Asymptotics Problem Sheet 2
Local Approximation to Linear ODEs

Solution to Problem 1

Academic Year 2025/2026

Problem 1: Leading Behaviours as $x \to 0^+$x → 0+

Context and Methodology: We seek the leading order behaviour of solutions to ODEs near $x = 0$. From Section 3.2 of the lecture notes, we know that $x = 0$ is an **irregular singular point** for these equations. The standard Frobenius series method fails at irregular singular points, so we employ the **controlling factor ansatz** combined with **dominant balance analysis**.

Core Strategy:

1. Use the ansatz $y(x) = e^{S(x)}$ where $S(x)$ contains the most rapidly changing behavior
2. Perform dominant balance analysis to find $S(x) \sim Cx^\beta$ as $x \to 0^+$
3. Determine β and C by balancing dominant terms in the ODE
4. The leading behavior is $y(x) \sim e^{S(x)}$ as $x \to 0^+$

Problem 1(a): $x^4 y''' = y$xy"' = y

Step 1: Identify the nature of $x = 0$.
Why do we do this? We must determine if $x = 0$ is an ordinary point, regular singular point, or irregular singular point to know which solution method to apply.

Rewrite the equation in standard form:

$$y''' = \frac{1}{x^4} y \tag{1}$$

What do we observe? The coefficient of y has the form $\frac{1}{x^4}$, which becomes unbounded as $x \to 0$.
Why does this matter? According to the classification in Section 3.1 of the lecture notes, for a third-order ODE written as

$$y''' + p_2(x)y'' + p_1(x)y' + p_0(x)y = 0, \tag{2}$$

the point $x = 0$ is a regular singular point only if $x^3 p_0(x)$, $x^2 p_1(x)$, and $xp_2(x)$ are all analytic at $x = 0$.
What is our conclusion? Here $p_0(x) = -\frac{1}{x^4}$, so $x^3 p_0(x) = -\frac{1}{x}$ is **not** analytic at $x = 0$. Therefore, $x = 0$ is an **irregular singular point**.

Step 2: Apply the controlling factor ansatz.
Why this ansatz? From Section 3.2.1 of the lecture notes, near irregular singular points, the solution exhibits exponential behavior. We therefore use:

$$y(x) = e^{S(x)} \tag{3}$$

What are the derivatives? We need to compute y'''. First:

$$y' = S'e^{S(x)} = S'y \tag{4}$$
$$y'' = S''e^{S(x)} + (S')^2e^{S(x)} = (S'' + (S')^2)y \tag{5}$$
$$y''' = S'''e^{S(x)} + 3S'S''e^{S(x)} + (S')^3e^{S(x)} \tag{6}$$

Why these forms? Each derivative of $y = e^{S(x)}$ produces a polynomial in derivatives of $S(x)$ multiplied by y itself.

Thus:

$$y''' = (S''' + 3S'S'' + (S')^3)y \tag{7}$$

Step 3: Substitute into the ODE.

Substituting $y''' = (S''' + 3S'S'' + (S')^3)y$ into $x^4y''' = y$:

$$x^4(S''' + 3S'S'' + (S')^3)y = y \tag{8}$$

Why can we cancel y? Since we seek non-trivial solutions, $y \neq 0$, and we can divide both sides by y:

$$x^4S''' + 3x^4S'S'' + x^4(S')^3 = 1 \tag{9}$$

Step 4: Perform dominant balance analysis.

What is dominant balance? From Section 3.2.2, we assume $S(x) \sim Cx^\beta$ as $x \to 0^+$ and determine which terms dominate.

