Asymptotics 2025/2026 Sheet 1
Problem 5: Detailed Solution

Problem 5

Problem Statement: Prove that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{z^n}$ is an asymptotic expansion of $\dfrac{1}{z-1}$ as $z \to \infty$.

1 Understanding What We Must Prove

1.1 What Do We Have?

We are given:

- A function: $f(z) = \dfrac{1}{z-1}$
- A formal infinite series: $\displaystyle\sum_{n=1}^{\infty} \frac{1}{z^n}$
- A limiting process: $z \to \infty$

1.2 What Must We Show?

We must prove that this series is an **asymptotic expansion** of $f(z)$. But what does this mean precisely?

1.2.1 Why We Need the Formal Definition

The phrase "asymptotic expansion" has a precise mathematical meaning that we cannot prove without invoking. We must consult the lecture notes Section 2.5, which provides:

Definition 1 (Asymptotic Series Expansion, from Lecture Notes p. 9)**.** *Given an asymptotic sequence $\{\phi_n(x)\}$, the formal series $\sum_{n=0}^{\infty} a_n\phi_n(x)$ is an asymptotic expansion of $f(x)$ if, for every N,*

$$\lim_{x\to x_0} \frac{f(x) - \sum_{n=0}^{N} a_n\phi_n(x)}{\phi_N(x)} = 0. \tag{1}$$

1.2.2 Why This Definition?

This definition captures the idea that:

1. The partial sums $\sum_{n=0}^{N} a_n\phi_n(x)$ approximate $f(x)$
2. The error $f(x) - \sum_{n=0}^{N} a_n\phi_n(x)$ is "small compared to the last term kept"
3. "Small compared to" means the ratio goes to zero

4. This must hold for *every* N, not just one particular truncation

The key insight is that the remainder $R_N = f(x) - \sum_{n=0}^{N} a_n \phi_n(x)$ must satisfy:

$$R_N = o(\phi_N(x)) \quad \text{as } x \to x_0.$$

This is equation (60) in the lecture notes.

2 Identifying the Components

2.1 What is the Asymptotic Sequence?

From our series $\sum_{n=1}^{\infty} \frac{1}{z^n}$, we can identify:

$$\phi_n(z) = \frac{1}{z^n} = z^{-n}, \quad n = 1, 2, 3, \ldots$$

2.1.1 Why Must We Verify This is an Asymptotic Sequence?

The definition of asymptotic expansion *requires* that $\{\phi_n(z)\}$ be an asymptotic sequence. From the lecture notes (p. 9):

Definition 2 (Asymptotic Sequence)**.** *A sequence of functions* $\{\phi_n(x)\}$, $n = 1, 2, \ldots$ *is an asymptotic sequence as* $x \to x_0$ *if, for all* n*,*

$$\phi_{n+1}(x) = o(\phi_n(x)) \quad \text{as } x \to x_0,$$

i.e., $\lim_{x \to x_0} \frac{\phi_{n+1}(x)}{\phi_n(x)} = 0.$

2.2 Verification That $\{z^{-n}\}$ is an Asymptotic Sequence

What we must check: For all n, does $\phi_{n+1}(z) = o(\phi_n(z))$ as $z \to \infty$?

Computation:

$$\frac{\phi_{n+1}(z)}{\phi_n(z)} = \frac{z^{-(n+1)}}{z^{-n}} = \frac{z^{-n-1}}{z^{-n}} = \frac{z^{-n} \cdot z^{-1}}{z^{-n}} = z^{-1} = \frac{1}{z}.$$

Why this computation? We divided numerator by denominator using properties of exponents.

Taking the limit:

$$\lim_{z \to \infty} \frac{\phi_{n+1}(z)}{\phi_n(z)} = \lim_{z \to \infty} \frac{1}{z} = 0.$$

Why does this limit equal zero? As z grows without bound, $1/z$ shrinks toward zero. This is a fundamental limit.

