Methods of Applied Mathematics - Part 1
Exercise Sheet 2: Question 4
Stability in 2D Systems

Complete Solution with XYZ Methodology

Problem Statement

Find the equilibria of the system:

$$\dot{x} = y - x^2 \tag{1}$$
$$\dot{y} = x - y^2 \tag{2}$$

and determine their stability.

1 Step 1: Find All Equilibria

Step 1A: Define Equilibrium Conditions

Solution 1.
- **STAGE X (What we need):** Equilibria are points (x^*, y^*) where the system doesn't change with time, i.e., where both $\dot{x} = 0$ and $\dot{y} = 0$ simultaneously.
- **STAGE Y (Why this method):** From Lecture Notes (Section 6, page 21), for a 2D system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$, equilibria satisfy $\mathbf{f}(\mathbf{x}^*) = \mathbf{0}$. This gives us two algebraic equations to solve simultaneously.
- **STAGE Z (Our approach):** Set both equations to zero, solve the resulting algebraic system, and verify all solutions.

Step 1B: Set Up the System of Equations

At equilibrium:

$$\dot{x} = 0 \quad \Rightarrow \quad y - x^2 = 0 \tag{3}$$
$$\dot{y} = 0 \quad \Rightarrow \quad x - y^2 = 0 \tag{4}$$

From equation (**??**):

$$y = x^2 \tag{5}$$

Step 1C: Solve for Equilibria

Substitute equation (??) into equation (??):

$$x - y^2 = 0 \tag{6}$$
$$x - (x^2)^2 = 0 \tag{7}$$
$$x - x^4 = 0 \tag{8}$$
$$x(1 - x^3) = 0 \tag{9}$$

This gives us two cases:

$$\text{Case 1: } x = 0 \tag{10}$$
$$\text{Case 2: } 1 - x^3 = 0 \quad \Rightarrow \quad x^3 = 1 \quad \Rightarrow \quad x = 1 \tag{11}$$

Explanation 1 (Why Only Real Solutions)**.** *The equation $x^3 = 1$ has three solutions in $\mathbb{C}$:*

$$x = 1, \quad x = e^{2\pi i/3}, \quad x = e^{4\pi i/3} \tag{12}$$

However, since we're working with real dynamical systems (real-valued x and y), we only consider the real solution $x = 1$.

Step 1D: Find Corresponding y-Values

Using $y = x^2$:

For $x = 0$:

$$y = 0^2 = 0 \quad \Rightarrow \quad \text{Equilibrium at } (0, 0) \tag{13}$$

For $x = 1$:

$$y = 1^2 = 1 \quad \Rightarrow \quad \text{Equilibrium at } (1, 1) \tag{14}$$

Step 1E: Verify the Solutions (ESSENTIAL)

Check $(0, 0)$:

$$\dot{x}\big|_{(0,0)} = 0 - 0^2 = 0 \quad \checkmark \tag{15}$$
$$\dot{y}\big|_{(0,0)} = 0 - 0^2 = 0 \quad \checkmark \tag{16}$$

Check $(1, 1)$:

$$\dot{x}\big|_{(1,1)} = 1 - 1^2 = 0 \quad \checkmark \tag{17}$$
$$\dot{y}\big|_{(1,1)} = 1 - 1^2 = 0 \quad \checkmark \tag{18}$$

Final Answer for Equilibria

$$\boxed{\text{The system has two equilibria: } (x_1^*, y_1^*) = (0, 0) \text{ and } (x_2^*, y_2^*) = (1, 1)} \tag{19}$$

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2 Step 2: Linearization and Jacobian Matrix

Step 2A: Theory of Linearization in 2D

Solution 2. • **STAGE X (What we need):** To determine stability, we must linearize the system around each equilibrium. From Lecture Notes (Section 9, pages 32-33), the linearization is given by the Jacobian matrix.

- **STAGE Y (Why the Jacobian):** Near an equilibrium $\mathbf{x}^*$, the system behaves like:

$$\dot{\mathbf{x}} \approx \mathbf{J}(\mathbf{x}^*) \cdot (\mathbf{x} - \mathbf{x}^*) \tag{20}$$

  where $\mathbf{J}$ is the Jacobian matrix of partial derivatives. The eigenvalues of $\mathbf{J}$ determine the stability and type of equilibrium.

