Exercise Sheet 4: Maps
Question 4 - Complete Solution

Methods of Applied Mathematics

Problem Statement

Newton's Method is a numerical tool for finding roots of functions (i.e., x such that $f(x) = 0$). Starting from an initial value x_0, we repeatedly apply the mapping:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

where $f' \equiv df/dx$, until the method converges on a root. We are iterating a map until it reaches a stable fixed point.

Tasks:

(a) Show that all fixed points of the map are points where $f(x) = 0$

(b) Show that certain roots cannot be found with this method, using the concept of local stability to derive the condition that must be met by a fixed point for it to be reachable by Newton's Method

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1 Step 1: Define the Newton Map

Map structure

Newton's Method defines a map $g : \mathbb{R} \to \mathbb{R}$ given by:

$$g(x) = x - \frac{f(x)}{f'(x)}$$

The iterative scheme is:

$$x_{n+1} = g(x_n)$$

XYZ Analysis of Newton Map Structure

- **STAGE X (What we have):** Newton's method is a discrete dynamical system (a map). Each iteration updates the current guess x_n to a new guess x_{n+1} using the function value $f(x_n)$ and its derivative $f'(x_n)$.
- **STAGE Y (Why this formula):** The Newton map comes from linear approximation. Near a point x_n, we approximate $f(x)$ by its tangent line:

$$f(x) \approx f(x_n) + f'(x_n)(x - x_n)$$

To find where this linear approximation crosses zero:

$$\begin{aligned}
0 &= f(x_n) + f'(x_n)(x_{n+1} - x_n) \\
f'(x_n)(x_{n+1} - x_n) &= -f(x_n) \\
x_{n+1} - x_n &= -\frac{f(x_n)}{f'(x_n)} \\
x_{n+1} &= x_n - \frac{f(x_n)}{f'(x_n)}
\end{aligned}$$

Geometrically: draw the tangent line at $(x_n, f(x_n))$, find where it hits the x-axis, and that's x_{n+1}.

- **STAGE Z (What this means):** Newton's method is trying to "chase" the root by following tangent lines. If it converges, it converges to a fixed point of the map g. Understanding when this works requires analyzing fixed points and their stability.

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2 Step 2: Part (a) - Fixed Points of Newton Map

Find fixed points

A fixed point x^* satisfies:

$$x^* = g(x^*) = x^* - \frac{f(x^*)}{f'(x^*)}$$

Solve for fixed point condition

Rearranging:

$$\begin{aligned}
x^* &= x^* - \frac{f(x^*)}{f'(x^*)} \\
0 &= -\frac{f(x^*)}{f'(x^*)} \\
0 &= f(x^*)
\end{aligned}$$

(assuming $f'(x^*) \neq 0$)

Conclusion

$$\boxed{\text{Fixed points of } g(x) \text{ are exactly the roots of } f(x)}$$

XYZ Analysis of Fixed Points

- **STAGE X (What we proved):** Every fixed point of the Newton map corresponds to a root of f, and every root of f (where $f' \neq 0$) is a fixed point of the Newton map.
- **STAGE Y (Why this is fundamental):** The equation $x^* = g(x^*)$ means the map doesn't change the value - we've converged. The algebra shows this happens precisely when $f(x^*) = 0$. This is beautiful: we've transformed the root-finding problem "find x such that $f(x) = 0$" into the fixed-point problem "find x^* such that $g(x^*) = x^*$".

  The key insight: Newton's method is designed so that its fixed points are exactly the objects we're looking for (roots of f). This is not accidental - the map was constructed with this property.

- **STAGE Z (What this means):** Part (a) tells us WHERE the method converges if it converges (to roots of f). But it doesn't tell us WHETHER it converges, or FROM WHICH initial conditions. That's what part (b) addresses through stability analysis.

