Problem 7, Question 4: When is the WKB Solution Exact?

Pedagogical Breakdown

Question Statement

For what choices of $q(x)$ in the equation

$$\varepsilon^2 y'' + q(x)y = 0 \tag{1}$$

is the WKB solution exact?

Solution

Step 1: Recall the Structure of the WKB Approximation

What are we doing? We begin by recalling the form of the leading-order WKB approximation as developed in Section 6.3.2 of the lecture notes.

Why? Before we can determine when the WKB solution is *exact*, we must understand what the WKB solution *is*. The WKB approximation provides an asymptotic solution; exactness means this asymptotic form satisfies the ODE without any residual error.

The WKB Solution: For $q(x) > 0$, the leading-order WKB approximation (equations 382–383, page 69 of the lecture notes) gives:

$$y_\pm(x) = \frac{A_\pm}{[q(x)]^{1/4}} \exp\left(\pm\frac{i}{\varepsilon}\int^x \sqrt{q(s)}\,\mathrm{d}s\right) \tag{2}$$

For $q(x) < 0$, the corresponding form is:

$$y_\pm(x) = \frac{B_\pm}{|q(x)|^{1/4}} \exp\left(\pm\frac{1}{\varepsilon}\int^x \sqrt{|q(s)|}\,\mathrm{d}s\right) \tag{3}$$

Key observation: The WKB solution has two components:

- An *amplitude factor*: $[q(x)]^{-1/4}$
- A *phase factor*: $\exp\left(\pm\frac{i}{\varepsilon}\int^x \sqrt{q(s)}\,\mathrm{d}s\right)$

Step 2: Criterion for Exactness – Direct Substitution

What are we doing? We substitute the WKB solution directly into the ODE $\varepsilon^2 y'' + q(x)y = 0$ to determine when it is satisfied exactly.

Why? The WKB solution is derived as an asymptotic approximation. It is *exact* if and only if substituting it into the ODE yields zero identically, not just to leading order in ε.

Setup: Let us write the WKB solution as:

$$y(x) = A(x)e^{\pm i\phi(x)} \tag{4}$$

where the amplitude is $A(x) = [q(x)]^{-1/4}$ and the phase is $\phi(x) = \frac{1}{\varepsilon}\int^x \sqrt{q(s)}\,\mathrm{d}s$.

Computing derivatives:

$$y' = \left(A' \pm i\phi' A\right) e^{\pm i\phi} \tag{5}$$
$$y'' = \left(A'' \pm 2i\phi' A' \pm i\phi'' A - (\phi')^2 A\right) e^{\pm i\phi} \tag{6}$$

Substituting into the ODE:

$$\varepsilon^2 y'' + qy = \varepsilon^2 \left(A'' \pm 2i\phi' A' \pm i\phi'' A - (\phi')^2 A\right) e^{\pm i\phi} + qA\, e^{\pm i\phi} \tag{7}$$

Using $\phi' = \frac{\sqrt{q}}{\varepsilon}$:

$$(\phi')^2 = \frac{q}{\varepsilon^2} \tag{8}$$

Therefore:

$$\varepsilon^2 y'' + qy = \left(\varepsilon^2 A'' \pm 2i\varepsilon^2 \phi' A' \pm i\varepsilon^2 \phi'' A - qA + qA\right) e^{\pm i\phi} \tag{9}$$

The terms $-qA + qA$ cancel, leaving:

$$\varepsilon^2 y'' + qy = \left(\varepsilon^2 A'' \pm 2i\varepsilon^2 \phi' A' \pm i\varepsilon^2 \phi'' A\right) e^{\pm i\phi} \tag{10}$$

Step 3: Condition for Exact Solution

What are we doing? We determine when the residual from Step 2 vanishes identically.

Why? For the WKB solution to be exact, we need $\varepsilon^2 y'' + qy = 0$, which requires the expression in parentheses to vanish.

The residual: For the WKB solution to be exact, we need:

$$\varepsilon^2 A'' \pm 2i\varepsilon^2 \phi' A' \pm i\varepsilon^2 \phi'' A = 0 \tag{11}$$

Separating real and imaginary parts: Since this must hold for both y_+ and y_-, the real and imaginary parts must separately vanish:

$$\text{Real part:} \quad \varepsilon^2 A'' = 0 \tag{12}$$
$$\text{Imaginary part:} \quad 2\varepsilon^2 \phi' A' + \varepsilon^2 \phi'' A = 0 \tag{13}$$

The key condition: Since $\varepsilon \neq 0$, the real part condition gives:

$$A''(x) = 0 \tag{14}$$

But wait – the imaginary part is automatically satisfied! To see this, note that:

$$2\phi' A' + \phi'' A = \frac{\mathrm{d}}{\mathrm{d}x}\left(\phi' A^2\right) \cdot \frac{1}{A} \tag{15}$$

Since $\phi' = \frac{\sqrt{q}}{\varepsilon}$ and $A = q^{-1/4}$, we have $\phi' A^2 = \frac{\sqrt{q}}{\varepsilon} \cdot q^{-1/2} = \frac{1}{\varepsilon}$, which is constant. Thus the imaginary condition is automatically satisfied.

