Asymptotics Problem 9.2: Complete Pedagogical Solution

Multiple Scales Method for the Lightly Damped Oscillator to Second Order

Problem 1. For the lightly damped linear oscillator

$$\ddot{x} + 2\varepsilon\dot{x} + \omega^2 x = 0$$

1. Use multiple-scale analysis to find the solution in second order, i.e. go one order beyond what was done in the lecture.

2. Compare your result with the exact solution.

Initial conditions: $x(0) = 1$, $\dot{x}(0) = 0$.

Solution: Step-by-Step Atomic Breakdown

Step 1: Understanding the Physical Problem

Strategy: *This is the canonical example of the multiple scales method from Lecture Notes §7.1.2, equations (394)–(406). The damped harmonic oscillator exhibits two distinct time scales:*

- **Fast oscillations** with period $\sim 2\pi/\omega$

- **Slow amplitude decay** with time scale $\sim 1/\varepsilon$

A regular perturbation expansion fails because it produces secular terms that grow like t, invalidating the expansion for $t = O(1/\varepsilon)$.

Justification: *Why does regular perturbation fail? If we try $x = x_0 + \varepsilon x_1 + \cdots$ with $x_0 = \cos(\omega t)$, the $O(\varepsilon)$ equation becomes $\ddot{x}_1 + \omega^2 x_1 = -2\dot{x}_0 = 2\omega\sin(\omega t)$. The right-hand side $\sin(\omega t)$ is a solution of the homogeneous equation, producing a secular term $t\cos(\omega t)$ in x_1. This is precisely the resonance phenomenon described in Lecture Notes §7.1.1, equation (393).*

Step 2: The Exact Solution (For Later Comparison)

Before applying multiple scales, let us record the exact solution for comparison in part (b).
Technique: *For the ODE $\ddot{x} + 2\varepsilon\dot{x} + \omega^2 x = 0$, try $x = e^{mt}$:*

$$m^2 + 2\varepsilon m + \omega^2 = 0.$$

Using the quadratic formula:

$$m = -\varepsilon \pm \sqrt{\varepsilon^2 - \omega^2} = -\varepsilon \pm i\sqrt{\omega^2 - \varepsilon^2}.$$

The general solution is:

$$x(t) = e^{-\varepsilon t}\left[A\cos\left(t\sqrt{\omega^2 - \varepsilon^2}\right) + B\sin\left(t\sqrt{\omega^2 - \varepsilon^2}\right)\right].$$

Applying initial conditions $x(0) = 1$ and $\dot{x}(0) = 0$:
From $x(0) = 1$: $A = 1$.
From $\dot{x}(0) = 0$: Differentiating and setting $t = 0$ gives $B = \frac{\varepsilon}{\sqrt{\omega^2 - \varepsilon^2}}$.

> **Exact Solution:**
> $$x(t) = e^{-\varepsilon t}\cos\left(t\sqrt{\omega^2 - \varepsilon^2}\right) + \frac{\varepsilon}{\sqrt{\omega^2 - \varepsilon^2}}e^{-\varepsilon t}\sin\left(t\sqrt{\omega^2 - \varepsilon^2}\right)$$

Step 3: Setting Up the Multiple Scales Framework

Step 3a: Defining the Time Scales

We introduce two time variables:

$$t_0 = t \quad \text{(fast time scale — captures oscillations)}$$
$$t_1 = \varepsilon t \quad \text{(slow time scale — captures amplitude decay)}$$

Justification: *The oscillations occur on the $O(1)$ time scale with frequency ω, so $t_0 = t$ is the natural fast variable. The damping causes the amplitude to decay on the time scale $1/\varepsilon$; when $t = O(1/\varepsilon)$, we have $t_1 = \varepsilon t = O(1)$. This is the standard two-scale setup from Lecture Notes §7.1.2.*

Step 3b: Transforming Derivatives

Treating $x = x(t_0, t_1)$ with both variables independent:

$$\frac{d}{dt} = \frac{\partial}{\partial t_0} + \varepsilon \frac{\partial}{\partial t_1}.$$

