Methods of Applied Mathematics - Part 1
Exercise Sheet 2: Question 3
Stability in 2D Systems

Complete Solution with XYZ Methodology

Problem Statement

Consider the 2D ODE:

$$\dot{x} = x - 4y \tag{1}$$
$$\dot{y} = y - x \tag{2}$$

with initial condition $x(0) = 1, y(0) = 0$.

1 Question 3(a): Equilibria, Stability, and Classification

Step 1: Find the Equilibria

Solution 1. • **STAGE X (What we need):** An equilibrium (x^*, y^*) is a point where the system doesn't change, i.e., where $\dot{x} = 0$ and $\dot{y} = 0$ simultaneously.

- **STAGE Y (Why this approach):** From Lecture Notes (Section 6, page 21), equilibria of a system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ occur where $\mathbf{f}(\mathbf{x}^*) = \mathbf{0}$. For our 2D system, we solve the coupled algebraic equations.

- **STAGE Z (What we'll do):** Set both equations to zero and solve for (x^*, y^*).

Step 1A: Set Up the Equilibrium Equations

$$\dot{x} = 0 \quad \Rightarrow \quad x - 4y = 0 \tag{3}$$
$$\dot{y} = 0 \quad \Rightarrow \quad y - x = 0 \tag{4}$$

Step 1B: Solve the System
From the second equation:

$$y - x = 0 \quad \Rightarrow \quad y = x \tag{5}$$

Substitute into the first equation:

$$x - 4y = 0 \tag{6}$$
$$x - 4x = 0 \tag{7}$$
$$-3x = 0 \tag{8}$$
$$x = 0 \tag{9}$$

Therefore: $x = 0$ and $y = 0$.

$$\boxed{\text{Unique equilibrium: } (x^*, y^*) = (0, 0)} \tag{10}$$

Explanation 1 (Verification)**.** *Check by substitution:*

$$\dot{x}\big|_{(0,0)} = 0 - 4(0) = 0 \quad \checkmark \tag{11}$$
$$\dot{y}\big|_{(0,0)} = 0 - 0 = 0 \quad \checkmark \tag{12}$$

Step 2: Write System in Matrix Form

To analyze stability, we write the system in matrix-vector form.

The system is already linear:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 1 & -4 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \tag{13}$$

So we have $\dot{\mathbf{x}} = A\mathbf{x}$ where:

$$A = \begin{pmatrix} 1 & -4 \\ -1 & 1 \end{pmatrix} \tag{14}$$

Explanation 2 (Why Matrix Form Matters)**.** *From Lecture Notes (Section 7, page 24): For a linear system $\dot{\mathbf{x}} = A\mathbf{x}$, the solution is $\mathbf{x}(t) = e^{At}\mathbf{x}_0$. The behavior near equilibrium is completely determined by the eigenvalues and eigenvectors of the matrix A.*

For a general nonlinear system, we would compute the Jacobian matrix at the equilibrium. Here, the system is already linear, so A is exactly the Jacobian everywhere.

Step 3: Compute Eigenvalues

- **STAGE X (What eigenvalues tell us):** From Lecture Notes (Section 8, pages 29-31), eigenvalues determine stability and equilibrium type:
  - Sign of real parts $\Rightarrow$ stable or unstable
  - Real vs. complex $\Rightarrow$ node/saddle vs. focus/center
- **STAGE Y (How to find them):** Solve the characteristic equation $\det(A - \lambda I) = 0$.
- **STAGE Z (What we expect):** Two eigenvalues λ_1, λ_2 (possibly complex conjugates).

