

Asymptotics Problem 7.2: Complete Pedagogical Solution

WKB Method via Double Transformation

Problem 1. For the equation $\varepsilon^2 y''(x) + q(x)y(x) = 0$, look for transformations of both the dependent and independent variables, $z = \phi(x)$, $\nu(z) = \psi(x)y(x)$ so that with suitable choice for the functions $\varphi(x)$ and $\psi(x)$, the ODE becomes $\varepsilon^2 \nu''(z) + \nu(z) = 0$ to the leading order as $\varepsilon \rightarrow 0$. Hence deduce the leading order solution and show it is equivalent to the WKB solution. (You may assume that $q(x)$ is a positive function.)

Solution: Step-by-Step Atomic Breakdown

Step 1: Understanding the Problem and Overall Strategy

Strategy: We are performing a double transformation:

- **Independent variable transformation:** $z = \phi(x)$ (change of coordinate)
- **Dependent variable transformation:** $\nu(z) = \psi(x)y(x)$ (amplitude rescaling)

The goal is to transform the ODE with variable coefficient $q(x)$ into the simplest possible form with constant coefficients: $\varepsilon^2 \nu'' + \nu = 0$.

Justification: Why do we want this particular form? Because $\varepsilon^2 \nu'' + \nu = 0$ is exactly solvable:

$$\nu(z) = A \cos\left(\frac{z}{\varepsilon}\right) + B \sin\left(\frac{z}{\varepsilon}\right),$$

which exhibits the rapid oscillations characteristic of WKB solutions. By transforming our original equation into this form, we can:

1. Solve it exactly in the transformed variables
2. Transform back to obtain the WKB approximation
3. Understand geometrically what the WKB method is doing

This approach appears in Lecture Notes §7.2.1, equations (479)-(511), where it's used to derive WKB from the averaging method.

Step 2: Setting Up the Transformation Framework

What we have:

$$\begin{aligned}\text{Original equation: } & \varepsilon^2 y''(x) + q(x)y(x) = 0 \\ \text{New independent variable: } & z = \phi(x) \\ \text{New dependent variable: } & \nu(z) = \psi(x)y(x) \\ \text{Target equation: } & \varepsilon^2 \nu''(z) + \nu(z) = 0\end{aligned}$$

Technique: To substitute into the ODE, we need to express y and its derivatives in terms of ν and its derivatives. This requires careful application of the chain rule for both transformations.

Step 3: Relating y to ν

From $\nu(z) = \psi(x)y(x)$, we can express y in terms of ν :

$$y(x) = \frac{\nu(z)}{\psi(x)} = \frac{\nu(\phi(x))}{\psi(x)}.$$

Justification: *This inverse relationship is crucial: we're expressing the old unknown $y(x)$ in terms of the new unknown $\nu(z)$. Note that ν is a function of z , which itself depends on x through $z = \phi(x)$.*

Step 4: Computing the First Derivative $y'(x)$

What we need: $\frac{dy}{dx}$ using $y = \frac{\nu(z)}{\psi(x)}$ and $z = \phi(x)$.

Technique: *Use the quotient rule combined with the chain rule. We have:*

$$y(x) = \frac{\nu(z(x))}{\psi(x)}.$$

Applying the quotient rule:

$$\frac{dy}{dx} = \frac{\frac{d\nu}{dz} \cdot \psi(x) - \nu(z) \cdot \frac{d\psi}{dx}}{\psi(x)^2}.$$

But ν depends on x through z , so by the chain rule:

$$\frac{d\nu}{dx} = \frac{d\nu}{dz} \cdot \frac{dz}{dx} = \nu'(z) \cdot \phi'(x).$$

Therefore:

$$y'(x) = \frac{\nu'(z)\phi'(x) \cdot \psi(x) - \nu(z) \cdot \psi'(x)}{\psi(x)^2}.$$

Simplifying:

$$y'(x) = \frac{\phi'(x)}{\psi(x)}\nu'(z) - \frac{\psi'(x)}{\psi(x)^2}\nu(z).$$

Step 5: Computing the Second Derivative $y''(x)$

What we need: Differentiate $y'(x)$ with respect to x .

Technique: *We have two terms to differentiate:*

$$y'(x) = \frac{\phi'(x)}{\psi(x)}\nu'(z) - \frac{\psi'(x)}{\psi(x)^2}\nu(z).$$

Each term requires the product rule and chain rule.