Let $S(x) \sim Cx^\beta$. Then:

$$S'(x) \sim C\beta x^{\beta-1} \tag{10}$$
$$S''(x) \sim C\beta(\beta-1)x^{\beta-2} \tag{11}$$
$$S'''(x) \sim C\beta(\beta-1)(\beta-2)x^{\beta-3} \tag{12}$$

What are the orders of each term? Substituting into equation (7):

$$x^4S''' \sim C\beta(\beta-1)(\beta-2)x^{\beta+1} \tag{13}$$
$$3x^4S'S'' \sim 3C^2\beta^2(\beta-1)x^{2\beta+2} \tag{14}$$
$$x^4(S')^3 \sim C^3\beta^3x^{3\beta+1} \tag{15}$$
$$1 \sim x^0 \tag{16}$$

Step 5: Determine which terms balance.

Strategy: We systematically check which pairs of terms can balance (have the same order) as $x \to 0^+$.

Case 1: Assume $x^4S''' \sim 1$, i.e., $x^{\beta+1} \sim x^0$.

What does this give? This implies $\beta + 1 = 0$, so $\beta = -1$.

Is this consistent? With $\beta = -1$:

$$x^4S''' \sim x^{-1+1} = x^0 \sim 1 \quad \checkmark \tag{17}$$
$$3x^4S'S'' \sim x^{-2+2} = x^0 \sim 1 \quad \text{(also order 1!)} \tag{18}$$
$$x^4(S')^3 \sim x^{-3+1} = x^{-2} \to \infty \quad \text{(dominant!)} \tag{19}$$

Why is this inconsistent? The term $x^4(S')^3$ becomes unbounded as $x \to 0^+$, dominating both other terms. We cannot neglect it. This violates our assumption.

Case 2: Assume $3x^4S'S'' \sim 1$, i.e., $x^{2\beta+2} \sim x^0$.
What does this give? This implies $2\beta + 2 = 0$, so $\beta = -1$.
Is this consistent? This is the same β as Case 1, leading to the same inconsistency.

Case 3: Assume $x^4(S')^3 \sim 1$, i.e., $x^{3\beta+1} \sim x^0$.
What does this give? This implies $3\beta + 1 = 0$, so:

$$\beta = -\frac{1}{3} \tag{20}$$

Is this consistent? With $\beta = -\frac{1}{3}$:

$$x^4S''' \sim x^{-1/3+1} = x^{2/3} \to 0 \quad \text{as } x \to 0^+ \tag{21}$$

$$3x^4S'S'' \sim x^{-2/3+2} = x^{4/3} \to 0 \quad \text{as } x \to 0^+ \tag{22}$$

$$x^4(S')^3 \sim x^{-1+1} = x^0 \sim 1 \quad \checkmark \tag{23}$$

Why is this consistent? Both x^4S''' and $3x^4S'S''$ vanish as $x \to 0^+$, while $x^4(S')^3 \sim 1$. The dominant balance equation becomes:

$$x^4(S')^3 \sim 1 \quad \text{as } x \to 0^+ \tag{24}$$

Step 6: Solve for C.

From $S(x) \sim Cx^{-1/3}$, we have $S'(x) \sim -\frac{C}{3}x^{-4/3}$. Therefore:

$$x^4(S')^3 \sim x^4\left(-\frac{C}{3}\right)^3 x^{-4} = -\frac{C^3}{27} \sim 1 \tag{25}$$

What does this tell us? Solving for C:

$$C^3 = -27 \quad \Rightarrow \quad C = -3 \tag{26}$$

Why the choice of sign? We take the real cube root. Thus:

$$S(x) \sim -3x^{-1/3} \quad \text{as } x \to 0^+ \tag{27}$$

Step 7: Write the leading behavior.
Final answer: The leading behavior as $x \to 0^+$ is:

$$\boxed{y(x) \sim e^{-3x^{-1/3}} \quad \text{as } x \to 0^+} \tag{28}$$

Interpretation: This solution exhibits extremely rapid decay as $x \to 0^+$ (the exponent $-3x^{-1/3} \to -\infty$), characteristic of solutions near irregular singular points.

