21 Bifurcations (for maps)

For every bifurcation in an ODE, there is a corresponding bifurcation in a map. Let λ be an eigenvalue in a n-d system, or just $\lambda = f'(x_*)$ for a 1d system. A bifurcation happens:

- In an ODE as λ passes through the imaginary axis where $\mathsf{Re}(\lambda) = 0$,
- In a map as λ passes through the unit circle where $|\lambda| = 1$.

The basic bifurcations are (taking just 1d, except for the last case):

In an ODE $\dot{x} = f(x)$:

Fold

- $f(x_*) = 0$ and $\lambda = 0$

Transcritical

- $f(x_*) = 0$ and $\lambda = 0$
- $\frac{\partial f}{\partial \beta} = 0$

Pitchfork

- $f(x_*) = 0$ and $\lambda = 0$
- $f(-x) = -f(x)$

- - no counterpart - -

Cusp

- $f(x_*) = 0$ and $\lambda = 0$
- $f''(x_*) = 0$

Hopf

- $f(x_*) = 0$ and $\mathsf{Re}(\lambda) = 0$
- 2 dims, $\lambda \in \mathbb{C}$

In a Map $x_{n+1} = f(x_n)$:

Fold

- $f(x_*) = x_*$ and $\lambda = 1$

Transcritical

- $f(x_*) = x_*$ and $\lambda = 1$
- $\frac{\partial f}{\partial \beta} = 0$

Pitchfork

- $f(x_*) = x_*$ and $\lambda = 1$
- $f(-x) = -f(x)$

Flip (or period doubling)

- $f(x_*) = x_*$ and $\lambda = -1$

Cusp

- $f(x_*) = x_*$ and $\lambda = 1$
- $f''(x_*) = 0$

Neimark-Sacker

- $f(x_*) = x_*$ and $|\lambda| = 1$
- 2 dims, $\lambda \in \mathbb{C}$

Let's look in more detail just at the familiar transcritical and fold, and then at the new one, the flip . . .

Transcritical bifurcation

Take the population model

$$x_{n+1} = rx_n(1 - x_n) \tag{21.1}$$

- This has fixed points $x_{*1} = 0$ and $x_{*2} = (r-1)/r$.
- These exist for any r, but at $r = 1$ clearly $x_{*1} = x_{*2} = 0$. As r passes through $r = 1$ the two equilibria pass through each other.
- Their stability is given by

$$\begin{aligned} \text{at } x_{*1} \quad &: \quad \left.\tfrac{dx_{n+1}}{dx_n}\right|_{*1} = r(1 - 2x_{*1}) = r \\ \text{at } x_{*2} \quad &: \quad \left.\tfrac{dx_{n+1}}{dx_n}\right|_{*2} = r(1 - 2x_{*2}) = 2 - r \end{aligned}$$

  so for :

  · $-1 < r < 1$ the fixed point x_{*1} is stable and x_{*2} unstable (as $|r| < 1$ and $|2 - r| > 1$)

  · $1 < r < 3$ the fixed point x_{*1} is unstable and x_{*2} stable (as $|r| > 1$ and $|2 - r| < 1$)

- So the two fixed points pass through each other, and in doing so swap stability (just as in ODEs).

Things are much the same in a two-dimensional map. Take the predator-prey model and discretize it as we did for the population model, to get [see Ex.Sht.]

$$\tfrac{x(t+\Delta t)-x(t)}{\Delta t} \approx \dot{x} = (\alpha - y)x \ , \qquad \tfrac{y(t+\Delta t)-y(t)}{\Delta t} \approx \dot{y} = (x - \gamma)y \ , \tag{21.2}$$

let $x(t + \Delta t) = x_{n+1}$ and $x(t) = x_n$, and take the time-one map so $\Delta t = 1$, giving a map

$$x_{n+1} = (1 + \alpha - y_n)x_n \ , \qquad y_{n+1} = (1 + x_n - \gamma)y_n \ , \tag{21.3}$$

- For simplicity let $\alpha = \gamma$, then a transcritical bifurcation occurs as α passes through zero [see Ex.Sht.]

