Exercise Sheet 4: Maps
Question 3 - Complete Solution

Methods of Applied Mathematics

Problem Statement

The logistic map is given by:

$$x_{n+1} = rx_n(1 - x_n)$$

with fixed points at:

$$x_1^* = 0 \quad \text{and} \quad x_2^* = \frac{r-1}{r}$$

Tasks:

- Derive the linearization of the map about each fixed point
- Show that x_1^* is unstable for $r > 1$
- Show that x_2^* is stable for $r > 1$

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1 Step 1: Verify Fixed Points

Definition of fixed point

A fixed point x^* of map $x_{n+1} = f(x_n)$ satisfies:

$$x^* = f(x^*)$$

For the logistic map, $f(x) = rx(1 - x)$.

Check $x_1^* = 0$

$$\begin{aligned} f(0) &= r \cdot 0 \cdot (1 - 0) \\ &= 0 \quad \checkmark \end{aligned}$$

Check $x_2^* = (r - 1)/r$

$$\begin{aligned} f\left(\frac{r-1}{r}\right) &= r \cdot \frac{r-1}{r} \cdot \left(1 - \frac{r-1}{r}\right) \\ &= (r-1) \cdot \left(\frac{r-(r-1)}{r}\right) \\ &= (r-1) \cdot \frac{1}{r} \\ &= \frac{r-1}{r} \quad \checkmark \end{aligned}$$

XYZ Analysis of Fixed Points

- **STAGE X (What we have):** Two fixed points: $x_1^* = 0$ (always exists) and $x_2^* = (r-1)/r$ (exists for all r, coincides with x_1^* at $r = 1$).
- **STAGE Y (Why these are the only fixed points):** Solving $x = rx(1-x)$:

  $$x = rx - rx^2$$
  $$0 = x(r(1-x) - 1)$$

  This gives $x = 0$ or $r(1-x) = 1 \Rightarrow x = (r-1)/r$.
- **STAGE Z (What this means):** $x_1^* = 0$ represents extinction. x_2^* represents steady-state population. For $r > 1$, we have $x_2^* > 0$ (positive equilibrium exists). Stability determines which state the system reaches.

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2 Step 2: Compute Derivative

Linearization formula

For map $x_{n+1} = f(x_n)$, Taylor expansion near x^*:

$$f(x^* + \epsilon) = f(x^*) + f'(x^*)\epsilon + O(\epsilon^2)$$

Since $f(x^*) = x^*$:

$$\boxed{x_{n+1} - x^* \approx f'(x^*)(x_n - x^*)}$$

Compute $f'(x)$

For $f(x) = rx(1-x) = rx - rx^2$:

$$f'(x) = r - 2rx = r(1-2x)$$

Therefore:

$$\boxed{f'(x) = r(1-2x)}$$

XYZ Analysis

- **STAGE X (What we computed):** Derivative $f'(x) = r(1-2x)$ is linear in x, depends on parameter r.
- **STAGE Y (Why this form):** The logistic map is a parabola with:
  - Maximum at $x = 1/2$ where $f'(1/2) = 0$
  - $f'(0) = r$ (slope at origin)
  - $f'(1) = -r$ (slope at boundary)

  The derivative measures local stretching/contraction - key for stability.
- **STAGE Z (What this means):** If $|f'(x^*)| < 1$: nearby points contract toward x^* (stable). If $|f'(x^*)| > 1$: nearby points stretch away from x^* (unstable).

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3 Step 3: Stability of $x_1^* = 0$

Evaluate derivative

$$\lambda_1 = f'(0) = r(1 - 0) = r$$

Linearized map

Let $\epsilon_n = x_n - 0 = x_n$:

$$\epsilon_{n+1} = r\epsilon_n$$

Solution by iteration:

$$\epsilon_n = r^n \epsilon_0 \quad \Rightarrow \quad x_n = r^n x_0$$

Stability criterion

Fixed point stable if $|\lambda| < 1$, unstable if $|\lambda| > 1$.
For x_1^*: $|\lambda_1| = r$ (assuming $r > 0$)

$$\begin{aligned} r < 1: &\quad |\lambda_1| < 1 \quad \Rightarrow \quad \text{STABLE} \\ r = 1: &\quad |\lambda_1| = 1 \quad \Rightarrow \quad \text{NEUTRAL} \\ r > 1: &\quad |\lambda_1| > 1 \quad \Rightarrow \quad \boxed{\text{UNSTABLE}} \end{aligned}$$

