Appendix

The last few pages below contain important notions you may want to be familiar with if you're going to model or study nonlinear dynamical systems. But we have run out of time in our course, so please take a look at these topics for future reference, but you won't be tested on them in this course.

A Some other useful notions

There are many other techniques people have invented to characterize the dynamics of systems, but we've covered the most fundamental. Others include:

A.1 Invariants of the motion

An invariant is any quantity that doesn't change during motion, that is, any quantity $V(x,t)$ such that

$$\dot{V}(x,t) = 0 \qquad \Rightarrow \qquad V(x,t) = \text{constant} \; . \tag{A.1}$$

- These are useful because $V(x,t)$ =constant is then often an algebraic equation (much simpler than the original ODE).
- E.g. in mechanics the total energy (kinetic + potential + elastic) is constant. It is a *constant* or *invariant of the motion.* Take a particle of mass m falling under gravitational acceleration g, with height x and speed $y = \dot{x}$, we have

  $$\dot{x} = y \qquad \dot{y} = -g \tag{A.2}$$

  which has a total energy

  $$\begin{aligned} & E = \text{KE} + \text{PE} = \tfrac{1}{2}my^2 + mgx \\ \Rightarrow \qquad & \dot{E} = my\dot{y} + mg\dot{x} = -myg + mgy = 0 \end{aligned} \tag{A.3}$$

  hence E is an invariant of the motion.

  These are also known as *conservative* systems (usually in physics), because they conserve some quantity.
- An important class are Hamiltonian systems, for which we can define a conserved energy and momentum.

A.2 Decreasing (Lyapunov) functions

Sometimes we can find scalar quantities that strictly decrease in the flow, that is, a quantity $V(x,t)$ such that

$$\dot{V}(x,t) < 0 \tag{A.4}$$

for any x, t.

- E.g. consider a simple focus

  $$\dot{x} = ax + y \qquad \dot{y} = ay - x \tag{A.5}$$

  which is stable for $a < 0$. A Lyapunov function for this is $V = \frac{1}{2}(x^2 + y^2)$, since

  $$\dot{V} = x\dot{x} + y\dot{y} = x(ax + y) + y(ay - x) = a(x^2 + y^2) < 0 \tag{A.6}$$

  for $a < 0$.

- Because V is strictly decreasing, if we plot the contours of V in our space (and this is useful for systems of any dimension), we know that solutions are always traveling down the gradient of V, inward across its contours.

A.3 Multiple timescales

Large and small quantities in a system can do strange things.

- If a particular constant is very small, say $\dot{x} = x + ay$, $\dot{y} = -y$, with $|a| \ll 1$, then the term it appears in usually won't have much affect and we can approximate $\dot{x} \approx x$, $\dot{y} \approx -y$.

- If a particular constant is very large then the term it appears in will usually dominate things, say $\dot{x} = x + ay$, $\dot{y} = -y$, with $|a| \gg 1$, then we can approximate $\dot{x} \approx ay$, $\dot{y} \approx -y$.

- If a small constant multiplies the highest order term in a system — the highest order power like x^n or highest order derivative like $\frac{d}{dt}$ in an ODE — then we have to be much more careful.

  E.g. $x = ax^2$ doesn't look like $x \approx 0$ when a is small. Solve it, and we see it has a pair of very different roots $x = 0$ and $x = 1/a$, the latter of which is very large when a is small (and we would miss this last one with the approximation $x \approx 0$).

  E.g. $a\dot{x} = x$ doesn't look like $x \approx 0$ when a is small. Solve it to get $x = x_0 e^{t/a}$, and we see that $x = 0$ is one special solution for $x_0 = 0$, but most orbits evolve very quickly as $x = x_0 e^{t/a}$ either towards (if $a < 0$) or away (if $a > 0$) from $x = 0$.

The key to systems like this is scaling. Take the system $a\dot{x} = x$ for small a.

- Take the extreme limit $a \to 0$, giving $x = 0$. This is clearly a valid solution of the system. It is the solution of the system on its natural t timescale. We often call this the **slow** timescale in such cases, for the following reason.

- Now try changing the timescale first. Let $\tau = t/a$, then $a\dot{x} = a\frac{dx}{dt} = \frac{dx}{d\tau}$, so the system becomes $\frac{dx}{d\tau} = x$, with simple solution $x = x_0 e^{\tau}$, that of a simple equilibrium. But one unit of time in τ is $1/a$ units of time in t, which is large if a is small, so τ is a **fast** timescale.

