Question 2: Perturbation Analysis of Cubic Equation
Complete Two-Term Expansions for All Roots

Asymptotics Course — Sheet 4

Problem Statement

Find two-term expansions as $\epsilon \to 0$ of all solutions of:

$$(1-\epsilon)x^3 + (\epsilon - 3)x^2 + 3x - 1 = 0$$

1 Step 1: Analyze the Unperturbed Problem

Set $\epsilon = 0$

$$x^3 - 3x^2 + 3x - 1 = 0$$

Recognize the Structure

This is a perfect cube:

$$x^3 - 3x^2 + 3x - 1 = (x-1)^3 = 0$$

Unperturbed root: $x = 1$ (with multiplicity 3)

- **STAGE X (What we have):** A degenerate triple root at $x = 1$ when $\epsilon = 0$.
- **STAGE Y (Why this matters):** For small $\epsilon \neq 0$, the cubic equation has three distinct roots. The triple root splits into three separate roots. This is a **singular perturbation problem** similar to Section 2.3 of the lecture notes.
- **STAGE Z (What this means):** We expect non-integer power expansions (likely $\epsilon^{1/2}$ or $\epsilon^{1/3}$) for some roots, as standard integer power expansions fail for degenerate roots.

2 Step 2: Reformulate the Equation

Expand and Rearrange

$$\begin{aligned}
(1-\epsilon)x^3 + (\epsilon - 3)x^2 + 3x - 1 &= 0 \\
x^3 - \epsilon x^3 + \epsilon x^2 - 3x^2 + 3x - 1 &= 0 \\
x^3 - 3x^2 + 3x - 1 &= \epsilon x^3 - \epsilon x^2 \\
(x-1)^3 &= \epsilon x^2 (x-1)
\end{aligned}$$

Factor Out Common Term

$$(x-1)^3 = \epsilon x^2(x-1)$$

This can be written as:

$$(x-1)\left[(x-1)^2 - \epsilon x^2\right] = 0$$

- **STAGE X (What we found):** The equation factors as $(x-1)$ times another quadratic expression.
- **STAGE Y (Why this helps):** This reveals that either $x = 1$ exactly, or $(x-1)^2 = \epsilon x^2$.
- **STAGE Z (Solution strategy):** We have one obvious root $x = 1$, and two other roots from solving $(x-1)^2 = \epsilon x^2$.

3 Step 3: ROOT 1 — The Persistent Root

Test $x = 1$ Directly

Substitute $x = 1$ into the original equation:

$$\begin{aligned}(1-\epsilon)(1)^3 + (\epsilon-3)(1)^2 + 3(1) - 1 &= 1 - \epsilon + \epsilon - 3 + 3 - 1 \\ &= 0 \quad \checkmark\end{aligned}$$

Conclusion: $x = 1$ is an exact root for all values of ϵ.

Two-Term Expansion

ROOT 1:

$$x_1 = 1 + 0 \cdot \epsilon + O(\epsilon^2) = 1$$

- **STAGE X (Special case):** This root does not move from $x = 1$ as ϵ varies.
- **STAGE Y (Why this happens):** The original cubic has a special structure where $x = 1$ satisfies the equation identically for all ϵ.
- **STAGE Z (Interpretation):** One of the three roots from the degenerate triple root remains at $x = 1$, while the other two separate.

4 Step 4: ROOTS 2 and 3 — Solve $(x-1)^2 = \epsilon x^2$

Expand the Equation

$$\begin{aligned}(x-1)^2 &= \epsilon x^2 \\ x^2 - 2x + 1 &= \epsilon x^2 \\ x^2(1-\epsilon) - 2x + 1 &= 0\end{aligned}$$

Apply Quadratic Formula

$$x = \frac{2 \pm \sqrt{4 - 4(1-\epsilon)}}{2(1-\epsilon)} = \frac{2 \pm \sqrt{4\epsilon}}{2(1-\epsilon)} = \frac{2 \pm 2\sqrt{\epsilon}}{2(1-\epsilon)}$$

Simplify:

$$x = \frac{1 \pm \sqrt{\epsilon}}{1-\epsilon}$$

- **STAGE X (What we found):** Two roots of the form $\frac{1\pm\sqrt{\epsilon}}{1-\epsilon}$.
- **STAGE Y (Why fractional powers):** The discriminant $\sqrt{4\epsilon} = 2\sqrt{\epsilon}$ introduces $\epsilon^{1/2}$, confirming our expectation of non-integer power expansions.
- **STAGE Z (Next step):** Expand for small ϵ using geometric series.

