# Asymptotics Problem 8.2: Complete Pedagogical Solution

Boundary Layer with Inhomogeneous ODE and Prandtl Matching

**Problem 1.** Obtain a one-term composite expansion for $\varepsilon \to 0$, for the solution of

$$\varepsilon \frac{d^2 f}{dx^2} - \frac{df}{dx} + \frac{f}{x+1} = 2, \quad 0 < x < 1, \quad \varepsilon > 0,$$

with boundary conditions $f(0) = 0$, $f(1) = 3$, using Prandtl's matching criterion.

## Solution: Step-by-Step Atomic Breakdown

### Step 1: Understanding the Problem Structure and Classification

**Strategy:** *We have a second-order linear ODE with:*

- *A small parameter $\varepsilon$ multiplying the highest derivative $f''$*
- *A first derivative term $-f'$ (coefficient is $-1$, which is* negative*)*
- *A variable coefficient term $f/(x+1)$*
- *An inhomogeneous term (RHS = 2)*
- *Two boundary conditions at $x = 0$ and $x = 1$*

*Our task is to find a one-term composite expansion using Prandtl's matching.*
**Justification:** *This is a singular perturbation problem because setting $\varepsilon = 0$ reduces the second-order ODE to a first-order ODE, which generically cannot satisfy two boundary conditions. Therefore, a boundary layer must form at one of the boundaries.*

*The key question is:* At which boundary does the layer form?

### Step 2: Determining the Boundary Layer Location

**Key Concept:** *From Lecture Notes §6.2.1, for an equation of the form $\varepsilon y'' + p(x)y' + q(x)y = r(x)$, the location of the boundary layer is determined by the* ***sign of the coefficient*** *$p(x)$:*

- *If $p(x) > 0$ throughout $[0, 1]$: boundary layer at $x = 0$*
- *If $p(x) < 0$ throughout $[0, 1]$: boundary layer at $x = 1$*

*The physical intuition: information "flows" in the direction of decreasing y along characteristics, and the layer forms where the outer solution cannot meet the imposed boundary condition.*
**Identifying $p(x)$ in our equation:**
Rewrite the equation in standard form:

$$\varepsilon f'' - f' + \frac{f}{x+1} = 2$$

Comparing with $\varepsilon f'' + p(x)f' + q(x)f = r(x)$:

$$p(x) = -1, \quad q(x) = \frac{1}{x+1}, \quad r(x) = 2$$

**Justification:** *Since $p(x) = -1 < 0$ for all $x \in [0, 1]$, the boundary layer is located at* $\boxed{x = 1}$.

*This means:*

- *The outer solution will satisfy the boundary condition at $x = 0$*
- *The inner solution (boundary layer) will be needed near $x = 1$ to satisfy $f(1) = 3$*
- *The boundary layer has width $O(\varepsilon)$*

## Step 3: Finding the Outer Solution

**What we do:** Neglect the $\varepsilon f''$ term to obtain the reduced (outer) equation.
**Technique:** *The outer expansion assumes $f(x, \varepsilon) = f_0(x) + \varepsilon f_1(x) + \cdots$ where $f_0$ satisfies the equation with $\varepsilon = 0$.*
Setting $\varepsilon = 0$:

$$-f_0' + \frac{f_0}{x+1} = 2$$

Rearranging:

$$f_0' - \frac{f_0}{x+1} = -2$$

### Step 3a: Solving the First-Order Linear ODE

**Technique:** *This is a first-order linear ODE of the form $f_0' + P(x)f_0 = Q(x)$ where $P(x) = -1/(x+1)$ and $Q(x) = -2$. Use the integrating factor method:*

$$\mu(x) = \exp\left(\int P(x)\, dx\right) = \exp\left(-\int \frac{dx}{x+1}\right) = \exp(-\ln(x+1)) = \frac{1}{x+1}$$

Multiply the ODE by $\mu(x) = 1/(x+1)$:

$$\frac{f_0'}{x+1} - \frac{f_0}{(x+1)^2} = \frac{-2}{x+1}$$

The left side is exactly $\frac{d}{dx}\left[\frac{f_0}{x+1}\right]$:

$$\frac{d}{dx}\left[\frac{f_0}{x+1}\right] = \frac{-2}{x+1}$$

Integrate both sides:

$$\frac{f_0}{x+1} = -2\ln(x+1) + C$$

Therefore:

$$f_0(x) = (x+1)\left[C - 2\ln(x+1)\right] = C(x+1) - 2(x+1)\ln(x+1)$$

### Step 3b: Applying the Boundary Condition at $x = 0$

**Justification:** *Since the boundary layer is at $x = 1$, the outer solution must satisfy the boundary condition at $x = 0$. (The boundary condition at $x = 1$ will be handled by the inner solution.)*
Apply $f_0(0) = 0$:

$$f_0(0) = C(0+1) - 2(0+1)\ln(0+1) = C - 2 \cdot 0 = C = 0$$

Therefore, the outer solution is:

$$\boxed{f_0(x) = -2(x+1)\ln(x+1)}$$

### Step 3c: Verifying the Outer Solution

**Technique:** *Always verify that the solution satisfies the original ODE:*

$$f_0(x) = -2(x+1)\ln(x+1)$$
$$f_0'(x) = -2\left[\ln(x+1) + (x+1)\cdot\frac{1}{x+1}\right] = -2\ln(x+1) - 2$$

Check the ODE $-f_0' + f_0/(x+1) = 2$:

$$\begin{aligned}
-f_0' + \frac{f_0}{x+1} &= -(-2\ln(x+1) - 2) + \frac{-2(x+1)\ln(x+1)}{x+1} \\
&= 2\ln(x+1) + 2 - 2\ln(x+1) \\
&= 2 \quad \checkmark
\end{aligned}$$

Check $f_0(0) = -2(1)\ln(1) = -2\cdot 0 = 0 \quad \checkmark$

### Step 3d: Evaluating the Outer Solution at $x = 1$

**Justification:** *We need to know the value of the outer solution at $x = 1$ for the matching process. This tells us how much the inner solution must "correct" to meet the actual boundary condition.*

$$f_0(1) = -2(1+1)\ln(1+1) = -2\cdot 2\cdot \ln(2) = -4\ln(2)$$

The boundary condition requires $f(1) = 3$, but the outer solution gives $f_0(1) = -4\ln(2) \approx -2.77$.
The **mismatch** is: $3 - (-4\ln 2) = 3 + 4\ln 2 \approx 5.77$.

## Step 4: Setting Up the Inner Solution at $x = 1$

**What we do:** Introduce a stretched coordinate near $x = 1$.
**Technique:** *For a boundary layer at $x = 1$ with width $O(\varepsilon)$, introduce the inner variable:*

$$X = \frac{x-1}{\varepsilon}$$

*Note: $X \leq 0$ for $x \in [0,1]$ since $x - 1 \leq 0$.*
*Define the inner function $F(X) = f(x)$.*

### Step 4a: Transforming the Derivatives

Using the chain rule:

$$\frac{df}{dx} = \frac{dF}{dX}\cdot\frac{dX}{dx} = \frac{1}{\varepsilon}F'$$
$$\frac{d^2 f}{dx^2} = \frac{1}{\varepsilon^2}F''$$

### Step 4b: Transforming the Equation

Also, we need to express $x$ in terms of $X$:

$$x = 1 + \varepsilon X \quad \Longrightarrow \quad x + 1 = 2 + \varepsilon X$$

Substitute into the original ODE $\varepsilon f'' - f' + f/(x+1) = 2$:

$$\varepsilon\cdot\frac{1}{\varepsilon^2}F'' - \frac{1}{\varepsilon}F' + \frac{F}{2+\varepsilon X} = 2$$

Simplifying:

$$\frac{1}{\varepsilon}F'' - \frac{1}{\varepsilon}F' + \frac{F}{2+\varepsilon X} = 2$$

Multiply through by $\varepsilon$:

$$F'' - F' + \frac{\varepsilon F}{2+\varepsilon X} = 2\varepsilon$$

#### Step 4c: Taking the Leading Order as $\varepsilon \to 0$

**Justification:** *As $\varepsilon \to 0$, the terms $\varepsilon F/(2+\varepsilon X) \to 0$ and $2\varepsilon \to 0$. The leading order inner equation is:*

$$F_0'' - F_0' = 0$$

*This is a homogeneous constant-coefficient ODE.*

### Step 5: Solving the Inner Equation

**The inner equation:** $F_0'' - F_0' = 0$
**Technique:** *Try $F_0 = e^{\lambda X}$:*

$$\lambda^2 e^{\lambda X} - \lambda e^{\lambda X} = 0 \quad \Longrightarrow \quad \lambda(\lambda - 1) = 0 \quad \Longrightarrow \quad \lambda = 0 \text{ or } \lambda = 1$$

The general solution is:

$$F_0(X) = A + Be^X$$

where $A$ and $B$ are constants to be determined.

#### Step 5a: Applying the Boundary Condition at $x = 1$

At $x = 1$, we have $X = (1-1)/\varepsilon = 0$. The boundary condition $f(1) = 3$ gives:

$$F_0(0) = A + Be^0 = A + B = 3$$

This gives us one equation: $A + B = 3$.

#### Step 5b: Rewriting the Solution in a Convenient Form

**Technique:** *It is useful to rewrite the general solution using the boundary condition. From $A + B = 3$, we have $B = 3 - A$. Substituting:*

$$F_0(X) = A + (3-A)e^X = A(1-e^X) + 3e^X$$

*Alternatively, rearranging:*

$$F_0(X) = 3e^X + A(1-e^X)$$

*This form clearly shows: $F_0(0) = 3 \cdot 1 + A \cdot 0 = 3$ ✓*

### Step 6: Applying Prandtl's Matching Criterion

**Key Concept:** *Prandtl's matching rule (Lecture Notes §6.1.2) states that the inner and outer solutions must agree in an overlap region. Formally:*

$$\lim_{x\to 1^-} f_0(x) = \lim_{X\to -\infty} F_0(X)$$

*The left side is the "inner limit of the outer solution" (approaching the boundary layer from outside). The right side is the "outer limit of the inner solution" (moving away from the boundary into the interior).*

**Step 6a: Computing the Inner Limit of the Outer Solution**

As $x \to 1^-$:

$$\lim_{x \to 1^-} f_0(x) = \lim_{x \to 1^-} [-2(x+1)\ln(x+1)] = -2(2)\ln(2) = -4\ln 2$$

**Step 6b: Computing the Outer Limit of the Inner Solution**

As $X \to -\infty$ (moving into the domain from $x = 1$):

$$F_0(X) = 3e^X + A(1 - e^X)$$

- $e^X \to 0$ as $X \to -\infty$
- $1 - e^X \to 1$ as $X \to -\infty$

Therefore:

$$\lim_{X \to -\infty} F_0(X) = 3 \cdot 0 + A \cdot 1 = A$$

**Step 6c: Applying the Matching Condition**

Prandtl's rule requires:

$$\lim_{x \to 1^-} f_0(x) = \lim_{X \to -\infty} F_0(X)$$

$$-4\ln 2 = A$$

Therefore:

$$\boxed{A = -4\ln 2}$$

## Step 7: Writing the Complete Inner Solution

With $A = -4\ln 2$:

$$F_0(X) = 3e^X + (-4\ln 2)(1 - e^X) = 3e^X - 4\ln 2 + 4(\ln 2)e^X$$

$$F_0(X) = (3 + 4\ln 2)e^X - 4\ln 2$$

Converting back to $x$-coordinates using $X = (x-1)/\varepsilon$:

$$\boxed{F_0 = (3 + 4\ln 2)\exp\left(\frac{x-1}{\varepsilon}\right) - 4\ln 2}$$

**Step 7a: Verifying the Inner Solution**

**Technique:** *Check boundary condition: At $x = 1$ ($X = 0$):*

$$F_0(0) = (3 + 4\ln 2) \cdot 1 - 4\ln 2 = 3 \quad \checkmark$$

*Check matching: As $X \to -\infty$:*

$$F_0 \to (3 + 4\ln 2) \cdot 0 - 4\ln 2 = -4\ln 2 = f_0(1) \quad \checkmark$$

### Step 8: Constructing the Composite Solution

**Technique:** *The composite solution is formed by adding the outer and inner solutions and subtracting their common limit to avoid double-counting (Lecture Notes §6.1.2 and §6.2.3):*

$$f_c(x) = f_0(x) + F_0(X) - (\textit{common limit})$$

*The common limit is the value both solutions approach in the overlap region, which is:*

$$\lim_{x \to 1} f_0(x) = \lim_{X \to -\infty} F_0(X) = -4 \ln 2$$

Therefore:

$$\begin{aligned} f_c(x) &= f_0(x) + F_0\left(\frac{x-1}{\varepsilon}\right) - (-4\ln 2) \\ &= -2(x+1)\ln(x+1) + \left[(3 + 4\ln 2)\exp\left(\frac{x-1}{\varepsilon}\right) - 4\ln 2\right] + 4\ln 2 \end{aligned}$$

Simplifying:

$$\boxed{f_c(x) = -2(x+1)\ln(x+1) + (3 + 4\ln 2)\exp\left(\frac{x-1}{\varepsilon}\right)}$$

### Step 9: Verifying the Composite Solution

#### Step 9a: Check Boundary Condition at $x = 0$

$$\begin{aligned} f_c(0) &= -2(1)\ln(1) + (3 + 4\ln 2)\exp\left(\frac{-1}{\varepsilon}\right) \\ &= 0 + (3 + 4\ln 2) \cdot e^{-1/\varepsilon} \end{aligned}$$

**Justification:** *For small $\varepsilon$, the term $e^{-1/\varepsilon}$ is exponentially small (essentially zero). Therefore:*

$$f_c(0) \approx 0 \quad \checkmark$$

*The boundary condition at $x = 0$ is satisfied up to exponentially small corrections.*

#### Step 9b: Check Boundary Condition at $x = 1$

$$\begin{aligned} f_c(1) &= -2(2)\ln(2) + (3 + 4\ln 2)\exp(0) \\ &= -4\ln 2 + 3 + 4\ln 2 \\ &= 3 \quad \checkmark \end{aligned}$$

#### Step 9c: Check Behavior in the Interior

For $0 < x < 1 - O(\varepsilon)$ (away from the boundary layer):

$$\exp\left(\frac{x-1}{\varepsilon}\right) \approx 0 \quad \text{(exponentially small)}$$

Therefore, in the interior:

$$f_c(x) \approx f_0(x) = -2(x+1)\ln(x+1) \quad \checkmark$$

## Step 10: Physical Interpretation and Summary

**Reflection:** *What have we learned from this problem?*

1. ***Boundary layer location:*** *The coefficient of* $f'$ *is* $p(x) = -1 < 0$*, so by the general theory (Lecture Notes §6.2.1), the boundary layer forms at* $x = 1$*.*

2. ***Outer solution behavior:*** *The outer solution* $f_0(x) = -2(x+1)\ln(x+1)$ *satisfies* $f_0(0) = 0$ *but gives* $f_0(1) = -4\ln 2 \neq 3$*. The "mismatch" of* $3 + 4\ln 2$ *must be corrected by the boundary layer.*

3. ***Inner solution structure:*** *The inner equation* $F_0'' - F_0' = 0$ *has solutions involving* $e^X$*. Only the decaying exponential (for* $X \to -\infty$*) can match to the outer solution.*

4. ***Prandtl matching:*** *The matching condition* $\lim_{x\to 1} f_0(x) = \lim_{X\to-\infty} F_0(X)$ *determines the free constant* $A = -4\ln 2$*.*

5. ***Composite solution:*** *The one-term composite expansion is:*

$$f_c(x) = -2(x+1)\ln(x+1) + (3 + 4\ln 2)\exp\left(\frac{x-1}{\varepsilon}\right)$$

6. ***Inhomogeneous term:*** *The presence of the RHS term "= 2" affects only the outer solution (making it nonzero). The inner solution at leading order is still homogeneous because the inhomogeneity is* $O(\varepsilon)$ *in the inner region.*

7. ***Boundary layer width:*** *The boundary layer has width* $O(\varepsilon)$*. For* $x < 1 - O(\varepsilon)$*, the exponential term is negligible, and the solution is well-approximated by the outer solution alone.*

**Complete Solution Summary:**

$$\begin{aligned} \text{Outer solution:}\quad & f_0(x) = -2(x+1)\ln(x+1) \\ \text{Inner solution:}\quad & F_0(X) = (3+4\ln 2)e^X - 4\ln 2, \quad X = \frac{x-1}{\varepsilon} \\ \text{Common limit:}\quad & -4\ln 2 \\ \text{Composite solution:}\quad & f_c(x) = -2(x+1)\ln(x+1) + (3+4\ln 2)\exp\left(\frac{x-1}{\varepsilon}\right) \end{aligned}$$

The boundary layer is at $x = 1$ with width $O(\varepsilon)$, determined by the negative coefficient of $f'$.

## Numerical Verification

**Reflection:** *For concreteness, let's compute some values. With* $\ln 2 \approx 0.693$*:*

- $3 + 4\ln 2 \approx 3 + 2.772 = 5.772$
- $-4\ln 2 \approx -2.772$

*At* $x = 0.5$ *(middle of domain), for small* $\varepsilon$*:*

$$\begin{aligned} f_c(0.5) &\approx -2(1.5)\ln(1.5) + 5.772 \cdot e^{-0.5/\varepsilon} \\ &\approx -3 \cdot 0.405 + (\textit{negligible}) \\ &\approx -1.22 \end{aligned}$$

*Near $x = 1$ (in the boundary layer), for $\varepsilon = 0.01$ and $x = 0.99$:*

$$
\begin{aligned}
f_c(0.99) &\approx -2(1.99)\ln(1.99) + 5.772 \cdot e^{-0.01/0.01} \\
&\approx -2.74 + 5.772 \cdot e^{-1} \\
&\approx -2.74 + 2.12 \\
&\approx -0.62
\end{aligned}
$$

*And at $x = 1$:* $f_c(1) = -2.77 + 5.77 = 3.00$ ✓