Asymptotics 2025/2026 Solutions for Sheet 7[1]

WKB method

Problem 1 (5 marks):

We insert $u = fy$ into the equation for u and obtain:

$$y''f + y'(2f' + pf) + y(f'' + pf' + qf) = 0 \, .$$

The requirement that the y'-term has to vanish leads to the condition

$$2f' + pf = 0 \qquad \Longrightarrow \qquad f(x) = const \cdot \exp\left(-\frac{1}{2}\int^x p(s)ds\right) \, ,$$

and the resulting equation for y is

$$y'' + \left(q - \frac{p'}{2} - \frac{p^2}{4}\right) y = 0 \, . \qquad 5$$

Problem 2 (10 marks):
With $z = \phi(x)$ and $\nu(z) = \psi(x)y(x)$ we obtain

$$y = \frac{\nu}{\psi} \, , \quad y' = \frac{\nu'\phi'\psi - \nu\psi'}{\psi^2} \, , \quad y'' = \frac{\nu''(\phi')^2\psi^2 + \nu'[\phi''\psi^2 - 2\phi'\psi'\psi] + \nu[2(\psi')^2 - \psi''\psi]}{\psi^3} \, .$$

We insert these expressions into the differential equation for y and take only the leading order expression, as $\varepsilon \to 0$, for each coefficient of ν, ν' and ν''. This leads to

$$\nu''\varepsilon^2(\phi')^2 + \nu'\varepsilon^2\frac{\phi''\psi - 2\phi'\psi'}{\psi} + \nu q = 0 \qquad 3.$$

To obtain the form $\varepsilon^2\nu'' + \nu = 0$ we require that the coefficient of ν' vanishes

$$\phi''\psi - 2\phi'\psi' = 0 \quad \Longrightarrow \quad \frac{\phi''}{\phi'} = 2\frac{\psi'}{\psi} \quad \Longrightarrow \quad \phi' = const \cdot [\psi]^2 \qquad 2 \, , \tag{1}$$

and that the quotient of the other two coefficients is ε^2:

$$(\phi')^2 = q \quad \Longrightarrow \quad \phi(x) = \pm \int^x \sqrt{q(s)}ds + const \, . \qquad 2$$

(We assume that $q(x) > 0$). From (1) it follows further that $\psi = const \cdot q^{1/4}$. Since the solutions of the differential equation for ν are $\exp(\pm iz/\varepsilon)$ we obtain the following solution for y (which agrees with the WKB solution):

$$\begin{aligned} y(x) &\approx \frac{\tilde{A}_+}{\psi(x)} \exp\left(+\frac{i\phi(x)}{\varepsilon}\right) + \frac{\tilde{A}_-}{\psi(x)} \exp\left(-\frac{i\phi(x)}{\varepsilon}\right) \\ &= \frac{A_+}{q(x)^{1/4}} \exp\left(+\frac{i}{\varepsilon}\int^x \sqrt{q(x)}ds\right) + \frac{A_-}{q(x)^{1/4}} \exp\left(-\frac{i}{\varepsilon}\int^x \sqrt{q(x)}ds\right) \qquad 3. \end{aligned}$$

Problem 3 (10 marks):
We insert $y = \exp[S(x,\varepsilon)]$ and $dS(x,\varepsilon)/dx = p(x,\varepsilon)$, into the equation for y and obtain the following equation for p: $\varepsilon^2(p' + p^2) + q + \chi r = 0$. Assuming that $p(x,\varepsilon) = p_0(x)\lambda_0(\varepsilon) + p_1(x)\lambda_1(\varepsilon) + \ldots$ leads to

$$\varepsilon^2(p_0'\lambda_0 + p_1'\lambda_1 + \ldots) + \varepsilon^2(p_0^2\lambda_0^2 + 2p_0p_1\lambda_0\lambda_1 + \ldots) + q + \chi r = 0\ .$$

We make a table in which the terms in each column have descending magnitude.

$\varepsilon^2 p_0'\lambda_0$	$\varepsilon^2 p_0^2\lambda_0^2$	q
$\varepsilon^2 p_1'\lambda_1$	$\varepsilon^2 2p_0p_1\lambda_0\lambda_1$	χr
$\ldots$	$\ldots$	

2

The difference to the standard WKB approximation is the $\chi(\varepsilon)r(x)$ term in the third column. The leading order WKB approximation is not changed by this term, i.e. $\varepsilon^2 p_0^2\lambda_0^2$ and q are of the same order, $\lambda_0(\varepsilon) = \varepsilon^{-1}$, and $p_0(x) = \pm i\sqrt{q(x)}$ (we assume that $q(x) > 0$). The next order terms may include

$$\varepsilon p_0'\ , \qquad \varepsilon 2p_0p_1\lambda_1(\varepsilon)\ , \qquad \chi(\varepsilon)r\ . \qquad 2$$

(We used $\lambda_0 = \varepsilon^{-1}$). For the dominant balance analysis we have to distinguish three cases