- **STAGE Z (What we'll compute):** Calculate the Jacobian matrix, evaluate it at each equilibrium, find eigenvalues, and classify stability.

Step 2B: Compute the Jacobian Matrix

For the system:

$$f(x, y) = y - x^2 \tag{21}$$
$$g(x, y) = x - y^2 \tag{22}$$

The Jacobian matrix is (Lecture Notes, equation 9.3):

$$\mathbf{J} = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix} \tag{23}$$

Compute Each Partial Derivative:

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(y - x^2) = -2x \tag{24}$$
$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(y - x^2) = 1 \tag{25}$$
$$\frac{\partial g}{\partial x} = \frac{\partial}{\partial x}(x - y^2) = 1 \tag{26}$$
$$\frac{\partial g}{\partial y} = \frac{\partial}{\partial y}(x - y^2) = -2y \tag{27}$$

Therefore:

$$\mathbf{J}(x, y) = \begin{pmatrix} -2x & 1 \\ 1 & -2y \end{pmatrix} \tag{28}$$

Explanation 2 (Understanding the Jacobian)**.** *The Jacobian encodes how the vector field changes near a point:*

- *The diagonal elements* $(-2x, -2y)$ *represent how each variable affects its own rate of change*

- *The off-diagonal elements* $(1, 1)$ *represent coupling: how* y *affects* $\dot{x}$ *and how* x *affects* $\dot{y}$
- *This coupling creates the interesting dynamics in this system*

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3 Step 3: Stability Analysis of Equilibrium $(0, 0)$

Step 3A: Evaluate Jacobian at $(0, 0)$

$$\mathbf{J}(0,0) = \begin{pmatrix} -2(0) & 1 \\ 1 & -2(0) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{29}$$

Step 3B: Find Eigenvalues

The eigenvalues λ satisfy the characteristic equation:

$$\det(\mathbf{J} - \lambda \mathbf{I}) = 0 \tag{30}$$

Compute the determinant:

$$\det \begin{pmatrix} 0-\lambda & 1 \\ 1 & 0-\lambda \end{pmatrix} = 0 \tag{31}$$

$$\det \begin{pmatrix} -\lambda & 1 \\ 1 & -\lambda \end{pmatrix} = 0 \tag{32}$$

$$(-\lambda)(-\lambda) - (1)(1) = 0 \tag{33}$$

$$\lambda^2 - 1 = 0 \tag{34}$$

$$(\lambda - 1)(\lambda + 1) = 0 \tag{35}$$

Therefore:

$$\boxed{\lambda_1 = +1, \quad \lambda_2 = -1} \tag{36}$$

Step 3C: Classify the Equilibrium

Solution 3. • **STAGE X (What we have):** Two real eigenvalues with opposite signs: one positive ($\lambda_1 = +1$) and one negative ($\lambda_2 = -1$).

- **STAGE Y (Why this determines type):** From Lecture Notes (Section 8, pages 29-31):
  - **Node**: Both eigenvalues real with same sign
  - **Saddle**: Both eigenvalues real with opposite signs
  - **Focus**: Complex conjugate eigenvalues
  - **Center**: Pure imaginary eigenvalues

  Since we have real eigenvalues with opposite signs, this is a **saddle point**.

- **STAGE Z (What this means physically):** The equilibrium is unstable. There exist stable and unstable manifolds - trajectories approach along one direction (stable manifold) and repel along another (unstable manifold).

Step 3D: Find Eigenvectors for Geometric Understanding

For $\lambda_1 = +1$ (unstable direction):

Solve $(\mathbf{J} - \lambda_1 \mathbf{I})\mathbf{v}_1 = \mathbf{0}$:

$$\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} v_x \\ v_y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{37}$$

From the first row: $-v_x + v_y = 0 \Rightarrow v_y = v_x$

Eigenvector: $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ (unstable manifold direction)

For $\lambda_2 = -1$ (stable direction):

Solve $(\mathbf{J} - \lambda_2 \mathbf{I})\mathbf{v}_2 = \mathbf{0}$:

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} v_x \\ v_y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{38}$$

