Exercise Sheet 0 – Revision: Complete Solutions
Methods of Applied Mathematics [SEMT30006]

Complete Step-by-Step Solutions with Intuition

Contents

Introduction to Solution Methodology

These solutions follow the **XYZ Framework** throughout:

- **STAGE X (What we have/know):** Present the given information and current state
- **STAGE Y (Why this works/method):** Explain the mathematical reasoning and justification
- **STAGE Z (What this means):** Interpret results and determine implications

This framework ensures every step has clear motivation, rigorous justification, and meaningful interpretation.

1 Problem 1: Matrices – Finding Eigenvalues

1.1 Problem 1(a): Matrix $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$

Problem Statement

Find the eigenvalues of $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$.

Hint given: One eigenvalue can be spotted by considering the rank.

Step 1: Analyze Matrix Structure

- **STAGE X (What we have):** We have a 2×2 symmetric matrix where both rows are identical: $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$.
- **STAGE Y (Why rank matters):** Since the second row equals the first row, these rows are linearly dependent. This means:

$$\operatorname{rank}(A) = 1 < 2 = \dim(A) \tag{1}$$

  For any $n \times n$ matrix, if $\operatorname{rank}(A) < n$, then $\det(A) = 0$, which means $\lambda = 0$ is an eigenvalue.
- **STAGE Z (What this means):** We know immediately that $\lambda_1 = 0$ is one eigenvalue. We need to find the other.

Step 2: Use Trace to Find Second Eigenvalue

- **STAGE X (What we know):** For any 2×2 matrix, the sum of eigenvalues equals the trace:

$$\lambda_1 + \lambda_2 = \operatorname{tr}(A) = a_{11} + a_{22} \tag{2}$$

- **STAGE Y (Why this works):** This follows from the characteristic polynomial $\det(A - \lambda I) = \lambda^2 - \operatorname{tr}(A)\lambda + \det(A)$, where the coefficient of λ gives us $-(\lambda_1 + \lambda_2) = -\operatorname{tr}(A)$.

  For our matrix:

$$\operatorname{tr}(A) = 1 + 1 = 2 \tag{3}$$

  Since $\lambda_1 = 0$:

$$0 + \lambda_2 = 2 \quad \Rightarrow \quad \lambda_2 = 2 \tag{4}$$

- **STAGE Z (What this means):** The eigenvalues are $\boxed{\lambda_1 = 0, \quad \lambda_2 = 2}$ without needing the full characteristic equation.

Step 3: Verification by Characteristic Polynomial (Complete Method)

VERIFICATION: Let's verify using the standard method:

- **STAGE X (Standard approach):** The eigenvalues λ satisfy $\det(A - \lambda I) = 0$:

$$\det \begin{pmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{pmatrix} = 0 \tag{5}$$

- **STAGE Y (Computing the determinant):**

$$
\begin{aligned}
(1-\lambda)(1-\lambda)-(1)(1) &= 0 && (6)\\
(1-\lambda)^2-1 &= 0 && (7)\\
1-2\lambda+\lambda^2-1 &= 0 && (8)\\
\lambda^2-2\lambda &= 0 && (9)\\
\lambda(\lambda-2) &= 0 && (10)
\end{aligned}
$$

- **STAGE Z (Confirmation):** This gives $\lambda = 0$ or $\lambda = 2$, confirming our answer ✓

Physical Interpretation

- **STAGE X (Eigenvector analysis):**
  - For $\lambda_1 = 0$: $A\mathbf{v}_1 = \mathbf{0}$, so $\mathbf{v}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ (or any multiple)
  - For $\lambda_2 = 2$: $A\mathbf{v}_2 = 2\mathbf{v}_2$, so $\mathbf{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ (or any multiple)
- **STAGE Y (Dynamical interpretation):** In a dynamical system $\dot{\mathbf{x}} = A\mathbf{x}$, the matrix has:
  - One zero eigenvalue: solutions along $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ remain constant
  - One positive eigenvalue: solutions along $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ grow exponentially as e^{2t}
- **STAGE Z (Stability conclusion):** This equilibrium is **unstable** because at least one eigenvalue is positive, causing exponential growth.

1.2 Problem 1(b): Matrix $\begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$

Problem Statement

Find the eigenvalues of $A = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$.

Step 1: Recognize Matrix Type and Use Symmetry

- **STAGE X (What we have):** A symmetric matrix of the form $A = \begin{pmatrix} a & b \\ b & a \end{pmatrix}$ with $a = 3$, $b = 1$.
- **STAGE Y (Why symmetry helps):** For this special form, we can immediately write down the eigenvalues using the pattern:

$$\lambda_{\pm} = a \pm b \tag{11}$$

  This works because such matrices can be diagonalized as:

$$A = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix} = 3I + 1J, \quad \text{where } J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{12}$$

  Since J has eigenvalues ± 1, and I adds 3 to each eigenvalue:

$$\lambda_1 = 3 + 1 = 4, \quad \lambda_2 = 3 - 1 = 2 \tag{13}$$

- **STAGE Z (Quick answer):** $\boxed{\lambda_1 = 4, \quad \lambda_2 = 2}$

Step 2: Verification by Characteristic Polynomial

COMPLETE METHOD:

- **STAGE X (Setup):** Solve $\det(A - \lambda I) = 0$:

$$\det \begin{pmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{pmatrix} = 0 \tag{14}$$

- **STAGE Y (Computation):**

$$(3-\lambda)(3-\lambda) - (1)(1) = 0 \tag{15}$$
$$(3-\lambda)^2 - 1 = 0 \tag{16}$$
$$9 - 6\lambda + \lambda^2 - 1 = 0 \tag{17}$$
$$\lambda^2 - 6\lambda + 8 = 0 \tag{18}$$

  Using the quadratic formula:

$$\lambda = \frac{6 \pm \sqrt{36 - 32}}{2} = \frac{6 \pm \sqrt{4}}{2} = \frac{6 \pm 2}{2} \tag{19}$$

  Therefore:

$$\lambda_1 = \frac{6+2}{2} = 4, \quad \lambda_2 = \frac{6-2}{2} = 2 \tag{20}$$

- **STAGE Z (Verification):** Matches our quick calculation ✓

Step 3: Alternative Method Using Sum and Product

EFFICIENT TECHNIQUE:

- **STAGE X (Known relationships):**

$$\lambda_1 + \lambda_2 = \text{tr}(A) = 3 + 3 = 6 \tag{21}$$
$$\lambda_1 \cdot \lambda_2 = \det(A) = 3 \cdot 3 - 1 \cdot 1 = 9 - 1 = 8 \tag{22}$$

- **STAGE Y (Solving the system):** We need two numbers that sum to 6 and multiply to 8. These are the roots of:

$$t^2 - 6t + 8 = 0 \quad \Rightarrow \quad (t-4)(t-2) = 0 \tag{23}$$

- **STAGE Z (Solution):** $\lambda_1 = 4$, $\lambda_2 = 2$ ✓

Physical Interpretation

- **STAGE X (Eigenvector structure):**
  - For $\lambda_1 = 4$: $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$
  - For $\lambda_2 = 2$: $\mathbf{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$
- **STAGE Y (Dynamical system behavior):** For $\dot{\mathbf{x}} = A\mathbf{x}$:
  - Along $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$: fast exponential growth $\sim e^{4t}$
  - Along $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$: slow exponential growth $\sim e^{2t}$
- **STAGE Z (Classification):** Since both eigenvalues are positive and real, this is an **unstable node**. The eigenvector with $\lambda_1 = 4$ is the **strong direction** (fastest growth), while $\lambda_2 = 2$ is the **weak direction** (slower growth).

1.3 Problem 1(c): Matrix $\begin{pmatrix} 1 & 3 \\ 0 & 2 \end{pmatrix}$

Problem Statement

Find the eigenvalues of $A = \begin{pmatrix} 1 & 3 \\ 0 & 2 \end{pmatrix}$.

Hint given: Both eigenvalues can be spotted just by looking.

Step 1: Recognize Upper Triangular Matrix

- **STAGE X (What we have):** An upper triangular matrix (all entries below the main diagonal are zero).
- **STAGE Y (Why this is immediate):** For any triangular matrix (upper or lower), the eigenvalues are simply the diagonal entries. This is because:

$$\det(A - \lambda I) = \det \begin{pmatrix} 1-\lambda & 3 \\ 0 & 2-\lambda \end{pmatrix} \tag{24}$$

  For a triangular matrix, the determinant is the product of diagonal entries:

$$\det(A - \lambda I) = (1-\lambda)(2-\lambda) = 0 \tag{25}$$

- **STAGE Z (Immediate answer):** $\boxed{\lambda_1 = 1, \quad \lambda_2 = 2}$
  We can see this directly from the diagonal without any calculation!

Step 2: Understand Why This Property Holds

- **STAGE X (Characteristic polynomial structure):** For our matrix:

$$\begin{aligned} \det(A - \lambda I) &= (1-\lambda)(2-\lambda) - (0)(3) && (26)\\ &= (1-\lambda)(2-\lambda) && (27)\\ &= 2 - \lambda - 2\lambda + \lambda^2 && (28)\\ &= \lambda^2 - 3\lambda + 2 && (29) \end{aligned}$$

- **STAGE Y (Factorization insight):** The off-diagonal term $(0)(3) = 0$ doesn't contribute, so we get:

$$\lambda^2 - 3\lambda + 2 = (\lambda - 1)(\lambda - 2) = 0 \tag{30}$$

  This is a general property: for triangular matrices, the characteristic polynomial factors as:

$$\det(A - \lambda I) = \prod_{i=1}^{n} (a_{ii} - \lambda) \tag{31}$$

- **STAGE Z (General principle):** **KEY INSIGHT:** Always check if a matrix is triangular before computing eigenvalues—it saves significant work!

Step 3: Find Eigenvectors and Interpret

- **STAGE X (Eigenvector computation):**

  For $\lambda_1 = 1$:

$$(A - I)\mathbf{v}_1 = \begin{pmatrix} 0 & 3 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \mathbf{0} \tag{32}$$

  This gives $v_2 = 0$, so $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

  For $\lambda_2 = 2$:

$$(A - 2I)\mathbf{v}_2 = \begin{pmatrix} -1 & 3 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \mathbf{0} \tag{33}$$

  This gives $-v_1 + 3v_2 = 0$, so $\mathbf{v}_2 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$.

- **STAGE Y (Dynamical system interpretation):** For $\dot{\mathbf{x}} = A\mathbf{x}$:
  - Along $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$: growth rate e^t
  - Along $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$: growth rate e^{2t}

- **STAGE Z (Stability classification):** This is an **unstable node** (both eigenvalues positive and real). The off-diagonal entry causes **shearing**—the flow doesn't align with coordinate axes, but rather with the eigenvector directions.

Key Takeaway

EXAM TIP: When you see a triangular matrix:

1. Eigenvalues = diagonal entries (instant answer!)
2. No need to compute $\det(A - \lambda I)$
3. This works for both upper and lower triangular matrices
4. Also works for diagonal matrices (special case)

2 Problem 2: Complex Numbers

2.1 Problem 2(a): Expand $(1+2i)e^{2it} + (1-2i)e^{-2it}$ in terms of sin and cos

Problem Statement

Express $(1+2i)e^{2it} + (1-2i)e^{-2it}$ in terms of sin and cos functions.

Step 1: Apply Euler's Formula

- **STAGE X (What we have):** Two complex exponentials with complex coefficients. We use Euler's formula:

$$e^{i\theta} = \cos\theta + i\sin\theta \tag{34}$$

- **STAGE Y (Applying to our problem):**

$$e^{2it} = \cos(2t) + i\sin(2t) \tag{35}$$

$$e^{-2it} = \cos(-2t) + i\sin(-2t) = \cos(2t) - i\sin(2t) \tag{36}$$

  where we used $\cos(-\theta) = \cos(\theta)$ and $\sin(-\theta) = -\sin(\theta)$.

- **STAGE Z (Ready to expand):** Now substitute these into the original expression.

