

15 Limit cycles and periodic orbits

Equilibria are points where a system is stationary, i.e. doesn't change. There are also points to which a system might return again and again, forming closed orbits or cycles like the one we saw in the Hopf bifurcation.

- A **periodic orbit** is the orbit of a solution that satisfies

$$x(t) = x(t + T) \quad (15.1)$$

and so returns to any point along its orbit after a time T , called its **period**.

- A **limit cycle** is a periodic orbit that is stable or unstable, i.e. attracts or repels surrounding orbits a bit like an equilibrium.
- We sometimes use the terms interchangeably, but be careful. A limit cycle is also a periodic orbit (because it is a cycle so it repeats), but a periodic orbit might not be a limit cycle (other orbits might not limit to/from it).
- E.g. In a center every orbit is a periodic orbit. Since this means no orbit is attracted to or repelled from any other, none of these are limit cycles.
- E.g. In the Hopf bifurcation we saw a limit cycle born from an equilibrium as it changed stability.

We usually cannot write the equations of a periodic orbit or cycle exactly. So we need a different way to study them, a simpler way.

But if an orbit always returns to the same point, then why bother studying the whole orbit, why not just study one point and the flow nearby? This is the idea of a **return map**, which we'll look at shortly below.

16 Maps from ODEs

All of the theory we did above was **local** — behaviour at or near equilibria.

- **Global** (or non-local) behaviour concerns orbits that travel large distances in a system. They might:
 - connect the unstable manifold of a saddle back to its stable manifold (a **homoclinic** connection),
 - connect the unstable manifold of one equilibrium to the stable manifold of another (a **heteroclinic** connection),
 - travel in closed repeating orbits called **periodic orbits**, on which

$$x(t + T) = x(T) \quad \text{for period } T$$

- become trapped in enclosed shapes that almost but never precisely repeat, giving **chaos**.

To study **global** behaviour in ODEs we often use maps.

- Poincaré maps are most commonly used to study periodic orbits and chaos (which we'll shortly get to).
- Stroboscopic maps are most commonly used in systems with an obvious imposed period, e.g. an electric circuit driven by an alternating current with a known frequency.

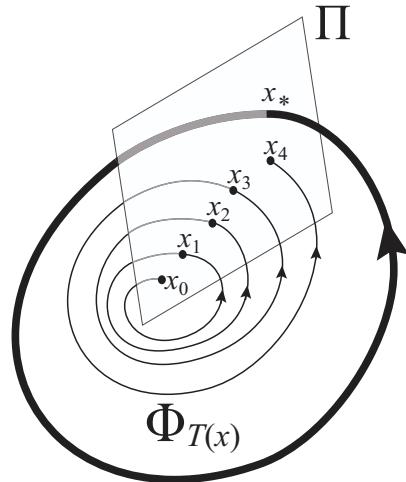
The rough idea is to take a cross-section Π through an ODEs phase portrait, then follow the orbits that pass through that section. There are two main types:

- Π is called a **Poincaré section** if the orbits return repeatedly through the same section Π . To find the map:
 - say an orbit crosses Π at a point x_0 , then evolves through the ODE and returns to Π at a point $x_1 = \Phi_{T(x_0)}(x_0)$ after a *return time* $T(x_0)$ (this Φ is the integral of the solution from x_0 to x_1),
 - assume we can write this as a function f where $x_1 = f(x_0) = \Phi_{T(x_0)}(x_0)$,
 - more generally we'll want to look at multiple returns through points $x_0, x_1 = \Phi_{T(x_0)}(x_0), x_2 = \Phi_{T(x_1)}(x_1), \dots$, so instead we write

$$x_{n+1} = f(x_n)$$

we call this the **Poincaré map** $f : \Pi \mapsto \Pi$.

For example in a pendulum you might define a section $\theta = 0$ corresponding to “pendulum is vertical”.



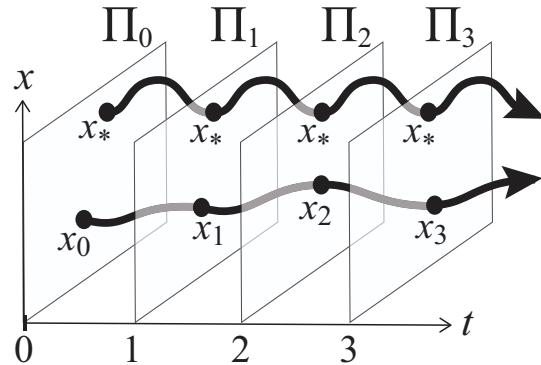
The picture shows a section Π , an orbit integrated through four returns, and also a periodic orbit that goes through x_* .