From the first row: $v_x + v_y = 0 \Rightarrow v_y = -v_x$

Eigenvector: $\mathbf{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ (stable manifold direction)

Step 3E: Geometric Picture

Explanation 3 (Phase Portrait Near $(0, 0)$)**.** *The linearized dynamics near $(0, 0)$ are:*

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} \approx c_1 e^t \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \tag{39}$$

Stable manifold *(along $(1, -1)$): Trajectories approach the origin as $t \to +\infty$, exponentially with rate $|\lambda_2| = 1$.*

Unstable manifold *(along $(1, 1)$): Trajectories repel from the origin as $t \to +\infty$, exponentially with rate $\lambda_1 = 1$.*

Most trajectories near the origin are initially attracted along the stable manifold but eventually repelled along the unstable manifold.

Final Answer for Equilibrium $(0, 0)$

$$\boxed{\begin{array}{l} \text{Equilibrium: } (0,0) \\ \text{Eigenvalues: } \lambda_1 = +1, \quad \lambda_2 = -1 \\ \text{Type: } \textbf{SADDLE POINT (Unstable)} \\ \text{Stable manifold: direction } (1,-1) \\ \text{Unstable manifold: direction } (1,1) \end{array}} \tag{40}$$

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4 Step 4: Stability Analysis of Equilibrium $(1, 1)$

Step 4A: Evaluate Jacobian at $(1, 1)$

$$\mathbf{J}(1, 1) = \begin{pmatrix} -2(1) & 1 \\ 1 & -2(1) \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} \tag{41}$$

Step 4B: Find Eigenvalues

The characteristic equation is:

$$\det(\mathbf{J} - \lambda\mathbf{I}) = 0 \tag{42}$$

Compute:

$$\det\begin{pmatrix} -2-\lambda & 1 \\ 1 & -2-\lambda \end{pmatrix} = 0 \tag{43}$$

$$(-2-\lambda)(-2-\lambda) - (1)(1) = 0 \tag{44}$$

$$(-2-\lambda)^2 - 1 = 0 \tag{45}$$

$$4 + 4\lambda + \lambda^2 - 1 = 0 \tag{46}$$

$$\lambda^2 + 4\lambda + 3 = 0 \tag{47}$$

Solve Using Quadratic Formula:

$$\lambda = \frac{-4 \pm \sqrt{16-12}}{2} \tag{48}$$

$$= \frac{-4 \pm \sqrt{4}}{2} \tag{49}$$

$$= \frac{-4 \pm 2}{2} \tag{50}$$

Therefore:

$$\lambda_1 = \frac{-4+2}{2} = \frac{-2}{2} = -1 \tag{51}$$

$$\lambda_2 = \frac{-4-2}{2} = \frac{-6}{2} = -3 \tag{52}$$

$$\boxed{\lambda_1 = -1, \quad \lambda_2 = -3} \tag{53}$$

Step 4C: Classify the Equilibrium

Solution 4. • **STAGE X (What we have):** Two real eigenvalues, both negative: $\lambda_1 = -1$ and $\lambda_2 = -3$.

- **STAGE Y (Why this determines type):** From Lecture Notes (Section 8, page 29):
  - Both eigenvalues are **real**
  - Both eigenvalues have the **same sign** (both negative)
  - This defines a **node**
  - Since both are negative, it's a **stable node** (attractor)
- **STAGE Z (What this means):** All trajectories starting near $(1, 1)$ will converge to $(1, 1)$ as $t \to \infty$. The approach is exponential, without oscillations.

Step 4D: Determine Strong and Weak Eigendirections

For a node, we characterize the approach by identifying:

- **Weak eigendirection**: Corresponding to $\lambda_1 = -1$ (smaller $|\lambda|$, slower decay)
- **Strong eigendirection**: Corresponding to $\lambda_2 = -3$ (larger $|\lambda|$, faster decay)

For $\lambda_1 = -1$ (weak, slower):
Solve $(\mathbf{J} - \lambda_1 \mathbf{I})\mathbf{v}_1 = \mathbf{0}$:

$$\begin{pmatrix} -2-(-1) & 1 \\ 1 & -2-(-1) \end{pmatrix} \begin{pmatrix} v_x \\ v_y \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} v_x \\ v_y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{54}$$

