Exercise Sheet 3: Bifurcations
Question 3 - Complete Solution

Methods of Applied Mathematics

Problem Statement

Consider the dynamical system:

$$\dot{x} = y - 3x$$
$$\dot{y} = \alpha x - x^2$$

for $-9/4 < \alpha < 9/4$.

Tasks:

(a) Compute and classify the stability/type of any equilibria

(b) What bifurcation happens in the system at $\alpha = 0$?

(c) Draw a bifurcation diagram with α on the horizontal axis, and x on the vertical. What would the diagram look like if you drew α against y?

---

1 Step 1: Find Equilibria

Set up equilibrium conditions

For equilibria, we require $\dot{x} = 0$ and $\dot{y} = 0$ simultaneously:

$$\begin{aligned} y - 3x = 0 \quad &\Rightarrow \quad y = 3x \\ \alpha x - x^2 = 0 \quad &\Rightarrow \quad x(\alpha - x) = 0 \end{aligned}$$

Solve for equilibrium points

From the second equation: either $x = 0$ or $x = \alpha$

Equilibrium 1: $x = 0$

$$y = 3(0) = 0 \quad \Rightarrow \quad \boxed{(x^*, y^*) = (0, 0)}$$

Equilibrium 2: $x = \alpha$

$$y = 3\alpha \quad \Rightarrow \quad \boxed{(x^*, y^*) = (\alpha, 3\alpha)}$$

Special case: $\alpha = 0$

When $\alpha = 0$: both equilibria coincide at $(0, 0)$

XYZ Analysis of Equilibrium Structure

- **STAGE X (What we found):** Two equilibria for $\alpha \neq 0$: one pinned at the origin, one that moves linearly with α along the line $y = 3x$. They collide at the origin when $\alpha = 0$.
- **STAGE Y (Why this structure):** The equation $\dot{y} = \alpha x - x^2 = x(\alpha - x)$ factors, giving two roots. The constraint $y = 3x$ from the first equation means equilibria must lie on this line in the phase plane. As α varies:
  - The origin $(0, 0)$ is always an equilibrium (pinned)
  - The second equilibrium $(\alpha, 3\alpha)$ moves along $y = 3x$:
    * For $\alpha < 0$: in third quadrant (left and below origin)
    * For $\alpha = 0$: at origin (collision)
    * For $\alpha > 0$: in first quadrant (right and above origin)
- **STAGE Z (What this means):** One equilibrium passing through another pinned equilibrium is the signature of a transcritical bifurcation. The collision at $\alpha = 0$ will involve an exchange of stability.

---

2 Step 2: Compute Jacobian Matrix

General Jacobian

For the system $\dot{x} = f(x, y)$, $\dot{y} = g(x, y)$:

$$J = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix}$$

With $f(x, y) = y - 3x$ and $g(x, y) = \alpha x - x^2$:

$$J = \begin{pmatrix} -3 & 1 \\ \alpha - 2x & 0 \end{pmatrix}$$

XYZ Analysis of Jacobian

- **STAGE X (What we have):** The Jacobian is simple - constant entries except for $\alpha - 2x$ in the lower-left, which depends on both the parameter and equilibrium location.
- **STAGE Y (Why this form):** The partial derivatives are:
  - $\partial f/\partial x = -3$: constant damping in $\dot{x}$ equation
  - $\partial f/\partial y = 1$: linear coupling from y to $\dot{x}$
  - $\partial g/\partial x = \alpha - 2x$: varies with position, reflects the $x(\alpha - x)$ structure
  - $\partial g/\partial y = 0$: no direct y-dependence in $\dot{y}$

  The $\alpha - 2x$ term is crucial: at origin it's α, at $(\alpha, 3\alpha)$ it becomes $-\alpha$.
- **STAGE Z (What this determines):** The eigenvalues of J at each equilibrium will determine stability. The symmetry in how α appears (positive at origin, negative at second equilibrium) suggests stability exchange.

