# Exercise Sheet 1: Question 1
# Rewriting Systems as First Order ODEs
# Methods of Applied Mathematics [SEMT30006]

Complete Solutions with XYZ Methodology

## Problem Statement

Rewrite the following systems as first order ODEs, making it clear what state variables you have chosen and what their state-space is.

(a) $\dfrac{d^3u}{dt^3} - \dfrac{du}{dt} + \sin(u) = 0$

(b) $\dfrac{d^2u}{dt^2} + \dfrac{du}{dt} + u - 2v = 0, \quad \dfrac{d^2v}{dt^2} + \dfrac{dv}{dt} + v - 2u = 0$

(c) $\dfrac{d^2u}{dt^2} + \dfrac{du}{dt} - u + u^3 - v = 0, \quad \dfrac{dv}{dt} = u - v$

**CONTEXT FROM COURSE:** In dynamical systems theory (lecture notes pages 6-13), we always work with **first-order** systems of ODEs. Higher-order equations must be converted to first-order form by introducing new state variables. This canonical form allows us to:

- Analyze phase space structure
- Find equilibria and study their stability
- Construct phase portraits
- Apply existence and uniqueness theorems

# 1 Problem 1(a): Third-Order Scalar ODE

### Problem Statement

Convert to first-order form:

$$\frac{d^3u}{dt^3} - \frac{du}{dt} + \sin(u) = 0 \tag{1}$$

### Step 1: Identify the Order and Required State Variables

- **STAGE X (What we have):** A third-order ODE involving $u$, $\frac{du}{dt}$, and $\frac{d^3u}{dt^3}$. The highest derivative is of order 3.
- **STAGE Y (Why we need 3 state variables):** To convert an $n$-th order ODE to first-order form, we need exactly $n$ state variables. Each variable represents one derivative level:
  - One variable for $u$ itself
  - One variable for $\frac{du}{dt}$
  - One variable for $\frac{d^2u}{dt^2}$

  The third derivative $\frac{d^3u}{dt^3}$ will be expressed in terms of these state variables using the original equation.
- **STAGE Z (Strategy):** Introduce new variables for each derivative up to order $n-1=2$, then solve the original equation for the highest derivative.

### Step 2: Define State Variables

- **STAGE X (Introducing state variables):** Define:

$$x_1 = u \tag{2}$$

$$x_2 = \frac{du}{dt} = \dot{u} \tag{3}$$

$$x_3 = \frac{d^2u}{dt^2} = \ddot{u} \tag{4}$$

- **STAGE Y (Why this choice):** This is the **standard canonical form**. By defining each variable as the derivative of the previous one, we create a "chain" structure:

$$\dot{x}_1 = x_2 \tag{5}$$

$$\dot{x}_2 = x_3 \tag{6}$$

$$\dot{x}_3 = ? \tag{7}$$

  The first two equations come directly from our definitions. The third equation comes from the original ODE.
- **STAGE Z (State space):** Our state vector is:

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} u \\ \dot{u} \\ \ddot{u} \end{pmatrix} \tag{8}$$

### Step 3: Express Highest Derivative from Original Equation

- **STAGE X (Solving for $\frac{d^3u}{dt^3}$):** From the original equation:

$$\frac{d^3u}{dt^3} - \frac{du}{dt} + \sin(u) = 0 \tag{9}$$

  Rearrange:

$$\frac{d^3u}{dt^3} = \frac{du}{dt} - \sin(u) \tag{10}$$

- **STAGE Y (Converting to state variables):** Express in terms of $x_1, x_2, x_3$:

$$\begin{align} \frac{d^3u}{dt^3} &= \frac{du}{dt} - \sin(u) \tag{11} \\ &= x_2 - \sin(x_1) \tag{12} \end{align}$$

  Since $\dot{x}_3 = \frac{d}{dt}\left(\frac{d^2u}{dt^2}\right) = \frac{d^3u}{dt^3}$:

$$\dot{x}_3 = x_2 - \sin(x_1) \tag{13}$$

- **STAGE Z (Complete third equation):** We now have all three first-order equations.