Conclusion: Since the limit is zero for all n, we have $\phi_{n+1}(z) = o(\phi_n(z))$ for all n, confirming that $\{z^{-n}\}$ is indeed an asymptotic sequence as $z \to \infty$. ✓

2.3 What are the Coefficients?

From the series $\sum_{n=1}^{\infty} \frac{1}{z^n}$, we have:

$$a_n = 1 \quad \text{for all } n \geq 1.$$

Why? Each term is simply $1 \cdot z^{-n}$, so the coefficient multiplying $\phi_n(z) = z^{-n}$ is $a_n = 1$.

3 Strategy for the Proof

3.1 What Must We Prove According to the Definition?

We must show that for every positive integer N:

$$\lim_{z\to\infty} \frac{f(z) - \sum_{n=1}^{N} a_n\phi_n(z)}{\phi_N(z)} = 0.$$

Substituting our specific functions:

$$\lim_{z\to\infty} \frac{\frac{1}{z-1} - \sum_{n=1}^{N} \frac{1}{z^n}}{\frac{1}{z^N}} = 0.$$

3.2 Why This Approach?

This is not merely *one way* to prove the result—it is *the definition* we must verify. We have no choice but to:

1. Compute the partial sum $\sum_{n=1}^{N} \frac{1}{z^n}$
2. Compute the remainder $R_N = f(z) - \sum_{n=1}^{N} \frac{1}{z^n}$
3. Show that $R_N/\phi_N(z) \to 0$ as $z \to \infty$

4 Computing the Partial Sum

4.1 What is the Partial Sum?

We need to evaluate:

$$S_N = \sum_{n=1}^{N} \frac{1}{z^n} = \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \cdots + \frac{1}{z^N}.$$

4.2 Why is This a Geometric Series?

Observation: Each term is obtained from the previous by multiplying by the common ratio $r = \frac{1}{z}$.

Indeed:

$$\frac{1}{z^{n+1}} = \frac{1}{z^n} \cdot \frac{1}{z}.$$

This is the defining property of a geometric series.

4.3 Geometric Series Formula

From elementary calculus/analysis, a geometric series with first term a and common ratio r has partial sum:

$$\sum_{k=0}^{N-1} ar^k = a \cdot \frac{1-r^N}{1-r}, \quad r \neq 1.$$

4.4 Applying the Formula

In our case:

- First term: $a = \frac{1}{z}$
- Common ratio: $r = \frac{1}{z}$
- Number of terms: N (from $n = 1$ to $n = N$)

Therefore:

$$\sum_{n=1}^{N} \frac{1}{z^n} = \frac{1}{z} \cdot \frac{1 - \left(\frac{1}{z}\right)^N}{1 - \frac{1}{z}}.$$

4.5 Simplifying the Expression

Simplify the numerator:

$$\frac{1}{z} \cdot \left(1 - \frac{1}{z^N}\right) = \frac{1}{z} - \frac{1}{z^{N+1}}.$$

Simplify the denominator:

$$1 - \frac{1}{z} = \frac{z-1}{z}.$$

Why these simplifications? We're expressing everything with a common denominator and combining fractions—standard algebraic manipulations.

Complete the division:

$$S_N = \frac{\frac{1}{z} - \frac{1}{z^{N+1}}}{\frac{z-1}{z}} = \frac{\frac{1}{z} - \frac{1}{z^{N+1}}}{\frac{z-1}{z}} = \left(\frac{1}{z} - \frac{1}{z^{N+1}}\right) \cdot \frac{z}{z-1}.$$

Distribute:

$$S_N = \frac{1}{z-1} - \frac{1}{z^N(z-1)}.$$

Alternative form: Factor out from numerator:

$$S_N = \frac{1}{z-1}\left(1 - \frac{1}{z^N}\right).$$

4.6 Why This Form is Useful

This form explicitly shows:

- The leading behavior: $\frac{1}{z-1}$ (which is our target function $f(z)$!)
- The correction term: $-\frac{1}{z^N(z-1)}$

This structure will make computing the remainder trivial.