---

3 Step 3: Analyze Equilibrium at Origin

Jacobian at $(0, 0)$

$$J(0,0) = \begin{pmatrix} -3 & 1 \\ \alpha & 0 \end{pmatrix}$$

Compute trace and determinant

$$\text{Trace: } \tau = -3 + 0 = -3$$
$$\text{Determinant: } \Delta = (-3)(0) - (1)(\alpha) = -\alpha$$

Find eigenvalues

Characteristic equation: $\lambda^2 - \tau\lambda + \Delta = 0$

$$\lambda^2 + 3\lambda - \alpha = 0$$

Quadratic formula:

$$\lambda = \frac{-3 \pm \sqrt{9 + 4\alpha}}{2}$$

Classify by parameter value

Case 1: $\alpha < 0$ **(and** $\alpha > -9/4$ **to keep discriminant real)**
Discriminant: $9 + 4\alpha > 0$ for $\alpha > -9/4$
Both eigenvalues are real. Since:

- $\Delta = -\alpha > 0$ (product of eigenvalues positive $\rightarrow$ same sign)
- $\tau = -3 < 0$ (sum of eigenvalues negative $\rightarrow$ both negative)

$$\boxed{\text{STABLE NODE}}$$

Eigenvalues: both negative, real, distinct (for $-9/4 < \alpha < 0$)
Case 2: $\alpha = 0$

$$\lambda = \frac{-3 \pm 3}{2} \quad \Rightarrow \quad \lambda_1 = 0, \quad \lambda_2 = -3$$

$$\boxed{\text{NEUTRAL (one zero eigenvalue)}}$$

This signals a bifurcation point.
Case 3: $\alpha > 0$ **(and** $\alpha < 9/4$ **to stay in given range)**

- $\Delta = -\alpha < 0$ (product of eigenvalues negative $\rightarrow$ opposite signs)

$$\boxed{\text{SADDLE POINT}}$$

Eigenvalues: one positive, one negative, real

XYZ Analysis of Origin Stability

- **STAGE X (What we found):** Origin changes from stable node ($\alpha < 0$) through neutral ($\alpha = 0$) to saddle ($\alpha > 0$).
- **STAGE Y (Why this transition):** The determinant $\Delta = -\alpha$ controls the product of eigenvalues:
  - For $\alpha < 0$: $\Delta > 0$ means eigenvalues have same sign; combined with $\tau < 0$ (negative sum), both must be negative $\rightarrow$ stable
  - For $\alpha = 0$: $\Delta = 0$ means one eigenvalue is zero $\rightarrow$ bifurcation
  - For $\alpha > 0$: $\Delta < 0$ means eigenvalues have opposite signs $\rightarrow$ saddle

  The trace $\tau = -3$ stays constant (always negative), but the determinant crossing zero at $\alpha = 0$ fundamentally changes the eigenvalue configuration from $(-, -)$ to $(-, +)$.
- **STAGE Z (What this means physically):** Before bifurcation, the origin is an attractor - all nearby trajectories spiral/flow inward. After bifurcation, it becomes a saddle - trajectories are repelled in one direction (unstable manifold) while attracted in another (stable manifold). The origin loses its basin of attraction.

---

4 Step 4: Analyze Equilibrium at $(\alpha, 3\alpha)$

Existence condition

This equilibrium only exists for $\alpha \neq 0$ (when $\alpha = 0$, it coincides with origin).

Jacobian at $(\alpha, 3\alpha)$

$$J(\alpha, 3\alpha) = \begin{pmatrix} -3 & 1 \\ \alpha - 2\alpha & 0 \end{pmatrix} = \begin{pmatrix} -3 & 1 \\ -\alpha & 0 \end{pmatrix}$$

Compute trace and determinant

$$\text{Trace: } \tau = -3 + 0 = -3$$
$$\text{Determinant: } \Delta = (-3)(0) - (1)(-\alpha) = \alpha$$

Find eigenvalues

Characteristic equation:

$$\lambda^2 + 3\lambda + \alpha = 0$$

Quadratic formula:

$$\lambda = \frac{-3 \pm \sqrt{9 - 4\alpha}}{2}$$

Classify by parameter value

Case 1: $\alpha < 0$

- $\Delta = \alpha < 0$ (product negative $\to$ opposite signs)

$$\boxed{\text{SADDLE POINT}}$$

Eigenvalues: one positive, one negative
Case 2: $0 < \alpha < 9/4$
Discriminant: $9 - 4\alpha > 0$ for $\alpha < 9/4 \to$ real eigenvalues

- $\Delta = \alpha > 0$ (product positive $\to$ same sign)
- $\tau = -3 < 0$ (sum negative $\to$ both negative)

$$\boxed{\text{STABLE NODE}}$$

Eigenvalues: both negative, real, distinct
Note: At $\alpha = 9/4$, the discriminant vanishes and eigenvalues become complex, but this is outside our detailed analysis.