Step 5a: Differentiating the First Term

For the term $\frac{\phi'}{\psi}\nu'$, apply the product rule:

$$\frac{d}{dx} \left[\frac{\phi'(x)}{\psi(x)} \nu'(z) \right] = \frac{d}{dx} \left[\frac{\phi'}{\psi} \right] \cdot \nu' + \frac{\phi'}{\psi} \cdot \frac{d\nu'}{dx}.$$

For the first factor, use the quotient rule:

$$\frac{d}{dx} \left[\frac{\phi'}{\psi} \right] = \frac{\phi''\psi - \phi'\psi'}{\psi^2}.$$

For the second factor, use the chain rule:

$$\frac{d\nu'}{dx} = \frac{d}{dx} \left[\frac{d\nu}{dz} \right] = \frac{d^2\nu}{dz^2} \cdot \frac{dz}{dx} = \nu''(z) \cdot \phi'(x).$$

Therefore:

$$\frac{d}{dx} \left[\frac{\phi'}{\psi} \nu' \right] = \frac{\phi''\psi - \phi'\psi'}{\psi^2} \nu' + \frac{\phi'}{\psi} \nu'' \phi' = \frac{\phi''\psi - \phi'\psi'}{\psi^2} \nu' + \frac{(\phi')^2}{\psi} \nu''.$$

Step 5b: Differentiating the Second Term

For the term $-\frac{\psi'}{\psi^2}\nu$, apply the product rule:

$$\frac{d}{dx} \left[-\frac{\psi'}{\psi^2}\nu \right] = -\frac{d}{dx} \left[\frac{\psi'}{\psi^2} \right] \cdot \nu - \frac{\psi'}{\psi^2} \cdot \frac{d\nu}{dx}.$$

For the first factor, use the quotient rule:

$$\frac{d}{dx} \left[\frac{\psi'}{\psi^2} \right] = \frac{\psi''\psi^2 - \psi' \cdot 2\psi\psi'}{\psi^4} = \frac{\psi''\psi - 2(\psi')^2}{\psi^3}.$$

For the second factor:

$$\frac{d\nu}{dx} = \nu'(z)\phi'(x).$$

Therefore:

$$\frac{d}{dx} \left[-\frac{\psi'}{\psi^2}\nu \right] = -\frac{\psi''\psi - 2(\psi')^2}{\psi^3}\nu - \frac{\psi'}{\psi^2}\nu'\phi'.$$

Step 5c: Combining Both Terms

Adding the results from Steps 5a and 5b:

$$y''(x) = \frac{\phi''\psi - \phi'\psi'}{\psi^2}\nu' + \frac{(\phi')^2}{\psi}\nu'' - \frac{\psi''\psi - 2(\psi')^2}{\psi^3}\nu - \frac{\psi'\phi'}{\psi^2}\nu'.$$

Collecting terms by derivative order of ν :

$$y''(x) = \frac{(\phi')^2}{\psi}\nu'' + \left[\frac{\phi''\psi - \phi'\psi'}{\psi^2} - \frac{\psi'\phi'}{\psi^2} \right]\nu' - \frac{\psi''\psi - 2(\psi')^2}{\psi^3}\nu.$$

Simplifying the coefficient of ν' :

$$\frac{\phi''\psi - \phi'\psi' - \psi'\phi'}{\psi^2} = \frac{\phi''\psi - 2\phi'\psi'}{\psi^2}.$$

Therefore:

$$y''(x) = \frac{(\phi')^2}{\psi}\nu'' + \frac{\phi''\psi - 2\phi'\psi'}{\psi^2}\nu' - \frac{\psi''\psi - 2(\psi')^2}{\psi^3}\nu.$$

Step 6: Substituting into the Original ODE

What we do: Substitute y and y'' into $\varepsilon^2 y'' + qy = 0$.

$$\begin{aligned} \varepsilon^2 y'' + q(x)y &= 0 \\ \varepsilon^2 \left[\frac{(\phi')^2}{\psi}\nu'' + \frac{\phi''\psi - 2\phi'\psi'}{\psi^2}\nu' - \frac{\psi''\psi - 2(\psi')^2}{\psi^3}\nu \right] + q(x)\frac{\nu}{\psi} &= 0. \end{aligned}$$

Multiply through by ψ to clear denominators:

$$\varepsilon^2(\phi')^2\nu'' + \varepsilon^2\frac{\phi''\psi - 2\phi'\psi'}{\psi}\nu' - \varepsilon^2\frac{\psi''\psi - 2(\psi')^2}{\psi^2}\nu + q(x)\nu = 0.$$

Step 7: Imposing Conditions for the Target Form

Our goal: We want $\varepsilon^2 \nu''(z) + \nu(z) = 0$, which means:

- Coefficient of ν'' : should be ε^2
- Coefficient of ν' : should be 0
- Coefficient of ν : should be 1

Strategy: We have three conditions and two unknown functions (ϕ and ψ). We'll impose the conditions strategically:

1. Make the coefficient of ν'' equal to ε^2
2. Make the coefficient of ν' equal to 0
3. Check what remains for the coefficient of ν

Step 7a: Condition from the ν'' Term

The coefficient of ν'' is $\varepsilon^2(\phi')^2$. We require:

$$\varepsilon^2(\phi')^2 = \varepsilon^2 \implies (\phi')^2 = 1 \implies \phi'(x) = \pm 1.$$

Justification: We choose the positive sign (the negative would just reverse the direction of z):

$$\phi'(x) = 1 \implies \phi(x) = x + \text{const.}$$

Without loss of generality, we can set the constant to zero, so $\phi(x) = x$, which means $z = x$.

Reflection: Wait! If $\phi(x) = x$, then we're not really changing the independent variable at all. This seems strange at first, but it makes sense: the essence of the WKB transformation is in the dependent variable transformation $\psi(x)$, not the independent variable. However, let's not assume this yet and continue more generally.

Step 7b: General Case - Condition from the ν' Term

Let's continue with general $\phi'(x)$ satisfying $(\phi')^2 = 1$. The coefficient of ν' is:

$$\varepsilon^2 \frac{\phi''\psi - 2\phi'\psi'}{\psi}.$$

We require this to vanish:

$$\phi''\psi - 2\phi'\psi' = 0 \implies \phi''\psi = 2\phi'\psi'.$$

Rearranging:

$$\frac{\phi''}{\phi'} = 2 \frac{\psi'}{\psi}.$$

Technique: Recognize that $\frac{\phi''}{\phi'} = \frac{d}{dx} \ln |\phi'|$ and $\frac{\psi'}{\psi} = \frac{d}{dx} \ln |\psi|$. Therefore:

$$\frac{d}{dx} \ln |\phi'| = 2 \frac{d}{dx} \ln |\psi| = \frac{d}{dx} \ln |\psi^2|.$$

Integrating:

$$\ln |\phi'| = \ln |\psi^2| + C \implies |\phi'| = K\psi^2,$$

where $K = e^C$ is a positive constant.

Combined with $(\phi')^2 = 1$, we have $|\phi'| = 1$, so:

$$\psi^2 = \frac{1}{K}.$$

Choosing $K = 1$ (absorbing the constant into ν), we get:

$$\psi(x) = 1.$$

Reflection: *This is surprising! The dependent variable transformation is trivial: $\psi(x) = 1$ means $\nu(z) = y(x)$. Combined with $\phi(x) = x$, we seem to have done nothing. But this can't be right - let's reconsider our approach.*

Step 8: Reconsidering the Transformation Strategy

Strategy: *The issue is that we've been too restrictive. Let's instead require that the coefficient of ν'' be ε^2 times something that could vary with x , but which we'll handle at leading order as $\varepsilon \rightarrow 0$. The key insight is that we want:*

$$\varepsilon^2(\phi')^2 = \varepsilon^2 \quad \text{to leading order.}$$

But actually, let's try a different approach: set $(\phi')^2 = q(x)$ instead!

Justification: *Why this choice? Because if $(\phi')^2 = q(x)$, then:*

$$\phi'(x) = \sqrt{q(x)},$$

and the coefficient of ν'' becomes $\varepsilon^2 q(x)$. If we then divide the entire equation by $q(x)$, we can potentially get the desired form. Let's try this!

Step 9: New Approach - Setting $\phi'(x) = \sqrt{q(x)}$

Choice: Let $\phi'(x) = \sqrt{q(x)}$, so:

$$z = \phi(x) = \int^x \sqrt{q(s)} ds.$$

Justification: *This is the natural choice because $\sqrt{q(x)}$ is the local "frequency" of oscillation in the WKB solution. By integrating it, we're creating a new coordinate z that measures accumulated phase. This connects directly to the WKB solution formula (Lecture Notes §6.3.2, equation (382)):*

$$y_{\pm}(x) \sim \frac{A_{\pm}}{|q(x)|^{1/4}} \exp\left(\pm \frac{i}{\varepsilon} \int^x \sqrt{q(s)} ds\right).$$

