Exercise Sheet 1, Question 5: 2D Phase Portraits
Complete Solution with XYZ Methodology
Methods of Applied Mathematics [SEMT30006]

Problem Statement

Sketch phase portraits for the following differential equations and classify the equilibria.

(a) $\frac{du}{dt} = v^2 - u$, $\frac{dv}{dt} = u^2 - v$

(b) $\frac{d^2u}{dt^2} + \frac{du}{dt} + \sin(u) = 0$

1 Foundational Concepts: 2D Phase Portraits

What is a 2D Phase Portrait?

- **STAGE X (Definition):** For a 2D autonomous system:

$$\frac{du}{dt} = f(u, v), \quad \frac{dv}{dt} = g(u, v) \tag{1}$$

  The **phase portrait** is a visualization in the (u, v) plane showing:

  - **Equilibria** (fixed points): where $f = 0$ and $g = 0$
  - **Nullclines**: curves where $\dot{u} = 0$ or $\dot{v} = 0$
  - **Vector field**: arrows (f, g) showing direction of flow
  - **Trajectories**: solution curves $(u(t), v(t))$ in the plane

- **STAGE Y (Key tools for analysis):**

  1. **Nullclines**:
     - u-nullcline: where $\dot{u} = 0$ (flow vertical)
     - v-nullcline: where $\dot{v} = 0$ (flow horizontal)
     - Equilibria: intersections of nullclines
  2. **Linearization**: Near equilibrium (u^*, v^*), approximate:

$$\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} \approx J \begin{pmatrix} u - u^* \\ v - v^* \end{pmatrix} \tag{2}$$

     where J is the Jacobian matrix.
  3. **Eigenvalues of J**: Determine local behavior

- **STAGE Z (Classification by eigenvalues):** For Jacobian with eigenvalues λ_1, λ_2:

  - **Node** (both λ real, same sign):

    * Stable if $\lambda_1, \lambda_2 < 0$
    * Unstable if $\lambda_1, \lambda_2 > 0$
  - **Saddle** (both λ real, opposite signs): Always unstable
  - **Focus/Spiral** ($\lambda = \alpha \pm i\beta$, complex conjugates):
    * Stable if $\alpha < 0$
    * Unstable if $\alpha > 0$
  - **Center** ($\lambda = \pm i\beta$, purely imaginary): Neutrally stable

FROM LECTURE NOTES (pages 29-34): The trace and determinant of the Jacobian determine equilibrium type:

$$\tau = \mathrm{tr}(J) = \lambda_1 + \lambda_2, \quad \Delta = \det(J) = \lambda_1 \lambda_2 \tag{3}$$

2 Part (a): $\frac{du}{dt} = v^2 - u$, $\frac{dv}{dt} = u^2 - v$

Step 1: Identify System Structure and Symmetry

- **STAGE X (System form):**

$$\dot{u} = v^2 - u = f(u, v) \tag{4}$$
$$\dot{v} = u^2 - v = g(u, v) \tag{5}$$

  This is a coupled nonlinear system with quadratic terms.

- **STAGE Y (Symmetry observation):** The system has a beautiful symmetry: swapping $u \leftrightarrow v$ leaves the system unchanged!

  If $(u(t), v(t))$ is a solution, then $(v(t), u(t))$ is also a solution.

  This means:

  - Phase portrait is symmetric about the line $v = u$
  - If (a, b) is an equilibrium, so is (b, a)
  - Trajectories are mirror images across $v = u$

- **STAGE Z (Implications):** We can exploit this symmetry when:

  - Finding equilibria
  - Sketching phase portraits
  - Understanding global dynamics

Step 2: Find Equilibria

- **STAGE X (Equilibrium conditions):** At equilibrium: $\dot{u} = 0$ and $\dot{v} = 0$:

$$v^2 - u = 0 \quad \Rightarrow \quad u = v^2 \tag{6}$$
$$u^2 - v = 0 \quad \Rightarrow \quad v = u^2 \tag{7}$$

- **STAGE Y (Solving the system):** Substitute $u = v^2$ into $v = u^2$:

$$v = (v^2)^2 = v^4 \tag{8}$$

  Rearranging:

$$v^4 - v = 0 \quad \Rightarrow \quad v(v^3 - 1) = 0 \tag{9}$$

  So $v = 0$ or $v^3 = 1$.