XYZ Analysis of Moving Equilibrium

- **STAGE X (What we found):** The equilibrium $(\alpha, 3\alpha)$ changes from saddle ($\alpha < 0$) to stable node ($\alpha > 0$).
- **STAGE Y (Why this transition):** The determinant $\Delta = \alpha$ (opposite sign to origin!) controls stability:
  - For $\alpha < 0$: $\Delta < 0 \to$ saddle (unstable)
  - For $\alpha > 0$: $\Delta > 0$ and $\tau < 0 \to$ stable node

  Notice the symmetry with the origin: when origin has $\Delta = -\alpha > 0$ (stable for $\alpha < 0$), this equilibrium has $\Delta = \alpha < 0$ (unstable). When origin has $\Delta = -\alpha < 0$ (saddle for $\alpha > 0$), this equilibrium has $\Delta = \alpha > 0$ (stable). They swap determinant signs, hence swap stability types.
- **STAGE Z (What this means):** As the moving equilibrium passes through the origin at $\alpha = 0$, it inherits the stability that the origin loses. Before: origin stable, moving point saddle. After: origin saddle, moving point stable. This is the stability exchange characteristic of transcritical bifurcation.

---

5 Step 5: Summary of Stability Analysis

Parameter	**Equilibrium** $(0,0)$	**Equilibrium** $(\alpha, 3\alpha)$
$-9/4 < \alpha < 0$	Stable node	Saddle
$\alpha = 0$	Neutral (one $\lambda = 0$)	Coincides with origin
$0 < \alpha < 9/4$	Saddle	Stable node

XYZ Analysis of Stability Exchange

- **STAGE X (What the table shows):** Complete reversal of stability roles across $\alpha = 0$. The stable equilibrium and saddle swap identities.
- **STAGE Y (Why this pattern):** Both Jacobians have the same trace ($\tau = -3$) but opposite-sign determinants ($\Delta = -\alpha$ vs $\Delta = \alpha$). Since:

$$\text{Stability requires: } \tau < 0 \text{ and } \Delta > 0$$
$$\text{Saddle requires: } \Delta < 0$$

  The sign of α determines which equilibrium satisfies which condition. The exchange happens precisely because the determinants are negatives of each other.

- **STAGE Z (What this means globally):** The system always has exactly one stable equilibrium (for $\alpha \neq 0$). The bifurcation doesn't create or destroy stability, it transfers it from one location to another. This is fundamentally different from fold bifurcations where stability is lost entirely.

---

6 Step 6: Identify Bifurcation Type

Observed characteristics at $\alpha = 0$

1. Two equilibria approach each other and collide at origin
2. One equilibrium (origin) is pinned for all α
3. Other equilibrium passes through the pinned one
4. Equilibria exchange stability (stable $\leftrightarrow$ saddle)
5. One eigenvalue crosses zero at bifurcation point
6. No equilibria created or destroyed (always exactly 2 for $\alpha \neq 0$)

Conclusion

$$\boxed{\text{TRANSCRITICAL BIFURCATION at } \alpha = 0}$$

XYZ Analysis of Bifurcation Classification

- **STAGE X (What identifies this):** The defining features all match transcritical: pinned equilibrium, passing equilibrium, stability exchange, no creation/destruction.
- **STAGE Y (Why transcritical and not others):**
  - **Not fold**: Equilibria don't annihilate. Number stays constant (2 equilibria before, 2 after, excluding the collision point).
  - **Not pitchfork**: Only 2 equilibria involved (not 1 splitting into 3), and no symmetry $f(-x) = -f(x)$ in the system.
  - **Not Hopf**: No periodic orbits created, no complex eigenvalues becoming real (eigenvalues stay real throughout for the range given).

