Exercise Sheet 4: Maps
Question 1 - Complete Solution

Methods of Applied Mathematics

Problem Statement

Derive the discrete population model:

$$N_{n+1} = N_n(1 + \beta - \gamma N_n)$$

from the solution of the nonlinear ODE population model.

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1 Step 1: Recall the Continuous Population Model

The ODE

The nonlinear population model is:

$$\frac{dN}{dt} = N(\beta - \gamma N)$$

where:

- $N(t)$ is the population at time t
- $\beta > 0$ is the birth rate
- $\gamma > 0$ is the death rate (proportional to population)

XYZ Analysis of the ODE

- **STAGE X (What we have):** A first-order nonlinear ODE with quadratic nonlinearity in N. The right-hand side has two contributions: linear growth βN and quadratic decay $-\gamma N^2$.
- **STAGE Y (Why this form):** The model assumes:
  - Birth rate is proportional to population: births $= \beta N$
  - Death rate is proportional to population squared: deaths $= \gamma N^2$

  The quadratic death term models resource competition - as population increases, individuals compete for limited resources, increasing mortality. This gives logistic-type growth with a carrying capacity at $N = \beta/\gamma$ where births balance deaths.
- **STAGE Z (What we need):** To derive a discrete map, we'll solve this ODE explicitly, then sample the solution at discrete time intervals $t = n\Delta t$ for integer n.

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2 Step 2: Solve the ODE

Separate variables

Rearrange the ODE:

$$\frac{dN}{N(\beta - \gamma N)} = dt$$

Partial fraction decomposition

Decompose the left side:

$$\frac{1}{N(\beta - \gamma N)} = \frac{A}{N} + \frac{B}{\beta - \gamma N}$$

Multiply both sides by $N(\beta - \gamma N)$:

$$1 = A(\beta - \gamma N) + BN$$

Setting $N = 0$: $1 = A\beta \Rightarrow A = \frac{1}{\beta}$
Setting $N = \beta/\gamma$: $1 = B \cdot \frac{\beta}{\gamma} \Rightarrow B = \frac{\gamma}{\beta}$
Therefore:

$$\frac{1}{N(\beta - \gamma N)} = \frac{1}{\beta N} + \frac{\gamma}{\beta(\beta - \gamma N)}$$

Integrate both sides

$$\begin{aligned}
\int \left[\frac{1}{\beta N} + \frac{\gamma}{\beta(\beta - \gamma N)}\right] dN &= \int dt \\
\frac{1}{\beta} \log|N| - \frac{\gamma}{\beta\gamma} \log|\beta - \gamma N| &= t + C \\
\frac{1}{\beta} \log|N| - \frac{1}{\beta} \log|\beta - \gamma N| &= t + C \\
\frac{1}{\beta} \log\left|\frac{N}{\beta - \gamma N}\right| &= t + C
\end{aligned}$$

Solve for $N(t)$

Exponentiate both sides:

$$\frac{N}{\beta - \gamma N} = Ke^{\beta t}$$

where $K = e^{\beta C}$ is determined by initial condition.

Solve for N:

$$\begin{aligned}
N &= (\beta - \gamma N)Ke^{\beta t} \\
N &= \beta K e^{\beta t} - \gamma N K e^{\beta t} \\
N + \gamma N K e^{\beta t} &= \beta K e^{\beta t} \\
N(1 + \gamma K e^{\beta t}) &= \beta K e^{\beta t} \\
N(t) &= \frac{\beta K e^{\beta t}}{1 + \gamma K e^{\beta t}}
\end{aligned}$$

Apply initial condition

At $t = 0$, let $N(0) = N_0$:

$$N_0 = \frac{\beta K}{1 + \gamma K}$$

Solve for K:

$$\begin{aligned} N_0(1 + \gamma K) &= \beta K \\ N_0 + \gamma N_0 K &= \beta K \\ N_0 &= K(\beta - \gamma N_0) \\ K &= \frac{N_0}{\beta - \gamma N_0} \end{aligned}$$

General solution

Substituting back:

$$\boxed{N(t) = \frac{\beta N_0 e^{\beta t}}{\beta - \gamma N_0 + \gamma N_0 e^{\beta t}}}$$

This can be rewritten as:

$$N(t) = \frac{N_0 e^{\beta t}}{1 + \frac{\gamma N_0}{\beta}(e^{\beta t} - 1)}$$

XYZ Analysis of the Solution

- **STAGE X (What we derived):** An explicit solution for $N(t)$ in terms of initial population N_0, parameters β, γ, and time t.
- **STAGE Y (Why this form):** The solution is a logistic curve:
  - As $t \to \infty$: $N(t) \to \beta/\gamma$ (carrying capacity)
  - Growth rate decreases as N approaches carrying capacity
  - The exponential $e^{\beta t}$ drives growth, but is tempered by the denominator

  The fraction structure emerges from the competition between linear growth and quadratic death terms in the original ODE.
- **STAGE Z (What's next):** We'll evaluate this solution at discrete time steps separated by interval Δt, then derive a relationship between successive populations.