From the first row: $-v_x + v_y = 0 \Rightarrow v_y = v_x$
Eigenvector: $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ (weak eigendirection)
For $\lambda_2 = -3$ (strong, faster):
Solve $(\mathbf{J} - \lambda_2 \mathbf{I})\mathbf{v}_2 = \mathbf{0}$:

$$\begin{pmatrix} -2-(-3) & 1 \\ 1 & -2-(-3) \end{pmatrix} \begin{pmatrix} v_x \\ v_y \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} v_x \\ v_y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{55}$$

From the first row: $v_x + v_y = 0 \Rightarrow v_y = -v_x$
Eigenvector: $\mathbf{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ (strong eigendirection)

Step 4E: Dynamics Near $(1, 1)$

Explanation 4 (Trajectory Behavior)**.** *The linearized solution near $(1, 1)$ is:*

$$\begin{pmatrix} x(t) - 1 \\ y(t) - 1 \end{pmatrix} \approx c_1 e^{-t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{-3t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \tag{56}$$

Short-term behavior *(t small):*

- *Both exponential terms present*
- *The e^{-3t} term (strong direction) decays 3 times faster*
- *Trajectories quickly align with the weak eigendirection $(1, 1)$*

Long-term behavior *(t large):*

- *The e^{-3t} term becomes negligible*
- *Only the e^{-t} term remains significant*
- *Trajectories approach $(1, 1)$ along the direction $(1, 1)$ (weak eigendirection)*

Graphical interpretation:

- *Near $(1, 1)$, trajectories initially move quickly toward the line through $(1, 1)$ with direction $(1, 1)$*
- *Once near this line, they approach $(1, 1)$ more slowly along this line*
- *The approach is **monotonic** (no spiraling) because eigenvalues are real*

Step 4F: Check for Hyperbolicity

Explanation 5 (Hyperbolicity and Hartman-Grobman Theorem)**.** *From Lecture Notes (Section 11, page 38), an equilibrium is* ***hyperbolic*** *if none of its eigenvalues have zero real part.*

For $(1,1)$*:*

- $Re(\lambda_1) = -1 \neq 0$ ✓
- $Re(\lambda_2) = -3 \neq 0$ ✓

Therefore $(1,1)$ *is hyperbolic. By the* ***Hartman-Grobman Theorem****, the nonlinear system near* $(1,1)$ *is topologically equivalent to its linearization. Our stability analysis based on the linearization is* ***guaranteed to be correct*** *for the full nonlinear system.*

Final Answer for Equilibrium $(1,1)$

$$\boxed{\begin{array}{l} \text{Equilibrium: } (1,1) \\ \text{Eigenvalues: } \lambda_1 = -1, \quad \lambda_2 = -3 \\ \text{Type: } \textbf{STABLE NODE (Attractor)} \\ \text{Weak eigendirection (slow decay): } (1,1) \\ \text{Strong eigendirection (fast decay): } (1,-1) \\ \text{Behavior: Monotonic approach along weak eigendirection} \end{array}} \tag{57}$$

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5 Step 5: Global Phase Portrait and Summary

Step 5A: Trace-Determinant Analysis

For additional insight, we can use the trace-determinant classification (Lecture Notes, Section 8). **For** $(0,0)$**:**

$$\mathrm{tr}(\mathbf{J}) = 0 + 0 = 0 \tag{58}$$
$$\det(\mathbf{J}) = (0)(0) - (1)(1) = -1 < 0 \tag{59}$$

Since $\det < 0$: **Saddle ✓For** $(1,1)$**:**

$$\mathrm{tr}(\mathbf{J}) = -2 + (-2) = -4 < 0 \tag{60}$$
$$\det(\mathbf{J}) = (-2)(-2) - (1)(1) = 4 - 1 = 3 > 0 \tag{61}$$

Since $\det > 0$ and $\mathrm{tr} < 0$: **Stable node** ✓

Explanation 6 (Trace-Determinant Diagram)**.** *The classification can be visualized in the* $(tr, \det)$ *plane:*

- $\det < 0$*: Saddle (one positive, one negative eigenvalue)*
- $\det > 0$ *and* $tr^2 > 4\det$*: Node (real eigenvalues, same sign)*

