Asymptotics Problem 8.3: Complete Pedagogical Solution

Boundary Layer with Van Dyke Matching (One and Two-Term)

Problem 1. Perform an asymptotic matching to obtain a uniformly valid one-term (optionally: two-term) composite expansion for the solution, $f(x)$, as $\varepsilon \to 0$ of

$$\varepsilon f'' + (2+x)f' + f = 1, \quad 0 < x < 1, \quad \varepsilon > 0,$$

with boundary conditions $f(0) = 2$, $f(1) = 0$.

Solution: Step-by-Step Atomic Breakdown

Step 1: Understanding the Problem Structure and Classification

Strategy: *We have a second-order linear ODE with:*

- *A small parameter ε multiplying the highest derivative f''*
- *A first derivative term $(2+x)f'$ with coefficient $p(x) = 2+x > 0$ for all $x \in [0,1]$*
- *A zeroth-order term f*
- *An inhomogeneous term (RHS = 1)*
- *Two boundary conditions at $x = 0$ and $x = 1$*

Our task is to find one-term and optionally two-term composite expansions.
Justification: *This is a singular perturbation problem because setting $\varepsilon = 0$ reduces the second-order ODE to a first-order ODE, which cannot generically satisfy two boundary conditions. A boundary layer must form at one of the boundaries.*

Step 2: Determining the Boundary Layer Location

Key Concept: *From Lecture Notes §6.2.1, for an equation of the form $\varepsilon y'' + p(x)y' + q(x)y = r(x)$:*

- *If $p(x) > 0$ throughout $[0,1]$: boundary layer at $x = 0$*
- *If $p(x) < 0$ throughout $[0,1]$: boundary layer at $x = 1$*

Identifying $p(x)$ in our equation:

$$\varepsilon f'' + (2+x)f' + f = 1$$

Comparing with $\varepsilon f'' + p(x)f' + q(x)f = r(x)$:

$$p(x) = 2+x, \quad q(x) = 1, \quad r(x) = 1$$

Justification: *Since $p(x) = 2+x > 0$ for all $x \in [0,1]$ (in fact, $p(x) \geq 2$ on this interval), the boundary layer is located at $\boxed{x = 0}$.*

This means:

- *The outer solution will satisfy the boundary condition at $x = 1$*
- *The inner solution (boundary layer) will be needed near $x = 0$ to satisfy $f(0) = 2$*
- *The boundary layer has width $O(\varepsilon)$*

Part I: One-Term Composite Expansion

Step 3: Finding the Leading-Order Outer Solution

What we do: Set $\varepsilon = 0$ and solve the reduced equation.
Technique: *The outer expansion assumes* $f(x, \varepsilon) = f_0(x) + \varepsilon f_1(x) + \cdots$ *where* f_0 *satisfies the equation with* $\varepsilon = 0$*.*
Setting $\varepsilon = 0$:

$$(2 + x)f_0' + f_0 = 1$$

Step 3a: Solving the First-Order Linear ODE

Technique: *This is a first-order linear ODE* $f_0' + P(x)f_0 = Q(x)$ *where* $P(x) = 1/(2 + x)$ *and* $Q(x) = 1/(2 + x)$*. Use the integrating factor method:*

$$\mu(x) = \exp\left(\int \frac{dx}{2 + x}\right) = \exp(\ln(2 + x)) = 2 + x$$

Multiply the ODE $(2+x)f_0'+f_0 = 1$ by the integrating factor... actually, the equation is already in the right form! Let's rewrite:

$$(2 + x)f_0' + f_0 = 1$$

Notice that the left side is:

$$\frac{d}{dx}\left[(2 + x)f_0\right] = (2 + x)f_0' + f_0$$

So:

$$\frac{d}{dx}\left[(2 + x)f_0\right] = 1$$

Integrate both sides:

$$(2 + x)f_0 = x + a_0$$

where a_0 is an integration constant.
Therefore:

$$f_0(x) = \frac{x + a_0}{x + 2}$$

Step 3b: Applying the Boundary Condition at $x = 1$

Justification: *Since the boundary layer is at* $x = 0$*, the outer solution must satisfy the boundary condition at* $x = 1$*.*
Apply $f_0(1) = 0$:

$$f_0(1) = \frac{1 + a_0}{1 + 2} = \frac{1 + a_0}{3} = 0 \quad\Longrightarrow\quad a_0 = -1$$

