Asymptotics 2025/2026 Solutions for Sheet 8 [1]

Boundary layers and asymptotic matching

Problem 1: For the leading order **outer solution** we neglect the ε^2-term in $\varepsilon^2 y'' - y = 0$, and obtain a vanishing outer solution

$$y_0(x) = 0 \ .$$

Inner solution at $x = 1$: With $x - 1 = \varepsilon X$ and $Y(X) = y(x)$ we obtain the inner equation $Y'' - Y = 0$ with the general solution $Y(X) = ae^X + be^{-X}$. From the boundary condition at $x = 1$ we obtain one condition for the parameters: $a + b = 1$. The second condition is obtained by matching the inner solution to the outer solution. Matching is only possible with the exponential decaying term for large negative X, so $a = 1$ and $b = 0$

$$Y(X) = e^X = \exp\left(\frac{x-1}{\varepsilon}\right) \ .$$

Inner solution at $x = -1$: With $x + 1 = \varepsilon V$ and $W(V) = y(x)$ we obtain the same inner equation $W'' - W = 0$ with the solution $W(V) = ae^V + be^{-V}$. By a similar argument as before we obtain $a = 0$ and $b = 1$, and so

$$W(V) = e^{-V} = \exp\left(-\frac{x+1}{\varepsilon}\right) \ .$$

The **composite solution** is the sum of the two inner solutions, since the function is only appreciably different from zero in the two boundary layer regions.

$$y_c(x) = Y(X) + W(V) = \exp\left(\frac{x-1}{\varepsilon}\right) + \exp\left(-\frac{x+1}{\varepsilon}\right) = \frac{\exp\left(\frac{x}{\varepsilon}\right) + \exp\left(-\frac{x}{\varepsilon}\right)}{\exp\left(\frac{1}{\varepsilon}\right)} \ .$$

It differs from the exact solution

$$y(x) = \frac{\cosh\left(\frac{x}{\varepsilon}\right)}{\cosh\left(\frac{1}{\varepsilon}\right)} = \frac{\exp\left(\frac{x}{\varepsilon}\right) + \exp\left(-\frac{x}{\varepsilon}\right)}{\exp\left(\frac{1}{\varepsilon}\right) + \exp\left(-\frac{1}{\varepsilon}\right)}$$

by an exponentially small term in the denominator.

Problem 2: From the analysis in the lecture we know that there is a boundary layer with size of $\mathcal{O}(\varepsilon)$ at $x = 1$ since the coefficient of f' is negative. (The fact that the differential equation is inhomogeous does not change the analysis.)
For the leading **outer solution** we neglect the term $\varepsilon f''$ and obtain the equation

$$-f_0' + f_0/(x+1) = 2 \ .$$

The general solution of this equation is $f_0(x) = c(x+1) - 2(x+1)\log(x+1)$. The boundary condition $f_0(0) = 0$ then requires $c = 0$, so that

$$f_0(x) = -2(x+1)\log(x+1)\ .$$

For the **inner solution** we set $X = (x-1)/\varepsilon$, $F(X) = f(x)$ and obtain the inner equation

$$\frac{1}{\varepsilon}F'' - \frac{1}{\varepsilon}F' + \frac{1}{2+\varepsilon X}F = 2\ .$$

In leading order this yields $F_0'' - F_0' = 0$ with the general solution $F_0(X) = A + Be^X$. The one which satisfies the boundary condition at $x = 1$, $F_0(0) = 3$, is

$$F_0(X) = 3e^X + A(1 - e^X)\ .$$

We perform the matching between inner and outer solution with Prandtl's rule

$$\lim_{x\to 1} f_0(x) = \lim_{X\to -\infty} F_0(X) \qquad \Longrightarrow \qquad A = -4\log 2\ ,$$

Finally, the composite solution is given by

$$f_c(x) = f_0(x) + F_0(X) + 4\log 2 = -2(x+1)\log(x+1) + (3 + 4\log 2)\exp\left(\frac{x-1}{\varepsilon}\right)\ .$$

