

$$N_{n+1} = bN_n - fN_n^2 + N_n \tag{17.9}$$
$$= N_n(1 + \beta - \gamma N_n) \tag{17.10}$$

  You can see that this is exactly the time-one stroboscopic map we obtained from the ODE.

- Applying the same scaling as we did then, letting $N = (1 + \beta)x/\gamma$ and $r = 1 + \beta$, gives the **logistic map**

$$x_{n+1} = rx_n(1 - x_n) \tag{17.11}$$

  For the model to be realistic we must have $\beta > 0$, which now means $r > 1$.

- Now this system cannot be solved. If you try to iterate you get

$$\begin{aligned} x_n &= rx_{n-1}(1 - x_{n-1}) \\ &= r[rx_{n-2}(1 - x_{n-2})](1 - [rx_{n-2}(1 - x_{n-2})]) \\ &= r[r[rx_{n-3}(1 - x_{n-3})](1 - [rx_{n-3}(1 - x_{n-3})])](1 - ... \end{aligned} \tag{17.12}$$

  . . . this goes on and on getting longer and longer and higher order in m.

- We're going to need some new tricks.

. . . local analysis

- In ODEs we learned that we can get a lot from local analysis.
- Whereas ODEs tell us rates of change, and are stationary at *equilibria*, maps tell us how a system updates, so they are stationary at **fixed points**.
- A fixed point is a place where the map just repeats the same value, $x_0 = x_1 = x_2 = ...$, so we find it by solving $x_{n+1} = x_n$, which from (17.11) gives

$$\begin{aligned} & x_n = r x_n (1 - x_n) \\ \Rightarrow \qquad & 0 = x_n (1 - r + r x_n) \end{aligned} \tag{17.13}$$

  with solutions

$$x_{*1} = 0 \qquad \& \qquad x_{*2} = (r-1)/r \tag{17.14}$$

- Like ODEs, we can linearize around these.
- Near $x = x_{*2} = 0$ the system is $x_{n+1} \approx r x_n$ with solution $x_n \approx r^n x_0$, so with each n the factor r^n grows (since we said $r > 1$), so the population x_n grows away from zero . . . the fixed point is *unstable* (a repeller).
- Near $x = x_{*2}$ the system is $x_{n+1} \approx x_{*2} + (2-r)(x_n - x_{*2})$ with a solution we can write as $x_n - x_{*2} = (2-r)^n (x_0 - x_{*2})$ [see Ex.Sht], so with each n the factor $(2-r)^n$ shrinks towards zero (since $r > 1$ implies $2 - r < 1$), dragging the population x_n closer to x_{*2} . . . the fixed point is *stable* (an attractor).
- So the behaviour looks very much consistent with the continuous time model (the ODE).

. . . ODE versus map

- In the continuous system, let's take an initial point $x_0 = 0.3$ and look at each timestep $\Delta t = 1$. Take $r = 3.6$ for example, and we get a steady and rapid convergence to the stable equilibrium: (to 2d.p.)

$$x(0) = 0.3, \quad x(1) = 0.65, \quad x(2) = 0.72, \quad x(3) = 0.72, \quad x(4) = 0.72, ...$$

- In the discrete system, let's do the same thing, with the same values, and we get: (to 2d.p.)

$$x(0) = 0.3, \quad x(1) = 0.9, \quad x(2) = 0.32, \quad x(3) = 0.79, \quad x(4) = 0.6, ...$$

  Firstly we're not getting the same kind of convergence, and certainly not to $x = (r-1)/r = 0.72$. Worse, these values are able to jump to above and below $x = 0.72$ rather than just tend monotonically towards it.

- So there is clearly more that can happen in the map model. Why?

- This is largely due to period orbits of the map. As well as the fixed point in the population model where $x_{n+1} = x_n$, given by

$$x_0 = rx_0(1 - x_0) \qquad \Rightarrow \qquad x_0 = \ 0 \text{ or } (r-1)/r \tag{17.15}$$

  the map might only return to a point after m iterates so that $x_{n+m} = x_n$. For example there is a period 2 orbit

$$\begin{aligned} x_0 &= [rx_0(1 - x_0)](1 - [rx_0(1 - x_0)])) \\ &\Rightarrow \qquad x_0 = \tfrac{1}{2r}(1 + r \pm \sqrt{(r+1)(r-3)}) \end{aligned} \tag{17.16}$$

  and a period 3 orbit and so on . . . these get more and more difficult to find, but we'll come back to them later.

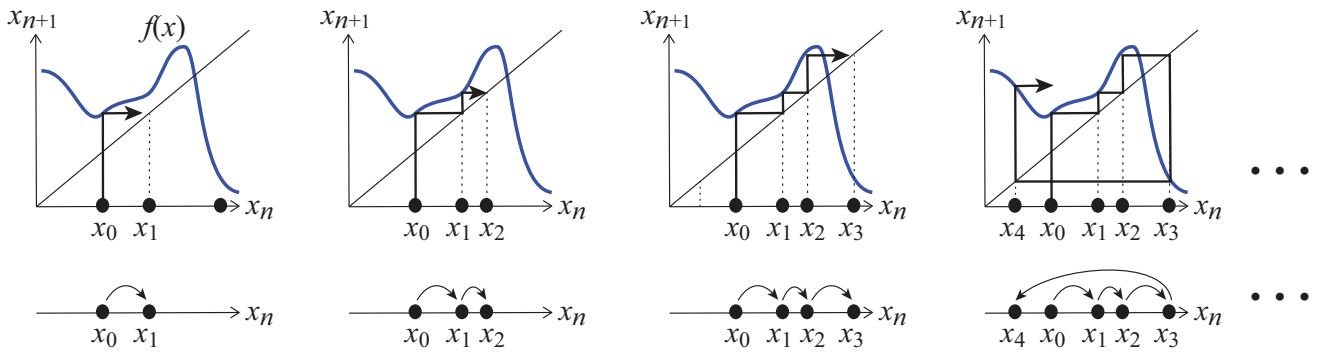
For ODEs we could sketch phase portraits. For maps we have an alternative but often more powerful tool, called a *cobweb diagram*.

18 Cobweb diagrams

So a map f generates a sequence of points $x_0, x_1, x_2, ..., x_n, ...$ that a system goes through as we iterate it through the 'process' by applying f again and again.