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3 Step 3: Discretize by Sampling at Fixed Time Intervals

Define discrete time points

Let $t_n = n\Delta t$ for $n = 0, 1, 2, \ldots$ where Δt is a fixed time step.

Define:

$$N_n = N(t_n) = N(n\Delta t)$$

Evaluate solution at t_n and t_{n+1}

At time $t_n = n\Delta t$:

$$N_n = \frac{\beta N_0 e^{\beta n\Delta t}}{\beta - \gamma N_0 + \gamma N_0 e^{\beta n\Delta t}}$$

At time $t_{n+1} = (n+1)\Delta t = n\Delta t + \Delta t$:

$$N_{n+1} = \frac{\beta N_0 e^{\beta(n+1)\Delta t}}{\beta - \gamma N_0 + \gamma N_0 e^{\beta(n+1)\Delta t}}$$

Factor out $e^{\beta n\Delta t}$ from numerator and denominator of N_{n+1}:

$$N_{n+1} = \frac{\beta N_0 e^{\beta n\Delta t} \cdot e^{\beta\Delta t}}{\beta - \gamma N_0 + \gamma N_0 e^{\beta n\Delta t} \cdot e^{\beta\Delta t}}$$

Relate N_{n+1} to N_n

From the expression for N_n, we can write:

$$\beta N_0 e^{\beta n\Delta t} = N_n(\beta - \gamma N_0 + \gamma N_0 e^{\beta n\Delta t})$$

Let $\alpha = e^{\beta\Delta t}$. Then:

$$N_{n+1} = \frac{\beta N_0 e^{\beta n\Delta t} \cdot \alpha}{\beta - \gamma N_0 + \gamma N_0 e^{\beta n\Delta t} \cdot \alpha}$$

Factor the denominator:

$$N_{n+1} = \frac{\beta N_0 e^{\beta n\Delta t} \cdot \alpha}{(\beta - \gamma N_0)(1-\alpha) + \beta - \gamma N_0 + \gamma N_0 e^{\beta n\Delta t}(\alpha - 1) + \beta - \gamma N_0 + \gamma N_0 e^{\beta n\Delta t}}$$

This is getting messy. Let's try a different approach.

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4 Step 3 (Alternative): Use Relation Between Consecutive Times

Direct ratio approach

From the solution:

$$N(t) = \frac{\beta N_0 e^{\beta t}}{\beta - \gamma N_0 + \gamma N_0 e^{\beta t}}$$

We can invert this to get:

$$\frac{1}{N(t)} = \frac{\beta - \gamma N_0 + \gamma N_0 e^{\beta t}}{\beta N_0 e^{\beta t}} = \frac{\beta - \gamma N_0}{\beta N_0 e^{\beta t}} + \frac{\gamma}{\beta}$$

Therefore:

$$\frac{1}{N(t)} - \frac{\gamma}{\beta} = \frac{\beta - \gamma N_0}{\beta N_0} e^{-\beta t}$$

At $t = n\Delta t$ and $t = (n+1)\Delta t$:

$$\frac{1}{N_n} - \frac{\gamma}{\beta} = \frac{\beta - \gamma N_0}{\beta N_0} e^{-\beta n\Delta t}$$
$$\frac{1}{N_{n+1}} - \frac{\gamma}{\beta} = \frac{\beta - \gamma N_0}{\beta N_0} e^{-\beta(n+1)\Delta t}$$

Divide the second by the first:

$$\frac{\frac{1}{N_{n+1}} - \frac{\gamma}{\beta}}{\frac{1}{N_n} - \frac{\gamma}{\beta}} = e^{-\beta\Delta t}$$

Let $\alpha = e^{-\beta\Delta t}$. Then:

$$\frac{1}{N_{n+1}} - \frac{\gamma}{\beta} = \alpha\left(\frac{1}{N_n} - \frac{\gamma}{\beta}\right)$$