### Step 4: Write Complete First-Order System

- **STAGE X (System of first-order ODEs):**

$$\begin{align} \frac{dx_1}{dt} &= x_2 \tag{14} \\ \frac{dx_2}{dt} &= x_3 \tag{15} \\ \frac{dx_3}{dt} &= x_2 - \sin(x_1) \tag{16} \end{align}$$

- **STAGE Y (Vector form):** More compactly, as a vector ODE:

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_2 \\ x_3 \\ x_2 - \sin(x_1) \end{pmatrix} \tag{17}$$

  Or using the notation $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) \quad \text{where} \quad \mathbf{f}(\mathbf{x}) = \begin{pmatrix} x_2 \\ x_3 \\ x_2 - \sin(x_1) \end{pmatrix} \tag{18}$$

- **STAGE Z (Key observations):**
  - This is a **3-dimensional** dynamical system
  - The system is **autonomous** (no explicit dependence on $t$)
  - The system is **nonlinear** due to $\sin(x_1)$

### Step 5: Specify State Space

- **STAGE X (State space definition):** The state space is the set of all possible values of $\mathbf{x} = (x_1, x_2, x_3)$.
- **STAGE Y (Determining constraints):** Since $u$ can be any real number and its derivatives can also be any real numbers (no physical constraints given), the state space is:

$$\mathcal{S} = \mathbb{R}^3 = \{(x_1, x_2, x_3) \in \mathbb{R}^3\} \tag{19}$$

  This is **three-dimensional Euclidean space**.

- **STAGE Z (Physical interpretation):** Each point $(x_1, x_2, x_3)$ in this space represents a complete state of the system:
  - $x_1$: position
  - $x_2$: velocity
  - $x_3$: acceleration

  Trajectories in this 3D space describe how the system evolves over time.

### Final Answer for Problem 1(a)

**Complete Solution**

**State Variables:**

$$x_1 = u \tag{20}$$
$$x_2 = \dot{u} \tag{21}$$
$$x_3 = \ddot{u} \tag{22}$$

**First-Order System:**

$$\boxed{\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ \dot{x}_3 &= x_2 - \sin(x_1) \end{aligned}} \tag{23}$$

**State Space:**

$$\boxed{\mathcal{S} = \mathbb{R}^3} \tag{24}$$

**Vector Form:**

$$\boxed{\dot{\mathbf{x}} = \begin{pmatrix} x_2 \\ x_3 \\ x_2 - \sin(x_1) \end{pmatrix}, \quad \mathbf{x} \in \mathbb{R}^3} \tag{25}$$

### Verification

**CHECK:** Let's verify by recovering the original equation.

- From $\dot{x}_1 = x_2$, we have $x_2 = \dot{u}$
- From $\dot{x}_2 = x_3$, we have $x_3 = \ddot{x}_2 = \ddot{u}$

- From $\dot{x}_3 = x_2 - \sin(x_1)$, we have:
$$\dddot{u} = \dot{u} - \sin(u) \tag{26}$$
  Rearranging:
$$\dddot{u} - \dot{u} + \sin(u) = 0 \quad \checkmark \tag{27}$$

# 2 Problem 1(b): Coupled Second-Order ODEs

### Problem Statement

Convert to first-order form:

$$\frac{d^2u}{dt^2} + \frac{du}{dt} + u - 2v = 0 \tag{28}$$

$$\frac{d^2v}{dt^2} + \frac{dv}{dt} + v - 2u = 0 \tag{29}$$

### Step 1: Count Total Order and Required State Variables

- **STAGE X (What we have):** Two coupled second-order ODEs. Each equation involves a variable and its first and second derivatives.
- **STAGE Y (Why we need 4 state variables):** We have:
  - Variable $u$ appears with derivatives up to order 2
  - Variable $v$ appears with derivatives up to order 2

  For each second-order variable, we need 2 state variables:
  - Total state variables needed: $2 + 2 = 4$
- **STAGE Z (Strategy):** Introduce state variables for $u$, $\dot{u}$, $v$, and $\dot{v}$. Express $\ddot{u}$ and $\ddot{v}$ from the original equations.

