Asymptotics Problem 9.1: Complete Pedagogical Solution

Multiple-Scale Method for First-Order Nonlinear ODEs

Problem 1. For the first-order nonlinear differential equation

$$\frac{df}{dt} - f = \varepsilon f^2 e^{-t}$$

with $\varepsilon \ll 1$ and initial condition $f(0) = 1$, determine an approximation by using the multiple-scale method. Show that the resulting expression is the exact solution.

Solution: Step-by-Step Atomic Breakdown

Step 1: Understanding the Problem and Motivation for Multiple Scales

Strategy: *We have a first-order nonlinear ODE with a small parameter ε. Our task is to:*

1. *Recognise why a regular perturbation expansion fails*
2. *Apply the multiple-scale method to obtain a uniformly valid approximation*
3. *Verify that the result is actually the exact solution*

Step 1a: Why Regular Perturbation Fails

Justification: *This problem was treated by regular perturbation expansion on Problem Sheet 5, yielding:*

$$f(t) = e^t + \varepsilon t e^t + O(\varepsilon^2).$$

*This expansion is **not uniformly valid** because the second term $\varepsilon t e^t$ grows relative to the first term e^t. Specifically:*

$$\frac{\textit{second term}}{\textit{first term}} = \frac{\varepsilon t e^t}{e^t} = \varepsilon t.$$

*When $t > 1/\varepsilon$, the "correction" term exceeds the "leading" term, invalidating the asymptotic ordering. This is a **secular term** — a term that grows unboundedly with time.*
Key Concept: ***Secular terms*** *are terms in perturbative solutions that grow unboundedly at long times, breaking uniform convergence and invalidating the solution approach for $t = O(1/\varepsilon)$ or larger. The multiple-scale method systematically eliminates secular terms by allowing the solution's amplitude to vary on a slow time scale. This is discussed in Lecture Notes §7.1.1, equations (393)–(405).*

Step 1b: The Idea Behind Multiple Scales

Justification: *The multiple-scale method assumes that the solution depends on two (or more) time scales:*

- *A **fast time** $t_0 = t$ capturing the rapid dynamics (here, the exponential growth $\sim e^t$)*
- *A **slow time** $t_1 = \varepsilon t$ capturing the gradual modulation of the amplitude due to the $O(\varepsilon)$ perturbation*

By treating these as independent variables and requiring that no secular terms appear at each order, we obtain conditions that determine the slow variation of the solution.

Step 2: Setting Up the Multiple-Scale Framework

Goal: Reformulate the ODE by treating f as a function of two independent time variables.

Step 2a: Introducing Two Time Scales

Technique: *Define:*

$$\begin{aligned} t_0 &= t \quad \textit{(fast time)} \\ t_1 &= \varepsilon t \quad \textit{(slow time)} \end{aligned}$$

We now treat f as a function of both: $f = f(t_0, t_1)$.

Step 2b: Transforming the Time Derivative

By the chain rule, the total derivative with respect to t becomes:

$$\frac{d}{dt} = \frac{\partial}{\partial t_0}\frac{dt_0}{dt} + \frac{\partial}{\partial t_1}\frac{dt_1}{dt} = \frac{\partial}{\partial t_0} \cdot 1 + \frac{\partial}{\partial t_1} \cdot \varepsilon.$$

Therefore:

$$\boxed{\frac{d}{dt} = \frac{\partial}{\partial t_0} + \varepsilon \frac{\partial}{\partial t_1}}$$

Justification: *This transformation is the key step of the multiple-scale method. By decomposing the time derivative, we separate the fast oscillation/growth (captured by $\partial/\partial t_0$) from the slow modulation (captured by $\varepsilon\,\partial/\partial t_1$). This appears in Lecture Notes §7.1.2, equation (406).*

Step 2c: Transforming the ODE

The original ODE is:

$$\frac{df}{dt} - f = \varepsilon f^2 e^{-t}.$$

Substituting the derivative transformation and noting that $e^{-t} = e^{-t_0}$ (since $t = t_0$):

$$\frac{\partial f}{\partial t_0} + \varepsilon \frac{\partial f}{\partial t_1} - f = \varepsilon f^2 e^{-t_0}.$$

Rearranging:

$$\boxed{\frac{\partial f}{\partial t_0} - f = \varepsilon f^2 e^{-t_0} - \varepsilon \frac{\partial f}{\partial t_1}}$$

Step 3: Expanding in Powers of ε

Goal: Expand f as a power series in ε and solve order by order.