Step 3A: Set Up Characteristic Equation

$$\det(A - \lambda I) = \det \begin{pmatrix} 1-\lambda & -4 \\ -1 & 1-\lambda \end{pmatrix} = 0 \tag{15}$$

Step 3B: Compute the Determinant

$$\det(A - \lambda I) = (1-\lambda)(1-\lambda) - (-4)(-1) \tag{16}$$
$$= (1-\lambda)^2 - 4 \tag{17}$$
$$= 1 - 2\lambda + \lambda^2 - 4 \tag{18}$$
$$= \lambda^2 - 2\lambda - 3 \tag{19}$$

Step 3C: Solve the Characteristic Equation

$$\lambda^2 - 2\lambda - 3 = 0 \tag{20}$$

Using the quadratic formula:

$$\begin{aligned}
\lambda &= \frac{2 \pm \sqrt{4+12}}{2} && (21)\\
&= \frac{2 \pm \sqrt{16}}{2} && (22)\\
&= \frac{2 \pm 4}{2} && (23)
\end{aligned}$$

Therefore:

$$\begin{aligned}
\lambda_1 &= \frac{2+4}{2} = 3 && (24)\\
\lambda_2 &= \frac{2-4}{2} = -1 && (25)
\end{aligned}$$

$$\boxed{\text{Eigenvalues: } \lambda_1 = 3, \quad \lambda_2 = -1} \tag{26}$$

Step 4: Determine Stability

Explanation 3 (Stability Criterion from Lecture Notes (Section 8, page 29))**.** *For a 2D linear system with eigenvalues λ_1, λ_2:*

- ***Stable****: Both $Re(\lambda_1) < 0$ and $Re(\lambda_2) < 0$*
- ***Unstable****: At least one $Re(\lambda_i) > 0$*
- ***Saddle****: One positive, one negative real eigenvalue*

Analysis of our eigenvalues:

- $\lambda_1 = 3 > 0$ (positive, real)
- $\lambda_2 = -1 < 0$ (negative, real)

Since we have one positive and one negative eigenvalue:

$$\boxed{\text{The equilibrium is a } \textbf{SADDLE} \text{ (unstable)}} \tag{27}$$

Step 5: Classify the Equilibrium Type

From Lecture Notes (Section 8, pages 29-31), classification scheme:

Eigenvalue Type	**Signs**	**Classification**
Both real, same sign	$\lambda_1, \lambda_2 > 0$	Unstable node
Both real, same sign	$\lambda_1, \lambda_2 < 0$	Stable node
Both real, opposite signs	$\lambda_1 > 0, \lambda_2 < 0$	**Saddle**
Complex conjugates	$\text{Re}(\lambda) > 0$	Unstable focus/spiral
Complex conjugates	$\text{Re}(\lambda) < 0$	Stable focus/spiral
Pure imaginary	$\text{Re}(\lambda) = 0$	Center

Our case: $\lambda_1 = 3 > 0$ and $\lambda_2 = -1 < 0$ (both real, opposite signs)

$$\boxed{\text{Classification: } \textbf{SADDLE POINT}} \tag{28}$$

Explanation 4 (Physical Meaning of a Saddle)**.** ***Behavior near the equilibrium:***

- *Along the* ***unstable eigendirection*** *(associated with $\lambda_1 = 3$): trajectories are repelled exponentially with rate e^{3t}*
- *Along the* ***stable eigendirection*** *(associated with $\lambda_2 = -1$): trajectories are attracted exponentially with rate e^{-t}*

Global behavior:

- *Most trajectories are repelled (because $|\lambda_1| > |\lambda_2|$, unstable direction dominates)*
- *Only trajectories starting exactly on the stable manifold approach the saddle*
- *The saddle is overall unstable*

Step 6: Compute Eigenvectors (for Part b)

To fully characterize the saddle and for solving in part (b), we need the eigenvectors.

Step 6A: Eigenvector for $\lambda_1 = 3$

Solve $(A - 3I)\mathbf{v}_1 = \mathbf{0}$:

$$\begin{pmatrix} 1-3 & -4 \\ -1 & 1-3 \end{pmatrix} \begin{pmatrix} v_{1x} \\ v_{1y} \end{pmatrix} = \begin{pmatrix} -2 & -4 \\ -1 & -2 \end{pmatrix} \begin{pmatrix} v_{1x} \\ v_{1y} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{29}$$

First row: $-2v_{1x} - 4v_{1y} = 0 \Rightarrow v_{1x} = -2v_{1y}$

Choose $v_{1y} = 1$, then $v_{1x} = -2$:

$$\mathbf{v}_1 = \begin{pmatrix} -2 \\ 1 \end{pmatrix} \quad \text{or normalized: } \mathbf{c}_1 = \begin{pmatrix} 2 \\ -1 \end{pmatrix} \tag{30}$$

