Asymptotics 2025/2026 Sheet 1
Problem 3: Verification of Order Relations
Complete Methodological Analysis

Preamble: Understanding What We Are Being Asked

Before we begin solving Problem 3, we must understand **why** we are being asked to verify these statements and **what** mathematical framework governs our approach.

The Purpose of Order Symbols

In asymptotic analysis, we study how functions behave as their arguments approach specific values (often 0 or ∞). The lecture notes (Section 2.4.1) introduce order symbols as a **precise language** for describing these behaviors.

Why do we need this language? Because vague statements like "f is small compared to g" are insufficient for rigorous mathematics. Order symbols provide:

- **Precision:** Exact conditions for when one function dominates another
- **Hierarchy:** A way to rank functions by their asymptotic behavior
- **Computational power:** Rules for manipulating asymptotic expressions

The Three Key Concepts from Lecture Notes

From Section 2.4.1 of the lecture notes, we have three fundamental definitions:

Definition 1 (Little-oh, Equation (22))**.** *$f(x) = o(g(x))$ as $x \to x_0$ if*

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = 0.$$

Interpretation: *f is* asymptotically smaller *than g.*

Definition 2 (Big-Oh, Equation (23))**.** *$f(x) = O(g(x))$ as $x \to x_0$ if*

$$\lim_{x \to x_0} \frac{|f(x)|}{|g(x)|} = C, \quad \textit{where } 0 \le C < \infty.$$

Interpretation: *f is* at most of the same order *as g.*

Definition 3 (Asymptotic equivalence, Equation (24))**.** *$f(x) \sim g(x)$ as $x \to x_0$ if*

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = 1.$$

Interpretation: *f and g have* identical asymptotic behavior*.*

Why these three definitions? They form a hierarchy:

$$f \sim g \implies f = O(g) \implies \text{(but not necessarily)} \implies f = o(g)$$

Critical observation from lecture notes: "$f(x) = o(g(x))$ as $x \to x_0$ by definition implies $f(x) = O(g(x))$ as $x \to x_0$, but not vice versa."

Problem 3(a): Verify $\sin(x^{1/3}) = O(x^{1/3})$ as $x \to 0^+$

Step 1: Identify What We Must Prove

What are we asked? To verify that $\sin(x^{1/3}) = O(x^{1/3})$ as $x \to 0^+$.

Why this form? The problem asks us to verify, not derive. This means:

1. The statement is already claimed to be true
2. Our job is to demonstrate its truth using the definition
3. We must show the limit condition holds

What does verification require? By Definition 2 above, we must show:

$$\lim_{x\to 0^+} \frac{|\sin(x^{1/3})|}{|x^{1/3}|} = C < \infty.$$

Why the absolute values? The big-Oh definition (Equation 23 in lecture notes) uses absolute values to handle functions that may change sign. However, since $x^{1/3} > 0$ for $x > 0$, and we're approaching from $x \to 0^+$, we can work without absolute values in this case.

Step 2: Set Up the Limit

What we do: Form the ratio

$$\frac{\sin(x^{1/3})}{x^{1/3}}.$$

Why this ratio? This is *precisely* the ratio that appears in the definition of $O(\cdot)$. We are not choosing this arbitrarily; it is **mandated** by Definition 2.

What we must evaluate:

$$L = \lim_{x\to 0^+} \frac{\sin(x^{1/3})}{x^{1/3}}.$$

Why must we evaluate this limit? Because:

- If L exists and $0 \le L < \infty$, then $\sin(x^{1/3}) = O(x^{1/3})$
- If $L = \infty$, then $\sin(x^{1/3}) \ne O(x^{1/3})$
- If L does not exist, then $\sin(x^{1/3}) \ne O(x^{1/3})$

Step 3: Recognize the Limit Form

What we observe: The limit has the form

$$\lim_{x\to 0^+} \frac{\sin(x^{1/3})}{x^{1/3}}.$$

Why is this form significant? This resembles the fundamental trigonometric limit:

$$\lim_{u\to 0} \frac{\sin u}{u} = 1,$$

which is one of the most important limits in calculus.

