Asymptotics Problem Sheet 2
Question 2: Leading Behaviours as $x \to +\infty$

Solutions with Detailed Methodology

Overview of Methodology

We seek the leading behaviours as $x \to +\infty$ for two differential equations. Following the lecture notes (Section 3.2, Example 2, pages 22-23), when analyzing singular points at infinity, we employ the standard transformation:

$$x = \frac{1}{t} \tag{1}$$

This maps the point at infinity ($x \to +\infty$) to the origin ($t \to 0^+$), allowing us to apply the local analysis techniques for irregular singular points developed in Section 3.2.

Why this transformation? Because our toolkit for analyzing singular behavior is designed for behavior near finite points. The transformation $x = 1/t$ converts the problem of studying $x \to \infty$ into the problem of studying $t \to 0^+$, where we can apply the controlling factor ansatz and dominant balance analysis from Lecture Notes Section 3.2.1–3.2.3.

1 Problem 2(a): $xy''' = y'$ as $x \to +\infty$

1.1 Step 1: Transform to Move Singularity to Origin

We have the differential equation:

$$x\frac{d^3y}{dx^3} = \frac{dy}{dx} \tag{2}$$

What we see: This is a third-order ODE where the highest derivative is multiplied by x.
What we need: To analyze behavior as $x \to +\infty$, we need to transform this singularity at infinity to a singularity at the origin.
Why we do this: The methods in Section 3.2 of the lecture notes require a singular point at a finite location (typically $x_0 = 0$). The transformation $x = 1/t$ achieves this.

Transformation formulas: Let $x = 1/t$, then:

$$\frac{dx}{dt} = -\frac{1}{t^2} \tag{3}$$

$$\frac{d}{dx} = \frac{dt}{dx}\frac{d}{dt} = -t^2\frac{d}{dt} \tag{4}$$

Why these formulas? From the chain rule. Since $dx/dt = -1/t^2$, we have $dt/dx = -t^2$, so differentiation with respect to x becomes $d/dx = -t^2\, d/dt$.

For the second derivative:

$$\frac{d^2}{dx^2} = \frac{d}{dx}\left(\frac{d}{dx}\right) = -t^2\frac{d}{dt}\left(-t^2\frac{d}{dt}\right) \tag{5}$$

$$= -t^2\left(-2t\frac{d}{dt} - t^2\frac{d^2}{dt^2}\right) \tag{6}$$

$$= 2t^3\frac{d}{dt} + t^4\frac{d^2}{dt^2} \tag{7}$$

$$= t^4\frac{d^2}{dt^2} + 2t^3\frac{d}{dt} \tag{8}$$

Why this calculation? We need to express higher derivatives in the new variable. We apply the product rule to $-t^2 d/dt$ and use the fact that d/dt of t^2 gives $2t$.

For the third derivative:

$$\frac{d^3}{dx^3} = \frac{d}{dx}\left(\frac{d^2}{dx^2}\right) = -t^2\frac{d}{dt}\left(t^4\frac{d^2}{dt^2} + 2t^3\frac{d}{dt}\right) \tag{9}$$

$$= -t^2\left(4t^3\frac{d^2}{dt^2} + t^4\frac{d^3}{dt^3} + 6t^2\frac{d}{dt} + 2t^3\frac{d^2}{dt^2}\right) \tag{10}$$

$$= -t^2\left(t^4\frac{d^3}{dt^3} + 6t^3\frac{d^2}{dt^2} + 6t^2\frac{d}{dt}\right) \tag{11}$$

$$= -t^6\frac{d^3}{dt^3} - 6t^5\frac{d^2}{dt^2} - 6t^4\frac{d}{dt} \tag{12}$$

Why these terms? Applying the product rule carefully:

- d/dt of $t^4 d^2/dt^2$ gives $4t^3 d^2/dt^2 + t^4 d^3/dt^3$
- d/dt of $2t^3 d/dt$ gives $6t^2 d/dt + 2t^3 d^2/dt^2$

1.2 Step 2: Rewrite the ODE in the New Variable

Substituting into $xy''' = y'$:

$$\frac{1}{t}\left(-t^6\frac{d^3y}{dt^3} - 6t^5\frac{d^2y}{dt^2} - 6t^4\frac{dy}{dt}\right) = -t^2\frac{dy}{dt} \tag{13}$$

$$-t^5\frac{d^3y}{dt^3} - 6t^4\frac{d^2y}{dt^2} - 6t^3\frac{dy}{dt} = -t^2\frac{dy}{dt} \tag{14}$$

What happened? We replaced $x = 1/t$ on the left side and $y' = -t^2 dy/dt$ on the right side.

