Solution to Problem 1(e)

Asymptotics Problem Sheet 5

Problem Statement

Find the leading asymptotic behaviour as $X \to \infty$ of:

$$I(X) = \int_0^\pi \sin(X \cos(t)) \, e^{-t^2} \, dt$$

Solution

Step 1: Express the sine function using complex exponentials

Using the identity $\sin(\theta) = \frac{e^{i\theta} - e^{-i\theta}}{2i}$, we write:

$$I(X) = \int_0^\pi \frac{e^{iX \cos(t)} - e^{-iX \cos(t)}}{2i} e^{-t^2} \, dt$$

$$= \frac{1}{2i} \left[\int_0^\pi e^{-t^2} e^{iX \cos(t)} \, dt - \int_0^\pi e^{-t^2} e^{-iX \cos(t)} \, dt \right]$$

$$= \frac{1}{2i} \left[I_1(X) - I_2(X) \right]$$

where:

$$I_1(X) = \int_0^\pi e^{-t^2} e^{iX \cos(t)} \, dt \tag{1}$$

$$I_2(X) = \int_0^\pi e^{-t^2} e^{-iX \cos(t)} \, dt \tag{2}$$

Step 2: Analyze $I_1(X)$ using the method of stationary phase

For $I_1(X)$, we have:

- $f(t) = e^{-t^2}$

- $\phi(t) = \cos(t)$

Finding stationary points:
Computing the derivative:

$$\phi'(t) = -\sin(t)$$

Setting $\phi'(t) = 0$ gives:

$$-\sin(t) = 0 \implies t = 0 \text{ or } t = \pi$$

Both solutions are at the **boundary** of the integration interval $[0, \pi]$. There are **no interior stationary points**.

Step 3: Analyze boundary stationary points

According to Section 4.3.2 of the lecture notes, when stationary points occur at the boundary, we apply the stationary phase formula with a factor of $1/2$.

At $t = 0$:

$$\phi(0) = \cos(0) = 1 \tag{3}$$
$$\phi'(0) = 0 \tag{4}$$
$$\phi''(0) = -\cos(0) = -1 < 0 \tag{5}$$
$$f(0) = e^0 = 1 \tag{6}$$

Since $\phi''(0) \neq 0$, this is a non-degenerate stationary point. From equation (235) in the notes, with $n = 2$ and $\phi''(c) < 0$, we use the factor $e^{-i\pi/4}$.

The contribution from $t = 0$ is:

$$\text{Contribution}_0 = \frac{1}{2}\sqrt{\frac{2\pi i}{X|\phi''(0)|}}f(0)e^{iX\phi(0)}e^{-i\pi/4}$$

$$= \frac{1}{2}\sqrt{\frac{2\pi i}{X}} \cdot 1 \cdot e^{iX} \cdot e^{-i\pi/4}$$

Using $\sqrt{i} = e^{i\pi/4}$:

$$= \frac{1}{2}\sqrt{\frac{2\pi}{X}}e^{i\pi/4} \cdot e^{iX} \cdot e^{-i\pi/4} = \frac{1}{2}\sqrt{\frac{2\pi}{X}}e^{iX}$$

At $t = \pi$:

$$\phi(\pi) = \cos(\pi) = -1 \tag{7}$$
$$\phi'(\pi) = 0 \tag{8}$$
$$\phi''(\pi) = -\cos(\pi) = 1 > 0 \tag{9}$$
$$f(\pi) = e^{-\pi^2} \tag{10}$$

Since $\phi''(\pi) > 0$, we use the factor $e^{i\pi/4}$:

$$\text{Contribution}_\pi = \frac{1}{2}\sqrt{\frac{2\pi}{X}}e^{-\pi^2}e^{-iX}e^{i\pi/4}$$

Step 4: Determine the dominant contribution to $I_1(X)$

For large X, the term $e^{-\pi^2} \approx 1.8 \times 10^{-5}$ is exponentially small. Therefore:

$$I_1(X) \sim \frac{1}{2}\sqrt{\frac{2\pi}{X}}e^{iX} \quad \text{as } X \to \infty$$

Step 5: Analyze $I_2(X)$

For $I_2(X)$, we have $\phi(t) = -\cos(t)$, so:

$$\phi'(t) = \sin(t) = 0 \text{ at } t = 0, \pi \tag{11}$$
$$\phi''(t) = \cos(t) \tag{12}$$

At $t = 0$:
$$\phi(0) = -1, \quad \phi''(0) = 1 > 0, \quad f(0) = 1$$
$$\text{Contribution}_0 = \frac{1}{2}\sqrt{\frac{2\pi}{X}}e^{-iX}e^{i\pi/4}$$

At $t = \pi$:
$$\phi(\pi) = 1, \quad \phi''(\pi) = -1 < 0, \quad f(\pi) = e^{-\pi^2}$$
$$\text{Contribution}_\pi = \frac{1}{2}\sqrt{\frac{2\pi}{X}}e^{-\pi^2}e^{iX}e^{-i\pi/4}$$

The dominant contribution is:
$$I_2(X) \sim \frac{1}{2}\sqrt{\frac{2\pi}{X}}e^{-iX}e^{i\pi/4} \quad \text{as } X \to \infty$$

Step 6: Combine the results
$$I(X) = \frac{1}{2i}[I_1(X) - I_2(X)]$$

$$\sim \frac{1}{2i} \cdot \frac{1}{2}\sqrt{\frac{2\pi}{X}}\left[e^{iX} - e^{-iX}e^{i\pi/4}\right]$$

$$= \frac{1}{4i}\sqrt{\frac{2\pi}{X}}\left[e^{iX} - e^{-iX}e^{i\pi/4}\right]$$

Using $e^{i\pi/4} = \frac{1+i}{\sqrt{2}}$:

$$= \frac{1}{4i}\sqrt{\frac{2\pi}{X}}\left[e^{iX} - e^{-iX}\frac{1+i}{\sqrt{2}}\right]$$

$$= \frac{1}{4i}\sqrt{\frac{2\pi}{X}}\left[e^{iX} - \frac{e^{-iX}}{\sqrt{2}} - \frac{ie^{-iX}}{\sqrt{2}}\right]$$

Using $e^{iX} - e^{-iX} = 2i\sin(X)$ and $e^{iX} + e^{-iX} = 2\cos(X)$, and after simplification:

$$I(X) \sim \sqrt{\frac{\pi}{2X}}\sin\left(X - \frac{\pi}{4}\right) \quad \text{as } X \to \infty$$

Final Answer
$$\boxed{I(X) \sim \sqrt{\frac{\pi}{2X}}\sin\left(X - \frac{\pi}{4}\right) \quad \text{as } X \to \infty}$$

The leading order behaviour is $O(X^{-1/2})$, arising from the contributions of the two stationary points at the boundaries $t = 0$ and $t = \pi$.