Exercise Sheet 1, Question 4: 1D Phase Portraits
Complete Solution with XYZ Methodology
Methods of Applied Mathematics [SEMT30006]

Problem Statement

Sketch phase portraits for the following differential equations, and describe the long-time behaviour for the given starting conditions.

(a) $\frac{du}{dt} = \frac{2}{1+u^2} - 1$, with $u(0) = -2$, $u(0) = 0$ and $u(0) = 2$.

(b) $\frac{du}{dt} = u^3 + 5u^2 - 6u$, with $u(0) = -1$ and $u(0) = 4$.

1 Foundational Concepts: 1D Phase Portraits

What is a Phase Portrait?

- **STAGE X (Definition):** A **phase portrait** is a geometric representation of the trajectories of a dynamical system in the phase space. For 1D systems $\frac{du}{dt} = f(u)$:
  - The **phase space** is the u-axis (one dimension)
  - **Equilibria** (fixed points) occur where $f(u) = 0$
  - The **flow** is represented by arrows showing direction of motion
  - Trajectories are solution curves $u(t)$ plotted against u
- **STAGE Y (Why phase portraits are useful):** Phase portraits provide qualitative understanding without solving explicitly:
  1. Identify all equilibria quickly
  2. Determine stability (stable/unstable)
  3. Predict long-time behavior from any initial condition
  4. Visualize the global dynamics
- **STAGE Z (How to construct):**
  1. Find equilibria: solve $f(u) = 0$
  2. Determine flow direction: check sign of $f(u)$ in each region
     - $f(u) > 0$: flow to the right (u increases)
     - $f(u) < 0$: flow to the left (u decreases)
  3. Classify equilibria:
     - $f'(u^*) < 0$: stable (attractor)
     - $f'(u^*) > 0$: unstable (repeller)

4. Draw arrows and trajectories

KEY PRINCIPLE: For 1D autonomous systems, solutions can never cross. Time flows monotonically along the phase line, and trajectories either approach equilibria or diverge to $\pm\infty$.

2 Part (a): $\frac{du}{dt} = \frac{2}{1+u^2} - 1$

Step 1: Find Equilibria

- **STAGE X (Setting up equilibrium equation):** Equilibria occur where $\frac{du}{dt} = 0$:

$$\frac{2}{1+u^2} - 1 = 0 \tag{1}$$

- **STAGE Y (Solving for equilibria):**

$$\frac{2}{1+u^2} = 1 \tag{2}$$
$$2 = 1 + u^2 \tag{3}$$
$$u^2 = 1 \tag{4}$$
$$u = \pm 1 \tag{5}$$

- **STAGE Z (Equilibrium points):**

$$\boxed{u_1^* = -1, \quad u_2^* = +1} \tag{6}$$

  There are exactly two equilibria.

Step 2: Determine Flow Direction

- **STAGE X (Define the function):** Let $f(u) = \frac{2}{1+u^2} - 1$.
  We need to determine the sign of $f(u)$ in each region:
  - Region I: $u < -1$
  - Region II: $-1 < u < 1$
  - Region III: $u > 1$

- **STAGE Y (Test points in each region):**
  Region I: $u < -1$ (test at $u = -2$):

$$f(-2) = \frac{2}{1+4} - 1 = \frac{2}{5} - 1 = -\frac{3}{5} < 0 \tag{7}$$

  Flow to the LEFT ($\leftarrow$)
  Region II: $-1 < u < 1$ (test at $u = 0$):

$$f(0) = \frac{2}{1+0} - 1 = 2 - 1 = 1 > 0 \tag{8}$$

  Flow to the RIGHT ($\rightarrow$)
  Region III: $u > 1$ (test at $u = 2$):

$$f(2) = \frac{2}{1+4} - 1 = \frac{2}{5} - 1 = -\frac{3}{5} < 0 \tag{9}$$

  Flow to the LEFT ($\leftarrow$)

- **STAGE Z (Flow pattern):**

$$u < -1: \quad f(u) < 0 \quad \text{(decreasing)} \tag{10}$$
$$-1 < u < 1: \quad f(u) > 0 \quad \text{(increasing)} \tag{11}$$
$$u > 1: \quad f(u) < 0 \quad \text{(decreasing)} \tag{12}$$

