Asymptotics Problem 8.7: Complete Pedagogical Solution

Interior Layer Analysis via Matched Asymptotic Expansions

Problem 1. Find an asymptotic expansion to leading order for the solution $y(x)$ to

$$\varepsilon y'' + xy' + xy = 0, \quad \text{in } -1 < x < 1 \text{ for } \varepsilon \to 0$$

with $y(-1) = e$ and $y(1) = 2e^{-1}$, given that the solution has an 'interior layer'.

Solution: Step-by-Step Atomic Breakdown

Step 1: Identifying Problem Type and Layer Location

Strategy: *We have a singularly perturbed second-order linear ODE of the form* $\varepsilon y'' + p(x)y' + q(x)y = 0$*. We must:*

1. *Identify where layers can occur by analysing the coefficient* $p(x)$
2. *Construct outer solutions in regions away from the layer*
3. *Construct an inner solution valid near the layer*
4. *Match these solutions using Prandtl's matching rule*
5. *Form a composite solution*

What we have: The ODE is

$$\varepsilon y'' + xy' + xy = 0,$$

so we identify $p(x) = x$ and $q(x) = x$.

Justification: *From the general theory of boundary layers (Lecture Notes §6.2.1, equations (340)–(353)), the sign of* $p(x)$ *at the boundaries determines where boundary layers can occur:*

- *At* $x = -1$*:* $p(-1) = -1 < 0$*. This means if we tried a boundary layer at* $x = -1$*, the inner solution would grow exponentially as we move into the domain, preventing matching.*
- *At* $x = 1$*:* $p(1) = 1 > 0$*. This means if we tried a boundary layer at* $x = 1$*, the inner solution would again grow exponentially into the domain.*

Since neither boundary can support a boundary layer, yet the outer equation cannot satisfy both boundary conditions (as we shall verify), the layer must occur at an ***interior point*** *where* $p(x) = 0$*.*

The coefficient $p(x) = x$ vanishes at $x = 0$. This is where the interior layer is located.

Key Concept: *An* ***interior layer*** *occurs at a point* x_0 *inside the domain where the coefficient* $p(x)$ *vanishes. Unlike boundary layers (which occur at domain endpoints), interior layers separate the domain into two regions, each requiring its own outer solution. The interior layer acts as a transition zone connecting these outer solutions. This is described in Lecture Notes §6.2.2, equations (354)–(356).*

Step 2: Constructing the Outer Solutions

Goal: Find the leading-order outer solutions valid in $x < 0$ and $x > 0$, away from the interior layer at $x = 0$.

Step 2a: Deriving the Outer Equation

Technique: *To obtain the outer expansion, we neglect the ε-dependent term in the ODE. This is justified because in the outer region (away from rapid transitions), the solution varies on the $O(1)$ length scale, so $y'' = O(1)$ and thus $\varepsilon y'' = O(\varepsilon) \ll O(1)$.*
Setting $\varepsilon = 0$ in the original ODE gives the leading-order outer equation:

$$xy_0' + xy_0 = 0.$$

For $x \neq 0$, we can divide by x to obtain:

$$y_0' + y_0 = 0.$$

Step 2b: Solving the Outer Equation

This is a first-order linear ODE. Separating variables:

$$\frac{dy_0}{y_0} = -dx.$$

Integrating both sides:

$$\ln |y_0| = -x + C'.$$

Exponentiating:

$$y_0(x) = ae^{-x},$$

where a is an arbitrary constant.
Justification: *This general solution is valid for $x \neq 0$. Since the interior layer at $x = 0$ separates the domain into two regions, we need different constants for the left ($x < 0$) and right ($x > 0$) outer solutions. Each outer solution must satisfy its respective boundary condition.*

Step 2c: Left Outer Solution ($x < 0$)