Step 3a: The Expansion Ansatz

Technique: *Assume:*

$$f(t_0, t_1) = f_0(t_0, t_1) + \varepsilon f_1(t_0, t_1) + \varepsilon^2 f_2(t_0, t_1) + \cdots$$

Step 3b: Expanding the Initial Condition

The initial condition $f(0) = 1$ becomes (at $t = 0$, we have $t_0 = 0$ and $t_1 = 0$):

$$f_0(0, 0) + \varepsilon f_1(0, 0) + \cdots = 1.$$

Matching powers of ε:

$$\begin{aligned} O(1): &\quad f_0(0,0) = 1 \\ O(\varepsilon): &\quad f_1(0,0) = 0 \\ &\quad \vdots \end{aligned}$$

Step 3c: Substituting the Expansion into the ODE

Substituting $f = f_0 + \varepsilon f_1 + \cdots$ into the transformed ODE:

$$\begin{aligned} &\frac{\partial}{\partial t_0}(f_0 + \varepsilon f_1 + \cdots) - (f_0 + \varepsilon f_1 + \cdots) \\ &\qquad = \varepsilon (f_0 + \varepsilon f_1 + \cdots)^2 e^{-t_0} - \varepsilon \frac{\partial}{\partial t_1}(f_0 + \varepsilon f_1 + \cdots). \end{aligned}$$

Expanding the square:

$$(f_0 + \varepsilon f_1 + \cdots)^2 = f_0^2 + 2\varepsilon f_0 f_1 + O(\varepsilon^2).$$

Collecting terms by powers of ε:

$$\begin{aligned} &\left(\frac{\partial f_0}{\partial t_0} - f_0\right) + \varepsilon\left(\frac{\partial f_1}{\partial t_0} - f_1\right) + O(\varepsilon^2) \\ &\qquad = \varepsilon f_0^2 e^{-t_0} - \varepsilon \frac{\partial f_0}{\partial t_1} + O(\varepsilon^2). \end{aligned}$$

Step 4: Solving at Leading Order $O(1)$

Goal: Find $f_0(t_0, t_1)$.

Step 4a: The $O(1)$ Equation

Equating $O(1)$ terms:

$$\frac{\partial f_0}{\partial t_0} - f_0 = 0.$$

With initial condition: $f_0(0, 0) = 1$.

Step 4b: Solving the $O(1)$ Equation

Technique: *This is a first-order linear PDE in t_0, treating t_1 as a parameter. It has the form:*

$$\frac{\partial f_0}{\partial t_0} = f_0.$$

The solution is:

$$f_0(t_0, t_1) = A(t_1)e^{t_0},$$

where $A(t_1)$ is an arbitrary function of the slow time t_1 (the "constant" of integration with respect to t_0).

Step 4c: Applying the Initial Condition

At $t_0 = 0$, $t_1 = 0$:

$$f_0(0,0) = A(0)e^0 = A(0) = 1.$$

Therefore: $A(0) = 1$.
Justification: *The function $A(t_1)$ is not fully determined at this order — we only know $A(0) = 1$. The full dependence $A(t_1)$ will be determined by the* ***solvability condition*** *at the next order: we require that no secular terms appear in f_1.*
The leading-order solution is:

$$\boxed{f_0(t_0, t_1) = A(t_1)e^{t_0}, \quad A(0) = 1}$$

Step 5: Solving at Order $O(\varepsilon)$ and Eliminating Secular Terms

Goal: Find f_1 and determine $A(t_1)$ by requiring no secular terms.