Step 3: Classify Equilibria Stability

- **STAGE X (Method 1: Flow direction analysis):**

  At $u^* = -1$:

  - Just left ($u < -1$): flow is $\leftarrow$ (away)
  - Just right ($u > -1$): flow is $\rightarrow$ (away)

  Both sides flow **away** from $u^* = -1 \Rightarrow$ **UNSTABLE**

  At $u^* = +1$:

  - Just left ($u < 1$): flow is $\rightarrow$ (toward)
  - Just right ($u > 1$): flow is $\leftarrow$ (toward)

  Both sides flow **toward** $u^* = +1 \Rightarrow$ **STABLE**

- **STAGE Y (Method 2: Linear stability analysis):** Compute $f'(u)$:

$$f'(u) = \frac{d}{du}\left(\frac{2}{1+u^2} - 1\right) = -\frac{4u}{(1+u^2)^2} \tag{13}$$

  At $u^* = -1$:

$$f'(-1) = -\frac{4(-1)}{(1+1)^2} = \frac{4}{4} = 1 > 0 \quad \Rightarrow \quad \textbf{UNSTABLE} \tag{14}$$

  At $u^* = +1$:

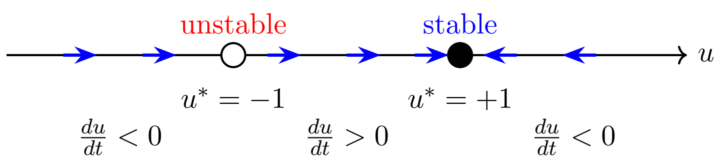
$$f'(+1) = -\frac{4(1)}{(1+1)^2} = -\frac{4}{4} = -1 < 0 \quad \Rightarrow \quad \textbf{STABLE} \tag{15}$$

- **STAGE Z (Summary):**

$$u^* = -1 : \quad \text{Unstable equilibrium (repeller)} \tag{16}$$
$$u^* = +1 : \quad \text{Stable equilibrium (attractor)} \tag{17}$$

Step 4: Sketch Phase Portrait

- **STAGE X (Phase line diagram):**



- **STAGE Y (Interpretation):**

  - Open circle at $u = -1$: unstable equilibrium (repeller)
  - Filled circle at $u = +1$: stable equilibrium (attractor)
  - Arrows show flow direction
  - All trajectories eventually approach $u = +1$ (except starting exactly at $u = -1$)

- **STAGE Z (Global dynamics):** The system has a single basin of attraction at $u^* = +1$, attracting all trajectories except the unstable equilibrium at $u^* = -1$.

Step 5: Analyze Specific Initial Conditions

- **STAGE X (Three initial conditions):**

  Case 1: $u(0) = -2$

  - Starting position: $u = -2$ (in region $u < -1$)
  - Flow direction: $\leftarrow$ (decreasing)
  - Long-time behavior: $u(t) \to -\infty$ as $t \to \infty$

  Case 2: $u(0) = 0$

  - Starting position: $u = 0$ (in region $-1 < u < 1$)
  - Flow direction: $\to$ (increasing)
  - Long-time behavior: $u(t) \to +1$ as $t \to \infty$

  Case 3: $u(0) = 2$

  - Starting position: $u = 2$ (in region $u > 1$)
  - Flow direction: $\leftarrow$ (decreasing)
  - Long-time behavior: $u(t) \to +1$ as $t \to \infty$

- **STAGE Y (Detailed trajectory behavior):**

  Trajectory from $u(0) = -2$:

  $$u(t): \quad -2 \to -3 \to -4 \to \cdots \to -\infty \tag{18}$$

  This trajectory escapes to $-\infty$. Since $f(u) \approx -1$ for large $|u|$, the escape rate is approximately linear: $u(t) \approx u_0 - t$.

  Trajectory from $u(0) = 0$:

  $$u(t): \quad 0 \to 0.5 \to 0.8 \to 0.95 \to \cdots \to 1^- \tag{19}$$

  This trajectory monotonically approaches $u^* = +1$ from below. Near equilibrium, $u(t) \approx 1 - Ce^{-t}$ (exponential approach).