- Π is called a **stroboscopic section** if it is one of a set of sections Π, Π_1, Π_2, \dots taken at fixed time intervals. To find the map:
 - say an orbit crosses Π at a point x_0 , then evolves through the ODE and crosses Π_1 at a point $x_1 = \Phi_T(x_0)$, then crosses Π_2 at a point $x_2 = \Phi_T(x_1)$, etc. . . . each after a *return time* T ,
 - if we can write each map as

$$x_{n+1} = f(x_n)$$

we call this the **stroboscopic map** $f : \Pi_n \mapsto \Pi_{n+1}$.

For example in the population model we took stroboscopic sections at time intervals $\Delta t = 1$.



The picture shows a section Π every period $T = 1$, an orbit integrated through three periods, and also a periodic orbit that goes through x_* .

17 Example: Maps for the population model

Let's derive the same population model, but in map form.

We'll do this two ways, first as a stroboscopic map, and then from first principles.

Stroboscopic map for the population model

Starting from the ODE, we had $\dot{N} = N(\beta - \gamma N)$. The derivative is defined as

$$\dot{N} = \frac{dN}{dt} = \lim_{\Delta t \rightarrow 0} \frac{N(t + \Delta t) - N(t)}{\Delta t} = N(t)(\beta - \gamma N(t)) \quad (17.1)$$

- So if Δt is small (but not zero) we can at least write

$$\begin{aligned} \frac{N(t + \Delta t) - N(t)}{\Delta t} &\approx N(t)(\beta - \gamma N(t)) \\ \Rightarrow N(t + \Delta t) - N(t) &\approx N(t)(\beta - \gamma N(t)) \Delta t \\ \Rightarrow N(t + \Delta t) &\approx N(t)(\beta - \gamma N(t)) \Delta t + N(t) \end{aligned} \quad (17.2)$$

Now let $N_{n+1} = N(t + \Delta t)$ and $N_n = N(t)$, and we obtain a map from each point N_n to the next point N_{n+1} in the population's growth. If we just let the time-step be $\Delta t = 1$ we have

$$N_{n+1} \approx N_n(1 + \beta - \gamma N_n) \quad (17.3)$$

sometimes called a '*time one map*'.

- You can also derive this from the solution $N(t)$ itself [see Ex.Sht.]
- Let's re-scale this discrete time system by letting $N = \frac{1+\beta}{\gamma}x$, and let $r = 1+\beta$, so (17.3) becomes

$$\frac{1+\beta}{\gamma}x_{n+1} = \frac{(1+\beta)^2}{\gamma}x_n(1 - x_n) \Rightarrow x_{n+1} = rx_n(1 - x_n) \quad (17.4)$$

which is an important system called the **logistic map**.

- The ODE and the map are different ways of approximating — and hence modeling — the population. But be careful, their behaviours aren't the same (more on this soon).

Discrete time population model from first principles

We don't only get maps from ODEs, they are often derived as models in their own right, when we want a model in steps rather than in continuous time.

For example, we might only be interested in the population level measured once a day. So let's re-derive the population model as a map in discrete time steps.

- An ODE gives the rate of change of a quantity, $\dot{x} = \dots$
- A *difference equation* gives the amount a quantity changes between two instants.
- Say a population N at some 'time' $n + 1$ relates to an earlier 'time' n as

$$N_{n+1} - N_n = B - D + M \quad (17.5)$$

and similar to the ODE, let's take: births $B = \beta N_n$, deaths $D = \delta N_n$, migration $M = 0$, so

$$N_{n+1} = (\beta - \delta + 1)N_n \quad (17.6)$$

- A solution of this would tell us N_n for any initial value N_0 .
- To find this try iterating:

$$\begin{aligned} N_n &= (\beta - \delta + 1)N_{n-1} && \text{just the difference eqn} \\ &= (\beta - \delta + 1)^2 N_{n-2} && \text{subbing in } N_{n-1} = (\beta - \delta + 1)N_{n-2} \\ &= (\beta - \delta + 1)^3 N_{n-3} && \text{subbing in } N_{n-2} = (\beta - \delta + 1)N_{n-3} \\ &\vdots \\ N_n &= (\beta - \delta + 1)^n N_0 && \dots \text{ keeping going until you reach } N_0 \end{aligned} \quad (17.7)$$

