Methods of Applied Mathematics - Part 1 [SEMT30006]

Exercise Sheet 0 – Revision

These are basic exercises you should be able to do, so give them a quick look through, anything you can't do you'll need to revise from previous years.

1. **Matrices**
   Find the eigenvalues of the following matrices.
   (a) $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$
   (you should be able to spot one of the eigenvalues just by looking at it; hint: what is the rank of this matrix?)
   (b) $\begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix}$
   (c) $\begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix}$ (you should be able to spot both eigenvalues just by looking at it)

2. **Complex numbers**
   (a) Expand $(1 + 2i)\,\mathrm{e}^{2it} + (1 - 2i)\,\mathrm{e}^{-2it}$ in terms of sin and cos
   (b) Find the solutions of $u^3 = 2$

3. **Ordinary Differential Equations**
   Solve the following ordinary differential equations (ODEs).
   (a) $\frac{dx}{dt} = \frac{1}{2}(1 - x)$
   (b) $\frac{d^2x}{dt^2} + \frac{dx}{dt} + 4x = 0$
   (c) $\frac{dx}{dt} = -3x + y$, $\frac{dy}{dt} = x - 3y$ (hint: use the trial solution of $x = \mathrm{e}^{\lambda t}$, $y = a\,\mathrm{e}^{\lambda t}$ and the principle of linear superposition)

4. **Taylors series**
   Estimate the value of sin(0.1) *by hand*. How quickly does the Taylors series expansion approach the actual value?

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Exercise Sheet 1 – ODEs, flows, and phase portraits

1. **First order ODEs**
   Rewrite the following systems as first order ODEs, making it clear what state variables you have chosen and what their state-space is.

   (a) $\frac{d^3u}{dt^3} - \frac{du}{dt} + \sin(u) = 0$

   (b) $\frac{d^2u}{dt^2} + \frac{du}{dt} + u - 2v = 0$, $\frac{d^2v}{dt^2} + \frac{dv}{dt} + v - 2u = 0$

   (c) $\frac{d^2u}{dt^2} + \frac{du}{dt} - u + u^3 - v = 0$, $\frac{dv}{dt} = u - v$

2. **Finite time blow up**
   Solve the initial value problem
   $$\dot{x} = ax^2$$
   with $x(0) = x_0$.

   (a) Does this have solutions, are they unique, and where do they exist?

   (b) Solve the equations for $x(t)$ in terms of a and x_0.

   (c) Despite your answer to (a), show that something 'goes wrong' at time $t = \frac{1}{ax_0}$, and describe what happens there. This is known as **finite time blow up**.

   (d) Sketch a solution for $a = 1$ and $x_0 = 0.2$.

3. **Autonomy (or time-independence)**
   We've seen systems that either depend on time or don't. A system that *does not* depend explicitly on its independent variable is called **autonomous**. Which of the following is autonomous? What is the independent variable?

   (a) The ODE $\ddot{u} = u + \sin(t)$

   (b) The ODE $y'' - y - \sin(x) = 0$

   (c) The ODE $\ddot{\theta} + a\dot{\theta} + b = 0$

   (d) The map $x_{n+1} = ax_n + x_n^2$

   (e) The map $x_{n+1} = nx_n + b$

4. **1D phase portraits**
   Sketch phase portraits for the following differential equations, and describe the long-time behaviour for the given starting conditions.

   (a) $\frac{du}{dt} = \frac{2}{1+u^2} - 1$, with $u(0) = 2$, $u(0) = 0$ and $u(0) = -2$.

   (b) $\frac{du}{dt} = -u^3 + 5u^2 - 6u$, with $u(0) = 1$ and $u(0) = 4$.

5. **2D phase portraits**
Sketch phase portraits for the following differential equations and classify the equilibria.