Problem 3: We know from the analysis in the lecture that there is a boundary layer with size of $\mathcal{O}(\varepsilon)$ at $x = 0$, since the coefficient of f' is positive in $[0,1]$.
To obtain the **outer solution** we insert $f(x) = f_0(x) + \varepsilon f_1(x) + \ldots$ into the differential equation and the boundary condition at $x = 1$

$$\varepsilon[f_0'' + \varepsilon f_1'' + \ldots] + (2+x)[f_0' + \varepsilon f_1' + \ldots] + [f_0 + \varepsilon f_1 + \ldots] = 1\ ,$$

$$f_0(1) + \varepsilon f_1(1) + \cdots = 0\ .$$

At $\mathcal{O}(\varepsilon^0)$, the problem is

$$(2+x)f_0' + f_0 = 1\ , \qquad f_0(1) = 0\ .$$

The general solution of the differential equation is $f_0(x) = (x + a_0)/(x+2)$ and from the boundary condition we obtain $a_0 = -1$. At $\mathcal{O}(\varepsilon^1)$, the problem is

$$(2+x)f_1' + f_1 = -f_0'' = \frac{6}{(x+2)^3}\ , \qquad f_1(1) = 0\ .$$

The general solution of the differential equation is $f_1(x) = -3/(x+2)^3 + a_1/(x+2)$ and from the boundary condition we obtain $a_1 = 1/3$, so the final result is

$$f(x) = \frac{x-1}{x+2} + \varepsilon\left[\frac{-3}{(x+2)^3} + \frac{1}{3(x+2)}\right] + \mathcal{O}(\varepsilon^2)\ .$$

For the **inner solution** we set $x = \varepsilon X$, $F(X) = f(x)$ and insert $F(X) = F_0(X) + \varepsilon F_1(X) + \ldots$ into the differential equation and the boundary condition at $x = 0$.

$$\frac{1}{\varepsilon}[F_0'' + \varepsilon F_1'' + \ldots] + \frac{(2+\varepsilon X)}{\varepsilon}[F_0' + \varepsilon F_1' + \ldots] + [F_0 + \varepsilon F_1 + \ldots] = 1 \ ,$$
$$F_0(0) + \varepsilon F_1(0) + \cdots = 2 \ .$$

At $\mathcal{O}(\varepsilon^{-1})$, the problem is

$$F_0'' + 2F_0' = 0 \ , \qquad F_0(0) = 2 \ .$$

The general solution of the differential equation is $F_0(X) = A_0 e^{-2X} + B_0$ and from the boundary condition we obtain $B_0 = 2 - A_0$ (or $A_0 = 2 - B_0$). At $\mathcal{O}(\varepsilon^0)$, the problem is

$$F_1'' + 2F_1' = 1 - F_0 - XF_0' = A_0 - 1 - A_0 e^{-2X} + 2A_0 X e^{-2X} \ , \qquad F_1(0) = 0 \ .$$

The general solution of the differential equation is

$$F_1(X) = -\frac{1}{2}A_0 X^2 e^{-2X} + \frac{1}{2}(A_0 - 1)X + A_1 e^{-2X} + B_1 \ ,$$

and from the boundary condition we obtain $B_1 = -1$, so the final result is

$$F(X) = 2 - A_0 + A_0 e^{-2X} + \varepsilon\left[-\frac{A_0}{2}X^2 e^{-2X} + \frac{A_0 - 1}{2}X + A_1(e^{-2X} - 1)\right] + \mathcal{O}(\varepsilon^2) \ .$$

To perform the **Van Dyke matching** we insert $x = \varepsilon X$ into the outer equation and determine then the first two leading orders in ε.

$$f(\varepsilon X) = \frac{\varepsilon X - 1}{\varepsilon X + 2} + \varepsilon\left[\frac{-3}{(\varepsilon X + 2)^3} + \frac{1}{3(\varepsilon X + 2)}\right] + \mathcal{O}(\varepsilon^2) = -\frac{1}{2} + \frac{3}{4}\varepsilon X - \frac{5}{24}\varepsilon + \mathcal{O}(\varepsilon^2) \ . \quad (1)$$

Vice versa, we insert $X = x/\varepsilon$ into the inner solution and determine again the first two leading orders in ε.