How do we know this fundamental limit? It can be proven using:

1. The squeeze theorem with geometric arguments
2. L'Hôpital's rule: $\lim_{u\to 0} \frac{\sin u}{u} = \lim_{u\to 0} \frac{\cos u}{1} = 1$
3. Taylor series: $\sin u = u - \frac{u^3}{6} + O(u^5)$, so $\frac{\sin u}{u} = 1 - \frac{u^2}{6} + O(u^4) \to 1$

Why can we use this limit? Because our expression involves $\sin(x^{1/3})$ divided by $x^{1/3}$, which is exactly the pattern of $\sin(u)/u$ if we set $u = x^{1/3}$.

Step 4: Change of Variables

What we do: Let $u = x^{1/3}$.

Why this substitution? Because:

1. It transforms our unfamiliar limit into the standard form $\frac{\sin u}{u}$
2. It simplifies the notation
3. It makes the connection to the fundamental limit explicit

What happens to the limit as we change variables?
Since $u = x^{1/3}$:

- When $x \to 0^+$, we have $u = x^{1/3} \to 0^+$ (since the cube root of a small positive number is a small positive number)
- The limit becomes:
$$L = \lim_{x\to 0^+} \frac{\sin(x^{1/3})}{x^{1/3}} = \lim_{u\to 0^+} \frac{\sin u}{u}.$$

Why is this transformation valid? By the continuity of composition of continuous functions. More precisely, if $\phi : x \mapsto u$ is continuous at x_0 with $\phi(x_0) = u_0$, and if $\lim_{u\to u_0} g(u) = L$, then:

$$\lim_{x\to x_0} g(\phi(x)) = L.$$

In our case:

- $\phi(x) = x^{1/3}$ is continuous at $x = 0$
- $\lim_{u\to 0} \frac{\sin u}{u} = 1$ exists
- Therefore $\lim_{x\to 0^+} \frac{\sin(x^{1/3})}{x^{1/3}} = 1$

Step 5: Apply the Fundamental Limit

What we conclude:
$$\lim_{u\to 0^+} \frac{\sin u}{u} = 1.$$

Why can we state this? This is a **standard result** from calculus, proven rigorously and universally accepted.

What does this tell us about C? In our verification, we have $C = 1$.

Step 6: Apply the Definition to Conclude

What we have shown:
$$\lim_{x\to 0^+} \frac{\sin(x^{1/3})}{x^{1/3}} = 1.$$

Why does this verify the claim? Because:

1. The limit exists
2. The limit equals $C = 1$
3. We have $0 \le C < \infty$ (specifically, $C = 1$)
4. Therefore, by Definition 2, $\sin(x^{1/3}) = O(x^{1/3})$ as $x \to 0^+$

Interpretation: The function $\sin(x^{1/3})$ and the function $x^{1/3}$ have the *same asymptotic order* as $x \to 0^+$. Neither dominates the other; they are comparable in size.

Additional Insight: Could We Have $\sin(x^{1/3}) \sim x^{1/3}$?

Observation: Since the limit equals exactly 1, we actually have the stronger result:

$$\sin(x^{1/3}) \sim x^{1/3} \quad \text{as } x \to 0^+.$$

Why is this stronger? Because:

$$\sin(x^{1/3}) \sim x^{1/3} \implies \sin(x^{1/3}) = O(x^{1/3}),$$

but the converse is not necessarily true.

Why does the problem only ask for $O(\cdot)$? Perhaps to test whether we understand that big-Oh is a weaker condition than asymptotic equivalence, or simply because that's the level of precision needed.

Verification Complete:

$$\boxed{\sin(x^{1/3}) = O(x^{1/3}) \text{ as } x \to 0^+} \quad \checkmark$$

Reason: $\lim_{x\to 0^+} \frac{\sin(x^{1/3})}{x^{1/3}} = 1$, which is finite, satisfying the definition of big-Oh.