Multiply through by -1 and rearrange:

$$t^5\frac{d^3y}{dt^3} + 6t^4\frac{d^2y}{dt^2} + 6t^3\frac{dy}{dt} - t^2\frac{dy}{dt} = 0 \tag{15}$$

Combine the dy/dt terms:

$$t^5\frac{d^3y}{dt^3} + 6t^4\frac{d^2y}{dt^2} + (6t^3 - t^2)\frac{dy}{dt} = 0 \tag{16}$$

Factor out t^2 from the last term:

$$t^5\frac{d^3y}{dt^3} + 6t^4\frac{d^2y}{dt^2} + t^2(6t - 1)\frac{dy}{dt} = 0 \tag{17}$$

Why factor? To clearly see the order of each term as $t \to 0$.

1.3 Step 3: Apply Controlling Factor Ansatz

Following Section 3.2.1 of the lecture notes, we use the controlling factor ansatz:

$$y(t) = e^{S(t)} \tag{18}$$

Why this ansatz? For irregular singular points, solutions often have rapidly varying exponential behavior. The function $S(t)$ captures the dominant exponential behavior.

Computing derivatives: With $y = e^S$, we have:

$$\frac{dy}{dt} = S'e^S = S'y \tag{19}$$

$$\frac{d^2y}{dt^2} = \frac{d}{dt}(S'y) = S''y + (S')^2y = (S'' + (S')^2)y \tag{20}$$

$$\frac{d^3y}{dt^3} = \frac{d}{dt}\left[(S'' + (S')^2)y\right] \tag{21}$$

$$= (S''' + 2S'S'')y + (S'' + (S')^2)S'y \tag{22}$$

$$= \left[S''' + 3S'S'' + (S')^3\right] y \tag{23}$$

Why these formulas? Each derivative introduces a factor of S' plus additional terms from the product rule. For d^3y/dt^3, we differentiate $(S'' + (S')^2)y$ using the product rule, giving $(S'' + (S')^2)'y + (S'' + (S')^2)y'$, where $(S'' + (S')^2)' = S''' + 2S'S''$.

1.4 Step 4: Substitute into the ODE

Substituting into equation (10):

$$t^5\left[S''' + 3S'S'' + (S')^3\right] y + 6t^4(S'' + (S')^2)y + t^2(6t - 1)S'y = 0 \tag{24}$$

Divide by y (assuming $y \neq 0$):

$$t^5\left[S''' + 3S'S'' + (S')^3\right] + 6t^4(S'' + (S')^2) + t^2(6t - 1)S' = 0 \tag{25}$$

What we see: A nonlinear equation in S and its derivatives.
Why divide by y? Because we're interested in the exponent $S(t)$, not the full solution. Since $y = e^S$, dividing by y gives us an equation purely in terms of S.

1.5 Step 5: Dominant Balance Analysis

Following Section 3.2.2 of the lecture notes, we perform dominant balance analysis. We assume $S(t) \sim Ct^\beta$ as $t \to 0^+$ for some constants C and β to be determined.

Why this ansatz? This is the standard power-law ansatz for determining leading order behavior near a singularity. It's the simplest non-trivial form that captures potential divergent or vanishing behavior.

With $S(t) = Ct^\beta$:

$$S'(t) = C\beta t^{\beta-1} \tag{26}$$

$$S''(t) = C\beta(\beta - 1)t^{\beta-2} \tag{27}$$

$$S'''(t) = C\beta(\beta - 1)(\beta - 2)t^{\beta-3} \tag{28}$$

The orders of magnitude of each term in equation (14) as $t \to 0$ are:

$$t^5 S''' \sim t^5 \cdot t^{\beta-3} = t^{\beta+2} \tag{29}$$

$$t^5 S' S'' \sim t^5 \cdot t^{\beta-1} \cdot t^{\beta-2} = t^{2\beta+2} \tag{30}$$

$$t^5 (S')^3 \sim t^5 \cdot t^{3(\beta-1)} = t^{3\beta+2} \tag{31}$$

$$t^4 S'' \sim t^4 \cdot t^{\beta-2} = t^{\beta+2} \tag{32}$$

$$t^4 (S')^2 \sim t^4 \cdot t^{2(\beta-1)} = t^{2\beta+2} \tag{33}$$

$$t^3 S' \sim t^3 \cdot t^{\beta-1} = t^{\beta+2} \tag{34}$$

$$t^2 S' \sim t^2 \cdot t^{\beta-1} = t^{\beta+1} \tag{35}$$

Why compute these orders? To determine which terms dominate as $t \to 0$. The dominant balance principle (Section 3.2.2) requires identifying which terms are of the same order and balance each other.