  For $v^3 = 1$: only real solution is $v = 1$.

  Therefore: $v \in \{0, 1\}$

  For each v, find corresponding u:

  - If $v = 0$: $u = v^2 = 0 \Rightarrow$ equilibrium at $(0, 0)$
  - If $v = 1$: $u = v^2 = 1 \Rightarrow$ equilibrium at $(1, 1)$

- **STAGE Z (Two equilibria):**

$$\boxed{(u^*, v^*) = (0, 0) \text{ and } (1, 1)} \tag{10}$$

  Note: Both lie on the symmetry line $v = u$ (as expected from symmetry).

Step 3: Find and Sketch Nullclines

- **STAGE X (u-nullcline):** Where $\dot{u} = 0$:

$$v^2 - u = 0 \quad \Rightarrow \quad u = v^2 \tag{11}$$

  This is a parabola opening to the right. On this curve, flow is purely vertical.

- **STAGE Y (v-nullcline):** Where $\dot{v} = 0$:

$$u^2 - v = 0 \quad \Rightarrow \quad v = u^2 \tag{12}$$

  This is a parabola opening upward. On this curve, flow is purely horizontal.

- **STAGE Z (Intersections):** The nullclines intersect where both $u = v^2$ and $v = u^2$, which gives our equilibria $(0, 0)$ and $(1, 1)$ as found above.

  The nullclines divide the plane into regions with different flow directions.

Step 4: Determine Flow Direction in Each Region

- **STAGE X (Sign analysis):** The nullclines divide the (u, v) plane into several regions. We determine the sign of $\dot{u}$ and $\dot{v}$ in each:

  - $\dot{u} = v^2 - u > 0$ when $u < v^2$ (left of u-nullcline)
  - $\dot{u} = v^2 - u < 0$ when $u > v^2$ (right of u-nullcline)
  - $\dot{v} = u^2 - v > 0$ when $v < u^2$ (below v-nullcline)
  - $\dot{v} = u^2 - v < 0$ when $v > u^2$ (above v-nullcline)

- **STAGE Y (Test points):**

  Region between parabolas (e.g., point $(0.5, 0.5)$):

$$\dot{u} = (0.5)^2 - 0.5 = -0.25 < 0 \quad \Rightarrow \quad \text{flow LEFT} \tag{13}$$
$$\dot{v} = (0.5)^2 - 0.5 = -0.25 < 0 \quad \Rightarrow \quad \text{flow DOWN} \tag{14}$$

  Flow is $\searrow$ (southwest)

  Region below both parabolas (e.g., point $(2, 0.5)$):

$$\dot{u} = (0.5)^2 - 2 = -1.75 < 0 \quad \Rightarrow \quad \text{flow LEFT} \tag{15}$$
$$\dot{v} = (2)^2 - 0.5 = 3.5 > 0 \quad \Rightarrow \quad \text{flow UP} \tag{16}$$

  Flow is $\nwarrow$ (northwest)

  Region above both parabolas (e.g., point $(0.5, 2)$):

$$\dot{u} = (2)^2 - 0.5 = 3.5 > 0 \quad \Rightarrow \quad \text{flow RIGHT} \tag{17}$$
$$\dot{v} = (0.5)^2 - 2 = -1.75 < 0 \quad \Rightarrow \quad \text{flow DOWN} \tag{18}$$

  Flow is $\searrow$ (southeast)

- **STAGE Z (Flow pattern):** The flow patterns suggest spiral or rotational behavior around equilibria, with the parabolas guiding the flow.

Step 5: Linearization at $(0, 0)$

- **STAGE X (Compute Jacobian):** The Jacobian matrix is:

$$J = \begin{pmatrix} \frac{\partial f}{\partial u} & \frac{\partial f}{\partial v} \\ \frac{\partial g}{\partial u} & \frac{\partial g}{\partial v} \end{pmatrix} = \begin{pmatrix} -1 & 2v \\ 2u & -1 \end{pmatrix} \tag{19}$$

  At $(u^*, v^*) = (0, 0)$:

$$J(0,0) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -I \tag{20}$$

- **STAGE Y (Find eigenvalues):** For $J = -I$, the characteristic equation is:

$$\det(J - \lambda I) = \det \begin{pmatrix} -1-\lambda & 0 \\ 0 & -1-\lambda \end{pmatrix} = (-1-\lambda)^2 = 0 \tag{21}$$

  Therefore:

$$\lambda_1 = \lambda_2 = -1 \tag{22}$$

  Both eigenvalues are real, negative, and equal (repeated eigenvalue).

