Local approximation to linear ODEs

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1. (a) The equation $x^4\, y''' = y$ has an irregular singular point at $x = 0$. We set $y(x) = \exp(S(x))$ and obtain the following equation for S

$$x^4\, S''' + 3\, x^4\, S''\, S' + x^4\, S'^3 - 1 = 0\ .$$

We try $S(x) \sim \alpha x^\beta$ as $x \to 0+$, and insert it into the equation

$$\alpha\, \beta\, (\beta - 1)\, (\beta - 2)\, x^{\beta+1} + 3\, \alpha^2\, \beta^2\, (\beta - 1)\, x^{2\beta+1} + \alpha^3\, \beta^3\, x^{3\beta+1} \sim 1 \quad \text{as} \quad x \to 0+\ .$$

If $\beta \geq 0$ then the left-hand side (LHS) is $O(x)$ as $x \to 0+$ and thus there is no balance with the RHS. So we assume that $\beta < 0$. Then the third term on the LHS is the dominant one, and we obtain

$$\alpha^3\, \beta^3\, x^{3\beta+1} \sim 1 \quad \text{as} \quad x \to 0+\ .$$

We conclude that $\beta = -1/3$ and $\alpha = -3\delta$, where $\delta^3 = 1$, i. e. $\delta = 1$ or $e^{\pm 2\pi i/3}$. For the next-order correction we set $S(x) = -3\delta x^{-1/3} + C(x)$, where $C(x) = o(x^{-1/3})$ as $x \to 0+$. We insert it into the equation for $S(x)$ and obtain

$$x^4 \left(\frac{28}{9} \delta x^{-10/3} + C''' \right) + 3x^4 \left(-\frac{4}{3} \delta x^{-7/3} + C'' \right) \left(\delta x^{-4/3} + C' \right) + x^4 \left(\delta x^{-4/3} + C' \right)^3 = 1.$$

We have that $C'(x) = o(x^{-4/3})$, $C''(x) = o(x^{-7/3})$, and $C'''(x) = o(x^{-10/3})$. For this reason, we can neglect all derivatives of $C(x)$ except in the last term on the LHS, because there the leading term cancels with the 1 on the RHS. We obtain

$$\frac{28}{9} \delta x^{2/3} - 4x^{1/3} \delta^2 \sim -3\delta^2 C' x^{4/3} \quad \text{as} \quad x \to 0+\ .$$

We conclude that $C'(x) \sim \frac{4}{3x}$, or $C(x) \sim \frac{4}{3} \log(x)$ as $x \to 0+$.
A similar calculation yields the next order correction. Altogether we find

$$S(x) \sim -3\, \delta\, x^{-1/3} + \frac{4}{3}\, \log(x) + d + \frac{8}{9\, \delta}\, x^{1/3} \quad \text{as} \quad x \to 0+\ .$$

Correspondingly, the leading order behaviour of $y(x)$ is

$$y(x) \sim c\, x^{4/3}\, \exp(-3\, \delta\, x^{-1/3}) \quad \text{as} \quad x \to 0+\ .$$

(b) The equation $y'' = (\cot x)^4 y$ has an irregular singular point at $x = 0$. We set $y(x) = \exp(S(x))$ and obtain the following equation for S

$$S'' + S'^2 = (\cot x)^4 \ .$$

To find the leading order behaviour of $S(x)$ we can use that $\cot x \sim \frac{1}{x} - \frac{x}{3} + \dots$ as $x \to 0+$ and set $S(x) \sim \alpha x^\beta$. Insertion into the equation yields

$$\alpha\,\beta\,(\beta-1)\,x^{\beta-2} + \alpha^2\,\beta^2\,x^{2\beta-2} \sim x^{-4} \quad \text{as} \quad x \to 0+ \ .$$

For $\beta \geq 0$ the LHS cannot balance the RHS as $x \to 0+$. So we assume $\beta < 0$. Then the second term on the LHS is the dominant one, and we obtain

$$\alpha^2\,\beta^2\,x^{2\beta-2} \sim x^{-4} \quad \text{as} \quad x \to 0+ \ .$$

We conclude that $\beta = -1$ and $\alpha = \pm 1$. For the next-order correction we set $S(x) = \pm x^{-1} + C(x)$, where $C(x) = o(x^{-1})$ as $x \to 0+$, and insert it into the equation for $S(x)$ to obtain

$$\pm 2x^{-3} + C'' + x^{-4} \mp 2x^{-2} C' + [C']^2 = \left(\frac{1}{x} - \frac{x}{3} + \dots\right)^4 = x^{-4} - \frac{4}{3} x^{-2} + \dots \ .$$