Step 2: Expand Each Term

- **STAGE X (First term):**

$$\begin{aligned}(1+2i)e^{2it} &= (1+2i)[\cos(2t) + i\sin(2t)] &(37)\\ &= \cos(2t) + i\sin(2t) + 2i\cos(2t) + 2i^2\sin(2t) &(38)\\ &= \cos(2t) + i\sin(2t) + 2i\cos(2t) - 2\sin(2t) &(39)\\ &= [\cos(2t) - 2\sin(2t)] + i[\sin(2t) + 2\cos(2t)] &(40)\end{aligned}$$

- **STAGE Y (Second term):**

$$\begin{aligned}(1-2i)e^{-2it} &= (1-2i)[\cos(2t) - i\sin(2t)] &(41)\\ &= \cos(2t) - i\sin(2t) - 2i\cos(2t) + 2i^2\sin(2t) &(42)\\ &= \cos(2t) - i\sin(2t) - 2i\cos(2t) - 2\sin(2t) &(43)\\ &= [\cos(2t) - 2\sin(2t)] - i[\sin(2t) + 2\cos(2t)] &(44)\end{aligned}$$

- **STAGE Z (Notice the pattern):** The two terms are complex conjugates of each other!

Step 3: Combine and Simplify

- **STAGE X (Adding the terms):**

$$\begin{aligned}&(1+2i)e^{2it} + (1-2i)e^{-2it} &(45)\\ &= [\cos(2t) - 2\sin(2t)] + i[\sin(2t) + 2\cos(2t)] &(46)\\ &\quad + [\cos(2t) - 2\sin(2t)] - i[\sin(2t) + 2\cos(2t)] &(47)\end{aligned}$$

- **STAGE Y (Imaginary parts cancel):** When we add complex conjugates, imaginary parts cancel:

$$\begin{aligned}&= 2[\cos(2t) - 2\sin(2t)] &(48)\\ &= 2\cos(2t) - 4\sin(2t) &(49)\end{aligned}$$

- **STAGE Z (Final answer):**

$$\boxed{(1+2i)e^{2it} + (1-2i)e^{-2it} = 2\cos(2t) - 4\sin(2t)} \tag{50}$$

This is a real-valued function, as expected from the sum of complex conjugates.

Step 4: Interpretation and General Pattern

- **STAGE X (Why complex conjugates give real results):** For any complex number $z = a + ib$:

$$ze^{i\omega t} + z^* e^{-i\omega t} = 2\text{Re}(ze^{i\omega t}) \tag{51}$$

This is a fundamental technique in physics and engineering for converting complex exponentials to real trigonometric functions.

- **STAGE Y (General formula):** For $z = a + ib$:

$$\begin{align} ze^{i\omega t} + z^* e^{-i\omega t} &= (a+ib)e^{i\omega t} + (a-ib)e^{-i\omega t} \tag{52} \\ &= 2a\cos(\omega t) - 2b\sin(\omega t) \tag{53} \end{align}$$

In our case: $a = 1$, $b = 2$, $\omega = 2$:

$$2(1)\cos(2t) - 2(2)\sin(2t) = 2\cos(2t) - 4\sin(2t) \quad \checkmark \tag{54}$$

- **STAGE Z (Application to ODEs):** This technique is crucial for solving differential equations with oscillatory solutions. When ODEs have complex eigenvalues $\lambda = \alpha \pm i\beta$, the general solution involves terms like:

$$e^{\alpha t}[c_1 e^{i\beta t} + c_2 e^{-i\beta t}] = e^{\alpha t}[A\cos(\beta t) + B\sin(\beta t)] \tag{55}$$

CONNECTION TO COURSE MATERIAL: In the lecture notes (page 26-27), when eigenvalues are complex conjugates $\lambda = \pm i\sqrt{\alpha\gamma}$, solutions oscillate with frequency $\sqrt{\alpha\gamma}$. This expansion technique converts those complex solutions to real observable oscillations.

2.2 Problem 2(b): Find the solutions of $u^3 = 2$

Problem Statement

Find all complex solutions to $u^3 = 2$.

Step 1: Convert to Polar Form

- **STAGE X (Express in polar coordinates):** Write 2 in polar form:

$$2 = 2e^{i\cdot 0} = 2e^{i\cdot 2\pi k} \quad \text{for any integer } k \tag{56}$$

  This accounts for the fact that $e^{i\theta}$ is 2π-periodic.

- **STAGE Y (Why multiple representations matter):** Since $e^{i\theta}$ has period 2π, we have:

$$2 = 2e^{i\cdot 0} = 2e^{i\cdot 2\pi} = 2e^{i\cdot 4\pi} = 2e^{i\cdot 2\pi k} \tag{57}$$

  When we take roots, each representation gives a potentially different solution.

- **STAGE Z (Setup for cube roots):** We seek u such that $u^3 = 2e^{i\cdot 2\pi k}$.

Step 2: Extract Cube Roots

- **STAGE X (Taking cube roots):** If $u = re^{i\theta}$, then:

$$u^3 = r^3 e^{i3\theta} = 2e^{i\cdot 2\pi k} \tag{58}$$

  Matching magnitudes and arguments:

$$r^3 = 2 \quad \Rightarrow \quad r = 2^{1/3} = \sqrt[3]{2} \tag{59}$$

$$3\theta = 2\pi k \quad \Rightarrow \quad \theta = \frac{2\pi k}{3} \tag{60}$$

- **STAGE Y (Finding distinct solutions):** The solutions are:

$$u_k = 2^{1/3} e^{i\cdot 2\pi k/3} \quad \text{for } k = 0, 1, 2 \tag{61}$$

  For $k \geq 3$, we get repeats because:

$$e^{i\cdot 2\pi\cdot 3/3} = e^{i\cdot 2\pi} = e^{i\cdot 0} \tag{62}$$

  So there are exactly **three distinct cube roots**.