Step 6B: Eigenvector for $\lambda_2 = -1$

Solve $(A + I)\mathbf{v}_2 = \mathbf{0}$:

$$\begin{pmatrix} 1+1 & -4 \\ -1 & 1+1 \end{pmatrix} \begin{pmatrix} v_{2x} \\ v_{2y} \end{pmatrix} = \begin{pmatrix} 2 & -4 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} v_{2x} \\ v_{2y} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{31}$$

First row: $2v_{2x} - 4v_{2y} = 0 \Rightarrow v_{2x} = 2v_{2y}$

Choose $v_{2y} = 1$, then $v_{2x} = 2$:

$$\mathbf{v}_2 = \mathbf{c}_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \tag{32}$$

Verification:

$$A\mathbf{v}_1 = \begin{pmatrix} 1 & -4 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 2+4 \\ -2-1 \end{pmatrix} = \begin{pmatrix} 6 \\ -3 \end{pmatrix} = 3\begin{pmatrix} 2 \\ -1 \end{pmatrix} \quad \checkmark \tag{33}$$

$$A\mathbf{v}_2 = \begin{pmatrix} 1 & -4 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 2-4 \\ -2+1 \end{pmatrix} = \begin{pmatrix} -2 \\ -1 \end{pmatrix} = -1\begin{pmatrix} 2 \\ 1 \end{pmatrix} \quad \checkmark \tag{34}$$

Step 7: Characterize Stable and Unstable Manifolds

From Lecture Notes (Section 10, pages 34-35):

- **Unstable manifold** $W^u(0,0)$: The set of points approaching $(0,0)$ as $t \to -\infty$.

$$W^u(0,0) = \text{span}\{\mathbf{v}_1\} = \left\{ c \begin{pmatrix} 2 \\ -1 \end{pmatrix} : c \in \mathbb{R} \right\} \tag{35}$$

  This is the line $y = -\frac{1}{2}x$ through the origin.

- **Stable manifold** $W^s(0,0)$: The set of points approaching $(0,0)$ as $t \to +\infty$.

$$W^s(0,0) = \text{span}\{\mathbf{v}_2\} = \left\{ c \begin{pmatrix} 2 \\ 1 \end{pmatrix} : c \in \mathbb{R} \right\} \tag{36}$$

  This is the line $y = \frac{1}{2}x$ through the origin.

Final Answer for Part (a)

$$\boxed{\begin{array}{l} \textbf{Equilibrium: } (x^*, y^*) = (0,0) \text{ (unique)} \\ \textbf{Eigenvalues: } \lambda_1 = 3 \text{ (unstable)}, \quad \lambda_2 = -1 \text{ (stable)} \\ \textbf{Eigenvectors: } \mathbf{c}_1 = \begin{pmatrix} 2 \\ -1 \end{pmatrix}, \quad \mathbf{c}_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \\ \textbf{Type: SADDLE POINT (unstable)} \\ \textbf{Stable manifold: } y = \dfrac{1}{2}x \quad \text{(along } \mathbf{c}_2\text{)} \\ \textbf{Unstable manifold: } y = -\dfrac{1}{2}x \quad \text{(along } \mathbf{c}_1\text{)} \end{array}} \tag{37}$$

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2 Question 3(b): Solve the System and Verify

Step 1: General Solution Form via Eigendecomposition

Solution 2. • **STAGE X (What we know):** From Lecture Notes (Section 7, page 28, equation 7.12), for a linear system $\dot{\mathbf{x}} = A\mathbf{x}$ with eigenvalues λ_1, λ_2 and eigenvectors $\mathbf{c}_1, \mathbf{c}_2$, the general solution is:

$$\mathbf{x}(t) = \alpha_1 \mathbf{c}_1 e^{\lambda_1 t} + \alpha_2 \mathbf{c}_2 e^{\lambda_2 t} \tag{38}$$

- **STAGE Y (Why this works):** Each eigenvector gives an independent solution. Along eigenvector $\mathbf{c}_i$, the system behaves like a 1D exponential with rate λ_i. The general solution is a linear combination of these two fundamental solutions.

