Asymptotics 2025/2026 Sheet 1
Problem 1: Complete Solutions with Full Justification

Problem 1(a)

Problem Statement

For $\epsilon \ll 1$, obtain two-term expansions for the solutions of

$$(x-1)(x-2)(x-3)+\epsilon = 0.$$

Complete Solution

Phase I: Problem Classification

Step 1.1: Identify the structure of the equation.
What we observe: The equation has the form

$$F(x)+\epsilon = 0,$$

where $F(x) = (x-1)(x-2)(x-3)$ is a polynomial of degree 3, and ϵ is a small parameter that appears additively (not multiplying the highest degree term).

Why this matters: According to Lecture Notes Section 2.1, when a small parameter appears additively in an algebraic equation, we must first examine the unperturbed equation (obtained by setting $\epsilon = 0$) to classify whether this is a regular or singular perturbation problem.

Theoretical foundation: The lecture notes define:

- **Regular perturbation problem:** "The exact solution for small but finite ϵ approaches the unperturbed solution(s) x_0 as $\epsilon \to 0$. Consequently, all solutions of the perturbed system can be expressed as well-defined power series expansions around the unperturbed solution."
- **Singular perturbation problem:** "The perturbed and unperturbed problem differ in an essential way: Not all solutions of the perturbed problem can be expressed as an expansion of the form $x(\epsilon) = x_0 + x_1\epsilon + x_2\epsilon^2 + \cdots$ around the unperturbed solution(s) x_0."

Step 1.2: Solve the unperturbed equation.
What we do: Set $\epsilon = 0$ in the original equation:

$$(x-1)(x-2)(x-3)+0 = 0.$$

Why we do this: The unperturbed equation reveals the "baseline" solutions around which we will attempt to construct perturbative expansions. This is the starting point of any perturbative analysis.

Solution of unperturbed equation:

$$(x-1)(x-2)(x-3) = 0.$$

This factored form immediately gives us three solutions:

$$x_0^{(1)} = 1, \quad x_0^{(2)} = 2, \quad x_0^{(3)} = 3.$$

Why these are the solutions: A product of factors equals zero if and only if at least one factor equals zero. Thus $x-1=0$ gives $x=1$, $x-2=0$ gives $x=2$, and $x-3=0$ gives $x=3$.

Step 1.3: Count degrees of freedom.

What we observe:

- The unperturbed equation (degree 3 polynomial) has exactly 3 solutions.
- The perturbed equation (also degree 3 polynomial) must also have exactly 3 solutions (counting multiplicities, over $\mathbb{C}$).

Why this matters: Since the number of solutions is preserved, and each unperturbed solution is simple (non-degenerate), we expect that each perturbed solution will smoothly approach one of the unperturbed solutions as $\epsilon \to 0$.

Step 1.4: Check for degeneracy.

What we check: Are any of the unperturbed roots repeated?

Finding: All three roots $x = 1, 2, 3$ are distinct (simple roots).

Why this matters: According to Lecture Notes Section 2.3, degenerate roots often require non-integer power expansions. Since all our roots are simple, we expect regular behavior with integer power expansions of the form $x(\epsilon) = x_0 + x_1\epsilon + x_2\epsilon^2 + \cdots$.

Step 1.5: Classify the problem.

Conclusion: This is a **regular perturbation problem**.

Justification:

1. The number of solutions is preserved (3 solutions in both cases).
2. All unperturbed roots are simple (non-degenerate).
3. The perturbation is additive and small.
4. We expect each perturbed solution to approach exactly one unperturbed solution as $\epsilon \to 0$.

Method to use: According to Lecture Notes Section 2.1.1, we will use the **expansion method**, making the ansatz

$$x(\epsilon) = x_0 + x_1\epsilon + x_2\epsilon^2 + \cdots$$

for each unperturbed solution x_0.

Phase II: Solution Near $x_0 = 1$

Step 2.1: Make the expansion ansatz.