For the second derivative, we apply this operator twice:

$$\frac{d^2}{dt^2} = \left(\frac{\partial}{\partial t_0} + \varepsilon \frac{\partial}{\partial t_1} \right)^2$$
$$= \frac{\partial^2}{\partial t_0^2} + 2\varepsilon \frac{\partial^2}{\partial t_0 \partial t_1} + \varepsilon^2 \frac{\partial^2}{\partial t_1^2}.$$

Derivative Transformations:

$$\frac{d}{dt} = \frac{\partial}{\partial t_0} + \varepsilon \frac{\partial}{\partial t_1}$$

$$\frac{d^2}{dt^2} = \frac{\partial^2}{\partial t_0^2} + 2\varepsilon \frac{\partial^2}{\partial t_0 \partial t_1} + \varepsilon^2 \frac{\partial^2}{\partial t_1^2}$$

Step 4: Transforming the ODE

Original ODE: $\ddot{x} + 2\varepsilon \dot{x} + \omega^2 x = 0$.
Substituting the derivative transformations:

$$\left(\frac{\partial^2}{\partial t_0^2} + 2\varepsilon \frac{\partial^2}{\partial t_0 \partial t_1} + \varepsilon^2 \frac{\partial^2}{\partial t_1^2} \right) x + 2\varepsilon \left(\frac{\partial}{\partial t_0} + \varepsilon \frac{\partial}{\partial t_1} \right) x + \omega^2 x = 0.$$

Expanding and collecting by powers of ε:

$$\frac{\partial^2 x}{\partial t_0^2} + \omega^2 x + 2\varepsilon \left(\frac{\partial^2 x}{\partial t_0 \partial t_1} + \frac{\partial x}{\partial t_0} \right) + \varepsilon^2 \left(\frac{\partial^2 x}{\partial t_1^2} + 2 \frac{\partial x}{\partial t_1} \right) = 0.$$

Step 5: Expanding x in Powers of ε

We expand:

$$x(t_0, t_1) = x_0(t_0, t_1) + \varepsilon x_1(t_0, t_1) + \varepsilon^2 x_2(t_0, t_1) + O(\varepsilon^3).$$

Each x_n depends on both time scales.

Step 6: Transforming Initial Conditions

At $t = 0$: $t_0 = 0$ and $t_1 = 0$.
Condition $x(0) = 1$:

$$x_0(0,0) + \varepsilon x_1(0,0) + \varepsilon^2 x_2(0,0) + \cdots = 1.$$

Therefore:

$$x_0(0,0) = 1,$$
$$x_n(0,0) = 0 \quad \text{for } n \geq 1.$$

Condition $\dot{x}(0) = 0$:
Using $\frac{dx}{dt} = \frac{\partial x}{\partial t_0} + \varepsilon \frac{\partial x}{\partial t_1}$:

$$\frac{\partial x_0}{\partial t_0}(0,0) + \varepsilon \left(\frac{\partial x_1}{\partial t_0}(0,0) + \frac{\partial x_0}{\partial t_1}(0,0) \right) + O(\varepsilon^2) = 0.$$

Therefore:

$$\frac{\partial x_0}{\partial t_0}(0,0) = 0,$$
$$\frac{\partial x_1}{\partial t_0}(0,0) = -\frac{\partial x_0}{\partial t_1}(0,0).$$

Step 7: The $O(1)$ Problem

At order ε^0:

$$\frac{\partial^2 x_0}{\partial t_0^2} + \omega^2 x_0 = 0.$$

With initial conditions: $x_0(0,0) = 1$ and $\frac{\partial x_0}{\partial t_0}(0,0) = 0$.

Step 7a: Solving the Leading-Order Equation

Technique: *This is the simple harmonic oscillator equation in t_0, with t_1 as a parameter. The general solution is:*

$$x_0(t_0, t_1) = A(t_1)\cos(\omega t_0) + B(t_1)\sin(\omega t_0),$$

where $A(t_1)$ and $B(t_1)$ are undetermined functions of the slow time.

Step 7b: Applying Initial Conditions

From $x_0(0,0) = 1$:

$$A(0)\cos(0) + B(0)\sin(0) = A(0) = 1.$$

From $\frac{\partial x_0}{\partial t_0}(0,0) = 0$:

$$\frac{\partial x_0}{\partial t_0} = -\omega A(t_1)\sin(\omega t_0) + \omega B(t_1)\cos(\omega t_0).$$

At $(0,0)$: $\omega B(0) = 0$, so $B(0) = 0$.