- Like an ODE we can describe this with a flow operator Φ as

$$N_n = \Phi_n(N_0) \quad \text{where} \quad \Phi_n(N_0) = ((\beta - \delta + 1)^n N_0) \quad (17.8)$$

- It's pretty easy to see what happens to this solution. If $|\beta - \delta + 1| > 1$ population size grows exponentially, if $|\beta - \delta + 1| < 1$ it shrinks asymptotically to zero. Assuming $\beta - \delta + 1$ is positive, these conditions just become $\beta > \delta$ and $\beta < \delta$ as in the continuous time model.
- Note if $\beta - \delta + 1 < 0$ then N_n will flip between being positive and negative on each iteration. Clearly this isn't physically realistic for the population model.

As we did for the ODE, let's improve the population model by making the death rate proportional to the population, introducing a cut-off.

- Assume that the death rate increases with the population a $\delta = \gamma N$, so

$$N_{n+1} = bN_n - fN_n^2 + N_n \quad (17.9)$$

$$= N_n(1 + \beta - \gamma N_n) \quad (17.10)$$

You can see that this is exactly the time-one stroboscopic map we obtained from the ODE.

- Applying the same scaling as we did then, letting $N = (1 + \beta)x/\gamma$ and $r = 1 + \beta$, gives the **logistic map**

$$x_{n+1} = rx_n(1 - x_n) \quad (17.11)$$

For the model to be realistic we must have $\beta > 0$, which now means $r > 1$.

- Now this system cannot be solved. If you try to iterate you get

$$\begin{aligned} x_n &= rx_{n-1}(1 - x_{n-1}) \\ &= r[rx_{n-2}(1 - x_{n-2})](1 - [rx_{n-2}(1 - x_{n-2})]) \\ &= r[r[rx_{n-3}(1 - x_{n-3})](1 - [rx_{n-3}(1 - x_{n-3})])] \dots \end{aligned} \quad (17.12)$$

. . . this goes on and on getting longer and longer and higher order in m .

- We're going to need some new tricks.

. . . local analysis

- In ODEs we learned that we can get a lot from local analysis.
- Whereas ODEs tell us rates of change, and are stationary at *equilibria*, maps tell us how a system updates, so they are stationary at **fixed points**.
- A fixed point is a place where the map just repeats the same value, $x_0 = x_1 = x_2 = \dots$, so we find it by solving $x_{n+1} = x_n$, which from (17.11) gives

$$\begin{aligned} x_n &= rx_n(1 - x_n) \\ \Rightarrow 0 &= x_n(1 - r + rx_n) \end{aligned} \tag{17.13}$$

with solutions

$$x_{*1} = 0 \quad \& \quad x_{*2} = (r - 1)/r \tag{17.14}$$

- Like ODEs, we can linearize around these.
- Near $x = x_{*2} = 0$ the system is $x_{n+1} \approx rx_n$ with solution $x_n \approx r^n x_0$, so with each n the factor r^n grows (since we said $r > 1$), so the population x_n grows away from zero . . . the fixed point is *unstable* (a repeller).
- Near $x = x_{*2}$ the system is $x_{n+1} \approx x_{*2} + (2 - r)(x_n - x_{*2})$ with a solution we can write as $x_n - x_{*2} = (2 - r)^n(x_0 - x_{*2})$ [see Ex.Sht], so with each n the factor $(2 - r)^n$ shrinks towards zero (since $r > 1$ implies $2 - r < 1$), dragging the population x_n closer to x_{*2} . . . the fixed point is *stable* (an attractor).
- So the behaviour looks very much consistent with the continuous time model (the ODE).