There's a useful way to visualize these, called a cobweb diagram:

- draw the graph of $f(x)$, and draw the diagonal $y = x$
- label the horizontal axis as x_n and the vertical axis as x_{n+1}
- to draw an orbit from an initial point x_0 you now:
  - *iterate the map*: draw a vertical line from the point x_0 on the horizontal axis until you reach the graph, arriving at $x_1 = f(x_0)$
  - *reset the map*: continue with a horizontal line across to the diagonal
  - *iterate again*: continue with a vertical line until you reach the graph again, arriving at $x_2 = f(x_1)$
  - *reset the map*: continue with a horizontal line across to the diagonal
  - and repeat



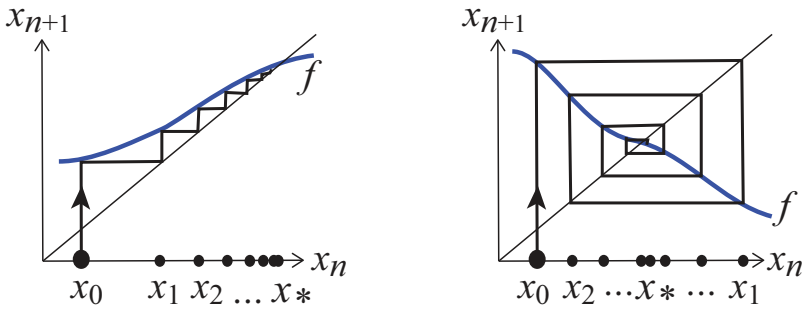
- You obtain a sequence of vertical lines at $x_n = x_0, x_1, x_2, ...$, joined by horizontal resets to the diagonal. This is the 'cobweb'.

- If the graph $f(x)$ crosses the diagonal $y = x$, it forms a **fixed point** of the map since it creates a point where

$$x_* = f(x_*)$$

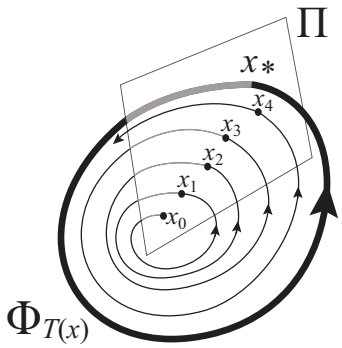
and hence $x_{n+1} = x_n$.

- A fixed point of a map corresponds to a **periodic orbit** of an ODE, as it says an orbit will always return to the same point $x_0 = x_1 = x_2 = ...$ each time it hits the section Π, at $x_0 = x_1 = x_2 = ...$



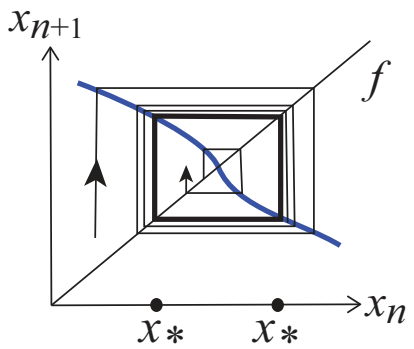
Note in the figure how iterates tend directly towards the fixed point if the gradient of f is positive at x_*, but oscillate around it if the gradient of f is negative at x_*.

If this was derived from the Poincaré map of an ODE then the corresponding flow might look like:



- The cobweb may form a closed orbit that repeats after m iterations. This is a **periodic orbit** of the map, where

$$x_m = x_0 \qquad \Rightarrow \qquad x_0 = f^m(x_0) = f(\overset{m}{\cdots} f(f(x_0))\cdots) \tag{18.1}$$



The picture shows a period two orbit.

19 ODE vs. maps — a comparison

Everything we've learnt about ODEs has a counterpart in maps. We'll summarize them here and then unpack these a little below . . .

ODEs

Rate of change

The rate x is changing is

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) = \lim_{\varepsilon \to 0} \frac{\mathbf{x}(t) - \mathbf{x}(t-\Delta t)}{\Delta t}$$

Equilibrium

$$\mathbf{f}(\mathbf{x}_*) = 0$$

Stability

$$\begin{array}{lcl} \mathsf{Re}(\lambda) < 0 & \Rightarrow & \text{stable} \\ \mathsf{Re}(\lambda) > 0 & \Rightarrow & \text{unstable} \end{array}$$

where $||\underline{\underline{A}} - \lambda \underline{\underline{1}}|| = 0$ and

$$\underline{\underline{A}} = \left. \frac{\partial \mathbf{f}}{\partial \mathbf{x}} \right|_{\mathbf{x}=\mathbf{x}_*}$$

Bifurcations

If $\mathsf{Re}(\lambda) = 0$ . . .

- fold (saddle-node)
- transcritical
- pitchfork
- cusp
- Hopf

Maps

Difference

From some x_{n-1} to x_n the difference is

$$\mathbf{x}_n = \mathbf{f}(\mathbf{x}_{n-1}) \qquad \left\{ \begin{array}{l} \mathbf{x}_n \quad = \mathbf{x}(t) \\ \mathbf{x}_{n-1} = \mathbf{x}(t - \Delta t) \end{array} \right.$$

Fixed Point

$$\mathbf{f}(\mathbf{x}_*) = \mathbf{x}_*$$

Stability

$$\begin{array}{lcl} |\lambda| < 1 & \Rightarrow & \text{stable} \\ |\lambda| > 1 & \Rightarrow & \text{unstable} \end{array}$$

where $||\underline{\underline{A}} - \lambda \underline{\underline{1}}|| = 0$ and

$$\underline{\underline{A}} = \left. \frac{\partial \mathbf{f}}{\partial \mathbf{x}} \right|_{\mathbf{x}=\mathbf{x}_*}$$

Bifurcations

If $|\lambda| = 1$ . . .

- fold (saddle-node)
- transcritical
- pitchfork or flip
- cusp
- Neimark-Sacker

20 Linear stability (for maps)

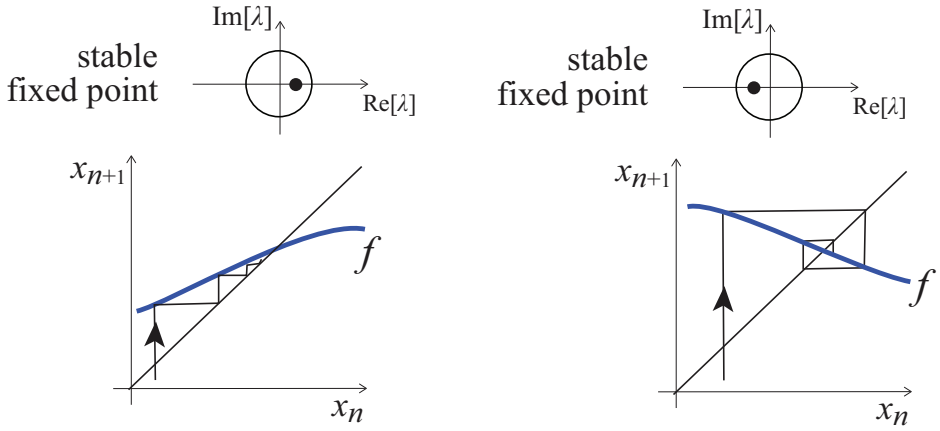
The fixed points of maps has a Jacobian $\frac{\partial \mathbf{f}(\mathbf{x})}{\partial \mathbf{x}}$, with eigenvalues λ and eigenvectors $\mathbf{v}$, just as in ODEs. But:

- whereas in an ODE the stability depends on whether an eigenvalue lies in the right or left half of the complex plane, in a map the stability depends on whether an eigenvalue has magnitude greater than or less than one.

