Asymptotics 2025/2026 Sheet 1
Problem 4: Detailed Solution

Problem 4

Problem Statement: Explain why the sequence $\{\phi_n(x) = x^{-n}\cos(nx)\}$, $n = 0, 1, \ldots$, is **not** an asymptotic sequence as $x \to \infty$.

1 Stage 1: Understanding What We Need to Prove

1.1 What is the Question Asking?

What we see: We are given a sequence of functions:

$$\phi_0(x) = x^0 \cos(0 \cdot x) = 1 \cdot 1 = 1 \tag{1}$$

$$\phi_1(x) = x^{-1}\cos(x) = \frac{\cos(x)}{x} \tag{2}$$

$$\phi_2(x) = x^{-2}\cos(2x) = \frac{\cos(2x)}{x^2} \tag{3}$$

$$\phi_3(x) = x^{-3}\cos(3x) = \frac{\cos(3x)}{x^3} \tag{4}$$

$$\vdots$$

Why: We list these explicitly because we need to see the *pattern* of how the sequence behaves. Each successive term has:

1. An **algebraic part**: x^{-n} that gets smaller as n increases (for fixed large x)
2. An **oscillatory part**: $\cos(nx)$ that oscillates with increasing frequency as n increases

The question is asking us to determine whether this sequence satisfies the formal definition of an "asymptotic sequence."

1.2 Recalling the Definition from the Lecture Notes

Definition 1 (Asymptotic Sequence)**.** *From Lecture Notes Section 2.5, page 9: A sequence of functions $\{\phi_n(x)\}$, $n = 0, 1, 2, \ldots$ is an **asymptotic sequence** as $x \to x_0$ if, for all n,*

$$\phi_{n+1}(x) = o(\phi_n(x)) \quad \text{as } x \to x_0. \tag{5}$$

Why: This definition is the *precise mathematical criterion* we must check. The notation $\phi_{n+1}(x) = o(\phi_n(x))$ means that each successive function in the sequence is "asymptotically smaller" than the previous one.

1.3 Understanding the "Little-oh" Notation

Definition 2 (Little-oh, from Lecture Notes Section 2.4.1, page 6)**.** *We write $f(x) = o(g(x))$ as $x \to x_0$ if*

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = 0. \tag{6}$$

Alternatively, we write $f(x) \ll g(x)$ as $x \to x_0$.

Why: The little-oh notation captures the idea that f is "negligible compared to g" near x_0. When we take the ratio and it goes to zero, it means f decays *faster* than g (or grows slower than g).

How we know: We translate the asymptotic sequence condition into a concrete limit we can evaluate:

$$\text{For } \{\phi_n(x)\} \text{ to be asymptotic, we need: } \lim_{x \to \infty} \frac{\phi_{n+1}(x)}{\phi_n(x)} = 0 \text{ for every } n. \tag{7}$$

2 Stage 2: Setting Up the Test

2.1 What We Must Verify

What we see: To determine if $\{\phi_n(x) = x^{-n}\cos(nx)\}$ is an asymptotic sequence as $x \to \infty$, we must check whether:

$$\lim_{x \to \infty} \frac{\phi_{n+1}(x)}{\phi_n(x)} = 0 \quad \text{for every } n = 0, 1, 2, \ldots \tag{8}$$

Why: If this limit equals zero for *all* n, then the sequence is asymptotic. If we can find even *one* value of n for which this limit either:

1. Does not exist, or
2. Exists but does not equal zero,

then the sequence **fails** to be asymptotic.

2.2 Computing the Ratio

What we see: Let us compute the ratio explicitly:

$$\frac{\phi_{n+1}(x)}{\phi_n(x)} = \frac{x^{-(n+1)}\cos((n+1)x)}{x^{-n}\cos(nx)} \tag{9}$$

$$= \frac{x^{-n-1}}{x^{-n}} \cdot \frac{\cos((n+1)x)}{\cos(nx)} \tag{10}$$

$$= x^{-n-1-(-n)} \cdot \frac{\cos((n+1)x)}{\cos(nx)} \tag{11}$$

$$= x^{-1} \cdot \frac{\cos((n+1)x)}{\cos(nx)} \tag{12}$$

$$= \frac{\cos((n+1)x)}{x\cos(nx)}. \tag{13}$$

Why: We separate the ratio into two parts:

1. **The algebraic part**: $x^{-1} = \frac{1}{x}$, which *does* go to zero as $x \to \infty$
2. **The trigonometric part**: $\frac{\cos((n+1)x)}{\cos(nx)}$, which we must analyze carefully

The key question is: does the product of these two parts have a well-defined limit as $x \to \infty$?