Therefore, the leading-order outer solution is:

$$\boxed{f_0(x) = \frac{x - 1}{x + 2}}$$

Step 3c: Verifying and Evaluating at $x = 0$

Technique: *Check the ODE:* $f_0' = \frac{(x+2)-(x-1)}{(x+2)^2} = \frac{3}{(x+2)^2}$

$$(2 + x)f_0' + f_0 = (2 + x) \cdot \frac{3}{(x + 2)^2} + \frac{x - 1}{x + 2} = \frac{3}{x + 2} + \frac{x - 1}{x + 2} = \frac{x + 2}{x + 2} = 1 \quad ✓$$

Value at $x = 0$:

$$f_0(0) = \frac{0 - 1}{0 + 2} = -\frac{1}{2}$$

The boundary condition requires $f(0) = 2$, but the outer solution gives $f_0(0) = -1/2$. The **mismatch** is $2 - (-1/2) = 5/2$.

Step 4: Setting Up the Leading-Order Inner Solution

What we do: Introduce a stretched coordinate near $x = 0$.
Technique: *For a boundary layer at $x = 0$ with width $O(\varepsilon)$, introduce the inner variable:*

$$X = \frac{x}{\varepsilon}$$

Note: $X \geq 0$ for $x \in [0, 1]$.
Define the inner function $F(X) = f(x)$.

Step 4a: Transforming the Derivatives

Using the chain rule:

$$\frac{df}{dx} = \frac{dF}{dX} \cdot \frac{dX}{dx} = \frac{1}{\varepsilon}F', \quad \frac{d^2 f}{dx^2} = \frac{1}{\varepsilon^2}F''$$

Step 4b: Transforming the Equation

Also, $x = \varepsilon X$, so $x + 2 = 2 + \varepsilon X$.
Substitute into $\varepsilon f'' + (2 + x)f' + f = 1$:

$$\varepsilon \cdot \frac{1}{\varepsilon^2}F'' + (2 + \varepsilon X) \cdot \frac{1}{\varepsilon}F' + F = 1$$

$$\frac{1}{\varepsilon}F'' + \frac{2 + \varepsilon X}{\varepsilon}F' + F = 1$$

$$\frac{1}{\varepsilon}\left[F'' + 2F' + \varepsilon X F'\right] + F = 1$$

Multiply through by ε:

$$F'' + 2F' + \varepsilon X F' + \varepsilon F = \varepsilon$$

Step 4c: Taking the Leading Order as $\varepsilon \to 0$

Justification: *At leading order ($O(\varepsilon^{-1})$ before multiplying by ε, or $O(1)$ after), we keep only the terms without ε:*

$$F_0'' + 2F_0' = 0$$

Step 5: Solving the Leading-Order Inner Equation

The inner equation: $F_0'' + 2F_0' = 0$
Technique: *This is a constant-coefficient ODE. Try $F_0 = e^{\lambda X}$:*

$$\lambda^2 e^{\lambda X} + 2\lambda e^{\lambda X} = 0 \quad \Longrightarrow \quad \lambda(\lambda + 2) = 0 \quad \Longrightarrow \quad \lambda = 0 \text{ or } \lambda = -2$$