- **STAGE Z (Classification):**

$$\boxed{\text{Stable node (star node)}} \tag{23}$$

  Since $J = -I$:

  - All directions are eigendirections
  - Trajectories approach origin along straight lines
  - This is called a **star node** or **proper node**
  - Decay rate: e^{-t} (time constant $\tau = 1$)

Step 6: Linearization at $(1, 1)$

- **STAGE X (Jacobian at $(1, 1)$):**

$$J(1,1) = \begin{pmatrix} -1 & 2(1) \\ 2(1) & -1 \end{pmatrix} = \begin{pmatrix} -1 & 2 \\ 2 & -1 \end{pmatrix} \tag{24}$$

- **STAGE Y (Find eigenvalues):** Characteristic equation:

$$\det(J - \lambda I) = \det \begin{pmatrix} -1-\lambda & 2 \\ 2 & -1-\lambda \end{pmatrix} = 0 \tag{25}$$

$$(-1-\lambda)^2 - 4 = 0 \tag{26}$$
$$\lambda^2 + 2\lambda + 1 - 4 = 0 \tag{27}$$
$$\lambda^2 + 2\lambda - 3 = 0 \tag{28}$$
$$(\lambda + 3)(\lambda - 1) = 0 \tag{29}$$

  Therefore:

$$\lambda_1 = -3, \quad \lambda_2 = +1 \tag{30}$$

  One negative, one positive (opposite signs).

- **STAGE Z (Classification):**

$$\boxed{\text{Saddle point (unstable)}} \tag{31}$$

Since eigenvalues have opposite signs:

  - One stable direction (along eigenvector for $\lambda_1 = -3$)
  - One unstable direction (along eigenvector for $\lambda_2 = +1$)
  - Saddle points are always unstable (hyperbolic)
  - $\det(J) = \lambda_1\lambda_2 = -3 < 0$ confirms saddle

Step 7: Find Eigenvectors for Saddle at $(1, 1)$

- **STAGE X (Eigenvector for $\lambda_1 = -3$):** Solve $(J - \lambda_1 I)\mathbf{v}_1 = 0$:

$$\begin{pmatrix} -1-(-3) & 2 \\ 2 & -1-(-3) \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \mathbf{0} \tag{32}$$

From first row: $2v_1 + 2v_2 = 0 \Rightarrow v_1 = -v_2$

Eigenvector: $\mathbf{v}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

Stable manifold: Along direction $(1, -1)$ from $(1, 1)$

- **STAGE Y (Eigenvector for $\lambda_2 = +1$):** Solve $(J - \lambda_2 I)\mathbf{v}_2 = 0$:

$$\begin{pmatrix} -1-1 & 2 \\ 2 & -1-1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} -2 & 2 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \mathbf{0} \tag{33}$$

From first row: $-2v_1 + 2v_2 = 0 \Rightarrow v_1 = v_2$

Eigenvector: $\mathbf{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Unstable manifold: Along direction $(1, 1)$ from $(1, 1)$

- **STAGE Z (Geometric interpretation):** At the saddle point $(1, 1)$:

  - Stable direction: slope -1 (line through origin and $(1, 1)$ with negative slope)
  - Unstable direction: slope $+1$ (the symmetry line $v = u$)
  - Trajectories approach along stable manifold, repel along unstable manifold

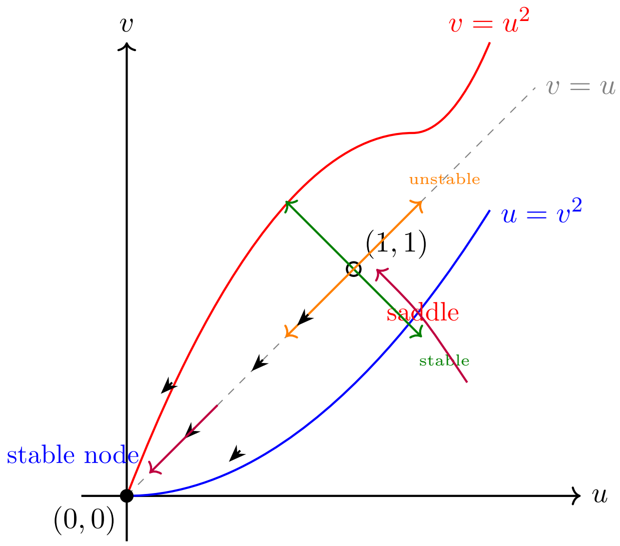
Note: The unstable direction lies exactly on the symmetry line $v = u$!