Step 5a: The $O(\varepsilon)$ Equation

Equating $O(\varepsilon)$ terms:

$$\frac{\partial f_1}{\partial t_0} - f_1 = f_0^2 e^{-t_0} - \frac{\partial f_0}{\partial t_1}.$$

Step 5b: Substituting the Leading-Order Solution

We have $f_0 = A(t_1)e^{t_0}$. Computing each term on the RHS:

$$f_0^2 e^{-t_0} = \left(A(t_1)e^{t_0}\right)^2 e^{-t_0} = A(t_1)^2 e^{2t_0} \cdot e^{-t_0} = A^2 e^{t_0},$$
$$\frac{\partial f_0}{\partial t_1} = \frac{\partial}{\partial t_1}\left(A(t_1)e^{t_0}\right) = \frac{dA}{dt_1}e^{t_0}.$$

Therefore, the $O(\varepsilon)$ equation becomes:

$$\frac{\partial f_1}{\partial t_0} - f_1 = A^2 e^{t_0} - \frac{dA}{dt_1}e^{t_0} = \left(A^2 - \frac{dA}{dt_1}\right)e^{t_0}.$$

Step 5c: Identifying Secular Terms

Key Concept: *The equation for f_1 has the form:*

$$\frac{\partial f_1}{\partial t_0} - f_1 = g(t_0, t_1).$$

This is an inhomogeneous first-order linear ODE in t_0. The homogeneous equation $\partial f_1/\partial t_0 - f_1 = 0$ has solutions $\propto e^{t_0}$.

A ***secular term*** *arises when the inhomogeneity $g(t_0, t_1)$ is itself a solution of the homogeneous equation. In our case, the RHS is $\propto e^{t_0}$, which is exactly the homogeneous solution!*

From the theory of linear ODEs (variation of parameters), when the forcing matches a homogeneous solution, the particular solution grows by an extra factor of t_0:

$$f_1 \sim t_0 e^{t_0} \quad \textit{(secular term!)}.$$

This would invalidate our asymptotic expansion for large t_0.

Step 5d: The Solvability Condition

Technique: *To prevent secular terms, we require the coefficient of e^{t_0} on the RHS to vanish:*

$$A^2 - \frac{dA}{dt_1} = 0.$$

*This is the **solvability condition** (also called the "secularity condition").*
Justification: *The solvability condition ensures that the forcing term in the $O(\varepsilon)$ equation is not resonant with the homogeneous solution. This is the central mechanism of the multiple-scale method: by allowing the amplitude A to vary slowly with t_1, we absorb what would otherwise be secular growth into a well-behaved slow modulation.*

Step 5e: Solving for $A(t_1)$

The solvability condition is:

$$\frac{dA}{dt_1} = A^2.$$

This is a separable ODE. Separating variables:

$$\frac{dA}{A^2} = dt_1.$$

Integrating both sides:

$$-\frac{1}{A} = t_1 + C,$$

where C is a constant of integration.
Solving for A:

$$A(t_1) = -\frac{1}{t_1 + C} = \frac{1}{-t_1 - C}.$$

Let us write this as:

$$A(t_1) = \frac{1}{c - t_1},$$

where $c = -C$ is a new constant.

Step 5f: Applying the Initial Condition for A

We require $A(0) = 1$:

$$A(0) = \frac{1}{c - 0} = \frac{1}{c} = 1 \implies c = 1.$$

Therefore:

$$\boxed{A(t_1) = \frac{1}{1 - t_1}}$$

Step 6: Constructing the Leading-Order Multiple-Scale Solution

Goal: Write the complete first-order approximation.

Step 6a: Combining Results

From the leading-order solution $f_0 = A(t_1)e^{t_0}$ with $A(t_1) = 1/(1 - t_1)$:

$$f_0(t_0, t_1) = \frac{1}{1 - t_1}e^{t_0}.$$

Step 6b: Converting Back to Original Variable t

Recall $t_0 = t$ and $t_1 = \varepsilon t$. Substituting:

$$f(t) \approx f_0(t, \varepsilon t) = \frac{1}{1-\varepsilon t} e^t.$$

The **multiple-scale approximation** is:

$$\boxed{f(t) = \frac{e^t}{1-\varepsilon t}}$$

Step 7: Verifying This is the Exact Solution

Goal: Show that the multiple-scale result satisfies the original ODE exactly.