We know that $C'(x) = o(x^{-2})$ and $C''(x) = o(x^{-3})$. For this reason, we find that the balance of the two leading terms is given by

$$\pm 2x^{-3} \sim \pm 2x^{-2} C'(x) \quad \text{as} \quad x \to 0+ \ .$$

We conclude that $C'(x) \sim x^{-1}$, and $C(x) \sim \log(x)$ as $x \to 0+$. To find the next term we insert $C(x) = \log(x) + D(x)$ into the equation for $C(x)$, where $D(x) = o(\log(x))$

$$-x^{-2} + D'' \mp 2x^{-2} D' + [x^{-1} + D']^2 = -\frac{4}{3} x^{-2} + \dots \ .$$

Using the property that $D'(x) = o(x^{-1})$ and $D''(x) = o(x^{-2})$, yields

$$\mp 2x^{-2} D' \sim -\frac{4}{3} x^{-2} \quad \text{as} \quad x \to 0+ \ ,$$

and so $D'(x) \sim \pm 2/3$, and $D(x) \sim d \pm \frac{2}{3} x$ as $x \to 0+$. Consequently, the expansion of $S(x)$ is given by

$$S(x) \sim \pm x^{-1} + \log(x) + d \pm \frac{2}{3} x \quad \text{as} \quad x \to 0+ \ ,$$

and the leading order behaviour of $y(x)$ is

$$y(x) \sim c\,x\,\exp(\pm x^{-1}) \quad \text{as} \quad x \to 0+ \ .$$

(c) The calculation is very similar as in question 1(a). We give here only the final result. The function $S(x)$ in $y(x) = \exp(S(x))$ has the asymptotic form

$$S(x) \sim 3\,\delta \left(\frac{x}{2}\right)^{-1/3} + \frac{4}{3} \log x + d - \frac{35}{9\,\delta} \left(\frac{x}{2}\right)^{1/3} \quad \text{as} \quad x \to 0+ \ ,$$

where d is an arbitary constant and $\delta^3 = 1$, i. e. $\delta = 1$, or $\exp(\pm 2\pi/3)$. Consequently, the leading order approximation for $y(x)$ is

$$y(x) \sim c\,x^{4/3} \exp\left(3\,\delta \left(\frac{x}{2}\right)^{-1/3}\right) \ .$$

(d) The equation $y'' = \sqrt{x}\, y$ has an irregular singular point at $x = 0$, because the square root has its branch point there. If we try a solution of the form $y(x) = \exp(S(x))$ we obtain the following equation for S

$$S'' + S'^2 = \sqrt{x}\ .$$

Setting $S(x) \sim \alpha x^\beta$ as $x \to 0+$ yields

$$\alpha\beta(\beta-1)x^{\beta-2} + \alpha^2\beta^2 x^{2\beta-2} \sim x^{1/2}\ .$$

Now we apply the method of dominant balance. A short check shows that there can be no balance between the leading order terms if $\beta < 0$. If $\beta > 0$, then the first term on the LHS dominates the second term as $x \to O+$, and we conclude that $\beta = 5/2$ and $\alpha = 4/15$. So $S(x) \sim \frac{4}{15}x^{5/2}$. We can add an arbitrary constant, since a constant is always a solution to the equation for S (and is dominant here as $x \to 0+$). We conclude that

$$y(x) \sim c\ \exp\left(\frac{4}{15}x^{5/2}\right) \sim c\left(1 + \frac{4}{15}x^{5/2}\right) \quad \text{as} \quad x \to 0+\ .$$

We have obtained only one solution, but need a second one for this second-order ODE. We notice that in the method of dominant balance above, the exponents $(\beta - 2)$ and $(2\beta - 2)$ match (and correspond to the dominant terms) if $\beta = 0$, but the derivation of the considered equation is not valid for $\beta = 0$. This is always an indication that $S(x)$ might have a $\log(x)$ dependence, i. e. that $y(x)$ depends on a power of x. Trying $y(x) \sim cy^\gamma + dy^\eta$, we find for the second solution

$$y(x) \sim c\left(x + \frac{4}{35}x^{7/2}\right) \quad \text{as} \quad x \to 0+\ .$$