. . . ODE versus map

- In the continuous system, let's take an initial point $x_0 = 0.3$ and look at each timestep $\Delta t = 1$. Take $r = 3.6$ for example, and we get a steady and rapid convergence to the stable equilibrium: (to 2d.p.)

$$x(0) = 0.3, \quad x(1) = 0.65, \quad x(2) = 0.72, \quad x(3) = 0.72, \quad x(4) = 0.72, \dots$$

- In the discrete system, let's do the same thing, with the same values, and we get: (to 2d.p.)

$$x(0) = 0.3, \quad x(1) = 0.9, \quad x(2) = 0.32, \quad x(3) = 0.79, \quad x(4) = 0.6, \dots$$

Firstly we're not getting the same kind of convergence, and certainly not to $x = (r - 1)/r = 0.72$. Worse, these values are able to jump to above and below $x = 0.72$ rather than just tend monotonically towards it.

- So there is clearly more that can happen in the map model. Why?
- This is largely due to period orbits of the map. As well as the fixed point in the population model where $x_{n+1} = x_n$, given by

$$x_0 = rx_0(1 - x_0) \quad \Rightarrow \quad x_0 = 0 \text{ or } (r - 1)/r \quad (17.15)$$

the map might only return to a point after m iterates so that $x_{n+m} = x_n$. For example there is a period 2 orbit

$$\begin{aligned} x_0 &= [rx_0(1 - x_0)](1 - [rx_0(1 - x_0)]) \\ &\Rightarrow x_0 = \frac{1}{2r}(1 + r \pm \sqrt{(r + 1)(r - 3)}) \end{aligned} \quad (17.16)$$

and a period 3 orbit and so on . . . these get more and more difficult to find, but we'll come back to them later.

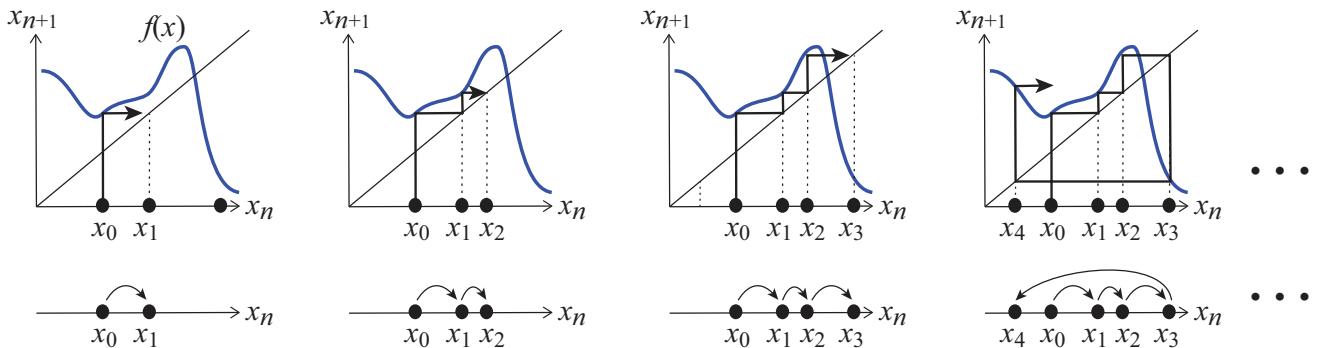
For ODEs we could sketch phase portraits. For maps we have an alternative but often more powerful tool, called a *cobweb diagram*.

18 Cobweb diagrams

So a map f generates a sequence of points $x_0, x_1, x_2, \dots, x_n, \dots$ that a system goes through as we iterate it through the ‘process’ by applying f again and again.

There’s a useful way to visualize these, called a cobweb diagram:

- draw the graph of $f(x)$, and draw the diagonal $y = x$
- label the horizontal axis as x_n and the vertical axis as x_{n+1}
- to draw an orbit from an initial point x_0 you now:
 - *iterate the map*: draw a vertical line from the point x_0 on the horizontal axis until you reach the graph, arriving at $x_1 = f(x_0)$
 - *reset the map*: continue with a horizontal line across to the diagonal
 - *iterate again*: continue with a vertical line until you reach the graph again, arriving at $x_2 = f(x_1)$
 - *reset the map*: continue with a horizontal line across to the diagonal
 - and repeat



