Asymptotics Problem 2(e)

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Find the leading asymptotic behaviour as $X \to \infty$ of:

$$I(X) = \int_{-1}^{\infty} \sqrt{1+t}\,\cos(Xt^2)\,e^{X(t-t^3/3)}\,dt$$

Solution

Step 1: Identify the integral structure

We have a complex integral that can be written as:

$$I(X) = \int_{-1}^{\infty} f(t)\,e^{X\phi(t)}\,dt$$

where the effective exponent combines both real and oscillatory parts. Note that:

- $f(t) = \sqrt{1+t}\,\cos(Xt^2)$
- The real part of the exponent: $u(t) = t - \frac{t^3}{3}$
- Oscillatory factor: $\cos(Xt^2)$

Step 2: Locate the critical point

Since $\cos(Xt^2)$ remains bounded and oscillatory, the dominant contribution comes from the maximum of the real part $u(t) = t - \frac{t^3}{3}$ on the integration domain $[-1, \infty)$.

Find stationary points:

$$u'(t) = 1 - t^2 = 0 \implies t = \pm 1$$

Evaluate $u(t)$ at critical points and boundaries:

$$\begin{aligned} u(-1) &= -1 - \frac{(-1)^3}{3} = -1 + \frac{1}{3} = -\frac{2}{3} \\ u(1) &= 1 - \frac{1}{3} = \frac{2}{3} \quad \text{(maximum)} \\ \lim_{t\to\infty} u(t) &= -\infty \end{aligned}$$

Check second derivative at $t = 1$:

$$u''(t) = -2t \implies u''(1) = -2 < 0$$

Therefore, $t = 1$ is a **maximum** of $u(t)$ and will give the dominant contribution as $X \to \infty$.

Step 3: Apply Laplace's method near the maximum

Near $t = 1$, expand $u(t)$:

$$\begin{aligned} u(t) &= u(1) + u'(1)(t-1) + \frac{1}{2}u''(1)(t-1)^2 + O((t-1)^3) \\ &= \frac{2}{3} + 0 + \frac{1}{2}(-2)(t-1)^2 + O((t-1)^3) \\ &= \frac{2}{3} - (t-1)^2 + O((t-1)^3) \end{aligned}$$

Evaluate other functions at $t = 1$:

$$\begin{aligned} \sqrt{1+t}\Big|_{t=1} &= \sqrt{2} \\ \cos(Xt^2)\Big|_{t=1} &= \cos(X) \end{aligned}$$

For the oscillatory term, near $t = 1$:

$$t^2 = 1 + 2(t-1) + (t-1)^2 \implies \cos(Xt^2) \approx \cos(X + 2X(t-1) + X(t-1)^2)$$

To leading order as $X \to \infty$, we can approximate $\cos(Xt^2) \approx \cos(X)$ near the maximum.

Step 4: Set up the Laplace approximation

Substitute $s = t - 1$, so $t = s + 1$ and $dt = ds$. The integration limits become $s \in [-2, \infty)$.

$$I(X) \sim \int_{-2}^{\infty} \sqrt{2}\,\cos(X)\,e^{X(2/3 - s^2)}\,ds$$

Factor out constants:

$$I(X) \sim \sqrt{2}\,e^{2X/3}\,\cos(X) \int_{-2}^{\infty} e^{-Xs^2}\,ds$$

Step 5: Evaluate the Gaussian integral

For large X, the integrand e^{-Xs^2} decays rapidly away from $s = 0$. The contribution from $s < -2$ is exponentially small, so we can extend the lower limit to $-\infty$:

$$\int_{-2}^{\infty} e^{-Xs^2}\,ds \sim \int_{-\infty}^{\infty} e^{-Xs^2}\,ds = \sqrt{\frac{\pi}{X}}$$

Step 6: Final result

Combining all terms:

$$I(X) \sim \sqrt{2}\,e^{2X/3}\,\cos(X) \cdot \sqrt{\frac{\pi}{X}}$$

$$\boxed{I(X) \sim \sqrt{\frac{2\pi}{X}}\,e^{2X/3}\,\cos(X) \quad \text{as } X \to \infty}$$

The leading asymptotic behaviour has:

- Exponential growth: $e^{2X/3}$ (from the maximum of $u(t)$ at $t = 1$)
- Algebraic decay: $X^{-1/2}$ (from the Gaussian integral near the maximum)
- Oscillation: $\cos(X)$ (from the $\cos(Xt^2)$ factor evaluated at $t = 1$)