Expanding:

$$\frac{1}{N_{n+1}} = \alpha \cdot \frac{1}{N_n} - \alpha \cdot \frac{\gamma}{\beta} + \frac{\gamma}{\beta} = \frac{\alpha}{N_n} + \frac{\gamma}{\beta}(1-\alpha)$$

Therefore:

$$N_{n+1} = \frac{1}{\frac{\alpha}{N_n} + \frac{\gamma}{\beta}(1-\alpha)} = \frac{N_n}{\alpha + \frac{\gamma N_n}{\beta}(1-\alpha)}$$

Simplify for $\Delta t = 1$

For unit time step $\Delta t = 1$:

$$\alpha = e^{-\beta}$$

Let's expand for small β (or equivalently, Taylor expand around the continuous limit). Actually, let's take a more direct approach.

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5 Step 3 (Direct Approach): From First Principles

Alternative derivation using ODE directly

The definition of derivative gives:

$$\frac{dN}{dt} = \lim_{\Delta t \to 0} \frac{N(t+\Delta t) - N(t)}{\Delta t}$$

For finite Δt, we approximate:

$$\frac{N(t+\Delta t) - N(t)}{\Delta t} \approx N(t)(\beta - \gamma N(t))$$

Rearranging:

$$N(t+\Delta t) \approx N(t) + \Delta t \cdot N(t)(\beta - \gamma N(t))$$

$$N(t+\Delta t) \approx N(t)[1 + \Delta t(\beta - \gamma N(t))]$$

Discretize with $t = n\Delta t$ and $\Delta t = 1$

Let $N_n = N(n \cdot 1) = N(n)$ and take $\Delta t = 1$:

$$N_{n+1} = N_n[1 + 1 \cdot (\beta - \gamma N_n)]$$

$$\boxed{N_{n+1} = N_n(1 + \beta - \gamma N_n)}$$

This is the desired discrete population map.

XYZ Analysis of Derivation

- **STAGE X (What we did):** Started from the ODE's definition as a derivative, approximated the derivative for finite time step Δt, then set $\Delta t = 1$ to get discrete time intervals.
- **STAGE Y (Why this works):** The continuous ODE $\dot{N} = N(\beta - \gamma N)$ tells us the instantaneous rate of change. For a small (but finite) time step Δt, the change in population is approximately:

  $$\Delta N = N(t + \Delta t) - N(t) \approx \dot{N} \cdot \Delta t = N(\beta - \gamma N)\Delta t$$

  This is a first-order Euler approximation. When we take $\Delta t = 1$ as our fundamental time unit (e.g., one day, one year), we get:

  $$N_{n+1} = N_n + N_n(\beta - \gamma N_n) = N_n(1 + \beta - \gamma N_n)$$

  The map represents: current population + change over one time unit.
- **STAGE Z (What this means):** This discrete map is:
  - An approximation to the continuous ODE when Δt is small
  - A model in its own right when time naturally occurs in discrete steps
  - Valid when $1 + \beta - \gamma N_n > 0$ (otherwise population becomes negative)
  - Simpler to iterate than solving the ODE repeatedly

  For β small (slow growth), the map closely approximates the ODE. For larger β, the map can exhibit different behavior including oscillations and chaos.

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6 Step 4: Verification from Exact Solution

Alternative verification

We can verify our discrete map by checking it's consistent with the exact solution in the limit of small Δt.

From the exact solution at t and $t + \Delta t$:

$$N(t) = \frac{\beta N_0 e^{\beta t}}{\beta - \gamma N_0 + \gamma N_0 e^{\beta t}}$$

Taking the ratio $N(t + \Delta t)/N(t)$:

$$\frac{N(t + \Delta t)}{N(t)} = \frac{e^{\beta(t+\Delta t)}}{e^{\beta t}} \cdot \frac{\beta - \gamma N_0 + \gamma N_0 e^{\beta t}}{\beta - \gamma N_0 + \gamma N_0 e^{\beta(t+\Delta t)}}$$

For small Δt, using $e^{\beta \Delta t} \approx 1 + \beta \Delta t$:

$$\frac{N(t + \Delta t)}{N(t)} \approx (1 + \beta \Delta t) \cdot \frac{\beta - \gamma N_0 + \gamma N_0 e^{\beta t}}{\beta - \gamma N_0 + \gamma N_0 e^{\beta t}(1 + \beta \Delta t)}$$