[Side Notes:] Transcritical bifurcation normal form

Let

$$x_{n+1} = f(x_n, \beta) \tag{21.4}$$

with $x, \beta \in \mathbb{R}$.
Then a transcritical bifurcation occurs at $x = x_*$ when $\beta = \beta_*$ if the following conditions hold:

(B1) $f(x_*, \beta) = x_*$ "fixed point at $x = x_*$",

(B2) $\frac{\partial f}{\partial x} = 1$ at $x = x_*$, $\beta = \beta_*$, "unit eigenvalue",

(B3) $\frac{\partial f}{\partial \beta} = 0$ at $x = x_*$, $\beta = \beta_*$, "zero speed of f w.r.t. β",

(G1) $\frac{\partial^2 f}{\partial x^2} \neq 0$ at $x = x_*$, $\beta = \beta_*$, "second order derivative nonzero",

(G2) $\frac{\partial}{\partial \beta}\frac{\partial f}{\partial x} \neq 0$ at $x = x_*$, $\beta = \beta_*$, "positive speed of $\frac{\partial f}{\partial x}$ in β",

then in a neighbourhood of (x_*, β_*) this system has the topological normal form

$$y_{n+1} = (1 + \beta) y_n \pm y_n^2,$$

in a neighbourhood of (x_*, β_*).

As in ODEs this is a special case.

Fold bifurcation

Let's tweak the population model to have a background of constant growth (and set birth rate to 1)

$$x_{n+1} = r + x_n(1 - x_n) \tag{21.5}$$

- This has fixed points $x_{*\pm} = \pm\sqrt{r}$
- There are two fixed points for $r > 0$, none for $r < 0$. At $r = 0$ they coincide.
- When they do exist for $r > 0$, their stability is given by

$$\begin{aligned} \text{at } x_{*+} \quad &: \quad \left.\tfrac{dx_{n+1}}{dx_n}\right|_{*+} = 1 - 2x_{*1} = 1 + 2\sqrt{r} \\ \text{at } x_{*-} \quad &: \quad \left.\tfrac{dx_{n+1}}{dx_n}\right|_{*-} = 1 - 2x_{*2} = 1 - 2\sqrt{r} \end{aligned}$$

  so clearly

$$\left.\tfrac{dx_{n+1}}{dx_n}\right|_{*-} < 1 < \left.\tfrac{dx_{n+1}}{dx_n}\right|_{*+}$$

  thus for $r > 0$ the fixed point x_{*+} is unstable and x_{*-} stable.

[Side Notes:] Fold normal form

Let

$$x_{n+1} = f(x_n, \beta) \tag{21.6}$$

with $x, \beta \in \mathbb{R}$.
Then a fold bifurcation occurs at $x = x_*$ when $\beta = \beta_*$ if the following conditions hold:

(B1) $f(x_*, \beta) = x_*$ “fixed point at $x = x_*$”,

(B2) $\frac{\partial f}{\partial x} = 1$ at $x = x_*$, $\beta = \beta_*$, “unit eigenvalue”,

(G1) $\frac{\partial^2 f}{\partial x^2} \neq 0$ at $x = x_*$, $\beta = \beta_*$, “second order derivative nonzero”,

(G2) $\frac{\partial f}{\partial \beta} \neq 0$ at $x = x_*$, $\beta = \beta_*$, “positive speed of f in β”,

then in a neighbourhood of (x_*, β_*) this system has the topological normal form

$$y_{n+1} = \beta + y_n \pm y_n^2.$$

Flip bifurcation

Take again the population model

$$x_{n+1} = rx_n(1 - x_n) \tag{21.7}$$

- We saw this has a fixed point $x_{*2} = (r-1)/r$ with stability is given by

$$\text{at } x_{*2} \quad : \quad \left.\frac{dx_{n+1}}{dx_n}\right|_{*2} = r(1 - 2x_{*2}) = 2 - r\ .$$

- We saw above that a transcritical bifurcation happened as r passed through $r = 1$.

- As r increases through $r = 2$ the stability changes from positive to negative. There is no bifurcation ($\left.\frac{dx_{n+1}}{dx_n}\right|_{*2} = 2 - r$ still lies inside the unit circle), but being negative means the orbit starts to oscillate or 'flip' about the fixed point as it tends towards it.

- As r increases through $r = 3$ there is a bifurcation, the fixed point becomes unstable as $\left.\frac{dx_{n+1}}{dx_n}\right|_{*2} = 2 - r$ leaves the unit circle at $\left.\frac{dx_{n+1}}{dx_n}\right|_{*2} = -1$.

- If there has been a stability change, then something else must have accompanied it to 'balance' the change. It cannot involve the fixed point $x_{*1} = 0$, which is not local to $x_{*2} = (r-1)/r = 2/3$ at $r = 3$.