5 Computing the Remainder

5.1 Definition of Remainder

The remainder after N terms is:

$$R_N = f(z) - S_N = \frac{1}{z-1} - \sum_{n=1}^{N} \frac{1}{z^n}.$$

5.2 Substituting Our Result

We found $S_N = \frac{1}{z-1} - \frac{1}{z^N(z-1)}$, therefore:

$$R_N = \frac{1}{z-1} - \left[\frac{1}{z-1} - \frac{1}{z^N(z-1)}\right].$$

5.3 Simplification

Distribute the negative sign:

$$R_N = \frac{1}{z-1} - \frac{1}{z-1} + \frac{1}{z^N(z-1)}.$$

Cancel like terms:

$$R_N = \frac{1}{z^N(z-1)}.$$

Why does this make sense? The partial sum S_N was designed to approximate $f(z) = \frac{1}{z-1}$, and we explicitly computed it as $\frac{1}{z-1}$ minus a correction. The remainder is precisely that correction term.

5.4 Alternative Expression

We can also write:

$$R_N = \frac{1}{z^N} \cdot \frac{1}{z-1}.$$

Why factor this way? Because $\phi_N(z) = \frac{1}{z^N}$ appears explicitly, which will be essential for computing the ratio $R_N/\phi_N(z)$.

6 Verifying the Asymptotic Condition

6.1 What Must We Show?

According to the definition, we must prove:

$$\lim_{z\to\infty} \frac{R_N}{\phi_N(z)} = 0.$$

6.2 Computing the Ratio

Substitute our expressions:

$$\frac{R_N}{\phi_N(z)} = \frac{\frac{1}{z^N(z-1)}}{\frac{1}{z^N}}.$$

Why this substitution? We use $R_N = \frac{1}{z^N(z-1)}$ and $\phi_N(z) = \frac{1}{z^N}$ directly from our previous work.

6.3 Simplifying the Complex Fraction

Method 1: Multiply numerator and denominator by z^N

Multiply both numerator and denominator by z^N:

$$\frac{R_N}{\phi_N(z)} = \frac{\frac{1}{z^N(z-1)} \cdot z^N}{\frac{1}{z^N} \cdot z^N} = \frac{\frac{z^N}{z^N(z-1)}}{1} = \frac{1}{z-1}.$$

Why multiply by z^N? This is the standard technique for simplifying complex fractions: multiply by the LCD of all denominators. Here, z^N appears in denominators of both the numerator and denominator of our complex fraction.

Method 2: Division of fractions

Recall that $\frac{a/b}{c/d} = \frac{a}{b} \cdot \frac{d}{c}$:

$$\frac{R_N}{\phi_N(z)} = \frac{1}{z^N(z-1)} \cdot \frac{z^N}{1} = \frac{z^N}{z^N(z-1)} = \frac{1}{z-1}.$$

Why does this work? The z^N factors cancel.

Both methods yield:

$$\frac{R_N}{\phi_N(z)} = \frac{1}{z-1}.$$

6.4 Taking the Limit

We must evaluate:

$$\lim_{z\to\infty} \frac{1}{z-1}.$$

Why can we take this limit? The function $\frac{1}{z-1}$ is defined for all $z \neq 1$, and we're considering $z \to \infty$, far from the point $z = 1$.

6.5 Evaluating the Limit

Intuition: As z grows large, $z - 1$ also grows large (approximately like z), so $\frac{1}{z-1}$ shrinks toward zero.

Formal argument:

Method 1 (Direct):

$$\lim_{z\to\infty} \frac{1}{z-1} = 0,$$

because the denominator grows without bound while the numerator remains constant.

Method 2 (Using limit laws):

$$\lim_{z\to\infty} \frac{1}{z-1} = \frac{\lim_{z\to\infty} 1}{\lim_{z\to\infty}(z-1)} = \frac{1}{\infty} = 0.$$

Method 3 (Rigorous ϵ-δ): For any $\epsilon > 0$, we need z large enough that $\left|\frac{1}{z-1}\right| < \epsilon$.