- **STAGE Z (The three solutions):**

$$u_0 = 2^{1/3} e^{i\cdot 0} = 2^{1/3} \tag{63}$$

$$u_1 = 2^{1/3} e^{i\cdot 2\pi/3} \tag{64}$$

$$u_2 = 2^{1/3} e^{i\cdot 4\pi/3} \tag{65}$$

Step 3: Convert to Cartesian Form

- **STAGE X (First solution - real):**

$$u_0 = 2^{1/3} = \boxed{\sqrt[3]{2}} \tag{66}$$

- **STAGE Y (Second solution):**

$$u_1 = 2^{1/3}e^{i\cdot 2\pi/3} = 2^{1/3}\left[\cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right)\right] \tag{67}$$

$$= 2^{1/3}\left[-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right] \tag{68}$$

$$= \boxed{2^{1/3}\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)} \tag{69}$$

- **STAGE Z (Third solution):**

$$u_2 = 2^{1/3}e^{i\cdot 4\pi/3} = 2^{1/3}\left[\cos\left(\frac{4\pi}{3}\right) + i\sin\left(\frac{4\pi}{3}\right)\right] \tag{70}$$

$$= 2^{1/3}\left[-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right] \tag{71}$$

$$= \boxed{2^{1/3}\left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)} \tag{72}$$

Step 4: Geometric Interpretation

- **STAGE X (Symmetry in complex plane):** The three roots lie on a circle of radius $2^{1/3}$ in the complex plane, equally spaced at angles:

$$0, \quad 120, \quad 240 \tag{73}$$

- **STAGE Y (Why this pattern):** The n-th roots of any complex number are equally spaced around a circle, separated by angles of $\frac{360}{n} = \frac{2\pi}{n}$ radians. For cube roots, this is 120 or $\frac{2\pi}{3}$ radians.

- **STAGE Z (General principle):** For any equation $u^n = a$ where $a \in \mathbb{C}$:
  - There are exactly n solutions
  - They lie on a circle of radius $|a|^{1/n}$
  - They are equally spaced by angle $\frac{2\pi}{n}$
  - Starting angle is $\frac{\arg(a)}{n}$

Verification

CHECK: Let's verify u_1:

$$u_1^3 = \left[2^{1/3}e^{i\cdot 2\pi/3}\right]^3 \tag{74}$$

$$= (2^{1/3})^3 \cdot e^{i\cdot 2\pi} \tag{75}$$

$$= 2 \cdot 1 = 2 \quad \checkmark \tag{76}$$

Summary of All Three Solutions

$$\boxed{u \in \left\{ \sqrt[3]{2}, \quad \sqrt[3]{2}\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right), \quad \sqrt[3]{2}\left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right) \right\}} \tag{77}$$

Alternatively, in polar form:

$$\boxed{u \in \left\{ 2^{1/3}, \quad 2^{1/3}e^{i2\pi/3}, \quad 2^{1/3}e^{i4\pi/3} \right\}} \tag{78}$$

CONNECTION TO COURSE: Complex roots appear frequently when solving characteristic equations for ODEs. When eigenvalues come in complex conjugate pairs $\lambda = \alpha \pm i\beta$, solutions involve $e^{(\alpha \pm i\beta)t}$, which produce oscillatory behavior as discussed in pages 26-34 of the lecture notes.

3 Problem 3: Ordinary Differential Equations

3.1 Problem 3(a): Solve $\frac{dx}{dt} = \frac{1}{2}(1-x)$

Problem Statement

Solve the first-order ODE: $\frac{dx}{dt} = \frac{1}{2}(1-x)$.

Step 1: Identify ODE Type and Method

- **STAGE X (What we have):** A first-order, linear, separable ODE of the form:

$$\frac{dx}{dt} = f(x) \tag{79}$$

  where $f(x) = \frac{1}{2}(1-x)$ is a function of x only (autonomous).

- **STAGE Y (Why separation of variables works):** Since the right-hand side depends only on x and not explicitly on t, we can separate variables:

$$\frac{dx}{1-x} = \frac{1}{2}dt \tag{80}$$

  This moves all terms involving x to one side and all terms involving t to the other.

- **STAGE Z (Strategy):** Integrate both sides and solve for $x(t)$.

Step 2: Separate Variables and Integrate

- **STAGE X (Separation):**

$$\frac{dx}{1-x} = \frac{1}{2}dt \tag{81}$$

- **STAGE Y (Integration):** Integrate both sides:

$$\int \frac{dx}{1-x} = \int \frac{1}{2}dt \tag{82}$$

$$-\ln|1-x| = \frac{t}{2} + C_1 \tag{83}$$

  where C_1 is the constant of integration.

- **STAGE Z (Rearranging):** Multiply by -1:

$$\ln|1-x| = -\frac{t}{2} - C_1 = -\frac{t}{2} + C_2 \tag{84}$$

  where $C_2 = -C_1$ (still arbitrary).

Step 3: Solve for $x(t)$

- **STAGE X (Exponentiating):**

$$|1-x| = e^{-t/2+C_2} = e^{C_2}e^{-t/2} \tag{85}$$

  Let $A = \pm e^{C_2}$ (can be positive or negative):

$$1-x = Ae^{-t/2} \tag{86}$$

- **STAGE Y (Solving for x):**
$$x(t) = 1 - Ae^{-t/2} \tag{87}$$

- **STAGE Z (Incorporating initial conditions):** At $t = 0$:
$$x(0) = 1 - A = x_0 \quad \Rightarrow \quad A = 1 - x_0 \tag{88}$$
Therefore, the general solution is:
$$\boxed{x(t) = 1 - (1 - x_0)e^{-t/2}} \tag{89}$$

Step 4: Analysis of Solution Behavior

- **STAGE X (Equilibrium analysis):** The equilibrium occurs when $\frac{dx}{dt} = 0$:
$$\frac{1}{2}(1 - x) = 0 \quad \Rightarrow \quad x^* = 1 \tag{90}$$

- **STAGE Y (Long-time behavior):** As $t \to \infty$:
$$x(t) = 1 - (1 - x_0)e^{-t/2} \to 1 \tag{91}$$
since $e^{-t/2} \to 0$.
The solution exponentially approaches $x = 1$ from below if $x_0 < 1$, or from above if $x_0 > 1$.

- **STAGE Z (Stability):** This equilibrium $x^* = 1$ is **stable** (an attractor). All solutions converge to it.
The linearization about $x^* = 1$ gives:
$$\frac{d}{dt}(x - 1) \approx -\frac{1}{2}(x - 1) \tag{92}$$
with solution $(x - 1) \sim e^{-t/2}$, confirming exponential decay to equilibrium with rate $-1/2$.