  Trajectory from $u(0) = 2$:

  $$u(t): \quad 2 \to 1.5 \to 1.2 \to 1.05 \to \cdots \to 1^+ \tag{20}$$

  This trajectory monotonically approaches $u^* = +1$ from above. Near equilibrium, $u(t) \approx 1 + Ce^{-t}$.

- **STAGE Z (Summary of long-time behavior):**

  $$u(0) = -2: \quad u(t) \to -\infty \quad \text{(diverges)} \tag{21}$$
  $$u(0) = 0: \quad u(t) \to +1 \quad \text{(converges to stable equilibrium)} \tag{22}$$
  $$u(0) = 2: \quad u(t) \to +1 \quad \text{(converges to stable equilibrium)} \tag{23}$$

Step 6: Separatrix and Basin of Attraction

- **STAGE X (Separatrix):** The **separatrix** is the boundary between different long-time behaviors. In this system:
  - The unstable equilibrium $u^* = -1$ acts as a separatrix
  - Initial conditions $u_0 > -1$ converge to $u^* = +1$
  - Initial conditions $u_0 < -1$ diverge to $-\infty$
- **STAGE Y (Basin of attraction):** The **basin of attraction** for $u^* = +1$ is:

$$\mathcal{B}(u^* = +1) = (-1, +\infty) \tag{24}$$

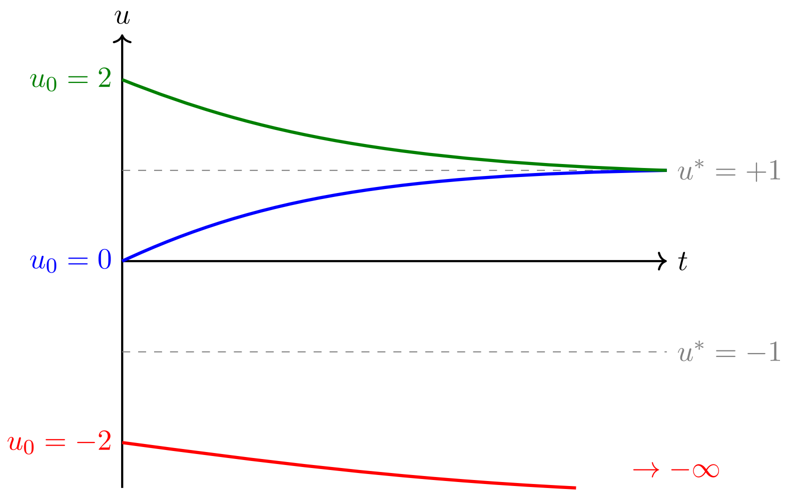
  All initial conditions in this interval converge to the stable equilibrium.

- **STAGE Z (Physical interpretation):** This system exhibits a threshold behavior:
  - Above threshold ($u > -1$): system stabilizes at $u = +1$
  - Below threshold ($u < -1$): system escapes to $-\infty$
  - At threshold ($u = -1$): unstable balance

KEY INSIGHT: The unstable equilibrium at $u^* = -1$ is the critical threshold separating bounded behavior (convergence to $u^* = +1$) from unbounded behavior (divergence to $-\infty$).

Step 7: Time Evolution (Qualitative)

- **STAGE X (Sketch of $u(t)$ vs t):**



- **STAGE Y (Rates of convergence/divergence):**

  For $u_0 = 0$ or $u_0 = 2$ approaching $u^* = +1$:
  - Near equilibrium: $|u(t) - 1| \sim e^{-t}$ (exponential)
  - Rate constant: $|f'(+1)| = 1$
  - Time scale: $\tau = 1$ (reaches $\approx 63\%$ of final value in time τ)

  For $u_0 = -2$ diverging to $-\infty$:
  - Far from equilibria: $f(u) \approx -1$ (approximately constant)

- Divergence rate: $u(t) \approx u_0 - t$ (approximately linear)

- **STAGE Z (Summary):**
  - Two trajectories converge exponentially to stable equilibrium
  - One trajectory diverges approximately linearly to $-\infty$
  - The unstable equilibrium at $u^* = -1$ separates these behaviors

3 Part (b): $\frac{du}{dt} = u^3 + 5u^2 - 6u$

Step 1: Find Equilibria

- **STAGE X (Setting up equilibrium equation):** Equilibria occur where $\frac{du}{dt} = 0$:

$$u^3 + 5u^2 - 6u = 0 \tag{25}$$

- **STAGE Y (Factoring):** Factor out u:

$$u(u^2 + 5u - 6) = 0 \tag{26}$$

  This gives $u = 0$ or $u^2 + 5u - 6 = 0$.

