Asymptotics 2025/2026
Solution to Problem Sheet 5, Question 1(a)

Question 1(a)

Use the method of stationary phase to obtain the leading asymptotic behaviour of

$$I(X) = \int_0^1 \tan(t)\, e^{iXt^4} \mathrm{d}t \quad \text{as } X \to \infty.$$

Solution

We have a Fourier-type integral of the form

$$I(X) = \int_0^1 f(t)\, e^{iX\phi(t)} \mathrm{d}t$$

where $f(t) = \tan(t)$ and $\phi(t) = t^4$.

Step 1: Identify and classify stationary points

Following the method of stationary phase (Section 4.3 of the lecture notes), we first identify points where $\phi'(t) = 0$.

We have

$$\phi'(t) = 4t^3.$$

Setting $\phi'(t) = 0$ gives

$$4t^3 = 0 \implies t = 0.$$

The only stationary point is at $t = 0$, which lies at the boundary of the integration interval $[0, 1]$.

Step 2: Determine the order of the stationary point

At $t = 0$, we compute the derivatives of $\phi(t)$:

$$\begin{aligned}
\phi'(0) &= 0, \\
\phi''(0) &= 12t^2\big|_{t=0} = 0, \\
\phi'''(0) &= 24t\big|_{t=0} = 0, \\
\phi^{(4)}(0) &= 24 \neq 0.
\end{aligned}$$

The first non-vanishing derivative is the fourth derivative, so this is a stationary point of order $n = 4$.

Step 3: Handle the vanishing amplitude

A crucial observation is that

$$f(0) = \tan(0) = 0.$$

Since the amplitude function vanishes at the stationary point, the standard stationary phase formula (Equation 236 from the notes) does not directly apply. We must expand $f(t)$ near $t = 0$.

Using the Taylor series of $\tan(t)$ as $t \to 0$:

$$\tan(t) = t + \frac{t^3}{3} + O(t^5).$$

Therefore, the integral becomes

$$I(X) = \int_0^1 \left(t + \frac{t^3}{3} + O(t^5) \right) e^{iXt^4} \mathrm{d}t.$$

The leading contribution comes from the first term:

$$I(X) \sim I_1(X) = \int_0^1 t\, e^{iXt^4} \mathrm{d}t \quad \text{as } X \to \infty.$$

Step 4: Evaluate the leading order integral

We perform a change of variables. Let $u = Xt^4$, so that

$$t = \left(\frac{u}{X}\right)^{1/4} \quad \text{and} \quad \mathrm{d}t = \frac{1}{4}\left(\frac{u}{X}\right)^{-3/4} \frac{1}{X}\mathrm{d}u = \frac{1}{4X}u^{-3/4}X^{3/4}\mathrm{d}u.$$

Substituting into $I_1(X)$:

$$\begin{aligned} I_1(X) &= \int_0^X \left(\frac{u}{X}\right)^{1/4} e^{iu} \cdot \frac{1}{4X}u^{-3/4}X^{3/4}\mathrm{d}u \\ &= \int_0^X \frac{u^{1/4}}{X^{1/4}} e^{iu} \cdot \frac{X^{3/4}}{4X} u^{-3/4}\mathrm{d}u \\ &= \frac{1}{4X^{1/2}} \int_0^X u^{-1/2} e^{iu}\mathrm{d}u. \end{aligned}$$

Step 5: Apply the asymptotic limit

As $X \to \infty$, the upper limit of integration tends to infinity. The integral

$$\int_0^\infty u^{-1/2} e^{iu}\mathrm{d}u$$

is a standard Fourier integral. Using the formula for oscillatory integrals with power-law singularities (which can be derived from contour integration or known special functions):

$$\int_0^\infty t^\alpha e^{it}\mathrm{d}t = \Gamma(\alpha+1)e^{i\pi(\alpha+1)/2}, \quad -1 < \alpha < 0.$$

With $\alpha = -1/2$:

$$\int_0^\infty u^{-1/2}e^{iu}\mathrm{d}u = \Gamma\left(\frac{1}{2}\right) e^{i\pi/4} = \sqrt{\pi}\, e^{i\pi/4}.$$

Therefore, the leading asymptotic behaviour is

$$I_1(X) \sim \frac{1}{4X^{1/2}} \cdot \sqrt{\pi}\, e^{i\pi/4} = \frac{\sqrt{\pi}}{4} X^{-1/2} e^{i\pi/4} \quad \text{as } X \to \infty.$$

Final Answer

$$\boxed{I(X) \sim \frac{\sqrt{\pi}}{4X^{1/2}}\, e^{i\pi/4} \quad \text{as } X \to \infty}$$

Alternatively, this can be written as

$$I(X) \sim \frac{1}{4}\sqrt{\frac{\pi}{X}}\left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right) = \frac{1}{4\sqrt{2}}\sqrt{\frac{\pi}{X}}(1+i) \quad \text{as } X \to \infty.$$

The asymptotic order is $O(X^{-1/2})$.