This requires: $\frac{1}{|z-1|} < \epsilon$, i.e., $|z-1| > \frac{1}{\epsilon}$.

For $z > \frac{1}{\epsilon} + 1$, this holds. So choosing $M = \frac{1}{\epsilon} + 1$, we have that $z > M$ implies $\left|\frac{1}{z-1}\right| < \epsilon$.

Conclusion:

$$\lim_{z\to\infty} \frac{R_N}{\phi_N(z)} = \lim_{z\to\infty} \frac{1}{z-1} = 0. \quad \checkmark$$

7 Final Verification and Conclusion

7.1 What Have We Proven?

We have shown that for *every* positive integer N:

$$\lim_{z\to\infty} \frac{f(z) - \sum_{n=1}^{N} \frac{1}{z^n}}{\frac{1}{z^N}} = 0.$$

7.2 Why Does This Complete the Proof?

This is *precisely* the definition of an asymptotic expansion from the lecture notes (Definition on p. 9, Equation 58). We have verified:

1. $\{z^{-n}\}$ is an asymptotic sequence as $z \to \infty$ (✓, Section 2.2)
2. For every N, the remainder $R_N = f(z) - \sum_{n=1}^{N} a_n z^{-n}$ satisfies $R_N = o(z^{-N})$ as $z \to \infty$ (✓, Section 6)

These are the two requirements in the definition.

7.3 Why Did This Work?

The key insight is that our function $f(z) = \frac{1}{z-1}$ can be written exactly as:

$$\frac{1}{z-1} = \sum_{n=1}^{N} \frac{1}{z^n} + \frac{1}{z^N(z-1)},$$

where the remainder $\frac{1}{z^N(z-1)}$ is smaller than $\frac{1}{z^N}$ by a factor of $\frac{1}{z-1} \to 0$.

7.4 Connection to Convergence

Important Note: The series $\sum_{n=1}^{\infty} \frac{1}{z^n}$ actually *converges* for $|z| > 1$:

$$\sum_{n=1}^{\infty} \frac{1}{z^n} = \frac{1/z}{1 - 1/z} = \frac{1}{z-1}.$$

So in this case, the asymptotic expansion is also a convergent series equal to $f(z)$.

Why mention this? To emphasize that being an asymptotic expansion is a *weaker* condition than convergence. As noted in the lecture notes (p. 9): "a convergent power series is asymptotic, while a power series can be asymptotic without being convergent."

Our proof of the asymptotic property did not require proving convergence—it only required showing the remainder is $o(\phi_N)$.

8 Summary

8.1 What We Proved

We have rigorously established:

Theorem 3. *The series* $\displaystyle\sum_{n=1}^{\infty} \frac{1}{z^n}$ *is an asymptotic expansion of* $\displaystyle\frac{1}{z-1}$ *as* $z \to \infty$*.*

8.2 Method Used

Following the methodology from Lecture Notes Section 2.5:

1. Identified the asymptotic sequence $\{\phi_n(z)\} = \{z^{-n}\}$
2. Verified it satisfies the asymptotic sequence condition
3. Computed the partial sum $S_N = \sum_{n=1}^{N} \frac{1}{z^n}$ using geometric series formula
4. Computed the remainder $R_N = f(z) - S_N$
5. Showed $R_N/\phi_N(z) \to 0$ as $z \to \infty$
6. Concluded this holds for all N, completing the proof

8.3 Why Each Step Was Necessary

- **Step 1-2:** The definition requires an asymptotic sequence
- **Step 3:** We need the explicit form of the partial sum
- **Step 4:** The remainder is what we must control
- **Step 5:** This is the heart of the definition—proving the limit is zero
- **Step 6:** The definition requires this for *every* N

Conclusion: We have proven that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{z^n}$ is an asymptotic expansion of $\dfrac{1}{z-1}$ as $z \to \infty$ by verifying the defining property: for every N, the remainder after N terms is asymptotically smaller than the N-th term. ■