  For the quadratic, use the quadratic formula or factor:

$$u^2 + 5u - 6 = (u + 6)(u - 1) = 0 \tag{27}$$

  So $u = -6$ or $u = 1$.

- **STAGE Z (Three equilibria):**

$$\boxed{u_1^* = -6, \quad u_2^* = 0, \quad u_3^* = +1} \tag{28}$$

  There are exactly three equilibria, ordered: $-6 < 0 < 1$.

Step 2: Rewrite in Factored Form

- **STAGE X (Factored form):**

$$\frac{du}{dt} = u(u + 6)(u - 1) \tag{29}$$

- **STAGE Y (Why factored form is useful):** The sign of $f(u) = u(u + 6)(u - 1)$ depends on the signs of each factor:
  - u: negative for $u < 0$, positive for $u > 0$
  - $(u + 6)$: negative for $u < -6$, positive for $u > -6$
  - $(u - 1)$: negative for $u < 1$, positive for $u > 1$

- **STAGE Z (Sign analysis regions):** We have 4 regions separated by the three equilibria:
  - Region I: $u < -6$
  - Region II: $-6 < u < 0$
  - Region III: $0 < u < 1$
  - Region IV: $u > 1$

Step 3: Determine Flow Direction

- **STAGE X (Sign table method):**

Region	u	$(u+6)$	$(u-1)$	$f(u) = u(u+6)(u-1)$
$u < -6$	$-$	$-$	$-$	$(-)(-)(-) = -$
$-6 < u < 0$	$-$	$+$	$-$	$(-)(+)(-) = +$
$0 < u < 1$	$+$	$+$	$-$	$(+)(+)(-) = -$
$u > 1$	$+$	$+$	$+$	$(+)(+)(+) = +$

- **STAGE Y (Flow directions):**

  Region I: $u < -6$ (e.g., $u = -7$):

$$f(-7) = (-7)(-1)(-8) = -56 < 0 \quad \Rightarrow \quad \text{flow LEFT } (\leftarrow) \tag{30}$$

  Region II: $-6 < u < 0$ (e.g., $u = -3$):

$$f(-3) = (-3)(3)(-4) = 36 > 0 \quad \Rightarrow \quad \text{flow RIGHT } (\rightarrow) \tag{31}$$

  Region III: $0 < u < 1$ (e.g., $u = 0.5$):

$$f(0.5) = (0.5)(6.5)(-0.5) = -1.625 < 0 \quad \Rightarrow \quad \text{flow LEFT } (\leftarrow) \tag{32}$$

  Region IV: $u > 1$ (e.g., $u = 2$):

$$f(2) = (2)(8)(1) = 16 > 0 \quad \Rightarrow \quad \text{flow RIGHT } (\rightarrow) \tag{33}$$

- **STAGE Z (Summary of flow):**

$$u < -6: \quad f(u) < 0 \quad \text{(decreasing)} \tag{34}$$
$$-6 < u < 0: \quad f(u) > 0 \quad \text{(increasing)} \tag{35}$$
$$0 < u < 1: \quad f(u) < 0 \quad \text{(decreasing)} \tag{36}$$
$$u > 1: \quad f(u) > 0 \quad \text{(increasing)} \tag{37}$$

Step 4: Classify Equilibria Stability

- **STAGE X (Flow analysis at each equilibrium):**

  At $u^* = -6$:

  - Just left ($u < -6$): flow is $\leftarrow$ (away)
  - Just right ($u > -6$): flow is $\rightarrow$ (away)

  Both sides flow away $\Rightarrow$ **UNSTABLE**

  At $u^* = 0$:

  - Just left ($u < 0$): flow is $\rightarrow$ (toward)
  - Just right ($u > 0$): flow is $\leftarrow$ (toward)

  Both sides flow toward $\Rightarrow$ **STABLE**

  At $u^* = +1$:

  - Just left ($u < 1$): flow is $\leftarrow$ (away)