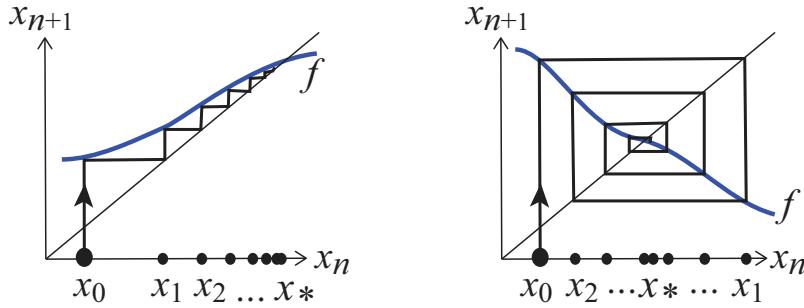
- You obtain a sequence of vertical lines at $x_n = x_0, x_1, x_2, \dots$, joined by horizontal resets to the diagonal. This is the ‘cobweb’.

- If the graph $f(x)$ crosses the diagonal $y = x$, it forms a **fixed point** of the map since it creates a point where

$$x_* = f(x_*)$$

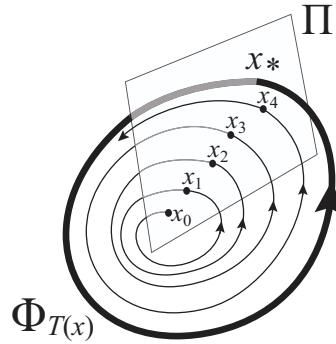
and hence $x_{n+1} = x_n$.

- A fixed point of a map corresponds to a **periodic orbit** of an ODE, as it says an orbit will always return to the same point $x_0 = x_1 = x_2 = \dots$ each time it hits the section Π , at $x_0 = x_1 = x_2 = \dots$



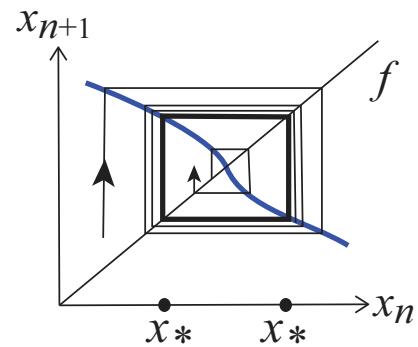
Note in the figure how iterates tend directly towards the fixed point if the gradient of f is positive at x_* , but oscillate around it if the gradient of f is negative at x_* .

If this was derived from the Poincaré map of an ODE then the corresponding flow might look like:



- The cobweb may form a closed orbit that repeats after m iterations. This is a **periodic orbit** of the map, where

$$x_m = x_0 \quad \Rightarrow \quad x_0 = f^m(x_0) = f(\overset{m}{\cdots} f(f(x_0)) \cdots) \quad (18.1)$$



The picture shows a period two orbit.

19 ODE vs. maps — a comparison

Everything we've learnt about ODEs has a counterpart in maps. We'll summarize them here and then unpack these a little below . . .

ODEs

Rate of change

The rate x is changing is

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) = \lim_{\varepsilon \rightarrow 0} \frac{\mathbf{x}(t) - \mathbf{x}(t - \Delta t)}{\Delta t}$$

Equilibrium

$$\mathbf{f}(\mathbf{x}_*) = 0$$

Stability

$$\operatorname{Re}(\lambda) < 0 \Rightarrow \text{stable}$$

$$\operatorname{Re}(\lambda) > 0 \Rightarrow \text{unstable}$$

where $\|\underline{\underline{A}} - \lambda \underline{\underline{1}}\| = 0$ and

$$\underline{\underline{A}} = \left. \frac{\partial \mathbf{f}}{\partial \mathbf{x}} \right|_{\mathbf{x}=\mathbf{x}_*}$$

Bifurcations

If $\operatorname{Re}(\lambda) = 0 \dots$

- fold (saddle-node)
- transcritical
- pitchfork
- cusp
- Hopf

Maps

Difference

From some x_{n-1} to x_n the difference is

$$\mathbf{x}_n = \mathbf{f}(\mathbf{x}_{n-1}) \quad \begin{cases} \mathbf{x}_n = \mathbf{x}(t) \\ \mathbf{x}_{n-1} = \mathbf{x}(t - \Delta t) \end{cases}$$

