

# Question 1(a): Asymptotic Analysis Leading Order Behavior of $\int_X^\infty e^{-t^3} dt$

Asymptotics Course — Sheet 4

## Problem Statement

Find the leading order asymptotic behavior as  $X \rightarrow \infty$  of:

$$I(X) = \int_X^\infty e^{-t^3} dt$$

## 1 Step 1: Problem Classification and Strategy

### Form Recognition

The integral has the form:

$$I(X) = \int_X^\infty e^{-t^3} dt$$

- **STAGE X (What we have):** An integral with exponential decay  $e^{-t^3}$  over  $[X, \infty)$  where the decay rate accelerates rapidly as  $t$  increases.
- **STAGE Y (Why this approach):** As  $X \rightarrow \infty$ , the exponential  $e^{-t^3}$  decays extremely rapidly. The main contribution to the integral comes from values of  $t$  near the lower limit  $X$ . This suggests using **integration by parts** (Section 4.1 of lecture notes).
- **STAGE Z (What this means):** The integral is exponentially small, and we seek its leading order behavior by extracting contributions from the boundary at  $t = X$ .

## 2 Step 2: Variable Transformation

To apply standard techniques, we first transform the integral using substitution.

### Substitution

Set:

$$u = t^3 \quad \Rightarrow \quad du = 3t^2 dt \quad \Rightarrow \quad dt = \frac{du}{3t^2}$$

Since  $t = u^{1/3}$ , we have  $t^2 = u^{2/3}$ , therefore:

$$dt = \frac{du}{3u^{2/3}}$$

### Transformed Limits

$$\begin{aligned} t = X &\Rightarrow u = X^3 \\ t = \infty &\Rightarrow u = \infty \end{aligned}$$

## Transformed Integral

$$I(X) = \int_{X^3}^{\infty} e^{-u} \cdot \frac{1}{3u^{2/3}} du$$

- **STAGE X (What we did):** Transformed to standard exponential form with  $e^{-u}$  and identified the prefactor  $f(u) = \frac{1}{3u^{2/3}}$ .
- **STAGE Y (Why this helps):** The integral now has the form  $\int_a^{\infty} f(u)e^{-u}du$  which is amenable to integration by parts as described in Section 4.1, Equation (165).
- **STAGE Z (Next step):** Apply integration by parts to extract the leading order contribution from the lower limit.

## 3 Step 3: Integration by Parts

### General Formula (Section 4.1)

For an integral of the form  $\int_a^{\infty} f(u)e^{-u}du$ , we write:

$$e^{-u} = \frac{d}{du}(-e^{-u})$$

Integration by parts gives:

$$\begin{aligned} \int_a^{\infty} f(u)e^{-u}du &= [-f(u)e^{-u}]_a^{\infty} + \int_a^{\infty} f'(u)e^{-u}du \\ &= f(a)e^{-a} + \int_a^{\infty} f'(u)e^{-u}du \end{aligned}$$

### Application to Our Integral

With  $f(u) = \frac{1}{3u^{2/3}}$  and  $a = X^3$ :

$$I(X) = \frac{1}{3(X^3)^{2/3}}e^{-X^3} + \int_{X^3}^{\infty} \frac{d}{du} \left( \frac{1}{3u^{2/3}} \right) e^{-u} du$$

Simplify the first term:

$$\frac{1}{3(X^3)^{2/3}} = \frac{1}{3X^2}$$

Therefore:

$$I(X) = \frac{e^{-X^3}}{3X^2} + \int_{X^3}^{\infty} \frac{d}{du} \left( \frac{1}{3u^{2/3}} \right) e^{-u} du$$

## 4 Step 4: Compute the Derivative

### Derivative Calculation

$$\frac{d}{du} \left( \frac{1}{3u^{2/3}} \right) = \frac{1}{3} \cdot \left( -\frac{2}{3} \right) u^{-5/3} = -\frac{2}{9} u^{-5/3}$$

### Updated Expression

$$I(X) = \frac{e^{-X^3}}{3X^2} - \frac{2}{9} \int_{X^3}^{\infty} u^{-5/3} e^{-u} du$$

## 5 Step 5: Asymptotic Order Analysis

### Order of the Remainder Term

The remainder integral satisfies:

$$\left| \int_{X^3}^{\infty} u^{-5/3} e^{-u} du \right| < (X^3)^{-5/3} \int_{X^3}^{\infty} e^{-u} du = X^{-5} e^{-X^3}$$

### Comparison of Terms

- Leading term:  $\frac{e^{-X^3}}{3X^2} = O(X^{-2}e^{-X^3})$  as  $X \rightarrow \infty$
- Remainder term:  $O(X^{-5}e^{-X^3})$  as  $X \rightarrow \infty$

### Dominance Relation

$$\frac{X^{-5}e^{-X^3}}{X^{-2}e^{-X^3}} = X^{-3} \rightarrow 0 \text{ as } X \rightarrow \infty$$

- **STAGE X (What we found):** The remainder term is of order  $X^{-5}e^{-X^3}$ , which is three powers of  $X$  smaller than the leading term.
- **STAGE Y (Why this matters):** The ratio  $X^{-3} \rightarrow 0$  confirms the remainder is asymptotically negligible compared to the first term. The leading order behavior is completely determined by the boundary contribution at  $u = X^3$  (equivalently  $t = X$ ).
- **STAGE Z (Conclusion):** We can neglect the remainder term in the asymptotic limit.

## 6 Step 6: Final Answer

Using the proper asymptotic notation (Definition, page 8 of notes):

**Leading Order Asymptotic:**

$$I(X) = \int_X^{\infty} e^{-t^3} dt \sim \frac{e^{-X^3}}{3X^2} \quad \text{as } X \rightarrow \infty$$

### Alternative Notation

This can also be written as:

$$I(X) = \frac{e^{-X^3}}{3X^2} \left[ 1 + O\left(\frac{1}{X^3}\right) \right] \quad \text{as } X \rightarrow \infty$$

## 7 Verification Checklist

*Following the standards of Section 4.1-4.2:*

- ✓ **Method identified:** Integration by parts (Section 4.1)
- ✓ **Transformation applied:** Substitution  $u = t^3$  to standard form
- ✓ **Leading term extracted:** Boundary contribution at lower limit

- ✓ **Remainder estimated:** Shown to be  $O(X^{-5}e^{-X^3})$
- ✓ **Dominance verified:** Ratio  $X^{-3} \rightarrow 0$  confirms asymptotic ordering
- ✓ **Proper notation:** Used  $\sim$  for asymptotic equivalence

## Physical Interpretation

- **STAGE X (Behavior):** The integral decays faster than any polynomial as  $X \rightarrow \infty$  due to the  $e^{-X^3}$  factor.
- **STAGE Y (Mechanism):** The rapid exponential decay means only a thin layer of width  $\sim O(X^{-2})$  near  $t = X$  contributes significantly to the integral.
- **STAGE Z (Result):** The leading order is determined entirely by the local behavior at the lower integration limit, with the asymptotic form  $\frac{e^{-X^3}}{3X^2}$ .