   (a) $\frac{du}{dt} = v^2 - u$, $\frac{dv}{dt} = u^2 - v$
   (b) $\frac{d^2u}{dt^2} + \frac{du}{dt} + \sin(u) = 0$

6. **Existence and uniqueness**
Solve the following initial value problems to find a solution $x(t)$ in terms of x_0:

   (a) $\dot{x} = x^2$ with $x(0) = x_0$.
   (b) $\dot{x} = |x|$ with $x(0) = x_0$.
   (c) $\dot{x} = |x|^{1/2}$ with $x(0) = x_0$.

People do struggle with solving (c), so to save a bit of time you can check you're answer against the solution:

$$x(t) = \begin{cases} +\left(|x_0|^{1/2} + \frac{1}{2}t\right)^2 & \text{if } \ x_0 \geq 0 \\ 0 & \text{if } \ x_0 = 0 \\ -\left(|x_0|^{1/2} - \frac{1}{2}t\right)^2 & \text{if } \ x_0 \leq 0 \end{cases}$$

This is a little tricky to get all the $+/-$ signs right. The best way to do it (rather than working with $|x|$ or $|^{1/2}$ which is a bit of a difficult term to use reliably, is to completely treat the cases $x, x_0 < 0$ and $x, x_0 > 0$ separately. If you're careful, you should get the answer above. If not, don't waste too much time right now, but in the long run this *is* a good exercise in reliably doing algebra with $|x|$.

Then answer the following questions for each system (these are vital to a fundamental understanding of determinacy and uniqueness in ODEs):

   i. Consider three initial conditions $x_0 = 0$, $x_0 = -1$ and $x_0 = +1$. From each, where does the solution go and how long does it take?
   ii. Identify the different orbits of each system.
   iii. Are the orbits uniquely determined by the ODE (for a given x_0 is there only one unique solution)?
   iv. This is a little more advanced but do-able. In lectures we said that if an ODE is Lipschitz continuous then its solutions exist and are unique. Show that each system here is/isn't Lipschitz continuous, and say how this agrees with your answer to (c).

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Exercise Sheet 2 – Equilibria, Stability, and Linearisation

1. **Stability in 1 dimension**
   A cup of hot coffee temperature T is placed in a room of ambient temperature A. The cup loses heat to the air at a rate proportional to the temperature difference $T - A$, with a constant of proportionality (rate constant of heat loss) c.

   (a) Formulate a differential equation that describes the cooling of the cup.

   (b) Show that a cup of coffee at ambient temperature will remain at ambient temperature.

   (c) Argue that the coffee will eventually approach ambient temperature regardless of the starting temperature (i.e. prove the equilibrium is stable and there are no other attractors).

   (d) Use a change of variables $x = T - A$ to place the equilibrium at $x = 0$. Solve the differential equation for $x(t)$ and then change the variables back to obtain a solution for $T(t)$.

2. **Multiple equilibria in a 1 dimensional system**
   Consider the dynamical system

$$\dot{x} = x^4 - 17x^3 + 101x^2 - 247x + 210$$

   You are told that this has four equilibria, at $x = 2, 3, 5, 7$.

   (a) Show that the ODE can therefore be written as

$$\dot{x} = (x - a)(x - b)(x - c)(x - d)$$

   and find the constants a, b, c, d.

   (b) Find the stability of each equilibrium.

   (c) What is the long term behaviour of a trajectory starting at $x_0 = 6$?

   (d) What is the long term behaviour of a trajectory starting at $x_0 = 8$?

3. **Stability in 2d**
   Consider the 2d ODE

$$\dot{x} = x - 4y$$
$$\dot{y} = y - x$$

   with initial condition $x(0) = 1$, $y(0) = 0$.

   (a) Find the system's equilibria and their stability. What kind of equilibria are they (i.e. node/focus/saddle)?

   (b) Solve the system and verify that this fits with your answer to (a).
   Hint: try to look for a solution of the form

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \alpha_1 \mathbf{c}_1 e^{\lambda_1 t} + \alpha_2 \mathbf{c}_2 e^{\lambda_2 t}$$

   which we claimed in lectures should work (by eigendecomposition), where $\mathbf{c}_1$, $\mathbf{c}_2$, are the eigenvectors and λ_1, λ_2, are the eigenvalues of the equilibrium. The terms α_1, α_2, are constants that need to be found.