(e) Inserting $y(x) = \exp(S(x))$ into the equation $x^5 y''' - 2xy' + y = 0$ yields

$$x^5\, S''' + 3\, x^5\, S''\, S' + x^5 S'^3 - 2\, x\, S' + 1 = 0\ .$$

Setting $S(x) \sim \alpha x^\beta$ as $x \to 0+$ results in

$$\alpha\beta(\beta-1)(\beta-2)x^{\beta+2} + 3\alpha^2\beta^2(\beta-1)x^{2\beta+2} + \alpha^3\beta^3 x^{3\beta+2} - 2\alpha\beta x^\beta \sim -1 \quad \text{as} \quad x \to 0+\ .$$

For positive β there is no balancing with the 1 on the RHS so we assume $\beta < 0$. Then the method of dominant balance yields

$$\alpha^3\beta^3 x^{3\beta+2} \sim 2\alpha\beta x^\beta \quad \text{as} \quad x \to 0+\ .$$

We conclude that $\beta = -1$ and $\alpha = \pm\sqrt{2}$. The higher-order terms are given here without derivation

$$S(x) \sim \pm\sqrt{2}x^{-1} + \frac{11}{4}\log(x) + d \pm \frac{63}{64}\sqrt{2}x \quad \text{as} \quad x \to 0+\ .$$

Consequently,

$$y(x) \sim cx^{11/4}\ \exp\left(\pm\frac{\sqrt{2}}{x}\right) \quad \text{as} \quad x \to 0+\ .$$

We still need the leading order behaviour for the third solution. Similarly as for problem 1(d) we note that in the method of dominant balance above, the exponents β on the LHS and 0 on the RHS match (and correspond to the dominant terms) if $\beta = 0$, although the derivation of the considered equation is not valid for $\beta = 0$. This indicates that $S(x)$ might have a $\log(x)$ dependence, i. e. that $y(x)$ depends on a power of x. Trying $y(x) \sim cx^\gamma$, we obtain

$$c\gamma(\gamma-1)(\gamma-2)x^{\gamma+2} - 2c\gamma x^\gamma \sim -cx^\gamma \ .$$

We conclude that c is arbitary and $\gamma = \frac{1}{2}$, and

$$y(x) \sim c\sqrt{x} \quad \text{as} \quad x \to 0+ \ .$$

2. (a) By making the substitution $y = e^S$ we have

$$x(S''' + 3\,S''\,S' + S'^3) = S' \ .$$

By assuming that $S''' = o(S'')$ and $S'' = o(S'^2)$ as $x \to \infty$, we obtain

$$xS'^3 \sim S' \quad \text{as} \quad x \to \infty \ ,$$

which leads to $S' = 0$ and $S' \sim \pm x^{-1/2}$ as $x \to \infty$. The first possibility leads to $y(x) = \text{const}$ which is a solution of the ODE. The other two possibilities give $S = \pm 2x^{1/2} + C(x)$, where $C = o(x^{1/2})$ as $x \to \infty$.

We consider first the solution with the plus sign. The other follows a similar pattern. First we have

$$\begin{aligned} S' &= x^{-1/2} + C', & C' &= o(x^{-1/2}) \\ S'' &= -\frac{1}{2}x^{-3/2} + C'', & C'' &= o(x^{-3/2}) \\ S''' &= \frac{3}{4}x^{-5/2} + C''', & C''' &= o(x^{-5/2}). \end{aligned}$$

By inserting the above expressions into the equation for $S(x)$ we get

$$x\left(\frac{3}{4}x^{-5/2} + C'''\right) + 3x\left(x^{-1/2} + C'\right)\left(-\frac{1}{2}x^{-3/2} + C''\right) + x\left(x^{-1/2} + C'\right)^3 = x^{-1/2} + C'.$$

Expanding this equation and using the order relation for $C(x)$ and its derivatives we obtain for $x \to \infty$

$$-\frac{3}{2}x^{-1} + 3C' \sim C' \quad \Longrightarrow \quad C' \sim \frac{3}{4}x^{-1} \quad \Longrightarrow \quad C = \frac{3}{4}\log x + D(x) \ ,$$

where $D(x) = o(\log x)$ as $x \to \infty$. Now,

$$\begin{aligned} S' &= x^{-1/2} + \frac{3}{4}x^{-1} + D', & D' &= o(x^{-1}) \\ S'' &= -\frac{1}{2}x^{-3/2} - \frac{3}{4}x^{-2} + D'', & D'' &= o(x^{-2}) \\ S''' &= \frac{3}{4}x^{-5/2} + \frac{3}{2}x^{-3} + D''', & D''' &= o(x^{-3}), \end{aligned}$$

as $x \to \infty$. Inserting these relations into the equation for $S(x)$, we have

$$x\left(\frac{3}{4}x^{-5/2}+\frac{3}{2}x^{-3}+D'''\right)+3x\left(x^{-1/2}+\frac{3}{4}x^{-1}+D'\right)\left(-\frac{1}{2}x^{-3/2}-\frac{3}{4}x^{-2}+D''\right)$$
$$+x\left(x^{-1/2}+\frac{3}{4}x^{-1}+D'\right)^3=x^{-1/2}+\frac{3}{4}x^{-1}+D'.$$