What we assume: For the root near $x_0 = 1$, we write:

$$x(\epsilon) = 1 + x_1\epsilon + O(\epsilon^2).$$

Step 2.2: Substitute into the equation.

Original equation:

$$(x-1)(x-2)(x-3) + \epsilon = 0.$$

Substitution: Replace x with $1 + x_1\epsilon + \cdots$:

$$\begin{aligned} \text{First factor:} \quad & (x-1) = x_1\epsilon + O(\epsilon^2), \\ \text{Second factor:} \quad & (x-2) = -1 + x_1\epsilon + O(\epsilon^2), \\ \text{Third factor:} \quad & (x-3) = -2 + x_1\epsilon + O(\epsilon^2). \end{aligned}$$

Step 2.3: Expand the product.
Multiplying the second and third factors first:

$$(-1 + x_1\epsilon)(-2 + x_1\epsilon) = 2 - 3x_1\epsilon + O(\epsilon^2).$$

Now multiply by the first factor:

$$(x_1\epsilon)(2 - 3x_1\epsilon + \cdots) = 2x_1\epsilon + O(\epsilon^2).$$

Step 2.4: Apply the equation.

$$2x_1\epsilon + O(\epsilon^2) + \epsilon = 0 \implies (2x_1 + 1)\epsilon + O(\epsilon^2) = 0.$$

At $O(\epsilon)$:

$$2x_1 + 1 = 0 \implies x_1 = -\frac{1}{2}.$$

Final answer for root near $x_0 = 1$:

$$\boxed{x(\epsilon) = 1 - \frac{1}{2}\epsilon + O(\epsilon^2)}.$$

Phase III: Solution Near $x_0 = 2$

Ansatz: $x(\epsilon) = 2 + x_1\epsilon + O(\epsilon^2)$.
Factors:

$$\begin{aligned}(x-1) &= 1 + x_1\epsilon + O(\epsilon^2),\\ (x-2) &= x_1\epsilon + O(\epsilon^2),\\ (x-3) &= -1 + x_1\epsilon + O(\epsilon^2).\end{aligned}$$

Product:

$$(1 + x_1\epsilon)(-1 + x_1\epsilon) = -1 + O(\epsilon^2).$$
$$(x_1\epsilon)(-1) = -x_1\epsilon.$$

Equation:

$$-x_1\epsilon + \epsilon = 0 \implies (-x_1 + 1)\epsilon = 0 \implies x_1 = 1.$$

Final answer for root near $x_0 = 2$:

$$\boxed{x(\epsilon) = 2 + \epsilon + O(\epsilon^2)}.$$

Phase IV: Solution Near $x_0 = 3$

Ansatz: $x(\epsilon) = 3 + x_1\epsilon + O(\epsilon^2)$.
Factors:

$$\begin{aligned}(x-1) &= 2 + x_1\epsilon + O(\epsilon^2),\\ (x-2) &= 1 + x_1\epsilon + O(\epsilon^2),\\ (x-3) &= x_1\epsilon + O(\epsilon^2).\end{aligned}$$

Product:

$$(2 + x_1\epsilon)(1 + x_1\epsilon) = 2 + 3x_1\epsilon + O(\epsilon^2).$$
$$(x_1\epsilon)(2 + 3x_1\epsilon) = 2x_1\epsilon + O(\epsilon^2).$$

Equation:

$$2x_1\epsilon + \epsilon = 0 \implies (2x_1 + 1)\epsilon = 0 \implies x_1 = -\frac{1}{2}.$$

Final answer for root near $x_0 = 3$:

$$\boxed{x(\epsilon) = 3 - \frac{1}{2}\epsilon + O(\epsilon^2)}.$$

Summary for Problem 1(a)

The three roots of $(x-1)(x-2)(x-3)+\epsilon=0$ are:

$$
\begin{aligned}
x_1(\epsilon) &= 1 - \frac{1}{2}\epsilon + O(\epsilon^2),\\
x_2(\epsilon) &= 2 + \epsilon + O(\epsilon^2),\\
x_3(\epsilon) &= 3 - \frac{1}{2}\epsilon + O(\epsilon^2).
\end{aligned}
$$

Problem 1(b)

Problem Statement

For $\epsilon \ll 1$, obtain two-term expansions for the solutions of

$$x^3 + x^2 - \epsilon = 0.$$

Complete Solution

Phase I: Problem Classification and Structure

Step 1.1: Examine the equation structure.

