Solution 5.1(e)

Problem: Find the asymptotic behavior of

$$I(X) = \int_0^\pi \sin(X \cos t)\, e^{-t^2}\, dt$$

as $X \to \infty$.

Solution:
Step 1: Express as imaginary part of complex integral.
We write the integral in complex exponential form:

$$I(X) = \operatorname{Im} \int_0^\pi e^{-t^2} e^{iX\cos t}\, dt.$$

Step 2: Identify the phase function and locate stationary points.
The phase function is $\Phi(t) = \cos t$, with derivatives:

$$\Phi'(t) = -\sin t, \qquad \Phi''(t) = -\cos t.$$

The stationary points occur where $\Phi'(t) = 0$, i.e., where $\sin t = 0$. Within the integration interval $[0, \pi]$, this gives stationary points at the two endpoints:

$$t = 0 \quad \text{and} \quad t = \pi.$$

Step 3: Evaluate the phase function at stationary points.
At the endpoints:

$$\begin{aligned} \Phi(0) &= \cos 0 = 1, & \Phi''(0) &= -\cos 0 = -1, \\ \Phi(\pi) &= \cos\pi = -1, & \Phi''(\pi) &= -\cos\pi = 1. \end{aligned}$$

Step 4: Apply the stationary phase formula for boundary stationary points.
For a stationary point at an endpoint $t = c$ of the integration interval, the contribution to the integral is half that of an interior stationary point. The asymptotic contribution from an endpoint stationary point is given by:

$$\frac{1}{2}\sqrt{\frac{2\pi i}{X|\Phi''(c)|}}\, e^{\pm i\pi/4}\, f(c)\, e^{iX\Phi(c)},$$

where the sign in the exponential $e^{\pm i\pi/4}$ depends on the sign of $\Phi''(c)$, and $f(t) = e^{-t^2}$ is the amplitude function.
Step 5: Compute the contribution from $t = 0$.
At $t = 0$:

- $f(0) = e^0 = 1$
- $\Phi(0) = 1$
- $\Phi''(0) = -1 < 0$

The contribution is:

$$\frac{1}{2}\sqrt{\frac{2\pi i}{X\cdot 1}}\cdot e^{-i\pi/4}\cdot 1\cdot e^{iX}=\frac{1}{2}\sqrt{\frac{2\pi}{X}}\cdot e^{i\pi/4}\cdot e^{-i\pi/4}\cdot e^{iX}=\frac{1}{2}\sqrt{\frac{2\pi}{X}}\,e^{iX}.$$

More carefully, using $\sqrt{i}=e^{i\pi/4}$ and accounting for $\Phi''(0)=-1$:

$$\sqrt{\frac{i}{\Phi''(0)}}=\sqrt{\frac{i}{-1}}=\sqrt{-i}=e^{-i\pi/4}.$$

Thus the contribution from $t=0$ is:

$$I_0(X)\sim\frac{1}{2}\sqrt{\frac{2\pi}{X}}\,e^{-i\pi/4}\,e^{iX}=\frac{1}{2}\sqrt{\frac{2\pi}{X}}\,e^{i(X-\pi/4)}.$$

Step 6: Compute the contribution from $t=\pi$.
At $t=\pi$:

- $f(\pi)=e^{-\pi^2}$
- $\Phi(\pi)=-1$
- $\Phi''(\pi)=1>0$

Using $\sqrt{i/\Phi''(\pi)}=\sqrt{i}=e^{i\pi/4}$:

$$I_\pi(X)\sim\frac{1}{2}\sqrt{\frac{2\pi}{X}}\,e^{i\pi/4}\,e^{-\pi^2}\,e^{-iX}=\frac{1}{2}\sqrt{\frac{2\pi}{X}}\,e^{-\pi^2}\,e^{-i(X-\pi/4)}.$$

Step 7: Combine contributions from both endpoints.
The total asymptotic contribution is:

$$\begin{aligned}\int_0^\pi e^{-t^2}e^{iX\cos t}\,dt&\sim I_0(X)+I_\pi(X)\\&=\frac{1}{2}\sqrt{\frac{2\pi}{X}}\left[e^{i(X-\pi/4)}+e^{-\pi^2}e^{-i(X-\pi/4)}\right].\end{aligned}$$

Step 8: Extract the imaginary part.
Taking the imaginary part:

$$\begin{aligned}I(X)&=\operatorname{Im}\left\{\frac{1}{2}\sqrt{\frac{2\pi}{X}}\left[e^{i(X-\pi/4)}+e^{-\pi^2}e^{-i(X-\pi/4)}\right]\right\}\\&=\frac{1}{2}\sqrt{\frac{2\pi}{X}}\left[\sin\left(X-\frac{\pi}{4}\right)-e^{-\pi^2}\sin\left(X-\frac{\pi}{4}\right)\right]\\&=\frac{1}{2}\sqrt{\frac{2\pi}{X}}\left(1-e^{-\pi^2}\right)\sin\left(X-\frac{\pi}{4}\right).\end{aligned}$$

Step 9: Final result.
Therefore, the asymptotic behavior is:

$$\boxed{I(X)\sim\sqrt{\frac{\pi}{2X}}\sin\left(X-\frac{\pi}{4}\right)\left(1-e^{-\pi^2}\right)\quad\text{as } X\to\infty.}$$

Remark: Although $e^{-\pi^2} \approx 5.2 \times 10^{-5}$ is numerically small, the factor $(1 - e^{-\pi^2})$ is $O(1)$ as $X \to \infty$ and must be retained in the asymptotic expression. Both endpoint stationary points contribute at the same asymptotic order $O(X^{-1/2})$, and their interference produces this factor. The amplitude e^{-t^2} evaluates to 1 at $t = 0$ and to $e^{-\pi^2}$ at $t = \pi$, giving rise to the difference in contributions.