Asymptotics Problem Sheet 6, Question 5

Eigenvalue Perturbation for a 2×2 Matrix

Following Lecture Notes Section 5.2.4:
Fredholm Alternative in Linear Algebra

Problem. Estimate the eigenvalues and eigenvectors of the matrix

$$A = \begin{bmatrix} 1 & 1-\epsilon \\ \epsilon - 1 & 1 \end{bmatrix}$$

for $\epsilon \ll 1$. Compare your results with the exact solution.

Overview and Strategy

Strategy 1. *This problem requires a* ***perturbative approach to eigenvalue problems*** *as developed in Lecture Notes Section 5.2.4. We identify that:*

i. The matrix can be decomposed as $A = C + \epsilon D$ (unperturbed + perturbation)

ii. This fits the framework: $(C + \epsilon D)x = \lambda x$

iii. We solve the unperturbed problem first, then compute corrections using the Fredholm alternative

iv. Since C is not *self-adjoint, we need both right and left eigenvectors*

1 Step 1: Matrix Decomposition

Why We Do This

Reason: To apply perturbation theory, we must separate the problem into an exactly solvable unperturbed part and a small perturbation.

What We Have

The given matrix is:

$$A = \begin{bmatrix} 1 & 1-\epsilon \\ \epsilon - 1 & 1 \end{bmatrix}$$

What We Do

Rewrite by collecting terms with and without ϵ:

$$\begin{aligned} A &= \begin{bmatrix} 1 & 1-\epsilon \\ \epsilon - 1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} + \epsilon \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \end{aligned}$$

Why This Form

This is precisely the form required by Lecture Notes Eq. (318):

$$Cx + \epsilon Dx = \lambda x$$

where we identify:

$$\boxed{C = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}}$$

2 Step 2: Solve the Unperturbed Problem

Why We Do This

Reason: The perturbative expansion begins with the $O(\epsilon^0)$ problem, which determines λ_0 and x_0 (Lecture Notes, page 50).

What We Need

We solve: $Cx_0 = \lambda_0 x_0$

Computing the Characteristic Polynomial

The characteristic equation is $\det(C - \lambda_0 I) = 0$:

$$\det \begin{bmatrix} 1 - \lambda_0 & 1 \\ -1 & 1 - \lambda_0 \end{bmatrix} = 0$$

Expanding the determinant:

$$\begin{aligned} (1 - \lambda_0)(1 - \lambda_0) - (1)(-1) &= 0 \\ (1 - \lambda_0)^2 + 1 &= 0 \\ 1 - 2\lambda_0 + \lambda_0^2 + 1 &= 0 \\ \lambda_0^2 - 2\lambda_0 + 2 &= 0 \end{aligned}$$

Why This Leads to Complex Eigenvalues

Using the quadratic formula:

$$\lambda_0 = \frac{2 \pm \sqrt{4 - 8}}{2} = \frac{2 \pm \sqrt{-4}}{2} = \frac{2 \pm 2i}{2} = 1 \pm i$$

Key Insight 1. *The discriminant is negative, yielding complex conjugate eigenvalues. This means the unperturbed matrix C represents a rotation with scaling, not a Hermitian operator.*

$$\boxed{\lambda_{0,1} = 1 + i, \quad \lambda_{0,2} = 1 - i}$$

3 Step 3: Find Right Eigenvectors x_0

For $\lambda_{0,1} = 1 + i$

What We Need: Solve $(C - \lambda_0 I)x_0 = 0$

Setting up the system:

$$\begin{bmatrix} 1-(1+i) & 1 \\ -1 & 1-(1+i) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -i & 1 \\ -1 & -i \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

From the first equation:

$$-ix_1 + x_2 = 0 \quad \Rightarrow \quad x_2 = ix_1$$

Why We Can Choose $x_1 = 1$: Eigenvectors are determined up to a scalar multiple. Setting $x_1 = 1$ gives:

$$\boxed{x_{0,1} = \begin{bmatrix} 1 \\ i \end{bmatrix}}$$

Verification

Why We Check: To ensure our eigenvector is correct.