Leading-Order Solution:

$$x_0(t_0, t_1) = A(t_1)\cos(\omega t_0) + B(t_1)\sin(\omega t_0)$$

with $A(0) = 1$ and $B(0) = 0$.

Step 8: The $O(\varepsilon)$ Problem

At order ε^1:

$$\frac{\partial^2 x_1}{\partial t_0^2} + \omega^2 x_1 = -2\frac{\partial^2 x_0}{\partial t_0 \partial t_1} - 2\frac{\partial x_0}{\partial t_0}.$$

Step 8a: Computing the Right-Hand Side

First term: $\frac{\partial x_0}{\partial t_0}$

$$\frac{\partial x_0}{\partial t_0} = -\omega A \sin(\omega t_0) + \omega B \cos(\omega t_0).$$

Second term: $\frac{\partial^2 x_0}{\partial t_0 \partial t_1}$

$$\frac{\partial^2 x_0}{\partial t_0 \partial t_1} = \frac{\partial}{\partial t_1}\left(\frac{\partial x_0}{\partial t_0}\right) = -\omega \frac{dA}{dt_1}\sin(\omega t_0) + \omega \frac{dB}{dt_1}\cos(\omega t_0).$$

The complete RHS:

$$\text{RHS} = -2\left(-\omega \frac{dA}{dt_1}\sin(\omega t_0) + \omega \frac{dB}{dt_1}\cos(\omega t_0)\right) - 2\left(-\omega A \sin(\omega t_0) + \omega B \cos(\omega t_0)\right)$$

$$= 2\omega \sin(\omega t_0)\left(\frac{dA}{dt_1} + A\right) - 2\omega \cos(\omega t_0)\left(\frac{dB}{dt_1} + B\right).$$

Step 8b: Identifying Secular Terms

Key Concept: *Secularity Condition for Oscillators (Lecture Notes §7.1.2): For the equation $\frac{\partial^2 x_1}{\partial t_0^2} + \omega^2 x_1 = f(t_0, t_1)$, secular terms arise from any component of f that is proportional to $\sin(\omega t_0)$ or $\cos(\omega t_0)$—the homogeneous solutions. To eliminate secular growth, we must set the coefficients of these resonant terms to zero.*

The RHS contains:

- Coefficient of $\sin(\omega t_0)$: $2\omega\left(\frac{dA}{dt_1} + A\right)$

- Coefficient of $\cos(\omega t_0)$: $-2\omega\left(\frac{dB}{dt_1} + B\right)$

Both are resonant with the homogeneous equation.

Step 8c: Eliminating Secular Terms

Setting both coefficients to zero:

$$\frac{dA}{dt_1} + A = 0, \tag{1}$$

$$\frac{dB}{dt_1} + B = 0. \tag{2}$$

Step 8d: Solving for $A(t_1)$ and $B(t_1)$

Equation for A: $\frac{dA}{dt_1} = -A$ with $A(0) = 1$.

$$A(t_1) = e^{-t_1}.$$

Equation for B: $\frac{dB}{dt_1} = -B$ with $B(0) = 0$.

$$B(t_1) = 0.$$

> **Slow-Time Functions (First Order):**
>
> $$A(t_1) = e^{-t_1}, \quad B(t_1) = 0$$

Step 8e: Solving for x_1

With secular terms eliminated, the RHS is zero:

$$\frac{\partial^2 x_1}{\partial t_0^2} + \omega^2 x_1 = 0.$$

The general solution is:

$$x_1(t_0, t_1) = C(t_1)\cos(\omega t_0) + D(t_1)\sin(\omega t_0),$$

where $C(t_1)$ and $D(t_1)$ are new undetermined functions.