With this choice, the coefficient of ν'' becomes:

$$\varepsilon^2(\phi')^2 = \varepsilon^2 q(x).$$

Step 10: Finding $\psi(x)$ with the New ϕ

Now we need to find $\psi(x)$ such that the ν' term vanishes. The coefficient of ν' is:

$$\varepsilon^2 \frac{\phi''\psi - 2\phi'\psi'}{\psi}.$$

Setting this to zero:

$$\phi''\psi - 2\phi'\psi' = 0 \implies \frac{\phi''}{\phi'} = 2\frac{\psi'}{\psi}.$$

With $\phi' = \sqrt{q}$:

$$\phi'' = \frac{d}{dx} \sqrt{q} = \frac{q'}{2\sqrt{q}} = \frac{q'}{2\phi'}.$$

Therefore:

$$\frac{\phi''}{\phi'} = \frac{q'}{2q}.$$

The condition becomes:

$$\frac{q'}{2q} = 2 \frac{\psi'}{\psi} \implies \frac{\psi'}{\psi} = \frac{q'}{4q}.$$

Technique: *Integrate:* $\ln |\psi| = \frac{1}{4} \ln |q| + C$, so:

$$\psi(x) = Kq(x)^{1/4},$$

where K is a constant. Choosing $K = 1$:

$$\psi(x) = q(x)^{1/4}.$$

Step 11: Verifying the Coefficient of ν

With $\phi' = \sqrt{q}$ and $\psi = q^{1/4}$, let's check the coefficient of ν in our transformed equation. The original coefficient (after multiplying by ψ) was:

$$-\varepsilon^2 \frac{\psi''\psi - 2(\psi')^2}{\psi^2} + q(x).$$

Technique: *We need to compute ψ' and ψ'' :*

$$\begin{aligned} \psi &= q^{1/4} \\ \psi' &= \frac{1}{4} q^{-3/4} q' = \frac{q'}{4q^{3/4}} \\ \psi'' &= \frac{d}{dx} \left[\frac{q'}{4q^{3/4}} \right] = \frac{1}{4} \left[\frac{q'' q^{3/4} - q' \cdot \frac{3}{4} q^{-1/4} q'}{q^{3/2}} \right] \\ &= \frac{1}{4q^{3/4}} \left[q'' - \frac{3(q')^2}{4q} \right]. \end{aligned}$$

Computing $\psi''\psi$:

$$\begin{aligned} \psi''\psi &= \frac{1}{4q^{3/4}} \left[q'' - \frac{3(q')^2}{4q} \right] \cdot q^{1/4} \\ &= \frac{1}{4q^{1/2}} \left[q'' - \frac{3(q')^2}{4q} \right]. \end{aligned}$$

Computing $(\psi')^2$:

$$(\psi')^2 = \frac{(q')^2}{16q^{3/2}}.$$

Therefore:

$$\begin{aligned} \psi''\psi - 2(\psi')^2 &= \frac{1}{4q^{1/2}} \left[q'' - \frac{3(q')^2}{4q} \right] - \frac{(q')^2}{8q^{3/2}} \\ &= \frac{1}{4q^{1/2}} q'' - \frac{3(q')^2}{16q^{3/2}} - \frac{2(q')^2}{16q^{3/2}} \\ &= \frac{1}{4q^{1/2}} q'' - \frac{5(q')^2}{16q^{3/2}}. \end{aligned}$$

And:

$$\frac{\psi''\psi - 2(\psi')^2}{\psi^2} = \frac{\frac{1}{4q^{1/2}}q'' - \frac{5(q')^2}{16q^{3/2}}}{q^{1/2}} = \frac{1}{4q}q'' - \frac{5(q')^2}{16q^2}.$$

The coefficient of ν is:

$$-\varepsilon^2 \left[\frac{1}{4q}q'' - \frac{5(q')^2}{16q^2} \right] + q = q \left[1 - \varepsilon^2 \left(\frac{q''}{4q^2} - \frac{5(q')^2}{16q^3} \right) \right].$$

Step 12: Taking the Leading Order as $\varepsilon \rightarrow 0$

Justification: As $\varepsilon \rightarrow 0$, the term multiplied by ε^2 becomes negligible. To leading order:

$$\text{Coefficient of } \nu \rightarrow q(x) \quad \text{as } \varepsilon \rightarrow 0.$$

So our transformed equation is (to leading order):

$$\varepsilon^2 q(x) \nu''(z) + q(x) \nu(z) = 0.$$

Dividing by $q(x)$:

$$\varepsilon^2 \nu''(z) + \nu(z) = 0.$$

Success! With the transformations:

$$z = \phi(x) = \int^x \sqrt{q(s)} ds$$

$$\nu(z) = \psi(x)y(x) = q(x)^{1/4}y(x)$$

the ODE $\varepsilon^2 y'' + q(x)y = 0$ becomes $\varepsilon^2 \nu'' + \nu = 0$ to leading order as $\varepsilon \rightarrow 0$.