$$\begin{aligned} Cx_{0,1} &= \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ i \end{bmatrix} \\ &= \begin{bmatrix} 1+i \\ -1+i \end{bmatrix} \\ &= (1+i) \begin{bmatrix} 1 \\ i \end{bmatrix} = \lambda_{0,1} x_{0,1} \quad \checkmark \end{aligned}$$

For $\lambda_{0,2} = 1 - i$

By similar calculation (or by complex conjugation):

$$\boxed{x_{0,2} = \begin{bmatrix} 1 \\ -i \end{bmatrix}}$$

4 Step 4: Find Left Eigenvectors y_0

Why We Need Left Eigenvectors

Key Insight 2. ***Critical point from Lecture Notes (page 50):*** *Since C is NOT self-adjoint (not Hermitian), we cannot use the simplified formula. We must find y_0 satisfying:*

$$C^* y_0 = \lambda_0 y_0$$

where $C^* = C^T$ for real matrices (Lecture Notes, Example on page 50).

Computing C^T

$$C^T = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

For $\lambda_{0,1} = 1 + i$

We solve: $(C^T - \lambda_0 I)y_0 = 0$

$$\begin{bmatrix} 1-(1+i) & -1 \\ 1 & 1-(1+i) \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -i & -1 \\ 1 & -i \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

From the first equation:

$$-iy_1 - y_2 = 0 \quad \Rightarrow \quad y_2 = -iy_1$$

Setting $y_1 = 1$:

$$\boxed{y_{0,1} = \begin{bmatrix} 1 \\ -i \end{bmatrix}}$$

For $\lambda_{0,2} = 1 - i$

Similarly:

$$\boxed{y_{0,2} = \begin{bmatrix} 1 \\ i \end{bmatrix}}$$

5 Step 5: Apply First-Order Perturbation Formula

The Formula (Lecture Notes, page 50)

For non-self-adjoint case:

$$\lambda_1 = \frac{\langle Dx_0, y_0 \rangle}{\langle x_0, y_0 \rangle}$$

where the inner product for real/complex vectors is: $\langle u, v \rangle = u_1 v_1 + u_2 v_2$ (standard dot product).

For $\lambda_{0,1} = 1 + i$

Step 5a: Compute Dx_0

What we compute:

$$Dx_{0,1} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ i \end{bmatrix}$$

Why this calculation matters: This gives us the perturbation D acting on the unperturbed eigenvector.

Computing entry by entry:

$$(Dx_{0,1})_1 = 0 \cdot 1 + (-1) \cdot i = -i$$
$$(Dx_{0,1})_2 = 1 \cdot 1 + 0 \cdot i = 1$$

$$\boxed{Dx_{0,1} = \begin{bmatrix} -i \\ 1 \end{bmatrix}}$$

Step 5b: Compute $\langle Dx_0, y_0 \rangle$

What we compute:

$$\langle Dx_{0,1}, y_{0,1} \rangle = \begin{bmatrix} -i \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -i \end{bmatrix}$$

Why: This is the numerator of our perturbation formula.
Computing:

$$= (-i)(1) + (1)(-i) = -i - i = -2i$$

$$\boxed{\langle Dx_{0,1}, y_{0,1} \rangle = -2i}$$

Step 5c: Compute $\langle x_0, y_0 \rangle$

What we compute:

$$\langle x_{0,1}, y_{0,1} \rangle = \begin{bmatrix} 1 \\ i \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -i \end{bmatrix}$$

Why: This is the denominator, ensuring proper normalization.
Computing:

$$= (1)(1) + (i)(-i) = 1 - i^2 = 1 - (-1) = 2$$

$$\boxed{\langle x_{0,1}, y_{0,1} \rangle = 2}$$

Step 5d: Compute λ_1

Applying the formula:

$$\lambda_1 = \frac{-2i}{2} = -i$$

Key Insight 3. *The first-order correction is purely imaginary, which will modify the imaginary part of the eigenvalue.*

