Asymptotics 2025/2026 — Sheet 8[1]

Boundary layers and asymptotic matching

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1. Consider the differential equation $\varepsilon^2 y'' - y = 0$, $-1 < x < 1$, $\varepsilon > 0$, with boundary conditions $y(1) = y(-1) = 1$. Determine outer and inner solutions in leading order, given that there are boundary layers at $x = 1$ and $x = -1$. Is matching needed here. Determine also a composite solution and compare it to the exact solution.

2. Obtain a one-term composite expansion for $\varepsilon \to 0$, for the solution of
$$\varepsilon \frac{d^2 f}{dx^2} - \frac{df}{dx} + \frac{f}{x+1} = 2 \, , \qquad 0 < x < 1 \, , \qquad \varepsilon > 0 \, ,$$
with boundary conditions $f(0) = 0$, $f(1) = 3$, using Prandtl's matching criterion.

3. Perform an asymptotic matching to obtain a uniformly valid one-term (optionally: two-term) composite expansion for the solution, $f(x)$, as $\varepsilon \to 0$ of
$$\varepsilon f'' + (2 + x) f' + f = 1 \, , \qquad 0 < x < 1 \, , \qquad \varepsilon > 0 \, ,$$
with boundary conditions $f(0) = 2$, $f(1) = 0$.

4. Find a first-order uniform expansion as $\varepsilon \to 0$ for $y(x)$ satisfying
$$\varepsilon y'' + x^2 y' - x^3 y = 0 \, , \qquad y(0) = \alpha \, , \qquad y(1) = \beta \, .$$

5. Carefully explain how successful Van Dyke matching of inner and outer asymptotic expansions, in one dimension, can lead to a composite expansion usable over the whole domain. Is this possible when the domain has a boundary layer at both ends of the domain?

6. For the o.d.e. $\varepsilon y'' + \sin(x) y' + \sin(2x) y = 0$, $0 \le x \le \pi$, and $y(0) = \pi$, $y(\pi) = 0$, consider asymptotic expansions for $\varepsilon \to 0$ with a boundary layer at either end of the interval, and find one or more asymptotic expansions for the solution $y(x)$ to leading order.

7. Find an asymptotic expansion to leading order for the solution $y(x)$ to

$$\varepsilon y'' + xy' + xy = 0\,, \qquad \text{in} \qquad -1 < x < 1 \qquad \text{for} \qquad \varepsilon \to 0$$

with $y(-1) = e$, and $y(1) = 2e^{-1}$, given that the solution has an 'interior layer'.