

# Exercise 5, Question 2: Sawtooth Map

## Periodic Orbits and Chaotic Dynamics

### Problem Statement

Sketch or graph the sawtooth map

$$x_{n+1} = \begin{cases} 2x_n & \text{for } 0 \leq x_n < 1/2 \\ 2x_n - 1 & \text{for } 1/2 < x_n \leq 1 \end{cases}$$

Either by hand or computer, investigate its dynamics with cobweb diagrams.

(a) Show there is a period two orbit with an iterate at  $x = 1/3$ , and find the other iterate.

(b) Show there is a period three orbit with an iterate at  $x = 1/7$ , and find the other iterates.

(c) Show that the orbits from (a)-(b) are unstable.

(d) Argue why this map cannot have any stable periodic orbits, and conjecture what kind of dynamics you will see from a typical initial point.

---

## 1 Understanding the Sawtooth Map

### Step 1: Recognize the Map Structure

The sawtooth map is a piecewise linear map on the interval  $[0, 1]$ .

**Explanation 1** (What is a piecewise map?). *The map has different formulas depending on which region  $x_n$  lies in. We need to track which piece applies at each iteration.*

### Step 2: Analyze the First Piece ( $0 \leq x_n < 1/2$ )

For  $x_n$  in  $[0, 1/2)$ :

$$x_{n+1} = 2x_n$$

This doubles the value. Since  $x_n < 1/2$ , we have  $x_{n+1} = 2x_n < 2(1/2) = 1$ .

Also,  $x_{n+1} \geq 2(0) = 0$ .

So if  $x_n \in [0, 1/2)$ , then  $x_{n+1} \in [0, 1)$ .

### Step 3: Analyze the Second Piece ( $1/2 < x_n \leq 1$ )

For  $x_n$  in  $(1/2, 1]$ :

$$x_{n+1} = 2x_n - 1$$

This doubles and then subtracts 1.

If  $x_n = 1/2$ :  $x_{n+1} = 2(1/2) - 1 = 0$ .

If  $x_n = 1$ :  $x_{n+1} = 2(1) - 1 = 1$ .

So if  $x_n \in (1/2, 1]$ , then  $x_{n+1} \in (0, 1]$ .

**Explanation 2** (Geometric picture). *The map "unfolds" the interval  $[0, 1]$ . It stretches each half by a factor of 2. The left half  $[0, 1/2)$  maps to  $[0, 1)$ , and the right half  $(1/2, 1]$  also maps to  $(0, 1]$ . This is why it's called a "sawtooth" - the graph has two rising linear pieces.*

### Step 4: Write a Compact Notation

We can write:

$$f(x) = \begin{cases} 2x & \text{if } x < 1/2 \\ 2x - 1 & \text{if } x \geq 1/2 \end{cases}$$

This can also be written as:

$$f(x) = 2x \mod 1$$

where " $\mod 1$ " means take the fractional part (subtract the integer part).

## 2 Part (a): Period Two Orbit with Iterate at $x = 1/3$

### Step 1: Understand What a Period Two Orbit Means

A period two orbit consists of two points  $\{x_1, x_2\}$  such that:

$$f(x_1) = x_2 \quad \text{and} \quad f(x_2) = x_1$$

This means  $f(f(x_1)) = x_1$ , i.e.,  $f^2(x_1) = x_1$ .

**Explanation 3** (Why period two?). *From lecture notes Section 22, page 80: A period  $m$  orbit satisfies  $x = f^m(x)$ . For  $m = 2$ , the orbit has two distinct points that map to each other.*

### Step 2: Apply the Map to $x_1 = 1/3$

We have  $x_1 = 1/3$ . Check which piece of the map applies:

Since  $1/3 < 1/2$ , we use  $f(x) = 2x$ :

$$x_2 = f(1/3) = 2 \cdot \frac{1}{3} = \frac{2}{3}$$

### Step 3: Apply the Map to $x_2 = 2/3$

Now check  $x_2 = 2/3$ . Which piece applies?

Since  $2/3 > 1/2$ , we use  $f(x) = 2x - 1$ :

$$f(2/3) = 2 \cdot \frac{2}{3} - 1 = \frac{4}{3} - 1 = \frac{4-3}{3} = \frac{1}{3}$$

So  $f(x_2) = f(2/3) = 1/3 = x_1$ .