- So on the fast $\tau = t/a$ timescale $a\dot{x} = x$ looks like a simple equilibrium for small a. This fast dynamics quickly evolves to or from $x = 0$. On the slow timescale, when the system sits at or near $x = 0$, it simply behaves like the trivial $x = 0$.

Setting $a = 0$ and only looking at the slow system is also called *coarse graining*. Things get more interesting with multiple variables. A general model in two variables is

$$\dot{x} = f(x, y) \qquad\qquad a\dot{y} = g(x, y) \tag{A.7}$$

for small a.

- The $\dot{x}$ term doesn't involve any small quantity and is *slow*. If we set $a \approx 0$ we see it satisfies a *differential-algebraic equation*, that is, an ODE $\dot{x} = f$, combined with an invariant $g \approx 0$, since

  $$\dot{x} = f(x, y) \qquad\qquad 0 \approx g(x, y) \tag{A.8}$$

  This is known as the slow or **reduced** subsystem, or the coarse-grained system (depending on application).

- The $\dot{y}$ term will usually be very large because $\dot{y} = g/a$ is large for small a (except where $g = 0$, but that is dealt with by the reduced system above). But scale time to $\tau = t/a$ and we get

  $$\tfrac{d}{d\tau}x = af(x, y) \approx 0 \qquad\qquad \tfrac{d}{d\tau}\dot{y} = g(x, y) \tag{A.9}$$

  This is known as the fast or **layer** subsystem. We see that the variable x is slow, since on this fast timescale it is constant, $\dot{x} \approx 0$. This subsystem then tells us how y varies, on its fast timescale, outside of $g = 0$. The equilibria of the fast dynamics for y in (A.9) are just the invariants (the $g = 0$ equation) of the slow dynamics in (A.8).

- (Strictly the terms 'reduced' and 'layer' refer to the exact equations above obtained in the limit $a \to 0$).

A.4 Nonsmooth dynamics

All of the stuff above concerns dynamical systems defined by differentiable equations, e.g. $\dot{x} = f(x)$ or $x_{n+1} = f(x_n)$ where f is differentiable in x.

- Many of these results apply if f is at least continuous in x, e.g. $f(x) = |x| + a$ is continuous but not differentiable in x.

  In many cases the theory is harder to prove in these cases, but equilibria turn out to be much the same. Bifurcations can be much more involved. Chaos is much the same. The powerful Sharkovskii ordering, for example, is true provided f is continuous.

- We believed until recently, and nearly a century of theory suggested, that almost none of these results still applied if f is discontinuous, e.g. $f(x) = H(x) + a$ where H is the Heaviside function (with value $H = +1$ if $x > 0$ and $H = -1$ if $x < 0$).

  I've spent the last decade showing that, with the right approach, all of the stuff above applies at least qualitatively to discontinuous systems. That is, equilibria behave the same, you can study their stability in similar ways. Periodic orbits and bifurcations behave the same. We have even shown (just in the past year) that the Sharkovskii ordering even applies to discontinuous systems.

- There's still a lot of work to be done in non-differentiable, non-continuous, or more simply non-smooth systems. They are a big area of growth in modern applications. Collisions and nonsmooth, neurons fire in nonsmooth impulses, decisions are nonsmooth, electronic switches are nonsmooth . . . dynamical systems will never be done.

A.5 Numerical methods

This course has shown you that we rarely care about the exact solutions of differential equations. Even if we can find them, they are often so complicated that they are of no use.
Instead we study qualitative aspects, geometric objects like equilibria or periodic orbits.
Alongside understanding those, we often want to see them brought to life to verify our qualitative predictions. Or else we have complicated data that we wish to probe by seeking the qualitative objects that create it.
There are some powerful tools for bringing the equations to life or creating data from equations.

- There are various numerical packages for solving ODEs or maps. The most commonly used are in MATLAB, with ODE solvers such as ode45, ode15, and others implementing Runge-Kutta numerical recipes or adaptive methods.