$$\begin{aligned} F\left(\frac{x}{\varepsilon}\right) &= 2 - A_0 + A_0 e^{-2x/\varepsilon} + \varepsilon\left[-\frac{A_0 x^2}{2\varepsilon^2}e^{-2x/\varepsilon} + \frac{(A_0 - 1)x}{2\varepsilon} + A_1(e^{-2x/\varepsilon} - 1)\right] + \mathcal{O}(\varepsilon^2) \\ &= 2 - A_0 + \frac{x}{2}(A_0 - 1) - \varepsilon A_1 + \mathcal{O}(\varepsilon^2) \ . \qquad (2) \end{aligned}$$

We get an exact agreement between equations (1) and (2) if we choose $A_0 = 5/2$ and $A_1 = 5/24$ (note that $x = \varepsilon X$). For the **composite solution** we add inner and outer solutions and subtract their common limit in the matching region. For the considered order

$$\begin{aligned} f_c(x) &= f_0(x) + \varepsilon f_1(x) + F_0(x/\varepsilon) + \varepsilon F_1(x/\varepsilon) - \left[-\frac{1}{2} + \frac{3}{4}x - \frac{5}{24}\varepsilon\right] \\ &= \frac{x - 1}{x + 2} + \varepsilon\left[\frac{-3}{(x+2)^3} + \frac{1}{3(x+2)}\right] + \left[\frac{5}{2} - \frac{5x^2}{4\varepsilon} + \frac{5\varepsilon}{24}\right]e^{-2x/\varepsilon} \ . \end{aligned}$$

Problem 4: Since the coefficient of y' is positive for $x > 0$ there can't be a boundary layer for $x > 0$, so the only possibility is a boundary layer at $x = 0$. However since the coefficient of y' is zero at $x = 0$ it can have a different size than $\mathcal{O}(\varepsilon)$.

For the first-order **outer solution** we neglect the ε-dependent term in the o.d.e. and obtain

$$y_0' - xy_0 = 0 \;, \qquad y_0(1) = \beta \qquad \Longrightarrow \qquad y_0(x) = \beta \exp\left(\frac{x^2-1}{2}\right) \;.$$

For the **inner solution** we set $x = \delta X$, $Y(X) = y(x)$ and obtain

$$\frac{\varepsilon}{\delta^2} Y'' + \frac{\delta^2 X^2}{\delta} Y' - \delta^3 X^3 Y = 0 \;.$$

The third term on the LHS is smaller than the second term (since we are looking for a δ of o(1)), so the dominant balance argument yields $\varepsilon/\delta^2 = \delta$ or $\delta = \varepsilon^{1/3}$. Then the equation for the first-order outer solution is

$$Y_0'' + X^2 Y_0' = 0 \quad \Longrightarrow \quad Y_0' = A \exp\left(-\frac{X^3}{3}\right) \quad \Longrightarrow \quad Y_0 = A \int_0^X \exp\left(-\frac{s^3}{3}\right) ds + B \;,$$

and from the boundary condition $Y_0(0) = \alpha$ we obtain $B = \alpha$.
The matching can be performed by Prandtl's rule since both functions approach a constant in the matching region as we will see

$$\lim_{x\to 0} y_0(x) = \beta e^{-1/2} \;,$$

$$\lim_{X\to\infty} Y_0(X) = \alpha + AI \quad \text{where} \quad I = \int_0^\infty e^{-s^3/3} ds \quad [= 3^{-2/3}\Gamma(1/3)] \;.$$

Thus the matching yields $A = (\beta e^{-1/2} - \alpha)/I$ and the composite solution is

$$y_c(x) = y_0(x) + Y_0\left(\frac{x}{\varepsilon^{1/3}}\right) - \frac{\beta}{\sqrt{e}} = \beta \exp\left(\frac{x^2-1}{2}\right) + \left(\frac{\beta}{\sqrt{e}} - \alpha\right)\left[\frac{\int_0^{x\varepsilon^{-1/3}} e^{-s^3/3} ds}{\int_0^\infty e^{-s^3/3} ds} - 1\right] \;.$$