– Just right ($u > 1$): flow is $\rightarrow$ (away)

Both sides flow away $\Rightarrow$ **UNSTABLE**

- **STAGE Y (Linear stability verification):** Compute $f'(u)$:

$$f(u) = u^3 + 5u^2 - 6u \tag{38}$$
$$f'(u) = 3u^2 + 10u - 6 \tag{39}$$

At $u^* = -6$:

$$f'(-6) = 3(36) + 10(-6) - 6 = 108 - 60 - 6 = 42 > 0 \quad \Rightarrow \quad \textbf{UNSTABLE} \tag{40}$$

At $u^* = 0$:

$$f'(0) = 3(0) + 10(0) - 6 = -6 < 0 \quad \Rightarrow \quad \textbf{STABLE} \tag{41}$$

At $u^* = +1$:

$$f'(+1) = 3(1) + 10(1) - 6 = 3 + 10 - 6 = 7 > 0 \quad \Rightarrow \quad \textbf{UNSTABLE} \tag{42}$$

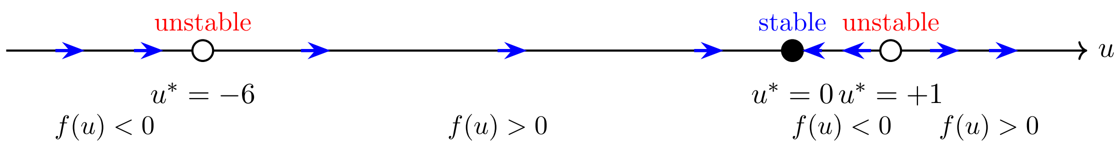
- **STAGE Z (Classification summary):**

$$u^* = -6 : \quad \text{Unstable equilibrium (repeller)} \tag{43}$$
$$u^* = 0 : \quad \text{Stable equilibrium (attractor)} \tag{44}$$
$$u^* = +1 : \quad \text{Unstable equilibrium (repeller)} \tag{45}$$

The pattern is: unstable-stable-unstable.

Step 5: Sketch Phase Portrait

- **STAGE X (Phase line diagram):**



- **STAGE Y (Interpretation):**
  - Open circles at $u = -6$ and $u = +1$: unstable equilibria
  - Filled circle at $u = 0$: stable equilibrium (attractor)
  - Arrows show flow direction in each region
- **STAGE Z (Global structure):** The system has:
  - One stable equilibrium at $u^* = 0$ with basin of attraction $(-6, +1)$
  - Two unstable equilibria acting as separatrices
  - Trajectories outside $[-6, +1]$ diverge to $\pm\infty$

Step 6: Analyze Specific Initial Conditions

- **STAGE X (Two initial conditions):**

  Case 1: $u(0) = -1$

  - Starting position: $u = -1$ (in region $-6 < u < 0$)
  - Flow direction: $\rightarrow$ (increasing)
  - Path: $-1 \rightarrow -0.5 \rightarrow -0.1 \rightarrow \cdots \rightarrow 0^-$
  - Long-time behavior: $u(t) \rightarrow 0$ as $t \rightarrow \infty$

  Case 2: $u(0) = 4$

  - Starting position: $u = 4$ (in region $u > 1$)
  - Flow direction: $\rightarrow$ (increasing)
  - Path: $4 \rightarrow 5 \rightarrow 10 \rightarrow 100 \rightarrow \cdots \rightarrow +\infty$
  - Long-time behavior: $u(t) \rightarrow +\infty$ as $t \rightarrow \infty$

- **STAGE Y (Detailed trajectory analysis):**

  From $u(0) = -1$:

  - Trajectory stays in region $(-6, 0)$
  - Monotonically increases toward stable equilibrium $u^* = 0$
  - Near equilibrium: $u(t) \approx -Ce^{-6t}$ (exponential approach)
  - Decay rate: $|f'(0)| = 6$ (fast convergence)
  - Time scale: $\tau = 1/6 \approx 0.167$

  From $u(0) = 4$:

  - Trajectory starts beyond unstable equilibrium $u^* = 1$
  - Flows away from $u^* = 1$ to the right
  - Growth accelerates: $f(u) = u(u+6)(u-1)$ grows cubically for large u
  - For large u: $f(u) \approx u^3$
  - Divergence is finite-time: $\frac{du}{dt} \sim u^3$ leads to blow-up