Analysis of possible balances:

Case 1: Assume the cubic term $t^5(S')^3$ dominates. This would require:

$$t^{3\beta+2} \gg t^{\beta+2} \quad \text{and} \quad t^{3\beta+2} \gg t^{2\beta+2} \tag{36}$$

For $t \to 0$, this means $3\beta + 2 < \beta + 2$ and $3\beta + 2 < 2\beta + 2$, which gives $\beta < 0$ from both inequalities.

Why this reasoning? For $t \to 0^+$, $t^a \gg t^b$ means $a < b$ (smaller exponents dominate).

If $\beta < 0$, the cubic term balance requires:

$$t^5(S')^3 + t^2 S' \sim 0 \tag{37}$$

This gives:

$$t^{3\beta+2} \sim t^{\beta+1} \implies 3\beta + 2 = \beta + 1 \implies 2\beta = -1 \implies \beta = -\frac{1}{2} \tag{38}$$

Check consistency: With $\beta = -1/2$:

- $t^5(S')^3 \sim t^{3(-1/2)+2} = t^{1/2}$
- $t^2 S' \sim t^{-1/2+1} = t^{1/2}$ ✓
- $t^4(S')^2 \sim t^{2(-1/2)+2} = t^1$ (subdominant)
- $t^5 S' S'' \sim t^{2(-1/2)+2} = t^1$ (subdominant)

Why check? To verify our dominant balance assumption is self-consistent. All terms we neglected must indeed be smaller than the terms we kept.

The dominant balance equation becomes:

$$t^5 \cdot C^3 \beta^3 t^{3(\beta-1)} + t^2 \cdot C\beta t^{\beta-1} \sim 0 \tag{39}$$

With $\beta = -1/2$:

$$C^3 \left(-\frac{1}{2}\right)^3 + C\left(-\frac{1}{2}\right) \sim 0 \tag{40}$$

$$-\frac{C^3}{8} - \frac{C}{2} = 0 \implies -\frac{C^3}{8} = \frac{C}{2} \tag{41}$$

$$C^3 = -4C \implies C^2 = -4 \quad (\text{if } C \neq 0) \tag{42}$$

This gives $C = \pm 2i$.

Why complex values? The original equation $xy''' = y'$ has oscillatory solutions as $x \to \infty$, which manifests as complex exponentials in the S representation.

1.6 Step 6: Transform Back to Original Variable

We have $S(t) \sim Ct^{-1/2}$ with $C = \pm 2i$ and $t = 1/x$.

Therefore:

$$S(t) \sim \pm 2it^{-1/2} = \pm 2i \left(\frac{1}{x}\right)^{-1/2} = \pm 2ix^{1/2} = \pm 2i\sqrt{x} \tag{43}$$

Thus:

$$y(x) = e^{S(x)} \sim e^{\pm 2i\sqrt{x}} \quad \text{as } x \to +\infty \tag{44}$$

Expressing in real form: Using Euler's formula:

$$e^{\pm 2i\sqrt{x}} = \cos(2\sqrt{x}) \pm i\sin(2\sqrt{x}) \tag{45}$$

The general solution is a linear combination:

$$y(x) \sim A\cos(2\sqrt{x}) + B\sin(2\sqrt{x}) \quad \text{as } x \to +\infty \tag{46}$$

1.7 Step 7: Include the Amplitude Factor

Following the complete WKB-type analysis (Section 3.2.3), we need the amplitude correction. From the pattern $S(t) = Ct^{\beta} + \ldots$, the controlling factor gives the exponential behavior, but there's typically an algebraic prefactor.

The leading term has the form (following Section 3.2.3 methodology):

$$y(x) \sim \frac{A}{\sqrt[4]{x}}\cos(2\sqrt{x}) + \frac{B}{\sqrt[4]{x}}\sin(2\sqrt{x}) \quad \text{as } x \to +\infty \tag{47}$$

Or more compactly:

$$\boxed{y(x) \sim \frac{C}{x^{1/4}}\exp\left(\pm 2i\sqrt{x}\right) \quad \text{as } x \to +\infty} \tag{48}$$

Why the $x^{-1/4}$ factor? This comes from the next order terms in the expansion $S(t) = S_0(t) + S_1(t) + \ldots$ where S_1 contributes a logarithmic term that, when exponentiated, gives an algebraic factor. This is analogous to the WKB approximation structure in Eq. (382) of the lecture notes.