Step 7a: Computing df/dt

Let $f(t) = \frac{e^t}{1-\varepsilon t}$. Using the quotient rule:

$$\frac{df}{dt} = \frac{\frac{d}{dt}(e^t) \cdot (1-\varepsilon t) - e^t \cdot \frac{d}{dt}(1-\varepsilon t)}{(1-\varepsilon t)^2}.$$

Computing the derivatives:

$$\frac{d}{dt}(e^t) = e^t,$$
$$\frac{d}{dt}(1-\varepsilon t) = -\varepsilon.$$

Therefore:

$$\frac{df}{dt} = \frac{e^t(1-\varepsilon t) - e^t(-\varepsilon)}{(1-\varepsilon t)^2} = \frac{e^t(1-\varepsilon t) + \varepsilon e^t}{(1-\varepsilon t)^2} = \frac{e^t(1-\varepsilon t+\varepsilon)}{(1-\varepsilon t)^2} = \frac{e^t}{(1-\varepsilon t)^2}.$$

Step 7b: Computing $df/dt - f$

$$\begin{aligned}
\frac{df}{dt} - f &= \frac{e^t}{(1-\varepsilon t)^2} - \frac{e^t}{1-\varepsilon t} \\
&= \frac{e^t}{(1-\varepsilon t)^2} - \frac{e^t(1-\varepsilon t)}{(1-\varepsilon t)^2} \\
&= \frac{e^t - e^t(1-\varepsilon t)}{(1-\varepsilon t)^2} \\
&= \frac{e^t - e^t + \varepsilon t e^t}{(1-\varepsilon t)^2} \\
&= \frac{\varepsilon t e^t}{(1-\varepsilon t)^2}.
\end{aligned}$$

Step 7c: Computing $\varepsilon f^2 e^{-t}$

$$\begin{aligned}
\varepsilon f^2 e^{-t} &= \varepsilon \left(\frac{e^t}{1-\varepsilon t}\right)^2 e^{-t} \\
&= \varepsilon \cdot \frac{e^{2t}}{(1-\varepsilon t)^2} \cdot e^{-t} \\
&= \frac{\varepsilon e^t}{(1-\varepsilon t)^2}.
\end{aligned}$$

Step 7d: Comparing Both Sides

We need to check if $\frac{df}{dt} - f = \varepsilon f^2 e^{-t}$:

$$\text{LHS} = \frac{\varepsilon t e^t}{(1-\varepsilon t)^2}, \quad \text{RHS} = \frac{\varepsilon e^t}{(1-\varepsilon t)^2}.$$

Wait — these are not equal! Let me recompute more carefully.

Step 7e: Recomputing $df/dt - f$

Actually, let me redo this calculation:

$$\frac{df}{dt} = \frac{d}{dt}\left(\frac{e^t}{1-\varepsilon t}\right) = \frac{e^t(1-\varepsilon t) + \varepsilon e^t}{(1-\varepsilon t)^2} = \frac{e^t[(1-\varepsilon t) + \varepsilon]}{(1-\varepsilon t)^2} = \frac{e^t[1-\varepsilon t + \varepsilon]}{(1-\varepsilon t)^2}.$$

Now:

$$\begin{aligned}
\frac{df}{dt} - f &= \frac{e^t[1-\varepsilon t+\varepsilon]}{(1-\varepsilon t)^2} - \frac{e^t}{1-\varepsilon t} \\
&= \frac{e^t[1-\varepsilon t+\varepsilon] - e^t(1-\varepsilon t)}{(1-\varepsilon t)^2} \\
&= \frac{e^t[(1-\varepsilon t+\varepsilon) - (1-\varepsilon t)]}{(1-\varepsilon t)^2} \\
&= \frac{e^t \cdot \varepsilon}{(1-\varepsilon t)^2} \\
&= \frac{\varepsilon e^t}{(1-\varepsilon t)^2}.
\end{aligned}$$

And:

$$\varepsilon f^2 e^{-t} = \varepsilon \cdot \frac{e^{2t}}{(1-\varepsilon t)^2} \cdot e^{-t} = \frac{\varepsilon e^t}{(1-\varepsilon t)^2}.$$

Therefore:

$$\frac{df}{dt} - f = \frac{\varepsilon e^t}{(1-\varepsilon t)^2} = \varepsilon f^2 e^{-t}. \quad \checkmark$$

Step 7f: Checking the Initial Condition

$$f(0) = \frac{e^0}{1-\varepsilon\cdot 0} = \frac{1}{1} = 1. \quad \checkmark$$

Conclusion: The multiple-scale approximation

$$f(t) = \frac{e^t}{1-\varepsilon t}$$

is the **exact solution** to the original differential equation!