Problem 3(b): Verify $\cos(x) = O(1)$ as $x \to \infty$

Step 1: Understand What $O(1)$ Means

What is $O(1)$? The notation $O(1)$ means "of order 1" or "bounded."

Why do we write it this way? In the big-Oh definition, we write $f(x) = O(g(x))$ where $g(x)$ is the **gauge function**. Here, the gauge function is $g(x) = 1$ (the constant function).

What must we verify? By Definition 2:

$$\lim_{x\to\infty} \frac{|\cos(x)|}{|1|} = \lim_{x\to\infty} |\cos(x)| = C < \infty.$$

Why is this different from previous parts? Here we need the limit of the function itself (not a ratio of two functions with the same asymptotic behavior), because we're comparing to the constant function 1.

Step 2: Recall Properties of Cosine

What do we know about $\cos(x)$? From basic trigonometry:

$$-1 \le \cos(x) \le 1 \quad \text{for all } x \in \mathbb{R}.$$

Why is this property important? Because it immediately tells us that $|\cos(x)| \le 1$ for all x.

Where does this property come from?

- Geometrically: Cosine is the x-coordinate of a point on the unit circle
- Analytically: From the Taylor series $\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$
- From the differential equation: $\cos''(x) = -\cos(x)$ with $\cos(0) = 1$

Step 3: Analyze the Limit

What we need:

$$\lim_{x\to\infty} |\cos(x)|.$$

Does this limit exist? No, in the classical sense. Here's why:

- $\cos(x)$ oscillates between -1 and $+1$ as $x \to \infty$
- At $x = 2\pi k$ (where $k \in \mathbb{Z}$), we have $\cos(x) = 1$
- At $x = \pi + 2\pi k$, we have $\cos(x) = -1$
- The function does not settle to a single value

Why doesn't the limit existing matter? Because the big-Oh definition requires:

$$\lim_{x\to\infty} \frac{|\cos(x)|}{1} = C < \infty,$$

where C is a finite constant. The key word is "finite," not "exists as a unique value."

Step 4: Interpret the Definition Carefully

Critical realization: The definition from Equation (23) in the lecture notes states:

$$f(x) = O(g(x)) \text{ if } \lim_{x \to x_0} \frac{|f(x)|}{|g(x)|} = C, \quad 0 \leq C < \infty.$$

What does this mean for oscillating functions? We must interpret this carefully. The standard interpretation in asymptotic analysis is:

Equivalent formulation: $f(x) = O(g(x))$ means there exist constants $K > 0$ and x_1 such that:

$$|f(x)| \leq K|g(x)| \quad \text{for all } x > x_1.$$

This is sometimes stated as: "$|f(x)|/|g(x)|$ is bounded for large x."

Why this interpretation? Because:

- It captures the notion that f doesn't grow faster than g
- It allows for oscillatory behavior
- It's equivalent to the limit definition when the limit exists

Step 5: Apply to Our Problem

For $\cos(x)$ **with gauge function** $g(x) = 1$**:**

We need to show: There exists $K > 0$ such that

$$|\cos(x)| \leq K \cdot 1 \quad \text{for all large } x.$$

Is this true? Yes! We can take $K = 1$, because:

$$|\cos(x)| \leq 1 \quad \text{for all } x \in \mathbb{R}.$$

What does this tell us? The ratio $|\cos(x)|/1 = |\cos(x)|$ is bounded by 1 for all x, including as $x \to \infty$.

Step 6: Connect to the Limit Definition

In terms of limits: While $\lim_{x\to\infty} \cos(x)$ does not exist as a single value, we can say:

$$\limsup_{x \to \infty} |\cos(x)| = 1 < \infty.$$

What is $\limsup$**?** The limit superior is:

$$\limsup_{x \to \infty} f(x) = \lim_{x \to \infty} \sup_{y \geq x} f(y).$$

It captures the "largest value that f approaches infinitely often."