The general solution is:

$$F_0(X) = A_0 e^{-2X} + B_0$$

Step 5a: Applying the Boundary Condition at $x = 0$

At $x = 0$, we have $X = 0$. The boundary condition $f(0) = 2$ gives:

$$F_0(0) = A_0 e^0 + B_0 = A_0 + B_0 = 2$$

This gives: $B_0 = 2 - A_0$.
So:

$$F_0(X) = A_0 e^{-2X} + (2 - A_0) = 2 - A_0 + A_0 e^{-2X}$$

Step 6: Applying Prandtl's Matching Criterion

Key Concept: *Prandtl's matching rule (Lecture Notes §6.1.2):*

$$\lim_{x\to 0^+} f_0(x) = \lim_{X\to +\infty} F_0(X)$$

Step 6a: Computing the Inner Limit of the Outer Solution

$$\lim_{x\to 0^+} f_0(x) = \lim_{x\to 0^+} \frac{x-1}{x+2} = \frac{-1}{2} = -\frac{1}{2}$$

Step 6b: Computing the Outer Limit of the Inner Solution

As $X \to +\infty$:

$$F_0(X) = 2 - A_0 + A_0 e^{-2X}$$

Since $e^{-2X} \to 0$ as $X \to +\infty$:

$$\lim_{X\to +\infty} F_0(X) = 2 - A_0$$

Step 6c: Applying the Matching Condition

$$-\frac{1}{2} = 2 - A_0 \quad \Longrightarrow \quad A_0 = 2 + \frac{1}{2} = \frac{5}{2}$$

Therefore:

$$\boxed{A_0 = \frac{5}{2}}$$

Step 7: Writing the Complete Leading-Order Inner Solution

With $A_0 = 5/2$:

$$F_0(X) = 2 - \frac{5}{2} + \frac{5}{2}e^{-2X} = -\frac{1}{2} + \frac{5}{2}e^{-2X}$$

Converting back to x-coordinates using $X = x/\varepsilon$:

$$\boxed{F_0 = -\frac{1}{2} + \frac{5}{2}\exp\left(-\frac{2x}{\varepsilon}\right)}$$

Step 7a: Verification

Technique: *Check boundary condition: At $x = 0$ ($X = 0$):*

$$F_0(0) = -\frac{1}{2} + \frac{5}{2} = 2 \quad \checkmark$$

Check matching: As $X \to +\infty$:

$$F_0 \to -\frac{1}{2} = f_0(0) \quad \checkmark$$

Step 8: Constructing the One-Term Composite Solution

Technique: *The composite solution is (Lecture Notes §6.1.2):*

$$f_c(x) = f_0(x) + F_0(X) - (\text{common limit})$$

The common limit is:

$$\lim_{x\to 0} f_0(x) = \lim_{X\to\infty} F_0(X) = -\frac{1}{2}$$

Therefore:

$$\begin{aligned} f_c(x) &= f_0(x) + F_0\left(\frac{x}{\varepsilon}\right) - \left(-\frac{1}{2}\right) \\ &= \frac{x-1}{x+2} + \left[-\frac{1}{2} + \frac{5}{2}e^{-2x/\varepsilon}\right] + \frac{1}{2} \end{aligned}$$

Simplifying:

$$\boxed{f_c(x) = \frac{x-1}{x+2} + \frac{5}{2}\exp\left(-\frac{2x}{\varepsilon}\right)}$$

Step 9: Verifying the One-Term Composite Solution

Step 9a: Check Boundary Condition at $x = 0$

$$f_c(0) = \frac{-1}{2} + \frac{5}{2}e^0 = -\frac{1}{2} + \frac{5}{2} = 2 \quad \checkmark$$

Step 9b: Check Boundary Condition at $x = 1$

$$f_c(1) = \frac{1-1}{1+2} + \frac{5}{2}e^{-2/\varepsilon} = 0 + \frac{5}{2}e^{-2/\varepsilon}$$

Justification: *For small ε, $e^{-2/\varepsilon}$ is exponentially small. Therefore $f_c(1) \approx 0$ up to exponentially small corrections.* ✓

Step 9c: Check Behavior in the Interior

For $x \gg \varepsilon$ (away from the boundary layer), $e^{-2x/\varepsilon} \approx 0$:

$$f_c(x) \approx f_0(x) = \frac{x-1}{x+2} \quad \checkmark$$

Part II: Two-Term Composite Expansion (Optional)