$\chi(\varepsilon) = o(\varepsilon)$: Then χr can be neglected in comparison with $\varepsilon p_0'$. Consequently, the first and the second term have to be of the same order, we have $\lambda_1(\varepsilon) = 1$, and the equation for p_1 is $p_0' + 2p_0p_1 = 0 \quad \Longrightarrow \quad p_1 = -p_0'/(2p_0) = -q'/(4q)$. This is the standard WKB approximation. 2

$\varepsilon = o(\chi(\varepsilon))$: Then $\varepsilon p_0'$ can be neglected in comparison with χr. Thus the second and the third term have to be of the same order, and we can choose $\lambda_1(\varepsilon) = \chi(\varepsilon)$, and obtain the following equation for p_1: $2p_0p_1 + r = 0 \quad \Rightarrow \quad p_1 = -r/(2p_0) = \pm ir/(2\sqrt{q})$. 2

$\chi(\varepsilon) = ord(\varepsilon)$: Then $\varepsilon p_0'$ is of the same order as χr. We have to assume that also the remaining term is of the same order (otherwise we would obtain an equation between quantities that are all known which, in general, would give a contradiction). We choose $\lambda_1(\varepsilon) = 1$, and we have that $\chi \sim c\epsilon$ as $\varepsilon \to 0$, where c is a non-vanishing constant. Then the equation for p_1 is:$p_0' + 2p_0p_1 + cr = 0 \quad \Longrightarrow \quad p_1 = -p_0'/(2p_0) - cr/(2p_0) = -q'/(4q) \pm icr/(2\sqrt{q})$. 2

Problem 4: The leading WKB approximation is $y = y_+ + y_-$ where

$$y_\pm = \frac{A_\pm}{q^{1/4}} \exp\left(\pm\frac{i}{\varepsilon}\int^x \sqrt{q(s)}ds\right) \qquad \text{or} \qquad y_\pm = \frac{B_\pm}{|q|^{1/4}} \exp\left(\pm\frac{1}{\varepsilon}\int^x \sqrt{|q(s)|}ds\right) .$$

Substitution into the differential equation for y gives in both cases

$$\left(-\frac{q''}{4q} + \frac{5(q')^2}{16q^2}\right) y = 0 .$$

We see that the differential equation for y is exactly satisfied by the WBK approximation if

$$\frac{q''}{4q} = \frac{5(q')^2}{16q^2} \quad\Longrightarrow\quad \frac{q''}{q'} = \frac{5q'}{4q} \quad\Longrightarrow\quad q' = aq^{5/4} .$$

A further integration yields $q(x) = (ax+b)^{-4}$.

Problem 5: The WKB solution with $q(x) = x^{-2}$ and $\varepsilon = \lambda^{-1}$ can be written in the form

$$y(x) \sim A_1\sqrt{x}\cos(\lambda\log x) + A_2\sqrt{x}\sin(\lambda\log x) . \tag{2}$$

From $y(1) = 0$ follows $A_1 = 0$, and $y(e) = 0$ requires

$$A_2\sqrt{e}\sin\lambda = 0 \quad\Longrightarrow\quad \lambda = n\pi, \quad n = 1, 2, \ldots .$$

The WKB approximation is expected to be good for large λ, i.e. large n.

For the exact solutions we try $y(x) = x^\alpha$, and after insertion into the o.d.e. we find $\alpha = 1/2 \pm i\sqrt{\lambda^2 - 1/4}$, so the general solution is (with $\mu = \sqrt{\lambda^2 - 1/4}$):

$$y(x) = \tilde{a}_1 x^{1/2+i\mu} + \tilde{a}_2 x^{1/2-i\mu} = a_1\sqrt{x}\cos(\mu\log x) + a_2\sqrt{x}\sin(\mu\log x) .$$

From the boundary conditions it follows similarly as before that $a_1 = 0$ and $\mu = n\pi$, $n = 1, 2, \ldots$. Consequently, $\lambda = \sqrt{n^2\pi^2 + 1/4}$.

(i) The WKB result agrees with the expectation since it yields the leading order approximation for large values of n for $\lambda = n\pi\sqrt{1 + 1/(4n^2\pi^2)} = n\pi + 1/(8n\pi) + \mathcal{O}(n^{-3})$.

(ii) For the next-order correction to the WBK approximation (see lecture) we have to add the following expression to the arguments of sine and cosine in (2):

$$-\frac{1}{\lambda}\int_1^x \left[\frac{q''}{8q^{3/2}} - \frac{5(q')^2}{32q^{5/2}}\right] ds = -\frac{1}{\lambda}\int_1^x \frac{1}{8s}ds = -\frac{\log(x)}{8\lambda} .$$

Then the new condition for the eigenvalue λ is: $\lambda - 1/(8\lambda) = n\pi \quad\Longrightarrow\quad \lambda \approx n\pi + 1/(8n\pi)$. Comparison with the exact result shows that this is the correct next order correction for λ.