Step 8: Discussion and Physical Interpretation

Step 8a: Why the Multiple-Scale Method Gives the Exact Solution

Reflection: *In this particular problem, the multiple-scale method yields the exact solution because:*

1. *The solvability condition $dA/dt_1 = A^2$ captures the exact nonlinear dynamics of the amplitude modulation*

2. *The separation into fast (e^{t_0}) and slow ($A(t_1)$) components is exact for this problem structure*

3. *No higher-order corrections (f_1, f_2, etc.) are needed because the leading-order approximation already satisfies the full equation*

This is a special property of this particular ODE; in general, the multiple-scale method provides an asymptotic approximation, not an exact solution.

Step 8b: Comparison with Regular Perturbation

The regular perturbation result was:

$$f_{\text{regular}}(t) = e^t + \varepsilon t e^t + O(\varepsilon^2) = e^t(1 + \varepsilon t + O(\varepsilon^2)).$$

The exact/multiple-scale result is:

$$f_{\text{exact}}(t) = \frac{e^t}{1 - \varepsilon t} = e^t(1 + \varepsilon t + \varepsilon^2 t^2 + \varepsilon^3 t^3 + \cdots).$$

Justification: *The regular perturbation expansion is the Taylor series of* $1/(1 - \varepsilon t)$ *truncated at first order. This truncation is valid only when* $\varepsilon t \ll 1$, *i.e.,* $t \ll 1/\varepsilon$. *For times* $t = O(1/\varepsilon)$ *or larger, all terms in the series become comparable and the truncation fails.*

The multiple-scale method "resums" this divergent series by recognising that the $1/(1 - \varepsilon t)$ *factor represents the slow modulation of the amplitude.*

Step 8c: Domain of Validity

The exact solution $f(t) = e^t/(1 - \varepsilon t)$ has a singularity at $t = 1/\varepsilon$, where the denominator vanishes and $f \to \infty$. This is a genuine feature of the solution, not an artifact of the method. For $t < 1/\varepsilon$, the solution is well-defined and the multiple-scale approximation is uniformly valid.

Final Summary

Complete Solution for Problem 9.1:

Given: $\dfrac{df}{dt} - f = \varepsilon f^2 e^{-t}$, with $f(0) = 1$ and $\varepsilon \ll 1$.

Method: Multiple scales with $t_0 = t$ (fast) and $t_1 = \varepsilon t$ (slow).

Key steps:

1. Transform derivative: $\dfrac{d}{dt} = \dfrac{\partial}{\partial t_0} + \varepsilon \dfrac{\partial}{\partial t_1}$
2. Leading order: $f_0 = A(t_1)e^{t_0}$ with $A(0) = 1$
3. Solvability condition (no secular terms): $\dfrac{dA}{dt_1} = A^2$
4. Solve for A: $A(t_1) = \dfrac{1}{1 - t_1}$

Result:

$$f(t) = \frac{e^t}{1 - \varepsilon t}$$

Verification: Direct substitution confirms this is the **exact solution**.

Connection to Lecture Notes

Reflection: *This problem illustrates the core concepts of the multiple-scale method from Lecture Notes §7.1:*

- ***§7.1.1 (Secular terms):*** *The regular perturbation expansion produces secular terms ($\varepsilon t e^t$) that grow unboundedly, motivating the multiple-scale approach.*
- ***§7.1.2 (Method setup):*** *The introduction of fast time $t_0 = t$ and slow time $t_1 = \varepsilon t$, with the derivative transformation via chain rule (equation (406)–(407)).*
- ***Solvability condition:*** *The requirement that secular terms vanish determines the slow-time evolution of the amplitude, converting what would be unbounded growth into a well-behaved amplitude modulation.*
- ***Uniform validity:*** *Unlike regular perturbation, the multiple-scale result remains valid for times $t = O(1/\varepsilon)$.*