Problem 1(b): $y'' = (\cot x)^4 y$y" = (cot x)y

Given hint: $\cot x \sim \frac{1}{x} - \frac{x}{3} + \dots$ as $x \to 0$

Step 1: Determine the leading behavior of the coefficient.
Why do we need this? To classify the singularity at $x = 0$ and perform dominant balance, we need the asymptotic form of $(\cot x)^4$ as $x \to 0$.

Using the given expansion:

$$\cot x \sim \frac{1}{x} - \frac{x}{3} + \dots \sim \frac{1}{x}\left(1 - \frac{x^2}{3} + \dots\right) \quad \text{as } x \to 0 \tag{29}$$

What is the leading term? As $x \to 0$:

$$(\cot x)^4 \sim \frac{1}{x^4} \tag{30}$$

Step 2: Identify the nature of $x = 0$.

The ODE becomes:

$$y'' \sim \frac{1}{x^4} y \quad \text{as } x \to 0 \tag{31}$$

Why is this an irregular singular point? For a second-order ODE $y'' + p_1(x)y' + p_0(x)y = 0$, the point $x = 0$ is regular singular only if $x^2 p_0(x)$ is analytic at $x = 0$. Here $p_0(x) = -\frac{1}{x^4}$, so:

$$x^2 p_0(x) = -\frac{1}{x^2} \tag{32}$$

which is **not analytic** at $x = 0$. Thus $x = 0$ is an **irregular singular point**.

Step 3: Apply the controlling factor ansatz.

Let $y(x) = e^{S(x)}$. Then:

$$y' = S'y \tag{33}$$
$$y'' = (S'' + (S')^2)y \tag{34}$$

Substituting into $y'' = (\cot x)^4 y$:

$$(S'' + (S')^2)y = (\cot x)^4 y \tag{35}$$

Canceling y (for non-trivial solutions):

$$S'' + (S')^2 = (\cot x)^4 \tag{36}$$

Step 4: Apply the standard assumption.

From the lecture notes (Section 3.2.2): Near irregular singular points, we often have $S'' = o((S')^2)$ as $x \to 0^+$, meaning S'' is subdominant to $(S')^2$.

Why make this assumption? This is a heuristic that works in many cases. We'll verify consistency after finding $S(x)$.

Assuming $S'' \ll (S')^2$, the dominant balance equation is:

$$(S')^2 \sim (\cot x)^4 \sim \frac{1}{x^4} \quad \text{as } x \to 0 \tag{37}$$

Step 5: Solve for S'.

Taking square roots:

$$S' \sim \pm\frac{1}{x^2} \quad \text{as } x \to 0 \tag{38}$$

What does integration give? Integrating:

$$S(x) \sim \pm \int \frac{1}{x^2} dx = \mp\frac{1}{x} + \text{const.} \tag{39}$$

Why does the constant not matter? Constants in $S(x)$ only contribute multiplicative constants to $y = e^{S(x)}$, which are absorbed into the general solution.

Step 6: Verify the assumption $S'' \ll (S')^2$.

With $S(x) \sim \mp\frac{1}{x}$:

$$S'(x) \sim \pm\frac{1}{x^2} \tag{40}$$

$$S''(x) \sim \mp\frac{2}{x^3} \tag{41}$$

$$(S')^2 \sim \frac{1}{x^4} \tag{42}$$

Is the assumption valid? As $x \to 0^+$:

$$\frac{|S''|}{|(S')^2|} \sim \frac{2/x^3}{1/x^4} = 2x \to 0 \tag{43}$$

Conclusion: Yes! We have $S'' = o((S')^2)$ as $x \to 0^+$, confirming our assumption.

Step 7: Write the leading behavior.
Final answer: The leading behaviors as $x \to 0^+$ are:

$$\boxed{y(x) \sim e^{\pm 1/x} \quad \text{as } x \to 0^+} \tag{44}$$

More explicitly, the general solution has the form:

$$y(x) \sim Ae^{1/x} + Be^{-1/x} \quad \text{as } x \to 0^+ \tag{45}$$

Interpretation: One solution grows extremely rapidly ($e^{1/x} \to \infty$) while the other decays extremely rapidly ($e^{-1/x} \to 0$) as $x \to 0^+$.

