Asymptotics 2025/2026 – Problem Sheet 3
Question 3: Watson's Lemma Application

Solution

Question 3

Problem: Use Watson's lemma to find an infinite asymptotic expansion of

$$I(X) = \int_1^\infty e^{-X(t^2+1)}\, dt.$$

Solution Strategy

Why this approach? Watson's lemma (Section 4.2.2 of lecture notes) applies to integrals of the form

$$\int_0^b f(t) e^{-Xt}\, dt$$

where $f(t)$ has an asymptotic expansion near $t = 0$. Our integral has limits $[1, \infty)$ and exponent $-X(t^2 + 1)$, so we must transform it into Watson's lemma standard form.

Step 1: Factor the Exponential

What we have:

$$I(X) = \int_1^\infty e^{-X(t^2+1)}\, dt$$

What we do: Factor $e^{-X(t^2+1)} = e^{-X} \cdot e^{-Xt^2}$:

$$I(X) = e^{-X} \int_1^\infty e^{-Xt^2}\, dt$$

Why? The factor e^{-X} is independent of the integration variable t. Factoring it out isolates the t-dependent exponential e^{-Xt^2}, which we'll address with a substitution. This prepares us to transform the integral into Watson's standard form.

Step 2: Change of Variable

What we have:

$$I(X) = e^{-X} \int_1^\infty e^{-Xt^2}\, dt$$

What we do: Substitute $s = t^2 - 1$, so $t^2 = s + 1$.

Why this substitution? We need the integral to start at 0 (Watson's lemma requirement) and the exponent to be linear in the new variable. Since the lower limit is $t = 1$, setting $s = t^2 - 1$ gives $s = 0$ when $t = 1$.

Computing the differential:

$$s = t^2 - 1$$
$$ds = 2t\, dt$$
$$dt = \frac{ds}{2t} = \frac{ds}{2\sqrt{s+1}}$$

Why this form? From $s = t^2 - 1$, we have $t = \sqrt{s+1}$ (taking positive root since $t \geq 1$). Thus $dt = ds/(2\sqrt{s+1})$.

Transforming limits:

- When $t = 1$: $s = 1^2 - 1 = 0$

- When $t \to \infty$: $s \to \infty$

Why check limits? Watson's lemma requires integration from 0 to some positive limit. We verify the transformation achieves this.

Step 3: Rewrite the Integral

What we have after substitution:

$$I(X) = e^{-X} \int_0^\infty e^{-X(s+1)} \cdot \frac{1}{2\sqrt{s+1}} \, ds$$

What we do: Factor the exponential:

$$I(X) = e^{-X} \cdot e^{-X} \int_0^\infty e^{-Xs} \cdot \frac{1}{2\sqrt{s+1}} \, ds$$

$$I(X) = e^{-2X} \int_0^\infty e^{-Xs} \cdot \frac{1}{2\sqrt{s+1}} \, ds$$

Why? We factor $e^{-X(s+1)} = e^{-Xs} \cdot e^{-X}$ to isolate e^{-Xs}, which is the required exponential form for Watson's lemma. The factor e^{-2X} (combining both e^{-X} terms) is pulled outside the integral.

Identifying Watson's lemma components:

$$I(X) = e^{-2X} \int_0^\infty f(s) e^{-Xs} \, ds$$

where

$$f(s) = \frac{1}{2\sqrt{s+1}} = \frac{1}{2}(1+s)^{-1/2}$$

Why identify $f(s)$? Watson's lemma requires $f(s)$ to have an asymptotic expansion as $s \to 0^+$. We've now achieved the standard form.

Step 4: Asymptotic Expansion of $f(s)$

What we need: An expansion of $f(s) = \frac{1}{2}(1+s)^{-1/2}$ as $s \to 0^+$.

What we do: Use the binomial series:

$$(1+s)^{-1/2} = \sum_{n=0}^\infty \binom{-1/2}{n} s^n$$

Why the binomial series? For $|s| < 1$, the generalized binomial theorem gives:

$$(1+s)^\alpha = \sum_{n=0}^\infty \binom{\alpha}{n} s^n$$

where $\binom{\alpha}{n} = \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}$.

Computing binomial coefficients:

$$\binom{-1/2}{n} = \frac{(-1/2)(-1/2-1)(-1/2-2)\cdots(-1/2-n+1)}{n!}$$

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$$= \frac{(-1/2)(-3/2)(-5/2) \cdots (-(2n-1)/2)}{n!}$$

$$= \frac{(-1)^n \cdot 1 \cdot 3 \cdot 5 \cdots (2n-1)}{2^n \cdot n!}$$

Why this form? Each factor in the numerator is $(-1/2 - k) = -(2k+1)/2$ for $k = 0, 1, \ldots, n-1$, giving us odd numbers with alternating sign.

Expressing with Gamma functions:

$$\binom{-1/2}{n} = \frac{(-1)^n}{2^n} \cdot \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{n!}$$

The product of odd numbers can be written as:

$$1 \cdot 3 \cdot 5 \cdots (2n-1) = \frac{(2n)!}{2^n \cdot n!}$$

Why? Because $(2n)! = 1 \cdot 2 \cdot 3 \cdots (2n) = [1 \cdot 3 \cdot 5 \cdots (2n-1)] \cdot [2 \cdot 4 \cdot 6 \cdots (2n)]$ and $2 \cdot 4 \cdots (2n) = 2^n \cdot n!$.

