Exercise Sheet 4: Maps
Question 2 - Complete Solution

Methods of Applied Mathematics

Problem Statement

Derive a discrete map for the predator-prey system, in a similar way we did for the 1d population model.

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1 Step 1: The Continuous Predator-Prey System

The Lotka-Volterra equations

The classical predator-prey model is:

$$\frac{dx}{dt} = ax - bxy \tag{1}$$
$$\frac{dy}{dt} = -cy + dxy \tag{2}$$

where:

- $x(t)$ is the prey population at time t
- $y(t)$ is the predator population at time t
- $a > 0$ is the prey birth rate (in absence of predators)
- $b > 0$ is the predation rate coefficient
- $c > 0$ is the predator death rate (in absence of prey)
- $d > 0$ is the predator growth rate from consumption

XYZ Analysis of the System

- **STAGE X (What we have):** A coupled system of two first-order nonlinear ODEs. Each equation has a linear term (natural growth/death) and a nonlinear interaction term (xy).
- **STAGE Y (Why this structure):**
  - **Prey equation** $\dot{x} = ax - bxy$:
    - ax: Prey reproduce exponentially when alone
    - $-bxy$: Prey are consumed proportional to encounter rate (product of populations)
  - **Predator equation** $\dot{y} = -cy + dxy$:
    - $-cy$: Predators die exponentially without food
    - $+dxy$: Predators grow proportional to prey consumed

The coupling through xy creates the predator-prey dynamic: predators need prey to survive, prey are limited by predators.

- **STAGE Z (What we need):** Derive discrete-time versions by approximating derivatives for finite time step Δt, analogous to the single population case.

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2 Step 2: Discretize Using Euler Approximation

Approximate derivatives

Recall the definition of derivative:

$$\frac{dx}{dt} = \lim_{\Delta t \to 0} \frac{x(t + \Delta t) - x(t)}{\Delta t}$$

For finite (small) Δt, we approximate:

$$\frac{x(t + \Delta t) - x(t)}{\Delta t} \approx ax(t) - bx(t)y(t)$$

Similarly for y:

$$\frac{y(t + \Delta t) - y(t)}{\Delta t} \approx -cy(t) + dx(t)y(t)$$

Rearrange to get update rules

From the prey equation:

$$\begin{aligned} x(t + \Delta t) - x(t) &\approx \Delta t[ax(t) - bx(t)y(t)] \\ x(t + \Delta t) &\approx x(t) + \Delta t \cdot x(t)[a - by(t)] \\ x(t + \Delta t) &\approx x(t)[1 + \Delta t(a - by(t))] \end{aligned}$$

From the predator equation:

$$\begin{aligned} y(t + \Delta t) - y(t) &\approx \Delta t[-cy(t) + dx(t)y(t)] \\ y(t + \Delta t) &\approx y(t) + \Delta t \cdot y(t)[-c + dx(t)] \\ y(t + \Delta t) &\approx y(t)[1 + \Delta t(-c + dx(t))] \end{aligned}$$

Set time step $\Delta t = 1$

Taking the fundamental time unit as $\Delta t = 1$ (e.g., one day, one generation), and using discrete notation $x_n = x(n)$, $y_n = y(n)$:

$$x_{n+1} = x_n[1 + a - by_n] \tag{3}$$
$$y_{n+1} = y_n[1 - c + dx_n] \tag{4}$$

XYZ Analysis of Discretization

- **STAGE X (What we derived):** A pair of coupled difference equations that map $(x_n, y_n) \to (x_{n+1}, y_{n+1})$.

- **STAGE Y (Why this works):** The Euler method approximates:

$$\text{Population at next step} = \text{Current population} + \text{Change over } \Delta t$$

  For $\Delta t = 1$:

  - **Prey:** $x_{n+1} = x_n + x_n(a - by_n) = x_n(1 + a - by_n)$
    * Factor $(1 + a)$ would give exponential growth alone
    * Factor by_n represents reduction due to predation
  - **Predator:** $y_{n+1} = y_n + y_n(-c + dx_n) = y_n(1 - c + dx_n)$
    * Factor $(1 - c)$ would give exponential decay alone
    * Factor dx_n represents growth from eating prey

- **STAGE Z (What this gives):** A 2D discrete dynamical system (map). Unlike the continuous ODE which requires solving differential equations, this map can be iterated directly:

$$(x_0, y_0) \to (x_1, y_1) \to (x_2, y_2) \to \cdots$$

  Each iteration is algebraic, making it computationally simple.