Problem 1(c): $x^4y''' - 3x^2y' + 2y = 0$xy"' - 3x²y' + 2y = 0

Step 1: Check if $x = 0$ is an irregular singular point.
Rewrite in standard form:

$$y''' = \frac{3x^2y' - 2y}{x^4} = \frac{3}{x^2}y' - \frac{2}{x^4}y \tag{46}$$

Classification: For $p_2(x) = 0$, $p_1(x) = -\frac{3}{x^2}$, $p_0(x) = \frac{2}{x^4}$:

$$x \cdot p_2(x) = 0 \quad \text{(analytic)} \tag{47}$$

$$x^2 \cdot p_1(x) = -3 \quad \text{(analytic)} \tag{48}$$

$$x^3 \cdot p_0(x) = \frac{2}{x} \quad \text{(NOT analytic)} \tag{49}$$

Therefore $x = 0$ is an **irregular singular point**.

Step 2: Try a power law solution first.
Why try this? Before using the exponential ansatz, we check if a simple power law $y = x^\alpha$ works, as this is sometimes sufficient for certain irregular singular points.

Let $y = x^\alpha$. Then:

$$y' = \alpha x^{\alpha-1} \tag{50}$$

$$y''' = \alpha(\alpha-1)(\alpha-2)x^{\alpha-3} \tag{51}$$

Substituting into the ODE:

$$x^4 \cdot \alpha(\alpha-1)(\alpha-2)x^{\alpha-3} - 3x^2 \cdot \alpha x^{\alpha-1} + 2x^\alpha = 0 \tag{52}$$

Simplifying:

$$\alpha(\alpha-1)(\alpha-2)x^{\alpha+1} - 3\alpha x^{\alpha+1} + 2x^{\alpha} = 0 \tag{53}$$

What must hold? Factor out x^{α}:

$$x^{\alpha}\left[\alpha(\alpha-1)(\alpha-2)x - 3\alpha x + 2\right] = 0 \tag{54}$$

Can this vanish for all x near 0? For this to hold as $x \to 0^+$, we need the coefficient of x and the constant term to vanish separately:

$$\alpha(\alpha-1)(\alpha-2) - 3\alpha = 0 \tag{55}$$

$$2 = 0 \quad \text{(Contradiction!)} \tag{56}$$

Conclusion: A pure power law solution does not work.

Step 3: Apply the controlling factor ansatz.

Let $y = e^{S(x)}$:

$$y' = S'y \tag{57}$$

$$y''' = (S''' + 3S'S'' + (S')^3)y \tag{58}$$

Substituting:

$$x^4(S''' + 3S'S'' + (S')^3)y - 3x^2S'y + 2y = 0 \tag{59}$$

Dividing by y:

$$x^4S''' + 3x^4S'S'' + x^4(S')^3 - 3x^2S' + 2 = 0 \tag{60}$$

Step 4: Dominant balance analysis.

Assume $S(x) \sim Cx^{\beta}$. The orders of each term as $x \to 0^+$ are:

$$x^4S''' \sim x^{\beta+1} \tag{61}$$

$$3x^4S'S'' \sim x^{2\beta+2} \tag{62}$$

$$x^4(S')^3 \sim x^{3\beta+1} \tag{63}$$

$$-3x^2S' \sim x^{\beta-1} \tag{64}$$

$$2 \sim x^0 \tag{65}$$

Step 5: Find which terms balance.

Strategy: We need at least two terms of the same order. Let's systematically check.

Try: $x^4(S')^3 \sim 2$, i.e., $3\beta + 1 = 0 \Rightarrow \beta = -\frac{1}{3}$.

With $\beta = -\frac{1}{3}$:

$$x^4S''' \sim x^{2/3} \to 0 \tag{66}$$

$$3x^4S'S'' \sim x^{4/3} \to 0 \tag{67}$$

$$x^4(S')^3 \sim x^0 \sim 1 \tag{68}$$

$$-3x^2S' \sim x^{-4/3} \to \infty \quad \text{(Dominant!)} \tag{69}$$

Problem: The term $-3x^2S'$ dominates, contradicting our assumption.

Try: $-3x^2S' \sim 2$, i.e., $\beta - 1 = 0 \Rightarrow \beta = 1$.

With $\beta = 1$:

$$x^4S''' \sim x^2 \to 0 \tag{70}$$

$$3x^4S'S'' \sim x^4 \to 0 \tag{71}$$

$$x^4(S')^3 \sim x^4 \to 0 \tag{72}$$

$$-3x^2S' \sim x^0 \sim 1 \quad \checkmark \tag{73}$$

Is this consistent? Yes! All other terms vanish as $x \to 0^+$. The dominant balance is:

$$-3x^2 S' + 2 \sim 0 \quad \text{as } x \to 0^+ \tag{74}$$

Step 6: Solve for C.

With $S(x) \sim Cx$, we have $S'(x) \sim C$. Therefore:

$$-3x^2 \cdot C + 2 \sim 0 \quad \text{as } x \to 0^+ \tag{75}$$

Wait, this is problematic! The term $-3Cx^2 \to 0$ as $x \to 0^+$, but the constant 2 does not. They cannot balance.

Re-examination: Let's reconsider. Perhaps the solution has a different structure. Try $y = x^\alpha e^{S(x)}$ where $S(x)$ contains the exponential behavior.

Actually, let me try $y = x^\alpha$ more carefully. From equation (44):

$$\alpha(\alpha-1)(\alpha-2)x^{\alpha+1} - 3\alpha x^{\alpha+1} + 2x^\alpha = 0 \tag{76}$$

Factor out x^α:

$$x^\alpha \left[x(\alpha(\alpha-1)(\alpha-2) - 3\alpha) + 2\right] = 0 \tag{77}$$

For this to be satisfied as $x \to 0^+$, we need:

$$2 = 0 \quad \text{or} \quad x \to 0 \text{ and coefficient of } x \text{ diverges} \tag{78}$$

Alternative approach: Perhaps there's a boundary layer. Or we need the logarithmic term. Let me try:

$$y = x^\alpha(\text{polynomial or log terms}) \tag{79}$$

Actually, looking at the structure, let me try $y \sim x^\alpha$ where we determine α from the constant term dominance:

$$2x^\alpha \sim \text{leading term} \tag{80}$$

From the ODE, if $y \sim x^\alpha$, the terms scale as:

$$x^4 y''' \sim \alpha^3 x^{\alpha+1} \tag{81}$$

$$-3x^2 y' \sim -3\alpha x^{\alpha+1} \tag{82}$$

$$2y \sim 2x^\alpha \tag{83}$$

For the constant term $2y$ to be the leading term, we need:

$$\alpha < \alpha + 1 \tag{84}$$

which is always true. So $2y$ is indeed the leading term if $\alpha < \alpha + 1$.

But we need at least two terms to balance. Let's balance $2y$ with $-3x^2y'$:

$$2x^\alpha \sim -3\alpha x^{\alpha+1} \tag{85}$$

This gives $\alpha = \alpha + 1$, which is impossible.

Let's balance $2y$ with $x^4 y'''$:

$$2x^\alpha \sim \alpha(\alpha-1)(\alpha-2)x^{\alpha+1} \tag{86}$$

Again, $\alpha = \alpha + 1$ is impossible.

Conclusion: The dominant balance suggests that as $x \to 0^+$, the constant term $2y$ must balance with itself being small, which means:

$$y(x) \to 0 \quad \text{as } x \to 0^+ \tag{87}$$

A more careful analysis (beyond basic dominant balance) would use modified Frobenius or study the full series structure. For our purposes:
Final answer:

$$\boxed{y(x) = O(1) \text{ or decays as } x \to 0^+} \tag{88}$$

More precisely, the solution likely has the form $y \sim x^\alpha$ with $\alpha > 0$, giving:

$$\boxed{y(x) \to 0 \quad \text{as } x \to 0^+} \tag{89}$$

Problem 1(d): $y'' = \sqrt{x}\, y$y" = x y

Step 1: Identify the singularity.
Rewrite as:

$$y'' - \sqrt{x}\, y = 0 \tag{90}$$

What is $p_0(x)$? Here $p_0(x) = -\sqrt{x}$. Check $x^2 p_0(x) = -x^2\sqrt{x} = -x^{5/2}$.
Is this analytic at $x = 0$? Yes! The function $x^{5/2}$ is analytic at $x = 0$ (though only infinitely differentiable from the right).
Wait, is $x = 0$ regular or irregular? Actually, for real x, $x^{5/2}$ is well-defined and smooth for $x > 0$. But $\sqrt{x}$ is not analytic in the complex sense at $x = 0$ (it's a branch point). For the purpose of this problem, we treat $x = 0$ as a **regular singular point** or at worst, a weak singularity.

Step 2: Try a Frobenius ansatz.
Why Frobenius? Since the singularity is mild, a modified Frobenius series might work:

$$y(x) = x^\alpha \sum_{n=0}^{\infty} a_n x^n \tag{91}$$

However, given the $\sqrt{x}$ coefficient, let's try:

$$y(x) = e^{S(x)} \tag{92}$$

Step 3: Apply controlling factor ansatz.
With $y = e^{S(x)}$:

$$y'' = (S'' + (S')^2)e^{S(x)} = (S'' + (S')^2)y \tag{93}$$

The ODE becomes:

$$S'' + (S')^2 = \sqrt{x} \tag{94}$$

Step 4: Dominant balance.
Assume $S(x) \sim Cx^\beta$ as $x \to 0^+$:

$$S' \sim C\beta x^{\beta-1} \tag{95}$$
$$S'' \sim C\beta(\beta-1)x^{\beta-2} \tag{96}$$
$$(S')^2 \sim C^2\beta^2 x^{2\beta-2} \tag{97}$$

The RHS has order $x^{1/2}$.
Try: $(S')^2 \sim x^{1/2}$, i.e., $2\beta - 2 = \frac{1}{2} \Rightarrow \beta = \frac{5}{4}$.
With $\beta = \frac{5}{4}$:

$$S'' \sim x^{5/4-2} = x^{-3/4} \to \infty \tag{98}$$
$$(S')^2 \sim x^{5/2-2} = x^{1/2} \tag{99}$$

Problem: S'' diverges, dominating $(S')^2$ as $x \to 0^+$. This is inconsistent.
Try: $S'' \sim x^{1/2}$, i.e., $\beta - 2 = \frac{1}{2} \Rightarrow \beta = \frac{5}{2}$.
With $\beta = \frac{5}{2}$:

$$S'' \sim x^{1/2} \tag{100}$$
$$(S')^2 \sim x^3 \tag{101}$$

Is this consistent? Yes! As $x \to 0^+$, $(S')^2 \to 0$ while $S'' \sim x^{1/2}$. The dominant balance is:

$$S'' \sim \sqrt{x} \quad \text{as } x \to 0^+ \tag{102}$$

Step 5: Solve for $S(x)$.
With $S(x) \sim Cx^{5/2}$, we have:

$$S''(x) \sim C \cdot \frac{5}{2} \cdot \frac{3}{2} x^{1/2} = \frac{15C}{4} x^{1/2} \tag{103}$$

Matching with $S'' \sim x^{1/2}$:

$$\frac{15C}{4} = 1 \quad \Rightarrow \quad C = \frac{4}{15} \tag{104}$$

Therefore:

$$S(x) \sim \frac{4}{15} x^{5/2} \quad \text{as } x \to 0^+ \tag{105}$$

Step 6: Write the leading behavior.
Final answer:

$$\boxed{y(x) \sim e^{\pm \frac{4}{15} x^{5/2}} \sim 1 + O(x^{5/2}) \quad \text{as } x \to 0^+} \tag{106}$$

Interpretation: Near $x = 0$, the exponential factor is close to 1, so the solution is approximately constant (or slightly varying) as $x \to 0^+$.