### Step 2: Define State Variables

- **STAGE X (Introducing state variables):** Define:

$$x_1 = u \tag{30}$$

$$x_2 = \frac{du}{dt} = \dot{u} \tag{31}$$

$$x_3 = v \tag{32}$$

$$x_4 = \frac{dv}{dt} = \dot{v} \tag{33}$$

- **STAGE Y (Why this ordering):** We group variables related to $u$ first $(x_1, x_2)$, then variables related to $v$ $(x_3, x_4)$. This is a natural organization, though any consistent ordering works.

  Alternative notation sometimes used: $(u, \dot{u}, v, \dot{v})$ or $(u, v, \dot{u}, \dot{v})$. The key is consistency.
- **STAGE Z (Initial equations):** From our definitions, we immediately have:

$$\dot{x}_1 = x_2 \quad \text{(definition of } x_2\text{)} \tag{34}$$

$$\dot{x}_3 = x_4 \quad \text{(definition of } x_4\text{)} \tag{35}$$

### Step 3: Solve Original Equations for Second Derivatives

- **STAGE X (First equation):** From $\frac{d^2u}{dt^2} + \frac{du}{dt} + u - 2v = 0$:

$$\frac{d^2u}{dt^2} = -\frac{du}{dt} - u + 2v \tag{36}$$

- **STAGE Y (Converting to state variables):** Express in terms of $x_1, x_2, x_3, x_4$:

$$\frac{d^2u}{dt^2} = -x_2 - x_1 + 2x_3 \tag{37}$$

Since $\dot{x}_2 = \frac{d}{dt}\left(\frac{du}{dt}\right) = \frac{d^2u}{dt^2}$:

$$\dot{x}_2 = -x_2 - x_1 + 2x_3 \tag{38}$$

- **STAGE Z (First half complete):** We now have $\dot{x}_1$ and $\dot{x}_2$.

### Step 4: Handle Second Equation

- **STAGE X (Second equation):** From $\frac{d^2v}{dt^2} + \frac{dv}{dt} + v - 2u = 0$:

$$\frac{d^2v}{dt^2} = -\frac{dv}{dt} - v + 2u \tag{39}$$

- **STAGE Y (Converting to state variables):** Express in terms of $x_1, x_2, x_3, x_4$:

$$\frac{d^2v}{dt^2} = -x_4 - x_3 + 2x_1 \tag{40}$$

Since $\dot{x}_4 = \frac{d}{dt}\left(\frac{dv}{dt}\right) = \frac{d^2v}{dt^2}$:

$$\dot{x}_4 = -x_4 - x_3 + 2x_1 \tag{41}$$

- **STAGE Z (System complete):** We now have all four equations for $\dot{x}_1, \dot{x}_2, \dot{x}_3, \dot{x}_4$.

### Step 5: Write Complete First-Order System

- **STAGE X (System of first-order ODEs):**

$$\frac{dx_1}{dt} = x_2 \tag{42}$$

$$\frac{dx_2}{dt} = -x_1 - x_2 + 2x_3 \tag{43}$$

$$\frac{dx_3}{dt} = x_4 \tag{44}$$

$$\frac{dx_4}{dt} = 2x_1 - x_3 - x_4 \tag{45}$$

- **STAGE Y (Vector form):**

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} x_2 \\ -x_1 - x_2 + 2x_3 \\ x_4 \\ 2x_1 - x_3 - x_4 \end{pmatrix} \tag{46}$$

- **STAGE Z (Matrix representation):** Since this system is **linear**, we can write it as $\dot{\mathbf{x}} = A\mathbf{x}$ where:

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & -1 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 2 & 0 & -1 & -1 \end{pmatrix} \tag{47}$$

This is a $4 \times 4$ constant coefficient system.