Step 8: Sketch Phase Portrait

- **STAGE X (Key features to include):**

  1. Both nullclines (parabolas)
  2. Two equilibria with classifications
  3. Eigendirections at saddle
  4. Flow directions in each region
  5. Representative trajectories
  6. Symmetry about $v = u$

- **STAGE Y (Phase portrait):**



- **STAGE Z (Description):**
  - Blue curve: u-nullcline $u = v^2$
  - Red curve: v-nullcline $v = u^2$
  - Black dot at $(0, 0)$: stable node
  - White dot at $(1, 1)$: saddle
  - Green arrows: stable manifold
  - Orange arrows: unstable manifold
  - Trajectories spiral into origin
  - Saddle creates hyperbolic structure at $(1, 1)$

Step 9: Global Dynamics

- **STAGE X (Long-time behavior):**
  1. Most trajectories converge to the stable node at $(0, 0)$
  2. The stable manifold of the saddle forms a boundary
  3. Trajectories on the unstable manifold leave $(1, 1)$ along $v = u$
  4. System exhibits strong attraction to origin
- **STAGE Y (Basin of attraction):** The basin of attraction for $(0, 0)$ is most of the positive quadrant, except:
  - Points exactly on unstable manifold of saddle
  - These go to infinity along $v = u$ direction
- **STAGE Z (Physical interpretation):** This could model:
  - Two coupled populations with quadratic growth terms
  - Symmetric competition with nonlinear effects
  - System with inherent damping ($-u$ and $-v$ terms) balanced by growth (v^2 and u^2)

The saddle at $(1, 1)$ represents an unstable coexistence state.

KEY INSIGHT: The symmetry $u \leftrightarrow v$ creates mirror-image dynamics across $v = u$. The saddle's unstable manifold lies exactly on this symmetry line, making it a special separatrix in the system.

3 Part (b): $\frac{d^2u}{dt^2} + \frac{du}{dt} + \sin(u) = 0$

Step 1: Convert to First-Order System

- **STAGE X (Standard procedure):** This is a second-order ODE. To analyze it in the phase plane, introduce:

$$v = \frac{du}{dt} \tag{34}$$

Then:

$$\frac{dv}{dt} = \frac{d^2u}{dt^2} \tag{35}$$

- **STAGE Y (Rewrite as system):** From the original equation:

$$\frac{d^2u}{dt^2} = -\frac{du}{dt} - \sin(u) = -v - \sin(u) \tag{36}$$

Therefore, the first-order system is:

$$\frac{du}{dt} = v \tag{37}$$
$$\frac{dv}{dt} = -v - \sin(u) \tag{38}$$

- **STAGE Z (Phase space interpretation):**
  - (u, v) is the phase space (position and velocity)
  - u: displacement (like angle of pendulum)
  - $v = \dot{u}$: velocity
  - This represents a damped nonlinear oscillator

Step 2: Physical Interpretation

- **STAGE X (Pendulum analogy):** The equation $\ddot{u} + \dot{u} + \sin(u) = 0$ models a damped pendulum:
  - $\sin(u)$: restoring force (gravity)
  - $\dot{u}$: damping (friction)
  - $\ddot{u}$: acceleration

  For small angles: $\sin(u) \approx u$, recovering the simple harmonic oscillator with damping.

- **STAGE Y (Energy consideration):** This system is dissipative (loses energy due to damping term $\dot{u}$). We expect:
  - No closed orbits (energy decreases)
  - Convergence to equilibria
  - No periodic solutions

- **STAGE Z (Periodicity in u):** Since $\sin(u)$ is periodic with period 2π:
  - Phase portrait repeats every 2π in the u-direction
  - Equilibria at $u = n\pi$ for integer n
  - We typically plot one or two periods: $u \in [-\pi, 3\pi]$

Step 3: Find Equilibria

- **STAGE X (Equilibrium conditions):** At equilibrium:

$$\dot{u} = v = 0 \tag{39}$$
$$\dot{v} = -v - \sin(u) = 0 \tag{40}$$

From first equation: $v = 0$

From second equation: $-0 - \sin(u) = 0 \Rightarrow \sin(u) = 0$

- **STAGE Y (Solving):** $\sin(u) = 0$ when $u = n\pi$ for any integer n.