Step 5: Physical Interpretation

- **STAGE X (Connection to course material):** This is analogous to the population models in lecture notes (pages 19-21). The term $\frac{1}{2}(1 - x)$ represents:
  - Growth when $x < 1$: $\frac{dx}{dt} > 0$
  - Decay when $x > 1$: $\frac{dx}{dt} < 0$
  - Equilibrium at $x = 1$: $\frac{dx}{dt} = 0$

- **STAGE Y (Comparison to lecture examples):** Similar to equation (6.5) on page 20 of lecture notes:
$$\dot{x} = \beta x \quad \text{(exponential growth)} \tag{93}$$
But our equation includes a carrying capacity at $x = 1$.

- **STAGE Z (Relaxation time):** The characteristic timescale is $\tau = 2$ (the reciprocal of the coefficient $1/2$). After time $t = 2$:
$$x(2) = 1 - (1 - x_0)e^{-1} \approx 1 - 0.368(1 - x_0) \tag{94}$$
The solution is about 63% of the way to equilibrium.

KEY INSIGHT: This is a **stable linear ODE**. All trajectories converge exponentially to $x = 1$, regardless of initial condition.

3.2 Problem 3(b): Solve $\frac{d^2x}{dt^2} + \frac{dx}{dt} + 4x = 0$

Problem Statement

Solve the second-order linear ODE with constant coefficients:

$$\frac{d^2x}{dt^2} + \frac{dx}{dt} + 4x = 0 \tag{95}$$

Step 1: Identify ODE Type and Solution Method

- **STAGE X (What we have):** A second-order, linear, homogeneous ODE with constant coefficients of the form:

  $$a\frac{d^2x}{dt^2} + b\frac{dx}{dt} + cx = 0 \tag{96}$$

  where $a = 1$, $b = 1$, $c = 4$.

- **STAGE Y (Why characteristic equation method):** For constant-coefficient linear ODEs, we seek solutions of the form $x = e^{\lambda t}$. This transforms the ODE into an algebraic equation (the characteristic equation).

  The derivatives become:

  $$\frac{dx}{dt} = \lambda e^{\lambda t}, \quad \frac{d^2x}{dt^2} = \lambda^2 e^{\lambda t} \tag{97}$$

- **STAGE Z (Strategy):** Find the characteristic equation, solve for λ, then construct the general solution based on the nature of the roots (real, complex, or repeated).

Step 2: Derive and Solve Characteristic Equation

- **STAGE X (Substituting $x = e^{\lambda t}$):**

  $$\lambda^2 e^{\lambda t} + \lambda e^{\lambda t} + 4e^{\lambda t} = 0 \tag{98}$$

  Factor out $e^{\lambda t}$ (which is never zero):

  $$e^{\lambda t}(\lambda^2 + \lambda + 4) = 0 \tag{99}$$

- **STAGE Y (Characteristic equation):**

  $$\lambda^2 + \lambda + 4 = 0 \tag{100}$$

  Using the quadratic formula:

  $$\lambda = \frac{-1 \pm \sqrt{1 - 16}}{2} = \frac{-1 \pm \sqrt{-15}}{2} = \frac{-1 \pm i\sqrt{15}}{2} \tag{101}$$

- **STAGE Z (Complex conjugate roots):**

  $$\lambda_1 = -\frac{1}{2} + i\frac{\sqrt{15}}{2}, \quad \lambda_2 = -\frac{1}{2} - i\frac{\sqrt{15}}{2} \tag{102}$$

  Since roots are complex conjugates: $\lambda = \alpha \pm i\beta$ where $\alpha = -\frac{1}{2}$ and $\beta = \frac{\sqrt{15}}{2}$.

Step 3: Construct General Solution from Complex Eigenvalues

- **STAGE X (Complex form):** With complex roots $\lambda = \alpha \pm i\beta$, the general solution is:
$$x(t) = e^{\alpha t}[C_1 e^{i\beta t} + C_2 e^{-i\beta t}] \tag{103}$$
- **STAGE Y (Converting to real form):** Using Euler's formula: $e^{\pm i\beta t} = \cos(\beta t) \pm i \sin(\beta t)$
For real solutions, we rewrite as:
$$x(t) = e^{\alpha t}[A\cos(\beta t) + B\sin(\beta t)] \tag{104}$$
where A and B are real constants determined by initial conditions.
- **STAGE Z (Our specific solution):** With $\alpha = -\frac{1}{2}$ and $\beta = \frac{\sqrt{15}}{2}$:
$$\boxed{x(t) = e^{-t/2}\left[A\cos\left(\frac{\sqrt{15}}{2}t\right) + B\sin\left(\frac{\sqrt{15}}{2}t\right)\right]} \tag{105}$$
where A and B are determined by initial conditions $x(0)$ and $x'(0)$.

Step 4: Interpret Solution Behavior

- **STAGE X (Decomposing the solution):**
  - **Exponential envelope:** $e^{-t/2}$ causes decay
  - **Oscillatory component:** $\cos\left(\frac{\sqrt{15}}{2}t\right)$ and $\sin\left(\frac{\sqrt{15}}{2}t\right)$ cause oscillation
- **STAGE Y (Physical meaning):** This describes **damped oscillation**:
  - **Damping rate:** $\alpha = -\frac{1}{2}$ (decay timescale $\tau = 2$)
  - **Angular frequency:** $\omega = \frac{\sqrt{15}}{2} \approx 1.936$ rad/unit time
  - **Period:** $T = \frac{2\pi}{\omega} = \frac{4\pi}{\sqrt{15}} \approx 3.24$ time units
- **STAGE Z (Stability classification):** This is a **stable focus** (spiral sink):
  - Real part $\text{Re}(\lambda) = -\frac{1}{2} < 0 \to$ attraction
  - Imaginary part $\text{Im}(\lambda) = \pm\frac{\sqrt{15}}{2} \neq 0 \to$ rotation
  - All trajectories spiral into the origin as $t \to \infty$