Fixed Point

$$\mathbf{f}(\mathbf{x}_*) = \mathbf{x}_*$$

Stability

$$|\lambda| < 1 \Rightarrow \text{stable}$$

$$|\lambda| > 1 \Rightarrow \text{unstable}$$

where $\|\underline{\underline{A}} - \lambda \underline{\underline{1}}\| = 0$ and

$$\underline{\underline{A}} = \left. \frac{\partial \mathbf{f}}{\partial \mathbf{x}} \right|_{\mathbf{x}=\mathbf{x}_*}$$

Bifurcations

If $|\lambda| = 1 \dots$

- fold (saddle-node)
- transcritical
- pitchfork or flip
- cusp
- Neimark-Sacker

20 Linear stability (for maps)

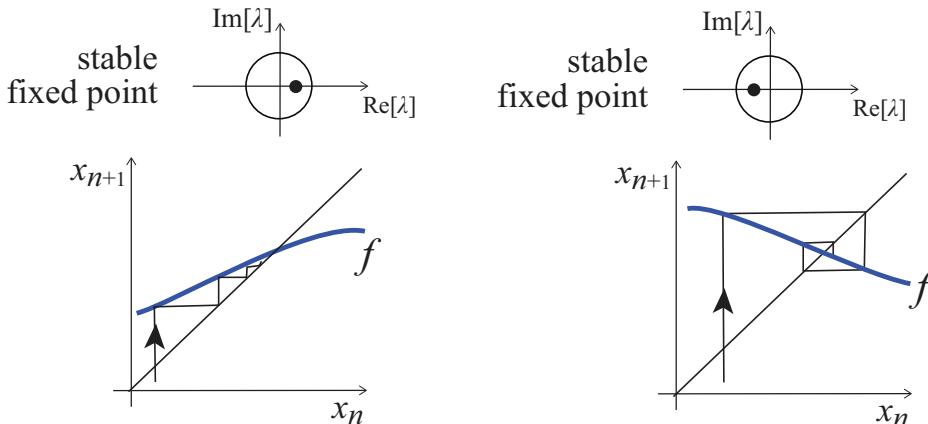
The fixed points of maps has a Jacobian $\frac{\partial \mathbf{f}(\mathbf{x})}{\partial \mathbf{x}}$, with eigenvalues λ and eigenvectors \mathbf{v} , just as in ODEs. But:

- whereas in an ODE the stability depends on whether an eigenvalue lies in the right or left half of the complex plane, in a map the stability depends on whether an eigenvalue has magnitude greater than or less than one.

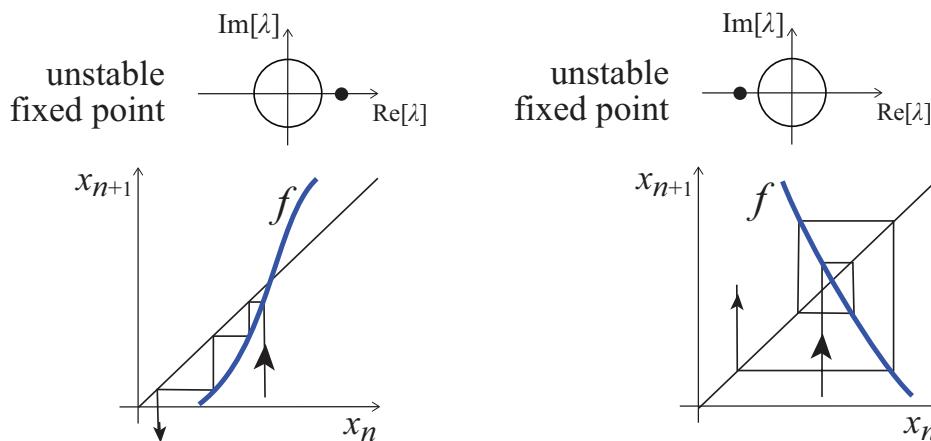
Let's take just a 1-dimensional map.

Then eigenvalue $\lambda = f'(x_*)$ is just the slope of f at a fixed point x_* . Now:

- If the slope at x_* has modulus less than one the fixed point is stable:



- If the slope at x_* is modulus greater than one the fixed point is unstable:



- To get complex eigenvalues we need more than one dimension, and as for ODEs they can only occur in conjugate pairs. If they are inside the unit circle they are stable, if they are outside they are unstable, if they are on the unit circle we have a center:

