Problem Sheet 1, Question 2

Problem Statement

Find two-term expansions for each root of

$$\epsilon^2 x^3 + x^2 + 2x + \epsilon = 0, \quad \epsilon \ll 1. \tag{1}$$

Complete Solution

Phase I: Problem Classification

Step 1.1: Identify the structure of the equation.

What we observe: The equation has the form $F(x;\epsilon) = 0$ where ϵ appears both as an additive perturbation (the $+\epsilon$ term) and multiplying the highest-degree term ($\epsilon^2 x^3$).

Why this matters: According to Lecture Notes Section 2.2, when a small parameter multiplies the highest-degree term, the problem is potentially singular because setting $\epsilon = 0$ reduces the degree of the equation.

Step 1.2: Solve the unperturbed equation.

Setting $\epsilon = 0$:

$$x^2 + 2x = x(x+2) = 0.$$

Solutions of unperturbed equation:

$$x_0^{(1)} = 0, \quad x_0^{(2)} = -2.$$

Step 1.3: Count degrees of freedom.

What we observe:

- The perturbed equation (1) is cubic (degree 3), so it has 3 roots.
- The unperturbed equation is quadratic (degree 2), with only 2 roots.

Why this matters: The mismatch in the number of roots confirms this is a **singular perturbation problem**. One root must "escape to infinity" as $\epsilon \to 0$.

Step 1.4: Classify the problem.

Conclusion: This is a **singular perturbation problem**.

Method to use:

1. For the two roots near finite values: use standard expansion method (Section 2.1.1)
2. For the "missing" third root: use dominant balance analysis (Section 2.2.2)

Phase II: Solution Near $x_0 = 0$

Step 2.1: Make the expansion ansatz.

What we assume: Since $x \to 0$ as $\epsilon \to 0$, and there is no constant term from the unperturbed root, we write:

$$x(\epsilon) = a_1\epsilon + a_2\epsilon^2 + O(\epsilon^3).$$

Step 2.2: Substitute into the equation.

The cubic term:

$$\epsilon^2 x^3 = \epsilon^2(a_1\epsilon + \cdots)^3 = a_1^3\epsilon^5 + O(\epsilon^6).$$

This is $O(\epsilon^5)$, negligible at the orders we need.

The quadratic term:

$$x^2 = (a_1\epsilon + a_2\epsilon^2 + \cdots)^2 = a_1^2\epsilon^2 + 2a_1a_2\epsilon^3 + O(\epsilon^4).$$

The linear term:

$$2x = 2a_1\epsilon + 2a_2\epsilon^2 + O(\epsilon^3).$$

The constant term: ϵ.

Step 2.3: Collect terms by powers of ϵ.

Adding all terms:

$$\underbrace{(2a_1 + 1)}_{O(\epsilon)}\epsilon + \underbrace{(a_1^2 + 2a_2)}_{O(\epsilon^2)}\epsilon^2 + O(\epsilon^3) = 0.$$

For this to hold for all small ϵ, each coefficient must vanish.

Step 2.4: Solve order by order.

At $O(\epsilon)$:

$$2a_1 + 1 = 0 \implies a_1 = -\frac{1}{2}.$$

At $O(\epsilon^2)$:

$$a_1^2 + 2a_2 = 0 \implies \frac{1}{4} + 2a_2 = 0 \implies a_2 = -\frac{1}{8}.$$

Final answer for root near $x_0 = 0$:

$$\boxed{x^{(1)}(\epsilon) = -\frac{1}{2}\epsilon - \frac{1}{8}\epsilon^2 + O(\epsilon^3)}.$$

Phase III: Solution Near $x_0 = -2$

Step 3.1: Make the expansion ansatz.

What we assume: Since $x \to -2$ as $\epsilon \to 0$:

$$x(\epsilon) = -2 + b_1\epsilon + O(\epsilon^2).$$

Step 3.2: Substitute into the equation.

