Methods of Applied Mathematics - Part 1 [SEMT30006]

Exercise Sheet 2 – Equilibria, Stability, and Linearisation

1. **Stability in 1 dimension**
   A cup of hot coffee temperature T is placed in a room of ambient temperature A. The cup loses heat to the air at a rate proportional to the temperature difference $T - A$, with a constant of proportionality (rate constant of heat loss) c.

   (a) Formulate a differential equation that describes the cooling of the cup.

   (b) Show that a cup of coffee at ambient temperature will remain at ambient temperature.

   (c) Argue that the coffee will eventually approach ambient temperature regardless of the starting temperature (i.e. prove the equilibrium is stable and there are no other attractors).

   (d) Use a change of variables $x = T - A$ to place the equilibrium at $x = 0$. Solve the differential equation for $x(t)$ and then change the variables back to obtain a solution for $T(t)$.

2. **Multiple equilibria in a 1 dimensional system**
   Consider the dynamical system

$$\dot{x} = x^4 - 17x^3 + 101x^2 - 247x + 210$$

   You are told that this has four equilibria, at $x = 2, 3, 5, 7$.

   (a) Show that the ODE can therefore be written as

$$\dot{x} = (x - a)(x - b)(x - c)(x - d)$$

   and find the constants a, b, c, d.

   (b) Find the stability of each equilibrium.

   (c) What is the long term behaviour of a trajectory starting at $x_0 = 6$?

   (d) What is the long term behaviour of a trajectory starting at $x_0 = 8$?

3. **Stability in 2d**
   Consider the 2d ODE

$$\dot{x} = x - 4y$$
$$\dot{y} = y - x$$

   with initial condition $x(0) = 1$, $y(0) = 0$.

   (a) Find the system's equilibria and their stability. What kind of equilibria are they (i.e. node/focus/saddle)?

   (b) Solve the system and verify that this fits with your answer to (a).
   Hint: try to look for a solution of the form

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \alpha_1 \mathbf{c}_1 e^{\lambda_1 t} + \alpha_2 \mathbf{c}_2 e^{\lambda_2 t}$$

   which we claimed in lectures should work (by eigendecomposition), where $\mathbf{c}_1$, $\mathbf{c}_2$, are the eigenvectors and λ_1, λ_2, are the eigenvalues of the equilibrium. The terms α_1, α_2, are constants that need to be found.

4. **Stability in 2d**
   Find the equilibria of the system

$$\dot{x} = y - x^2$$
$$\dot{y} = x - y^2$$

   and determine their stability.

5. **Classification of equilibria in 3D**
   Classify all hyperbolic equilibria of a linear vector field in three dimensions, i.e., draw phase portraits for all topologically different cases when the origin is a hyperbolic equilibrium of the vector field. (Hint: start from the 2D cases, e.g., attracting node, attracting spiral, saddle, etc, and bear in mind that a 3D system has 3 eigenvalues; where in the complex plane can they be?)

6. **Topological equivalence**

Consider the two linear systems

$$\begin{aligned} \dot{x}_1 &= -x_1, \\ \dot{x}_2 &= -x_2, \end{aligned} \qquad \text{and} \qquad \begin{aligned} \dot{y}_1 &= -y_1 - y_2, \\ \dot{y}_2 &= y_1 - y_2. \end{aligned}$$

(a) Sketch their 2D phase portraits.

(b) Follow the steps below to show that the two systems are topologically equivalent.

i. Write both systems in polar coordinates such that $x_1 = r\cos\theta$, $x_2 = r\sin\theta$ and $y_1 = \rho\cos\phi$, $y_2 = \rho\sin\phi$.

ii. Hence show that $\dot{r} = -r$, and find an expression for $\dot{\theta}$.

iii. Similarly, show that $\dot{\rho} = -\rho$, and find an expression for $\dot{\phi}$.

iv. Solve each of the resulting systems with initial conditions $r(0) = r_0$, $\theta(0) = \theta_0$ and $\rho(0) = \rho_0$, $\phi(0) = \phi_0$.

v. Hence show that the solution of one system can be transformed (or rather *mapped*) to the solution of the other by defining $\rho = r$ and $\phi = \theta - \ln(r)$. This shows that the solutions of one system can be mapped to those of the other by a continuous function, a *homeomorphism*, so these systems are *topologically equivalent*.