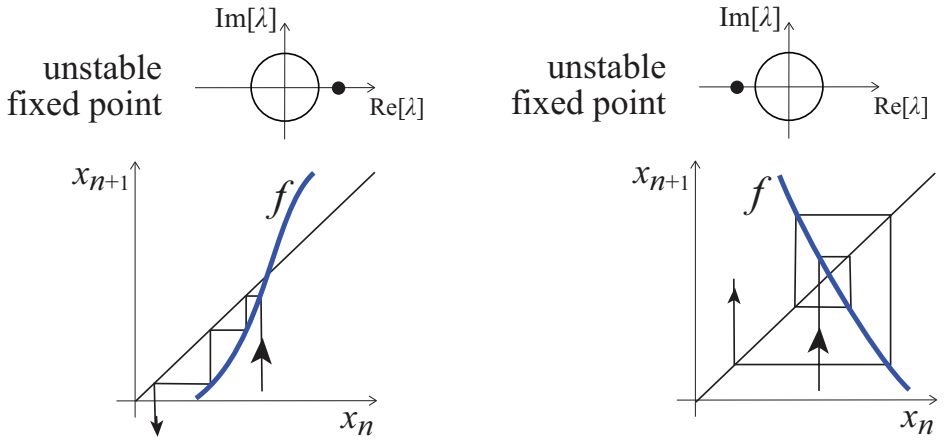
Let's take just a 1-dimensional map.

Then eigenvalue $\lambda = f'(x_*)$ is just the slope of f at a fixed point x_*. Now:

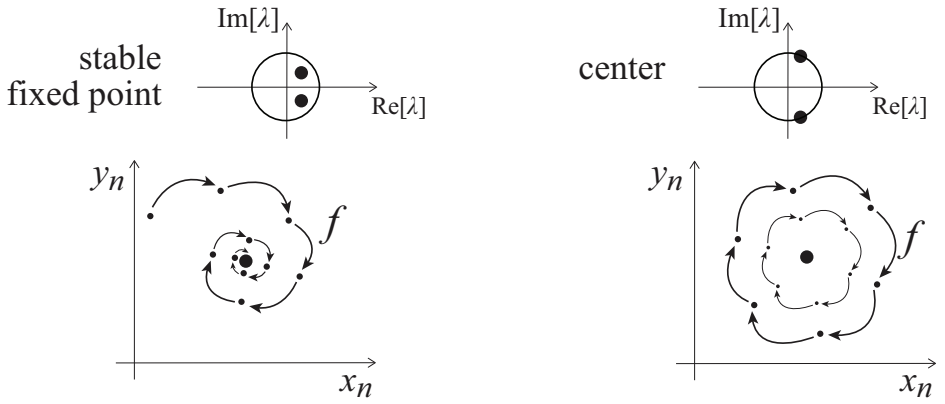
- If the slope at x_* has modulus less than one the fixed point is stable:



- If the slope at x_* is modulus greater than one the fixed point is unstable:



- To get complex eigenvalues we need more than one dimension, and as for ODEs they can only occur in conjugate pairs. If they are inside the unit circle they are stable, if they are outside they are unstable, if they are on the unit circle we have a center:



21 Bifurcations (for maps)

For every bifurcation in an ODE, there is a corresponding bifurcation in a map. Let λ be an eigenvalue in a n-d system, or just $\lambda = f'(x_*)$ for a 1d system. A bifurcation happens:

- In an ODE as λ passes through the imaginary axis where $\mathsf{Re}(\lambda) = 0$,
- In a map as λ passes through the unit circle where $|\lambda| = 1$.

The basic bifurcations are (taking just 1d, except for the last case):

In an ODE $\dot{x} = f(x)$:

Fold

- $f(x_*) = 0$ and $\lambda = 0$

Transcritical

- $f(x_*) = 0$ and $\lambda = 0$
- $\frac{\partial f}{\partial \beta} = 0$

Pitchfork

- $f(x_*) = 0$ and $\lambda = 0$
- $f(-x) = -f(x)$

- - no counterpart - -

Cusp

- $f(x_*) = 0$ and $\lambda = 0$
- $f''(x_*) = 0$

Hopf

- $f(x_*) = 0$ and $\mathsf{Re}(\lambda) = 0$
- 2 dims, $\lambda \in \mathbb{C}$

In a Map $x_{n+1} = f(x_n)$:

Fold

- $f(x_*) = x_*$ and $\lambda = 1$

Transcritical

- $f(x_*) = x_*$ and $\lambda = 1$
- $\frac{\partial f}{\partial \beta} = 0$

Pitchfork

- $f(x_*) = x_*$ and $\lambda = 1$
- $f(-x) = -f(x)$

Flip (or period doubling)

- $f(x_*) = x_*$ and $\lambda = -1$

Cusp

- $f(x_*) = x_*$ and $\lambda = 1$
- $f''(x_*) = 0$

Neimark-Sacker

- $f(x_*) = x_*$ and $|\lambda| = 1$
- 2 dims, $\lambda \in \mathbb{C}$

Let's look in more detail just at the familiar transcritical and fold, and then at the new one, the flip . . .

Transcritical bifurcation

Take the population model

$$x_{n+1} = r x_n (1 - x_n) \tag{21.1}$$

- This has fixed points $x_{*1} = 0$ and $x_{*2} = (r-1)/r$.
- These exist for any r, but at $r = 1$ clearly $x_{*1} = x_{*2} = 0$. As r passes through $r = 1$ the two equilibria pass through each other.
- Their stability is given by

$$\begin{aligned} \text{at } x_{*1} \quad &: \quad \left.\frac{dx_{n+1}}{dx_n}\right|_{*1} = r(1 - 2x_{*1}) = r \\ \text{at } x_{*2} \quad &: \quad \left.\frac{dx_{n+1}}{dx_n}\right|_{*2} = r(1 - 2x_{*2}) = 2 - r \end{aligned}$$

  so for :

  - $-1 < r < 1$ the fixed point x_{*1} is stable and x_{*2} unstable (as $|r| < 1$ and $|2 - r| > 1$)
  - $1 < r < 3$ the fixed point x_{*1} is unstable and x_{*2} stable (as $|r| > 1$ and $|2 - r| < 1$)

- So the two fixed points pass through each other, and in doing so swap stability (just as in ODEs).

Things are much the same in a two-dimensional map. Take the predator-prey model and discretize it as we did for the population model, to get [see Ex.Sht.]

$$\tfrac{x(t+\Delta t) - x(t)}{\Delta t} \approx \dot{x} = (\alpha - y)x \ , \qquad \tfrac{y(t+\Delta t) - y(t)}{\Delta t} \approx \dot{y} = (x - \gamma)y \ , \tag{21.2}$$

let $x(t + \Delta t) = x_{n+1}$ and $x(t) = x_n$, and take the time-one map so $\Delta t = 1$, giving a map

$$x_{n+1} = (1 + \alpha - y_n)x_n \ , \qquad y_{n+1} = (1 + x_n - \gamma)y_n \ , \tag{21.3}$$

- For simplicity let $\alpha = \gamma$, then a transcritical bifurcation occurs as α passes through zero [see Ex.Sht.]

