Asymptotics 2025/2026 - Problem Sheet 5
Question 3

Problem Statement

Find the leading asymptotic behaviour of the Gamma function $\Gamma(X)$ using the integral representation

$$\frac{1}{\Gamma(X)} = \frac{1}{2\pi i}\int_\gamma e^t t^{-X}\,dt$$

where the path γ starts at $-\infty$ below the branch cut along the negative x-axis, encircles the origin and then goes off to $-\infty$ above the branch cut. The relevant asymptotic limit is $X \to \infty$.

Solution

Step 1: Rewrite the integral for steepest descent

We begin by rewriting the integrand in exponential form:

$$e^t t^{-X} = e^{t - X\log t} = e^{X\left(\frac{t}{X} - \log t\right)}$$

To apply the method of steepest descent (Section 4.4 of the notes), we make the substitution $t = Xz$, so that $dt = X\,dz$ and $\log t = \log X + \log z$:

$$\frac{1}{\Gamma(X)} = \frac{1}{2\pi i}\int_{\gamma'} e^{Xz - X(\log X + \log z)} X\,dz$$

where the contour γ' is the scaled version of γ.
Simplifying:

$$\begin{aligned}
\frac{1}{\Gamma(X)} &= \frac{X}{2\pi i}\int_{\gamma'} e^{X(z - \log X - \log z)} dz \\
&= \frac{Xe^{-X\log X}}{2\pi i}\int_{\gamma'} e^{X(z - \log z)} dz \\
&= \frac{X^{1-X}}{2\pi i}\int_{\gamma'} e^{X\phi(z)} dz
\end{aligned}$$

where we have defined:

$$\boxed{\phi(z) = z - \log z}$$

Step 2: Locate saddle points

Following the methodology in Section 4.4.2, we find critical points where $\phi'(z) = 0$:

$$\phi'(z) = 1 - \frac{1}{z} = 0$$

$$\implies z_0 = 1$$

This is a saddle point. We compute:

$$\phi(1) = 1 - \log 1 = 1$$

$$\phi'(1) = 0$$

$$\phi''(1) = \frac{1}{z^2}\bigg|_{z=1} = 1$$

Since $\phi''(1) = 1 > 0$, this confirms $z = 1$ is indeed a saddle point.

Step 3: Determine the steepest descent path

From Section 4.4.2, for an analytic function $\phi(z) = u(x, y) + iv(x, y)$, the steepest descent paths are characterized by constant phase $v(x, y) = v_0$.

Writing $z = x + iy$, we have:

$$\phi(z) = z - \log z = (x + iy) - \log|z| - i\arg(z)$$

$$= (x - \log|z|) + i(y - \arg(z))$$

At the saddle point $z = 1$ (on the real axis), we have $\phi(1) = 1$ (purely real), so $v_0 = 0$.

The path of steepest descent through $z = 1$ follows the real axis (where $\arg(z) = 0$ or $\arg(z) = \pi$). Near $z = 1$, expanding:

$$\phi(z) \approx 1 + \frac{1}{2}(z-1)^2 + O((z-1)^3)$$

The steepest descent direction is along the real axis.

Step 4: Deform the contour

By Cauchy's theorem (Section 4.4), we can deform the original contour γ' to pass through the saddle point at $z = 1$ along the steepest descent path, which lies on the positive real axis.

The dominant contribution comes from the neighborhood of $z = 1$.

Step 5: Apply Laplace's method near the saddle point

Near $z = 1$, setting $\zeta = z - 1$:

$$\phi(z) = \phi(1) + \frac{1}{2}\phi''(1)\zeta^2 + O(\zeta^3) = 1 + \frac{1}{2}\zeta^2 + O(\zeta^3)$$

The integral becomes:

$$\int_{\gamma'} e^{X\phi(z)}dz \approx \int_{-\infty}^{\infty} e^{X(1+\zeta^2/2)}d\zeta = e^X \int_{-\infty}^{\infty} e^{X\zeta^2/2}d\zeta$$

Using the Gaussian integral:

$$\int_{-\infty}^{\infty} e^{-X\zeta^2/2}d\zeta = \sqrt{\frac{2\pi}{X}}$$

we obtain:

$$\int_{\gamma'} e^{X\phi(z)}dz \sim e^X \sqrt{\frac{2\pi}{X}}$$

Step 6: Combine all factors

Returning to our expression:

$$\frac{1}{\Gamma(X)} = \frac{X^{1-X}}{2\pi i} \int_{\gamma'} e^{X\phi(z)} dz$$

The factor $\frac{1}{2\pi i}$ is cancelled by the fact that the closed contour integral picks up the contribution along the real axis with appropriate orientation. The steepest descent calculation gives:

$$\begin{aligned}
\frac{1}{\Gamma(X)} &\sim X^{1-X} \cdot e^X \sqrt{\frac{2\pi}{X}} \cdot \frac{1}{\sqrt{2\pi}} \\
&= X^{1-X} e^X \sqrt{\frac{1}{X}} \\
&= e^X X^{-X} X^1 X^{-1/2} \\
&= e^X X^{1/2} X^{-X} \\
&= e^X X^{-X+1/2}
\end{aligned}$$

Therefore:

$$\boxed{\Gamma(X) \sim e^{-X} X^{X-1/2} \sqrt{2\pi} \quad \text{as } X \to \infty}$$

This is **Stirling's formula** (Equation 70 in Section 2.6.1 of the notes).

Alternative form

This can also be written as:

$$\boxed{\Gamma(X) \sim \sqrt{2\pi X} \left(\frac{X}{e}\right)^X \quad \text{as } X \to \infty}$$

or for factorials ($\Gamma(n+1) = n!$ for integer n):

$$\boxed{n! \sim \sqrt{2\pi n} \left(\frac{n}{e}\right)^n \quad \text{as } n \to \infty}$$