3 Stage 3: Analyzing the Limit

3.1 The Behavior of the Trigonometric Ratio

What we see: We need to understand:

$$\lim_{x\to\infty} \frac{\cos((n+1)x)}{x\cos(nx)} = \lim_{x\to\infty} \frac{1}{x} \cdot \frac{\cos((n+1)x)}{\cos(nx)}. \tag{14}$$

Why: The factor $\frac{1}{x} \to 0$ as $x \to \infty$, which is *good* for our purposes (we want the limit to be zero). However, the behavior of $\frac{\cos((n+1)x)}{\cos(nx)}$ as $x \to \infty$ is **problematic**.

3.2 Why the Trigonometric Ratio is Problematic

3.2.1 Observation 1: Oscillatory Behavior

What we see: Both $\cos((n+1)x)$ and $\cos(nx)$ are **periodic functions** that oscillate between -1 and $+1$ as x increases.

- $\cos(nx)$ oscillates with period $T_n = \frac{2\pi}{n}$
- $\cos((n+1)x)$ oscillates with period $T_{n+1} = \frac{2\pi}{n+1}$

Why: As $x \to \infty$, both functions continue to oscillate *indefinitely*. They do not settle down to any particular value. This means:

$$\lim_{x\to\infty} \cos(nx) \textbf{ does not exist}, \quad \lim_{x\to\infty} \cos((n+1)x) \textbf{ does not exist}. \tag{15}$$

How we know: We know this from basic analysis: a function that oscillates between two values indefinitely cannot have a limit. For example, $\lim_{x\to\infty} \sin(x)$ does not exist because $\sin(x)$ takes all values in $[-1,1]$ infinitely often as $x \to \infty$.

3.2.2 Observation 2: The Denominator Can Vanish

What we see: The denominator $\cos(nx)$ equals zero whenever:

$$nx = \frac{\pi}{2} + k\pi \quad \text{for } k \in \mathbb{Z}, \tag{16}$$

which occurs at:

$$x = \frac{\pi(2k+1)}{2n} \quad \text{for } k = 0, 1, 2, \ldots \tag{17}$$

Why: This is a **critical issue**. As $x \to \infty$, there are *infinitely many* values of x where $\cos(nx) = 0$. At these points:

$$\frac{\cos((n+1)x)}{\cos(nx)} = \frac{\cos((n+1)x)}{0}, \tag{18}$$

which is **undefined** (if $\cos((n+1)x) \neq 0$) or an indeterminate form $\frac{0}{0}$ (if both vanish simultaneously).

How we know: We can verify this by substituting. For example, take $n = 1$ and $x = \frac{\pi}{2}$:

$$\cos(x) = \cos\left(\frac{\pi}{2}\right) = 0 \tag{19}$$

$$\cos(2x) = \cos(\pi) = -1 \tag{20}$$

$$\frac{\cos(2x)}{\cos(x)} = \frac{-1}{0} = \text{undefined}. \tag{21}$$

3.3 Constructing a Sequence to Show Non-Existence

What we see: Let us construct a specific sequence of x-values to demonstrate that the limit does not exist.

3.3.1 Case 1: When $\cos(nx)$ is Near Zero

Choose: $x_k = \frac{\pi(2k+1)}{2n}$ for large integers k.

Why: At these values, $\cos(nx_k) = 0$ exactly.

What we see: Then:

$$\frac{\cos((n+1)x_k)}{x_k \cos(nx_k)} = \frac{\cos\left((n+1) \cdot \frac{\pi(2k+1)}{2n}\right)}{x_k \cdot 0}. \tag{22}$$

Why: If $\cos((n+1)x_k) \neq 0$, this ratio is **unbounded** (either $+\infty$ or $-\infty$).

How we know: We can verify that $\cos((n+1)x_k) \neq 0$ for *most* values of k (in general, the numerator and denominator zeros do not coincide). Therefore, along this sequence $\{x_k\}$:

$$\left|\frac{\cos((n+1)x_k)}{x_k \cos(nx_k)}\right| \to \infty. \tag{23}$$

3.3.2 Case 2: When Both Functions Are Non-Zero

Choose: $x_j = \frac{2\pi j}{n}$ for large integers j.

Why: At these values, $\cos(nx_j) = \cos(2\pi j) = 1$.

What we see: Then:

$$\frac{\cos((n+1)x_j)}{x_j \cos(nx_j)} = \frac{\cos\left((n+1) \cdot \frac{2\pi j}{n}\right)}{x_j \cdot 1} \tag{24}$$

$$= \frac{\cos\left(2\pi j + \frac{2\pi j}{n}\right)}{x_j} \tag{25}$$

$$= \frac{\cos\left(\frac{2\pi j}{n}\right)}{x_j}. \tag{26}$$

Why: Using the periodicity of cosine: $\cos(2\pi j + \theta) = \cos(\theta)$.