Therefore:

$$\text{Equilibria: } (u^*, v^*) = (n\pi, 0) \text{ for } n \in \mathbb{Z} \tag{41}$$

- **STAGE Z (Infinite equilibria):**

$$\ldots, (-2\pi, 0), (-\pi, 0), (0, 0), (\pi, 0), (2\pi, 0), (3\pi, 0), \ldots \tag{42}$$

For analysis, we focus on:

  - $(0, 0)$: corresponds to pendulum hanging down (stable)
  - $(\pi, 0)$: corresponds to pendulum pointing up (unstable)
  - Pattern repeats every 2π

Step 4: Find Nullclines

- **STAGE X (u-nullcline):** Where $\dot{u} = 0$:

$$v = 0 \tag{43}$$

This is the entire u-axis (horizontal line through $v = 0$).

- **STAGE Y (v-nullcline):** Where $\dot{v} = 0$:

$$-v - \sin(u) = 0 \quad \Rightarrow \quad v = -\sin(u) \tag{44}$$

This is a sinusoidal curve.

- **STAGE Z (Intersections):** Nullclines intersect where $v = 0$ and $v = -\sin(u)$:

$$0 = -\sin(u) \quad \Rightarrow \quad \sin(u) = 0 \quad \Rightarrow \quad u = n\pi \tag{45}$$

Confirming our equilibria at $(n\pi, 0)$.

Step 5: Linearization and Classification

- **STAGE X (Jacobian):** For the system:

$$\dot{u} = v, \quad \dot{v} = -v - \sin(u) \tag{46}$$

The Jacobian is:

$$J = \begin{pmatrix} 0 & 1 \\ -\cos(u) & -1 \end{pmatrix} \tag{47}$$

- **STAGE Y (At equilibria $u = n\pi$):**

  At even multiples: $u = 0, \pm 2\pi, \pm 4\pi, \ldots$

$$J(2n\pi, 0) = \begin{pmatrix} 0 & 1 \\ -\cos(2n\pi) & -1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} \tag{48}$$

  At odd multiples: $u = \pm\pi, \pm 3\pi, \ldots$

$$J((2n+1)\pi, 0) = \begin{pmatrix} 0 & 1 \\ -\cos((2n+1)\pi) & -1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ +1 & -1 \end{pmatrix} \tag{49}$$

- **STAGE Z (Two distinct types):** We need to analyze eigenvalues for each type.

Step 6: Classify Equilibria at $u = 2n\pi$ (e.g., $(0,0)$)

- **STAGE X (Eigenvalues):** For $J = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$:

  Characteristic equation:

$$\det(J - \lambda I) = \det\begin{pmatrix} -\lambda & 1 \\ -1 & -1-\lambda \end{pmatrix} = \lambda(\lambda+1) + 1 = 0 \tag{50}$$

$$\lambda^2 + \lambda + 1 = 0 \tag{51}$$

  Using quadratic formula:

$$\lambda = \frac{-1 \pm \sqrt{1-4}}{2} = \frac{-1 \pm \sqrt{-3}}{2} = \frac{-1 \pm i\sqrt{3}}{2} \tag{52}$$

- **STAGE Y (Complex eigenvalues):**

$$\lambda = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2} \tag{53}$$

  These are complex conjugates with:

  - Real part: $\alpha = -\frac{1}{2} < 0$ (stable)
  - Imaginary part: $\beta = \pm\frac{\sqrt{3}}{2}$ (oscillations)

- **STAGE Z (Classification):**

$$\boxed{\text{Stable focus (spiral sink)}} \tag{54}$$

  Equilibria at $(0,0), (\pm 2\pi, 0), (\pm 4\pi, 0), \ldots$ are stable spirals.

  Physical meaning: pendulum hanging down—stable equilibrium with damped oscillations.