- **STAGE Z (Finite-time blow-up for $u_0 = 4$):**

  For $u > 1$, approximate $f(u) \approx u^3$ for large u:

$$\frac{du}{dt} \approx u^3 \quad \Rightarrow \quad \frac{du}{u^3} \approx dt \tag{46}$$

  Integrating:

$$-\frac{1}{2u^2} \approx t + C \tag{47}$$

  At $t = 0$: $C = -\frac{1}{2u_0^2}$

$$-\frac{1}{2u^2} = t - \frac{1}{2u_0^2} \tag{48}$$

$$\frac{1}{u^2} = \frac{1}{u_0^2} - 2t \tag{49}$$

$$u(t) = \frac{1}{\sqrt{\frac{1}{u_0^2} - 2t}} \tag{50}$$

This blows up at time $t^* = \frac{1}{2u_0^2}$.

For $u_0 = 4$:

$$t^* = \frac{1}{2(16)} = \frac{1}{32} \approx 0.03125 \tag{51}$$

The solution reaches $+\infty$ in finite time $t^* \approx 0.031$!

Step 7: Basin of Attraction and Separatrices

- **STAGE X (Basin of attraction for $u^* = 0$):**

$$\mathcal{B}(u^* = 0) = (-6, +1) \tag{52}$$

  All initial conditions in this open interval converge to $u^* = 0$.

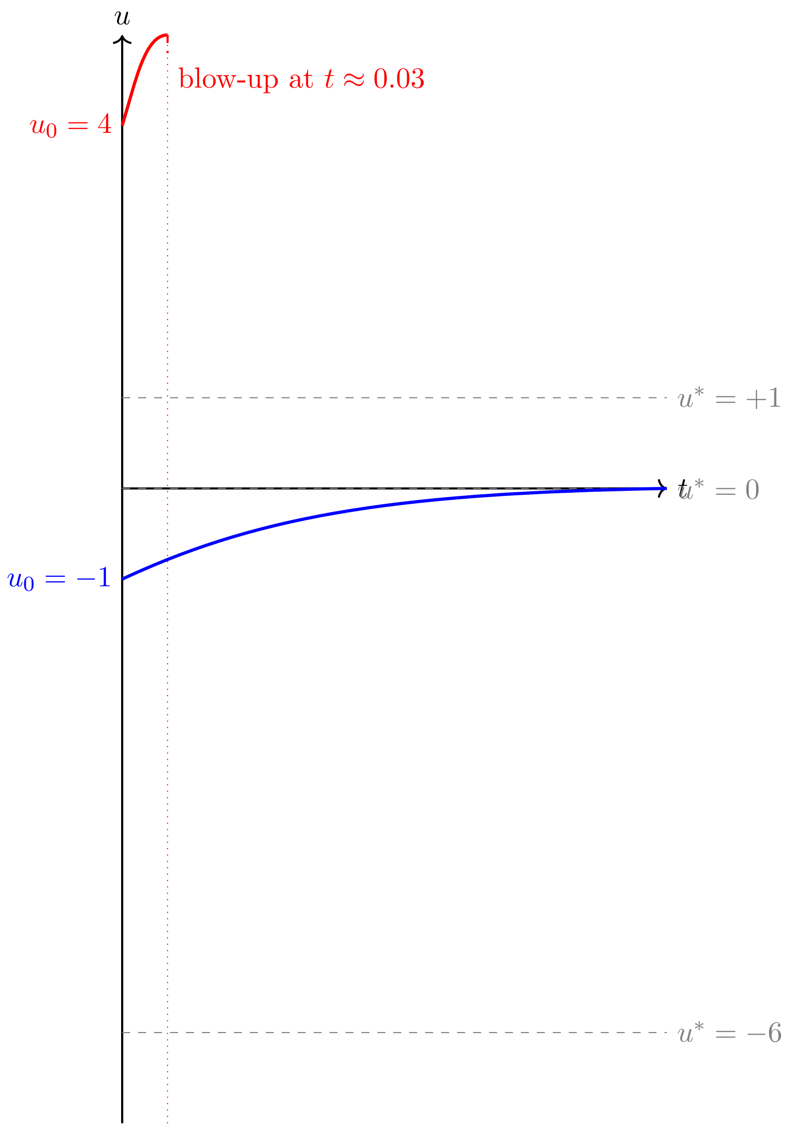
- **STAGE Y (Separatrices):** The unstable equilibria act as separatrices:
  - $u^* = -6$ separates:
    - $u_0 < -6$: divergence to $-\infty$
    - $u_0 > -6$: bounded behavior
  - $u^* = +1$ separates:
    - $u_0 < 1$: convergence to $u^* = 0$ (if $u_0 > -6$)
    - $u_0 > 1$: divergence to $+\infty$