### Step 6: Specify State Space

- **STAGE X (State space definition):** The state space consists of all possible values of $\mathbf{x} = (x_1, x_2, x_3, x_4)$.
- **STAGE Y (Determining constraints):** The variables $u, \dot{u}, v, \dot{v}$ can all take any real values (no constraints given), so:

$$\mathcal{S} = \mathbb{R}^4 = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4\} \tag{48}$$

- **STAGE Z (Interpretation):** This is a **four-dimensional** phase space. Each point represents:
  - $x_1 = u$: position/state of first variable
  - $x_2 = \dot{u}$: velocity of first variable
  - $x_3 = v$: position/state of second variable
  - $x_4 = \dot{v}$: velocity of second variable

### Step 7: Analyze System Structure

- **STAGE X (Coupling structure):** Notice the coupling:
  - $\dot{x}_2$ depends on $x_1$ (from $u$), $x_2$ (from $\dot{u}$), and $x_3$ (from $v$)
  - $\dot{x}_4$ depends on $x_1$ (from $u$), $x_3$ (from $v$), and $x_4$ (from $\dot{v}$)
- **STAGE Y (Symmetry observation):** The original equations have a special symmetry:

$$\ddot{u} + \dot{u} + u - 2v = 0 \tag{49}$$
$$\ddot{v} + \dot{v} + v - 2u = 0 \tag{50}$$

  If we swap $u \leftrightarrow v$, the system remains unchanged. This is a **permutation symmetry**.

  In the matrix form:

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & -1 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 2 & 0 & -1 & -1 \end{pmatrix} \tag{51}$$

  The $(2, 1)$ and $(4, 3)$ entries are both $-1$ (self-coupling), while $(2, 3) = 2$ and $(4, 1) = 2$ (cross-coupling).

- **STAGE Z (Physical interpretation):** This could model two coupled oscillators where:
  - Each oscillator has damping (the $\dot{u}$ and $\dot{v}$ terms)
  - Each oscillator has restoring force (the $u$ and $v$ terms)
  - They are coupled with strength 2 (the $-2v$ and $-2u$ terms)

### Final Answer for Problem 1(b)

**Complete Solution**

**State Variables:**

$$x_1 = u \tag{52}$$
$$x_2 = \dot{u} \tag{53}$$
$$x_3 = v \tag{54}$$
$$x_4 = \dot{v} \tag{55}$$

**First-Order System:**

$$\boxed{\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 - x_2 + 2x_3 \\ \dot{x}_3 &= x_4 \\ \dot{x}_4 &= 2x_1 - x_3 - x_4 \end{aligned}} \tag{56}$$

**State Space:**

$$\boxed{\mathcal{S} = \mathbb{R}^4} \tag{57}$$

**Matrix Form:**

$$\boxed{\dot{\mathbf{x}} = A\mathbf{x}, \quad A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & -1 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 2 & 0 & -1 & -1 \end{pmatrix}, \quad \mathbf{x} \in \mathbb{R}^4} \tag{58}$$

### Verification

**CHECK:** Verify by recovering the original equations.

- From $\dot{x}_1 = x_2$: we have $x_2 = \dot{u}$
- From $\dot{x}_2 = -x_1 - x_2 + 2x_3$:

$$\ddot{u} = -u - \dot{u} + 2v \quad \Rightarrow \quad \ddot{u} + \dot{u} + u - 2v = 0 \quad \checkmark \tag{59}$$

- From $\dot{x}_3 = x_4$: we have $x_4 = \dot{v}$
- From $\dot{x}_4 = 2x_1 - x_3 - x_4$:

$$\ddot{v} = 2u - v - \dot{v} \quad \Rightarrow \quad \ddot{v} + \dot{v} + v - 2u = 0 \quad \checkmark \tag{60}$$

