Asymptotics 2025/2026 — Sheet 7[1]

WKB method

Please hand in questions 1,2 and 3 by Thursday 27 November 2025 at 12pm

1. (**5 marks**)
   Consider the ordinary differential equation $u'' + p(x)u' + q(x)u = 0$. Set $u(x) = f(x)\,y(x)$ and find a function $f(x)$ such that the resulting o.d.e. for $y(x)$ has no term in $y'(x)$.

2. (**10 marks**)
   For the equation $\varepsilon^2 y''(x) + q(x)y(x) = 0$, look for transformations of both the dependent and independent variables, $z = \phi(x)$, $\nu(z) = \psi(x)y(x)$ so that with suitable choice for the functions $\phi(x)$ and $\psi(x)$, the o.d.e. becomes $\varepsilon^2\nu''(z) + \nu(z) = 0$ to the leading order as $\varepsilon \to 0$. Hence deduce the leading order solution and show it is equivalent to the WKB solution. (You may assume that $q(x)$ is a positive function.)

3. (**10 marks**)
   For the equation $\varepsilon^2 y''(x) + [q(x) + \chi(\varepsilon)r(x)]y(x) = 0$ deduce the leading order solution as $\varepsilon \to 0$ for those cases where $\chi(\varepsilon)$ is o(1) and yet not so small that the standard WKB solution is appropriate. Consider all relevant cases.

4. For what choices of $q(x)$ in the equation $\varepsilon^2 y'' + q(x)y = 0$ is the WKB solution exact?

5. Use the WKB approximation to estimate the large eigenvalues, λ, of the eigenvalue problem $y'' + \lambda^2 y/x^2 = 0$, $y(1) = 0$, $y(e) = 0$. Find also the exact solutions and the exact eigenvalues. (Try $y(x) = x^\alpha$.) Consider the two sets of eigenvalues:

   (i) Are the discrepancies between them consistent with the approximation made? If so, explain briefly why.

   (ii) Will more terms of the WKB approximation give a better result? If your answer is yes, determine the form of the next term in the approximation to $y(x)$ and show how this gives a better result for the eigenvalues.

6. The Bessel functions $J_n(z)$ are the solutions, $w(z)$ of $z^2w'' + zw' + (z^2 - n^2)w = 0$ which are regular at the origin.

   (a) Change variables to $W = z^{1/2}w$ and $t = z/(n^2 - 1/4)^{1/2}$ and hence show that the WKB solutions for large n are

$$w \sim \frac{A_\pm}{z^{1/2}} \left[\frac{z^2}{z^2-n^2}\right]^{1/4} \exp\left\{\pm i\left[(z^2-n^2)^{1/2} - n\cos^{-1}(n/z)\right]\right\} \qquad \text{for} \qquad z > n\,,$$

and
$$w \sim \frac{B_\pm}{z^{1/2}} \left[\frac{z^2}{n^2-z^2}\right]^{1/4} \exp\left\{\pm \left[(n^2-z^2)^{1/2} - n\cosh^{-1}(n/z)\right]\right\} \qquad \text{for} \qquad z < n\,.$$

(b) Compare your results with the asymptotic expansions you may find in works of reference (e.g. Abramowitz & Stegun, 1964, Handbook of Mathematical Functions) to find the values of $A_\pm$ and $B_\pm$ which correspond to $J_n(z)$.

(c) Plot both the approximations and $J_n(z)$ for $n = 5$. Where is the approximation poor?