We denote the left outer solution by $y_{0,a}(x)$. Applying the boundary condition at $x = -1$:

$$y_{0,a}(-1) = e.$$

Substituting $y_{0,a}(x) = ae^{-x}$:

$$ae^{-(-1)} = ae^1 = e.$$

Solving for a:

$$a = 1.$$

Therefore, the **left outer solution** is:

$$\boxed{y_{0,a}(x) = e^{-x}, \quad x < 0}$$

Step 2d: Right Outer Solution ($x > 0$)

We denote the right outer solution by $y_{0,b}(x)$. Applying the boundary condition at $x = 1$:

$$y_{0,b}(1) = 2e^{-1}.$$

Substituting $y_{0,b}(x) = ae^{-x}$:

$$ae^{-1} = 2e^{-1}.$$

Solving for a:

$$a = 2.$$

Therefore, the **right outer solution** is:

$$\boxed{y_{0,b}(x) = 2e^{-x}, \quad x > 0}$$

Step 2e: Verification of Inconsistency

Justification: *Notice that the two outer solutions have different limits as $x \to 0$:*

$$\lim_{x \to 0^-} y_{0,a}(x) = e^0 = 1,$$
$$\lim_{x \to 0^+} y_{0,b}(x) = 2e^0 = 2.$$

These limits do not match! This discontinuity confirms that an interior layer at $x = 0$ is necessary to connect the two outer solutions. The inner solution must smoothly transition from $y \to 1$ on the left to $y \to 2$ on the right.

Step 3: Setting Up the Inner Problem

Goal: Derive the inner equation valid in a neighbourhood of $x = 0$ of width $O(\delta(\varepsilon))$, where $\delta \to 0$ as $\varepsilon \to 0$.

Step 3a: Introducing the Inner Variable

Technique: *For an interior layer at $x = 0$, we introduce a rescaled "inner" variable:*

$$X = \frac{x}{\delta},$$

where $\delta = \delta(\varepsilon)$ is the boundary layer width to be determined. We also define the inner solution:

$$Y(X) = y(x) = y(\delta X).$$

Step 3b: Transforming Derivatives

We need to express the derivatives of y in terms of derivatives of Y.
Since $x = \delta X$:

$$\frac{dy}{dx} = \frac{dY}{dX} \cdot \frac{dX}{dx} = \frac{1}{\delta}\frac{dY}{dX} = \frac{1}{\delta}Y'.$$

Similarly:

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{1}{\delta}Y'\right) = \frac{1}{\delta^2}Y''.$$

Step 3c: Substituting into the ODE

The original equation $\varepsilon y'' + xy' + xy = 0$ becomes:

$$\varepsilon \cdot \frac{1}{\delta^2} Y'' + (\delta X) \cdot \frac{1}{\delta} Y' + (\delta X) \cdot Y = 0.$$

Simplifying:

$$\frac{\varepsilon}{\delta^2} Y'' + XY' + \delta XY = 0.$$

Step 4: Dominant Balance to Determine Layer Width

Goal: Determine the scaling $\delta(\varepsilon)$ by requiring the most important terms in the inner equation to balance.

Step 4a: Identifying Terms

Our transformed equation is:

$$\frac{\varepsilon}{\delta^2} Y'' + XY' + \delta XY = 0.$$

The three terms have coefficients:

- Term 1 (Y''): coefficient ε/δ^2
- Term 2 (XY'): coefficient 1
- Term 3 (XY): coefficient δ

Step 4b: Comparing Term Sizes

Justification: *We seek a distinguished limit where the leading terms balance. Since $\delta \to 0$ as $\varepsilon \to 0$, we have:*

- *The third term δXY is* ***smaller*** *than the second term XY' (since $\delta \ll 1$).*
- *For a distinguished limit, the first term $(\varepsilon/\delta^2)Y''$ must balance the second term XY'.*

Setting the coefficients of the first two terms equal:

$$\frac{\varepsilon}{\delta^2} = 1.$$

Solving for δ:

$$\delta^2 = \varepsilon \implies \boxed{\delta = \sqrt{\varepsilon}}$$

Key Concept: *The interior layer has width $O(\sqrt{\varepsilon})$, not $O(\varepsilon)$ as in standard boundary layers. This is a consequence of the coefficient $p(x) = x$ vanishing linearly at $x = 0$. From Lecture Notes §6.2.2, when $p(x_0) = 0$ and $p'(x_0) \neq 0$, the boundary layer width is $\delta = \sqrt{\varepsilon}$ rather than $\delta = \varepsilon$.*