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3 Step 3: Part (b) - Stability Analysis

Compute derivative of Newton map

To determine stability of fixed points, we need $g'(x)$ where:

$$g(x) = x - \frac{f(x)}{f'(x)}$$

Using the quotient rule:

$$\begin{aligned}
g'(x) &= \frac{d}{dx}\left[x - \frac{f(x)}{f'(x)}\right] \\
&= 1 - \frac{d}{dx}\left[\frac{f(x)}{f'(x)}\right] \\
&= 1 - \frac{f'(x)\cdot f'(x) - f(x)\cdot f''(x)}{[f'(x)]^2} \\
&= 1 - \frac{[f'(x)]^2 - f(x)f''(x)}{[f'(x)]^2} \\
&= 1 - 1 + \frac{f(x)f''(x)}{[f'(x)]^2} \\
&= \frac{f(x)f''(x)}{[f'(x)]^2}
\end{aligned}$$

Therefore:

$$\boxed{g'(x) = \frac{f(x)f''(x)}{[f'(x)]^2}}$$

Evaluate at fixed point

At a fixed point x^* where $f(x^*) = 0$:

$$g'(x^*) = \frac{f(x^*)f''(x^*)}{[f'(x^*)]^2} = \frac{0\cdot f''(x^*)}{[f'(x^*)]^2} = 0$$

(assuming $f'(x^*) \neq 0$)

Stability criterion from map theory

From lecture notes (Section 20), a fixed point x^* of a map is:

- **Stable** (attracting) if $|g'(x^*)| < 1$
- **Unstable** (repelling) if $|g'(x^*)| > 1$
- **Marginal** if $|g'(x^*)| = 1$

Apply to Newton's method

At a simple root x^* (where $f(x^*) = 0$ and $f'(x^*) \neq 0$):

$$|g'(x^*)| = 0 < 1$$

Therefore:

$$\boxed{\text{All simple roots are STABLE fixed points of Newton's method}}$$

XYZ Analysis of Stability

- **STAGE X (What we found):** The derivative of the Newton map at any simple root is zero: $g'(x^*) = 0$. This is less than 1, so simple roots are stable (attracting) fixed points.
- **STAGE Y (Why this works):** The calculation reveals why Newton's method converges so fast (quadratically) near simple roots. The derivative $g'(x^*) = 0$ means the map is "super-stable" at simple roots - not just $|g'| < 1$ but actually $g' = 0$.

  Geometrically, near a simple root, the Newton map has a horizontal tangent at the fixed point. The cobweb diagram shows iterates converging very rapidly - much faster than if $0 < |g'| < 1$.

  The formula $g'(x) = \frac{f(x)f''(x)}{[f'(x)]^2}$ tells us:

  - Numerator has $f(x)$: this vanishes at roots, giving $g' = 0$
  - Denominator has $[f'(x)]^2$: this is always positive (when $f' \neq 0$)
  - Factor $f''(x)$: the second derivative, which we'll analyze below
- **STAGE Z (What this means):** Simple roots (where $f'(x^*) \neq 0$) are always reachable by Newton's method, provided we start close enough. But what about roots where $f'(x^*) = 0$? Those are next.

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4 Step 4: Special Case - Multiple Roots

What if $f'(x^*) = 0$?

If x^* is a root where $f(x^*) = 0$ AND $f'(x^*) = 0$, then x^* is a **multiple root** (at least double). Near such a root, write $f(x) = (x - x^*)^m h(x)$ where $m \geq 2$ and $h(x^*) \neq 0$.