- Continuation is a powerful tool that is something of a specialism among a number of staff in Bristol. There are a number of numerical packages for it such as AUTO, or MathCont. Continuation takes what you've learnt about bifurcations and uses it to follow or 'continue' bifurcation curves through bifurcations, to find even the branches of unstable equilibria or periodic orbits (which are hard to find because numerical simulations tend to evolve away from unstable objects).

- Event detection is a useful tool in packages like ode45, to create return maps, or to encode switches in a discontinuous system.

All these techniques and more are easily looked up online, and searches will usually turn up useful forums to help you in coding up numerical simulations of any given problem.

A.6 ODEs from applications: mass action

We have focussed on the methods to study ODEs here. There are a lot of different ways that ODEs arise in physical and biological modelling. There are some nice examples here: https://people.tamu.edu/∼phoward/m442/modode.pdf

In the population model we had a quantity N increasing or decreasing at certain rates, proportional to how much 'N' was already in the system.

The Law of Mass Action is a big generalization of this.

- Let X represent a particular 'species', which goes through some 'process' that changes how much of X there is.

  For example:

  - X might represent a person in a particular population, and the process might be the passing of an infection between individuals.
  - X might represent a molecule of a particular chemical, and the process might be a chemical reaction like photosynthesis.

- If there is an amount A of X before the process, and amount B of X after, and the process takes place at a rate k, we write

$$AX \xrightarrow{k} BX \tag{A.10}$$

  called a mass action equation.

- From this we can derive an ODE, if we define a variable $x_i = [X_i]$ representing the concentration of X_i , then this concentration changes as

$$\dot{x} = k(B - A)x^A \, . \tag{A.11}$$

- But a process rarely involves just one 'species'. . .

- Usually a process involves multiple parts or species.

  E.g. disease spread involves infected individuals X_1 and uninfected individuals X_2 (at least), photosynthesis involves four species, X_1, X_2, X_3, X_4, representing carbon dioxide, water, oxygen, and glucose.

- For these the same formula applies but $X = (X_1, X_2, ...)$ is a vector and A is a square matrix (with components we'll call a_{ij}), and if we have a number of processes with rates $k_1, k_2, ...$ we can write

$$\begin{aligned} a_{11}X_1 + a_{12}X_2 + ... &\xrightarrow{k_1} b_{11}X_1 + b_{12}X_2 + ... \\ a_{21}X_1 + a_{22}X_2 + ... &\xrightarrow{k_2} b_{21}X_1 + b_{22}X_2 + ... \end{aligned}$$

  which in vector form is

$$\begin{pmatrix} a_{11} & a_{12} & \dots \\ a_{21} & a_{22} & \dots \\ \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ \vdots \end{pmatrix} \xrightarrow{\begin{pmatrix} k_1 \\ k_2 \\ \vdots \end{pmatrix}} \begin{pmatrix} b_{11} & b_{12} & \dots \\ b_{21} & b_{22} & \dots \\ \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ \vdots \end{pmatrix} \tag{A.12}$$

- The matrix $S = B - A$ is called the *stochiometry matrix*, giving the difference between the amount of X before and after.

- We can derive an ODE from such a mass action equation, if we assume a "closed well mixed system". Define the concentration of species X_i as a variable $x_i = [X_i]$, then

$$\dot{x}_i = \sum_{j=1}^{n} s_{ji} k_j \; x_1^{a_{j1}} ... x_n^{a_{jn}} \tag{A.13}$$

  or in vector form

$$\frac{d}{dt} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \end{pmatrix} = \begin{pmatrix} s_{11} & s_{21} & \dots \\ s_{12} & s_{22} & \dots \\ \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} k_1 x_1^{a_{11}} ... x_n^{a_{1n}} \\ k_2 x_1^{a_{21}} ... x_n^{a_{2n}} \\ \vdots \end{pmatrix} \tag{A.14}$$

There's a tidier form if we let $x = (x_1, x_2, ..., x_n)$ be a column vector, then

$$\dot{x} = S^{\mathsf{T}} . \left(x^A \cdot k \right) \tag{A.15}$$

if we define

$$x^A = e^{A \ln x} \quad \text{where} \quad \ln x = (\ln x_1, \ln x_2, ...) \quad \text{and} \quad e^x = (e^{x_1}, e^{x_2}, ...) \, .$$

- These are tough expressions, as general formulae often are.
- They are much more intuitive if you look at some simple examples, you'll see some in the textbooks.