Step 4c: Verifying Neglected Term

With $\delta = \sqrt{\varepsilon}$, check that the third term is indeed negligible:

$$\text{Third term coefficient} = \delta = \sqrt{\varepsilon} \to 0 \text{ as } \varepsilon \to 0. \quad \checkmark$$

Step 5: Solving the Leading-Order Inner Equation

Goal: Solve for the leading-order inner solution $Y_0(X)$.

Step 5a: The Inner ODE

With $\delta = \sqrt{\varepsilon}$, neglecting the $O(\delta)$ term, the leading-order inner equation is:

$$Y_0'' + XY_0' = 0.$$

Step 5b: Reducing the Order

Technique: *This is a second-order ODE with no explicit Y_0 term. We can reduce the order by setting $P = Y_0'$:*

$$P' + XP = 0.$$

Step 5c: Solving for $P = Y_0'$

Separating variables:

$$\frac{dP}{P} = -X\,dX.$$

Integrating:

$$\ln|P| = -\frac{X^2}{2} + C_1.$$

Exponentiating:

$$P = Y_0' = A\exp\left(-\frac{X^2}{2}\right),$$

where A is an arbitrary constant.

Step 5d: Integrating to Find Y_0

Integrating Y_0':

$$Y_0(X) = A\int_0^X \exp\left(-\frac{s^2}{2}\right) ds + B,$$

where B is another integration constant, and we have chosen the lower limit of integration as 0 for convenience.

Therefore, the **general inner solution** is:

$$\boxed{Y_0(X) = A\int_0^X \exp\left(-\frac{s^2}{2}\right) ds + B}$$

Step 6: Matching the Inner and Outer Solutions

Goal: Determine the constants A and B by requiring that the inner solution matches the outer solutions as $X \to \pm\infty$.

Step 6a: Prandtl's Matching Rule

Technique: *Prandtl's matching rule states that the inner limit of the outer solution must equal the outer limit of the inner solution. For an interior layer, we have two matching conditions:*

$$\begin{aligned}
\textit{Left matching:} \quad & \lim_{x\to 0^-} y_{0,a}(x) = \lim_{X\to-\infty} Y_0(X), \\
\textit{Right matching:} \quad & \lim_{x\to 0^+} y_{0,b}(x) = \lim_{X\to+\infty} Y_0(X).
\end{aligned}$$

Step 6b: Computing the Outer Limits

From Step 2:

$$\lim_{x\to 0^-} y_{0,a}(x) = \lim_{x\to 0^-} e^{-x} = e^0 = 1,$$
$$\lim_{x\to 0^+} y_{0,b}(x) = \lim_{x\to 0^+} 2e^{-x} = 2e^0 = 2.$$

Step 6c: Computing the Inner Limits

For the inner solution $Y_0(X) = A\int_0^X e^{-s^2/2}ds + B$, we need the limits as $X \to \pm\infty$.
Technique: *Recall the Gaussian integral:*

$$\int_0^\infty e^{-s^2/2}ds = \sqrt{\frac{\pi}{2}}.$$

By symmetry of the Gaussian:

$$\int_0^{-\infty} e^{-s^2/2}ds = -\int_{-\infty}^0 e^{-s^2/2}ds = -\sqrt{\frac{\pi}{2}}.$$

Therefore:

$$\lim_{X\to -\infty} Y_0(X) = A\cdot\left(-\sqrt{\frac{\pi}{2}}\right) + B = B - A\sqrt{\frac{\pi}{2}},$$
$$\lim_{X\to +\infty} Y_0(X) = A\cdot\sqrt{\frac{\pi}{2}} + B = B + A\sqrt{\frac{\pi}{2}}.$$