Recompute Newton map behavior

For a multiple root of order m:

$$\begin{aligned}
f(x) &= (x - x^*)^m h(x) \\
f'(x) &= m(x - x^*)^{m-1} h(x) + (x - x^*)^m h'(x) \\
&= (x - x^*)^{m-1}[mh(x) + (x - x^*)h'(x)]
\end{aligned}$$

Near x^*:

$$\frac{f(x)}{f'(x)} = \frac{(x - x^*)^m h(x)}{(x - x^*)^{m-1}[mh(x) + (x - x^*)h'(x)]} \approx \frac{(x - x^*)h(x^*)}{mh(x^*)} = \frac{x - x^*}{m}$$

Therefore:

$$g(x) = x - \frac{f(x)}{f'(x)} \approx x - \frac{x - x^*}{m} = x^* + \left(1 - \frac{1}{m}\right)(x - x^*)$$

The derivative at the fixed point:

$$g'(x^*) = 1 - \frac{1}{m}$$

Stability of multiple roots

For a root of multiplicity $m \geq 2$:

$$|g'(x^*)| = \left|1 - \frac{1}{m}\right| = \frac{m-1}{m}$$

Analysis:

- $m = 1$ (simple root): $g'(x^*) = 0 \Rightarrow$ stable
- $m = 2$ (double root): $g'(x^*) = 1/2 \Rightarrow$ stable, but slower convergence
- $m = 3$ (triple root): $g'(x^*) = 2/3 \Rightarrow$ stable, but even slower
- As $m \to \infty$: $g'(x^*) \to 1 \Rightarrow$ marginally stable

All multiple roots remain stable ($|g'| < 1$), but convergence slows down as multiplicity increases.

XYZ Analysis of Multiple Roots

- **STAGE X (What we found):** Multiple roots are still attracting fixed points, but with $g'(x^*) = (m-1)/m$ instead of 0. Convergence is linear rather than quadratic.
- **STAGE Y (Why convergence slows):** For a simple root, the numerator $f(x)$ in Newton's formula vanishes faster than the denominator $f'(x)$ as $x \to x^*$, giving a large correction step. For multiple roots, both numerator and denominator vanish at the same rate, so the correction step is smaller - only $(x - x^*)/m$ instead of $(x - x^*)$.

  The derivative $g' = (m-1)/m$ measures the "contraction rate": each iteration reduces the error by a factor of $(m-1)/m$. For $m = 2$, error is halved each step. For $m = 10$, error is only reduced by 10
- **STAGE Z (What this means practically):** Multiple roots can still be found but require many more iterations. Modified Newton's method (multiplying by m if multiplicity is known) can restore quadratic convergence: $x_{n+1} = x_n - m\frac{f(x_n)}{f'(x_n)}$.

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5 Step 5: When Newton's Method Fails

Conditions for failure

Newton's method can fail to reach a root when:

(1) Division by zero: If $f'(x_n) = 0$ at any iteration but $f(x_n) \neq 0$, the map is undefined:

$$x_{n+1} = x_n - \frac{f(x_n)}{0} = \text{undefined}$$

(2) Unstable dynamics: Even if all roots are stable fixed points, the initial condition x_0 might lie in a basin of repulsion or lead to chaotic behavior.

(3) Cycles: The iterates might enter a periodic orbit that doesn't include any root.

Example of failure

Consider $f(x) = x^{1/3}$ with root at $x^* = 0$.

Near $x = 0$:

$$f(x) = x^{1/3}$$
$$f'(x) = \frac{1}{3}x^{-2/3}$$

Newton map:

$$g(x) = x - \frac{x^{1/3}}{(1/3)x^{-2/3}} = x - 3x = -2x$$

Therefore:

$$\begin{aligned} x_1 &= -2x_0 \\ x_2 &= -2x_1 = 4x_0 \\ x_3 &= -2x_2 = -8x_0 \\ &\vdots \end{aligned}$$

The iterates diverge: $|x_n| = 2^n|x_0| \to \infty$!

Why this fails

The derivative at the root:

$$g'(0) = \lim_{x \to 0} \frac{d}{dx}[-2x] = -2$$

Since $|g'(0)| = 2 > 1$, the root $x^* = 0$ is an **unstable** fixed point.

Why? Because $f'(0)$ doesn't exist (or is infinite), violating our assumption of a simple root.