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3 Step 3: Standard Form of the Discrete Map

Present as a map

We can write the discrete predator-prey system as:

$$\boxed{\begin{aligned} x_{n+1} &= x_n(1 + a - by_n) \\ y_{n+1} &= y_n(1 - c + dx_n) \end{aligned}}$$

or in vector form as $\mathbf{z}_{n+1} = \mathbf{F}(\mathbf{z}_n)$ where $\mathbf{z}_n = (x_n, y_n)$ and:

$$\mathbf{F}(x, y) = \begin{pmatrix} x(1 + a - by) \\ y(1 - c + dx) \end{pmatrix}$$

Alternative formulation

We can also write this emphasizing the change:

$$\begin{aligned} x_{n+1} - x_n &= x_n(a - by_n) \\ y_{n+1} - y_n &= y_n(-c + dx_n) \end{aligned}$$

This makes clear that:

- Prey increase when $a > by_n$ (birth rate exceeds predation rate)
- Predators increase when $dx_n > c$ (consumption exceeds death rate)

XYZ Analysis of Form

- **STAGE X (What the form shows):** The map is *multiplicative* - each population is multiplied by a growth factor that depends on the other population.
- **STAGE Y (Why multiplicative):** Because the original ODEs are:
  - Linear in each variable separately: $\dot{x} = x(\cdots)$ and $\dot{y} = y(\cdots)$
  - This "factorizable" structure is preserved under Euler discretization
  - Each population's next value is current value $\times$ (1 + change rate)

  If either population is zero, it remains zero (extinction is permanent). The interaction terms xy couple the equations but maintain the multiplicative structure.
- **STAGE Z (What this means for analysis):**
  - The map has fixed points where $x_{n+1} = x_n$ and $y_{n+1} = y_n$
  - We can analyze stability using the Jacobian matrix
  - For small time steps, behavior mimics continuous system
  - For larger time steps, discrete map can show different dynamics

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4 Step 4: Properties of the Discrete Map

Fixed points

Fixed points satisfy $x_{n+1} = x_n$ and $y_{n+1} = y_n$:

$$x^* = x^*(1 + a - by^*)$$
$$y^* = y^*(1 - c + dx^*)$$

This gives:

$$0 = x^*(a - by^*)$$
$$0 = y^*(-c + dx^*)$$

Fixed point 1: $(x^*, y^*) = (0, 0)$ - extinction of both species
Fixed point 2: $x^* = 0$, $y^* \neq 0$ gives $0 = y^*(-c) \Rightarrow$ no solution (unless $y^* = 0$)
Fixed point 3: $y^* = 0$, $x^* \neq 0$ gives $0 = x^* \cdot a \Rightarrow$ no solution (unless $x^* = 0$)
Fixed point 4: $x^*, y^* \neq 0$ requires:

$$a - by^* = 0 \quad \Rightarrow \quad y^* = \frac{a}{b}$$
$$-c + dx^* = 0 \quad \Rightarrow \quad x^* = \frac{c}{d}$$

Therefore: $\boxed{(x^*, y^*) = \left(\frac{c}{d}, \frac{a}{b}\right)}$ - coexistence equilibrium

Jacobian matrix

The Jacobian of $\mathbf{F}(x, y)$ is:

$$J = \begin{pmatrix} \frac{\partial}{\partial x}[x(1 + a - by)] & \frac{\partial}{\partial y}[x(1 + a - by)] \\ \frac{\partial}{\partial x}[y(1 - c + dx)] & \frac{\partial}{\partial y}[y(1 - c + dx)] \end{pmatrix}$$

Computing derivatives:

$$J = \begin{pmatrix} 1 + a - by & -bx \\ dy & 1 - c + dx \end{pmatrix}$$

At the coexistence fixed point $(x^*, y^*) = (c/d, a/b)$:

$$J^* = \begin{pmatrix} 1 + a - b \cdot \frac{a}{b} & -b \cdot \frac{c}{d} \\ d \cdot \frac{a}{b} & 1 - c + d \cdot \frac{c}{d} \end{pmatrix} = \begin{pmatrix} 1 & -\frac{bc}{d} \\ \frac{ad}{b} & 1 \end{pmatrix}$$

Eigenvalue analysis

For the Jacobian $J^* = \begin{pmatrix} 1 & -\frac{bc}{d} \\ \frac{ad}{b} & 1 \end{pmatrix}$:

Characteristic equation:

$$\det(J^* - \lambda I) = (1 - \lambda)^2 + \frac{bc}{d} \cdot \frac{ad}{b} = 0$$

$$(1 - \lambda)^2 + ac = 0$$

$$\lambda = 1 \pm i\sqrt{ac}$$

The eigenvalues are complex with:

- Real part: $\text{Re}(\lambda) = 1$
- Imaginary part: $\text{Im}(\lambda) = \pm\sqrt{ac}$
- Modulus: $|\lambda| = \sqrt{1 + ac}$

Since $|\lambda| = \sqrt{1 + ac} > 1$ (for $a, c > 0$), the fixed point is **unstable**.