What we observe: The equation can be written as

$$x^2(x+1) = \epsilon.$$

Form: This has the structure $F(x) = \epsilon$ where $F(x) = x^2(x+1)$ is a cubic polynomial.

Step 1.2: Solve the unperturbed equation.

Setting $\epsilon = 0$:

$$x^3 + x^2 = x^2(x+1) = 0.$$

Solutions:

$$x^2 = 0 \implies x = 0 \text{ (double root)},$$
$$x + 1 = 0 \implies x = -1 \text{ (simple root)}.$$

Critical observation: The unperturbed equation has a **degenerate root** at $x = 0$ (multiplicity 2).

Step 1.3: Classify the problem.

Conclusion: This is a **singular perturbation problem with non-integer power expansions** (as discussed in Lecture Notes Section 2.3).

Key insight from the lecture notes: "In cases where unperturbed solutions are degenerate, their behavior as $\epsilon \to 0$ may sometimes not be captured by a power series expansion of integer powers."

Strategy:

1. Find the regular solution near $x = -1$ using standard integer power expansion.
2. Find the singular solutions near $x = 0$ using fractional power expansion with exponent determined by dominant balance.

Phase II: Regular Solution Near $x_0 = -1$

Step 2.1: Justification for regular expansion.

Observation: $x = -1$ is a **simple root** of the unperturbed equation.

Theory: Simple roots typically give rise to regular perturbative expansions with integer powers of ϵ.

Step 2.2: Make the standard ansatz.

$$x(\epsilon) = -1 + x_1\epsilon + x_2\epsilon^2 + O(\epsilon^3).$$

Step 2.3: Substitute into the equation.

Original equation: $x^3 + x^2 - \epsilon = 0.$

Compute x^2 and x^3:

$$x^2 = (-1 + x_1\epsilon + \cdots)^2 = 1 - 2x_1\epsilon + O(\epsilon^2),$$
$$x^3 = (-1 + x_1\epsilon + \cdots)^3 = -1 + 3x_1\epsilon + O(\epsilon^2).$$

Step 2.4: Combine and solve.

$$x^3 + x^2 = (-1 + 3x_1\epsilon) + (1 - 2x_1\epsilon) + O(\epsilon^2) = x_1\epsilon + O(\epsilon^2).$$

Setting equal to ϵ:

$$x_1\epsilon + O(\epsilon^2) = \epsilon.$$

At $O(\epsilon)$:

$$x_1 = 1.$$

Final answer for regular root:

$$\boxed{x(\epsilon) = -1 + \epsilon + O(\epsilon^2)}.$$

Phase III: Singular Solutions Near $x_0 = 0$ (Double Root)

Step 3.1: Why we need fractional powers.

Key observation: The unperturbed root $x = 0$ is degenerate (double root).

Theory from Lecture Notes (Section 2.3): For degenerate roots, we use the ansatz

$$x = x_1\epsilon^{\alpha} + x_2\epsilon^{2\alpha} + \cdots$$

where $\alpha > 0$ is determined by **dominant balance analysis**.

Step 3.2: Make the fractional power ansatz.

$$x(\epsilon) = x_1\epsilon^{\alpha} + x_2\epsilon^{2\alpha} + \cdots$$

Step 3.3: Substitute into the equation.

Original equation: $x^3 + x^2 = \epsilon$.