XYZ Analysis of Failure Modes

- **STAGE X (What we discovered):** Not all roots can be found by Newton's method. Roots where $f'(x^*) = 0$ or where f' doesn't exist can be unstable fixed points.
- **STAGE Y (Why instability occurs):** The formula $g'(x^*) = \frac{f(x^*)f''(x^*)}{[f'(x^*)]^2}$ only gives $g' = 0$ when:
  - $f(x^*) = 0$ (it's a root)
  - $f'(x^*) \neq 0$ (denominator is finite)

  If $f'(x^*) = 0$, the formula is indeterminate 0/0. More careful analysis (like the multiple root case) is needed. If f' has a singularity or vanishes in pathological ways, g' can exceed 1 in magnitude, making the fixed point repelling.

  The $x^{1/3}$ example: $f'(0)$ is infinite, so the denominator in g' vanishes, and the Newton map becomes $g(x) = -2x$ with slope -2 at the origin.
- **STAGE Z (What this means for applications):** Before applying Newton's method:
  1. Check that $f'(x) \neq 0$ near roots (no critical points)
  2. Start with a good initial guess (in the basin of attraction)
  3. Monitor for cycles or divergence
  4. Consider modified Newton for multiple roots

The method is not universally convergent, but when it works (simple roots, good initial guess), it's exceptionally fast.

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6 Step 6: Reachability Condition

Formal statement

A root x^* of $f(x)$ is **reachable** by Newton's method if and only if:

$$\boxed{|g'(x^*)| < 1 \quad \text{where} \quad g'(x) = \frac{f(x)f''(x)}{[f'(x)]^2}}$$

For a simple root (where $f(x^*) = 0$ and $f'(x^*) \neq 0$):

$$|g'(x^*)| = 0 < 1 \quad \Rightarrow \quad \boxed{\text{REACHABLE}}$$

For a multiple root of order m (where $f(x^*) = f'(x^*) = \cdots = f^{(m-1)}(x^*) = 0$ and $f^{(m)}(x^*) \neq 0$):

$$|g'(x^*)| = \frac{m-1}{m} < 1 \quad \Rightarrow \quad \boxed{\text{REACHABLE (but slow)}}$$

For pathological roots (where f' is undefined or behaves badly):

$$|g'(x^*)| \geq 1 \quad \Rightarrow \quad \boxed{\text{NOT REACHABLE}}$$

Basin of attraction

Even for reachable roots, convergence depends on initial condition x_0 being in the **basin of attraction**:

$$\mathcal{B}(x^*) = \{x_0 : \lim_{n\to\infty} x_n = x^* \text{ under Newton iteration}\}$$

The basin structure can be complex when multiple roots exist.

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7 Summary

Part (a): Fixed Points Are Roots

Starting from $x^* = g(x^*)$ where $g(x) = x - f(x)/f'(x)$:

$$x^* = x^* - \frac{f(x^*)}{f'(x^*)} \quad \Rightarrow \quad f(x^*) = 0$$

Conclusion: Fixed points of Newton map $\Leftrightarrow$ roots of f

Part (b): Stability Condition for Reachability

The derivative of the Newton map:

$$g'(x) = \frac{f(x)f''(x)}{[f'(x)]^2}$$

At a simple root ($f(x^*) = 0$, $f'(x^*) \neq 0$):

$$g'(x^*) = 0 < 1 \quad \Rightarrow \quad \text{STABLE (reachable)}$$

Reachability condition: A root is reachable if $|g'(x^*)| < 1$
This holds for:

- All simple roots: $g'(x^*) = 0$
- All multiple roots: $g'(x^*) = (m-1)/m < 1$

This fails when:

- $f'(x^*)$ vanishes or is undefined in pathological ways
- f has singularities near the root

Key insight: Newton's method is designed so roots are automatically stable fixed points (when f' is well-behaved), explaining its widespread success in practice.