Problem 5: For an inner expansion $Y(X) \sim \sum_{n=0}^N \chi_n(\varepsilon) Y_n(X)$ and an outer expansion $y(x) \sim \sum_{m=0}^M \psi_m(\varepsilon) y_m(x)$ we assume that there exists a matching region in which both functions are asymptotic to $u(x) \sim \sum_{k=0}^K \phi_k(\varepsilon) y_k(x)$. Thus we consider the composite expansion $y_c = y + Y - u$. In the inner region: $Y = Y$, $y \to u$, $u = u \Longrightarrow y_c \to Y$. In the outer region $y = y$, $Y \to u$, $u = u \Longrightarrow y_c \to y$, and hence the expansion is uniformly valid. For two "inner" regions suppose that the two inner solutions are Y and Z, and that they match to the outer solution y with expansions u and v, respectively. Then $y_c = y + Y + Z - u - v$ is an effective composite expansion: In one inner region: $Y = Y$, $y \to u$, $Z \to v$, $u = u$, $v = v \Longrightarrow y_c \to Y$. In the second inner region: $Z = Z$, $y \to v$, $Y \to u$, $u = u$, $v = v \Longrightarrow y_c \to Z$. Finally, in the inner region: $y = y$, $Y \to u$, $Z \to v$, $u = u$, $v = v \Longrightarrow y_c \to y$. Again we see that the expansion is uniformly valid.

Problem 6: The coefficient of y' vanishes at both boundaries, for that reason we do not know at which boundary we have boundary layers. We try an inner solution at either side and see if we can match it to the outer solution.

The **outer solution** is determined by neglecting the ε-dependent term in the o.d.e.

$$\sin(x)y' + \sin(2x)y = 0 \quad \Longrightarrow \quad y' + 2\cos(x)y = 0 \quad \Longrightarrow \quad y(x) = a\exp(-2\sin(x)) \;.$$

We do not insert a boundary condition yet, since we do not know the location of the boundary layer(s). We note, however, that the outer solution approaches a constant as x either goes to $x = 0$ or $x = \pi$.

For an **inner solution** at $x = 0$ we set $x = \delta X$, and $Y(X) = y(x)$. To perform a dominant balance, we take only the leading terms for each coefficient of Y, Y' and Y'':

$$\frac{\varepsilon}{\delta^2} Y'' + \frac{\sin(\delta X)}{\delta} Y' + \sin(2\delta X) Y = 0 \qquad \Longrightarrow \qquad \frac{\varepsilon}{\delta^2} Y'' + XY' + 2\delta XY \approx 0 \; .$$

We see that we can neglect the third term on the LHS in comparison to the second one, and for a distinguished limit we choose $\varepsilon/\delta^2 = 1$ or $\delta = \varepsilon^{1/2}$. Then the equation for the first-order approximation $Y_{0,a}(X)$ is

$$Y''_{0,a} + XY'_{0,a} = 0 \quad \Longrightarrow \quad Y'_{0,a} = A \exp\left(-\frac{1}{2}X^2\right) \quad \Longrightarrow \quad Y_{0,a} = A \int_0^X \exp\left(-\frac{1}{2}s^2\right) ds + B \; .$$

and with the boundary condition $Y_0(0) = \pi$ we obtain $B = \pi$. As X goes to infinity this solution approaches a constant, so a matching with the outer solution is possible.

For an **inner solution** at $x = \pi$ we set $x - \pi = \delta X$, and $Y(X) = y(x)$, and again we take only the leading terms for each coefficient of Y, Y' and Y'':

$$\frac{\varepsilon}{\delta^2} Y'' + \frac{\sin(\pi + \delta X)}{\delta} Y' + \sin(2\pi + 2\delta X) Y = 0 \qquad \Longrightarrow \qquad \frac{\varepsilon}{\delta^2} Y'' - XY' + 2\delta XY \approx 0 \; .$$

The arguments are analogous to before, but the coefficient of Y' has a different sign. We thus get $\delta = \varepsilon^{1/2}$ and

$$Y''_{0,b} - XY'_{0,b} = 0 \quad \Longrightarrow \quad Y'_{0,b} = C \exp\left(\frac{1}{2}X^2\right) \quad \Longrightarrow \quad Y_{0,b} = C \int_0^X \exp\left(\frac{1}{2}s^2\right) ds + D \; ,$$

and with the boundary condition $Y_0(0) = 0$ we obtain $D = 0$. However, as X goes to minus infinity this solution increases exponentially and no matching is possible. We conclude that there is no boundary layer at $x = \pi$. Consequently, we can apply the boundary condition at $x = \pi$ to the outer solution, $y_0(\pi) = 0$, with the result that $a = 0$ and the outer solution vanishes.