The key distinguishing feature is the pinned equilibrium at the origin: it exists for all α, and another equilibrium passes through it. This is the transcritical signature.

- **STAGE Z (What this means):** Transcritical bifurcations occur when there's a natural constraint fixing one equilibrium. Here, the origin is always an equilibrium because when $(x, y) = (0, 0)$, both $\dot{x} = 0 - 0 = 0$ and $\dot{y} = 0 - 0 = 0$ regardless of α. This structural constraint forces the bifurcation to be transcritical rather than fold.

---

7 Step 7: Bifurcation Diagram (α vs x)

Equilibrium branches in (α, x) space

Branch 1: Origin equilibrium

$$x = 0 \quad \text{for all } \alpha$$

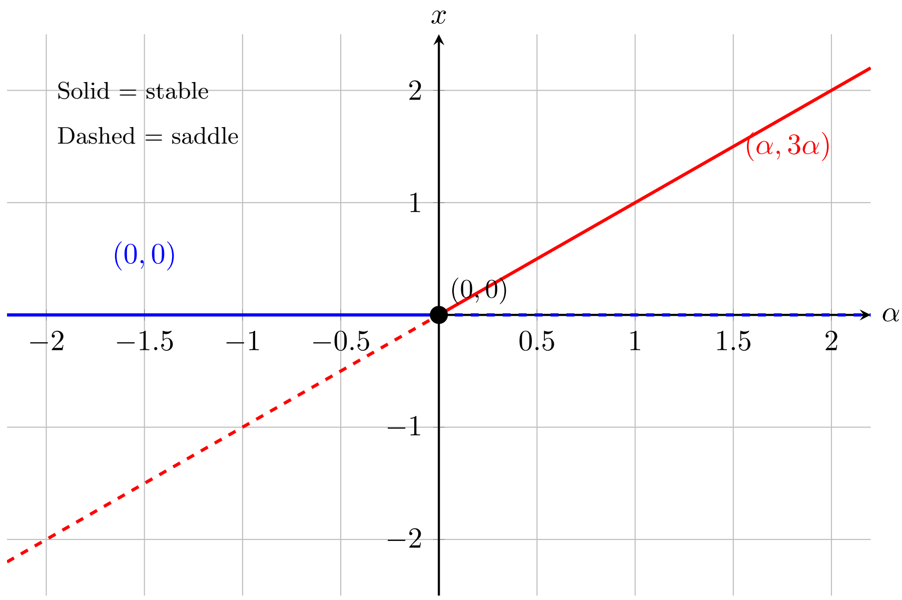
- Stable for $\alpha < 0$ (solid line)
- Unstable (saddle) for $\alpha > 0$ (dashed line)

Branch 2: Moving equilibrium

$$x = \alpha \quad \text{(diagonal line through origin)}$$

- Unstable (saddle) for $\alpha < 0$ (dashed line)
- Stable for $\alpha > 0$ (solid line)

Bifurcation diagram: α vs x



XYZ Analysis of (α, x) Diagram

- **STAGE X (What the diagram shows):** Two straight lines crossing at the origin. Horizontal line (pinned equilibrium) and diagonal line (moving equilibrium) exchange line styles at the crossing point.
- **STAGE Y (Why this structure):**

- The horizontal line $x = 0$ reflects that origin is always an equilibrium
- The diagonal line $x = \alpha$ shows the moving equilibrium's x-coordinate equals the parameter value
- They intersect at $\alpha = 0$ where both have $x = 0$ (collision point)
- The line style swap (solid $\leftrightarrow$ dashed) encodes the stability exchange

Unlike a fold where branches meet in a parabola and terminate, here branches continue through each other as straight lines - the transcritical signature of "passing through" rather than "colliding and annihilating."

- **STAGE Z (What this tells us):** Reading left to right as α increases:
  - Far left ($\alpha \ll 0$): stable equilibrium at origin, saddle far in third quadrant
  - Approaching zero: saddle moves toward origin along $y = 3x$ line
  - At zero: equilibria merge, both neutral
  - Past zero: origin becomes saddle, stable equilibrium moves into first quadrant
  - Far right ($\alpha \gg 0$): saddle at origin, stable equilibrium far in first quadrant

  The system's attractor "jumps" from origin to the first quadrant as α crosses zero.

