Asymptotics 2025/2026 Sheet 1[1]

Algebraic Equations and Asymptotic Expansions

Please hand in questions 1(a),1(b), 2, 3(c),3(d) on Thursday 2 October 2025 at 12pm

1. For $\epsilon \ll 1$, obtain two-term expansions for the solutions of the following equations

   (a) $(x-1)(x-2)(x-3)+\epsilon = 0$.

   (b) $x^3 + x^2 - \epsilon = 0$.

   (c) $\epsilon x^3 + x^2 + 2x + 1 = 0$.

   (d) $\sqrt{2}\sin(x+\pi/4) - 1 - x + \frac{1}{2}x^2 = -\frac{1}{6}\epsilon$, (consider only the solution near $x = 0$).

2. Find two terms in the expansion for each root of $\epsilon^2 x^3 + x^2 + 2x + \epsilon = 0$.

3. Verify the following statements

   (a) $\sin x^{1/3} = O(x^{1/3})$, $\quad x \to 0+$.

   (b) $\cos(x) = O(1)$, $\quad x \to \infty$.

   (c) $\sin x = O(x\cos x)$, $\quad x \to 0$.

   (d) $\log(\log \frac{1}{x}) = o(\log(x))$, $\quad x \to 0+$.

4. Explain why the sequence $\{\phi_n(x) = x^{-n}\cos(nx)\}$, $n = 0, 1, \ldots$, is not an asymptotic sequence as $x \to \infty$.

5. Prove that $\sum_{n=1} \frac{1}{z^n}$ is an asymptotic expansion of $\frac{1}{z-1}$ as $z \to \infty$.

6. Show that $f(x) \sim g(x)$ as $x \to x_0$ does not necessarily imply that

$$\exp(f(x)) \sim \exp(g(x)) \quad \text{as} \quad x \to x_0$$

   by finding a counterexample.