- **STAGE Z (Our task):** Substitute our eigenvalues and eigenvectors, then use initial conditions to find α_1 and α_2.

Step 2: Write the General Solution

From part (a):

- $\lambda_1 = 3$, $\mathbf{c}_1 = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$
- $\lambda_2 = -1$, $\mathbf{c}_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

General solution:

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \alpha_1 \begin{pmatrix} 2 \\ -1 \end{pmatrix} e^{3t} + \alpha_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{-t} \tag{39}$$

In component form:

$$x(t) = 2\alpha_1 e^{3t} + 2\alpha_2 e^{-t} \tag{40}$$
$$y(t) = -\alpha_1 e^{3t} + \alpha_2 e^{-t} \tag{41}$$

Step 3: Apply Initial Conditions

Given: $x(0) = 1$ and $y(0) = 0$

At $t = 0$:

$$x(0) = 2\alpha_1 e^0 + 2\alpha_2 e^0 = 2\alpha_1 + 2\alpha_2 = 1 \tag{42}$$
$$y(0) = -\alpha_1 e^0 + \alpha_2 e^0 = -\alpha_1 + \alpha_2 = 0 \tag{43}$$

This gives us the system:

$$2\alpha_1 + 2\alpha_2 = 1 \tag{44}$$
$$-\alpha_1 + \alpha_2 = 0 \tag{45}$$

Step 3A: Solve for α_1 and α_2

From the second equation:

$$\alpha_2 = \alpha_1 \tag{46}$$

Substitute into the first equation:

$$2\alpha_1 + 2\alpha_1 = 1 \tag{47}$$
$$4\alpha_1 = 1 \tag{48}$$
$$\alpha_1 = \frac{1}{4} \tag{49}$$

Therefore:

$$\alpha_2 = \alpha_1 = \frac{1}{4} \tag{50}$$

$$\boxed{\alpha_1 = \frac{1}{4}, \quad \alpha_2 = \frac{1}{4}} \tag{51}$$

Step 4: Write the Particular Solution

Substituting $\alpha_1 = \alpha_2 = \frac{1}{4}$ into the general solution:

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 2 \\ -1 \end{pmatrix} e^{3t} + \frac{1}{4}\begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{-t} \tag{52}$$

In component form:

$$x(t) = \frac{1}{4}(2e^{3t} + 2e^{-t}) = \frac{1}{2}(e^{3t} + e^{-t}) \tag{53}$$

$$y(t) = \frac{1}{4}(-e^{3t} + e^{-t}) = \frac{1}{4}(e^{-t} - e^{3t}) \tag{54}$$

$$\boxed{\begin{aligned} x(t) &= \frac{1}{2}(e^{3t} + e^{-t}) \\ y(t) &= \frac{1}{4}(e^{-t} - e^{3t}) \end{aligned}} \tag{55}$$

Step 5: Verify the Solution (ESSENTIAL)

We must verify that our solution satisfies:

1. The original ODEs
2. The initial conditions

Step 5A: Verify Initial Conditions

$$x(0) = \frac{1}{2}(e^0 + e^0) = \frac{1}{2}(1 + 1) = 1 \quad \checkmark \tag{56}$$

$$y(0) = \frac{1}{4}(e^0 - e^0) = \frac{1}{4}(1 - 1) = 0 \quad \checkmark \tag{57}$$