Therefore, the condition for exactness is simply:

$$\boxed{A''(x) = 0 \quad \text{where} \quad A(x) = [q(x)]^{-1/4}} \tag{16}$$

Step 4: Computing $A''(x)$

What are we doing? We compute the second derivative of $A(x) = [q(x)]^{-1/4}$ explicitly.

Why? This will give us an explicit condition on $q(x)$.

First derivative:

$$A = q^{-1/4} \quad \Rightarrow \quad A' = -\frac{1}{4}q^{-5/4} \cdot q' = -\frac{q'}{4q^{5/4}} \tag{17}$$

Second derivative:

$$\begin{aligned}
A'' &= -\frac{1}{4}\frac{\mathrm{d}}{\mathrm{d}x}\left(q' \cdot q^{-5/4}\right) && (18)\\
&= -\frac{1}{4}\left(q'' \cdot q^{-5/4} + q' \cdot \left(-\frac{5}{4}\right) q^{-9/4} \cdot q'\right) && (19)\\
&= -\frac{1}{4}\left(\frac{q''}{q^{5/4}} - \frac{5(q')^2}{4q^{9/4}}\right) && (20)\\
&= -\frac{q''}{4q^{5/4}} + \frac{5(q')^2}{16q^{9/4}} && (21)
\end{aligned}$$

Factoring out $q^{-9/4}$:

$$A'' = q^{-9/4}\left(-\frac{q''q}{4} + \frac{5(q')^2}{16}\right) = \frac{1}{q^{9/4}}\left(\frac{5(q')^2}{16} - \frac{q''q}{4}\right) \tag{22}$$

Step 5: The Differential Equation for $q(x)$

What are we doing? We set $A'' = 0$ and derive the resulting condition on $q(x)$.

Why? This gives us the explicit differential equation that $q(x)$ must satisfy for the WKB solution to be exact.

Setting $A'' = 0$: Since $q^{-9/4} \neq 0$ (assuming $q \neq 0$), we require:

$$\frac{5(q')^2}{16} - \frac{q''q}{4} = 0 \tag{23}$$

Rearranging:

$$\frac{q''}{4q} = \frac{5(q')^2}{16q^2} \tag{24}$$

Simplifying:

$$\frac{q''}{q'} = \frac{5q'}{4q} \tag{25}$$

This can be written as:

$$\frac{\mathrm{d}}{\mathrm{d}x}\ln|q'| = \frac{5}{4} \cdot \frac{\mathrm{d}}{\mathrm{d}x}\ln|q| \tag{26}$$

Step 6: First Integration

What are we doing? We integrate the differential equation $\frac{q''}{q'} = \frac{5q'}{4q}$ once.

Why? This reduces the second-order ODE for $q(x)$ to a first-order ODE.

Integrating both sides:

$$\int \frac{q''}{q'}\,\mathrm{d}x = \int \frac{5q'}{4q}\,\mathrm{d}x \tag{27}$$

$$\ln|q'| = \frac{5}{4}\ln|q| + C_1 \tag{28}$$

Exponentiating:

$$|q'| = e^{C_1}|q|^{5/4} \tag{29}$$

Writing with a constant K:

$$q' = Kq^{5/4} \tag{30}$$

where $K = \pm e^{C_1}$ is an arbitrary nonzero constant.

Step 7: Second Integration – Separation of Variables

What are we doing? We solve the first-order ODE $q' = Kq^{5/4}$ by separation of variables.

Why? This gives us the explicit form of $q(x)$.

Separating variables:

$$\frac{\mathrm{d}q}{q^{5/4}} = K\,\mathrm{d}x \tag{31}$$

Integrating the left side:

$$\int q^{-5/4}\,\mathrm{d}q = \frac{q^{-5/4+1}}{-5/4+1} = \frac{q^{-1/4}}{-1/4} = -4q^{-1/4} \tag{32}$$

Integrating the right side:

$$\int K\,\mathrm{d}x = Kx + C_2 \tag{33}$$

Combining:

$$-4q^{-1/4} = Kx + C_2 \tag{34}$$

Step 8: Solving for $q(x)$

What are we doing? We solve the equation $-4q^{-1/4} = Kx + C_2$ for $q(x)$.

Why? This gives us the final answer.

Isolating $q^{-1/4}$:

$$q^{-1/4} = -\frac{Kx + C_2}{4} \tag{35}$$

Relabeling constants: Let $a = -K/4$ and $b = -C_2/4$. Then:

$$q^{-1/4} = ax + b \tag{36}$$

Raising both sides to the power -4:

$$q = (ax + b)^{-4} \tag{37}$$

Equivalently:

$$\boxed{q(x) = (ax + b)^{-4}} \tag{38}$$

where a and b are arbitrary constants (with $a \neq 0$ for a non-constant solution).

