Complete Taylor Series Expansion Reference

General Taylor Series Formula

For a function $f(x)$ expanded about $x = a$:

$$f(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \frac{f'''(a)}{3!}(x - a)^3 + \cdots$$

f(x) = f(a) + f′(a)(x − a) + 2!f′′(a)(x − a)2 + 3!f′′′(a)(x − a)3 + ⋯

Maclaurin Series (special case with $a = 0$):

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots$$

f(x) = f(0) + f′(0)x + 2!f′′(0)x2 + 3!f′′′(0)x3 + ⋯

Elementary Functions

Exponential Function

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

ex = 1 + x + 2!x2 + 3!x3 + 4!x4 + ⋯ = n=0∑∞ n!xn

Convergence: All $x \in R$

Natural Logarithm

$$\ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n}$$

ln(1 + x) = x − 2x2 + 3x3 − 4x4 + ⋯ = n=1∑∞ n(−1)n+1xn

Convergence: $-1 < x \le 1$

$$\ln(1 - x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \cdots = -\sum_{n=1}^{\infty} \frac{x^n}{n}$$

ln(1 − x) = −x − 2x2 − 3x3 − 4x4 − ⋯ = − n=1∑∞ nxn

Convergence: $-1 \le x < 1$

Trigonometric Functions

Sine

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

sin x = x − 3!x3 + 5!x5 − 7!x7 + ⋯ = n=0∑∞ (2n + 1)!(−1)nx2n+1

Convergence: All $x \in R$

Cosine

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

cos x = 1 − 2!x2 + 4!x4 − 6!x6 + ⋯ = n=0∑∞ (2n)!(−1)nx2n

Convergence: All $x \in R$

Tangent

$$\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} + \cdots$$

tan x = x + 3x3 + 152x5 + 31517x7 + ⋯

Convergence: $|x| < \frac{\pi}{2}$ |x| < 2π

Inverse Trigonometric Functions

Arcsine

$$\arcsin x = x + \frac{x^3}{6} + \frac{3x^5}{40} + \frac{5x^7}{112} + \cdots = \sum_{n=0}^{\infty} \frac{(2n)!}{4^n (n!)^2 (2n+1)} x^{2n+1}$$

arcsin x = x + 6x3 + 403x5 + 1125x7 + ⋯ = n=0∑∞ 4n(n!)2(2n + 1)(2n)!x2n+1

Convergence: $|x| \le 1$|x| ≤ 1

Arctangent

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$$

arctan x = x − 3x3 + 5x5 − 7x7 + ⋯ = n=0∑∞ 2n + 1(−1)nx2n+1

Convergence: $|x| \le 1$|x| ≤ 1

Hyperbolic Functions

Hyperbolic Sine

$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$$

sinh x = x + 3!x3 + 5!x5 + 7!x7 + ⋯ = n=0∑∞ (2n + 1)!x2n+1

Convergence: All $x \in R$x ∈ R

Hyperbolic Cosine

$$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$$

cosh x = 1 + 2!x2 + 4!x4 + 6!x6 + ⋯ = n=0∑∞ (2n)!x2n

Convergence: All $x \in R$x ∈ R

Hyperbolic Tangent

$$\tanh x = x - \frac{x^3}{3} + \frac{2x^5}{15} - \frac{17x^7}{315} + \cdots$$

tanh x = x − 3x3 + 152x5 − 31517x7 + ⋯

Convergence: $|x| < \frac{\pi}{2}$|x| < 2π

Binomial Series (Your Example!)

$$(1+x)^{\alpha} = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3 + \cdots$$

(1 + x)α = 1 + αx + 2!α(α − 1)x2 + 3!α(α − 1)(α − 2)x3 + ⋯

Convergence: $|x| < 1$|x| < 1 (for non-integer αα)

Special Cases:

- $\alpha = 1/2$α = 1/2: $\sqrt{1+x} = 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} - \frac{5x^4}{128} + \cdots$ 1 + x = 1 + 2x − 8x2 + 16x3 − 1285x4 + ⋯
- $\alpha = -1$α = −1: $\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - \cdots$ 1+x1 = 1 − x + x2 − x3 + x4 − ⋯
- $\alpha = -2$α = −2: $\frac{1}{(1+x)^2} = 1 - 2x + 3x^2 - 4x^3 + 5x^4 - \cdots$ (1+x)21 = 1 − 2x + 3x2 − 4x3 + 5x4 − ⋯
- $\alpha = -1/2$α = −1/2: $\frac{1}{\sqrt{1+x}} = 1 - \frac{x}{2} + \frac{3x^2}{8} - \frac{5x^3}{16} + \frac{35x^4}{128} - \cdots$ 1+x 1 = 1 − 2x + 83x2 − 165x3 + 12835x4 − ⋯

Geometric Series

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots = \sum_{n=0}^{\infty} x^n$$

1 − x1 = 1 + x + x2 + x3 + x4 + ⋯ = n=0∑∞ xn

Convergence: $|x| < 1$|x| < 1

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - \cdots = \sum_{n=0}^{\infty} (-1)^n x^n$$

1 + x1 = 1 − x + x2 − x3 + x4 − ⋯ = n=0∑∞(−1)nxn

Convergence: $|x| < 1$|x| < 1

Error Functions and Special Functions

Error Function

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}}\left(x - \frac{x^3}{3} + \frac{x^5}{10} - \frac{x^7}{42} + \cdots\right)$$

erf(x) = π
√
2 (x − 3x3 + 10x5 − 42x7 + ⋯)

Bessel Function (First Kind, Order 0)

$$J_0(x) = 1 - \frac{x^2}{4} + \frac{x^4}{64} - \frac{x^6}{2304} + \cdots$$

J0(x) = 1 − 4x2 + 64x4 − 2304x6 + ⋯

Multivariate Extensions

Two Variables (about origin)

$$f(x, y) = f(0, 0) + x f_x(0, 0) + y f_y(0, 0) + \frac{1}{2!}(x^2 f_{xx} + 2xy f_{xy} + y^2 f_{yy}) + \cdots$$

f (x, y) = f (0, 0) + xfx(0, 0) + yfy(0, 0) + 2!1 (x2fxx + 2xyfxy + y2fyy) + ⋯

Practical Tips

1. **Radius of convergence:** Check where series converges (ratio test, root test)
2. **Error estimation:** For alternating series, error < first neglected term
3. **Composition:** Can substitute one series into another (e.g., $e^{\sin x}$)
4. **Differentiation/Integration:** Can differentiate/integrate term-by-term within radius
5. **Asymptotic expansions:** Some series diverge but are still useful for approximations

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Common Techniques

- **Substitution:** Replace x with $-x$, x^2, etc.
- **Addition/Subtraction:** Combine series (e.g., $\cosh x = \frac{e^x + e^{-x}}{2}$)
- **Multiplication:** Multiply series using Cauchy product
- **Reversion:** Find inverse function series
- **Padé approximants:** Rational function approximations (often better than polynomials)