### Step 4: Verify the Period Two Orbit

We have:

$$f(1/3) = 2/3 \quad \text{and} \quad f(2/3) = 1/3$$

This confirms a period two orbit:  $1/3 \rightarrow 2/3 \rightarrow 1/3 \rightarrow \dots$

### Step 5: Verify Using $f^2$

Check that  $f^2(1/3) = 1/3$ :

$$f^2(1/3) = f(f(1/3)) = f(2/3) = 1/3 \quad \checkmark$$

Also check  $f^2(2/3) = 2/3$ :

$$f^2(2/3) = f(f(2/3)) = f(1/3) = 2/3 \quad \checkmark$$

**Answer:**

Period 2 orbit:  $\{1/3, 2/3\}$

The other iterate is  $x_2 = 2/3$ .

---

## 3 Part (b): Period Three Orbit with Iterate at $x = 1/7$

### Step 1: Understand What a Period Three Orbit Means

A period three orbit consists of three points  $\{x_1, x_2, x_3\}$  such that:

$$f(x_1) = x_2, \quad f(x_2) = x_3, \quad f(x_3) = x_1$$

This means  $f^3(x_1) = x_1$ .

### Step 2: Apply the Map to $x_1 = 1/7$

We have  $x_1 = 1/7$ . Check which piece applies: Since  $1/7 \approx 0.143 < 1/2$ , we use  $f(x) = 2x$ :

$$x_2 = f(1/7) = 2 \cdot \frac{1}{7} = \frac{2}{7}$$

### Step 3: Apply the Map to $x_2 = 2/7$

Check  $x_2 = 2/7$ . Which piece applies?

Since  $2/7 \approx 0.286 < 1/2$ , we use  $f(x) = 2x$ :

$$x_3 = f(2/7) = 2 \cdot \frac{2}{7} = \frac{4}{7}$$

### Step 4: Apply the Map to $x_3 = 4/7$

Check  $x_3 = 4/7$ . Which piece applies?

Since  $4/7 \approx 0.571 > 1/2$ , we use  $f(x) = 2x - 1$ :

$$f(4/7) = 2 \cdot \frac{4}{7} - 1 = \frac{8}{7} - 1 = \frac{8-7}{7} = \frac{1}{7}$$

So  $f(x_3) = 1/7 = x_1$ .

### Step 5: Verify the Period Three Orbit

We have:

$$f(1/7) = 2/7, \quad f(2/7) = 4/7, \quad f(4/7) = 1/7$$

This confirms a period three orbit:  $1/7 \rightarrow 2/7 \rightarrow 4/7 \rightarrow 1/7 \rightarrow \dots$

### Step 6: Verify Using $f^3$

Check that  $f^3(1/7) = 1/7$ :

$$f^3(1/7) = f(f(f(1/7))) = f(f(2/7)) = f(4/7) = 1/7 \quad \checkmark$$

**Answer:**

Period 3 orbit:  $\{1/7, 2/7, 4/7\}$

The other iterates are  $x_2 = 2/7$  and  $x_3 = 4/7$ .

**Explanation 4** (Pattern observation). *Notice the pattern: starting from  $1/7$ , each iterate doubles the numerator:*

$$\frac{1}{7} \rightarrow \frac{2}{7} \rightarrow \frac{4}{7} \rightarrow \frac{8}{7} = \frac{1}{7} + 1 \equiv \frac{1}{7} \pmod{1}$$

*This reflects the structure  $f(x) = 2x \pmod{1}$ .*

## 4 Part (c): Stability of Periodic Orbits

### Overview of Stability for Periodic Orbits

From lecture notes Section 22, page 82: For a period  $p$  orbit, stability is determined by the product of derivatives at each point in the orbit:

$$\frac{dx_{n+p}}{dx_n} = f'(x_{n+p-1}) \cdots f'(x_{n+1}) f'(x_n)$$

The orbit is stable if this product has absolute value less than 1.

## Step 1: Compute the Derivative of $f$

For the sawtooth map:

$$f(x) = \begin{cases} 2x & \text{if } x < 1/2 \\ 2x - 1 & \text{if } x \geq 1/2 \end{cases}$$

The derivative is:

$$f'(x) = \begin{cases} 2 & \text{if } x < 1/2 \\ 2 & \text{if } x > 1/2 \end{cases}$$

**Explanation 5** (Key observation). *The derivative is constant and equals 2 everywhere (except at  $x = 1/2$  where the map is not differentiable, but we won't evaluate it there).*

## Step 2: Apply Chain Rule for Period Two Orbit

For the period two orbit  $\{1/3, 2/3\}$ , the stability is:

$$\left. \frac{dx_{n+2}}{dx_n} \right|_{x=1/3} = f'(2/3) \cdot f'(1/3)$$

## Step 3: Evaluate Derivatives at Orbit Points

At  $x = 1/3 < 1/2$ :  $f'(1/3) = 2$

At  $x = 2/3 > 1/2$ :  $f'(2/3) = 2$

Therefore:

$$\left. \frac{dx_{n+2}}{dx_n} \right|_{x=1/3} = 2 \cdot 2 = 4$$

## Step 4: Determine Stability of Period Two Orbit

Since  $|4| = 4 > 1$ , the period two orbit is **unstable**.

**Explanation 6** (Why unstable?). *A small perturbation away from the orbit grows by a factor of 4 after two iterations. The orbit repels nearby trajectories.*

Period 2 orbit is unstable: $\left  \frac{dx_{n+2}}{dx_n} \right  = 4 > 1$
--

## Step 5: Apply Chain Rule for Period Three Orbit

For the period three orbit  $\{1/7, 2/7, 4/7\}$ , the stability is:

$$\left. \frac{dx_{n+3}}{dx_n} \right|_{x=1/7} = f'(4/7) \cdot f'(2/7) \cdot f'(1/7)$$

## Step 6: Evaluate Derivatives at Orbit Points

At  $x = 1/7 < 1/2$ :  $f'(1/7) = 2$

At  $x = 2/7 < 1/2$ :  $f'(2/7) = 2$

At  $x = 4/7 > 1/2$ :  $f'(4/7) = 2$

Therefore:

$$\left. \frac{dx_{n+3}}{dx_n} \right|_{x=1/7} = 2 \cdot 2 \cdot 2 = 8$$

## Step 7: Determine Stability of Period Three Orbit

Since  $|8| = 8 > 1$ , the period three orbit is **unstable**.

Period 3 orbit is unstable: $\left  \frac{dx_{n+3}}{dx_n} \right  = 8 > 1$
--

**Explanation 7** (General pattern). *For any period  $p$  orbit, the stability multiplier is:*

$$\prod_{i=1}^p f'(x_i) = 2^p$$

*Since  $2^p > 1$  for all  $p \geq 1$ , all periodic orbits are unstable.*

---

## 5 Part (d): No Stable Periodic Orbits and Chaotic Dynamics

### Step 1: Generalize the Stability Argument

Consider any period  $p$  orbit with points  $\{x_1, x_2, \dots, x_p\}$ . The stability is determined by:

$$\left| \frac{dx_{n+p}}{dx_n} \right| = |f'(x_p) \cdot f'(x_{p-1}) \cdots f'(x_1)|$$

### Step 2: Evaluate the Product of Derivatives

Since  $f'(x) = 2$  for all  $x \neq 1/2$  (and periodic orbits generically avoid the non-differentiable point):

$$|f'(x_p) \cdot f'(x_{p-1}) \cdots f'(x_1)| = |2 \cdot 2 \cdots 2| = 2^p$$

### Step 3: Check Stability Condition

For stability, we need  $2^p < 1$ .

But  $2^p \geq 2^1 = 2 > 1$  for all  $p \geq 1$ .

Therefore,  $2^p > 1$  for all positive integers  $p$ .

## Step 4: Conclude About Periodic Orbits

Since every periodic orbit has stability multiplier  $2^p > 1$ , there are **no stable periodic orbits**.

**Explanation 8** (Why this happens). *The sawtooth map stretches by a factor of 2 everywhere. Any small perturbation gets amplified, so no orbit can be attracting. This is a key property of expanding maps.*

The sawtooth map has no stable periodic orbits.

## Step 5: Connect to Lyapunov Exponents

From lecture notes Section 23, page 85-86, the Lyapunov exponent is:

$$\lambda = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} \ln |f'(x_i)|$$

For the sawtooth map,  $f'(x) = 2$  almost everywhere, so:

$$\lambda = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} \ln(2) = \ln(2) > 0$$

**Explanation 9** (Positive Lyapunov exponent). *A positive Lyapunov exponent indicates chaos. Nearby trajectories diverge exponentially at rate  $e^\lambda = e^{\ln 2} = 2$  per iteration.*

## Step 6: Identify Chaotic Properties

From lecture notes Section 23, page 86, chaotic systems have:

**Solution 5. Sensitive dependence on initial conditions:** Small changes in  $x_0$  lead to exponentially different trajectories

2. **Topological transitivity:** The orbit can visit any region of  $[0, 1]$

3. **Dense periodic orbits:** Every point is arbitrarily close to a periodic orbit

The sawtooth map satisfies all three properties.