Step 9: Verification

What are we doing? We verify that $q(x) = (ax + b)^{-4}$ satisfies the condition $\frac{q''}{4q} = \frac{5(q')^2}{16q^2}$.

Why? It is good practice to check our answer by substitution.

Computing derivatives: Let $u = ax + b$. Then $q = u^{-4}$.

$$q' = -4u^{-5} \cdot a = -4au^{-5} \tag{39}$$

$$q'' = -4a \cdot (-5)u^{-6} \cdot a = 20a^2u^{-6} \tag{40}$$

Computing the left-hand side:

$$\frac{q''}{4q} = \frac{20a^2u^{-6}}{4u^{-4}} = \frac{20a^2}{4}u^{-2} = 5a^2u^{-2} \tag{41}$$

Computing the right-hand side:

$$\frac{5(q')^2}{16q^2} = \frac{5 \cdot 16a^2u^{-10}}{16u^{-8}} = 5a^2u^{-2} \tag{42}$$

Comparison: Both sides equal $5a^2u^{-2}$. ✓

Step 10: The Explicit Exact Solution

What are we doing? We write out the exact WKB solution when $q(x) = (ax + b)^{-4}$.

Why? Having identified when the WKB approximation is exact, we should state the explicit form of this exact solution.

For $q(x) = (ax + b)^{-4}$:

The amplitude factor is:

$$[q(x)]^{-1/4} = \left[(ax + b)^{-4}\right]^{-1/4} = (ax + b)^1 = ax + b \tag{43}$$

The phase integral is:

$$\int^x \sqrt{q(s)}\, \mathrm{d}s = \int^x (as + b)^{-2}\, \mathrm{d}s = -\frac{1}{a(ax + b)} \tag{44}$$

The exact solution:

$$y_\pm(x) = A_\pm(ax + b) \exp\left(\mp\frac{i}{\varepsilon \cdot a(ax + b)}\right) \tag{45}$$

Or in real form:

$$y_1(x) = (ax + b) \cos\left(\frac{1}{\varepsilon \cdot a(ax + b)}\right) \tag{46}$$

$$y_2(x) = (ax + b) \sin\left(\frac{1}{\varepsilon \cdot a(ax + b)}\right) \tag{47}$$

Verification by direct substitution: One can verify that these functions satisfy $\varepsilon^2y'' + (ax + b)^{-4}y = 0$ exactly.

Step 11: Special Case – Constant q

What are we doing? We check whether constant $q(x)$ is included in our answer.

Why? When q is constant, the ODE $\varepsilon^2 y'' + qy = 0$ has exact sinusoidal solutions. Is this consistent with our result?

Analysis: A constant $q = c$ corresponds to the limiting case $a \to 0$ in $(ax+b)^{-4}$. More precisely:

$$\lim_{a\to 0}(ax+b)^{-4} = b^{-4} = \text{constant} \tag{48}$$

For constant $q = c$, the exact solutions are:

$$y(x) = A\cos\left(\frac{\sqrt{c}}{\varepsilon}x\right) + B\sin\left(\frac{\sqrt{c}}{\varepsilon}x\right) \tag{49}$$

The WKB approximation for constant q gives:

$$y_{\mathrm{WKB}}(x) = \frac{A}{c^{1/4}}\exp\left(\pm\frac{i\sqrt{c}}{\varepsilon}x\right) \tag{50}$$

These are indeed exact solutions (up to the constant prefactor), confirming that constant q is a special case.

Step 12: Summary and Final Answer

The WKB solution to $\varepsilon^2 y'' + q(x)y = 0$ is exact if and only if:

$$\boxed{q(x) = (ax+b)^{-4}} \tag{51}$$

where a and b are arbitrary constants.

Equivalent characterizations:

1. **Differential equation form:** $\dfrac{q''}{4q} = \dfrac{5(q')^2}{16q^2}$, or equivalently, $q' = Kq^{5/4}$
2. **Amplitude condition:** The amplitude factor $A(x) = [q(x)]^{-1/4}$ satisfies $A''(x) = 0$, meaning $A(x)$ is linear in x
3. **Algebraic form:** $q(x) = (ax+b)^{-4}$ (inverse fourth power of a linear function)

Physical interpretation: The inverse fourth power form $q(x) \propto (ax+b)^{-4}$ represents a very special "potential" in the corresponding Schrödinger-like equation. For this particular form, the WKB phase integral and amplitude modulation combine in precisely the right way to yield an exact solution.

Note on signs and domains: For $q(x) > 0$ (oscillatory case), we need $(ax+b)^{-4} > 0$, which is satisfied for all $x \neq -b/a$. The solution is valid on any interval not containing the singularity at $x = -b/a$.