Expanding the above equation and the order relation for $D(x)$ and its derivatives we have

$$\frac{3}{4}x^{-3/2}-\frac{9}{4}x^{-3/2}-\frac{9}{8}x^{-3/2}+\frac{27}{16}x^{-3/2}+3D'\sim D' \quad\Longrightarrow\quad D'\sim\frac{15}{32}x^{-3/2}$$

as $x \to \infty$. We conclude that

$$D\sim-\frac{15}{16}x^{-1/2}+d \quad \text{as} \quad x\to\infty\,,$$

where d is an arbitrary constant. Including the corresponding result for the second solution we obtain finally

$$S(x)\sim\pm 2x^{1/2}+\frac{3}{4}\log x+d\mp\frac{15}{16}x^{-1/2} \quad \text{as} \quad x\to\infty\,.$$

Therefore, the leading order behaviours are

$$y(x)\sim cx^{3/4}\exp\left(\pm 2x^{1/2}\right) \quad \text{as} \quad x\to\infty.$$

(b) Let us set $y=e^S$, then the equation becomes

$$S''+S'^2=\sqrt{x}\,.$$

By assuming $S''=o(S'^2)$ as $x\to\infty$ we obtain $S'\sim\pm x^{1/4}$. Consequently, we set

$$\begin{aligned}
S(x)&=\pm\frac{4}{5}x^{5/4}+C(x), && C(x)=o(x^{5/4}),\\
S'(x)&=\pm x^{1/4}+C'(x), && C'(x)=o(x^{1/4}),\\
S''(x)&=\pm\frac{1}{4}x^{-3/4}+C''(x), && C''(x)=o(x^{-3/4}),
\end{aligned}$$

as $x\to\infty$. Note that indeed $S''=o(S'^2)$ as $x\to\infty$. Inserting these relations into the equation for $S(x)$ we have

$$\left(\pm\frac{1}{4}x^{-3/4}+C''\right)+\left(\pm x^{1/4}+C'\right)^2=x^{1/2}$$
$$\pm\frac{1}{4}x^{-3/4}+C''\pm 2x^{1/4}C'+C'^2=0.$$

By using the order relations for $C(x)$ and its derivatives, we get

$$\pm 2x^{1/4}C'\sim\mp\frac{1}{4}x^{-3/4} \quad\Longrightarrow\quad C'\sim-\frac{1}{8}x^{-1} \quad\Longrightarrow\quad C\sim-\frac{1}{8}\log x$$

as $x\to\infty$. Therefore we set

$$\begin{aligned}
S(x)&=\pm\frac{4}{5}x^{5/4}-\frac{1}{8}\log x+D(x), && D(x)=o(\log x),\\
S'(x)&=\pm x^{1/4}-\frac{1}{8}x^{-1}+D'(x), && D'(x)=o(x^{-1}),\\
S''(x)&=\pm\frac{1}{4}x^{-3/4}+\frac{1}{8}x^{-2}+D''(x), && D''(x)=o(x^{-2}),
\end{aligned}$$

as $x \to \infty$. Inserting the above formulas into the equation for $S(x)$ we get

$$\frac{1}{8}x^{-2} + D'' \pm 2x^{1/4}D' + \frac{1}{64}x^{-2} + D'^2 - \frac{1}{4}x^{-1}D' = 0 \ .$$

Using the order relations for $D(x)$ and its derivatives results in

$$\frac{9}{64}x^{-2} \sim \mp 2x^{1/4}D' \quad \Longrightarrow \quad D' \sim \mp\frac{9}{128}x^{-9/4} \quad \Longrightarrow \quad D \sim d \pm \frac{9}{160}x^{-5/4} \ ,$$

as $x \to \infty$ where d is a constant. Therefore

$$S(x) \sim \pm\frac{4}{5}x^{5/4} - \frac{1}{8}\log x + d \pm \frac{9}{160}x^{-5/4}, \qquad x \to \infty \ .$$

Finally, the leading order behaviour of $y(x)$ is given by

$$y(x) \sim cx^{-1/8}\exp\left(\pm\frac{4}{5}x^{5/4}\right) \ , \qquad x \to \infty.$$