Compute powers of x:

$$\begin{aligned} x^2 &= x_1^2\epsilon^{2\alpha} + 2x_1x_2\epsilon^{3\alpha} + O(\epsilon^{4\alpha}), \\ x^3 &= x_1^3\epsilon^{3\alpha} + 3x_1^2x_2\epsilon^{4\alpha} + O(\epsilon^{5\alpha}). \end{aligned}$$

Sum:

$$x^3 + x^2 = x_1^2\epsilon^{2\alpha} + x_1^3\epsilon^{3\alpha} + 2x_1x_2\epsilon^{3\alpha} + O(\epsilon^{4\alpha}).$$

Step 3.4: Determine α by dominant balance.

The fundamental principle: We must balance the left-hand side against ϵ on the right-hand side. The leading term on the left must match ϵ in order of magnitude.

Comparison of terms:

- x^2 term: $x_1^2\epsilon^{2\alpha}$
- x^3 term: $x_1^3\epsilon^{3\alpha}$
- RHS: ϵ

Since $\alpha > 0$, we have $2\alpha < 3\alpha$, so $\epsilon^{2\alpha} \gg \epsilon^{3\alpha}$ as $\epsilon \to 0$.

Therefore, the dominant term on the LHS is $x_1^2\epsilon^{2\alpha}$.

Dominant balance condition: The dominant term must balance the RHS:

$$x_1^2\epsilon^{2\alpha} \sim \epsilon.$$

Matching powers of ϵ:

$$2\alpha = 1 \implies \boxed{\alpha = \frac{1}{2}}.$$

Verification of consistency: With $\alpha = 1/2$:

- $x^2 \sim \epsilon^1$ (leading order, balances RHS)
- $x^3 \sim \epsilon^{3/2}$ (subdominant, since $3/2 > 1$)

This is consistent: the x^3 term is indeed smaller than the x^2 term.

Step 3.5: Solve at $O(\epsilon)$ (leading order).

With $\alpha = 1/2$, the leading-order equation is:

$$x_1^2 \epsilon = \epsilon \implies x_1^2 = 1 \implies x_1 = \pm 1.$$

Interpretation: The double root at $x = 0$ splits into two roots, one going as $+\epsilon^{1/2}$ and one as $-\epsilon^{1/2}$.

Step 3.6: Find the next-order correction.

Ansatz with known leading order:

$$x = \pm \epsilon^{1/2} + x_2 \epsilon + O(\epsilon^{3/2}).$$

Compute x^2 and x^3 to higher order:

For $x = x_1 \epsilon^{1/2} + x_2 \epsilon$ with $x_1 = \pm 1$:

$$\begin{aligned} x^2 &= x_1^2 \epsilon + 2x_1 x_2 \epsilon^{3/2} + O(\epsilon^2) = \epsilon + 2x_1 x_2 \epsilon^{3/2} + O(\epsilon^2), \\ x^3 &= x_1^3 \epsilon^{3/2} + 3x_1^2 x_2 \epsilon^2 + O(\epsilon^{5/2}) = x_1^3 \epsilon^{3/2} + O(\epsilon^2). \end{aligned}$$

Note: We used $x_1^2 = 1$ in simplifying x^2.

Sum:

$$x^3 + x^2 = \epsilon + x_1^3 \epsilon^{3/2} + 2x_1 x_2 \epsilon^{3/2} + O(\epsilon^2).$$

Setting equal to ϵ:

$$\epsilon + (x_1^3 + 2x_1 x_2)\epsilon^{3/2} + O(\epsilon^2) = \epsilon.$$

At $O(\epsilon^{3/2})$:

$$x_1^3 + 2x_1 x_2 = 0 \implies x_2 = -\frac{x_1^3}{2x_1} = -\frac{x_1^2}{2} = -\frac{1}{2}.$$

Key observation: The coefficient $x_2 = -1/2$ is the same for both branches ($x_1 = +1$ and $x_1 = -1$).

