Exercise Sheet 3: Bifurcations
Question 4 - Complete Solution

Methods of Applied Mathematics

Problem Statement

Consider the dynamical system:

$$\dot{x} = \alpha x - x^3$$
$$\dot{y} = -y$$

Tasks:

(a) Compute and classify the stability/type of any equilibria

(b) What bifurcation happens in the system at $\alpha = 0$?

(c) Draw a bifurcation diagram with α on the horizontal axis, and x on the vertical. What would the diagram look like if you drew α against y?

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1 Step 1: Observe System Structure

Decoupled system

The system has a special structure:

$$\dot{x} = \alpha x - x^3 \quad \text{(depends only on } x \text{ and } \alpha)$$
$$\dot{y} = -y \quad \text{(depends only on } y)$$

XYZ Analysis of Decoupling

- **STAGE X (What we have):** The equations are completely decoupled - $\dot{x}$ doesn't depend on y, and $\dot{y}$ doesn't depend on x. We can analyze each equation independently.
- **STAGE Y (Why this matters):** The decoupling means:
  - The y-dynamics are trivial: $\dot{y} = -y$ has exponential decay to zero regardless of x or α
  - The x-dynamics contain all the interesting bifurcation behavior
  - Any equilibrium must have $y^* = 0$ (from $\dot{y} = 0$), so equilibria lie on the x-axis
  - The stability in the y-direction is always the same (stable, $\lambda = -1$)
- **STAGE Z (What this means):** We essentially have a 1D bifurcation problem (in x) embedded in 2D space. The second dimension adds stable decay but doesn't affect the bifurcation structure. We can analyze $\dot{x} = \alpha x - x^3$ as a 1D system, then note that all equilibria have $y = 0$.

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2 Step 2: Find Equilibria

Equilibrium conditions

For equilibria, we require $\dot{x} = 0$ and $\dot{y} = 0$:

$$\alpha x - x^3 = 0$$
$$-y = 0$$

From the second equation: $y = 0$ (all equilibria lie on x-axis)
From the first equation:

$$x(\alpha - x^2) = 0$$

Solve for equilibrium points

This factors into: $x = 0$ or $x^2 = \alpha$
Equilibrium 1: $x = 0$

$$\boxed{(x^*, y^*) = (0, 0)} \quad \text{(exists for all } \alpha)$$

Equilibria 2 and 3: $x^2 = \alpha$
For real solutions, we need $\alpha \geq 0$:

$$\boxed{(x^*, y^*) = (\pm\sqrt{\alpha}, 0)} \quad \text{(exist only for } \alpha > 0)$$

Summary by parameter value

$$\begin{aligned}
\alpha < 0 :& \quad \text{One equilibrium: } (0, 0) \\
\alpha = 0 :& \quad \text{One equilibrium: } (0, 0) \\
\alpha > 0 :& \quad \text{Three equilibria: } (0, 0),\ (\sqrt{\alpha}, 0),\ (-\sqrt{\alpha}, 0)
\end{aligned}$$

XYZ Analysis of Equilibrium Structure

- **STAGE X (What we found):** The number of equilibria changes from 1 to 3 as α crosses zero. The origin always exists, and two new equilibria emerge symmetrically at $x = \pm\sqrt{\alpha}$ when $\alpha > 0$.
- **STAGE Y (Why this structure):** The equation $x^3 = \alpha x$ can be rewritten as $x^3 - \alpha x = x(x^2 - \alpha) = 0$. This factors completely:
  - One root at $x = 0$ always present (independent of α)
  - Two roots at $x = \pm\sqrt{\alpha}$ appear when $\alpha > 0$

  The symmetric pairing $\pm\sqrt{\alpha}$ reflects the system's symmetry: if we replace $x \to -x$, the equation $\dot{x} = \alpha x - x^3$ changes to $\dot{x} = -\alpha x + x^3 = -(\alpha x - x^3)$. So $-x$ satisfies $\dot{(-x)} = -\dot{x}$, meaning if $x(t)$ is a solution, so is $-x(t)$. This symmetry forces equilibria to appear in $\pm$ pairs (except at $x = 0$).
- **STAGE Z (What this means):** This is a supercritical pitchfork: one equilibrium "splits" into three as α increases through zero. The name "pitchfork" comes from the bifurcation diagram shape (see Step 7). The "supercritical" designation will be confirmed when we find the new equilibria are stable.

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3 Step 3: Compute Jacobian Matrix

General Jacobian

For the system $\dot{x} = f(x, y)$, $\dot{y} = g(x, y)$:

$$J = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix}$$

With $f(x, y) = \alpha x - x^3$ and $g(x, y) = -y$:

$$J = \begin{pmatrix} \alpha - 3x^2 & 0 \\ 0 & -1 \end{pmatrix}$$

XYZ Analysis of Jacobian

- **STAGE X (What we have):** A diagonal Jacobian - the off-diagonal terms are zero due to the decoupled structure.
- **STAGE Y (Why this form):** The diagonal structure directly reflects the decoupling:
  - Upper-left: $\partial(\alpha x - x^3)/\partial x = \alpha - 3x^2$ (how $\dot{x}$ responds to changes in x)
  - Lower-right: $\partial(-y)/\partial y = -1$ (how $\dot{y}$ responds to changes in y)
  - Off-diagonal zeros: no cross-coupling between x and y dynamics
- **STAGE Z (What this determines):** For a diagonal matrix, eigenvalues are simply the diagonal entries:

  $$\lambda_1 = \alpha - 3x^2 \quad \text{(controls stability in } x\text{-direction)}$$
  $$\lambda_2 = -1 \quad \text{(controls stability in } y\text{-direction)}$$

  The y-direction is always stable ($\lambda_2 = -1 < 0$). All bifurcation behavior comes from λ_1 changing sign.