[Side Notes:] Transcritical bifurcation normal form

Let

$$x_{n+1} = f(x_n, \beta) \tag{21.4}$$

with $x, \beta \in \mathbb{R}$.
Then a transcritical bifurcation occurs at $x = x_*$ when $\beta = \beta_*$ if the following conditions hold:

(B1) $f(x_*, \beta) = x_*$ "fixed point at $x = x_*$",

(B2) $\frac{\partial f}{\partial x} = 1$ at $x = x_*$, $\beta = \beta_*$, "unit eigenvalue",

(B3) $\frac{\partial f}{\partial \beta} = 0$ at $x = x_*$, $\beta = \beta_*$, "zero speed of f w.r.t. β",

(G1) $\frac{\partial^2 f}{\partial x^2} \neq 0$ at $x = x_*$, $\beta = \beta_*$, "second order derivative nonzero",

(G2) $\frac{\partial}{\partial \beta}\frac{\partial f}{\partial x} \neq 0$ at $x = x_*$, $\beta = \beta_*$, "positive speed of $\frac{\partial f}{\partial x}$ in β",

then in a neighbourhood of (x_*, β_*) this system has the topological normal form

$$y_{n+1} = (1 + \beta)y_n \pm y_n^2,$$

in a neighbourhood of (x_*, β_*).

As in ODEs this is a special case.

Fold bifurcation

Let's tweak the population model to have a background of constant growth (and set birth rate to 1)

$$x_{n+1} = r + x_n(1 - x_n) \tag{21.5}$$

- This has fixed points $x_{*\pm} = \pm\sqrt{r}$
- There are two fixed points for $r > 0$, none for $r < 0$. At $r = 0$ they coincide.
- When they do exist for $r > 0$, their stability is given by

$$\begin{aligned} \text{at } x_{*+} \quad &: \quad \left.\frac{dx_{n+1}}{dx_n}\right|_{*+} = 1 - 2x_{*1} = 1 + 2\sqrt{r} \\ \text{at } x_{*-} \quad &: \quad \left.\frac{dx_{n+1}}{dx_n}\right|_{*-} = 1 - 2x_{*2} = 1 - 2\sqrt{r} \end{aligned}$$

  so clearly

$$\left.\frac{dx_{n+1}}{dx_n}\right|_{*-} < 1 < \left.\frac{dx_{n+1}}{dx_n}\right|_{*+}$$

  thus for $r > 0$ the fixed point x_{*+} is unstable and x_{*-} stable.

[Side Notes:] Fold normal form

Let

$$x_{n+1} = f(x_n, \beta) \tag{21.6}$$

with $x, \beta \in \mathbb{R}$.
Then a fold bifurcation occurs at $x = x_*$ when $\beta = \beta_*$ if the following conditions hold:

(B1) $f(x_*, \beta) = x_*$ "fixed point at $x = x_*$",

(B2) $\frac{\partial f}{\partial x} = 1$ at $x = x_*$, $\beta = \beta_*$, "unit eigenvalue",

(G1) $\frac{\partial^2 f}{\partial x^2} \neq 0$ at $x = x_*$, $\beta = \beta_*$, "second order derivative nonzero",

(G2) $\frac{\partial f}{\partial \beta} \neq 0$ at $x = x_*$, $\beta = \beta_*$, "positive speed of f in β",

then in a neighbourhood of (x_*, β_*) this system has the topological normal form

$$y_{n+1} = \beta + y_n \pm y_n^2.$$

Flip bifurcation

Take again the population model

$$x_{n+1} = r x_n (1 - x_n) \tag{21.7}$$

- We saw this has a fixed point $x_{*2} = (r-1)/r$ with stability is given by

$$\text{at } x_{*2} \quad : \quad \left.\frac{dx_{n+1}}{dx_n}\right|_{*2} = r(1 - 2x_{*2}) = 2 - r \ .$$

- We saw above that a transcritical bifurcation happened as r passed through $r = 1$.

- As r increases through $r = 2$ the stability changes from positive to negative. There is no bifurcation ($\left.\frac{dx_{n+1}}{dx_n}\right|_{*2} = 2 - r$ still lies inside the unit circle), but being negative means the orbit starts to oscillate or 'flip' about the fixed point as it tends towards it.

- As r increases through $r = 3$ there is a bifurcation, the fixed point becomes unstable as $\left.\frac{dx_{n+1}}{dx_n}\right|_{*2} = 2 - r$ leaves the unit circle at $\left.\frac{dx_{n+1}}{dx_n}\right|_{*2} = -1$.

- If there has been a stability change, then something else must have accompanied it to 'balance' the change. It cannot involve the fixed point $x_{*1} = 0$, which is not local to $x_{*2} = (r-1)/r = 2/3$ at $r = 3$.

- So the only other place to look is in higher order objects . . .

- Look for a period 2 orbit, and call it $x^{(2)}_{*\pm}$ (with $\pm$ labeling its two iterates)

$$x_n = r[rx_n(1-x_n)](1-[rx_n(1-x_n)])$$
$$\Rightarrow \qquad x^{(2)}_{*\pm} = \tfrac{1}{2r}(1+r \pm \sqrt{(r+1)(r-3)}) \qquad (21.8)$$

  where the superscript denotes that this has period 2.

  You can check that the two $\pm$ solutions are just two iterates of the same orbit, (i.e. $x^{(2)}_{*+} = f(x^{(2)}_{*-})$ as $x^{(2)}_{*+} = rx^{(2)}_{*-}(1-x^{(2)}_{*0})$ and vice versa).

- For $r < 3$ this doesn't exist, it appears just for $r > 3$, when $x^{(2)}_{*\pm} = 2/3$, i.e. being 'born' from x_{*2} as it changes stability.

- What is the stability of the orbit?

- We could directly calculate $\frac{dx_{n+2}}{dx_n}$ at $x^{(2)}_{*\pm}$, but the map $x_{n+2} = f^2(x_n)$ is fairly complicated. When we get to higher periods these expressions will be even more difficult to differentiate.

- Instead, by the chain rule

$$\frac{dx_{n+2}}{dx_n} = \frac{dx_{n+2}}{dx_{n+1}}\frac{dx_{n+1}}{dx_n} = \frac{df(x_{n+1})}{dx_{n+1}}\frac{df(x_n)}{dx_n} \tag{21.9}$$

  This says the stability of the orbit is given by multiplying the slopes of f at each of the orbit's iterates.