---

8 Step 8: Bifurcation Diagram (α vs y)

Equilibrium branches in (α, y) space

Branch 1: Origin equilibrium

$$y = 0 \quad \text{for all } \alpha$$

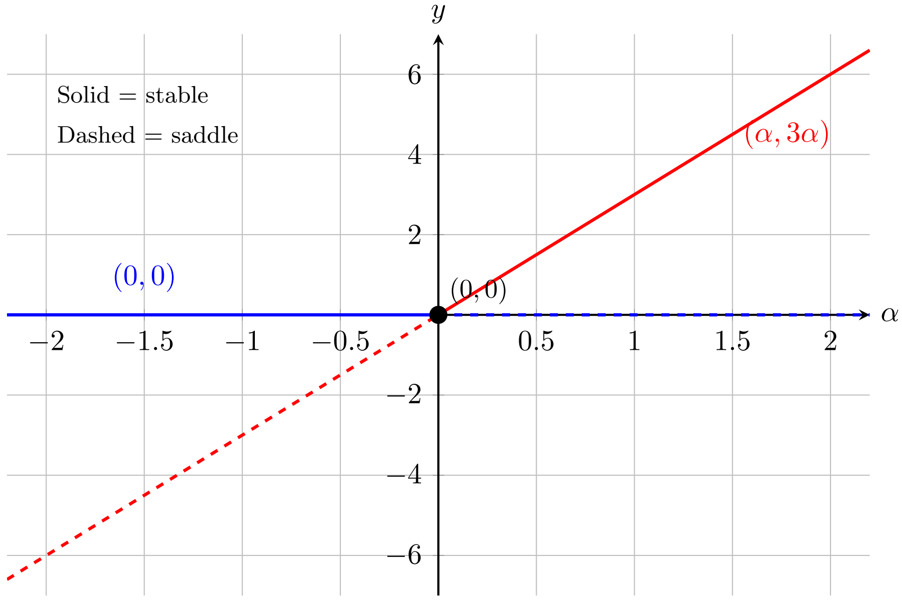
- Stable for $\alpha < 0$ (solid line)
- Unstable (saddle) for $\alpha > 0$ (dashed line)

Branch 2: Moving equilibrium

$$y = 3\alpha \quad \text{(diagonal line through origin, slope = 3)}$$

- Unstable (saddle) for $\alpha < 0$ (dashed line)
- Stable for $\alpha > 0$ (solid line)

Bifurcation diagram: α vs y



Comparison of diagrams

Feature	α **vs** x	α **vs** y
Branch 1 (origin)	$x = 0$ (horizontal)	$y = 0$ (horizontal)
Branch 2 (moving)	$x = \alpha$ (slope 1)	$y = 3\alpha$ (slope 3)
Qualitative structure	Same	Same
Only difference	Scale of diagonal	Scale of diagonal

XYZ Analysis of (α, y) Diagram

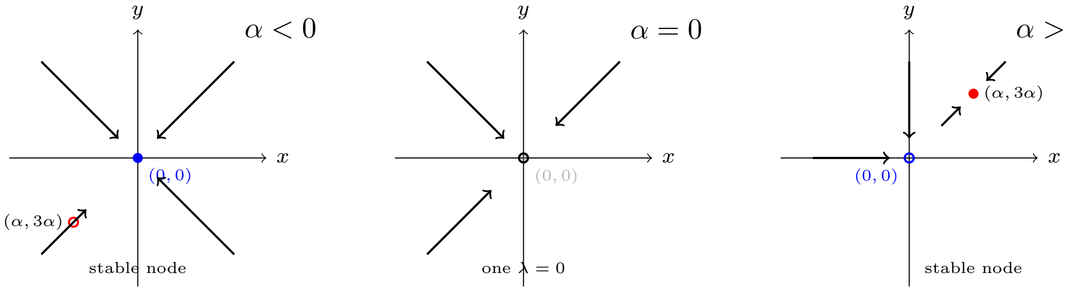
- **STAGE X (What changes):** The diagram looks similar but the moving equilibrium branch is steeper: slope 3 instead of 1. The stability patterns are identical.
- **STAGE Y (Why the difference):** Since the equilibrium coordinates are related by $y = 3x$, when we plot y instead of x, we're simply rescaling the vertical axis by factor 3. The moving equilibrium has:
  - x-coordinate: α (gives slope 1 in α-x plot)
  - y-coordinate: 3α (gives slope 3 in α-y plot)