Conclusion

$$\boxed{\text{For } r > 1: \quad x_1^* = 0 \text{ is UNSTABLE}}$$

XYZ Analysis

- **STAGE X (What we found):** Eigenvalue $\lambda_1 = r$. For $r > 1$: $|\lambda_1| > 1 \Rightarrow$ unstable.
- **STAGE Y (Why instability):** Solution $x_n = r^n x_0$ shows exponential growth. Each iteration multiplies by r. For $r > 1$, births exceed deaths, so any nonzero population grows away from extinction. Geometrically, parabola slope at origin ($f'(0) = r$) exceeds diagonal slope (1), pushing trajectories away.
- **STAGE Z (Biological meaning):** For $r > 1$, extinction is unstable - any small population grows. Makes sense: reproduction rate exceeds replacement, so population cannot stay at zero. System escapes toward x_2^*.

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4 Step 4: Stability of $x_2^* = (r - 1)/r$

Evaluate derivative

$$\begin{aligned} \lambda_2 = f'\left(\frac{r-1}{r}\right) &= r\left(1 - 2 \cdot \frac{r-1}{r}\right) \\ &= r\left(\frac{r - 2(r-1)}{r}\right) = r \cdot \frac{2-r}{r} = 2 - r \end{aligned}$$

$$\boxed{\lambda_2 = 2 - r}$$

Linearized map

Let $\epsilon_n = x_n - x_2^*$:

$$\epsilon_{n+1} = (2 - r)\epsilon_n$$

Solution:

$$\epsilon_n = (2 - r)^n \epsilon_0$$

Stability analysis

Need $|\lambda_2| = |2 - r| < 1$.

For $1 < r < 2$**:** $\lambda_2 = 2 - r \in (0, 1)$

$$|\lambda_2| < 1 \quad \Rightarrow \quad \text{STABLE (monotonic)}$$

For $r = 2$**:** $\lambda_2 = 0$

$$|\lambda_2| = 0 \quad \Rightarrow \quad \text{SUPERSTABLE}$$

For $2 < r < 3$**:** $\lambda_2 = 2 - r \in (-1, 0)$

$$|\lambda_2| < 1 \quad \Rightarrow \quad \text{STABLE (oscillatory)}$$

For $r = 3$**:** $\lambda_2 = -1$

$$|\lambda_2| = 1 \quad \Rightarrow \quad \text{BIFURCATION}$$

For $r > 3$**:** $\lambda_2 < -1$

$$|\lambda_2| > 1 \quad \Rightarrow \quad \text{UNSTABLE}$$

Conclusion

$$\boxed{\text{For } 1 < r < 3: \quad x_2^* = \frac{r-1}{r} \text{ is STABLE}}$$

XYZ Analysis

- **STAGE X (What we found):** Eigenvalue $\lambda_2 = 2 - r$. For $1 < r < 3$: $|\lambda_2| < 1 \Rightarrow$ stable.
- **STAGE Y (Why stability):** Solution $\epsilon_n = (2 - r)^n \epsilon_0$ shows exponential decay toward x_2^*:
  - For $1 < r < 2$: $\lambda_2 \in (0, 1)$ positive. Monotonic convergence (no oscillation).
  - For $2 < r < 3$: $\lambda_2 \in (-1, 0)$ negative. Oscillatory convergence (alternates above/below).
  - At $r = 2$: $\lambda_2 = 0$. Instantaneous convergence (superstable).
- **STAGE Z (Dynamic meaning):** For $1 < r < 3$, population converges to equilibrium $x_2^* = 1 - 1/r$. Bifurcations occur at:
  - $r = 1$: Transcritical bifurcation (extinction destabilizes, x_2^* born)
  - $r = 3$: Flip bifurcation (x_2^* destabilizes, period-2 orbit created)

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5 Summary

Main Results

For logistic map $x_{n+1} = rx_n(1 - x_n)$:
Linearization: $f'(x) = r(1 - 2x)$
Eigenvalues:

$$\lambda_1 = r$$
$$\lambda_2 = 2 - r$$

Stability for $r > 1$:

Fixed Point	Eigenvalue	Stability
$x_1^* = 0$	$\lambda_1 = r > 1$	Unstable
$x_2^* = \frac{r-1}{r}$	$\lambda_2 = 2 - r$	Stable ($1 < r < 3$)