- So to calculate the stability of the period 2 orbit

$$\begin{aligned}\left.\frac{dx_{n+2}}{dx_n}\right|_{x^{(2)}_{*\pm}} &= \left.\frac{df(x_{n+1})}{dx_{n+1}}\right|_{x^{(2)}_{*+}} \left.\frac{df(x_n)}{dx_n}\right|_{x^{(2)}_{*-}} \\ &= r(1-2x_{n+1})|_{x^{(2)}_{*+}}\, r(1-2x_n)|_{x^{(2)}_{*-}} \\ &= (1+\sqrt{(r+1)(r-3)})(1-\sqrt{(r+1)(r-3)}) \\ &= 1-(r+1)(r-3)\end{aligned} \tag{21.10}$$

  where I've used

$$r(1-2x^{(2)}_{*\pm}) = r - 2\tfrac{1}{2}(1+r\pm\sqrt{(r+1)(r-3)}) = -1\mp\sqrt{(r+1)(r-3)}$$

  Now we could simplify this expression further but it's in a good form. We can see from this that at $r=3$ we have $\left.\frac{dx_{n+2}}{dx_n}\right|_{x^{(2)}_{*\pm}} = 1$, consistent with a bifurcation occurring, and for $r>3$ we have $\left.\frac{dx_{n+2}}{dx_n}\right|_{x^{(2)}_{*\pm}} < 1$.

- So the period 2 orbit that appears is stable.

- This event, in which a stable period 1 orbit becomes unstable, and creates a stable period 2 orbit, is called a **flip bifurcation**.

[Side Notes:] **Flip normal form**

Let

$$x_{n+1} = f(x_n, \beta) \tag{21.11}$$

with $x, \beta \in \mathbb{R}$.
Then a flip bifurcation occurs at $x = x_*$ when $\beta = \beta_*$ if the following conditions hold:

(B1) $f(x_*, \beta) = x_*$ "fixed point at $x = x_*$",

(B2) $\frac{\partial f}{\partial x} = -1$ at $x = x_*$, $\beta = \beta_*$, '-1 eigenvalue",

(G1) $2\frac{\partial^3 f}{\partial x^3} + 3(\frac{\partial^2 f}{\partial x^2})^2 \neq 0$ at $x = x_*$, $\beta = \beta_*$,

(G2) $c = \frac{\partial f}{\partial \beta}\frac{\partial^2 f}{\partial x^2} + 2\frac{\partial}{\partial \beta}\frac{\partial f}{\partial x} \neq 0$ at $x = x_*$, $\beta = \beta_*$,

then in a neighbourhood of (x_*, β_*) this system has the topological normal form

$$y_{n+1} = -(1 + \beta)y_n \pm y_n^3, \tag{21.12}$$

The $\pm$ signs determine whether the period 2 orbit involved is stable or unstable (in which case it surrounds the unstable or stable fixed point, respectively).

This is otherwise known as **period doubling**, because when a flip bifurcation happens typically:

- a stable fixed point becomes unstable, and gives birth to a period two orbit, or
- a stable period p orbits becomes unstable, and gives birth to a period $2p$ orbit.

22 Finding periodic orbits

There are two important tricks for finding periodic orbits and their stability.

1. Factorizing

- When we solve $x = f^m(x)$ to find a period m orbit, this expression may also have 'false roots', because any lower period orbit will also satisfy this expression, e.g. the fixed point $x = f(x)$ satisfies $x = f^m(x)$ for any m.
- So we have to be careful to factorize out these solutions.
- E.g. in the population map

$$x_{n+1} = r x_n (1 - x_n) \tag{22.1}$$

fixed points $x_{*1} = 0$ and $x_{*2} = (r-1)/r$.

look for period two orbits. These are solutions of

$$\begin{aligned} 0 &= x - f^2(x) \\ &= x - r^2 x(1-x)(1 - rx(1-x)) \\ &= x - r^2 x(1-x) + r^3 x^2 (1-x)^2 \end{aligned} \tag{22.2}$$

Now, fixed points must also be solutions of this, so this must be divisible by factors $x - x_{*1}$ which is just x, and $x - x_{*2}$ which is $x - x_{*2} = x - (r-1)/r$, so we must be able to write (22.2) as

$$0 = x(x - (r-1)/r)(ax^2 + bx + c) \tag{22.3}$$

for some a, b, c.

To find these you can either:

- use polynomial division of (22.2) divided by $x(x-(r-1)/r)$,
- compare coefficients.

I'll use comparing coefficients here. If the two expressions above are the same then we have

$$x(x-(r-1)/r)(ax^2+bx+c) \;=\; x - r^2x(1-x) + r^3x^2(1-x)^2 \qquad (22.4)$$

then expanding out both sides (and optionally dividing by x) we have

$$\tfrac{1-r}{r}c + (c - \tfrac{r-1}{r}b)x + (b - \tfrac{r-1}{r}a)x^2 + ax^3 \;=\; 1 - r^2 + r^2(1+r)x - 2r^3x^2 + r^3x^3 \qquad (22.5)$$

implying

$$\begin{aligned} \tfrac{1-r}{r}c = 1-r^2 \qquad &\Rightarrow \qquad c = r(1+r) \\ c - \tfrac{r-1}{r}b = r^2(1+r) \qquad &\Rightarrow \qquad b = -r^2(r+1) \\ b - \tfrac{r-1}{r}a = -2r^3 \qquad &\Rightarrow \qquad a = r^3 \end{aligned} \qquad (22.6)$$

so period two orbits are solutions of

$$0 = x(x-(r-1)/r)(r^2x^2 - r(r+1)x + 1 + r)r \qquad (22.7)$$

so the first two factors on the righthand side give the period 1 solutions, and the last factor gives the period two solutions as

$$x^{(2)}_{*\pm} = (1 + r \pm \sqrt{(r+1)(r-3)})/2r \qquad (22.8)$$

exactly as we found before.

2. Chain rule

As we said above, for nonlinear maps, calculating the derivative $\frac{dx_{n+p}}{dx_p}$ at a given $x_*^{(p)}$ of a period p orbit can be difficult, but we can get around that using the chain rule.

- Essentially, this says that rather than working out what $x_{n+p} = f^p(x_n)$ is finding its stability, we can instead work out the stability (the slope) of $f(x_n)$ at each point the orbit visits and multiply the slopes.
- So for a period 2 orbit, which is a solution of $x_n = f^2(x_n)$, we can work out

$$\begin{aligned} \frac{dx_{n+2}}{dx_n} &= \frac{dx_{n+2}}{dx_{n+1}}\frac{dx_{n+1}}{dx_n} \\ &= \frac{df(x_{n+1})}{dx_{n+1}}\frac{df(x_n)}{dx_n} \;=\; f'(x_{n+1})f'(x_n) \end{aligned} \tag{22.9}$$

- For a period p orbit, which is a solution of $x_n = f^p(x_n)$, we can work out

$$\begin{aligned} \frac{dx_{n+p}}{dx_n} &= \frac{dx_{n+p}}{dx_{n+p-1}} \cdots \frac{dx_{n+1}}{dx_n} \\ &= \frac{df(x_{n+p-1})}{dx_{n+p-1}} \cdots \frac{df(x_n)}{dx_n} \;=\; f'(x_{n+p-1}) \ldots f'(x_n) \end{aligned} \tag{22.10}$$

[Further Reading Only:] Period doubling cascades

There's a lot more to period doubling that we have time to go into here. The most important is a phenomenon that was first identified in the logistic map, but then understood to apply to any nonlinear maps. You should at least make yourselves familiar with these concepts:

- Cascades – if a nonlinear map has a turning point that depends on one parameter, it can exhibit period doubling. This tends not to happen just once, but again, and again, and again, . . . in an infinite sequence called a **period doubling cascade**. So as a parameter β is changed, a fixed point creates a period two orbit (in a flip bifurcation), which creates a period 4 orbit (another flip), then a period 4, and so on, until the period reaches infinity.