Step 8f: Initial Conditions for x_1

From $x_1(0,0) = 0$:
$$C(0) = 0.$$

From $\frac{\partial x_1}{\partial t_0}(0,0) = -\frac{\partial x_0}{\partial t_1}(0,0)$:
We need $\frac{\partial x_0}{\partial t_1}$:

$$\frac{\partial x_0}{\partial t_1} = \frac{dA}{dt_1}\cos(\omega t_0) + \frac{dB}{dt_1}\sin(\omega t_0).$$

At $(0,0)$: $\frac{\partial x_0}{\partial t_1}(0,0) = \frac{dA}{dt_1}(0) = -A(0) = -1$.
And $\frac{\partial x_1}{\partial t_0} = -\omega C\sin(\omega t_0) + \omega D\cos(\omega t_0)$.
At $(0,0)$: $\omega D(0) = -(-1) = 1$, so $D(0) = \frac{1}{\omega}$.

Initial Conditions for x_1:

$$C(0) = 0, \quad D(0) = \frac{1}{\omega}$$

Step 9: The $O(\varepsilon^2)$ Problem

At order ε^2:

$$\frac{\partial^2 x_2}{\partial t_0^2} + \omega^2 x_2 = -2\frac{\partial^2 x_1}{\partial t_0 \partial t_1} - 2\frac{\partial x_1}{\partial t_0} - \frac{\partial^2 x_0}{\partial t_1^2} - 2\frac{\partial x_0}{\partial t_1}.$$

Step 9a: Computing Terms Involving x_0

With $A(t_1) = e^{-t_1}$ and $B(t_1) = 0$, we have $x_0 = e^{-t_1}\cos(\omega t_0)$.
Term: $\frac{\partial x_0}{\partial t_1}$

$$\frac{\partial x_0}{\partial t_1} = -e^{-t_1}\cos(\omega t_0).$$

Term: $\frac{\partial^2 x_0}{\partial t_1^2}$

$$\frac{\partial^2 x_0}{\partial t_1^2} = e^{-t_1}\cos(\omega t_0).$$

Contribution from x_0 terms:

$$-\frac{\partial^2 x_0}{\partial t_1^2} - 2\frac{\partial x_0}{\partial t_1} = -e^{-t_1}\cos(\omega t_0) - 2(-e^{-t_1}\cos(\omega t_0)) = e^{-t_1}\cos(\omega t_0).$$

Step 9b: Computing Terms Involving x_1

With $x_1 = C(t_1)\cos(\omega t_0) + D(t_1)\sin(\omega t_0)$:
Term: $\frac{\partial x_1}{\partial t_0}$

$$\frac{\partial x_1}{\partial t_0} = -\omega C \sin(\omega t_0) + \omega D \cos(\omega t_0).$$

Term: $\frac{\partial^2 x_1}{\partial t_0 \partial t_1}$

$$\frac{\partial^2 x_1}{\partial t_0 \partial t_1} = -\omega \frac{dC}{dt_1} \sin(\omega t_0) + \omega \frac{dD}{dt_1} \cos(\omega t_0).$$

Contribution from x_1 terms:

$$-2\frac{\partial^2 x_1}{\partial t_0 \partial t_1} - 2\frac{\partial x_1}{\partial t_0} = -2\left(-\omega \frac{dC}{dt_1} \sin(\omega t_0) + \omega \frac{dD}{dt_1} \cos(\omega t_0)\right)$$
$$- 2\left(-\omega C \sin(\omega t_0) + \omega D \cos(\omega t_0)\right)$$
$$= 2\omega \sin(\omega t_0)\left(\frac{dC}{dt_1} + C\right) - 2\omega \cos(\omega t_0)\left(\frac{dD}{dt_1} + D\right).$$

Step 9c: The Complete RHS of the $O(\varepsilon^2)$ Equation

$$\text{RHS} = 2\omega \sin(\omega t_0)\left(\frac{dC}{dt_1} + C\right) - 2\omega \cos(\omega t_0)\left(\frac{dD}{dt_1} + D\right) + e^{-t_1}\cos(\omega t_0).$$

Grouping by trigonometric functions:

$$\text{RHS} = 2\omega\left(\frac{dC}{dt_1} + C\right)\sin(\omega t_0) + \left(e^{-t_1} - 2\omega\left(\frac{dD}{dt_1} + D\right)\right)\cos(\omega t_0).$$