Let us use Van Dyke's matching here, although we can use also the simpler Prandtl rule. Set $X = x\varepsilon^{-1/2}$ in $Y_{0,a}$ and take the leading term as $\varepsilon \to 0$

$$Y_{0,a} = A \int_0^{x\varepsilon^{-1/2}} e^{-s^2/2} ds + \pi = A \left[\int_0^\infty - \int_{x\varepsilon^{-1/2}}^\infty \right] e^{-s^2/2} ds + \pi = A\sqrt{\frac{\pi}{2}} + \pi + o(1) \; .$$

Matching to the outer solution $y_0(x) = 0$ requires $A = -\sqrt{2\pi}$. The composite solution is identical to the inner solution (since the outer solution vanishes) and has the form

$$y_c(x) = \pi - \sqrt{2\pi} \int_0^{x\varepsilon^{-1/2}} \exp\left(-\frac{1}{2}s^2\right) ds = \pi \left[1 - \operatorname{erf}\left(\frac{x}{\sqrt{2\varepsilon}}\right) \right] \; .$$

Problem 7: Since the coefficient of y' is negative at the left boundary and positive at the right boundary there can be no boundary layer. The only possibility is an interior layer at $x = 0$ where the coefficient vanishes. Then one has two outer solutions, right and left to the interior layer.

The leading **outer equation** is obtained by neglecting the ε-dependent term in the o.d.e. and is given by

$$y_0' + y_0 = 0 \qquad \Longrightarrow \qquad y_0(x) = ae^{-x} \; .$$

For the left outer solution we require $y_{0,a}(-1) = e$ and obtain $y_{0,a}(x) = e^{-x}$, and for the right outer solution we require $y_{0,b}(1) = 2e^{-1}$ and obtain $y_{0,b}(x) = 2e^{-x}$.

For the first-order **inner solution** we set $x = \delta X$, $Y(X) = y(x)$ and obtain

$$\frac{\varepsilon}{\delta^2} Y'' + XY' + \delta XY = 0 \; .$$

We can neglect the third term on the LHS with respect to the second term, and we require for a distinguished limit that the other two terms are of the same order, which can be achieved by $\varepsilon/\delta^2 = 1$ or $\delta = \sqrt{\varepsilon}$. Then the leading-order inner solution is

$$Y_0'' + XY_0' = 0 \quad \Longrightarrow \quad Y_0' = A \exp\left(-\frac{1}{2}X^2\right) \quad \Longrightarrow \quad Y_0 = A \int_0^X \exp\left(-\frac{1}{2}s^2\right) ds + B \; .$$

We can perform here the simpler Prandtl's matching. The matching in the region to the left of $x = 0$ is done by requiring

$$\lim_{x\to 0} y_{0,a}(x) = \lim_{X\to -\infty} Y_0(X) \quad \Longrightarrow \quad B - A\sqrt{\frac{\pi}{2}} = 1 \; ,$$

and in the region to the right of $x = 0$ by requiring

$$\lim_{x\to 0} y_{0,b}(x) = \lim_{X\to +\infty} Y_0(X) \quad \Longrightarrow \quad B + A\sqrt{\frac{\pi}{2}} = 2 \; .$$

The solutions for A and B are $A = 1/\sqrt{2\pi}$ and $B = 3/2$, and the inner solution is

$$Y_0 = \frac{3}{2} + \frac{1}{\sqrt{2\pi}} \int_0^X \exp\left(-\frac{1}{2}s^2\right) ds = \frac{3}{2} + \frac{1}{2}\,\mathrm{erf}\left(\frac{X}{\sqrt{2}}\right) \; .$$

The composite solution has here not the form of a sum of the inner and the two outer solutions minus the limites in the two matching regions (since either outer solution does not vanish in the region where the other is valid). Instead one can form a composite solution by

$$y_c(x) = \left[\frac{3}{2} + \frac{1}{2}\,\mathrm{erf}\left(\frac{x}{\sqrt{2\varepsilon}}\right)\right] e^{-x} \; .$$