- Feigenbaum's constants – Importantly, this to infinite period happens only with a finite change the parameter β. So the interval of β values between each flip, i.e. the range for which each period exists, shrinks as the period increases. It shrinks at a rate called **Feigenbaum's (first) constant**. The distances between iterates also shrink at a rate called **Feigenbaum's second constant**, so the orbits remain finite in size.

  E.g. For the logistic map the Feigenbaum constants are

$$\text{1st Feigenbaum constant} = \delta = \lim_{n\to\infty} \frac{\beta_n - \beta_{n-1}}{\beta_{n+1} - \beta_n} = 4.669...$$

$$\text{2nd Feigenbaum constant} = \alpha = \lim_{n\to\infty} \frac{\Delta x_n - \Delta x_{n-1}}{\Delta x_{n+1} - \Delta x_n} = 2.503...$$

  where $\beta_1, \beta_2, ...$ are the parameter values of each successive flip bifurcation, and Δx characterizes the width of each fork in the bifurcation diagram.

- Universality – these appear to be fundamental constants of mathematics (like π or e), meaning the same constants apply to *any* differentiable one-dimensional map. Their exact values remain unknown. Their universality is proven by taking a small region around the (locally quadratic) hump in a map f where a period doubling occurs, *renormalizing* by re-scaling to

magnify x to cover only this region, then doing the same around a hump in f^2, then f^3, etc. obtaining the bifurcation values of β each time.

- Chaos – at the end of a period doubling cascade the map will become chaotic. We'll say more about chaos below, but basically it means the map remains trapped in some region but never repeats.

- Periodic windows – note that a period doubling cascade only involves even period orbits (because it arises by *doubling*). But in the midst of the chaos that follows a cascade we find windows of odd periods 3,5,7,... See if you can spot them in the figure above.

- In fact odd period orbits are rare to see, and always come with baggage like the chaos that surrounds them in the cascade. An almost incredible theorem by Sharkovkii says:

  If $x_{n+1} = f(x_n)$ is a continuous map with a period p orbit, then it also has a period m orbit for every $m \prec p$, meaning every m to the left of p in the sequence of numbers

$$\begin{array}{l} 1 \prec 2 \prec 2^2 \prec 2^3 \prec \cdots \prec 2^n \prec \ldots \\ \cdots \prec 7\cdot 2^n \prec 5\cdot 2^n \prec 3\cdot 2^n \prec \ldots \\ \vdots \\ \cdots \prec 7\cdot 2 \prec 5\cdot 2 \prec 3\cdot 2 \prec \ldots \\ \cdots \prec 9 \prec 7 \prec 5 \prec 3 \end{array} \tag{22.11}$$

  called the **Sharkovkii ordering**.

  (The Sharkovkii ordering consists of the sequence 2^k with k ticking up from $k = 0$ to $k = \infty$, followed by all the odd numbers in decreasing order multiplied by 2^k where k ticks down from ∞ to 1).

  Note that the odd numbers (except 1) only appear in the last row of this sequence, so if an odd period exists, then all of the infinitely many periods that appear to the left in the Sharkovkii ordering also exist. A consequence of this — the existence of infinitely many periodic orbits — is chaos. Two guys called Yorke and Li discovered (a decade after Sharkovskii's then little known result) one small hint of this, well known in the dynamical systems community as "*period three implies chaos*".

23 Chaos

"Chaos theory" isn't so much a theory as a bunch of observations and theorems that try to characterize how and where chaos can be found.
In short **chaos** is behaviour that repeats qualitatively but not quantitatively, evolving around inside a restricted area of phase space but never precisely repeating.
This has certain consequences.

- Two orbits very closely together travel far apart at later times.
- This means that a small change in the initial condition $\mathbf{x}_0 = \mathbf{x}(0)$ of an orbit can lead to huge differences in state at later times. As a result, evolution in chaotic systems is hugely unpredictable (which is why they have been of such interest in applications like weather prediction and neuroscience).
- Chaos often occurs on a region of space called a **chaotic attractor**. Orbits converge on it from the surrounding flow, then become chaotic. They are often *strange attractors* (attractors with a fractal structure).
- In fact orbits don't just travel apart in chaotic systems, they diverge exponentially. That means there is a fairly simple test that implies the existence of chaos. Given two nearby points x_0 and $x_0+\delta x_0$, the distance between them on each iteration of the map $x_{n+1} = f(x_n)$ should grow like $|\delta x_n| \approx |\delta_0|e^{\lambda n}$ where λ is something called the Lyapunov exponent.
- With this approximate idea we can derive a formula for the Lyapunov exponent. The distance after n iterates is $\delta x_n = f^n(x_0 + \delta_0) - f^n(x_0)$, so given $|\delta x_n| \approx |\delta_0|e^{\lambda n}$ we calculate

$$\lambda = \frac{1}{n} \ln \left| \frac{\delta_n}{\delta_0} \right| = \tfrac{1}{n} \ln \left| \frac{f^n(x_0 + \delta_0) - f^n(x_0)}{\delta_0} \right| \tag{23.1}$$

  For $\delta_0 \to 0$ the term inside the ln becomes the derivative $|(f^n(x_0))'|$, which can be expanded by the chain rule

$$(f^n(x_0))' = \prod_{i=0}^{n-1} f'(x_i) \tag{23.2}$$

then notice that

$$\ln|(f^n(x_0))'| = \ln\left|\prod_{i=0}^{n-1} f'(x_i)\right| = \sum_{i=0}^{n-1} \ln|f'(x_i)| \tag{23.3}$$

so we define the Lyapunov exponent as this in the limit $n \to \infty$

$$\lambda = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \ln|f'(x_i)| \tag{23.4}$$

but numerically we often approximate it by calculating it for a large, but of course finite, n.

- There is no single cause or test for chaos, or even a single completely general definition. But to have chaos a system must
  - be sensitive to initial conditions (orbits change a large amount with a small change in ICs),
  - be topologically transitive (any two regions of space overlap in the system's flow),
  - have dense periodic orbits (every point is infinitely close to a periodic orbit).

  These are often not easy to prove, but in certain cases one or two of these imply all three.

- Partly due to these criteria, an ODE must have at least 3 dimensions for chaos. A map only needs 1 dimension for chaos.