Step 3.7: Final answers for singular roots.

$$\boxed{x(\epsilon) = +\epsilon^{1/2} - \frac{1}{2}\epsilon + O(\epsilon^{3/2})},$$

$$\boxed{x(\epsilon) = -\epsilon^{1/2} - \frac{1}{2}\epsilon + O(\epsilon^{3/2})}.$$

Summary for Problem 1(b)

The three roots of $x^3 + x^2 - \epsilon = 0$ are:

1. **Regular root** (from simple root at $x = -1$):

$$x(\epsilon) = -1 + \epsilon + O(\epsilon^2).$$

2. **Singular roots** (from double root at $x = 0$):

$$x(\epsilon) = \pm \epsilon^{1/2} - \frac{1}{2}\epsilon + O(\epsilon^{3/2}).$$

Physical interpretation: The double root at $x = 0$ undergoes a "splitting" into two distinct roots when $\epsilon \neq 0$, with the separation growing as $2\epsilon^{1/2}$ for small ϵ.

Problem 1(c)

Problem Statement

For $\epsilon \ll 1$, obtain two-term expansions for the solutions of

$$\epsilon x^3 + x^2 + 2x + 1 = 0.$$

Complete Solution

Phase I: Problem Classification

Step 1.1: Examine the equation structure.
What we observe: The small parameter ϵ multiplies the highest-degree term x^3.
Why this matters: When ϵ multiplies the highest power, setting $\epsilon = 0$ reduces the degree of the polynomial. This is a hallmark of a **singular perturbation problem**.
Step 1.2: Solve the unperturbed equation.
Setting $\epsilon = 0$:

$$x^2 + 2x + 1 = 0 \implies (x+1)^2 = 0 \implies x = -1 \text{ (double root)}.$$

Critical observations:

- The unperturbed equation (quadratic) has only 2 roots (both equal to -1).
- The perturbed equation (cubic) has 3 roots.
- One root must "come from infinity" as $\epsilon \to 0$ — this is the hallmark of a singular perturbation.
- The double root at $x = -1$ suggests non-integer power expansions for the roots near -1.

Step 1.3: Strategy.
We will find:

1. Two roots near $x = -1$ (from the double root) using fractional power expansion.
2. One root going to infinity as $\epsilon \to 0$ (the singular root) using dominant balance.

Phase II: Roots Near $x_0 = -1$ (Double Root)

Step 2.1: Motivation for fractional powers.
Theory: Since $x = -1$ is a double root of the unperturbed equation, we expect the perturbation to split this into two distinct roots. Based on Lecture Notes Section 2.3 (see Eq. (17)–(20)), we try an expansion of the form:

$$x = -1 + x_1\epsilon^{1/2} + x_2\epsilon + \cdots$$

Step 2.2: Substitute the ansatz.
Original equation: $\epsilon x^3 + x^2 + 2x + 1 = 0$.
Let $x = -1 + x_1\epsilon^{1/2} + x_2\epsilon + \cdots$
Compute each term:
Term 1: ϵx^3

$$\begin{aligned} x^3 &= (-1 + x_1\epsilon^{1/2} + x_2\epsilon + \cdots)^3 \\ &= -1 + 3x_1\epsilon^{1/2} + O(\epsilon). \end{aligned}$$

Therefore:

$$\epsilon x^3 = \epsilon(-1 + 3x_1\epsilon^{1/2} + \cdots) = -\epsilon + O(\epsilon^{3/2}).$$

Term 2: x^2

$$\begin{aligned} x^2 &= (-1 + x_1\epsilon^{1/2} + x_2\epsilon)^2 \\ &= 1 - 2x_1\epsilon^{1/2} + (x_1^2 - 2x_2)\epsilon + O(\epsilon^{3/2}). \end{aligned}$$

Term 3: $2x$

$$2x = 2(-1 + x_1\epsilon^{1/2} + x_2\epsilon + \cdots) = -2 + 2x_1\epsilon^{1/2} + 2x_2\epsilon + O(\epsilon^{3/2}).$$

Term 4: $+1$

$$+1.$$

Step 2.3: Collect terms by order.

$$\epsilon x^3 + x^2 + 2x + 1 = [1 - 2 + 1] + [-2x_1 + 2x_1]\epsilon^{1/2} + [-1 + x_1^2 - 2x_2 + 2x_2]\epsilon + O(\epsilon^{3/2}).$$