Step 10: Finding the $O(\varepsilon)$ Outer Solution

Technique: *Insert $f = f_0 + \varepsilon f_1 + O(\varepsilon^2)$ into the ODE and collect $O(\varepsilon)$ terms.*
The ODE is $\varepsilon f'' + (2+x)f' + f = 1$. At $O(\varepsilon)$:

$$f_0'' + (2+x)f_1' + f_1 = 0$$
$$(2+x)f_1' + f_1 = -f_0''$$

Step 10a: Computing f_0''

We have $f_0 = (x-1)/(x+2)$, so:

$$f_0' = \frac{(x+2)-(x-1)}{(x+2)^2} = \frac{3}{(x+2)^2}$$

$$f_0'' = -\frac{6}{(x+2)^3}$$

Step 10b: Solving for f_1

The equation is:

$$(2+x)f_1' + f_1 = \frac{6}{(x+2)^3}$$

This has the form $\frac{d}{dx}[(x+2)f_1] = \frac{6}{(x+2)^3}$. Integrating:

$$(x+2)f_1 = \int \frac{6}{(x+2)^3}\,dx = -\frac{3}{(x+2)^2} + a_1$$

$$f_1(x) = -\frac{3}{(x+2)^3} + \frac{a_1}{x+2}$$

Step 10c: Applying Boundary Condition $f_1(1) = 0$

$$f_1(1) = -\frac{3}{27} + \frac{a_1}{3} = -\frac{1}{9} + \frac{a_1}{3} = 0 \quad \Longrightarrow \quad a_1 = \frac{1}{3}$$

Therefore:

$$\boxed{f_1(x) = -\frac{3}{(x+2)^3} + \frac{1}{3(x+2)}}$$

Step 11: Finding the $O(\varepsilon)$ Inner Solution

Technique: *Insert $F = F_0 + \varepsilon F_1 + O(\varepsilon^2)$ into the inner equation and collect $O(1)$ terms (after the ε^{-1} rescaling).*
From the inner equation $F'' + 2F' + \varepsilon XF' + \varepsilon F = \varepsilon$, at $O(1)$:

$$F_1'' + 2F_1' = 1 - F_0 - XF_0'$$

Step 11a: Computing the RHS

With $F_0 = -1/2 + (5/2)e^{-2X}$:

$$F_0' = -5e^{-2X}$$

The RHS is:

$$\begin{aligned}
1 - F_0 - XF_0' &= 1 - \left(-\frac{1}{2} + \frac{5}{2}e^{-2X}\right) - X(-5e^{-2X}) \\
&= 1 + \frac{1}{2} - \frac{5}{2}e^{-2X} + 5Xe^{-2X} \\
&= \frac{3}{2} - \frac{5}{2}e^{-2X} + 5Xe^{-2X}
\end{aligned}$$

Step 11b: Solving the $O(\varepsilon)$ Inner Equation

The equation is:

$$F_1'' + 2F_1' = \frac{3}{2} - \frac{5}{2}e^{-2X} + 5Xe^{-2X}$$

Technique: *The homogeneous solution is $F_1^{(h)} = C_1 + C_2e^{-2X}$.*
For particular solutions:

- *For the constant $3/2$: try $F_1^{(p1)} = aX$. Then $2a = 3/2$, so $a = 3/4$.*
- *For e^{-2X}: this is part of the homogeneous solution, so try $F_1^{(p2)} = bXe^{-2X}$.*
- *For Xe^{-2X}: try $F_1^{(p3)} = cX^2e^{-2X}$.*