# 3 Problem 1(c): Mixed-Order Coupled System

## Problem Statement

Convert to first-order form:

$$\frac{d^2u}{dt^2} + \frac{du}{dt} - u + u^3 - v = 0 \tag{61}$$

$$\frac{dv}{dt} = u - v \tag{62}$$

## Step 1: Analyze the System Structure

- **STAGE X (What we have):** A **mixed-order** system:
  - First equation: second-order in $u$, involves $u, \dot{u}, \ddot{u}, v$
  - Second equation: first-order in $v$, involves $u, v, \dot{v}$
- **STAGE Y (Why this is different):** Unlike problem 1(b) where both equations were second-order, here we have:
  - Variable $u$: appears up to order 2 $\Rightarrow$ need 2 state variables
  - Variable $v$: appears up to order 1 $\Rightarrow$ need 1 state variable

  Total state variables needed: $2 + 1 = 3$

  The second equation is **already first-order**, so it will transfer directly with minimal modification.
- **STAGE Z (Strategy):** Introduce state variables for $u$, $\dot{u}$, and $v$. The second equation is already in the right form.

## Step 2: Define State Variables

- **STAGE X (Introducing state variables):** Define:

$$x_1 = u \tag{63}$$

$$x_2 = \frac{du}{dt} = \dot{u} \tag{64}$$

$$x_3 = v \tag{65}$$

- **STAGE Y (Why only 3 variables):** Since $v$ only appears as $v$ and $\dot{v}$ (never $\ddot{v}$), we don't need a separate variable for $\dot{v}$. The variable $x_3 = v$ is sufficient.

  Note: If $v$ had appeared with higher derivatives, we would need additional state variables.
- **STAGE Z (First equation from definition):** From the definition of $x_2$:

$$\dot{x}_1 = x_2 \tag{66}$$

## Step 3: Convert First Original Equation

- **STAGE X (Solving for $\ddot{u}$):** From $\frac{d^2u}{dt^2} + \frac{du}{dt} - u + u^3 - v = 0$:

$$\frac{d^2u}{dt^2} = -\frac{du}{dt} + u - u^3 + v \tag{67}$$

- **STAGE Y (Converting to state variables):** Express in terms of $x_1, x_2, x_3$:

$$\frac{d^2u}{dt^2} = -x_2 + x_1 - x_1^3 + x_3 \tag{68}$$

Since $\dot{x}_2 = \frac{d^2u}{dt^2}$:

$$\dot{x}_2 = -x_2 + x_1 - x_1^3 + x_3 \tag{69}$$

- **STAGE Z (Nonlinearity):** Note the **cubic nonlinearity** $x_1^3$. This makes the system **nonlinear**, which means:
  - Cannot be written as $\dot{\mathbf{x}} = A\mathbf{x}$ (no constant matrix $A$)
  - More complex dynamics possible (multiple equilibria, limit cycles, etc.)
  - Linearization will be needed for stability analysis

### Step 4: Convert Second Original Equation

- **STAGE X (Already first-order):** The equation $\frac{dv}{dt} = u - v$ is already first-order.
- **STAGE Y (Direct substitution):** Simply replace $v$ with $x_3$ and $u$ with $x_1$:

$$\frac{dx_3}{dt} = x_1 - x_3 \tag{70}$$

Or:

$$\dot{x}_3 = x_1 - x_3 \tag{71}$$

- **STAGE Z (Linear coupling):** This equation is linear in the state variables, even though the overall system is nonlinear due to the first equation.