The cubic term:

$$x^3 = (-2 + b_1\epsilon + \cdots)^3 = -8 + 12b_1\epsilon + O(\epsilon^2).$$
$$\epsilon^2x^3 = -8\epsilon^2 + O(\epsilon^3).$$

The quadratic term:

$$x^2 = (-2)^2 + 2(-2)(b_1\epsilon) + O(\epsilon^2) = 4 - 4b_1\epsilon + O(\epsilon^2).$$

The linear term:

$$2x = -4 + 2b_1\epsilon + O(\epsilon^2).$$

The constant term: ϵ.

Step 3.3: Collect terms by powers of ϵ.

Adding all terms:

$$\underbrace{(4 - 4)}_{O(1)} + \underbrace{(-4b_1 + 2b_1 + 1)}_{O(\epsilon)}\epsilon + O(\epsilon^2) = 0.$$

Step 3.4: Solve order by order.

At $O(1)$:

$$4 - 4 = 0. \quad \checkmark$$

This confirms $x_0 = -2$ is a root of the unperturbed equation.

At $O(\epsilon)$:

$$-4b_1 + 2b_1 + 1 = 0 \implies -2b_1 + 1 = 0 \implies b_1 = \frac{1}{2}.$$

Final answer for root near $x_0 = -2$:

$$\boxed{x^{(2)}(\epsilon) = -2 + \frac{1}{2}\epsilon + O(\epsilon^2)}.$$

Note: For a two-term expansion, the two terms are -2 and $\frac{1}{2}\epsilon$.

Phase IV: Singular Solution via Dominant Balance

Step 4.1: Why dominant balance is needed.

The situation: We have found 2 roots, but a cubic equation has 3 roots. The third root cannot be found by expanding around any finite unperturbed value—it must escape to infinity as $\epsilon \to 0$.

The question: How does this root scale with ϵ? That is, what power of ϵ describes its size?

Step 4.2: The dominant balance principle.

Key insight: For an equation to be satisfied, terms cannot simply "blow up" to infinity—they must **cancel**. When $|x| \to \infty$, at least two terms must be of the same order of magnitude and opposite in sign, while all other terms are smaller (subdominant).

The method: Assume the singular root scales as

$$x \sim \epsilon^{-\alpha} \quad \text{for some } \alpha > 0.$$

Then determine α by requiring that:

(i) At least two terms have the same order in ϵ (they balance).

(ii) These balanced terms are the **largest** terms in the equation.

(iii) All other terms are smaller (subdominant).

Step 4.3: Compute the order of each term.

The equation:

$$\epsilon^2 x^3 + x^2 + 2x + \epsilon = 0.$$

With $x = O(\epsilon^{-\alpha})$, compute the ϵ-order of each term.

The rule: If $x = O(\epsilon^{-\alpha})$, then $x^n = O(\epsilon^{-n\alpha})$ (powers multiply). For a general term $\epsilon^m x^n$:

$$\epsilon^m x^n = O(\epsilon^m) \cdot O(\epsilon^{-n\alpha}) = O(\epsilon^{m-n\alpha}).$$

Applying this rule to each term:

$$\begin{aligned}
\epsilon^2 x^3 &= O(\epsilon^2) \cdot O(\epsilon^{-3\alpha}) = O(\epsilon^{2-3\alpha}), \\
x^2 &= O(\epsilon^{-2\alpha}), \\
2x &= O(\epsilon^{-\alpha}), \\
\epsilon &= O(\epsilon^1).
\end{aligned}$$

Step 4.4: Determine which terms can balance.

Systematic analysis: For large $|x|$ (i.e., $\alpha > 0$), rank the terms from largest to smallest. The exponent of ϵ determines size: **more negative = larger**.