What we see: Now, as $j \to \infty$ (and hence $x_j \to \infty$):

$$\frac{\cos\left(\frac{2\pi j}{n}\right)}{x_j} = \frac{\cos\left(\frac{2\pi j}{n}\right)}{\frac{2\pi j}{n}} \to 0 \text{ as } j \to \infty, \tag{27}$$

provided $\cos\left(\frac{2\pi j}{n}\right)$ remains bounded (which it does, oscillating between -1 and 1).

Why: Along *this* sequence, the ratio does approach zero because the denominator grows like x_j while the numerator is bounded.

3.4 The Contradiction: Limit Does Not Exist

What we see: We have shown that:

1. Along the sequence $x_k = \frac{\pi(2k+1)}{2n}$, the ratio $\to \infty$ (unbounded)
2. Along the sequence $x_j = \frac{2\pi j}{n}$, the ratio $\to 0$

Why: For a limit $\lim_{x\to\infty} f(x)$ to exist, $f(x)$ must approach the **same value** along *every* sequence $\{x_n\}$ with $x_n \to \infty$.

How we know: This is the **sequential criterion for limits**:

$$\lim_{x\to x_0} f(x) = L \iff \lim_{n\to\infty} f(x_n) = L \text{ for every sequence } x_n \to x_0. \tag{28}$$

Since we have found two sequences giving different limiting behaviors (one unbounded, one zero), the limit:

$$\lim_{x\to\infty} \frac{\cos((n+1)x)}{x\cos(nx)} \quad \textbf{does not exist}. \tag{29}$$

4 Stage 4: Conclusion

4.1 Applying the Definition

What we see: We needed to verify:

$$\lim_{x\to\infty} \frac{\phi_{n+1}(x)}{\phi_n(x)} = 0 \quad \text{for all } n. \tag{30}$$

What we see: We have shown that for **any** $n \geq 1$:

$$\lim_{x\to\infty} \frac{\phi_{n+1}(x)}{\phi_n(x)} = \lim_{x\to\infty} \frac{\cos((n+1)x)}{x\cos(nx)} \quad \textbf{does not exist}. \tag{31}$$

Why: Since the limit does not exist, we cannot have $\phi_{n+1}(x) = o(\phi_n(x))$.

How we know: By the definition of an asymptotic sequence from the lecture notes (Section 2.5), if even *one* of the ratios fails the little-oh condition, the entire sequence is **not** asymptotic.

4.2 The Fundamental Reason

Why the sequence fails: The **oscillatory factors** $\cos(nx)$ and $\cos((n+1)x)$ do not decay as $x \to \infty$. Instead, they oscillate indefinitely. When we form the ratio $\frac{\cos((n+1)x)}{\cos(nx)}$, the denominator passes through zero infinitely often, causing the ratio to become unbounded along certain subsequences. This prevents the overall limit from existing, violating the requirement that $\phi_{n+1} = o(\phi_n)$.

4.3 Contrasting with a True Asymptotic Sequence

What we see: Compare with the standard asymptotic sequence $\{x^{-n}\}$ (without the cosine factors):

$$\lim_{x\to\infty} \frac{x^{-(n+1)}}{x^{-n}} = \lim_{x\to\infty} \frac{1}{x} = 0 \quad \checkmark \tag{32}$$

Why: Here, there are **no oscillatory factors**, just pure decay. The ratio has a well-defined limit of zero, so $\{x^{-n}\}$ **is** an asymptotic sequence as $x \to \infty$.

5 Final Answer

The sequence $\{\phi_n(x) = x^{-n}\cos(nx)\}$ is **not** an asymptotic sequence as $x \to \infty$ because:

1. The ratio $\frac{\phi_{n+1}(x)}{\phi_n(x)} = \frac{\cos((n+1)x)}{x\cos(nx)}$ does not have a well-defined limit as $x \to \infty$.
2. The denominator $\cos(nx)$ vanishes at infinitely many points as $x \to \infty$, causing the ratio to become unbounded along certain sequences.
3. Along other sequences where $\cos(nx) \neq 0$, the ratio may approach zero, but the limit must be the same along *all* sequences for the limit to exist.
4. Since the limit does not exist, we cannot have $\phi_{n+1}(x) = o(\phi_n(x))$, violating the definition of an asymptotic sequence (Lecture Notes, Section 2.5).