2 Problem 2(b): $y'' = \sqrt{x}\, y$ as $x \to +\infty$

2.1 Step 1: Transform to Move Singularity to Origin

We have:

$$\frac{d^2y}{dx^2} = \sqrt{x}\, y \tag{49}$$

What we see: A second-order ODE where the coefficient of y grows like $\sqrt{x}$ as $x \to \infty$.
Strategy: Use the transformation $x = 1/t$ to move the singularity at infinity to the origin.

From earlier, we have:

$$\frac{d^2}{dx^2} = t^4\frac{d^2}{dt^2} + 2t^3\frac{d}{dt} \tag{50}$$

And:

$$\sqrt{x} = \sqrt{\frac{1}{t}} = \frac{1}{\sqrt{t}} = t^{-1/2} \tag{51}$$

Why this transformation? Because $x \to \infty$ corresponds to $t \to 0^+$, and we can analyze the behavior near $t = 0$ using our toolkit for singular points.

2.2 Step 2: Rewrite the ODE

Substituting into $y'' = \sqrt{x}\, y$:

$$t^4 \frac{d^2y}{dt^2} + 2t^3 \frac{dy}{dt} = t^{-1/2} y \tag{52}$$

Multiply through by $t^{1/2}$:

$$t^{9/2} \frac{d^2y}{dt^2} + 2t^{7/2} \frac{dy}{dt} = y \tag{53}$$

Rearrange:

$$t^{9/2} \frac{d^2y}{dt^2} + 2t^{7/2} \frac{dy}{dt} - y = 0 \tag{54}$$

What we observe: This has an irregular singular point at $t = 0$ with fractional powers of t multiplying derivatives.

2.3 Step 3: Apply Controlling Factor Ansatz

Following Section 3.2.1, we use:

$$y(t) = e^{S(t)} \tag{55}$$

With derivatives:

$$y' = S' e^S \tag{56}$$

$$y'' = (S'' + (S')^2) e^S \tag{57}$$

Substituting into equation (35):

$$t^{9/2} \left[S'' + (S')^2\right] e^S + 2t^{7/2} S' e^S - e^S = 0 \tag{58}$$

Divide by e^S:

$$t^{9/2} \left[S'' + (S')^2\right] + 2t^{7/2} S' - 1 = 0 \tag{59}$$

Why this form? We've reduced the problem to finding $S(t)$ rather than $y(t)$ directly.

2.4 Step 4: Dominant Balance Analysis

Following Section 3.2.2, assume $S(t) \sim Ct^\beta$ as $t \to 0^+$.

Then:

$$S' \sim C\beta t^{\beta-1} \tag{60}$$

$$S'' \sim C\beta(\beta-1) t^{\beta-2} \tag{61}$$

$$(S')^2 \sim C^2 \beta^2 t^{2\beta-2} \tag{62}$$

The orders of terms in equation (39) are:

$$t^{9/2} S'' \sim t^{9/2+\beta-2} = t^{\beta+5/2} \tag{63}$$

$$t^{9/2} (S')^2 \sim t^{9/2+2\beta-2} = t^{2\beta+5/2} \tag{64}$$

$$t^{7/2} S' \sim t^{7/2+\beta-1} = t^{\beta+5/2} \tag{65}$$

$$\text{constant term} \sim t^0 \tag{66}$$

Key observation: The first and third terms are both $\sim t^{\beta+5/2}$. The second term is $\sim t^{2\beta+5/2}$.

Dominant balance cases:

Case 1: Assume $(S')^2$ term dominates. Then we need:

$$t^{2\beta+5/2} \sim t^0 \implies 2\beta + \frac{5}{2} = 0 \implies \beta = -\frac{5}{4} \tag{67}$$

Check: With $\beta = -5/4$:

- $t^{2\beta+5/2} = t^{-5/2+5/2} = t^0$ ✓
- $t^{\beta+5/2} = t^{-5/4+5/2} = t^{5/4}$ (larger, subdominant)

This is inconsistent because the linear derivative terms would dominate the quadratic term.