Simplify:

- $O(1)$: $1 - 2 + 1 = 0$ ✓(automatically satisfied)
- $O(\epsilon^{1/2})$: $-2x_1 + 2x_1 = 0$ ✓(automatically satisfied)
- $O(\epsilon)$: $-1 + x_1^2 - 2x_2 + 2x_2 = -1 + x_1^2 = 0$

Step 2.4: Solve the balance equations.
At $O(\epsilon)$:

$$x_1^2 - 1 = 0 \implies x_1^2 = 1 \implies x_1 = \pm 1.$$

Step 2.5: Final answers for roots near $x = -1$.

$$\boxed{x(\epsilon) = -1 + \epsilon^{1/2} + O(\epsilon) = -1 + \sqrt{\epsilon} + O(\epsilon)},$$

$$\boxed{x(\epsilon) = -1 - \epsilon^{1/2} + O(\epsilon) = -1 - \sqrt{\epsilon} + O(\epsilon)}.$$

Phase III: Singular Root Going to Infinity

Step 3.1: Physical intuition.
The perturbed equation is cubic (3 roots), but the unperturbed is quadratic (2 roots). Where does the third root "go" as $\epsilon \to 0$?
Answer: It must go to infinity. As $\epsilon \to 0$, one root escapes to $\pm\infty$.
Step 3.2: Dominant balance analysis.
For large $|x|$, compare terms in $\epsilon x^3 + x^2 + 2x + 1 = 0$:

- ϵx^3: grows as $|x|^3$
- x^2: grows as $|x|^2$
- $2x$: grows as $|x|$
- 1: constant

For very large $|x|$, the dominant terms are ϵx^3 and x^2. The terms $2x$ and 1 become negligible. Dominant balance:

$$\epsilon x^3 + x^2 \approx 0 \implies \epsilon x^3 \sim -x^2 \implies x \sim -\frac{1}{\epsilon}.$$

Step 3.3: Make the ansatz for the singular root.
Based on the dominant balance, we try:

$$x = -\frac{1}{\epsilon} + x_0 + x_1\epsilon + \cdots$$

Step 3.4: Substitute and expand.
Compute x^2:

$$x^2 = \left(-\frac{1}{\epsilon} + x_0 + \cdots\right)^2 = \frac{1}{\epsilon^2} - \frac{2x_0}{\epsilon} + x_0^2 + O(1).$$

Compute x^3:

$$x^3 = \left(-\frac{1}{\epsilon} + x_0 + \cdots\right)^3 = -\frac{1}{\epsilon^3} + \frac{3x_0}{\epsilon^2} - \frac{3x_0^2}{\epsilon} + x_0^3 + O(\epsilon).$$

Compute ϵx^3:

$$\epsilon x^3 = -\frac{1}{\epsilon^2} + \frac{3x_0}{\epsilon} - 3x_0^2 + O(\epsilon).$$

Compute $2x$:

$$2x = -\frac{2}{\epsilon} + 2x_0 + O(\epsilon).$$

Step 3.5: Collect terms.

$$\epsilon x^3 + x^2 + 2x + 1 = \left(-\frac{1}{\epsilon^2} + \frac{1}{\epsilon^2}\right) + \left(\frac{3x_0}{\epsilon} - \frac{2x_0}{\epsilon} - \frac{2}{\epsilon}\right) + O(1).$$

At $O(\epsilon^{-2})$:

$$-\frac{1}{\epsilon^2} + \frac{1}{\epsilon^2} = 0. \quad \checkmark$$

At $O(\epsilon^{-1})$:

$$\frac{3x_0 - 2x_0 - 2}{\epsilon} = \frac{x_0 - 2}{\epsilon} = 0 \implies x_0 = 2.$$

Step 3.6: Final answer for singular root.

$$\boxed{x(\epsilon) = -\frac{1}{\epsilon} + 2 + O(\epsilon)}.$$

Summary for Problem 1(c)