Therefore:

$$\binom{-1/2}{n} = \frac{(-1)^n}{2^n} \cdot \frac{(2n)!}{2^n \cdot n! \cdot n!} = \frac{(-1)^n (2n)!}{2^{2n} (n!)^2}$$

Using Gamma function notation:

$$\binom{-1/2}{n} = \frac{(-1)^n}{\sqrt{\pi}} \cdot \frac{\Gamma(n+1/2)}{n!}$$

Why Gamma functions? The identity $\Gamma(n+1/2) = \frac{(2n)!\sqrt{\pi}}{2^{2n}n!}$ (from lecture notes example 2.6.2) simplifies our coefficients.

Step 5: Apply Watson's Lemma

What we have:

$$f(s) = \frac{1}{2} \sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{\pi}} \cdot \frac{\Gamma(n+1/2)}{n!} s^n$$

Identifying Watson's lemma parameters:

- $\alpha = 0$ (no leading power of s)

- $\beta = 1$ (integer powers)

- $a_n = \frac{1}{2} \cdot \frac{(-1)^n}{\sqrt{\pi}} \cdot \frac{\Gamma(n+1/2)}{n!}$

Why these values? Watson's lemma (Eq. 177 in notes) assumes $f(t) \sim t^\alpha \sum_{n=0}^{\infty} a_n t^{n\beta}$. Our expansion has no leading power ($\alpha = 0$) and integer increments ($\beta = 1$).

Applying Watson's lemma formula (Eq. 177):

$$\int_0^b f(s)e^{-Xs}ds \sim \sum_{n=0}^{\infty} a_n \frac{\Gamma(\alpha + n\beta + 1)}{X^{\alpha + n\beta + 1}}$$

With our values ($\alpha = 0$, $\beta = 1$):

$$\int_0^\infty f(s)e^{-Xs}ds \sim \sum_{n=0}^{\infty} a_n \frac{\Gamma(n+1)}{X^{n+1}} = \sum_{n=0}^{\infty} a_n \frac{n!}{X^{n+1}}$$

Why $\Gamma(n+1) = n!$? This is a fundamental property of the Gamma function (Eq. 69 in notes).

Step 6: Substitute Coefficients

What we do:

$$\int_0^\infty f(s)e^{-Xs}ds \sim \sum_{n=0}^\infty \frac{1}{2\sqrt{\pi}} \cdot \frac{(-1)^n \Gamma(n+1/2)}{n!} \cdot \frac{n!}{X^{n+1}}$$

Simplifying:

$$= \frac{1}{2\sqrt{\pi}} \sum_{n=0}^\infty \frac{(-1)^n \Gamma(n+1/2)}{X^{n+1}}$$

Why does $n!$ cancel? The $n!$ in the denominator of a_n cancels with the $n! = \Gamma(n+1)$ from Watson's lemma formula.

Step 7: Final Result

Recalling the full expression:

$$I(X) = e^{-2X} \int_0^\infty f(s)e^{-Xs}ds$$

Therefore:

$$\boxed{I(X) \sim \frac{e^{-2X}}{2\sqrt{\pi}} \sum_{n=0}^\infty \frac{(-1)^n \Gamma(n+1/2)}{X^{n+1}} \quad \text{as } X \to \infty}$$

Alternative form with explicit first terms:
Computing $\Gamma(n+1/2)$ for first few terms:

$$\Gamma(1/2) = \sqrt{\pi}$$
$$\Gamma(3/2) = \frac{1}{2}\sqrt{\pi}$$
$$\Gamma(5/2) = \frac{3}{4}\sqrt{\pi}$$
$$\Gamma(7/2) = \frac{15}{8}\sqrt{\pi}$$

Why these values? Using $\Gamma(z+1) = z\Gamma(z)$ recursively: $\Gamma(3/2) = \frac{1}{2}\Gamma(1/2) = \frac{\sqrt{\pi}}{2}$, etc.

$$\boxed{I(X) \sim \frac{e^{-2X}}{2\sqrt{\pi}} \left[\frac{\sqrt{\pi}}{X} - \frac{\sqrt{\pi}/2}{X^2} + \frac{3\sqrt{\pi}/4}{X^3} - \frac{15\sqrt{\pi}/8}{X^4} + \cdots \right]}$$

$$\boxed{I(X) \sim \frac{e^{-2X}}{2} \left[\frac{1}{X} - \frac{1}{2X^2} + \frac{3}{4X^3} - \frac{15}{8X^4} + \cdots \right] \quad \text{as } X \to \infty}$$

Verification of Conditions

Why is Watson's lemma applicable?

1. **Convergence at origin:** $f(s) = \frac{1}{2\sqrt{1+s}}$ behaves as $s^{-1/2} \cdot s^{1/2} = O(1)$ near $s = 0$, so the integral converges (since $\alpha = 0 > -1$).

2. **Convergence at infinity:** The integrand $e^{-Xs}/\sqrt{1+s}$ decays exponentially as $s \to \infty$ for $X > 0$, ensuring convergence.

3. **Asymptotic sequence:** $\{X^{-(n+1)}\}$ is an asymptotic sequence as $X \to \infty$ since $X^{-(n+2)}/X^{-(n+1)} = 1/X \to 0$.