Step 13: Solving the Transformed Equation

The equation: $\varepsilon^2 \nu''(z) + \nu(z) = 0$ is a constant-coefficient ODE.

Technique: Try $\nu(z) = e^{\lambda z}$:

$$\varepsilon^2 \lambda^2 e^{\lambda z} + e^{\lambda z} = 0 \implies \lambda^2 = -\frac{1}{\varepsilon^2} \implies \lambda = \pm \frac{i}{\varepsilon}.$$

The general solution is:

$$\nu(z) = A \exp\left(\frac{iz}{\varepsilon}\right) + B \exp\left(-\frac{iz}{\varepsilon}\right),$$

or equivalently:

$$\nu(z) = C \cos\left(\frac{z}{\varepsilon}\right) + D \sin\left(\frac{z}{\varepsilon}\right).$$

Step 14: Transforming Back to $y(x)$

From $\nu(z) = \psi(x)y(x) = q(x)^{1/4}y(x)$, we have:

$$y(x) = \frac{\nu(z)}{q(x)^{1/4}} = \frac{\nu(\phi(x))}{q(x)^{1/4}}.$$

Substituting $\nu(z) = A \exp(iz/\varepsilon) + B \exp(-iz/\varepsilon)$ and $z = \int^x \sqrt{q(s)} ds$:

$$y(x) = \frac{1}{q(x)^{1/4}} \left[A \exp\left(\frac{i}{\varepsilon} \int^x \sqrt{q(s)} ds\right) + B \exp\left(-\frac{i}{\varepsilon} \int^x \sqrt{q(s)} ds\right) \right].$$

Step 15: Comparing with the WKB Solution

Justification: From Lecture Notes §6.3.2, equation (382), the WKB solution for $\varepsilon^2 y'' + q(x)y = 0$ with $q(x) > 0$ is:

$$y_{\pm}(x) \sim \frac{A_{\pm}}{|q(x)|^{1/4}} \exp\left(\pm \frac{i}{\varepsilon} \int^x \sqrt{q(s)} ds + O(\varepsilon)\right) \quad \text{as } \varepsilon \rightarrow 0.$$

Since $q(x) > 0$, we have $|q(x)| = q(x)$, so this becomes:

$$y_{\pm}(x) \sim \frac{A_{\pm}}{q(x)^{1/4}} \exp\left(\pm \frac{i}{\varepsilon} \int^x \sqrt{q(s)} ds\right).$$

Our solution from the transformation method is:

$$y(x) = \frac{A}{q(x)^{1/4}} \exp\left(\frac{i}{\varepsilon} \int^x \sqrt{q(s)} ds\right) + \frac{B}{q(x)^{1/4}} \exp\left(-\frac{i}{\varepsilon} \int^x \sqrt{q(s)} ds\right).$$

Conclusion: The solution obtained by the double transformation method is *exactly* the leading order WKB solution! This proves that the WKB approximation can be understood as a systematic coordinate and amplitude transformation that converts the variable-coefficient ODE into a constant-coefficient one.

Final Summary and Physical Interpretation

Reflection: *What have we learned?*

1. **The transformation** $z = \int^x \sqrt{q(s)} ds$ converts the spatial variable x into a “phase variable” z that measures accumulated oscillations. Where $q(x)$ is large, oscillations are rapid and $dz/dx = \sqrt{q}$ is large.
2. **The transformation** $\nu = q^{1/4}y$ rescales the amplitude. The factor $q^{1/4}$ appears because it's the unique power that eliminates the first derivative term after the coordinate transformation.
3. **Geometric insight:** In the (z, ν) coordinates, solutions are pure sinusoids with wavelength $\sim \varepsilon$. When transformed back to (x, y) coordinates, these become:
 - Amplitude-modulated by $q(x)^{-1/4}$
 - Phase-modulated by $\int \sqrt{q(s)} ds$
4. **Connection to §7.2.1:** This is exactly the approach used in Lecture Notes equations (479)-(511) to derive WKB from the averaging method. The key equations there are:
 - Equation (488): $\dot{h}(t) = \sqrt{q(\varepsilon t)}$ (same as our ϕ')
 - Equation (510): $R(T) = c_2/q(\tau)^{1/4}$ (same as our ψ^{-1})
 - Equation (511): The final WKB form