5 Step 5: Expand Using Geometric Series

Expand the Denominator

For $|\epsilon| < 1$:

$$\frac{1}{1-\epsilon} = 1 + \epsilon + \epsilon^2 + \epsilon^3 + \cdots$$

Compute the Product

$$\begin{aligned} x &= (1 \pm \sqrt{\epsilon})(1 + \epsilon + \epsilon^2 + \cdots) \\ &= 1 \pm \sqrt{\epsilon} + \epsilon \pm \epsilon\sqrt{\epsilon} + \epsilon^2 \pm \epsilon^2\sqrt{\epsilon} + \cdots \\ &= 1 \pm \sqrt{\epsilon} + \epsilon \pm \epsilon^{3/2} + \epsilon^2 \pm \cdots \end{aligned}$$

Order the Terms

The asymptotic sequence is: $1, \epsilon^{1/2}, \epsilon, \epsilon^{3/2}, \epsilon^2, \ldots$

For two-term expansions (keeping terms up to order ϵ):

$$x = 1 \pm \epsilon^{1/2} + \epsilon + O(\epsilon^{3/2})$$

6 Step 6: State All Three Roots

Complete Two-Term Expansions

ROOT 1 (Persistent):

$$x_1 = 1$$

ROOT 2 (Positive Branch):

$$x_2 = 1 + \epsilon^{1/2} + \epsilon + O(\epsilon^{3/2})$$

ROOT 3 (Negative Branch):

$$x_3 = 1 - \epsilon^{1/2} + \epsilon + O(\epsilon^{3/2})$$

- **STAGE X (Summary):** The degenerate triple root at $x = 1$ splits into one exact root at $x = 1$ and two roots that separate symmetrically with leading correction $\pm\epsilon^{1/2}$.

- **STAGE Y (Verification strategy):** The symmetry $x_2 + x_3 = 2 + 2\epsilon + \cdots$ and $x_2 - x_3 = 2\epsilon^{1/2} + \cdots$ confirms the structure.
- **STAGE Z (Physical interpretation):** As ϵ increases from zero, two of the three coincident roots move away from $x = 1$ at a rate proportional to $\sqrt{\epsilon}$, while the third remains fixed.

7 Step 7: Verification

Check Using Vieta's Formulas

For the cubic $(1-\epsilon)x^3 + (\epsilon - 3)x^2 + 3x - 1 = 0$, divide by $(1-\epsilon)$:

$$x^3 + \frac{\epsilon - 3}{1-\epsilon}x^2 + \frac{3}{1-\epsilon}x - \frac{1}{1-\epsilon} = 0$$

Sum of roots:

$$x_1 + x_2 + x_3 = -\frac{\epsilon - 3}{1-\epsilon} = \frac{3-\epsilon}{1-\epsilon} = (3-\epsilon)(1 + \epsilon + \cdots) = 3 + 2\epsilon + O(\epsilon^2)$$

From our solution:

$$1 + (1 + \epsilon^{1/2} + \epsilon) + (1 - \epsilon^{1/2} + \epsilon) = 3 + 2\epsilon + O(\epsilon^{3/2}) \quad \checkmark$$

Direct Verification for Root 2 (Leading Terms)

Substitute $x = 1 + \epsilon^{1/2}$ into the original equation and verify to leading order:

$$\begin{aligned}
(x-1)^3 &= (\epsilon^{1/2})^3 = \epsilon^{3/2} \\
\epsilon x^2(x-1) &= \epsilon(1+\epsilon^{1/2})^2(\epsilon^{1/2}) = \epsilon(1 + 2\epsilon^{1/2} + \epsilon)(\epsilon^{1/2}) \\
&= \epsilon^{3/2}(1 + 2\epsilon^{1/2} + \epsilon) = \epsilon^{3/2} + O(\epsilon^2)
\end{aligned}$$

To leading order: $\epsilon^{3/2} \approx \epsilon^{3/2}$ ✓

8 Verification Checklist

✓ **Unperturbed problem solved:** Found $(x-1)^3 = 0$, triple root at $x = 1$

✓ **Equation reformulated:** Factored as $(x-1)[(x-1)^2 - \epsilon x^2] = 0$

✓ **All three roots identified:** One exact, two from quadratic

✓ **Correct asymptotic sequence:** Used $\{1, \epsilon^{1/2}, \epsilon, \epsilon^{3/2}, \ldots\}$

✓ **Two terms for each root:** Included terms up to order ϵ

✓ **Vieta's formulas checked:** Sum of roots verified

✓ **Direct substitution:** Verified leading order balance

This solution follows the methodology of Section 2.3 (non-integer power expansions) and Section 2.2 (singular perturbations) of the lecture notes, systematically finding all roots with proper two-term asymptotic expansions.