Term	Order	Exponent
$\epsilon^2 x^3$	$O(\epsilon^{2-3\alpha})$	$2-3\alpha$
x^2	$O(\epsilon^{-2\alpha})$	-2α
$2x$	$O(\epsilon^{-\alpha})$	$-\alpha$
ϵ	$O(\epsilon)$	1

For the two largest terms to balance: Set their exponents equal. The natural candidates are $\epsilon^2 x^3$ and x^2 (both involve powers of x):

$$2 - 3\alpha = -2\alpha \implies \alpha = 2.$$

Step 4.5: Verify the balance is consistent.
With $\alpha = 2$, compute all exponents:

$$\begin{aligned} \epsilon^2 x^3 &= O(\epsilon^{2-6}) = O(\epsilon^{-4}), \\ x^2 &= O(\epsilon^{-4}), \\ 2x &= O(\epsilon^{-2}), \\ \epsilon &= O(\epsilon). \end{aligned}$$

Check the hierarchy:

$$\underbrace{O(\epsilon^{-4})}_{\epsilon^2 x^3,\, x^2} \gg \underbrace{O(\epsilon^{-2})}_{2x} \gg \underbrace{O(\epsilon)}_{\epsilon}.$$

Conclusion: The balance $\epsilon^2 x^3 \sim x^2$ is **consistent**—these are indeed the two largest terms, and $2x$ and ϵ are subdominant. The scaling $x \sim \epsilon^{-2}$ is correct.

Step 4.6: Extract the leading coefficient.
From the dominant balance:

$$\epsilon^2 x^3 + x^2 \approx 0 \implies x^2(\epsilon^2 x + 1) = 0.$$

Since $x \neq 0$ for this root:

$$\epsilon^2 x + 1 = 0 \implies x = -\frac{1}{\epsilon^2}.$$

Step 4.7: Find the next-order correction.
Ansatz: Based on dominant balance, write:

$$x = -\frac{1}{\epsilon^2} + c_0 + O(\epsilon),$$

where c_0 is a constant to be determined.
Substitute and expand each term:
Compute x^3:

$$x^3 = \left(-\frac{1}{\epsilon^2} + c_0\right)^3 = -\frac{1}{\epsilon^6} + \frac{3c_0}{\epsilon^4} - \frac{3c_0^2}{\epsilon^2} + c_0^3.$$

Compute $\epsilon^2 x^3$:

$$\epsilon^2 x^3 = -\frac{1}{\epsilon^4} + \frac{3c_0}{\epsilon^2} - 3c_0^2 + O(\epsilon^2).$$

Compute x^2:

$$x^2 = \frac{1}{\epsilon^4} - \frac{2c_0}{\epsilon^2} + c_0^2.$$

Compute $2x$:

$$2x = -\frac{2}{\epsilon^2} + 2c_0.$$

Step 4.8: Collect terms by powers of ϵ.
Adding all terms:

$$\underbrace{\left(-\frac{1}{\epsilon^4}+\frac{1}{\epsilon^4}\right)}_{O(\epsilon^{-4})}+\underbrace{\left(\frac{3c_0-2c_0-2}{\epsilon^2}\right)}_{O(\epsilon^{-2})}+\underbrace{(-3c_0^2+c_0^2+2c_0)}_{O(1)}+O(\epsilon)=0.$$

Step 4.9: Solve order by order.
At $O(\epsilon^{-4})$:

$$-\frac{1}{\epsilon^4}+\frac{1}{\epsilon^4}=0. \quad \checkmark$$

This confirms the leading term $-1/\epsilon^2$ is correct.
At $O(\epsilon^{-2})$:

$$\frac{3c_0-2c_0-2}{\epsilon^2}=\frac{c_0-2}{\epsilon^2}=0 \implies c_0=2.$$

Final answer for singular root:

$$\boxed{x^{(3)}(\epsilon)=-\frac{1}{\epsilon^2}+2+O(\epsilon)}.$$

Summary

The three roots of $\epsilon^2x^3+x^2+2x+\epsilon=0$ are:

$$x^{(1)}(\epsilon)=-\frac{1}{2}\epsilon-\frac{1}{8}\epsilon^2+O(\epsilon^3),$$
$$x^{(2)}(\epsilon)=-2+\frac{1}{2}\epsilon+O(\epsilon^2),$$
$$x^{(3)}(\epsilon)=-\frac{1}{\epsilon^2}+2+O(\epsilon).$$

Classification:

- Root 1: Regular solution near $x_0=0$. Two terms: $-\frac{1}{2}\epsilon$ and $-\frac{1}{8}\epsilon^2$.
- Root 2: Regular solution near $x_0=-2$. Two terms: -2 and $+\frac{1}{2}\epsilon$.
- Root 3: Singular solution (escapes to $-\infty$ as $\epsilon\to 0$). Two terms: $-\frac{1}{\epsilon^2}$ and $+2$.