Problem 1(e): $x^5 y''' - 2xy' + y = 0$xy"' - 2xy' + y = 0

Step 1: Classify the singularity.
In standard form:

$$y''' = \frac{2xy' - y}{x^5} \tag{107}$$

Is $x = 0$ irregular? We have $p_0(x) = -\frac{1}{x^5}$, so:

$$x^3 p_0(x) = -\frac{1}{x^2} \tag{108}$$

which is not analytic at $x = 0$. Thus $x = 0$ is an **irregular singular point**.

Step 2: Try a power law first.
Let $y = x^\alpha$:

$$y' = \alpha x^{\alpha - 1} \tag{109}$$
$$y''' = \alpha(\alpha - 1)(\alpha - 2) x^{\alpha - 3} \tag{110}$$

Substituting:

$$x^5 \alpha(\alpha - 1)(\alpha - 2) x^{\alpha - 3} - 2x \cdot \alpha x^{\alpha - 1} + x^\alpha = 0 \tag{111}$$

Simplifying:

$$\alpha(\alpha - 1)(\alpha - 2) x^{\alpha + 2} - 2\alpha x^\alpha + x^\alpha = 0 \tag{112}$$

Factor out x^α:

$$x^\alpha \left[\alpha(\alpha-1)(\alpha-2)x^2 - 2\alpha + 1\right] = 0 \tag{113}$$

For what α does this hold as $x \to 0^+$? The constant terms must vanish:

$$-2\alpha + 1 = 0 \quad \Rightarrow \quad \alpha = \frac{1}{2} \tag{114}$$

Check: With $\alpha = \frac{1}{2}$:

$$\alpha(\alpha-1)(\alpha-2) = \frac{1}{2} \cdot \left(-\frac{1}{2}\right) \cdot \left(-\frac{3}{2}\right) = \frac{3}{8} \tag{115}$$

$$-2\alpha + 1 = -1 + 1 = 0 \quad \checkmark \tag{116}$$

The equation becomes:

$$x^{1/2}\left[\frac{3}{8}x^2\right] = 0 \tag{117}$$

which is satisfied as $x \to 0^+$.

Step 3: Verify there are other solutions.
Why check? A third-order ODE should have three linearly independent solutions. We've found one power law solution; there may be others.

The controlling factor ansatz $y = e^{S(x)}$ would give:

$$x^5(S''' + 3S'S'' + (S')^3) - 2xS' + 1 = 0 \tag{118}$$

Dominant balance: Assume $S \sim Cx^\beta$:

$$x^5(S')^3 \sim x^{3\beta+2} \tag{119}$$

$$-2xS' \sim x^{\beta-1} \tag{120}$$

$$1 \sim x^0 \tag{121}$$

Try: $-2xS' \sim 1$, i.e., $\beta - 1 = 0 \Rightarrow \beta = 1$.

With $\beta = 1$:

$$x^5(S')^3 \sim x^5 \to 0 \tag{122}$$

$$-2xS' \sim x^0 \sim 1 \tag{123}$$

This gives $-2xC + 1 \sim 0$, which means $C \sim \frac{1}{2x}$, not a constant. This suggests $S \sim \log x$, but that's not of the form Cx^β.
Conclusion: The dominant behavior is given by the power law solution.

Final answer:

$$\boxed{y(x) \sim x^{1/2} \quad \text{as } x \to 0^+} \tag{124}$$

Additional solutions: There are likely other solutions involving $x^{1/2}\log x$ or exponential factors, but the leading power law behavior is $y \sim \sqrt{x}$.