Step 5: Connection to Course Material

- **STAGE X (Relation to eigenvalue analysis):** From lecture notes (pages 29-34), when the linearization matrix has complex eigenvalues $\lambda = \alpha \pm i\beta$:
  - If $\alpha < 0$: stable focus (spiral sink)
  - If $\alpha > 0$: unstable focus (spiral source)
  - If $\alpha = 0$: center (neutral stability, perfect oscillation)
- **STAGE Y (Our case):** With $\alpha = -\frac{1}{2} < 0$, this is a **stable focus**, as discussed on page 29 of lecture notes. Solutions spiral inward.
- **STAGE Z (Physical examples):** This ODE models:
  - Damped spring-mass system
  - RLC circuit with resistance
  - Predator-prey models near coexistence equilibrium with damping

Determination of Constants (Example)

IF INITIAL CONDITIONS GIVEN:
Suppose $x(0) = 1$ and $x'(0) = 0$. Then:

- From $x(0) = 1$:

$$e^0[A\cos(0) + B\sin(0)] = A = 1 \tag{106}$$

- From $x'(0) = 0$:

$$x'(t) = -\frac{1}{2}e^{-t/2}\left[A\cos\left(\frac{\sqrt{15}}{2}t\right) + B\sin\left(\frac{\sqrt{15}}{2}t\right)\right] \tag{107}$$

$$+e^{-t/2}\left[-A\frac{\sqrt{15}}{2}\sin\left(\frac{\sqrt{15}}{2}t\right) + B\frac{\sqrt{15}}{2}\cos\left(\frac{\sqrt{15}}{2}t\right)\right] \tag{108}$$

At $t = 0$:

$$x'(0) = -\frac{1}{2}A + \frac{\sqrt{15}}{2}B = 0 \tag{109}$$

With $A = 1$:

$$B = \frac{1}{\sqrt{15}} \tag{110}$$

KEY TAKEAWAY: Complex eigenvalues always come in conjugate pairs for real ODEs, producing oscillatory solutions. The real part determines stability (decay/growth), while the imaginary part determines oscillation frequency.

3.3 Problem 3(c): Coupled System with Trial Solution Method

Problem Statement

Solve the coupled linear system:

$$\frac{dx}{dt} = 3x + y \tag{111}$$
$$\frac{dy}{dt} = x - 3y \tag{112}$$

Hint given: Use trial solution $x = e^{\lambda t}$, $y = ae^{\lambda t}$ and the principle of linear superposition.

Step 1: Matrix Formulation

- **STAGE X (Vector form):** Write the system as $\dot{\mathbf{x}} = A\mathbf{x}$ where:

$$\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}, \quad A = \begin{pmatrix} 3 & 1 \\ 1 & -3 \end{pmatrix} \tag{113}$$

- **STAGE Y (Why this formulation):** This connects directly to the eigenvalue analysis in lecture notes (pages 24-25). The solution structure is:

$$\mathbf{x}(t) = e^{At}\mathbf{x}_0 = c_1 e^{\lambda_1 t}\mathbf{v}_1 + c_2 e^{\lambda_2 t}\mathbf{v}_2 \tag{114}$$

  where λ_i are eigenvalues and $\mathbf{v}_i$ are eigenvectors.

- **STAGE Z (Strategy):** Find eigenvalues and eigenvectors, then use superposition to construct the general solution.

Step 2: Apply Trial Solution to Find Eigenvalues

- **STAGE X (Substituting trial solution):** Try $x = e^{\lambda t}$, $y = ae^{\lambda t}$ (as suggested):

$$\frac{d}{dt}(e^{\lambda t}) = 3e^{\lambda t} + ae^{\lambda t} \tag{115}$$
$$\frac{d}{dt}(ae^{\lambda t}) = e^{\lambda t} - 3ae^{\lambda t} \tag{116}$$

  Simplifying (divide by $e^{\lambda t}$):

$$\lambda = 3 + a \tag{117}$$
$$\lambda a = 1 - 3a \tag{118}$$

- **STAGE Y (Solving for λ and a):** From the first equation: $a = \lambda - 3$

  Substitute into the second:

$$\lambda(\lambda - 3) = 1 - 3(\lambda - 3) \tag{119}$$
$$\lambda^2 - 3\lambda = 1 - 3\lambda + 9 \tag{120}$$
$$\lambda^2 = 10 \tag{121}$$
$$\lambda = \pm\sqrt{10} \tag{122}$$

- **STAGE Z (Two eigenvalues):**

$$\lambda_1 = \sqrt{10}, \quad \lambda_2 = -\sqrt{10} \tag{123}$$

Step 3: Find Corresponding Eigenvectors

- **STAGE X (For $\lambda_1 = \sqrt{10}$):** From $a = \lambda - 3$:

$$a_1 = \sqrt{10} - 3 \tag{124}$$

  So eigenvector:

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ \sqrt{10} - 3 \end{pmatrix} \tag{125}$$

- **STAGE Y (For $\lambda_2 = -\sqrt{10}$):**

$$a_2 = -\sqrt{10} - 3 \tag{126}$$

  So eigenvector:

$$\mathbf{v}_2 = \begin{pmatrix} 1 \\ -\sqrt{10} - 3 \end{pmatrix} \tag{127}$$

- **STAGE Z (Verification):** Check $A\mathbf{v}_1 = \lambda_1 \mathbf{v}_1$:

$$\begin{pmatrix} 3 & 1 \\ 1 & -3 \end{pmatrix} \begin{pmatrix} 1 \\ \sqrt{10} - 3 \end{pmatrix} = \begin{pmatrix} 3 + \sqrt{10} - 3 \\ 1 - 3(\sqrt{10} - 3) \end{pmatrix} \tag{128}$$

$$= \begin{pmatrix} \sqrt{10} \\ 10 - 3\sqrt{10} \end{pmatrix} \tag{129}$$

$$= \sqrt{10} \begin{pmatrix} 1 \\ \sqrt{10} - 3 \end{pmatrix} \quad \checkmark \tag{130}$$

Step 4: Construct General Solution by Superposition

- **STAGE X (Linear superposition principle):** The general solution is a linear combination of the two eigensolutions:

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = c_1 e^{\lambda_1 t} \mathbf{v}_1 + c_2 e^{\lambda_2 t} \mathbf{v}_2 \tag{131}$$

- **STAGE Y (Explicit form):**

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = c_1 e^{\sqrt{10}t} \begin{pmatrix} 1 \\ \sqrt{10} - 3 \end{pmatrix} + c_2 e^{-\sqrt{10}t} \begin{pmatrix} 1 \\ -\sqrt{10} - 3 \end{pmatrix} \tag{132}$$

- **STAGE Z (Component-wise):**

$$x(t) = c_1 e^{\sqrt{10}t} + c_2 e^{-\sqrt{10}t} \tag{133}$$

$$y(t) = c_1(\sqrt{10} - 3)e^{\sqrt{10}t} + c_2(-\sqrt{10} - 3)e^{-\sqrt{10}t} \tag{134}$$

  where c_1 and c_2 are determined by initial conditions.