Step 7: Classify Equilibria at $u = (2n+1)\pi$ (e.g., $(\pi, 0)$)

- **STAGE X (Eigenvalues):** For $J = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}$:

  Characteristic equation:

$$\det(J - \lambda I) = \det \begin{pmatrix} -\lambda & 1 \\ 1 & -1-\lambda \end{pmatrix} = \lambda(\lambda+1) - 1 = 0 \tag{55}$$

$$\lambda^2 + \lambda - 1 = 0 \tag{56}$$

  Using quadratic formula:

$$\lambda = \frac{-1 \pm \sqrt{1+4}}{2} = \frac{-1 \pm \sqrt{5}}{2} \tag{57}$$

- **STAGE Y (Real eigenvalues):**

$$\lambda_1 = \frac{-1-\sqrt{5}}{2} \approx -1.618 < 0 \tag{58}$$

$$\lambda_2 = \frac{-1+\sqrt{5}}{2} \approx +0.618 > 0 \tag{59}$$

  One negative, one positive (opposite signs).

- **STAGE Z (Classification):**

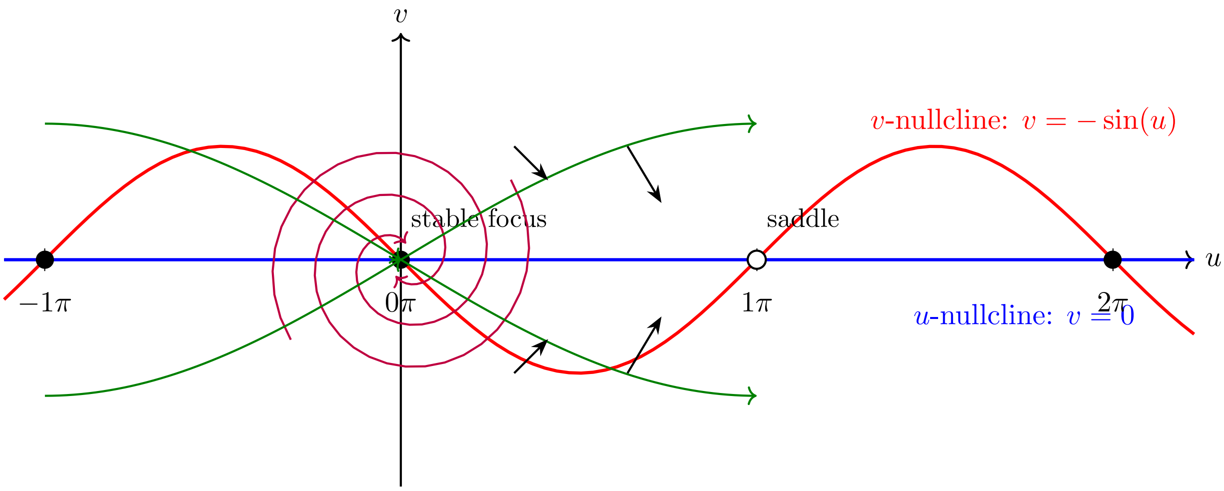
$$\boxed{\text{Saddle point (unstable)}} \tag{60}$$

  Equilibria at $(\pi, 0), (\pm 3\pi, 0), (\pm 5\pi, 0), \dots$ are saddles.

  Physical meaning: pendulum pointing up—unstable equilibrium (toppling point).

Step 8: Sketch Phase Portrait

- **STAGE X (Key features):**
  1. Stable foci at $u = 0, \pm 2\pi, \dots$
  2. Saddles at $u = \pm\pi, \pm 3\pi, \dots$
  3. u-nullcline: $v = 0$ (horizontal axis)
  4. v-nullcline: $v = -\sin(u)$ (sine curve)
  5. Heteroclinic connections between saddles
- **STAGE Y (Phase portrait):**



- **STAGE Z (Description):**
  - Filled circles: stable foci (pendulum down positions)
  - Open circles: saddles (pendulum up positions)
  - Purple spirals: trajectories converging to stable foci
  - Green curves: heteroclinic orbits connecting saddles
  - Pattern repeats every 2π horizontally

Step 9: Heteroclinic Connections

- **STAGE X (Special trajectories):** The stable and unstable manifolds of adjacent saddles connect, forming **heteroclinic orbits**.

  These are trajectories that:
  - Start at one saddle (as $t \to -\infty$)
  - End at an adjacent saddle (as $t \to +\infty$)
  - Form the boundary between different basins of attraction

- **STAGE Y (Physical meaning):** A heteroclinic orbit represents:
  - Pendulum starting at upright position (unstable)
  - Swinging all the way around
  - Approaching upright position again (but never quite reaching it in finite time)

  These require infinite time to complete and represent idealized "separatrix" motions.