Step 6d: Applying the Matching Conditions

Left matching ($X \to -\infty$, matching with left outer solution):

$$B - A\sqrt{\frac{\pi}{2}} = 1. \tag{M1}$$

Right matching ($X \to +\infty$, matching with right outer solution):

$$B + A\sqrt{\frac{\pi}{2}} = 2. \tag{M2}$$

Step 6e: Solving for A and B

Adding equations (M1) and (M2):

$$2B = 1 + 2 = 3 \implies \boxed{B = \frac{3}{2}}$$

Subtracting (M1) from (M2):

$$2A\sqrt{\frac{\pi}{2}} = 2 - 1 = 1 \implies A = \frac{1}{2\sqrt{\pi/2}} = \frac{1}{\sqrt{2\pi}}.$$

$$\boxed{A = \frac{1}{\sqrt{2\pi}}}$$

Step 6f: The Matched Inner Solution

Substituting A and B into the inner solution:

$$Y_0(X) = \frac{1}{\sqrt{2\pi}} \int_0^X \exp\left(-\frac{s^2}{2}\right) ds + \frac{3}{2}.$$

Technique: *This can be written in terms of the error function. Recall:*

$$\mathit{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2} dt.$$

With the substitution $t = s/\sqrt{2}$, we have $dt = ds/\sqrt{2}$, so:

$$\int_0^X e^{-s^2/2} ds = \sqrt{2} \int_0^{X/\sqrt{2}} e^{-t^2} dt = \sqrt{2} \cdot \frac{\sqrt{\pi}}{2} \mathit{erf}\left(\frac{X}{\sqrt{2}}\right) = \sqrt{\frac{\pi}{2}} \mathit{erf}\left(\frac{X}{\sqrt{2}}\right).$$

Therefore:

$$Y_0(X) = \frac{1}{\sqrt{2\pi}} \cdot \sqrt{\frac{\pi}{2}} \mathrm{erf}\left(\frac{X}{\sqrt{2}}\right) + \frac{3}{2} = \frac{1}{2} \mathrm{erf}\left(\frac{X}{\sqrt{2}}\right) + \frac{3}{2}.$$

The matched inner solution is:

$$\boxed{Y_0(X) = \frac{3}{2} + \frac{1}{2} \mathrm{erf}\left(\frac{X}{\sqrt{2}}\right)}$$

Step 7: Constructing the Composite Solution

Goal: Form a uniformly valid composite solution across the entire domain $-1 < x < 1$.

Step 7a: Standard Composite Formula

Key Concept: *For problems with a single boundary or interior layer, the standard composite solution is:*

$$y_c(x) = y_{outer}(x) + Y_{inner}\left(\frac{x}{\delta}\right) - (\mathit{common\ limit}).$$

However, for an interior layer with two outer solutions, the situation is more subtle because neither outer solution vanishes in the region where the other is valid.

Step 7b: Special Form for Interior Layers

Justification: *From the solution to Problem 8.7 and as noted in the solutions, when we have two outer solutions that match to a single inner solution, the composite solution takes a special form. The key observation is that both outer solutions have the form ae^{-x} for different values of a. The inner solution smoothly transitions between the left limit ($a = 1$) and right limit ($a = 2$).*

We can combine these by noting:

$$y_c(x) = \left(\frac{3}{2} + \frac{1}{2} \mathit{erf}\left(\frac{x}{\sqrt{2\varepsilon}}\right)\right) e^{-x}.$$

This works because:

- *As $x \to -1$ (with ε small): $\mathit{erf}(x/\sqrt{2\varepsilon}) \to -1$, so $y_c \to (3/2 - 1/2)e^{-x} = e^{-x} = y_{0,a}(x)$.* ✓
- *As $x \to +1$ (with ε small): $\mathit{erf}(x/\sqrt{2\varepsilon}) \to +1$, so $y_c \to (3/2 + 1/2)e^{-x} = 2e^{-x} = y_{0,b}(x)$.* ✓
- *Near $x = 0$: The solution transitions smoothly via the error function.*