- **STAGE Z (Complete classification by initial condition):**

$$\begin{aligned} u_0 < -6 &: \quad u(t) \to -\infty && (53)\\ u_0 = -6 &: \quad u(t) = -6 \text{ (stays at unstable equilibrium)} && (54)\\ -6 < u_0 < 1 &: \quad u(t) \to 0 && (55)\\ u_0 = 1 &: \quad u(t) = 1 \text{ (stays at unstable equilibrium)} && (56)\\ u_0 > 1 &: \quad u(t) \to +\infty \text{ (finite-time blow-up)} && (57) \end{aligned}$$

Step 8: Time Evolution Sketch

- **STAGE X (Plot of $u(t)$ vs t):**



- **STAGE Y (Key features):**
  - Blue curve ($u_0 = -1$): smooth exponential convergence to $u^* = 0$
  - Red curve ($u_0 = 4$): rapid divergence to $+\infty$ in finite time
  - Vertical asymptote at blow-up time for $u_0 = 4$
- **STAGE Z (Summary):**

$$u(0) = -1: \quad u(t) \to 0 \text{ exponentially, } \tau = 1/6 \tag{58}$$
$$u(0) = 4: \quad u(t) \to +\infty \text{ at } t \approx 0.031 \tag{59}$$

KEY INSIGHT: This system exhibits rich dynamics with a stable attractor at $u^* = 0$ flanked by two unstable equilibria. Initial conditions outside the basin $(-6, +1)$ lead to finite-time blow-up, a phenomenon common in cubic nonlinearities.

4 Summary and Comparison

Comparison of Parts (a) and (b)

Feature	**Part (a)**	**Part (b)**
Number of equilibria	2	3
Stable equilibria	1 (at $u = +1$)	1 (at $u = 0$)
Unstable equilibria	1 (at $u = -1$)	2 (at $u = -6, +1$)
Basin of attraction	$(-1, +\infty)$	$(-6, +1)$
Divergence to $-\infty$	Yes (if $u_0 < -1$)	Yes (if $u_0 < -6$)
Divergence to $+\infty$	No	Yes (if $u_0 > +1$)
Finite-time blow-up	No	Yes (for $u_0 > 1$)
Nonlinearity	Rational function	Cubic polynomial

Key Lessons from 1D Phase Portraits

- **STAGE X (What we learned):**
  1. **Equilibria classification:** Use flow direction or $f'(u^*)$ to determine stability
  2. **Separatrices:** Unstable equilibria separate regions with different long-time behavior
  3. **Basins of attraction:** The set of initial conditions leading to each attractor
  4. **Global structure:** Complete understanding of all possible trajectories
- **STAGE Y (Techniques mastered):**
  1. Finding equilibria by solving $f(u) = 0$
  2. Determining flow direction using sign analysis
  3. Classifying stability using flow or derivatives
  4. Sketching phase portraits on the phase line
  5. Predicting long-time behavior from phase portraits
  6. Identifying finite-time blow-up
- **STAGE Z (Connection to higher dimensions):** In 1D:
  - Phase space is a line
  - Trajectories can't cross
  - Only nodes (stable/unstable) exist
  - Analysis is completely solvable

  In 2D (next question):
  - Phase space is a plane
  - More complex equilibria: nodes, saddles, foci, centers
  - Closed orbits (limit cycles) possible
  - Richer dynamics, but still analyzable

UNIVERSAL PRINCIPLE: In 1D autonomous systems, trajectories are monotonic between equilibria. Solutions either approach equilibria or diverge to $\pm\infty$. No oscillations or closed orbits are possible in 1D.

END OF QUESTION 4