Methods of Applied Mathematics - Part 1 [SEMT30006]

Exercise Sheet 5 – Map bifurcations, period doubling, and chaos

1. Consider the two-dimensional map

$$
\begin{aligned}
x_{n+1} &= x_n^2 - c y_n \\
y_{n+1} &= \tfrac{1}{2}(x_n - y_n)
\end{aligned}
$$

   (a) Find the fixed points of the map.
   (b) Determine the stability of the fixed points and conjecture the bifurcation(s) that occur(s) as their stability changes.

2. Sketch or graph the sawtooth map

$$
x_{n+1} = \begin{cases} 2x_n & \text{for } 0 \le x_n < 1/2 \\ 2x_n - 1 & \text{for } 1/2 < x_n \le 1 \end{cases}
$$

   Either by hand or computer, investigate its dynamics with cobweb diagrams.

   (a) Show there is a period two orbit with an iterate at $x = 1/3$, and find the other iterate.
   (b) Show there is a period three orbit with an iterate at $x = 1/7$, and find the other iterates.
   (c) Show that the orbits from (a)-(b) are unstable.
   (d) Argue why this map cannot have any stable periodic orbits, and conjecture what kind of dynamics you will see from a typical initial point.

3. Take a look at the webpage
   http://www.complexity-explorables.org/flongs/logistic/
   which will help you explore the **logistic map**

$$
x_{n+1} = r x_n (1 - x_n)
$$

   Using the interactive figure in panel 3 on the site, can you find a period 2 orbit, and a period 4 orbit?

   As you do this, try to work out the following and check them against the simulation:

   (a) Find any fixed points (period one orbits) and the values of r for which they: (i) exist, (ii) are stable.
   (b) Find any period two orbits and the values of r for which they: (i) exist, (ii) are stable.
   (c) Find any period four orbits and the values of r for which they: (i) exist, (ii) are stable.
   (d) Sketch or simulate (e.g. in Matlab) a cobweb diagram showing a stable period one, two, or three orbit, for different suitable example values of r.
   (e) Sketch a bifurcation diagram showing the change from (a) to (b), and identify the bifurcation that occurs.