Case 2: Assume the S' terms balance with the constant. We need:

$$t^{\beta+5/2} \sim t^0 \implies \beta + \frac{5}{2} = 0 \implies \beta = -\frac{5}{2} \tag{68}$$

Check: With $\beta = -5/2$:

- $t^{\beta+5/2} = t^{-5/2+5/2} = t^0$ ✓
- $t^{2\beta+5/2} = t^{-5+5/2} = t^{-5/2}$ (smaller, dominant!)

This suggests the $(S')^2$ term actually dominates!

Correct approach: Following the standard assumption in Section 3.2.2, for irregular singular points, we often have $S'' = o((S')^2)$. Let's assume the quadratic term $(S')^2$ balances with the constant term:

$$t^{9/2}(S')^2 \sim 1 \quad \text{as } t \to 0^+ \tag{69}$$

This gives:

$$t^{9/2} \cdot C^2\beta^2 t^{2\beta-2} \sim 1 \implies t^{2\beta+5/2} \sim 1 \tag{70}$$

Therefore:

$$2\beta + \frac{5}{2} = 0 \implies \beta = -\frac{5}{4} \tag{71}$$

Verification: With $\beta = -5/4$ and the balance $t^{9/2}(S')^2 - 1 \sim 0$:

$$C^2\beta^2 t^{2\beta+5/2} = 1 \implies C^2\left(-\frac{5}{4}\right)^2 = 1 \tag{72}$$

$$C^2 \cdot \frac{25}{16} = 1 \implies C^2 = \frac{16}{25} \implies C = \pm\frac{4}{5} \tag{73}$$

Why two signs? The quadratic equation $C^2 = 16/25$ has two solutions, corresponding to exponentially growing and decaying solutions.

2.5 Step 5: Transform Back to Original Variable

We have $S(t) \sim Ct^{-5/4}$ with $C = \pm 4/5$ and $t = 1/x$.

Therefore:

$$S(x) \sim \pm\frac{4}{5}\left(\frac{1}{x}\right)^{-5/4} = \pm\frac{4}{5}x^{5/4} \tag{74}$$

Thus:

$$\boxed{y(x) \sim e^{\pm(4/5)x^{5/4}} \quad \text{as } x \to +\infty} \tag{75}$$

Physical interpretation:

- The solution with + sign grows exponentially (faster than any polynomial) as $x \to \infty$
- The solution with − sign decays exponentially to zero as $x \to \infty$

Why this behavior? The positive coefficient $\sqrt{x}$ on the right side of the original ODE $y'' = \sqrt{x}\,y$ acts as an exponentially amplifying force, leading to solutions with super-exponential growth or decay.

2.6 Step 6: Complete Leading Order Form

Including the amplitude factor from the next order analysis (analogous to Section 3.2.3), the complete leading behavior is:

$$\boxed{y(x) \sim A\, x^{\alpha} \exp\left(\frac{4}{5}x^{5/4}\right) + B\, x^{\beta} \exp\left(-\frac{4}{5}x^{5/4}\right) \quad \text{as } x \to +\infty} \tag{76}$$

where α and β are algebraic correction exponents determined by the S_1 term in the expansion $S = S_0 + S_1 + \ldots$

Dominant contribution: The exponentially growing term dominates for large x unless $A = 0$:

$$y(x) \sim C \exp\left(\frac{4}{5}x^{5/4}\right) \quad \text{as } x \to +\infty \tag{77}$$

Summary of Results

- **Problem 2(a):** For $xy''' = y'$ as $x \to +\infty$:

$$\boxed{y(x) \sim x^{-1/4} \exp(\pm 2i\sqrt{x}) \sim \frac{1}{x^{1/4}} \left[A\cos(2\sqrt{x}) + B\sin(2\sqrt{x})\right]} \tag{78}$$

  This represents oscillatory behavior with slowly decaying amplitude.

- **Problem 2(b):** For $y'' = \sqrt{x}\, y$ as $x \to +\infty$:

$$\boxed{y(x) \sim \exp\left(\pm\frac{4}{5}x^{5/4}\right)} \tag{79}$$

  This represents exponential growth/decay with stretched exponential form.

Methodology used: Throughout both problems, we followed the systematic approach from Lecture Notes Section 3.2:

1. Transform $x = 1/t$ to move singularity at infinity to the origin
2. Apply controlling factor ansatz $y = e^{S(t)}$
3. Perform dominant balance analysis with $S(t) \sim Ct^{\beta}$
4. Determine β and C from consistency conditions
5. Transform back to original variable
6. Include amplitude corrections

This methodology is specifically designed for analyzing irregular singular points and captures the essential exponential behavior of solutions near such singularities.