Step 5B: Verify the ODEs
Compute $\dot{x}(t)$:

$$\dot{x}(t) = \frac{d}{dt}\left[\frac{1}{2}(e^{3t} + e^{-t})\right] \tag{58}$$

$$= \frac{1}{2}(3e^{3t} - e^{-t}) \tag{59}$$

Compute $\dot{y}(t)$:

$$\dot{y}(t) = \frac{d}{dt}\left[\frac{1}{4}(e^{-t} - e^{3t})\right] \tag{60}$$

$$= \frac{1}{4}(-e^{-t} - 3e^{3t}) \tag{61}$$

Now check the first ODE: $\dot{x} = x - 4y$

Right-hand side:

$$x - 4y = \frac{1}{2}(e^{3t} + e^{-t}) - 4 \cdot \frac{1}{4}(e^{-t} - e^{3t}) \tag{62}$$
$$= \frac{1}{2}(e^{3t} + e^{-t}) - (e^{-t} - e^{3t}) \tag{63}$$
$$= \frac{1}{2}e^{3t} + \frac{1}{2}e^{-t} - e^{-t} + e^{3t} \tag{64}$$
$$= \frac{3}{2}e^{3t} - \frac{1}{2}e^{-t} \tag{65}$$
$$= \frac{1}{2}(3e^{3t} - e^{-t}) = \dot{x} \quad \checkmark \tag{66}$$

Check the second ODE: $\dot{y} = y - x$
Right-hand side:

$$y - x = \frac{1}{4}(e^{-t} - e^{3t}) - \frac{1}{2}(e^{3t} + e^{-t}) \tag{67}$$
$$= \frac{1}{4}e^{-t} - \frac{1}{4}e^{3t} - \frac{1}{2}e^{3t} - \frac{1}{2}e^{-t} \tag{68}$$
$$= -\frac{1}{4}e^{-t} - \frac{3}{4}e^{3t} \tag{69}$$
$$= \frac{1}{4}(-e^{-t} - 3e^{3t}) = \dot{y} \quad \checkmark \tag{70}$$

Both ODEs are satisfied!

Step 6: Analyze Long-Term Behavior

- **STAGE X (What happens as $t \to \infty$):** The solution contains e^{3t} (growing) and e^{-t} (decaying) terms.
- **STAGE Y (Why e^{3t} dominates):** As $t \to \infty$, $e^{3t} \to \infty$ much faster than $e^{-t} \to 0$. The unstable mode dominates.
- **STAGE Z (What this means):** The trajectory is repelled from the saddle point along the unstable manifold direction.

Asymptotic Behavior as $t \to \infty$:

$$x(t) = \frac{1}{2}(e^{3t} + e^{-t}) \sim \frac{1}{2}e^{3t} \tag{71}$$
$$y(t) = \frac{1}{4}(e^{-t} - e^{3t}) \sim -\frac{1}{4}e^{3t} \tag{72}$$

The ratio:

$$\frac{y(t)}{x(t)} \sim \frac{-\frac{1}{4}e^{3t}}{\frac{1}{2}e^{3t}} = -\frac{1}{2} \tag{73}$$

This means the trajectory asymptotically approaches the line $y = -\frac{1}{2}x$, which is exactly the **unstable manifold** we found in part (a)!

$$\boxed{\text{As } t \to \infty: \quad (x(t), y(t)) \to \infty \text{ along the line } y = -\frac{1}{2}x} \tag{74}$$

Asymptotic Behavior as $t \to -\infty$:
For large negative t, $e^{3t} \to 0$ and $e^{-t} \to \infty$:

$$x(t) \sim \frac{1}{2}e^{-t} \tag{75}$$

$$y(t) \sim \frac{1}{4}e^{-t} \tag{76}$$

The ratio:

$$\frac{y(t)}{x(t)} \sim \frac{\frac{1}{4}e^{-t}}{\frac{1}{2}e^{-t}} = \frac{1}{2} \tag{77}$$

This means as we go backward in time, the trajectory came from the line $y = \frac{1}{2}x$, which is the **stable manifold**!

$$\boxed{\text{As } t \to -\infty: \quad (x(t), y(t)) \to \infty \text{ along the line } y = \frac{1}{2}x} \tag{78}$$

Step 7: Geometric Interpretation

Explanation 5 (Trajectory Behavior Confirms Saddle Structure)**.** ***The solution verifies our part (a) analysis:***

1. ***Starting point****:* $(1, 0)$ *does NOT lie on either eigendirection, so the trajectory is a mixture of both modes.*

2. ***Initial direction****: At $t = 0$, both exponentials contribute. Since $\alpha_1 = \alpha_2 = 1/4$, the trajectory initially combines both stable and unstable components.*