## Step 7: Analyze the Binary Representation

The sawtooth map has a beautiful interpretation in binary:

Any  $x \in [0, 1]$  can be written in binary as:

$$x = 0.b_1b_2b_3b_4 \dots \quad (\text{binary})$$

where  $b_i \in \{0, 1\}$ .

Applying  $f$  (which is  $2x \bmod 1$ ) shifts the binary digits left:

$$f(x) = 0.b_2b_3b_4b_5 \dots$$

**Explanation 10** (Shift map interpretation). *The sawtooth map is equivalent to a left shift on binary sequences. This is the Bernoulli shift map, a canonical example of chaos.*

## Step 8: Describe Typical Orbit Behavior

For a typical initial point  $x_0$ :

- The orbit  $\{x_0, f(x_0), f^2(x_0), \dots\}$  will **never settle** to a fixed point or periodic orbit
- The orbit will appear to **randomly** visit different parts of  $[0, 1]$
- The orbit will **eventually come arbitrarily close** to any given point in  $[0, 1]$  (ergodicity)
- Two orbits starting very close together will **diverge exponentially** and become uncorrelated after a few iterations

## Step 9: Connect to Sharkovskii's Theorem

From lecture notes Section 22, page 84: The existence of a period 3 orbit implies the existence of periodic orbits of all periods (Sharkovskii's theorem).

We found a period 3 orbit in part (b), which means:

The sawtooth map has periodic orbits of every period.

But from Li-Yorke's theorem (page 84): "period three implies chaos."

## Step 10: Final Characterization

Answer to part (d):

### 1. Why no stable periodic orbits:

The map has constant derivative  $f'(x) = 2 > 1$  everywhere. Any periodic orbit of period  $p$  has stability multiplier  $2^p > 1$ , making it unstable. There is no contracting region, only expansion.

### 2. Typical dynamics:

From a typical initial point, the orbit exhibits **deterministic chaos**:

- The orbit never repeats
- It densely fills the interval  $[0, 1]$
- It has sensitive dependence on initial conditions
- The Lyapunov exponent is  $\lambda = \ln(2) > 0$
- The dynamics appear random despite being deterministic

The sawtooth map is chaotic with Lyapunov exponent  $\lambda = \ln(2)$

**Explanation 11** (Physical meaning). *The sawtooth map is a simple example of deterministic chaos. Despite having a simple, explicit formula, its long-term behavior is unpredictable. This map models situations where a system repeatedly stretches and folds, like kneading dough or mixing fluids.*

*The map serves as a prototype for understanding chaos in more complicated systems. Its analysis demonstrates that:*

- *Chaos can arise in very simple systems*
- *Expansion (derivative  $> 1$ ) prevents stable orbits*
- *Dense periodic orbits coexist with chaotic trajectories*
- *Predictability breaks down due to sensitivity to initial conditions*

*This connects to the broader theory of chaos covered in Section 23 of the lecture notes, including the period-doubling route to chaos and strange attractors.*

---

## 6 Summary

### Complete Results

**Part (a):** Period 2 orbit:  $\{1/3, 2/3\}$

**Part (b):** Period 3 orbit:  $\{1/7, 2/7, 4/7\}$

**Part (c):** Both orbits are unstable:

- Period 2:  $\left| \frac{dx_{n+2}}{dx_n} \right| = 4 > 1$
- Period 3:  $\left| \frac{dx_{n+3}}{dx_n} \right| = 8 > 1$

**Part (d):** No stable periodic orbits exist because  $f'(x) = 2$  everywhere implies all stability multipliers satisfy  $2^p > 1$ .

Typical dynamics: **Deterministic chaos** with Lyapunov exponent  $\lambda = \ln(2) > 0$ .

### Connection to Course Material

This problem illustrates key concepts from Sections 22-23:

- Finding periodic orbits by iteration (Section 22)
- Using the chain rule for stability (Section 22, page 82)
- Identifying chaos through Lyapunov exponents (Section 23, page 85-86)
- Sharkovskii ordering and "period three implies chaos" (Section 22, page 84)
- Sensitive dependence on initial conditions (Section 23, page 85)