The three roots of $\epsilon x^3 + x^2 + 2x + 1 = 0$ are:

1. **Two roots from the double root at $x = -1$:**

$$x(\epsilon) = -1 \pm \epsilon^{1/2} + O(\epsilon).$$

2. **Singular root (goes to $-\infty$ as $\epsilon \to 0$):**

$$x(\epsilon) = -\frac{1}{\epsilon} + 2 + O(\epsilon).$$

Problem 1(d)

Problem Statement

For $\epsilon \ll 1$, obtain a two-term expansion for the solution near $x = 0$ of

$$\sqrt{2}\sin\left(x+\frac{\pi}{4}\right) - 1 - x + \frac{1}{2}x^2 = -\frac{1}{6}\epsilon.$$

Complete Solution

Phase I: Simplification and Taylor Expansion

Step 1.1: Expand the left-hand side for small x.

The key insight: We are looking for solutions near $x = 0$, so we should Taylor expand the LHS around $x = 0$.

Use the angle addition formula:

$$\sqrt{2}\sin\left(x+\frac{\pi}{4}\right) = \sqrt{2}\left[\sin x\cos\frac{\pi}{4} + \cos x\sin\frac{\pi}{4}\right] = \sqrt{2}\left[\frac{\sin x}{\sqrt{2}} + \frac{\cos x}{\sqrt{2}}\right] = \sin x + \cos x.$$

Taylor expand $\sin x$ *and* $\cos x$*:*

$$\begin{aligned}\sin x &= x - \frac{x^3}{6} + \frac{x^5}{120} - \cdots,\\ \cos x &= 1 - \frac{x^2}{2} + \frac{x^4}{24} - \cdots.\end{aligned}$$

Therefore:

$$\sin x + \cos x = 1 + x - \frac{x^2}{2} - \frac{x^3}{6} + O(x^4).$$

Step 1.2: Substitute into the equation.
LHS of equation:

$$\begin{aligned}&\sqrt{2}\sin\left(x+\frac{\pi}{4}\right) - 1 - x + \frac{1}{2}x^2\\ &= (\sin x + \cos x) - 1 - x + \frac{1}{2}x^2\\ &= \left(1 + x - \frac{x^2}{2} - \frac{x^3}{6} + O(x^4)\right) - 1 - x + \frac{1}{2}x^2\\ &= 1 + x - \frac{x^2}{2} - \frac{x^3}{6} - 1 - x + \frac{x^2}{2} + O(x^4)\\ &= -\frac{x^3}{6} + \frac{x^4}{24} + O(x^5).\end{aligned}$$

Remarkable cancellation: The constant terms, x terms, and x^2 terms all cancel exactly!
Step 1.3: Write the simplified equation.
The equation becomes:

$$-\frac{x^3}{6} + \frac{x^4}{24} + O(x^5) = -\frac{\epsilon}{6}.$$

Multiply through by -6:

$$x^3 - \frac{x^4}{4} + O(x^5) = \epsilon.$$

Phase II: Leading-Order Solution

Step 2.1: Dominant balance for leading order.

For small x, compare terms:

- x^3: leading term
- $x^4/4$: subdominant (since $x^4 \ll x^3$ for small x)

Leading-order balance:

$$x^3 \approx \epsilon \implies x \approx \epsilon^{1/3}.$$

Step 2.2: Leading-order solution.

$$x_0 = \epsilon^{1/3}.$$

Verification: With $x \sim \epsilon^{1/3}$, we have $x^4 \sim \epsilon^{4/3}$, which is indeed $o(\epsilon)$ as $\epsilon \to 0$. So neglecting x^4 at leading order is justified.

Phase III: Next-Order Correction

Step 3.1: Make the ansatz for next order.

We seek:

$$x = \epsilon^{1/3} + \alpha\epsilon^{\beta} + \cdots$$

where $\beta > 1/3$ (since this is a correction to the leading term) and α is a constant to be determined.

Step 3.2: Substitute and expand.