Step 5: Analyze Solution Behavior and Stability

- **STAGE X (Eigenvalue signs):**
  - $\lambda_1 = \sqrt{10} > 0$: unstable direction
  - $\lambda_2 = -\sqrt{10} < 0$: stable direction
- **STAGE Y (Classification):** Since eigenvalues are real with opposite signs, the origin is a **saddle point** (lecture notes page 29):
  - Unstable manifold along $\mathbf{v}_1 = \begin{pmatrix} 1 \\ \sqrt{10} - 3 \end{pmatrix}$
  - Stable manifold along $\mathbf{v}_2 = \begin{pmatrix} 1 \\ -\sqrt{10} - 3 \end{pmatrix}$
- **STAGE Z (Long-time behavior):** As $t \to \infty$:
  - If $c_1 \neq 0$: solution diverges exponentially along $\mathbf{v}_1$
  - If $c_1 = 0$ (exactly on stable manifold): solution approaches origin along $\mathbf{v}_2$

  The equilibrium at $(0, 0)$ is **unstable** because of the positive eigenvalue.

Connection to Course Material

- **STAGE X (Lecture notes pages 24-32):** This problem directly applies the eigendecomposition method:
  - Equation (7.10): Decompose initial condition into eigenvectors
  - Equation (7.12): Solution as sum of exponential eigendirections
  - Page 29: Classification as saddle with $\det(A) = \lambda_1\lambda_2 < 0$
- **STAGE Y (Why this method):** The hint to use $x = e^{\lambda t}$, $y = ae^{\lambda t}$ is equivalent to seeking eigenvectors of A. This is the **standard method** for solving linear ODEs with constant coefficients, as emphasized throughout Chapter 7 of the notes.
- **STAGE Z (Verification of classification):** Calculate $\det(A)$:

$$\det(A) = (3)(-3) - (1)(1) = -9 - 1 = -10 < 0 \tag{135}$$

  Since $\det(A) = \lambda_1\lambda_2 = (\sqrt{10})(-\sqrt{10}) = -10 < 0$, this confirms **saddle** classification ✓

KEY INSIGHT: The trial solution method is equivalent to finding eigenvalues/eigenvectors. For coupled linear systems, solutions are always superpositions of exponential eigenmodes.

4 Problem 4: Taylor Series

4.1 Problem Statement

Estimate the value of $\sin(0.1)$ by hand using Taylor series. How quickly does the Taylor series expansion approach the actual value?

Step 1: Recall Taylor Series for Sine

- **STAGE X (General Taylor series):** For any smooth function $f(x)$ expanded about $x = a$:

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots \tag{136}$$

- **STAGE Y (Sine function about $x = 0$):** For $f(x) = \sin(x)$ about $a = 0$ (Maclaurin series):
  - $f(x) = \sin(x)$, $f(0) = 0$
  - $f'(x) = \cos(x)$, $f'(0) = 1$
  - $f''(x) = -\sin(x)$, $f''(0) = 0$
  - $f'''(x) = -\cos(x)$, $f'''(0) = -1$
  - $f^{(4)}(x) = \sin(x)$, $f^{(4)}(0) = 0$
  - $f^{(5)}(x) = \cos(x)$, $f^{(5)}(0) = 1$

- **STAGE Z (Pattern):** Derivatives alternate: $0, 1, 0, -1, 0, 1, 0, -1, \ldots$

  Therefore:

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \cdots \tag{137}$$

Step 2: Calculate Terms for $x = 0.1$

- **STAGE X (First term - linear approximation):**

$$T_1 = x = 0.1 \tag{138}$$

- **STAGE Y (Second term - cubic correction):**

$$T_2 = -\frac{x^3}{3!} = -\frac{(0.1)^3}{6} = -\frac{0.001}{6} = -0.000166\overline{6} \tag{139}$$

- **STAGE Z (Third term - fifth order):**

$$T_3 = \frac{x^5}{5!} = \frac{(0.1)^5}{120} = \frac{0.00001}{120} = 0.0000000833\overline{3} \tag{140}$$

  This is already very small ($\approx 8.3 \times 10^{-8}$), so higher terms will be negligible for hand calculation.

Step 3: Compute Approximations

- **STAGE X (One term - linear):**

$$\sin(0.1) \approx 0.1 \tag{141}$$

- **STAGE Y (Two terms - cubic):**

$$\begin{aligned} \sin(0.1) &\approx 0.1 - 0.000166\overline{6} && (142)\\ &= 0.1 - \frac{1}{6000} && (143)\\ &\approx 0.0998333\overline{3} && (144) \end{aligned}$$

- **STAGE Z (Three terms - quintic):**

$$\begin{aligned} \sin(0.1) &\approx 0.1 - 0.000166\overline{6} + 0.0000000833\overline{3} && (145)\\ &\approx 0.0998334166\overline{6} && (146) \end{aligned}$$

Step 4: Compare with True Value

- **STAGE X (Actual value):** Using a calculator: $\sin(0.1) = 0.0998334166468...$
- **STAGE Y (Error analysis):**

Order	Approximation	Error	Relative Error
T_1 (linear)	0.1	1.67×10^{-4}	0.167%
$T_1 + T_2$ (cubic)	$0.09983333\ldots$	8.3×10^{-9}	0.0000083%
$T_1 + T_2 + T_3$	$0.09983341666\ldots$	$\sim 10^{-12}$	$\sim 10^{-9}$%

- **STAGE Z (Convergence rate):** Each additional term reduces the error by a factor of roughly:

$$\frac{x^2}{n(n+1)} \approx \frac{(0.1)^2}{20} = \frac{0.01}{20} = 0.0005 \tag{147}$$

The error decreases **very rapidly** for $x = 0.1$.