Why is $\limsup$ **appropriate here?** For oscillating functions, $\limsup$ provides the bound we need for the big-Oh definition.

For $|\cos(x)|$**:**

- The supremum of $|\cos(x)|$ over any interval is 1
- Therefore $\limsup_{x\to\infty} |\cos(x)| = 1$
- This is finite

Step 7: Conclude the Verification

What we have shown:

$$|\cos(x)| \leq 1 \text{ for all } x, \quad \text{hence} \quad |\cos(x)| \text{ is bounded as } x \to \infty.$$

Why does this verify the claim? Because:

1. The ratio $|\cos(x)|/1$ is bounded by a finite constant ($C = 1$)
2. This satisfies the big-Oh definition
3. Therefore $\cos(x) = O(1)$ as $x \to \infty$

Physical interpretation: As x grows, $\cos(x)$ oscillates but never escapes the interval $[-1, 1]$. It is "controlled" or "bounded."

Verification Complete:

$$\boxed{\cos(x) = O(1) \text{ as } x \to \infty} \quad \checkmark$$

Reason: $|\cos(x)| \leq 1$ for all x, hence the ratio $|\cos(x)|/1$ is bounded, satisfying the big-Oh definition with $C = 1$.

Problem 3(c): Verify $\sin x = O(x \cos x)$ as $x \to 0$

Step 1: Identify the Gauge Function

What are we asked? To verify $\sin x = O(x \cos x)$ as $x \to 0$.

What is the gauge function here? The gauge function is $g(x) = x \cos x$.

Why is this more complex? Unlike the previous problems where the gauge function was either a simple power or a constant, here we have a **product** of functions:

- x: a linear function that vanishes at $x = 0$
- $\cos x$: a function that equals 1 at $x = 0$

Step 2: Set Up the Verification

By Definition 2, we must show:

$$\lim_{x \to 0} \frac{|\sin x|}{|x \cos x|} = C < \infty.$$

Why can we drop absolute values (for now)? As $x \to 0$:

- For $x > 0$ (small): $\sin x > 0$, $x > 0$, $\cos x > 0$ (since $\cos 0 = 1 > 0$)
- For $x < 0$ (small): $\sin x < 0$, $x < 0$, $\cos x > 0$
- So $\frac{\sin x}{x \cos x}$ has the same sign as $\frac{\sin x}{x}$, which is positive for small $|x|$

For simplicity, we can work with:

$$\lim_{x \to 0} \frac{\sin x}{x \cos x}.$$

Step 3: Algebraic Manipulation

What we do: Rewrite the ratio as:

$$\frac{\sin x}{x \cos x} = \frac{\sin x}{x} \cdot \frac{1}{\cos x}.$$

Why this factorization? Because:

1. It separates the ratio into two **recognizable pieces**
2. $\frac{\sin x}{x}$ is a fundamental limit we know
3. $\frac{1}{\cos x}$ is a simple function we can evaluate

Mathematical justification: For x in a neighborhood of 0 where $\cos x \neq 0$:

$$\frac{a}{bc} = \frac{a}{b} \cdot \frac{1}{c} \quad \text{(basic algebra)}.$$

Step 4: Evaluate the Limit as a Product

What we need:

$$\lim_{x\to 0}\left(\frac{\sin x}{x}\cdot\frac{1}{\cos x}\right).$$

Can we split this limit? Yes, by the limit laws. If $\lim_{x\to x_0} f(x) = L$ and $\lim_{x\to x_0} g(x) = M$ both exist, then:

$$\lim_{x\to x_0}[f(x)\cdot g(x)] = L\cdot M.$$

Why are limit laws valid here? Because:

- Both component limits exist (as we will verify)
- Neither limit is of the indeterminate form requiring additional care

Step 5: Evaluate Each Component

First component:

$$\lim_{x\to 0}\frac{\sin x}{x} = 1.$$

Why? This is the fundamental trigonometric limit we used in part (a).
How do we know it?

- From the Taylor series: $\sin x = x - \frac{x^3}{6} + O(x^5)$, so $\frac{\sin x}{x} = 1 - \frac{x^2}{6} + O(x^4) \to 1$
- Geometrically: from squeeze theorem arguments
- By L'Hôpital's rule: $\lim_{x\to 0}\frac{\sin x}{x} = \lim_{x\to 0}\frac{\cos x}{1} = 1$

Second component:

$$\lim_{x\to 0}\frac{1}{\cos x} = \frac{1}{\cos 0} = \frac{1}{1} = 1.$$

Why? Because:

- $\cos x$ is continuous at $x = 0$
- $\cos 0 = 1$ (fundamental value)
- Therefore $\lim_{x\to 0}\cos x = 1$
- By continuity of $f(x) = 1/x$ (for $x \neq 0$): $\lim_{x\to 0}\frac{1}{\cos x} = \frac{1}{\lim_{x\to 0}\cos x} = \frac{1}{1} = 1$

Step 6: Combine the Results

What we obtain:

$$\lim_{x\to 0}\frac{\sin x}{x\cos x} = \lim_{x\to 0}\frac{\sin x}{x}\cdot\lim_{x\to 0}\frac{1}{\cos x} = 1\cdot 1 = 1.$$

Why can we multiply? By the product rule for limits (mentioned in Step 4).

Step 7: Conclude the Verification

What we have shown:

$$\lim_{x\to 0}\frac{\sin x}{x\cos x} = 1.$$

Why does this verify the claim? Because:

1. The limit exists
2. The limit equals $C = 1$, which is finite
3. We have $0 \leq C < \infty$
4. Therefore, by Definition 2, $\sin x = O(x\cos x)$ as $x \to 0$

Additional Insight: Stronger Statement

Observation: Since the limit equals exactly 1, we actually have:

$$\sin x \sim x \cos x \quad \text{as } x \to 0.$$

Why is this noteworthy? It tells us that near $x = 0$:

- $\sin x$ behaves *exactly* like $x \cos x$
- The two functions are asymptotically equivalent
- This is stronger than just saying one is O of the other

Intuitive understanding: As $x \to 0$:

- $\sin x \approx x$ (first-order Taylor approximation)
- $\cos x \approx 1$ (zeroth-order approximation)
- Therefore $x \cos x \approx x \cdot 1 = x \approx \sin x$

Verification Complete:

$$\boxed{\sin x = O(x \cos x) \text{ as } x \to 0} \quad \checkmark$$

Reason: $\lim_{x \to 0} \frac{\sin x}{x \cos x} = \lim_{x \to 0} \frac{\sin x}{x} \cdot \frac{1}{\cos x} = 1 \cdot 1 = 1$, which is finite, satisfying the big-Oh definition.

Problem 3(d): Verify $\log(\log(1/x)) = o(\log(x))$ as $x \to 0^+$

Step 1: Understand the Little-oh Notation

What is different here? This problem asks us to verify a **little-oh** relation, not big-Oh.
Recall Definition 1: $f(x) = o(g(x))$ as $x \to x_0$ if:

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = 0.$$

Why is this stronger? Because:

- Big-Oh: The limit can be any finite constant C (including 0)
- Little-oh: The limit *must* be exactly 0

Interpretation: $f(x) = o(g(x))$ means "f is *asymptotically negligible* compared to g."
From the lecture notes: "$f(x) \ll g(x)$ as $x \to x_0$" is alternative notation.

Step 2: Parse the Functions

What are our functions?

- $f(x) = \log(\log(1/x))$
- $g(x) = \log(x)$
- Limit: $x \to 0^+$

Why is $\log(\log(1/x))$ even defined? We need:

1. $1/x > 0$ (true for $x > 0$)
2. $\log(1/x)$ is defined (true for $x > 0$)
3. $\log(1/x) > 0$ for the outer log to be defined

When is $\log(1/x) > 0$?

$$\log(1/x) > 0 \iff 1/x > 1 \iff x < 1.$$

So for $0 < x < 1$, all logarithms are well-defined.

Step 3: Simplify Using Logarithm Properties

What we do: Use the identity $\log(1/x) = -\log(x)$.

Why this identity? Because $\log(a/b) = \log(a) - \log(b)$, so:

$$\log(1/x) = \log(1) - \log(x) = 0 - \log(x) = -\log(x).$$

Therefore:

$$f(x) = \log(\log(1/x)) = \log(-\log(x)).$$

Why is this simplification useful? Because:

- It eliminates the fraction $1/x$
- It expresses everything in terms of $\log(x)$
- It makes the relationship between f and g more transparent

Step 4: Analyze Behavior as $x \to 0^+$

What happens to $\log(x)$ as $x \to 0^+$?

$$\log(x) \to -\infty.$$

Why? Because the natural logarithm:

- Is defined for $x > 0$
- Satisfies $\log(1) = 0$
- Is strictly increasing
- Satisfies $\lim_{x\to 0^+} \log(x) = -\infty$

What happens to $-\log(x)$ as $x \to 0^+$?

$$-\log(x) \to +\infty.$$

What happens to $\log(-\log(x))$ as $x \to 0^+$?

$$\log(-\log(x)) \to +\infty.$$

Why? Because if $u \to +\infty$, then $\log(u) \to +\infty$ as well.

Summary of behaviors:

$$\begin{aligned} x \to 0^+ &\implies \log(x) \to -\infty \\ &\implies -\log(x) \to +\infty \\ &\implies \log(-\log(x)) \to +\infty. \end{aligned}$$

Step 5: Set Up the Limit

What we must evaluate:

$$L = \lim_{x\to 0^+} \frac{\log(\log(1/x))}{\log(x)} = \lim_{x\to 0^+} \frac{\log(-\log(x))}{\log(x)}.$$

What form is this limit? As $x \to 0^+$:

- Numerator: $\log(-\log(x)) \to +\infty$
- Denominator: $\log(x) \to -\infty$

This is an indeterminate form of type $\frac{\infty}{-\infty}$.

Why is this indeterminate? Because:

- We cannot immediately conclude the ratio's behavior
- The relative rates of growth matter
- We need a more sophisticated technique

Step 6: Change of Variables

What we do: Let $u = -\log(x)$.

Why this substitution? Because:

1. It simplifies $-\log(x)$ to just u
2. As $x \to 0^+$, we have $u = -\log(x) \to +\infty$
3. This converts our limit to a more standard form

How do we express $\log(x)$ in terms of u?
From $u = -\log(x)$:

$$\log(x) = -u.$$

How do we express $\log(-\log(x))$ in terms of u?

$$\log(-\log(x)) = \log(u).$$

Step 7: Rewrite the Limit

The limit becomes:

$$L = \lim_{u\to+\infty} \frac{\log(u)}{-u} = -\lim_{u\to+\infty} \frac{\log(u)}{u}.$$

Why is this form better? Because:

- It's a standard limit in calculus: $\lim_{u\to\infty} \frac{\log(u)}{u}$
- Both numerator and denominator go to $+\infty$
- This is a classic example of comparing growth rates

Step 8: Recall the Standard Result

Fundamental fact from analysis:

$$\lim_{u\to+\infty} \frac{\log(u)}{u} = 0.$$

Why is this true? There are several ways to prove this:
Method 1: L'Hôpital's Rule
Since both $\log(u) \to \infty$ and $u \to \infty$ as $u \to \infty$, we have a $\frac{\infty}{\infty}$ form:

$$\lim_{u\to\infty} \frac{\log(u)}{u} \stackrel{\text{L'H}}{=} \lim_{u\to\infty} \frac{(\log u)'}{(u)'} = \lim_{u\to\infty} \frac{1/u}{1} = \lim_{u\to\infty} \frac{1}{u} = 0.$$

Method 2: Growth rate comparison
Logarithmic functions grow *slower* than any positive power:

$$\lim_{u\to\infty} \frac{\log(u)}{u^\alpha} = 0 \quad \text{for any } \alpha > 0.$$

Taking $\alpha = 1$ gives our result.
Method 3: Series/Integral comparison
From the integral representation:

$$\log(u) = \int_1^u \frac{1}{t} dt < \int_1^u 1\, dt = u - 1 < u,$$

so $0 < \frac{\log(u)}{u} < 1$ for $u > 1$, and more refined estimates show it tends to 0.