For $F = bXe^{-2X}$:

$$F' = be^{-2X} - 2bXe^{-2X}, \quad F'' = -4be^{-2X} + 4bXe^{-2X}$$

$$F'' + 2F' = -4be^{-2X} + 4bXe^{-2X} + 2be^{-2X} - 4bXe^{-2X} = -2be^{-2X}$$

So $-2b = -5/2$, giving $b = 5/4$.
For $F = cX^2e^{-2X}$:

$$F' = 2cXe^{-2X} - 2cX^2e^{-2X}$$

$$F'' = 2ce^{-2X} - 8cXe^{-2X} + 4cX^2e^{-2X}$$

$$F'' + 2F' = 2ce^{-2X} - 8cXe^{-2X} + 4cX^2e^{-2X} + 4cXe^{-2X} - 4cX^2e^{-2X} = 2ce^{-2X} - 4cXe^{-2X}$$

Matching Xe^{-2X}: $-4c = 5$, so $c = -5/4$.
General solution:

$$F_1(X) = A_1 + B_1e^{-2X} + \frac{3}{4}X + \frac{5}{4}Xe^{-2X} - \frac{5}{4}X^2e^{-2X}$$

Step 11c: Applying Boundary Condition $F_1(0) = 0$

$$F_1(0) = A_1 + B_1 = 0 \implies B_1 = -A_1$$

So:

$$F_1(X) = A_1(1 - e^{-2X}) + \frac{3}{4}X + \frac{5}{4}Xe^{-2X} - \frac{5}{4}X^2e^{-2X}$$

Step 12: Van Dyke Matching for Two-Term Expansion

Key Concept: *Van Dyke's matching rule (Lecture Notes §6.1.3): The n-term inner expansion of the m-term outer expansion equals the m-term outer expansion of the n-term inner expansion (written in the same variables).*

For two-term matching:

1. *Write outer solution in inner variables ($x = \varepsilon X$), expand to $O(\varepsilon)$*
2. *Write inner solution in outer variables ($X = x/\varepsilon$), expand to $O(\varepsilon)$*
3. *Equate the two expansions*

Step 12a: Outer Solution in Inner Variables

Substitute $x = \varepsilon X$ into $f_0(x) + \varepsilon f_1(x)$:

$$f_0(\varepsilon X) = \frac{\varepsilon X - 1}{\varepsilon X + 2}$$

Expand for small ε:

$$\frac{\varepsilon X - 1}{\varepsilon X + 2} = \frac{-1 + \varepsilon X}{2 + \varepsilon X} = \frac{-1}{2} \cdot \frac{1 - \varepsilon X}{1 + \varepsilon X/2}$$

$$= -\frac{1}{2}(1 - \varepsilon X)\left(1 - \frac{\varepsilon X}{2} + O(\varepsilon^2)\right) = -\frac{1}{2}\left(1 - \varepsilon X - \frac{\varepsilon X}{2} + O(\varepsilon^2)\right)$$

$$= -\frac{1}{2} + \frac{3\varepsilon X}{4} + O(\varepsilon^2)$$

Similarly:

$$\varepsilon f_1(\varepsilon X) = \varepsilon\left[-\frac{3}{(2 + \varepsilon X)^3} + \frac{1}{3(2 + \varepsilon X)}\right] = \varepsilon\left[-\frac{3}{8} + \frac{1}{6}\right] + O(\varepsilon^2) = -\frac{5\varepsilon}{24} + O(\varepsilon^2)$$

Total outer expansion in inner variables:

$$f(\varepsilon X) = -\frac{1}{2} + \frac{3}{4}\varepsilon X - \frac{5}{24}\varepsilon + O(\varepsilon^2)$$

Step 12b: Inner Solution in Outer Variables

Substitute $X = x/\varepsilon$ into $F_0 + \varepsilon F_1$. As $\varepsilon \to 0$ with x fixed, $X \to \infty$ and $e^{-2X} \to 0$:

$$F_0(x/\varepsilon) \to -\frac{1}{2}$$

$$\varepsilon F_1(x/\varepsilon) \to \varepsilon \left[A_1 + \frac{3}{4} \cdot \frac{x}{\varepsilon}\right] = \varepsilon A_1 + \frac{3x}{4}$$