Step 9d: Eliminating Secular Terms at $O(\varepsilon^2)$

To avoid secular terms, both coefficients must vanish:
Coefficient of $\sin(\omega t_0)$:

$$\frac{dC}{dt_1} + C = 0, \quad C(0) = 0.$$

Solution: $C(t_1) = 0$.
Coefficient of $\cos(\omega t_0)$:

$$e^{-t_1} - 2\omega\left(\frac{dD}{dt_1} + D\right) = 0.$$

$$\frac{dD}{dt_1} + D = \frac{e^{-t_1}}{2\omega}, \quad D(0) = \frac{1}{\omega}.$$

Step 9e: Solving for $D(t_1)$

Technique: *This is a first-order linear ODE: $\frac{dD}{dt_1} + D = \frac{e^{-t_1}}{2\omega}$.*
The integrating factor is $\mu(t_1) = e^{t_1}$.
Multiply through: $e^{t_1}\frac{dD}{dt_1} + e^{t_1}D = \frac{1}{2\omega}$.
The LHS is $\frac{d}{dt_1}(e^{t_1}D)$.
Integrate: $e^{t_1}D = \frac{t_1}{2\omega} + K$.
Therefore: $D(t_1) = \frac{t_1}{2\omega}e^{-t_1} + Ke^{-t_1}$.
Applying $D(0) = \frac{1}{\omega}$:

$$D(0) = 0 + K = \frac{1}{\omega} \quad \Longrightarrow \quad K = \frac{1}{\omega}.$$

Therefore:
$$D(t_1) = \frac{1}{\omega}e^{-t_1} + \frac{t_1}{2\omega}e^{-t_1} = \frac{e^{-t_1}}{\omega}\left(1 + \frac{t_1}{2}\right).$$

> **Slow-Time Functions (Second Order):**
> $$C(t_1) = 0, \quad D(t_1) = \frac{1}{\omega}e^{-t_1} + \frac{t_1}{2\omega}e^{-t_1}$$

Step 10: Assembling the Complete Solution

Step 10a: The Solution in Terms of t_0 and t_1

We have:
$$x_0(t_0, t_1) = e^{-t_1}\cos(\omega t_0),$$
$$x_1(t_0, t_1) = D(t_1)\sin(\omega t_0) = \left(\frac{1}{\omega}e^{-t_1} + \frac{t_1}{2\omega}e^{-t_1}\right)\sin(\omega t_0).$$

The approximation through $O(\varepsilon)$ is:
$$x(t_0, t_1) = x_0 + \varepsilon x_1 + O(\varepsilon^2).$$

Step 10b: Converting to the Original Variable t

Substituting $t_0 = t$ and $t_1 = \varepsilon t$:
$$x(t) = e^{-\varepsilon t}\cos(\omega t) + \varepsilon\left(\frac{1}{\omega}e^{-\varepsilon t} + \frac{\varepsilon t}{2\omega}e^{-\varepsilon t}\right)\sin(\omega t) + O(\varepsilon^2)$$
$$= e^{-\varepsilon t}\cos(\omega t) + \frac{\varepsilon}{\omega}e^{-\varepsilon t}\sin(\omega t) + \frac{\varepsilon^2 t}{2\omega}e^{-\varepsilon t}\sin(\omega t) + O(\varepsilon^2).$$

> **Multiple Scales Approximation (Second Order):**
> $$x(t) = e^{-\varepsilon t}\cos(\omega t) + \left(\frac{\varepsilon}{\omega} + \frac{\varepsilon^2 t}{2\omega}\right)e^{-\varepsilon t}\sin(\omega t) + O(\varepsilon^2)$$

Step 11: Comparison with the Exact Solution

Step 11a: Expanding the Exact Solution

The exact solution is:
$$x_{\text{exact}}(t) = e^{-\varepsilon t}\cos\left(t\sqrt{\omega^2 - \varepsilon^2}\right) + \frac{\varepsilon}{\sqrt{\omega^2 - \varepsilon^2}}e^{-\varepsilon t}\sin\left(t\sqrt{\omega^2 - \varepsilon^2}\right).$$