### Step 5: Write Complete First-Order System

- **STAGE X (System of first-order ODEs):**

$$\frac{dx_1}{dt} = x_2 \tag{72}$$

$$\frac{dx_2}{dt} = -x_2 + x_1 - x_1^3 + x_3 \tag{73}$$

$$\frac{dx_3}{dt} = x_1 - x_3 \tag{74}$$

- **STAGE Y (Vector form):**

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_2 \\ -x_2 + x_1 - x_1^3 + x_3 \\ x_1 - x_3 \end{pmatrix} \tag{75}$$

Or using the notation $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) \quad \text{where} \quad \mathbf{f}(\mathbf{x}) = \begin{pmatrix} x_2 \\ -x_2 + x_1 - x_1^3 + x_3 \\ x_1 - x_3 \end{pmatrix} \tag{76}$$

- **STAGE Z (System classification):**
  - **Dimension:** 3 (three state variables)
  - **Autonomy:** Autonomous (no explicit $t$ dependence)
  - **Linearity:** Nonlinear (due to $x_1^3$ term)
  - **Coupling:** All three variables are coupled together

### Step 6: Specify State Space

- **STAGE X (State space definition):** The state space consists of all possible values of $\mathbf{x} = (x_1, x_2, x_3)$.
- **STAGE Y (Determining constraints):** No constraints are specified on $u$, $\dot{u}$, or $v$, so:

$$\mathcal{S} = \mathbb{R}^3 = \{(x_1, x_2, x_3) \in \mathbb{R}^3\} \tag{77}$$

- **STAGE Z (Geometric picture):** This is a **three-dimensional** phase space where:
  - $x_1 = u$: first dynamical variable
  - $x_2 = \dot{u}$: rate of change of first variable
  - $x_3 = v$: second dynamical variable

  Trajectories are curves in $\mathbb{R}^3$ that never cross (by uniqueness of solutions).

### Step 7: Find Equilibria

- **STAGE X (Equilibrium definition):** Equilibria occur where $\dot{\mathbf{x}} = \mathbf{0}$:

$$x_2 = 0 \tag{78}$$
$$-x_2 + x_1 - x_1^3 + x_3 = 0 \tag{79}$$
$$x_1 - x_3 = 0 \tag{80}$$

- **STAGE Y (Solving the system):** From the first equation: $x_2 = 0$

  From the third equation: $x_3 = x_1$

  Substitute into the second equation:

$$-0 + x_1 - x_1^3 + x_1 = 0 \tag{81}$$
$$2x_1 - x_1^3 = 0 \tag{82}$$
$$x_1(2 - x_1^2) = 0 \tag{83}$$

  This gives:

$$x_1 = 0 \quad \text{or} \quad x_1 = \pm\sqrt{2} \tag{84}$$

- **STAGE Z (Three equilibria):** The equilibrium points are:

$$\mathbf{x}_1^* = (0, 0, 0) \tag{85}$$
$$\mathbf{x}_2^* = (\sqrt{2}, 0, \sqrt{2}) \tag{86}$$
$$\mathbf{x}_3^* = (-\sqrt{2}, 0, -\sqrt{2}) \tag{87}$$

  In terms of original variables:

$$(u, \dot{u}, v) = (0, 0, 0) \tag{88}$$
$$(u, \dot{u}, v) = (\sqrt{2}, 0, \sqrt{2}) \tag{89}$$
$$(u, \dot{u}, v) = (-\sqrt{2}, 0, -\sqrt{2}) \tag{90}$$

  The nonlinearity creates multiple equilibria!

### Step 8: Physical Interpretation

- **STAGE X (System structure):** The term $u - u^3$ is a classic **Duffing-type** nonlinearity, suggesting this could model:
  - A nonlinear oscillator with cubic restoring force
  - Damping from the $\dot{u}$ term
  - Coupling to another variable $v$
- **STAGE Y (The role of $v$):** The variable $v$ satisfies $\dot{v} = u - v$, which means:
  - $v$ is "driven" by $u$
  - $v$ decays exponentially toward $u$ with rate 1
  - $v$ acts as a "filtered" or "smoothed" version of $u$
- **STAGE Z (Feedback loop):** There's a feedback structure:
  - $u$ influences $v$ through $\dot{v} = u - v$
  - $v$ influences $u$ through the $+v$ term in $\ddot{u}$
  - This creates a bidirectional coupling