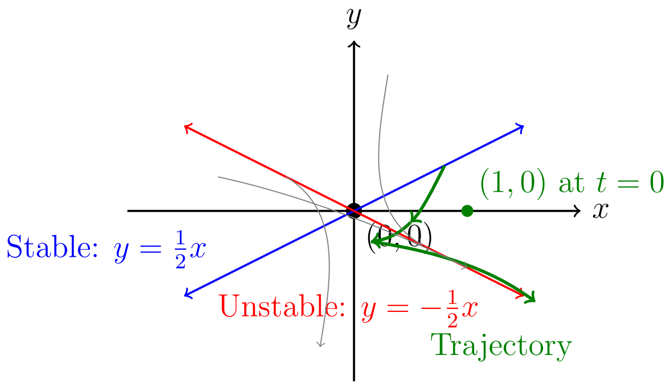
3. ***Forward evolution*** *($t > 0$): The unstable mode e^{3t} grows rapidly, dominating the solution. The trajectory is repelled from the origin along the unstable manifold direction $y = -\frac{1}{2}x$.*

4. ***Backward evolution*** *($t < 0$): The stable mode e^{-t} dominates (since e^{-t} grows as we go backward in time). The trajectory approaches the origin from the stable manifold direction $y = \frac{1}{2}x$.*

5. ***Saddle geometry****: The trajectory forms a hyperbolic path, approaching the stable manifold in the past and departing along the unstable manifold in the future.*

This is the characteristic behavior of a saddle point.

Step 8: Sketch the Phase Portrait (Qualitative)



Final Answer for Part (b)

$$\boxed{\begin{array}{l}\textbf{Particular Solution:}\\ x(t) = \dfrac{1}{2}(e^{3t} + e^{-t})\\ y(t) = \dfrac{1}{4}(e^{-t} - e^{3t})\\ \\ \textbf{Verification of Part (a):}\\ \text{1. As } t \to \infty: \text{ trajectory escapes to } \infty \text{ along unstable manifold } y = -\dfrac{1}{2}x\\ \text{2. As } t \to -\infty: \text{ trajectory came from stable manifold } y = \dfrac{1}{2}x\\ \text{3. Trajectory is repelled from saddle (confirms unstable equilibrium)}\\ \text{4. Solution exhibits characteristic saddle behavior:}\\ \quad \text{mixture of } e^{3t} \text{ (unstable, grows) and } e^{-t} \text{ (stable, decays)}\end{array}} \tag{79}$$

Summary: Complete Analysis of the 2D System

Key Results

1. **Equilibrium**: $(0, 0)$ is the unique equilibrium
2. **Matrix form**: $\dot{\mathbf{x}} = A\mathbf{x}$ where $A = \begin{pmatrix} 1 & -4 \\ -1 & 1 \end{pmatrix}$
3. **Eigenvalues**: $\lambda_1 = 3$ (unstable), $\lambda_2 = -1$ (stable)
4. **Eigenvectors**:
   - $\mathbf{c}_1 = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$ (unstable direction)
   - $\mathbf{c}_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ (stable direction)

5. **Classification**: SADDLE POINT (unstable)
6. **Manifolds**:
   - Stable: $y = \frac{1}{2}x$
   - Unstable: $y = -\frac{1}{2}x$
7. **Solution**: $x(t) = \frac{1}{2}(e^{3t} + e^{-t})$, $y(t) = \frac{1}{4}(e^{-t} - e^{3t})$
8. **Trajectory behavior**: Starts at $(1, 0)$, initially moves toward origin along stable component, then rapidly repelled along unstable manifold toward infinity

Connection to Lecture Notes

This problem exemplifies the complete theory from Sections 7-8 of the lecture notes:

- **Eigendecomposition** (Section 7): Solution is a linear combination of eigenmodes
- **Stability classification** (Section 8): Eigenvalue signs determine stability and type
- **Stable/unstable manifolds** (Section 10): Eigenvectors define invariant manifolds
- **Linear theory is exact**: For linear systems, the linearization IS the full dynamics