Technique: *We need to expand $\sqrt{\omega^2 - \varepsilon^2}$ and the trigonometric functions for small ε. First, expand $\sqrt{\omega^2 - \varepsilon^2}$:*
$$\sqrt{\omega^2 - \varepsilon^2} = \omega\sqrt{1 - \frac{\varepsilon^2}{\omega^2}} = \omega\left(1 - \frac{\varepsilon^2}{2\omega^2} + O(\varepsilon^4)\right) = \omega - \frac{\varepsilon^2}{2\omega} + O(\varepsilon^4).$$

Step 11b: Expanding the Coefficient

The coefficient of the sine term:
$$\frac{\varepsilon}{\sqrt{\omega^2 - \varepsilon^2}} = \frac{\varepsilon}{\omega - \frac{\varepsilon^2}{2\omega} + O(\varepsilon^4)} = \frac{\varepsilon}{\omega}\left(1 - \frac{\varepsilon^2}{2\omega^2}\right)^{-1} = \frac{\varepsilon}{\omega}\left(1 + \frac{\varepsilon^2}{2\omega^2} + O(\varepsilon^4)\right).$$

To order ε^2:
$$\frac{\varepsilon}{\sqrt{\omega^2 - \varepsilon^2}} = \frac{\varepsilon}{\omega} + O(\varepsilon^3).$$

Step 11c: Expanding the Trigonometric Arguments

The argument of the trig functions is:

$$t\sqrt{\omega^2 - \varepsilon^2} = t\left(\omega - \frac{\varepsilon^2}{2\omega}\right) + O(\varepsilon^4) = \omega t - \frac{\varepsilon^2 t}{2\omega} + O(\varepsilon^4).$$

Using Taylor expansions around ωt:

$$\cos\left(\omega t - \frac{\varepsilon^2 t}{2\omega}\right) = \cos(\omega t)\cos\left(\frac{\varepsilon^2 t}{2\omega}\right) + \sin(\omega t)\sin\left(\frac{\varepsilon^2 t}{2\omega}\right)$$

$$= \cos(\omega t)\left(1 + O(\varepsilon^4)\right) + \sin(\omega t)\left(\frac{\varepsilon^2 t}{2\omega} + O(\varepsilon^4)\right)$$

$$= \cos(\omega t) + \frac{\varepsilon^2 t}{2\omega}\sin(\omega t) + O(\varepsilon^4).$$

$$\sin\left(\omega t - \frac{\varepsilon^2 t}{2\omega}\right) = \sin(\omega t)\cos\left(\frac{\varepsilon^2 t}{2\omega}\right) - \cos(\omega t)\sin\left(\frac{\varepsilon^2 t}{2\omega}\right)$$

$$= \sin(\omega t) - \frac{\varepsilon^2 t}{2\omega}\cos(\omega t) + O(\varepsilon^4).$$

Step 11d: Assembling the Expanded Exact Solution

$$x_{\text{exact}}(t) = e^{-\varepsilon t}\left[\cos(\omega t) + \frac{\varepsilon^2 t}{2\omega}\sin(\omega t)\right]$$

$$+ \frac{\varepsilon}{\omega}e^{-\varepsilon t}\left[\sin(\omega t) - \frac{\varepsilon^2 t}{2\omega}\cos(\omega t)\right] + O(\varepsilon^3)$$

$$= e^{-\varepsilon t}\cos(\omega t) + \frac{\varepsilon}{\omega}e^{-\varepsilon t}\sin(\omega t) + \frac{\varepsilon^2 t}{2\omega}e^{-\varepsilon t}\sin(\omega t) + O(\varepsilon^3).$$

Step 11e: Comparison

Multiple scales result:

$$x_{\text{MS}}(t) = e^{-\varepsilon t}\cos(\omega t) + \frac{\varepsilon}{\omega}e^{-\varepsilon t}\sin(\omega t) + \frac{\varepsilon^2 t}{2\omega}e^{-\varepsilon t}\sin(\omega t) + O(\varepsilon^2).$$

Expanded exact solution:

$$x_{\text{exact}}(t) = e^{-\varepsilon t}\cos(\omega t) + \frac{\varepsilon}{\omega}e^{-\varepsilon t}\sin(\omega t) + \frac{\varepsilon^2 t}{2\omega}e^{-\varepsilon t}\sin(\omega t) + O(\varepsilon^3).$$

> **Conclusion:** The multiple scales approximation through second order **exactly matches** the expansion of the exact solution through order ε^2. The methods agree term by term!

