First Assessed Homework

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1. (a) (**5 marks**)
The function $\Phi(t) = t^3$ in the exponent has only one stationary point at $t = 0$ which is outside the integration range. Therefore the main contribution comes from the lower boundary of the integral. Integration by parts yields

$$I(X) = \int_X^\infty e^{-t^3}\,dt = \left[-\frac{1}{3t^2}e^{-t^3}\right]_X^\infty - \int_X^\infty \frac{2}{3t^3}e^{-t^3}\,dt \sim \frac{1}{3X^2}e^{-X^3} \quad \text{as} \quad X \to \infty\,. \quad (5)$$

(b) (**5 marks**)
Again the stationary point of the exponent at $t = 0$ lies outside the integration range, and we obtain the leading asymptotic form by integration by parts

$$I(X) = \int_3^6 e^{-Xt^2}\sqrt{1+t^2}\,dt = \left[-\frac{\sqrt{1+t^2}}{2tX}e^{-Xt^2}\right]_3^6 - \int_3^6 e^{-Xt^2}\frac{1}{2t^2X\sqrt{1+t^2}}\,dt\,. \quad (3)$$

We conclude that

$$I(X) \sim \frac{\sqrt{10}}{6X}e^{-9X} \quad \text{as} \quad X \to \infty\,. \quad (2)$$

(c) (**5 marks**)
The function in the exponent

$$\Phi(t) = -\sin t - \cos t = -\sqrt{2}\left(\sin t\,\sin\frac{\pi}{4} + \cos t\,\cos\frac{\pi}{4}\right) = -\sqrt{2}\cos\left(t - \frac{\pi}{4}\right) \quad (1)$$

has its minimum at $t = \pi/4$ where $\Phi(\pi/4) = -\sqrt{2}$ and $\Phi''(\pi/4) = \sqrt{2}$. The leading order form of the integral follows by expanding the exponent around the minimum up to second order, the exponential prefactor up to zeroth order, and extending the integration range to $\pm\infty$

$$I(X) = \int_0^{\pi/2} e^{X(\sin t + \cos t)}\sqrt{t}\,dt \sim \int_{-\infty}^{\infty} \exp\left(X\sqrt{2} - \frac{X}{\sqrt{2}}(t-\pi/4)^2\right)\sqrt{\frac{\pi}{4}}\,dt \quad (3)$$

The evaluation of the Gaussian integral leads to

$$I(X) \sim \frac{\pi\,2^{1/4}}{2\sqrt{X}}e^{X\sqrt{2}} \quad \text{as} \quad X \to \infty\,. \quad (1)$$

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2. (**10 marks**)
3 marks for arguing that the correct asymptotic expansion is in terms of fractional powers of ϵ.
The unperturbed equation is

$$x^3 - 3x^2 + 3x - 1 = (x-1)^3 = 0, \qquad 1 \qquad (1)$$

whose solution is $x = 1$ with multiplicity 3. Note that $x = 1$ is also an exact solution of the perturbed equation. Therefore, we only need to find the two-term expansion of two solutions. This suggests that the correct expansion is in terms of fractional powers of x, namely

$$x(\epsilon) = 1 + x_{1/2}\epsilon^{1/2} + x_1\epsilon + O(\epsilon^{3/2}), \quad \epsilon \to 0. \qquad 2 \qquad (2)$$

Factorising $(x-1)$ leads to

$$(x-1)\left[-\epsilon x^2 + (x-1)^2\right] \qquad (3)$$

Substituting (2) into (3) and collecting powers of ϵ gives

$$x_{1/2}(x_{1/2}^2 - 1)\epsilon^{3/2} + \left(3x_1x_{1/2}^2 - 2x_{1/2}^2 - x_1\right)\epsilon^2 + \cdots = 0. \qquad 4 \qquad (4)$$

The coefficient of $\epsilon^{3/2}$ gives

$$x_{1/2} = 0 \quad \text{and} \quad x_{1/2} = \pm 1. \qquad (5)$$

The value $x_{1/2} = 0$ is consistent with the fact that $x = 1$ is an exact solution of the equation; the others give

$$x = 1 \pm \epsilon^{1/2} + O(\epsilon). \qquad 3 \qquad (6)$$

Note that x_1 cannot be determined at the next order, as it is to be expected with expansions in terms of non-integer powers.

(a) (**5 marks**)
By setting $x = 1/z$ the differential equation becomes

$$\frac{d^2y}{dz^2} + \frac{2}{z}\frac{dy}{dz} - \left(\frac{1}{z^4} + \frac{1}{z^3}\right)y = 0. \qquad 3$$

This equation equation has an irregular singular point at $z = 0$ due to the $1/z^3$ and $1/z^4$ terms. Consequently, it has an irregular singular point at $x = 0$. 2

(b) (**10 marks.**)

We now look for solutions of the form $y(x) = \exp(S(x))$. The leading order term is given by those contributions to $S(x)$ that do not vanish as $x \to \infty$.
In terms of $S(x)$ the original equation becomes

$$S'' + S'^2 - 1 - \frac{1}{x} = 0. \qquad (7)$$

We first try $S(x) \sim Cx^\beta$ as $x \to \infty$, which yields

$$C\beta(\beta-1)x^{\beta-2} + C^2\beta^2x^{2(\beta-1)} \sim 1, \quad x \to \infty. \qquad (8)$$

The method of the dominant balance gives

$$C = \pm 1 \quad \text{and} \quad \beta = 1.$$

Therefore,

$$S(x) = \pm x + D(x), \quad D = o(x), \quad x \to \infty \qquad 2 \tag{9}$$

and $D'(x) = o(1)$. By substituting (9) into (7) we obtain the asymptotic equation

$$D'' + D'^2 \pm 2D' \sim \frac{1}{x}, \quad x \to \infty. \tag{10}$$

The above relation suggests a solution of the form $D' \sim x^{-\alpha}$ as $x \to \infty$ with $\alpha > 0$. This assumption gives

$$D'' = o(D') \quad \text{and} \quad D'^2 = o(D'), \quad x \to \infty.$$

Therefore, we have

$$D' \sim \pm\frac{1}{2x}, \qquad D \sim \pm\frac{1}{2}\log x, \quad x \to \infty.$$

Write

$$D(x) = \pm\frac{1}{2}\log x + E(x), \qquad 3 \tag{11}$$

where

$$E(x) = o(\log x), \quad E'(x) = o(1/x), \quad E''(x) = o(1/x^2), \quad x \to \infty \tag{12}$$

Substituting (11) into (10) leads to

$$E'' + E'^2 + \frac{1}{2x^2}\left(\frac{1}{2} \mp 1\right) \sim \mp 2E' \mp \frac{1}{x}E', \quad x \to \infty. \tag{13}$$

From this equation and the fact that $E'' = o(E')$ and $E'^2 = o(E')$, we arrive at

$$E'(x) \sim \frac{1}{8x^2}, \qquad E(x) \sim -\frac{1}{8x}, \quad x \to \infty \qquad 3$$

It follows that the general solution at leading order is

$$y(x) \sim A\sqrt{x}e^{x} + \frac{B}{\sqrt{x}}e^{-x}, \quad x \to \infty. \qquad 2$$