- The classic example of a chaotic system comes from model of atmospheric convection by Edward Lorenz,

$$\begin{aligned} \dot{x} &= \sigma(y - x) \\ \dot{y} &= \rho x - xz - y \\ \dot{z} &= xy - \beta z \end{aligned} \tag{23.5}$$

(try simulating this for $\rho = 28$, $\sigma = 10$, $\beta = 8/3$, in which chaos was dubbed in popular culture as the "*butterfly effect*".

24 Transient versus Asymptotic behaviour

When simulating a system it is important to distinguish between transient behaviour, which can only be seen for short times, and asymptotic behaviour which will determine the system's fate over long times.

Asymptotic behaviour describes what a system or orbit does as $t \to \infty$, typically either:

- settling onto an equilibrium or periodic orbit,
- becoming chaotic, or
- diverging off to infinity.

Transient dynamics is the behaviour that happens before that, between setting off a trajectory from a typical initial point $\mathbf{x}_0$ and it tending towards some attractor or long time state. This might involve:

- decaying towards an equilibrium or periodic orbit,
- an episode of almost-periodic behaviour before becoming chaotic, or
- a seemingly chaotic episode before settling to an equilibrium or periodic orbit.

Appendix

The last few pages below contain important notions you may want to be familiar with if you're going to model or study nonlinear dynamical systems. But we have run out of time in our course, so please take a look at these topics for future reference, but you won't be tested on them in this course.

A Some other useful notions

There are many other techniques people have invented to characterize the dynamics of systems, but we've covered the most fundamental. Others include:

A.1 Invariants of the motion

An invariant is any quantity that doesn't change during motion, that is, any quantity $V(x,t)$ such that

$$\dot{V}(x,t) = 0 \qquad \Rightarrow \qquad V(x,t) = \text{constant} \; . \tag{A.1}$$

- These are useful because $V(x,t)$ =constant is then often an algebraic equation (much simpler than the original ODE).
- E.g. in mechanics the total energy (kinetic + potential + elastic) is constant. It is a *constant* or *invariant of the motion.* Take a particle of mass m falling under gravitational acceleration g, with height x and speed $y = \dot{x}$, we have

  $$\dot{x} = y \qquad \dot{y} = -g \tag{A.2}$$

  which has a total energy

  $$\begin{aligned} & E = \text{KE} + \text{PE} = \tfrac{1}{2}my^2 + mgx \\ \Rightarrow \qquad & \dot{E} = my\dot{y} + mg\dot{x} = -myg + mgy = 0 \end{aligned} \tag{A.3}$$

  hence E is an invariant of the motion.

  These are also known as *conservative* systems (usually in physics), because they conserve some quantity.
- An important class are Hamiltonian systems, for which we can define a conserved energy and momentum.

A.2 Decreasing (Lyapunov) functions

Sometimes we can find scalar quantities that strictly decrease in the flow, that is, a quantity $V(x,t)$ such that

$$\dot{V}(x,t) < 0 \tag{A.4}$$

for any x, t.

- E.g. consider a simple focus

  $$\dot{x} = ax + y \qquad \dot{y} = ay - x \tag{A.5}$$

  which is stable for $a < 0$. A Lyapunov function for this is $V = \frac{1}{2}(x^2 + y^2)$, since

  $$\dot{V} = x\dot{x} + y\dot{y} = x(ax + y) + y(ay - x) = a(x^2 + y^2) < 0 \tag{A.6}$$

  for $a < 0$.

- Because V is strictly decreasing, if we plot the contours of V in our space (and this is useful for systems of any dimension), we know that solutions are always traveling down the gradient of V, inward across its contours.

A.3 Multiple timescales

Large and small quantities in a system can do strange things.

- If a particular constant is very small, say $\dot{x} = x + ay$, $\dot{y} = -y$, with $|a| \ll 1$, then the term it appears in usually won't have much affect and we can approximate $\dot{x} \approx x$, $\dot{y} \approx -y$.

- If a particular constant is very large then the term it appears in will usually dominate things, say $\dot{x} = x + ay$, $\dot{y} = -y$, with $|a| \gg 1$, then we can approximate $\dot{x} \approx ay$, $\dot{y} \approx -y$.

- If a small constant multiplies the highest order term in a system — the highest order power like x^n or highest order derivative like $\frac{d}{dt}$ in an ODE — then we have to be much more careful.

  E.g. $x = ax^2$ doesn't look like $x \approx 0$ when a is small. Solve it, and we see it has a pair of very different roots $x = 0$ and $x = 1/a$, the latter of which is very large when a is small (and we would miss this last one with the approximation $x \approx 0$).

  E.g. $a\dot{x} = x$ doesn't look like $x \approx 0$ when a is small. Solve it to get $x = x_0 e^{t/a}$, and we see that $x = 0$ is one special solution for $x_0 = 0$, but most orbits evolve very quickly as $x = x_0 e^{t/a}$ either towards (if $a < 0$) or away (if $a > 0$) from $x = 0$.

The key to systems like this is scaling. Take the system $a\dot{x} = x$ for small a.

- Take the extreme limit $a \to 0$, giving $x = 0$. This is clearly a valid solution of the system. It is the solution of the system on its natural t timescale. We often call this the **slow** timescale in such cases, for the following reason.

- Now try changing the timescale first. Let $\tau = t/a$, then $a\dot{x} = a\frac{dx}{dt} = \frac{dx}{d\tau}$, so the system becomes $\frac{dx}{d\tau} = x$, with simple solution $x = x_0 e^{\tau}$, that of a simple equilibrium. But one unit of time in τ is $1/a$ units of time in t, which is large if a is small, so τ is a **fast** timescale.

- So on the fast $\tau = t/a$ timescale $a\dot{x} = x$ looks like a simple equilibrium for small a. This fast dynamics quickly evolves to or from $x = 0$. On the slow timescale, when the system sits at or near $x = 0$, it simply behaves like the trivial $x = 0$.

Setting $a = 0$ and only looking at the slow system is also called *coarse graining*. Things get more interesting with multiple variables. A general model in two variables is

$$\dot{x} = f(x, y) \qquad\qquad a\dot{y} = g(x, y) \tag{A.7}$$

for small a.

- The $\dot{x}$ term doesn't involve any small quantity and is *slow*. If we set $a \approx 0$ we see it satisfies a *differential-algebraic equation*, that is, an ODE $\dot{x} = f$, combined with an invariant $g \approx 0$, since

  $$\dot{x} = f(x, y) \qquad\qquad 0 \approx g(x, y) \tag{A.8}$$

  This is known as the slow or **reduced** subsystem, or the coarse-grained system (depending on application).