The denominator:

$$\beta - \gamma N_0 + \gamma N_0 e^{\beta t} + \gamma N_0 e^{\beta t} \beta \Delta t$$

Factor out $(\beta - \gamma N_0 + \gamma N_0 e^{\beta t})$:

$$(\beta - \gamma N_0 + \gamma N_0 e^{\beta t}) \left(1 + \frac{\gamma N_0 e^{\beta t} \beta \Delta t}{\beta - \gamma N_0 + \gamma N_0 e^{\beta t}}\right)$$

But from the solution: $N(t) = \frac{\beta N_0 e^{\beta t}}{\beta - \gamma N_0 + \gamma N_0 e^{\beta t}}$
So: $\beta - \gamma N_0 + \gamma N_0 e^{\beta t} = \frac{\beta N_0 e^{\beta t}}{N(t)}$
And: $\frac{\gamma N_0 e^{\beta t}}{\beta - \gamma N_0 + \gamma N_0 e^{\beta t}} = \frac{\gamma N(t)}{\beta}$
Therefore:

$$\frac{N(t+\Delta t)}{N(t)} \approx (1+\beta\Delta t) \cdot \frac{1}{1 + \frac{\gamma N(t)}{\beta}\beta\Delta t} \approx (1+\beta\Delta t)(1-\gamma N(t)\Delta t)$$

Expanding:

$$\frac{N(t+\Delta t)}{N(t)} \approx 1 + \beta\Delta t - \gamma N(t)\Delta t + O(\Delta t^2)$$

Thus:

$$N(t+\Delta t) \approx N(t)[1 + \Delta t(\beta - \gamma N(t))]$$

Setting $\Delta t = 1$ and using discrete notation:

$$\boxed{N_{n+1} = N_n(1 + \beta - \gamma N_n)} \quad \checkmark$$

XYZ Analysis of Verification

- **STAGE X (What we verified):** The discrete map derived from the derivative approximation matches the small-Δt limit of the exact solution ratio.
- **STAGE Y (Why this consistency):** Both approaches use the same fundamental ODE. The Euler approximation (derivative approach) is exactly the first-order Taylor expansion of the exact solution. They must agree to $O(\Delta t)$.
- **STAGE Z (What this tells us):**
  - The map $N_{n+1} = N_n(1 + \beta - \gamma N_n)$ is the natural discrete-time analog of the continuous ODE
  - For small time steps, map and ODE give nearly identical results
  - For larger time steps, the map and ODE can diverge significantly - the map is an independent model
  - The map is computationally simpler: no integration needed, just iteration

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7 Summary

Derivation pathway

1. **Start:** Continuous ODE $\dot{N} = N(\beta - \gamma N)$
2. **Approximate derivative:** For finite Δt:
$$\frac{N(t+\Delta t) - N(t)}{\Delta t} \approx N(t)(\beta - \gamma N(t))$$
3. **Rearrange:**
$$N(t+\Delta t) \approx N(t) + \Delta t \cdot N(t)(\beta - \gamma N(t))$$
4. **Factor:**
$$N(t+\Delta t) \approx N(t)[1 + \Delta t(\beta - \gamma N(t))]$$
5. **Discretize with $\Delta t = 1$:**
$$\boxed{N_{n+1} = N_n(1 + \beta - \gamma N_n)}$$

Physical interpretation

The discrete map says:

$$\text{Next population} = \text{Current population} \times \text{Growth factor}$$

where the growth factor is:

$$1 + \beta - \gamma N_n = 1 + \underbrace{\beta}_{\text{birth rate}} - \underbrace{\gamma N_n}_{\text{death rate}}$$

- If $\beta > \gamma N_n$: growth factor > 1, population increases
- If $\beta < \gamma N_n$: growth factor < 1, population decreases
- If $\beta = \gamma N_n$: growth factor $= 1$, population stable

Fixed point: $N^* = N^*(1 + \beta - \gamma N^*)$ gives $N^* = 0$ or $N^* = \beta/\gamma$ (carrying capacity).

Connection to logistic map

With rescaling $N_n = \frac{1+\beta}{\gamma} x_n$ and $r = 1 + \beta$:

$$N_{n+1} = N_n(1 + \beta - \gamma N_n) \quad \Rightarrow \quad x_{n+1} = r x_n (1 - x_n)$$

This is the famous logistic map studied in chaos theory.