Step 7c: Final Composite Solution

The uniformly valid **composite solution** to leading order is:

$$\boxed{y_c(x) = \left(\frac{3}{2} + \frac{1}{2}\text{erf}\left(\frac{x}{\sqrt{2\varepsilon}}\right)\right) e^{-x}}$$

Step 8: Verification and Interpretation

Step 8a: Checking Boundary Conditions

At $x = -1$: For small ε, we have $x/\sqrt{2\varepsilon} = -1/\sqrt{2\varepsilon} \to -\infty$, so $\text{erf}(x/\sqrt{2\varepsilon}) \to -1$:

$$y_c(-1) \approx \left(\frac{3}{2} - \frac{1}{2}\right) e^{-(-1)} = 1 \cdot e = e. \quad \checkmark$$

At $x = 1$: For small ε, we have $x/\sqrt{2\varepsilon} = 1/\sqrt{2\varepsilon} \to +\infty$, so $\text{erf}(x/\sqrt{2\varepsilon}) \to +1$:

$$y_c(1) \approx \left(\frac{3}{2} + \frac{1}{2}\right) e^{-1} = 2e^{-1}. \quad \checkmark$$

Step 8b: Physical Interpretation

Reflection: *The solution exhibits the following behaviour:*

1. ***In the region* $x < 0$ *(away from* $x = 0$*):*** *The solution follows* $y \approx e^{-x}$*, an exponentially decaying function as* x *increases.*
2. ***In the region* $x > 0$ *(away from* $x = 0$*):*** *The solution follows* $y \approx 2e^{-x}$*, which has the same exponential decay but with twice the amplitude.*
3. ***Near* $x = 0$ *(the interior layer):*** *The solution rapidly transitions from amplitude* 1 *to amplitude* 2 *over a narrow region of width* $O(\sqrt{\varepsilon})$*. This transition is mediated by the error function, which provides a smooth "sigmoid-like" interpolation.*
4. ***The layer width* $\delta = \sqrt{\varepsilon}$*:*** *This is larger than the* $O(\varepsilon)$ *width of standard boundary layers because the coefficient* $p(x) = x$ *vanishes at the layer location. The linear vanishing of* $p(x)$ *leads to the square-root scaling, as predicted by dominant balance.*

Final Summary

Complete Solution for Problem 8.7:

Given: $\varepsilon y'' + xy' + xy = 0$ on $(-1, 1)$ with $y(-1) = e$, $y(1) = 2e^{-1}$.

Layer location: Interior layer at $x = 0$ (where $p(x) = x$ vanishes).

Layer width: $\delta = \sqrt{\varepsilon}$ (determined by dominant balance).

Left outer solution ($x < 0$): $y_{0,a}(x) = e^{-x}$

Right outer solution ($x > 0$): $y_{0,b}(x) = 2e^{-x}$

Inner solution: $Y_0(X) = \dfrac{3}{2} + \dfrac{1}{2}\text{erf}\left(\dfrac{X}{\sqrt{2}}\right)$

Composite solution:

$$y_c(x) = \left(\frac{3}{2} + \frac{1}{2}\text{erf}\left(\frac{x}{\sqrt{2\varepsilon}}\right)\right) e^{-x}$$

Connection to Lecture Notes

Reflection: *This problem illustrates several key concepts from the lecture notes:*

- ***§6.2.2 (Singular points inside the domain):*** *The theory of interior layers when $p(x_0) = 0$ for some $x_0 \in (0, 1)$.*
- ***§6.2.1 (Matching problem for singular points):*** *The use of Prandtl's matching rule to connect inner and outer solutions.*
- ***Dominant balance (§2.2.2):*** *The determination of layer width $\delta = \sqrt{\varepsilon}$ from requiring leading terms to balance.*
- ***Example 3 in §6.2.3:*** *The lecture notes present a similar interior layer problem $\varepsilon y'' + \sin(x)y' + \sin(x)y = 0$ with $p(x) = \sin(x)$ vanishing at $x = 0$, leading to analogous analysis.*