- So the only other place to look is in higher order objects . . .

- Look for a period 2 orbit, and call it $x_{*\pm}^{(2)}$ (with $\pm$ labeling its two iterates)

$$x_n = r[rx_n(1-x_n)](1-[rx_n(1-x_n)]))$$
$$\Rightarrow \qquad x_{*\pm}^{(2)} = \tfrac{1}{2r}(1+r\pm\sqrt{(r+1)(r-3)}) \qquad (21.8)$$

  where the superscript denotes that this has period 2.

  You can check that the two $\pm$ solutions are just two iterates of the same orbit, (i.e. $x_{*+}^{(2)} = f(x_{*-}^{(2)})$ as $x_{*+}^{(2)} = rx_{*-}^{(2)}(1-x_{*0}^{(2)})$ and vice versa).

- For $r < 3$ this doesn't exist, it appears just for $r > 3$, when $x_{*\pm}^{(2)} = 2/3$, i.e. being 'born' from x_{*2} as it changes stability.

- What is the stability of the orbit?

- We could directly calculate $\frac{dx_{n+2}}{dx_n}$ at $x_{*\pm}^{(2)}$, but the map $x_{n+2} = f^2(x_n)$ is fairly complicated. When we get to higher periods these expressions will be even more difficult to differentiate.

- Instead, by the chain rule

$$\frac{dx_{n+2}}{dx_n} = \frac{dx_{n+2}}{dx_{n+1}}\frac{dx_{n+1}}{dx_n} = \frac{df(x_{n+1})}{dx_{n+1}}\frac{df(x_n)}{dx_n} \tag{21.9}$$

  This says the stability of the orbit is given by multiplying the slopes of f at each of the orbit's iterates.

- So to calculate the stability of the period 2 orbit

$$\begin{aligned}\left.\frac{dx_{n+2}}{dx_n}\right|_{x^{(2)}_{*\pm}} &= \left.\frac{df(x_{n+1})}{dx_{n+1}}\right|_{x^{(2)}_{*+}} \left.\frac{df(x_n)}{dx_n}\right|_{x^{(2)}_{*-}} \\ &= r(1-2x_{n+1})|_{x^{(2)}_{*+}}\, r(1-2x_n)|_{x^{(2)}_{*-}} \\ &= (1+\sqrt{(r+1)(r-3)})(1-\sqrt{(r+1)(r-3)}) \\ &= 1-(r+1)(r-3) \end{aligned} \tag{21.10}$$

  where I've used

$$r(1-2x^{(2)}_{*\pm}) = r - 2\tfrac{1}{2}(1+r\pm\sqrt{(r+1)(r-3)}) = -1 \mp \sqrt{(r+1)(r-3)}$$

  Now we could simplify this expression further but it's in a good form. We can see from this that at $r=3$ we have $\left.\frac{dx_{n+2}}{dx_n}\right|_{x^{(2)}_{*\pm}} = 1$, consistent with a bifurcation occurring, and for $r>3$ we have $\left.\frac{dx_{n+2}}{dx_n}\right|_{x^{(2)}_{*\pm}} < 1$.

- So the period 2 orbit that appears is stable.

- This event, in which a stable period 1 orbit becomes unstable, and creates a stable period 2 orbit, is called a **flip bifurcation**.

[Side Notes:] Flip normal form

Let

$$x_{n+1} = f(x_n, \beta) \tag{21.11}$$

with $x, \beta \in \mathbb{R}$.
Then a flip bifurcation occurs at $x = x_*$ when $\beta = \beta_*$ if the following conditions hold:

(B1) $f(x_*, \beta) = x_*$ "fixed point at $x = x_*$",

(B2) $\frac{\partial f}{\partial x} = -1$ at $x = x_*$, $\beta = \beta_*$, '-1 eigenvalue",

(G1) $2\frac{\partial^3 f}{\partial x^3} + 3(\frac{\partial^2 f}{\partial x^2})^2 \neq 0$ at $x = x_*$, $\beta = \beta_*$,

(G2) $c = \frac{\partial f}{\partial \beta}\frac{\partial^2 f}{\partial x^2} + 2\frac{\partial}{\partial \beta}\frac{\partial f}{\partial x} \neq 0$ at $x = x_*$, $\beta = \beta_*$,

then in a neighbourhood of (x_*, β_*) this system has the topological normal form

$$y_{n+1} = -(1+\beta)y_n \pm y_n^3, \tag{21.12}$$

The $\pm$ signs determine whether the period 2 orbit involved is stable or unstable (in which case it surrounds the unstable or stable fixed point, respectively).