XYZ Analysis of Fixed Points

- **STAGE X (What we found):** Two fixed points: $(0, 0)$ (trivial) and $(c/d, a/b)$ (coexistence). The coexistence point has complex eigenvalues with modulus > 1.
- **STAGE Y (Why this instability):**
  - In the continuous Lotka-Volterra system, the coexistence point is a *center* with purely imaginary eigenvalues - neutral stability with closed orbits
  - The discrete map shifts eigenvalues: $\lambda_{\text{map}} \approx e^{\lambda_{\text{ODE}} \Delta t}$
  - For the ODE with $\lambda_{\text{ODE}} = \pm i\sqrt{ac}$, we get $\lambda_{\text{map}} = e^{\pm i\sqrt{ac}}$ which has $|\lambda| = 1$
  - However, our Euler discretization introduces additional terms that push $|\lambda|$ slightly above 1
  - The spiral instability means orbits slowly diverge outward from the fixed point

- **STAGE Z (What this means):** The discrete map has fundamentally different stability than the continuous system:
  - Continuous: Neutral stability, periodic orbits (conservative system)
  - Discrete (with $\Delta t = 1$): Unstable spiral (trajectories diverge)

  This illustrates that discrete maps are *not* just approximations to ODEs - they can exhibit genuinely different dynamics. For predator-prey, the discretization breaks the conservation law that existed in the continuous case.

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5 Step 5: Comparison with Continuous System

Continuous system properties

The Lotka-Volterra ODE has:

- Fixed point at $(c/d, a/b)$ with purely imaginary eigenvalues $\pm i\sqrt{ac}$
- This is a **center** - neutrally stable
- Solutions are closed periodic orbits around the fixed point
- System is **conservative**: has a conserved quantity $H(x, y) = dx - c \log x + by - a \log y$

Discrete system properties

The discrete map has:

- Same fixed point at $(c/d, a/b)$
- But eigenvalues $1 \pm i\sqrt{ac}$ have modulus $\sqrt{1 + ac} > 1$
- This is an **unstable spiral**
- Solutions spiral outward (for $\Delta t = 1$)
- System is **not conservative**: no preserved quantity

Why the difference?

1. **Euler method is first-order**: It only captures behavior to $O(\Delta t)$
2. **Time step too large**: For $\Delta t = 1$, discrete approximation introduces significant error
3. **Conservation broken**: Euler method doesn't preserve the Hamiltonian structure
4. **Eigenvalue transformation**: The map $\lambda_{\text{map}} = e^{\lambda_{\text{ODE}}\Delta t}$ takes $\pm i\omega \to e^{\pm i\omega}$ which has $|e^{\pm i\omega}| = 1$, but the Euler approximation $\lambda \approx 1 + \lambda_{\text{ODE}}\Delta t$ gives $1 \pm i\omega$ with $|1 \pm i\omega| = \sqrt{1 + \omega^2} > 1$

XYZ Analysis of Comparison

- **STAGE X (What differs):** The continuous system has periodic orbits (center), while the naive discrete system has spiraling unstable orbits.
- **STAGE Y (Why this happens):**
  - The continuous predator-prey system is *Hamiltonian* - it conserves energy-like quantities
  - Euler discretization is *not symplectic* - it doesn't preserve Hamiltonian structure
  - Each iteration adds a small numerical dissipation/excitation
  - Over many iterations, these errors accumulate, causing spiraling
  - The magnitude $|\lambda| = \sqrt{1+ac}$ quantifies the "per-iteration drift"

  Better discretization schemes (like symplectic integrators) can preserve the center structure.
- **STAGE Z (What we learn):**
  1. **Maps $\neq$ ODEs**: Discrete maps are independent models with their own dynamics
  2. **Discretization matters**: Choice of scheme affects qualitative behavior
  3. **Time step critical**: Smaller Δt improves agreement with ODE
  4. **Both are valid**: The map models discrete-time processes (generations), the ODE models continuous time

  In applications where time is naturally discrete (insect populations with distinct generations), the map may be more appropriate than the ODE.

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6 Step 6: Improved Discretization (Optional)

Better time step

For smaller time step $\Delta t \ll 1$, the discrete map becomes:

$$x_{n+1} = x_n[1 + \Delta t(a - by_n)]$$
$$y_{n+1} = y_n[1 + \Delta t(-c + dx_n)]$$

At the fixed point $(c/d, a/b)$, eigenvalues:

$$\lambda \approx 1 \pm i\sqrt{ac}\,\Delta t$$

with modulus:

$$|\lambda| = \sqrt{1 + ac(\Delta t)^2} \approx 1 + \frac{ac(\Delta t)^2}{2}$$

As $\Delta t \to 0$, we have $|\lambda| \to 1$, recovering the center behavior.

Stroboscopic interpretation

The discrete map with $\Delta t = 1$ can be viewed as:

- A **stroboscopic map** of the continuous system
- Sampling the ODE solution at times $t = 0, 1, 2, 3, \ldots$
- Each "flash" captures the instantaneous populations
- The sequence $\{(x_n, y_n)\}$ traces out points on the continuous trajectory

For systems with natural periodicity (seasonal breeding), stroboscopic sampling at appropriate intervals gives meaningful discrete models.

XYZ Analysis of Improvements

- **STAGE X (What improves):** Using smaller Δt makes the discrete map better approximate the continuous ODE's qualitative behavior.
- **STAGE Y (Why smaller is better):** Taylor expansion shows:

$$x(t + \Delta t) = x(t) + \dot{x}(t)\Delta t + \frac{1}{2}\ddot{x}(t)(\Delta t)^2 + O(\Delta t^3)$$

  Euler method only uses first two terms, so error is $O(\Delta t^2)$ per step. Over time interval T, we take $n = T/\Delta t$ steps, accumulating error $\sim n \cdot (\Delta t)^2 = T \cdot \Delta t$. Thus error $\to 0$ as $\Delta t \to 0$.
- **STAGE Z (What this teaches):**
  - **Resolution vs. accuracy**: Smaller Δt requires more iterations but gives better accuracy
  - **Map as model**: When Δt is naturally determined (breeding season), accept the map's own dynamics
  - **Map as algorithm**: When approximating ODE, choose Δt carefully

  The "right" choice depends on whether we're modeling inherently discrete-time processes or approximating continuous-time processes.

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7 Summary

Derivation

Starting from continuous Lotka-Volterra equations:

$$\dot{x} = ax - bxy, \quad \dot{y} = -cy + dxy$$

Euler approximation with $\Delta t = 1$ gives discrete predator-prey map:

$$\boxed{\begin{aligned} x_{n+1} &= x_n(1 + a - by_n) \\ y_{n+1} &= y_n(1 - c + dx_n) \end{aligned}}$$

Key properties

- **Fixed points:**
  - Extinction: $(0, 0)$
  - Coexistence: $(c/d, a/b)$
- **Stability:** Coexistence point has eigenvalues $\lambda = 1 \pm i\sqrt{ac}$ with $|\lambda| = \sqrt{1 + ac} > 1$ (unstable spiral)
- **Dynamics:** Unlike continuous system (neutral center with closed orbits), discrete system has spiraling trajectories

Biological interpretation

The discrete map models populations measured at regular intervals:

- x_n = number of prey at generation n
- y_n = number of predators at generation n
- Updates depend on current populations through:
  - Prey growth rate: $1 + a - by_n$ (high predators $\Rightarrow$ low growth)
  - Predator growth rate: $1 - c + dx_n$ (high prey $\Rightarrow$ high growth)

Continuous vs. discrete

Property	**Continuous ODE**	**Discrete Map** ($\Delta t = 1$)
Fixed point	$(c/d, a/b)$	$(c/d, a/b)$
Eigenvalues	$\pm i\sqrt{ac}$	$1 \pm i\sqrt{ac}$
$\lvert\lambda\rvert$	$\sqrt{ac}$	$\sqrt{1 + ac}$
Stability type	Center (neutral)	Unstable spiral
Orbits	Closed periodic	Outward spiraling
Conservative?	Yes	No

Conclusion: The discrete predator-prey map is a valid model in its own right for discrete-generation populations, but exhibits different long-term behavior than the continuous model due to broken conservation and numerical artifacts of Euler discretization.