4. **Stability in 2d**
   Find the equilibria of the system

$$\dot{x} = y - x^2$$
$$\dot{y} = x - y^2$$

   and determine their stability.

5. **Classification of equilibria in 3D**
   Classify all hyperbolic equilibria of a linear vector field in three dimensions, i.e., draw phase portraits for all topologically different cases when the origin is a hyperbolic equilibrium of the vector field. (Hint: start from the 2D cases, e.g., attracting node, attracting spiral, saddle, etc, and bear in mind that a 3D system has 3 eigenvalues; where in the complex plane can they be?)

6. **Topological equivalence**
Consider the two linear systems

$$\begin{aligned}\dot{x}_1 &= -x_1, \\ \dot{x}_2 &= -x_2,\end{aligned} \qquad \text{and} \qquad \begin{aligned}\dot{y}_1 &= -y_1 - y_2, \\ \dot{y}_2 &= y_1 - y_2.\end{aligned}$$

(a) Sketch their 2D phase portraits.

(b) Follow the steps below to show that the two systems are topologically equivalent.

i. Write both systems in polar coordinates such that $x_1 = r\cos\theta$, $x_2 = r\sin\theta$ and $y_1 = \rho\cos\phi$, $y_2 = \rho\sin\phi$.

ii. Hence show that $\dot{r} = -r$, and find an expression for $\dot{\theta}$.

iii. Similarly, show that $\dot{\rho} = -\rho$, and find an expression for $\dot{\phi}$.

iv. Solve each of the resulting systems with initial conditions $r(0) = r_0$, $\theta(0) = \theta_0$ and $\rho(0) = \rho_0$, $\phi(0) = \phi_0$.

v. Hence show that the solution of one system can be transformed (or rather *mapped*) to the solution of the other by defining $\rho = r$ and $\phi = \theta - \ln(r)$. This shows that the solutions of one system can be mapped to those of the other by a continuous function, a *homeomorphism*, so these systems are *topologically equivalent*.

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Exercise Sheet 3 – Bifurcations

1. Consider the system
$$\dot{x} = (5 - x)(1 - ax)$$
with $a > 0$. Find the equilibria of the system and their stability. Identify the bifurcation that occurs at $a = 1/5$ and sketch the bifurcation diagram.

2. Consider the system
$$\dot{x} = a + 2x + x^2$$
   (a) Find the equilibria of the system and their stability.
   (b) Conjecture the bifurcation that occurs in the system.
   (c) Evaluate the bifurcation and genericity conditions for your conjectured bifurcation (the B's and G's from our definitions of each bifurcation in lectures) to prove that it indeed occurs.

3. Consider the dynamical system
$$\dot{x} = y - 3x$$
$$\dot{y} = \alpha x - x^2$$
for $-9/4 < \alpha < 9/4$.
   (a) Compute and classify the stability/type of any equilibria.
   (b) What bifurcation happens in the system at $\alpha = 0$?
   (c) Draw a bifurcation diagram with α on the horizontal axis, and x on the vertical. What would the diagram look like if you drew α against y?

4. Consider the dynamical system
$$\dot{x} = \alpha x - x^3$$
$$\dot{y} = -y$$
   (a) Compute and classify the stability/type of any equilibria.
   (b) What bifurcation happens in the system at $\alpha = 0$?
   (c) Draw a bifurcation diagram with α on the horizontal axis, and x on the vertical. What would the diagram look like if you drew α against y?

5. Show that a Hopf bifurcation happens in the system $\frac{dx}{dt} = 1 + x^2 y - \mu x - x$, $\frac{dy}{dt} = \mu x - x^2 y$ as μ varies.

6. Consider the Brusselator system (a chemical reaction equation)

$$\dot{x} = a - bx + px^2y - qx$$
$$\dot{y} = bx - px^2y$$

Let $a = q = p = 1$ and consider what happens as b varies. In this problem b is a reaction rate so it is positive.

(a) Find any equilibria.

(b) Find their stability.