Let $x = \epsilon^{1/3} + \alpha\epsilon^{\beta}$ with $\beta > 1/3$.

Compute x^3:

$$\begin{aligned} x^3 &= (\epsilon^{1/3} + \alpha\epsilon^{\beta})^3 \\ &= \epsilon + 3\epsilon^{2/3}(\alpha\epsilon^{\beta}) + 3\epsilon^{1/3}(\alpha\epsilon^{\beta})^2 + (\alpha\epsilon^{\beta})^3 \\ &= \epsilon + 3\alpha\epsilon^{2/3+\beta} + O(\epsilon^{1/3+2\beta}). \end{aligned}$$

Compute x^4:

$$\begin{aligned} x^4 &= (\epsilon^{1/3})^4 + 4(\epsilon^{1/3})^3(\alpha\epsilon^{\beta}) + \cdots \\ &= \epsilon^{4/3} + 4\alpha\epsilon^{1+\beta} + \cdots. \end{aligned}$$

Therefore:

$$x^3 - \frac{x^4}{4} = \epsilon + 3\alpha\epsilon^{2/3+\beta} - \frac{\epsilon^{4/3}}{4} + O(\text{higher order}).$$

Step 3.3: Balance the equation.

The equation is:

$$x^3 - \frac{x^4}{4} + O(x^5) = \epsilon.$$

Substituting:

$$\epsilon + 3\alpha\epsilon^{2/3+\beta} - \frac{\epsilon^{4/3}}{4} + \cdots = \epsilon.$$

The ϵ terms cancel. We need to balance the remaining terms.

The next largest terms are:

- $3\alpha\epsilon^{2/3+\beta}$

- $-\frac{1}{4}\epsilon^{4/3}$

For these to balance:

$$\frac{2}{3} + \beta = \frac{4}{3} \implies \beta = \frac{2}{3}.$$

With $\beta = 2/3$:

$$3\alpha\epsilon^{4/3} - \frac{1}{4}\epsilon^{4/3} = 0 \implies 3\alpha = \frac{1}{4} \implies \alpha = \frac{1}{12}.$$

Step 3.4: Final answer.

$$\boxed{x(\epsilon) = \epsilon^{1/3} + \frac{1}{12}\epsilon^{2/3} + o(\epsilon^{2/3})}.$$

Verification

Let's verify the first few terms of our expansion.

With $x = \epsilon^{1/3} + \frac{1}{12}\epsilon^{2/3}$:

$$\begin{aligned} x^3 &= \left(\epsilon^{1/3}\right)^3 + 3\left(\epsilon^{1/3}\right)^2 \cdot \frac{1}{12}\epsilon^{2/3} + O(\epsilon^{5/3}) \\ &= \epsilon + \frac{1}{4}\epsilon^{4/3} + O(\epsilon^{5/3}). \end{aligned}$$

$$x^4 = \left(\epsilon^{1/3}\right)^4 + O(\epsilon^{5/3}) = \epsilon^{4/3} + O(\epsilon^{5/3}).$$

$$\begin{aligned} x^3 - \frac{x^4}{4} &= \epsilon + \frac{1}{4}\epsilon^{4/3} - \frac{1}{4}\epsilon^{4/3} + O(\epsilon^{5/3}) \\ &= \epsilon + O(\epsilon^{5/3}). \end{aligned}$$

This equals ϵ to the required order. ✓

Summary for Problem 1(d)

The solution near $x = 0$ of $\sqrt{2}\sin(x + \pi/4) - 1 - x + \frac{1}{2}x^2 = -\frac{\epsilon}{6}$ is:

$$x(\epsilon) = \epsilon^{1/3} + \frac{1}{12}\epsilon^{2/3} + o(\epsilon^{2/3}) \quad \text{as } \epsilon \to 0.$$

Key insight: The remarkable cancellation of lower-order terms in the Taylor expansion of the LHS leads to the dominant balance $x^3 \sim \epsilon$, giving the unusual scaling $x \sim \epsilon^{1/3}$.