Step 5e: Assemble the Perturbed Eigenvalue

The perturbative expansion (Lecture Notes, page 49):

$$\lambda(\epsilon) = \lambda_0 + \epsilon\lambda_1 + O(\epsilon^2)$$

Therefore:

$$\boxed{\lambda_1(\epsilon) = (1+i) + \epsilon(-i) + O(\epsilon^2) = 1 + i(1-\epsilon) + O(\epsilon^2)}$$

For $\lambda_{0,2} = 1 - i$

By similar calculation (or by symmetry):

$$Dx_{0,2} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -i \end{bmatrix} = \begin{bmatrix} i \\ 1 \end{bmatrix}$$

$$\langle Dx_{0,2}, y_{0,2} \rangle = \begin{bmatrix} i \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ i \end{bmatrix} = i + i = 2i$$

$$\langle x_{0,2}, y_{0,2} \rangle = \begin{bmatrix} 1 \\ -i \end{bmatrix} \cdot \begin{bmatrix} 1 \\ i \end{bmatrix} = 1 + 1 = 2$$

$$\lambda_1 = \frac{2i}{2} = i$$

$$\boxed{\lambda_2(\epsilon) = (1-i) + \epsilon(i) + O(\epsilon^2) = 1 - i(1-\epsilon) + O(\epsilon^2)}$$

6 Step 6: Exact Solution for Comparison

Why We Compute This

Reason: To verify that our perturbative approach gives the correct leading-order behavior (Lecture Notes methodology: always compare approximate with exact when possible).

Setting Up the Characteristic Equation

$$\det(A - \lambda I) = \det \begin{bmatrix} 1-\lambda & 1-\epsilon \\ \epsilon - 1 & 1-\lambda \end{bmatrix} = 0$$

Computing the Determinant

Expanding:

$$\begin{aligned} &(1-\lambda)(1-\lambda) - (1-\epsilon)(\epsilon-1) \\ &= (1-\lambda)^2 - (1-\epsilon)(\epsilon-1) \end{aligned}$$

Why we simplify $(1-\epsilon)(\epsilon-1)$:

$$(1-\epsilon)(\epsilon-1) = -(1-\epsilon)(1-\epsilon) = -(1-\epsilon)^2$$

Therefore:

$$\begin{aligned} &= (1-\lambda)^2 + (1-\epsilon)^2 \\ &= 1 - 2\lambda + \lambda^2 + 1 - 2\epsilon + \epsilon^2 \\ &= \lambda^2 - 2\lambda + (2 - 2\epsilon + \epsilon^2) \end{aligned}$$

Solving the Quadratic

$$\lambda = \frac{2 \pm \sqrt{4 - 4(2 - 2\epsilon + \epsilon^2)}}{2}$$

Simplifying the discriminant:

$$\begin{aligned} 4 - 4(2 - 2\epsilon + \epsilon^2) &= 4 - 8 + 8\epsilon - 4\epsilon^2 \\ &= -4 + 8\epsilon - 4\epsilon^2 \\ &= -4(1 - 2\epsilon + \epsilon^2) \\ &= -4(1-\epsilon)^2 \end{aligned}$$

Taking the square root:

$$\sqrt{-4(1-\epsilon)^2} = 2i\sqrt{(1-\epsilon)^2} = 2i|1-\epsilon|$$

For $\epsilon \ll 1$, we have $|1-\epsilon| = 1-\epsilon$, so:

$$= 2i(1-\epsilon)$$

Final Exact Eigenvalues

$$\lambda = \frac{2 \pm 2i(1-\epsilon)}{2} = 1 \pm i(1-\epsilon)$$

$$\boxed{\lambda_{\text{exact},1} = 1 + i(1-\epsilon), \quad \lambda_{\text{exact},2} = 1 - i(1-\epsilon)}$$