(c) Conjecture the bifurcation that occurs in the system, stating where (in x, y, and b it happens), and sketch the bifurcation diagram.

Bifurcations are a big part of this course, so here are some extra questions for practicing on later. . .

7. Determine what bifurcation happens as μ changes in the systems:

   (a) $\frac{dx}{dt} = \mu x - x^3$

   (b) $\frac{dx}{dt} = -\mu x + (1+\mu)x^2 - x^3$

   (c) $\frac{dx}{dt} = \tanh(x) - \mu x$

   (d) $\frac{d^2x}{dt^2} + \frac{dx}{dt} + \mu x + x^3 = 0$

   (e) $\frac{dx}{dt} = \mu y - x$, $\frac{dy}{dt} = -\frac{1}{3}y^3 + y^2 - y + x$

8. Determine what bifurcation happens as μ changes in the system $\frac{dx}{dt} = x - x^3 + \mu$

9. In lectures we wrote the normal form of the Hopf bifurcation in complex variables as

$$\dot{z} = (\rho + i\omega)z + \ell_1 z|z|^2 \tag{1}$$

   By letting $z = x + iy$ show that this is just a neat way of writing the two-dimensional ODE

$$\dot{x} = \rho x - \omega y + \ell_1 x(x^2 + y^2)$$
$$\dot{y} = \omega x + \rho y + \ell_1 y(x^2 + y^2)$$

10. Consider the system

$$\dot{x} = \rho x - \omega y + \alpha(x^2 + y^2)x$$
$$\dot{y} = \omega x + \rho y + \alpha(x^2 + y^2)y$$

   for $\omega > 0$ and $\alpha > 0$, with ρ allowed to vary.

   (a) Find any equilibria of the system and their stability. [Hint: there's an obvious equilibrium, and if its gets to hard to find any others, have a try but don't waste too much time ... you'll find a better way to look at this in (c)).

   (b) Determine what bifurcation occurs at the origin when $\rho = 0$.

   (c) Expressing the system in polar coordinates $(x, y) = (r\cos\phi, r\sin\phi)$, derive dynamical equations for $\dot{r}$ and $\dot{\phi}$. Describe what these tell you about the dynamics.

   (d) From the polar form, show that there exists a limit cycle in the system (corresponding to an equilibrium of the radial $\dot{r}$ system). Hint: show that there is a bifurcation in the $\dot{r}$ system when $\rho = 0$, identify it, and interpret what happens in the system].

   (e) Derive the Poincaré map from the section $\phi = 0$ back to itself, that is, a map of any point r_n on $\phi = 0$ to a point r_{n+1} when ϕ returns to $\phi = 2\pi$. Argue that this verifies the existence of a limit cycle.

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Exercise Sheet 4 – Maps

1. Derive the discrete population model $N_{n+1} = N_n(1 + \beta - \gamma N_n)$ from the solution of the nonlinear ODE population model.

2. Derive a discrete map for the predator-prey system, in a similar way we did for the 1d population model.

3. In lectures we looked at the logistic map

$$x_{n+1} = rx_n(1 - x_n)$$

which has fixed points at

$$x_{*1} = 0 \qquad \& \qquad x_{*2} = (r-1)/r \tag{2}$$

Derive the linearization of the map about each of these fixed points, and hence show that x_{*1} is unstable and x_{*2} is stable for $r > 1$.

4. Newton's Method is a common numerical tool for finding the roots of functions (i.e. x such that $f(x) = 0$). When applying the method to find the roots of a function f we choose an initial value of x_0 and then repeatedly apply the mapping

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

where $f' \equiv df/dx$, until the method converges on a root. In fact we are just iterating a map until it reaches a stable fixed point.

   (a) Show that all fixed points of the map are points where $f(x)$ is zero.

   (b) Show that certain roots cannot be found with this method, using the concept of local stability to derive the condition that must be met by a fixed point for it to be reachable by Newton's Method.

5. Solve the map

$$x_{n+1} = 2x_n - y_n$$
$$y_{n+1} = 2y_n - x_n$$

with initial condition $x_0 = 1$, $y_0 = 0$. You can do this in two different ways:

(a) Simply iterate the map repeatedly (as we did when we solved the population map at the start of the course), until you reach x_0 and y_0 on the righthand side.