This is otherwise known as **period doubling**, because when a flip bifurcation happens typically:

- a stable fixed point becomes unstable, and gives birth to a period two orbit, or
- a stable period p orbits becomes unstable, and gives birth to a period $2p$ orbit.

22 Finding periodic orbits

There are two important tricks for finding periodic orbits and their stability.

1. Factorizing

- When we solve $x = f^m(x)$ to find a period m orbit, this expression may also have 'false roots', because any lower period orbit will also satisfy this expression, e.g. the fixed point $x = f(x)$ satisfies $x = f^m(x)$ for any m.
- So we have to be careful to factorize out these solutions.
- E.g. in the population map

  $$x_{n+1} = rx_n(1 - x_n) \tag{22.1}$$

  fixed points $x_{*1} = 0$ and $x_{*2} = (r-1)/r$.

  look for period two orbits. These are solutions of

  $$\begin{aligned} 0 &= x - f^2(x) \\ &= x - r^2x(1-x)(1-rx(1-x)) \\ &= x - r^2x(1-x) + r^3x^2(1-x)^2 \end{aligned} \tag{22.2}$$

  Now, fixed points must also be solutions of this, so this must be divisible by factors $x - x_{*1}$ which is just x, and $x - x_{*2}$ which is $x - x_{*2} = x - (r-1)/r$, so we must be able to write (22.2) as

  $$0 = x(x - (r-1)/r)(ax^2 + bx + c) \tag{22.3}$$

  for some a, b, c.

To find these you can either:

- use polynomial division of (22.2) divided by $x(x - (r-1)/r)$,
- compare coefficients.

I'll use comparing coefficients here. If the two expressions above are the same then we have

$$x(x - (r-1)/r)(ax^2 + bx + c) \;=\; x - r^2x(1-x) + r^3x^2(1-x)^2 \tag{22.4}$$

then expanding out both sides (and optionally dividing by x) we have

$$\tfrac{1-r}{r}c + (c - \tfrac{r-1}{r}b)x + (b - \tfrac{r-1}{r}a)x^2 + ax^3 \;=\; 1 - r^2 + r^2(1+r)x - 2r^3x^2 + r^3x^3 \tag{22.5}$$

implying

$$\begin{aligned} \tfrac{1-r}{r}c = 1 - r^2 \qquad & \Rightarrow \qquad c = r(1+r) \\ c - \tfrac{r-1}{r}b = r^2(1+r) \qquad & \Rightarrow \qquad b = -r^2(r+1) \\ b - \tfrac{r-1}{r}a = -2r^3 \qquad & \Rightarrow \qquad a = r^3 \end{aligned} \tag{22.6}$$

so period two orbits are solutions of

$$0 = x(x - (r-1)/r)(r^2x^2 - r(r+1)x + 1 + r)r \tag{22.7}$$

so the first two factors on the righthand side give the period 1 solutions, and the last factor gives the period two solutions as

$$x^{(2)}_{*\pm} = (1 + r \pm \sqrt{(r+1)(r-3)})/2r \tag{22.8}$$

exactly as we found before.

2. Chain rule

As we said above, for nonlinear maps, calculating the derivative $\frac{dx_{n+p}}{dx_p}$ at a given $x_*^{(p)}$ of a period p orbit can be difficult, but we can get around that using the chain rule.

- Essentially, this says that rather than working out what $x_{n+p} = f^p(x_n)$ is finding its stability, we can instead work out the stability (the slope) of $f(x_n)$ at each point the orbit visits and multiply the slopes.