Problem 6: (a) From $W(t) = w(z)\sqrt{z}$ and $z = t\mu$ with $\mu = \sqrt{n^2 - 1/4}$ follows

$$w = \frac{W}{t^{1/2}\mu^{1/2}} , \qquad w' = \frac{W'}{t^{1/2}\mu^{3/2}} - \frac{W}{2t^{3/2}\mu^{3/2}} , \qquad w'' = \frac{W''}{t^{1/2}\mu^{5/2}} - \frac{W'}{t^{3/2}\mu^{5/2}} + \frac{3W}{4t^{5/2}\mu^{5/2}} .$$

We insert these expressions into the o.d.e. and obtain the following equation for W:

$$W'' + \mu^2(1-t^{-2})W = 0$$

. For large n we can set $\mu \approx n$, and the WBK approximation follows with $\varepsilon = n^{-1}$ and $q(t) = (1-t^{-2})$. For $t > 1$ we have

$$W_\pm(t) \sim \frac{A_\pm}{(1-t^{-2})^{1/4}} \exp\left(\pm in \int_1^t \sqrt{1-s^{-2}}ds\right) = \frac{A_\pm}{(1-t^{-2})^{1/4}} \exp\left(\pm in\left[\sqrt{t^2-1} - n\cos^{-1}\frac{1}{t}\right]\right) ,$$

and for $t < 1$:

$$W_\pm(t) \sim \frac{B_\pm}{(t^{-2}-1)^{1/4}} \exp\left(\pm n \int_1^t \sqrt{s^{-2}-1}ds\right) = \frac{B_\pm}{(t^{-2}-1)^{1/4}} \exp\left(\pm n\left[\sqrt{1-t^2} - \cosh^{-1}\frac{1}{t}\right]\right) .$$

The integrals can be evaluated for $t > 1$ by setting $t^2 - 1 = u^2$, and for $t < 1$ by setting $1-t^2 = v^2$, but you can also look them up in tables. If we translate the obtained approximations for W back to w we obtain the results that are given on the problem sheet.

(b) Asymptotic formulas for large n are equations 9.3.2 and 9.3.3 in Abramowitz/Stegun:

$$J_n(n\mathrm{sech}\alpha) \sim \frac{e^{n(\tanh\alpha-\alpha)}}{\sqrt{2\pi n\tanh\alpha}} , \qquad \alpha > 0 ,$$

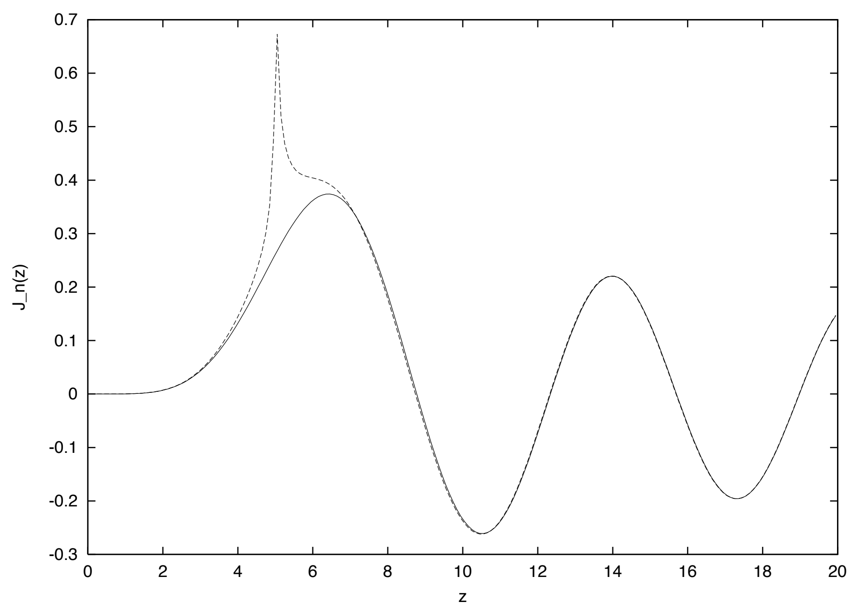
$$J_n(n\sec\alpha) \sim \frac{2\cos(n\tan\beta - n\beta - \pi/4)}{\sqrt{2\pi n\tan\beta}} , \qquad \alpha > 0 .$$

They are consistent with our approximation with $\alpha = \cosh^{-1}(n/z)$ and $\beta = \cos^{-1}(n/z)$, and

$$B_+ = \frac{1}{\sqrt{2\pi}} , \qquad B_- = 0 , \qquad A_+ = \frac{1}{\sqrt{2\pi}}e^{-i\pi/4} , \qquad A_- = \frac{1}{\sqrt{2\pi}}e^{+i\pi/4} .$$

Note that these constants satisfy the connection formula.

(c) A plot of $J_n(z)$ and its approximation for $n = 5$ is shown here: Except for the region near the turning point $z \approx n$ the approximation is excellent, even though n is not very large.


0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
-0.1
-0.2
-0.3
J_n(z)
0
2
4
6
8
10
12
14
16
18
20
z