- **STAGE Z (Global structure):** The phase portrait consists of:
  1. Stable foci at $u = 2n\pi$ attracting nearby trajectories
  2. Saddles at $u = (2n+1)\pi$ acting as gateways
  3. Heteroclinic connections forming boundaries
  4. Spiral convergence into each stable focus

Step 10: Damping and Energy

- **STAGE X (Energy dissipation):** The damping term $\dot{u}$ in $\ddot{u}+\dot{u}+\sin(u)=0$ causes energy loss:

  Define energy-like function:

  $$E = \frac{1}{2}v^2 + (1 - \cos(u)) \tag{61}$$

  (This is kinetic + potential energy for a pendulum)

- **STAGE Y (Rate of energy change):**

  $$\frac{dE}{dt} = v\frac{dv}{dt} + \sin(u)\frac{du}{dt} = v(-v - \sin(u)) + \sin(u) \cdot v = -v^2 \leq 0 \tag{62}$$

  Energy decreases (unless $v = 0$)!

- **STAGE Z (Implications):**
  - No closed orbits possible (energy always decreasing)
  - All trajectories eventually approach equilibria
  - System is dissipative
  - Contrasts with undamped pendulum ($\ddot{u} + \sin(u) = 0$), which has closed orbits

KEY INSIGHT: The damped pendulum exhibits alternating stable foci and saddles. The damping term prevents periodic orbits and causes all motion to eventually decay to the nearest stable equilibrium (pendulum hanging down).

4 Summary and Comparison

Comparison of Parts (a) and (b)

Feature	**Part (a)**	**Part (b)**
System	$\dot{u} = v^2 - u$, $\dot{v} = u^2 - v$	$\dot{u} = v$, $\dot{v} = -v - \sin(u)$
Equilibria	2 isolated: $(0,0)$, $(1,1)$	Infinite: $(n\pi, 0)$, $n \in \mathbb{Z}$
Stable equilibria	1 stable node at $(0,0)$	Stable foci at $u = 2n\pi$
Unstable equilibria	1 saddle at $(1,1)$	Saddles at $u = (2n+1)\pi$
Periodicity	None	Period 2π in u
Symmetry	Mirror across $v = u$	Translational in u
Special orbits	None	Heteroclinic connections
Physical model	Coupled populations	Damped pendulum
Energy	Not conservative	Dissipative ($dE/dt \leq 0$)

Key Techniques Demonstrated

- **STAGE X (Analysis methods):**
  1. Finding equilibria: solve $\dot{u} = 0$, $\dot{v} = 0$ simultaneously
  2. Computing nullclines: sets where $\dot{u} = 0$ or $\dot{v} = 0$
  3. Linearization: Jacobian matrix at equilibria
  4. Eigenvalue analysis: classify by λ_1, λ_2
  5. Finding eigenvectors: determine manifold directions
  6. Sketching phase portraits: combine all information
- **STAGE Y (Classification scheme):**

$$\begin{cases} \text{Real } \lambda, \text{ same sign} & \Rightarrow \text{Node} \\ \text{Real } \lambda, \text{ opposite signs} & \Rightarrow \text{Saddle} \\ \text{Complex } \lambda = \alpha \pm i\beta & \Rightarrow \text{Focus/Spiral} \\ \text{Pure imaginary } \lambda = \pm i\beta & \Rightarrow \text{Center} \end{cases} \tag{63}$$

  Stability determined by real part:

  - $\mathrm{Re}(\lambda) < 0 \Rightarrow$ stable
  - $\mathrm{Re}(\lambda) > 0 \Rightarrow$ unstable
- **STAGE Z (Connection to course material):** Both problems illustrate concepts from lecture notes (pages 19-34):
  - Equilibrium finding and classification
  - Linearization and Jacobian matrices
  - Eigenvalue-based stability analysis
  - Phase portrait construction
  - Nullcline analysis
  - Global dynamics and manifolds

  UNIVERSAL PRINCIPLES:

  1. Equilibria occur at nullcline intersections

2. Local behavior determined by linearization
3. Eigenvalues classify equilibrium type and stability
4. Global structure built from local information
5. Nullclines guide flow direction
6. Symmetries simplify analysis

END OF QUESTION 5