General Method: Finding Singular Roots via Dominant Balance

For any polynomial equation where $\epsilon\to 0$ causes the degree to drop (losing roots to infinity):

1. **Assume scaling:** Let $x\sim\epsilon^{-\alpha}$ for unknown $\alpha>0$.
2. **Compute orders:** For each term $\epsilon^m x^n$, apply the rule:

$$\epsilon^m x^n=O(\epsilon^{m-n\alpha}).$$

   The exponent $m-n\alpha$ determines the size of the term.

3. **Find α:** Set the exponents of the two largest terms equal and solve for α. Terms are "largest" when their exponent is most negative.

4. **Verify consistency:** Confirm these two terms are indeed the largest (most negative exponent), and all others are subdominant (less negative or positive exponent).

5. **Extract leading behavior:** From the balanced terms, solve for the leading coefficient of x.

6. **Iterate:** Substitute $x = (\text{leading}) + c_0 + \cdots$ and collect terms to find corrections.

This method works for **any** singular perturbation problem where roots escape to infinity, regardless of the specific equation.

Verification

Verification of Root 1:
Substitute $x = -\frac{1}{2}\epsilon - \frac{1}{8}\epsilon^2$ into $\epsilon^2 x^3 + x^2 + 2x + \epsilon$:

$$\begin{aligned}
\epsilon^2 x^3 &= O(\epsilon^5), \\
x^2 &= \frac{\epsilon^2}{4} + \frac{\epsilon^3}{8} + O(\epsilon^4), \\
2x &= -\epsilon - \frac{\epsilon^2}{4}, \\
\epsilon &= \epsilon.
\end{aligned}$$

Sum: $\frac{\epsilon^2}{4} - \frac{\epsilon^2}{4} - \epsilon + \epsilon + O(\epsilon^3) = O(\epsilon^3)$. ✓
Verification of Root 2:
Substitute $x = -2 + \frac{1}{2}\epsilon$ into the equation:

$$\begin{aligned}
\epsilon^2 x^3 &= -8\epsilon^2 + O(\epsilon^3), \\
x^2 &= 4 - 2\epsilon + O(\epsilon^2), \\
2x &= -4 + \epsilon, \\
\epsilon &= \epsilon.
\end{aligned}$$

Sum: $(4 - 4) + (-2\epsilon + \epsilon + \epsilon) + O(\epsilon^2) = O(\epsilon^2)$. ✓
Verification of Root 3:
Substitute $x = -\frac{1}{\epsilon^2} + 2$ into the equation:

$$\begin{aligned}
\epsilon^2 x^3 &= -\frac{1}{\epsilon^4} + \frac{6}{\epsilon^2} - 12 + O(\epsilon^2), \\
x^2 &= \frac{1}{\epsilon^4} - \frac{4}{\epsilon^2} + 4, \\
2x &= -\frac{2}{\epsilon^2} + 4, \\
\epsilon &= \epsilon.
\end{aligned}$$

At $O(\epsilon^{-4})$: $-\frac{1}{\epsilon^4} + \frac{1}{\epsilon^4} = 0$. ✓
At $O(\epsilon^{-2})$: $\frac{6}{\epsilon^2} - \frac{4}{\epsilon^2} - \frac{2}{\epsilon^2} = 0$. ✓
At $O(1)$: $-12 + 4 + 4 = -4 \neq 0$.
The $O(1)$ residual confirms we need higher-order corrections beyond the two-term expansion.
✓