  The constraint $y = 3x$ means the equilibria lie on a line in the (x, y) phase plane. Projecting this line onto either coordinate axis gives the bifurcation diagram branches.
- **STAGE Z (What's invariant):** The qualitative bifurcation structure is independent of which coordinate we plot:
  - Two straight lines crossing at origin
  - One horizontal (pinned equilibrium)
  - One diagonal (moving equilibrium)
  - Stability exchange at crossing

  The choice of coordinate only affects quantitative details (slopes), not the bifurcation type or stability pattern. This reflects that bifurcations are coordinate-invariant phenomena - they're about the system's dynamics, not our choice of variables.

9 Step 9: Phase Portraits

Three scenarios



Notation: Filled circle = stable node, hollow circle = saddle

XYZ Analysis of Phase Portrait Evolution

- **STAGE X (What we see):** The stable attractor moves from origin to first quadrant as α increases through zero. The saddle point correspondingly moves from third quadrant to origin.
- **STAGE Y (Why these flows):** The flow directions are determined by the vector field:
  - **For** $\alpha < 0$: Origin is stable node - all nearby trajectories flow inward (both eigenvalues negative). The saddle at $(\alpha, 3\alpha)$ has one stable direction (along eigenvector) and one unstable direction.
  - **For** $\alpha = 0$: One eigenvalue is zero at origin, creating a line of equilibria in that eigenvalue's direction (actually just one degenerate point here, but effectively neutral in one direction).
  - **For** $\alpha > 0$: Origin is saddle - trajectories approach along stable manifold but are repelled along unstable manifold. The equilibrium at $(\alpha, 3\alpha)$ is now the stable node, attracting nearby trajectories.
- **STAGE Z (What this means globally):** The basin of attraction fundamentally changes:
  - Before bifurcation: Most trajectories near origin are attracted to $(0, 0)$
  - After bifurcation: Most trajectories near origin are attracted to $(\alpha, 3\alpha)$ in first quadrant

  The saddle at origin post-bifurcation acts as a "separatrix" - its stable manifold divides phase space into regions flowing to different fates. This is a qualitative change in global dynamics, not just local stability.

10 Summary

Part (a): Equilibria and Stability

Equilibrium 1: $(0, 0)$

- Always exists (pinned)

- Stable node for $-9/4 < \alpha < 0$
- Neutral for $\alpha = 0$ (eigenvalues: $0, -3$)
- Saddle for $0 < \alpha < 9/4$

Equilibrium 2: $(\alpha, 3\alpha)$

- Exists for $\alpha \neq 0$
- Saddle for $-9/4 < \alpha < 0$
- Coincides with origin at $\alpha = 0$
- Stable node for $0 < \alpha < 9/4$

Part (b): Bifurcation at $\alpha = 0$

$$\boxed{\text{TRANSCRITICAL BIFURCATION}}$$

Characteristics:

- Pinned equilibrium at origin passes through by moving equilibrium
- Equilibria exchange stability (stable node $\leftrightarrow$ saddle)
- One eigenvalue crosses zero
- No creation/destruction of equilibria

Part (c): Bifurcation Diagrams

α vs x diagram:

- Horizontal line: $x = 0$ (origin)
- Diagonal line: $x = \alpha$ (slope 1)
- Cross at $(\alpha, x) = (0, 0)$
- Stability exchange at crossing

α vs y diagram:

- Horizontal line: $y = 0$ (origin)
- Diagonal line: $y = 3\alpha$ (slope 3)
- Cross at $(\alpha, y) = (0, 0)$
- Same qualitative structure, steeper slope

Key insight: Both diagrams show the same transcritical structure - two straight lines crossing with stability exchange. Only quantitative difference is the slope of the moving equilibrium branch (factor of 3 from $y = 3x$ constraint).