- So for a period 2 orbit, which is a solution of $x_n = f^2(x_n)$, we can work out

$$\begin{aligned} \frac{dx_{n+2}}{dx_n} &= \frac{dx_{n+2}}{dx_{n+1}}\frac{dx_{n+1}}{dx_n} \\ &= \frac{df(x_{n+1})}{dx_{n+1}}\frac{df(x_n)}{dx_n} \ = \ f'(x_{n+1})f'(x_n) \end{aligned} \tag{22.9}$$

- For a period p orbit, which is a solution of $x_n = f^p(x_n)$, we can work out

$$\begin{aligned} \frac{dx_{n+p}}{dx_n} &= \frac{dx_{n+p}}{dx_{n+p-1}} \cdots \frac{dx_{n+1}}{dx_n} \\ &= \frac{df(x_{n+p-1})}{dx_{n+p-1}} \cdots \frac{df(x_n)}{dx_n} \ = \ f'(x_{n+p-1}) \dots f'(x_n) \end{aligned} \tag{22.10}$$

[Further Reading Only:] Period doubling cascades

There's a lot more to period doubling that we have time to go into here. The most important is a phenomenon that was first identified in the logistic map, but then understood to apply to any nonlinear maps. You should at least make yourselves familiar with these concepts:

- Cascades – if a nonlinear map has a turning point that depends on one parameter, it can exhibit period doubling. This tends not to happen just once, but again, and again, and again, . . . in an infinite sequence called a **period doubling cascade**. So as a parameter β is changed, a fixed point creates a period two orbit (in a flip bifurcation), which creates a period 2 orbit (another flip), then a period 4, and so on, until the period reaches infinity.

- Feigenbaum's constants – Importantly, this to infinite period happens only with a finite change the parameter β. So the interval of β values between each flip, i.e. the range for which each period exists, shrinks as the period increases. It shrinks at a rate called **Feigenbaum's (first) constant**. The distances between iterates also shrink at a rate called **Feigenbaum's second constant**, so the orbits remain finite in size.

  E.g. For the logistic map the Feigenbaum constants are

$$\begin{aligned}\text{1st Feigenbaum constant} &= \delta = \lim_{n\to\infty}\frac{\beta_n-\beta_{n-1}}{\beta_{n+1}-\beta_n} = 4.669...\\ \text{2nd Feigenbaum constant} &= \alpha = \lim_{n\to\infty}\frac{\Delta x_n-\Delta x_{n-1}}{\Delta x_{n+1}-\Delta x_n} = 2.503...\end{aligned}$$

  where $\beta_1, \beta_2, ...$ are the parameter values of each successive flip bifurcation, and Δx characterizes the width of each fork in the bifurcation diagram.

- Universality – these appear to be fundamental constants of mathematics (like π or e), meaning the same constants apply to *any* differentiable one-dimensional map. Their exact values remain unknown. Their universality is proven by taking a small region around the (locally quadratic) hump in a map f where a period doubling occurs, *renormalizing* by re-scaling to magnify x to cover only this region, then doing the same around a hump in f^2, then f^3, etc. obtaining the bifurcation values of β each time.

- Chaos – at the end of a period doubling cascade the map will become chaotic. We'll say more about chaos below, but basically it means the map remains trapped in some region but never repeats.

- Periodic windows – note that a period doubling cascade only involves even period orbits (because it arises by *doubling*). But in the midst of the chaos that follows a cascade we find windows of odd periods 3,5,7,... See if you can spot them in the figure above.

- In fact odd period orbits are rare to see, and always come with baggage like the chaos that surrounds them in the cascade. An almost incredible theorem by Sharkovkii says:

  If $x_{n+1} = f(x_n)$ is a continuous map with a period p orbit, then it also has a period m orbit for every $m \prec p$, meaning every m to the left of p in the sequence of numbers

$$\begin{aligned}
&1 \prec 2 \prec 2^2 \prec 2^3 \prec \cdots \prec \ 2^n \ \prec \ldots \\
&\cdots \prec \ 7 \cdot 2^n \ \prec \ 5 \cdot 2^n \ \prec \ 3 \cdot 2^n \ \prec \ldots \\
&\vdots \\
&\cdots \prec \ 7 \cdot 2 \ \prec \ 5 \cdot 2 \ \prec \ 3 \cdot 2 \ \prec \ldots \\
&\cdots \prec \ 9 \ \prec \ 7 \ \prec \ 5 \ \prec \ 3
\end{aligned} \tag{22.11}$$

  called the **Sharkovkii ordering**.

  (The Sharkovkii ordering consists of the sequence 2^k with k ticking up from $k = 0$ to $k = \infty$, followed by all the odd numbers in decreasing order multiplied by 2^k where k ticks down from ∞ to 1).