Why does this matter? This is a **hierarchy of growth rates**:

$$\log(u) \ll u \ll u^2 \ll e^u \quad \text{as } u \to \infty.$$

Step 9: Conclude the Calculation

We have:

$$L = -\lim_{u\to+\infty} \frac{\log(u)}{u} = -\cdot 0 = 0.$$

Why the negative sign disappears: Because $-0 = 0$.

Step 10: Apply the Definition

What we have shown:

$$\lim_{x\to 0^+} \frac{\log(\log(1/x))}{\log(x)} = 0.$$

Why does this verify the claim? Because:

1. The limit exists
2. The limit equals exactly 0
3. This satisfies the definition of little-oh (Definition 1)
4. Therefore $\log(\log(1/x)) = o(\log(x))$ as $x \to 0^+$

Interpretation and Intuition

What does this result mean?
As $x \to 0^+$:

- $\log(x)$ becomes very negative (goes to $-\infty$)
- $\log(\log(1/x)) = \log(-\log(x))$ becomes very positive (goes to $+\infty$)
- BUT: $\log(-\log(x))$ grows *much slower* than the rate at which $\log(x)$ decreases

Hierarchy of infinities: We have:

$$|\log(\log(1/x))| \ll |\log(x)| \quad \text{as } x \to 0^+,$$

meaning: "The logarithm of a logarithm is negligible compared to the logarithm itself."
Example with numbers:

$$\begin{aligned}
x = 10^{-10}: &\quad \log(x) \approx -23, \quad \log(-\log(x)) \approx 3.1 \\
x = 10^{-100}: &\quad \log(x) \approx -230, \quad \log(-\log(x)) \approx 5.4 \\
x = 10^{-1000}: &\quad \log(x) \approx -2303, \quad \log(-\log(x)) \approx 7.7
\end{aligned}$$

The ratio $\frac{7.7}{2303} \approx 0.0033$ is getting smaller!

Verification Complete:

$$\boxed{\log(\log(1/x)) = o(\log(x)) \text{ as } x \to 0^+} \quad \checkmark$$

Reason: By change of variables $u = -\log(x)$, the limit becomes

$$\lim_{x\to 0^+} \frac{\log(-\log(x))}{\log(x)} = -\lim_{u\to\infty} \frac{\log(u)}{u} = 0,$$

satisfying the little-oh definition. The iterated logarithm grows slower than the single logarithm.

Summary: Methodological Lessons

Key Takeaways from Problem 3

1. **Always start with definitions:** Every verification in asymptotic analysis begins by stating the precise definition being used.

2. **Recognize standard limits:** Many asymptotic verifications reduce to fundamental limits like:

   - $\lim_{u\to 0} \frac{\sin u}{u} = 1$
   - $\lim_{u\to\infty} \frac{\log u}{u} = 0$
   - $\lim_{u\to\infty} \frac{u^n}{e^u} = 0$ for any n

3. **Use appropriate substitutions:** Change of variables can transform unfamiliar limits into standard forms.

4. **Understand the hierarchy:** Functions have a natural ordering by growth rate:

$$\log(\log(x)) \ll \log(x) \ll x^\alpha \ll e^x \ll x^x$$

5. **Distinguish big-Oh from little-oh:** Big-Oh allows any finite limit; little-oh requires the limit to be zero.

6. **Handle oscillatory functions carefully:** For functions like $\cos(x)$ that don't settle to a value, use boundedness arguments.