Total inner expansion in outer variables:

$$F(x/\varepsilon) = -\frac{1}{2} + \frac{3x}{4} + \varepsilon A_1 + O(\varepsilon^2)$$

Step 12c: Matching

Equating (with $x = \varepsilon X$):

$$-\frac{1}{2} + \frac{3}{4}\varepsilon X - \frac{5}{24}\varepsilon = -\frac{1}{2} + \frac{3}{4}\varepsilon X + \varepsilon A_1$$

This gives:

$$-\frac{5}{24}\varepsilon = \varepsilon A_1 \quad \Longrightarrow \quad \boxed{A_1 = -\frac{5}{24}}$$

Step 13: Two-Term Composite Solution

Technique: *The two-term composite solution is:*

$$f_c(x) = [f_0(x) + \varepsilon f_1(x)] + [F_0(X) + \varepsilon F_1(X)] - (\textit{common part})$$

where the common part is $-1/2 + (3/4)x - (5/24)\varepsilon$.
Substituting all components:

$$f_c(x) = \frac{x-1}{x+2} + \varepsilon \left[-\frac{3}{(x+2)^3} + \frac{1}{3(x+2)}\right] + \left[\frac{5}{2}e^{-2x/\varepsilon} - \frac{5x}{2\varepsilon}e^{-2x/\varepsilon} - \frac{5x^2}{4\varepsilon^2}e^{-2x/\varepsilon} + \frac{5\varepsilon}{24}e^{-2x/\varepsilon}\right]$$

After simplification, grouping the exponential terms:

$$\boxed{f_c(x) = \frac{x-1}{x+2} + \varepsilon \left[-\frac{3}{(x+2)^3} + \frac{1}{3(x+2)}\right] + \left[\frac{5}{2} - \frac{5x}{2\varepsilon} + \frac{5\varepsilon}{24}\right] e^{-2x/\varepsilon}}$$

Final Summary

Reflection: *What have we learned from this problem?*

1. ***Boundary layer location:*** *The coefficient of* f' *is* $p(x) = 2 + x > 0$, *so by the general theory (Lecture Notes §6.2.1), the boundary layer forms at* $x = 0$.

2. ***One-term solution:*** *The leading-order outer solution is* $f_0(x) = (x-1)/(x+2)$, *and the leading-order inner solution is* $F_0(X) = -1/2 + (5/2)e^{-2X}$. *Prandtl matching determines* $A_0 = 5/2$.

3. ***Two-term solution:*** *Van Dyke matching determines the higher-order constant* $A_1 = -5/24$ *by requiring consistency between the inner expansion of the outer solution and the outer expansion of the inner solution.*

4. ***Composite solutions:***

   - *One-term:* $f_c(x) = \frac{x-1}{x+2} + \frac{5}{2}e^{-2x/\varepsilon}$
   - *Two-term: includes* $O(\varepsilon)$ *corrections to both outer and inner parts*

5. ***Physical interpretation:*** *Near* $x = 0$*, the solution must rise rapidly from the boundary value* $f(0) = 2$ *to approach the outer solution value* $f_0(0) = -1/2$*. This transition occurs over a thin layer of width* $O(\varepsilon)$ *and involves an exponential decay on the scale* $e^{-2x/\varepsilon}$*.*

Complete Solution Summary:
One-Term Composite Expansion:

$$f_c(x) = \frac{x-1}{x+2} + \frac{5}{2}\exp\left(-\frac{2x}{\varepsilon}\right)$$

Two-Term Composite Expansion:

$$f_c(x) = \frac{x-1}{x+2} + \varepsilon\left[-\frac{3}{(x+2)^3} + \frac{1}{3(x+2)}\right] + \left[\frac{5}{2} - \frac{5x}{2\varepsilon} + \frac{5\varepsilon}{24}\right]e^{-2x/\varepsilon}$$

The boundary layer is at $x = 0$ with width $O(\varepsilon)$, determined by the positive coefficient $p(x) = 2 + x > 0$.