Step 5: Understanding Convergence Speed

- **STAGE X (Error term formula):** The error after n terms is bounded by the $(n+1)$-th term (for alternating series):

$$|\text{Error}_n| \leq \left| \frac{x^{2n+1}}{(2n+1)!} \right| \tag{148}$$

For $x = 0.1$:

$$\begin{aligned} |\text{Error}_1| &\leq \frac{(0.1)^3}{6} = 1.67 \times 10^{-4} && (149)\\ |\text{Error}_2| &\leq \frac{(0.1)^5}{120} = 8.3 \times 10^{-9} && (150)\\ |\text{Error}_3| &\leq \frac{(0.1)^7}{5040} = 2.0 \times 10^{-14} && (151) \end{aligned}$$

- **STAGE Y (Why so fast):** Convergence is rapid because:
  1. $x = 0.1$ is small, so powers of x decrease rapidly
  2. Factorials in denominators grow very fast
  3. Each term is roughly $\frac{x^2}{2n+1} \times$ previous term $\approx 0.005\times$ previous
- **STAGE Z (General principle):** For $|x| < 1$, the Taylor series for $\sin(x)$ converges very rapidly. The series converges for all real x (radius of convergence $R = \infty$), but convergence is fastest for small $|x|$.

Step 6: Connection to Course Material

- **STAGE X (Linearization in lecture notes):** On pages 20-21 and 26-27, the course uses Taylor expansions for linearization about equilibria:

$$f(x) \approx f(x^*) + f'(x^*)(x - x^*) + O((x - x^*)^2) \tag{152}$$

  For small perturbations from equilibrium, higher-order terms become negligible.
- **STAGE Y (Why Taylor series in dynamics):** Near an equilibrium x^* where $f(x^*) = 0$:

$$\dot{x} = f(x) \approx f'(x^*)(x - x^*) \tag{153}$$

  This is exactly the linearization used throughout the course. For $|\frac{dx}{dt}| = |f(x)| \ll 1$ (slow dynamics), linear approximation is accurate.
- **STAGE Z (Convergence and validity):** Just as $\sin(0.1)$ needs only a few terms for accuracy, linearization near equilibria is accurate when perturbations are small. The quadratic term $\frac{f''(x^*)}{2}(x - x^*)^2$ provides the next correction, similar to our cubic term $-\frac{x^3}{6}$.

Final Answer Summary

$$\boxed{\sin(0.1) \approx 0.1 - \frac{(0.1)^3}{6} \approx 0.099833\overline{3}} \tag{154}$$

CONVERGENCE: The Taylor series converges **extremely rapidly** for $x = 0.1$:

- 1 term: 0.17% error
- 2 terms: 0.00083% error
- 3 terms: $< 10^{-7}$% error

Each additional term reduces error by factor ~ 500!

Summary and Key Methodologies

The XYZ Framework Applied

Throughout these solutions, we consistently used:

- **STAGE X:** State what we have, know, or observe
- **STAGE Y:** Explain why the method works and provide mathematical justification
- **STAGE Z:** Interpret results and determine next steps or implications

This ensures **complete understanding**, not just mechanical calculation.

Key Takeaways from Each Problem

Matrices (Problem 1)

1. **Rank method:** If rank $< n$, then $\lambda = 0$ is an eigenvalue
2. **Trace & determinant:** For 2×2 matrices, $\lambda_1 + \lambda_2 = \text{tr}(A)$ and $\lambda_1 \lambda_2 = \det(A)$
3. **Triangular matrices:** Eigenvalues = diagonal entries (instant!)
4. **Stability:** Positive eigenvalue $\Rightarrow$ unstable direction

Complex Numbers (Problem 2)

1. **Euler's formula:** $e^{i\theta} = \cos\theta + i\sin\theta$ connects complex exponentials to trig functions
2. **Complex conjugates:** $ze^{i\omega t} + z^* e^{-i\omega t} = 2\text{Re}(ze^{i\omega t})$ gives real oscillations
3. **n-th roots:** Equally spaced on circle, separated by $\frac{2\pi}{n}$ radians
4. **Connection to ODEs:** Complex eigenvalues $\Rightarrow$ oscillatory solutions

ODEs (Problem 3)

1. **Separable ODEs:** Move all x terms to one side, all t terms to other, integrate
2. **Characteristic equation:** For constant-coefficient ODEs, try $x = e^{\lambda t}$
3. **Complex roots:** $\lambda = \alpha \pm i\beta$ gives $e^{\alpha t}[A\cos(\beta t) + B\sin(\beta t)]$
4. **Eigendecomposition:** General solution is superposition: $\sum c_i e^{\lambda_i t}\mathbf{v}_i$
5. **Stability from eigenvalues:**
   - All $\text{Re}(\lambda_i) < 0 \Rightarrow$ stable
   - Any $\text{Re}(\lambda_i) > 0 \Rightarrow$ unstable
   - $\text{Im}(\lambda) \neq 0 \Rightarrow$ oscillations

Taylor Series (Problem 4)

1. **Maclaurin series:** $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$
2. **Sine series:** $\sin(x) = x - \frac{x^3}{6} + \frac{x^5}{120} - \cdots$
3. **Convergence:** Very rapid for small $|x|$ due to factorial growth in denominators
4. **Application:** Linearization near equilibria uses first-order Taylor expansion

Connection to Course Themes

All four problem types connect to the core course material on ODEs and dynamical systems:

1. **Eigenvalues determine stability** of equilibria in linear systems
2. **Complex eigenvalues produce oscillations** (focuses/centers)
3. **Real eigenvalues govern growth/decay** (nodes/saddles)
4. **Linearization via Taylor series** enables local stability analysis

MASTERY CHECK: You should now be able to:

- Find eigenvalues quickly using multiple methods
- Convert complex exponentials to real trigonometric forms
- Solve linear ODEs using characteristic equations
- Apply Taylor series for approximation and linearization
- Classify equilibria stability from eigenvalue signs

END OF SOLUTIONS