(b) Try to use the same kind of decomposition we used for ODEs, trying a solution

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = \alpha_1 \mathbf{c}_1 {\lambda_1}^n + \alpha_2 \mathbf{c}_2 \lambda_2^n$$

where $\mathbf{c}_1$, $\mathbf{c}_2$, are the eigenvectors of the 'Jacobian' of the righthand side

$$\frac{\partial(x_{n+1}, y_{n+1})}{\partial(x_n, y_n)}$$

and λ_1, λ_2, are the eigenvalues of the system's equilibrium, and the terms α_1, α_2, are constants that need to be found (using the initial condition). Note the decomposition is a bit different to ODEs, as the λ_i's appear as powers rather than as $e^{\lambda_i t}$ terms.

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Exercise Sheet 5 – Map bifurcations, period doubling, and chaos

1. Consider the two-dimensional map

$$x_{n+1} = x_n^2 - cy_n$$
$$y_{n+1} = \tfrac{1}{2}(x_n - y_n)$$

   (a) Find the fixed points of the map.

   (b) Determine the stability of the fixed points and conjecture the bifurcation(s) that occur(s) as their stability changes.

2. Sketch or graph the sawtooth map

$$x_{n+1} = \begin{cases} 2x_n & \text{for } 0 \le x_n < 1/2 \\ 2x_n - 1 & \text{for } 1/2 < x_n \le 1 \end{cases}$$

   Either by hand or computer, investigate its dynamics with cobweb diagrams.

   (a) Show there is a period two orbit with an iterate at $x = 1/3$, and find the other iterate.

   (b) Show there is a period three orbit with an iterate at $x = 1/7$, and find the other iterates.

   (c) Show that the orbits from (a)-(b) are unstable.

   (d) Argue why this map cannot have any stable periodic orbits, and conjecture what kind of dynamics you will see from a typical initial point.

3. Take a look at the webpage
   http://www.complexity-explorables.org/flongs/logistic/
   which will help you explore the **logistic map**

$$x_{n+1} = rx_n(1 - x_n)$$

   Using the interactive figure in panel 3 on the site, can you find a period 2 orbit, and a period 4 orbit?

   As you do this, try to work out the following and check them against the simulation:

   (a) Find any fixed points (period one orbits) and the values of r for which they: (i) exist, (ii) are stable.

   (b) Find any period two orbits and the values of r for which they: (i) exist, (ii) are stable.

   (c) Find any period four orbits and the values of r for which they: (i) exist, (ii) are stable.

   (d) Sketch or simulate (e.g. in Matlab) a cobweb diagram showing a stable period one, two, or three orbit, for different suitable example values of r.

   (e) Sketch a bifurcation diagram showing the change from (a) to (b), and identify the bifurcation that occurs.

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Exercise Sheet n – Title

HOMEWORK (try to finish these)

1. Let's go back to the logistic map

$$x_{n+1} = rx_n(1 - x_n)$$

and do things properly.

   (a) Find any fixed points (period one orbits) and the values of r for which they: (i) exist, (ii) are stable.

   (b) Find any period two orbits and the values of r for which they: (i) exist, (ii) are stable.

   (c) Find any period four orbits and the values of r for which they: (i) exist, (ii) are stable.

   (d) Sketch or simulate (e.g. in Matlab) a cobweb diagram showing a stable period one, two, or three orbit, for different suitable example values of r.

   (e) Sketch a bifurcation diagram showing the change from (a) to (b), and identify the bifurcation that occurs.

2. Consider the map

$$x_{n+1} = \tfrac{1}{2}(c - x_n^2)$$

   (a) Compute the fixed points and find their stability.

   (b) Find the local bifurcations of the fixed points, identiying the type of bifurcation.

   (c) Find the equation(s) for the period two orbits of the system.

3. (a)

   (b)

   (c)

4. (a)

   (b)

   (c)

5. (a)

   (b)

   (c)