7 Step 7: Comparison and Validation

Perturbative Results

$$\lambda_1(\epsilon) = 1 + i(1-\epsilon) + O(\epsilon^2)$$
$$\lambda_2(\epsilon) = 1 - i(1-\epsilon) + O(\epsilon^2)$$

Exact Results

$$\lambda_{\text{exact},1} = 1 + i(1-\epsilon)$$
$$\lambda_{\text{exact},2} = 1 - i(1-\epsilon)$$

Key Insight 4. ***Remarkable observation:*** *The perturbative expansion is* exact to all orders*! This happens because the characteristic polynomial happens to have a special structure where higher-order terms in ϵ exactly cancel.*

Why This Agreement Is Significant

Validating the method: The agreement confirms that:

1. Our identification of C and D was correct
2. The Fredholm alternative formula (Lecture Notes, page 50) works perfectly
3. The computation of left eigenvectors was essential and correct

8 Step 8: Finding Perturbed Eigenvectors (Optional)

Using the Lecture Notes Framework

From the Lecture Notes (page 49-50), we have at $O(\epsilon)$:

$$(C - \lambda_0 I)x_1 = (\lambda_1 I - D)x_0$$

For $\lambda_{0,1} = 1 + i$, $\lambda_1 = -i$:
The right-hand side:

$$(\lambda_1 I - D)x_0 = \left(-i\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} - \begin{bmatrix}0 & -1\\1 & 0\end{bmatrix}\right)\begin{bmatrix}1\\i\end{bmatrix}$$
$$= \begin{bmatrix}-i & 1\\-1 & -i\end{bmatrix}\begin{bmatrix}1\\i\end{bmatrix} = \begin{bmatrix}-i+i\\-1+1\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$

This means: $(C - \lambda_0 I)x_1 = 0$, so x_1 can be any multiple of x_0. We typically choose $x_1 = 0$ for normalization.

Therefore, the eigenvector to first order is:

$$\boxed{x(\epsilon) = \begin{bmatrix}1\\i\end{bmatrix} + O(\epsilon)}$$

9 Summary and Conclusions

Main Results

Eigenvalues (Perturbative):

$$\lambda_1(\epsilon) = 1 + i(1-\epsilon) + O(\epsilon^2)$$

$$\lambda_2(\epsilon) = 1 - i(1-\epsilon) + O(\epsilon^2)$$

Eigenvalues (Exact):

$$\lambda_{\text{exact},1} = 1 + i(1-\epsilon)$$

$$\lambda_{\text{exact},2} = 1 - i(1-\epsilon)$$

Agreement: Perfect match to all orders in ϵ!
Eigenvectors (to leading order):

$$x_1 = \begin{bmatrix} 1 \\ i \end{bmatrix} + O(\epsilon), \quad x_2 = \begin{bmatrix} 1 \\ -i \end{bmatrix} + O(\epsilon)$$

Key Methodological Points

1. **Matrix decomposition:** Identifying $A = C + \epsilon D$ was crucial for applying perturbation theory
2. **Unperturbed problem:** Solving $Cx_0 = \lambda_0 x_0$ gave complex eigenvalues, indicating non-normality
3. **Left eigenvectors:** Because $C \neq C^T$, we needed y_0 from $C^T y_0 = \lambda_0 y_0$ (Lecture Notes, page 50)
4. **Perturbation formula:** $\lambda_1 = \langle Dx_0, y_0 \rangle / \langle x_0, y_0 \rangle$ (non-self-adjoint case)
5. **Verification:** Exact solution confirmed our perturbative result exactly

Connection to Course Material

This problem demonstrates:

- **Section 5.2.4:** Fredholm alternative for eigenvalue problems
- **Regular perturbation:** Smooth dependence on ϵ (no singular behavior)
- **Complex eigenvalues:** Handled naturally within the framework
- **Validation through exact solution:** Essential step in asymptotic analysis