  Note that the odd numbers (except 1) only appear in the last row of this sequence, so if an odd period exists, then all of the infinitely many periods that appear to the left in the Sharkovkii ordering also exist. A consequence of this — the existence of infinitely many periodic orbits — is chaos. Two guys called Yorke and Li discovered (a decade after Sharkovskii's then little known result) one small hint of this, well known in the dynamical systems community as "*period three implies chaos*".

23 Chaos

"Chaos theory" isn't so much a theory as a bunch of observations and theorems that try to characterize how and where chaos can be found.
In short **chaos** is behaviour that repeats qualitatively but not quantitatively, evolving around inside a restricted area of phase space but never precisely repeating.
This has certain consequences.

- Two orbits very closely together travel far apart at later times.
- This means that a small change in the initial condition $\mathbf{x}_0 = \mathbf{x}(0)$ of an orbit can lead to huge differences in state at later times. As a result, evolution in chaotic systems is hugely unpredictable (which is why they have been of such interest in applications like weather prediction and neuroscience).
- Chaos often occurs on a region of space called a **chaotic attractor**. Orbits converge on it from the surrounding flow, then become chaotic. They are often *strange attractors* (attractors with a fractal structure).
- In fact orbits don't just travel apart in chaotic systems, they diverge exponentially. That means there is a fairly simple test that implies the existence of chaos. Given two nearby points x_0 and $x_0+\delta x_0$, the distance between them on each iteration of the map $x_{n+1} = f(x_n)$ should grow like $|\delta x_n| \approx |\delta_0| e^{\lambda n}$ where λ is something called the Lyapunov exponent.
- With this approximate idea we can derive a formula for the Lyapunov exponent. The distance after n iterates is $\delta x_n = f^n(x_0 + \delta_0) - f^n(x_0)$, so given $|\delta x_n| \approx |\delta_0| e^{\lambda n}$ we calculate

  $$\lambda = \frac{1}{n} \ln \left| \frac{\delta_n}{\delta_0} \right| = \tfrac{1}{n} \ln \left| \frac{f^n(x_0 + \delta_0) - f^n(x_0)}{\delta_0} \right| \tag{23.1}$$

  For $\delta_0 \to 0$ the term inside the ln becomes the derivative $|(f^n(x_0))'|$, which can be expanded by the chain rule

  $$(f^n(x_0))' = \prod_{i=0}^{n-1} f'(x_i) \tag{23.2}$$

then notice that

$$\ln|(f^n(x_0))'| = \ln\left|\prod_{i=0}^{n-1} f'(x_i)\right| = \sum_{i=0}^{n-1} \ln|f'(x_i)| \tag{23.3}$$

so we define the Lyapunov exponent as this in the limit $n \to \infty$

$$\lambda = \lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} \ln|f'(x_i)| \tag{23.4}$$

but numerically we often approximate it by calculating it for a large, but of course finite, n.

- There is no single cause or test for chaos, or even a single completely general definition. But to have chaos a system must
  - be sensitive to initial conditions (orbits change a large amount with a small change in ICs),
  - be topologically transitive (any two regions of space overlap in the system's flow),
  - have dense periodic orbits (every point is infinitely close to a periodic orbit).

  These are often not easy to prove, but in certain cases one or two of these imply all three.

- Partly due to these criteria, an ODE must have at least 3 dimensions for chaos. A map only needs 1 dimension for chaos.

- The classic example of a chaotic system comes from model of atmospheric convection by Edward Lorenz,

$$\begin{aligned} \dot{x} &= \sigma(y - x) \\ \dot{y} &= \rho x - xz - y \\ \dot{z} &= xy - \beta z \end{aligned} \tag{23.5}$$

  (try simulating this for $\rho = 28$, $\sigma = 10$, $\beta = 8/3$, in which chaos was dubbed in popular culture as the "*butterfly effect*".

24 Transient versus Asymptotic behaviour

When simulating a system it is important to distinguish between transient behaviour, which can only be seen for short times, and asymptotic behaviour which will determine the system's fate over long times.

Asymptotic behaviour describes what a system or orbit does as $t \to \infty$, typically either:

- settling onto an equilibrium or periodic orbit,
- becoming chaotic, or
- diverging off to infinity.

Transient dynamics is the behaviour that happens before that, between setting off a trajectory from a typical initial point $\mathbf{x}_0$ and it tending towards some attractor or long time state. This might involve:

- decaying towards an equilibrium or periodic orbit,
- an episode of almost-periodic behaviour before becoming chaotic, or
- a seemingly chaotic episode before settling to an equilibrium or periodic orbit.