### Final Answer for Problem 1(c)

**Complete Solution**

**State Variables:**

$$x_1 = u \tag{91}$$
$$x_2 = \dot{u} \tag{92}$$
$$x_3 = v \tag{93}$$

**First-Order System:**

$$\boxed{\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_2 + x_1 - x_1^3 + x_3 \\ \dot{x}_3 &= x_1 - x_3 \end{aligned}} \tag{94}$$

**State Space:**

$$\boxed{\mathcal{S} = \mathbb{R}^3} \tag{95}$$

**Vector Form:**

$$\boxed{\dot{\mathbf{x}} = \begin{pmatrix} x_2 \\ -x_2 + x_1 - x_1^3 + x_3 \\ x_1 - x_3 \end{pmatrix}, \quad \mathbf{x} \in \mathbb{R}^3} \tag{96}$$

**Equilibria:**

$$\boxed{(u, \dot{u}, v) \in \{(0,0,0),\ (\sqrt{2}, 0, \sqrt{2}),\ (-\sqrt{2}, 0, -\sqrt{2})\}} \tag{97}$$

### Verification

**CHECK:** Verify by recovering the original equations.

- From $\dot{x}_1 = x_2$: we have $\dot{u} = x_2$

- From $\dot{x}_2 = -x_2 + x_1 - x_1^3 + x_3$:

$$\ddot{u} = -\dot{u} + u - u^3 + v \tag{98}$$

$$\ddot{u} + \dot{u} - u + u^3 - v = 0 \quad \checkmark \tag{99}$$

- From $\dot{x}_3 = x_1 - x_3$:

$$\dot{v} = u - v \quad \checkmark \tag{100}$$

## Summary: Converting Higher-Order ODEs to First-Order Form

### General Procedure (The XYZ Method)

1. **STAGE X (Identify):**
   - Count the variables and their highest derivative orders
   - Determine total number of state variables needed
   - Recognize any equations already in first-order form
2. **STAGE Y (Define):**
   - Introduce state variables for each variable up to order $n-1$
   - Standard choice: $x_1 = u$, $x_2 = \dot{u}$, $x_3 = \ddot{u}$, etc.
   - Write immediate relations: $\dot{x}_i = x_{i+1}$
3. **STAGE Z (Solve):**
   - Solve original equations for highest derivatives
   - Express in terms of state variables only
   - Verify by substituting back into original equations

### Key Principles from Course Material

**From lecture notes (pages 6-13):**

1. **Phase space dimension:** For $m$ variables with maximum orders $n_1, n_2, \ldots, n_m$, the phase space has dimension $N = n_1 + n_2 + \cdots + n_m$.
2. **Autonomy:** A system is autonomous if $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ with no explicit $t$ dependence. All three problems here are autonomous.
3. **Linearity:** System is linear if $\mathbf{f}(\mathbf{x}) = A\mathbf{x}$ for some matrix $A$. Only problem (b) is linear.
4. **Trajectories:** Solutions are curves $\mathbf{x}(t)$ in phase space that cannot cross (by uniqueness).
5. **Equilibria:** Found by solving $\mathbf{f}(\mathbf{x}^*) = \mathbf{0}$.

### Comparison Table

| Problem | Original Orders | State Dim. | Linear? | Equilibria |
|---|---|---|---|---|
| 1(a) | $u$: 3rd order | 3 | No | Need analysis |
| 1(b) | $u, v$: 2nd order | 4 | Yes | $(0,0,0,0)$ only |
| 1(c) | $u$: 2nd, $v$: 1st | 3 | No | Three: see above |

### Connection to Future Analysis

Once in first-order form, we can:

- Find equilibria by solving $\dot{\mathbf{x}} = \mathbf{0}$
- Linearize about equilibria using Jacobian matrix
- Analyze stability using eigenvalues
- Construct phase portraits

- Apply existence and uniqueness theorems

These techniques will be explored in subsequent exercise problems.

**END OF QUESTION 1 SOLUTIONS**