Step 12: Discussion of the Secular Term

Reflection: *The term $\frac{\varepsilon^2 t}{2\omega}e^{-\varepsilon t}\sin(\omega t)$ in our solution appears to be a secular term because of the factor of t. However, this is* **not** *a problematic secular term because:*

1. *The factor $e^{-\varepsilon t}$ dominates for large t, ensuring the entire term decays.*

2. *This term arises from the expansion of the exact frequency $\sqrt{\omega^2 - \varepsilon^2} \approx \omega - \frac{\varepsilon^2}{2\omega}$. The exact solution oscillates at a slightly different frequency, and expanding this frequency shift produces apparent secular terms.*

3. *From Lecture Notes §7.1.2, the solution notes that to avoid this secular term, we would need to introduce a third time scale $t_2 = \varepsilon^2 t$ or modify the fast time scale to $t_0 = \delta(\varepsilon)t$ where δ captures the frequency correction.*

The appearance of this secular term at $O(\varepsilon^2)$ indicates the limitation of the two-scale expansion. For longer times or higher accuracy, additional slow time scales would be needed.

Final Summary

Complete Solution for Problem 9.2:

Given: $\ddot{x} + 2\varepsilon\dot{x} + \omega^2 x = 0$ with $x(0) = 1$, $\dot{x}(0) = 0$.

Time scales: $t_0 = t$ (fast), $t_1 = \varepsilon t$ (slow).

Leading order ($O(1)$):

$$x_0 = A(t_1)\cos(\omega t_0) + B(t_1)\sin(\omega t_0)$$

Secularity conditions at $O(\varepsilon)$:

$$A(t_1) = e^{-t_1}, \quad B(t_1) = 0$$

First correction ($O(\varepsilon)$):

$$x_1 = C(t_1)\cos(\omega t_0) + D(t_1)\sin(\omega t_0)$$

Secularity conditions at $O(\varepsilon^2)$:

$$C(t_1) = 0, \quad D(t_1) = \frac{1}{\omega}e^{-t_1} + \frac{t_1}{2\omega}e^{-t_1}$$

Second-order approximation:

$$x(t) = e^{-\varepsilon t}\cos(\omega t) + \left(\frac{\varepsilon}{\omega} + \frac{\varepsilon^2 t}{2\omega}\right)e^{-\varepsilon t}\sin(\omega t)$$

Exact solution:

$$x(t) = e^{-\varepsilon t}\cos(t\sqrt{\omega^2 - \varepsilon^2}) + \frac{\varepsilon}{\sqrt{\omega^2 - \varepsilon^2}}e^{-\varepsilon t}\sin(t\sqrt{\omega^2 - \varepsilon^2})$$

Verification: Expanding the exact solution to $O(\varepsilon^2)$ reproduces the multiple scales result exactly.

Connection to Lecture Notes

Reflection: *This problem extends the analysis in Lecture Notes §7.1.2, equations (394)–(406), which covers the damped oscillator to first order. The key additional insights from going to second order are:*

1. ***Secularity conditions propagate:*** *At each order, new undetermined functions appear, and secularity conditions at the next order determine them.*

2. **Frequency corrections appear at** $O(\varepsilon^2)$**:** *The exact frequency* $\sqrt{\omega^2 - \varepsilon^2}$ *differs from* ω *by* $O(\varepsilon^2)$. *This correction manifests as the secular-looking term* $\varepsilon^2 t$ *in our expansion.*

3. **Limitations of two-scale expansions:** *As noted in the solutions, the secular term at* $O(\varepsilon^2)$ *could be removed by introducing a modified fast time scale that incorporates the frequency correction. This would require* $t_0 = \delta(\varepsilon)t$ *with* $\delta = \omega - \frac{\varepsilon^2}{2\omega} + O(\varepsilon^4)$.

4. **Physical interpretation:** *The* $e^{-\varepsilon t}$ *decay captures amplitude attenuation, while the modified frequency* $\sqrt{\omega^2 - \varepsilon^2}$ *shows that damping also slightly reduces the oscillation frequency.*