- $\det > 0$ *and* $tr^2 < 4\det$*: Focus (complex eigenvalues)*
- $tr < 0$*: Stable (negative real parts)*
- $tr > 0$*: Unstable (positive real parts)*

For $(1, 1)$*:* $tr^2 = 16$ *and* $4\det = 12$*, so* $tr^2 > 4\det$ *confirming it's a node (not a focus).*

Step 5B: System Symmetry

Explanation 7 (Symmetry Analysis)**.** *The system has a special symmetry: if we swap* $x \leftrightarrow y$*:*

$$\dot{y} = x - y^2 \tag{62}$$
$$\dot{x} = y - x^2 \tag{63}$$

This is identical to the original system! The system is symmetric under reflection across the line $y = x$*.*

Consequences:

- *Both equilibria lie on the line* $y = x$ *(indeed,* $(0, 0)$ *and* $(1, 1)$ *satisfy* $y = x$*)*
- *Phase portraits are symmetric about the line* $y = x$
- *If* $(x(t), y(t))$ *is a solution, so is* $(y(t), x(t))$

Step 5C: Global Behavior

Explanation 8 (Complete Phase Portrait Description)**.** ***Key features:***

1. ***Stable attractor at*** $(1, 1)$*: This is the "destination" for most trajectories in the positive quadrant.*
2. ***Saddle at*** $(0, 0)$*: Acts as a "gateway" with:*
    - *Stable manifold along* $(1, -1)$*: Trajectories in this direction approach origin*
    - *Unstable manifold along* $(1, 1)$*: Trajectories in this direction repel from origin*
3. ***Basin of attraction****: The stable manifolds of the saddle at* $(0, 0)$ *likely form boundaries (separatrices) between different behavior regions.*
4. ***Symmetry****: Everything is symmetric across* $y = x$*.*
5. ***Bounded vs. unbounded trajectories****:*
    - *Near* $(1, 1)$*: Trajectories converge (bounded)*
    - *Far from equilibria: Need to analyze nullclines to determine if trajectories escape to infinity or return*

Nullclines provide additional structure:

- $\dot{x} = 0$*: parabola* $y = x^2$ *(vertical motion on this curve)*
- $\dot{y} = 0$*: parabola* $x = y^2$ *(horizontal motion on this curve)*
- *These intersect at our equilibria* $(0, 0)$ *and* $(1, 1)$

Final Summary

Equilibrium	Eigenvalues	Type	Stability
$(0,0)$	$\lambda = +1, -1$	Saddle	Unstable
$(1,1)$	$\lambda = -1, -3$	Stable Node	Stable

(64)

Physical Interpretation:

- The system has one stable equilibrium at $(1,1)$ that attracts nearby trajectories
- The saddle at $(0,0)$ is unstable with mixed stability properties
- The system exhibits rich dynamics with symmetry about the line $y = x$
- Most trajectories in the first quadrant eventually converge to $(1,1)$

Key Concepts from Lecture Notes

Methodology Applied

1. **Finding equilibria** (Section 6): Set $\dot{\mathbf{x}} = \mathbf{0}$ and solve algebraically
2. **Linearization** (Section 9): Compute Jacobian matrix $\mathbf{J} = \frac{\partial \mathbf{f}}{\partial \mathbf{x}}$ at each equilibrium
3. **Eigenvalue analysis** (Section 7-8): Find eigenvalues from characteristic equation $\det(\mathbf{J} - \lambda \mathbf{I}) = 0$
4. **Classification** (Section 8, page 29-31):
   - Real eigenvalues, same sign $\to$ Node
   - Real eigenvalues, opposite signs $\to$ Saddle
   - Complex eigenvalues $\to$ Focus
   - Sign of real parts determines stability
5. **Hartman-Grobman** (Section 11, page 38): For hyperbolic equilibria, linearization captures true behavior
6. **Eigenvectors** (Section 7): Provide geometric understanding of flow directions

Critical Insights

- A 2D system can have multiple equilibria with different stability types
- Hyperbolicity ($\text{Re}(\lambda) \neq 0$) ensures linearization is reliable
- Saddle points create separatrices that organize the global phase portrait
- Symmetries simplify analysis and provide consistency checks