Problem 2(d): Method of Steepest Descent

Problem Statement

Find the leading asymptotic behaviour as $X \to \infty$ of:

$$I(X) = \int_0^1 \frac{e^{iXt^2}}{t^2 - t + \frac{5}{16}}\, dt$$

Solution

Step 1: Identify the complex phase function

We have a complex integral of the form:

$$I(X) = \int_C f(z)\, e^{X\phi(z)}\, dz$$

where:

$$\phi(z) = iz^2$$
$$f(z) = \frac{1}{z^2 - z + \frac{5}{16}}$$

The original contour C is from 0 to 1 along the real axis.

Step 2: Find saddle points

Following Section 4.4 of the lecture notes, we find critical points where $\phi'(z) = 0$:

$$\phi'(z) = 2iz = 0 \implies z_0 = 0$$

At the saddle point:

$$\phi(0) = 0$$
$$\phi'(0) = 0$$
$$\phi''(0) = 2i = 2e^{i\pi/2}$$

Thus $\alpha = \frac{\pi}{2}$.

Step 3: Determine steepest descent directions

From the lecture notes (Section 4.4.2), the steepest descent directions are:

$$\theta_d = -\frac{\alpha}{n} + \frac{(2p+1)\pi}{n}, \quad p = 0, 1, \ldots, n-1$$

For $n = 2$ (since $\phi''(0) \neq 0$):

$$\theta_1 = -\frac{\pi}{4} + \frac{\pi}{2} = \frac{\pi}{4}, \quad \theta_2 = -\frac{\pi}{4} + \frac{3\pi}{2} = \frac{5\pi}{4}$$

The steepest descent paths from $z = 0$ go in directions $e^{i\pi/4}$ and $e^{i5\pi/4}$.

Step 4: Verify steepest descent property

Along the path $z = se^{i\pi/4}$ with $s > 0$:

$$\phi(se^{i\pi/4}) = i(se^{i\pi/4})^2 = is^2 e^{i\pi/2} = -s^2$$

Therefore, $\text{Re}[\phi] = -s^2 < 0$ for $s > 0$, confirming this is indeed a steepest descent path (the real part decreases as we move away from the saddle point).

Step 5: Check for singularities

The function $f(z)$ has singularities where the denominator vanishes:

$$z^2 - z + \frac{5}{16} = 0$$

Using the quadratic formula:

$$z = \frac{1 \pm \sqrt{1 - \frac{5}{4}}}{2} = \frac{1 \pm \sqrt{-\frac{1}{4}}}{2} = \frac{1 \pm \frac{i}{2}}{2}$$

The singularities are at:

$$z_1 = \frac{1}{2} + \frac{i}{4}, \quad z_2 = \frac{1}{2} - \frac{i}{4}$$

These are both in the complex plane and not on our contour, so we can deform the contour without crossing any singularities.

Step 6: Deform contour and parameterize steepest descent path

We deform the original contour near $z = 0$ to follow the steepest descent path in direction $e^{i\pi/4}$.

Parameterizing the steepest descent path: $z = se^{i\pi/4}$ with $s \geq 0$, so $dz = e^{i\pi/4}\, ds$.

The integral becomes (considering the portion near the saddle point dominates):

$$I(X) \sim e^{i\pi/4} \int_0^\infty f(se^{i\pi/4})\, e^{-Xs^2}\, ds$$

Step 7: Evaluate f at the saddle point

Near $s = 0$ (i.e., near the saddle point $z = 0$):

$$f(0) = \frac{1}{0 - 0 + \frac{5}{16}} = \frac{16}{5}$$

To leading order:

$$f(se^{i\pi/4}) \approx f(0) = \frac{16}{5}$$

Step 8: Apply Laplace's method

Following Section 4.2.3, the integral along the steepest descent path is:

$$I(X) \sim e^{i\pi/4} \cdot \frac{16}{5} \int_0^\infty e^{-Xs^2}\, ds$$

Using the standard Gaussian integral:

$$\int_0^\infty e^{-Xs^2}\, ds = \frac{1}{2}\sqrt{\frac{\pi}{X}}$$

Therefore:

$$I(X) \sim e^{i\pi/4} \cdot \frac{16}{5} \cdot \frac{1}{2}\sqrt{\frac{\pi}{X}} = \frac{8e^{i\pi/4}}{5}\sqrt{\frac{\pi}{X}}$$

Step 9: Simplify the result

Since $e^{i\pi/4} = \cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}(1+i)$:

$$I(X) \sim \frac{8}{5\sqrt{2}}(1+i)\sqrt{\frac{\pi}{X}} = \frac{4\sqrt{2}}{5}(1+i)\sqrt{\frac{\pi}{X}}$$

This can be written as:

$$I(X) \sim \frac{4(1+i)}{5}\sqrt{\frac{2\pi}{X}}$$

Final Answer

$$\boxed{I(X) \sim \frac{4(1+i)}{5}\sqrt{\frac{2\pi}{X}} \quad \text{as } X \to \infty}$$

Alternatively:

$$\boxed{I(X) \sim \frac{4\sqrt{2\pi}}{5\sqrt{X}}(1+i) \quad \text{as } X \to \infty}$$

The leading order is $O(X^{-1/2})$, arising from the contribution near the saddle point at $z = 0$ along the steepest descent path.