- The $\dot{y}$ term will usually be very large because $\dot{y} = g/a$ is large for small a (except where $g = 0$, but that is dealt with by the reduced system above). But scale time to $\tau = t/a$ and we get

  $$\tfrac{d}{d\tau}x = af(x, y) \approx 0 \qquad\qquad \tfrac{d}{d\tau}\dot{y} = g(x, y) \tag{A.9}$$

  This is known as the fast or **layer** subsystem. We see that the variable x is slow, since on this fast timescale it is constant, $\dot{x} \approx 0$. This subsystem then tells us how y varies, on its fast timescale, outside of $g = 0$. The equilibria of the fast dynamics for y in (A.9) are just the invariants (the $g = 0$ equation) of the slow dynamics in (A.8).

- (Strictly the terms 'reduced' and 'layer' refer to the exact equations above obtained in the limit $a \to 0$).

A.4 Nonsmooth dynamics

All of the stuff above concerns dynamical systems defined by differentiable equations, e.g. $\dot{x} = f(x)$ or $x_{n+1} = f(x_n)$ where f is differentiable in x.

- Many of these results apply if f is at least continuous in x, e.g. $f(x) = |x| + a$ is continuous but not differentiable in x.

  In many cases the theory is harder to prove in these cases, but equilibria turn out to be much the same. Bifurcations can be much more involved. Chaos is much the same. The powerful Sharkovskii ordering, for example, is true provided f is continuous.

- We believed until recently, and nearly a century of theory suggested, that almost none of these results still applied if f is discontinuous, e.g. $f(x) = H(x) + a$ where H is the Heaviside function (with value $H = +1$ if $x > 0$ and $H = -1$ if $x < 0$).

  I've spent the last decade showing that, with the right approach, all of the stuff above applies at least qualitatively to discontinuous systems. That is, equilibria behave the same, you can study their stability in similar ways. Periodic orbits and bifurcations behave the same. We have even shown (just in the past year) that the Sharkovskii ordering even applies to discontinuous systems.

- There's still a lot of work to be done in non-differentiable, non-continuous, or more simply non-smooth systems. They are a big area of growth in modern applications. Collisions and nonsmooth, neurons fire in nonsmooth impulses, decisions are nonsmooth, electronic switches are nonsmooth . . . dynamical systems will never be done.

A.5 Numerical methods

This course has shown you that we rarely care about the exact solutions of differential equations. Even if we can find them, they are often so complicated that they are of no use.
Instead we study qualitative aspects, geometric objects like equilibria or periodic orbits.
Alongside understanding those, we often want to see them brought to life to verify our qualitative predictions. Or else we have complicated data that we wish to probe by seeking the qualitative objects that create it.
There are some powerful tools for bringing the equations to life or creating data from equations.

- There are various numerical packages for solving ODEs or maps. The most commonly used are in MATLAB, with ODE solvers such as ode45, ode15, and others implementing Runge-Kutta numerical recipes or adaptive methods.

- Continuation is a powerful tool that is something of a specialism among a number of staff in Bristol. There are a number of numerical packages for it such as AUTO, or MathCont. Continuation takes what you've learnt about bifurcations and uses it to follow or 'continue' bifurcation curves through bifurcations, to find even the branches of unstable equilibria or periodic orbits (which are hard to find because numerical simulations tend to evolve away from unstable objects).

- Event detection is a useful tool in packages like ode45, to create return maps, or to encode switches in a discontinuous system.

All these techniques and more are easily looked up online, and searches will usually turn up useful forums to help you in coding up numerical simulations of any given problem.

A.6 ODEs from applications: mass action

We have focussed on the methods to study ODEs here. There are a lot of different ways that ODEs arise in physical and biological modelling. There are some nice examples here: https://people.tamu.edu/∼phoward/m442/modode.pdf

In the population model we had a quantity N increasing or decreasing at certain rates, proportional to how much 'N' was already in the system.

The Law of Mass Action is a big generalization of this.

- Let X represent a particular 'species', which goes through some 'process' that changes how much of X there is.

  For example:

  - X might represent a person in a particular population, and the process might be the passing of an infection between individuals.
  - X might represent a molecule of a particular chemical, and the process might be a chemical reaction like photosynthesis.

- If there is an amount A of X before the process, and amount B of X after, and the process takes place at a rate k, we write

$$AX \xrightarrow{k} BX \tag{A.10}$$

  called a mass action equation.

- From this we can derive an ODE, if we define a variable $x_i = [X_i]$ representing the concentration of X_i , then this concentration changes as

$$\dot{x} = k(B - A)x^A \, . \tag{A.11}$$

- But a process rarely involves just one 'species'. . .

- Usually a process involves multiple parts or species.

  E.g. disease spread involves infected individuals X_1 and uninfected individuals X_2 (at least), photosynthesis involves four species, X_1, X_2, X_3, X_4, representing carbon dioxide, water, oxygen, and glucose.

- For these the same formula applies but $X = (X_1, X_2, ...)$ is a vector and A is a square matrix (with components we'll call a_{ij}), and if we have a number of processes with rates $k_1, k_2, ...$ we can write

$$\begin{aligned} a_{11}X_1 + a_{12}X_2 + ... &\xrightarrow{k_1} b_{11}X_1 + b_{12}X_2 + ... \\ a_{21}X_1 + a_{22}X_2 + ... &\xrightarrow{k_2} b_{21}X_1 + b_{22}X_2 + ... \end{aligned}$$

  which in vector form is

$$\begin{pmatrix} a_{11} & a_{12} & \dots \\ a_{21} & a_{22} & \dots \\ \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ \vdots \end{pmatrix} \xrightarrow{\begin{pmatrix} k_1 \\ k_2 \\ \vdots \end{pmatrix}} \begin{pmatrix} b_{11} & b_{12} & \dots \\ b_{21} & b_{22} & \dots \\ \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ \vdots \end{pmatrix} \tag{A.12}$$

- The matrix $S = B - A$ is called the *stochiometry matrix*, giving the difference between the amount of X before and after.

- We can derive an ODE from such a mass action equation, if we assume a "closed well mixed system". Define the concentration of species X_i as a variable $x_i = [X_i]$, then

$$\dot{x}_i = \sum_{j=1}^{n} s_{ji} k_j \; x_1^{a_{j1}} ... x_n^{a_{jn}} \tag{A.13}$$

  or in vector form

$$\frac{d}{dt} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \end{pmatrix} = \begin{pmatrix} s_{11} & s_{21} & \dots \\ s_{12} & s_{22} & \dots \\ \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} k_1 x_1^{a_{11}} ... x_n^{a_{1n}} \\ k_2 x_1^{a_{21}} ... x_n^{a_{2n}} \\ \vdots \end{pmatrix} \tag{A.14}$$

There's a tidier form if we let $x = (x_1, x_2, ..., x_n)$ be a column vector, then

$$\dot{x} = S^{\mathsf{T}} . \left(x^A \cdot k \right) \tag{A.15}$$

if we define

$$x^A = e^{A \ln x} \quad \text{where} \quad \ln x = (\ln x_1, \ln x_2, ...) \quad \text{and} \quad e^x = (e^{x_1}, e^{x_2}, ...) \, .$$

- These are tough expressions, as general formulae often are.
- They are much more intuitive if you look at some simple examples, you'll see some in the textbooks.