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4 Step 4: Analyze Equilibrium at Origin

Jacobian at $(0, 0)$

$$J(0, 0) = \begin{pmatrix} \alpha - 3(0)^2 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} \alpha & 0 \\ 0 & -1 \end{pmatrix}$$

Eigenvalues

$$\lambda_1 = \alpha, \quad \lambda_2 = -1$$

Classify by parameter value

Case 1: $\alpha < 0$
Both eigenvalues negative: $\lambda_1 = \alpha < 0$, $\lambda_2 = -1 < 0$

$$\boxed{\text{STABLE NODE}}$$

Case 2: $\alpha = 0$

One eigenvalue zero, one negative: $\lambda_1 = 0$, $\lambda_2 = -1 < 0$

$$\boxed{\text{NEUTRAL (bifurcation point)}}$$

Case 3: $\alpha > 0$
One eigenvalue positive, one negative: $\lambda_1 = \alpha > 0$, $\lambda_2 = -1 < 0$

$$\boxed{\text{SADDLE POINT}}$$

XYZ Analysis of Origin Stability

- **STAGE X (What we found):** Origin transitions from stable node ($\alpha < 0$) through neutral ($\alpha = 0$) to saddle ($\alpha > 0$). One eigenvalue crosses zero while the other stays at -1.
- **STAGE Y (Why this transition):** The eigenvalue $\lambda_1 = \alpha$ varies linearly with the parameter:
  - For $\alpha < 0$: $\lambda_1 < 0$, so both eigenvalues negative $\rightarrow$ stable node (flows toward origin in both directions)
  - For $\alpha = 0$: $\lambda_1 = 0 \rightarrow$ one zero eigenvalue (marginal stability in x-direction, stable in y-direction)
  - For $\alpha > 0$: $\lambda_1 > 0 \rightarrow$ saddle (unstable in x-direction, stable in y-direction)

  The critical moment is when λ_1 passes through zero - this is where the x-direction loses stability. The y-direction remains stable throughout since $\lambda_2 = -1$ never changes.
- **STAGE Z (What this means physically):** Before bifurcation, the origin is an attractor - all nearby trajectories flow toward it. After bifurcation, it becomes a saddle - trajectories are repelled in the x-direction (along the unstable manifold $y = 0$) but still attracted in the y-direction. The origin loses its basin of attraction in the plane, though it remains stable along vertical lines.

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5 Step 5: Analyze Equilibria at $(\pm\sqrt{\alpha}, 0)$

Existence

These equilibria only exist for $\alpha > 0$.

Jacobian at $(\pm\sqrt{\alpha}, 0)$

$$J(\pm\sqrt{\alpha}, 0) = \begin{pmatrix} \alpha - 3(\pm\sqrt{\alpha})^2 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} \alpha - 3\alpha & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} -2\alpha & 0 \\ 0 & -1 \end{pmatrix}$$

Eigenvalues

$$\lambda_1 = -2\alpha, \quad \lambda_2 = -1$$

Stability for $\alpha > 0$

Since $\alpha > 0$:

- $\lambda_1 = -2\alpha < 0$ (negative)
- $\lambda_2 = -1 < 0$ (negative)

Both eigenvalues negative:

$$\boxed{\text{STABLE NODES}}$$

Symmetry

By symmetry of the system, both $(\sqrt{\alpha}, 0)$ and $(-\sqrt{\alpha}, 0)$ have identical stability properties.

XYZ Analysis of Emergent Equilibria

- **STAGE X (What we found):** The two new equilibria that emerge at $\alpha = 0$ are both stable nodes for $\alpha > 0$. They move away from the origin along the x-axis as $\sqrt{\alpha}$ (distance grows like square root of parameter).
- **STAGE Y (Why they're stable):** At positions $x = \pm\sqrt{\alpha}$, the eigenvalue $\lambda_1 = \alpha - 3x^2 = \alpha - 3\alpha = -2\alpha$ is negative for $\alpha > 0$. This can be understood from the 1D dynamics $\dot{x} = \alpha x - x^3$:
  - Near $x = \sqrt{\alpha}$: slightly to the right ($x > \sqrt{\alpha}$) gives $x^2 > \alpha$, so $\dot{x} = x(\alpha - x^2) < 0$ (flow leftward toward equilibrium)
  - Near $x = \sqrt{\alpha}$: slightly to the left ($x < \sqrt{\alpha}$) gives $x^2 < \alpha$, so $\dot{x} = x(\alpha - x^2) > 0$ for $x > 0$ (flow rightward toward equilibrium)

  The equilibrium at $x = \sqrt{\alpha}$ is locally attracting in the x-direction. The same argument applies by symmetry to $x = -\sqrt{\alpha}$. The y-direction is always stable.
- **STAGE Z (What this means):** This is a **supercritical** pitchfork: when the origin loses stability, two stable equilibria emerge to "catch" nearby trajectories. Contrast with subcritical pitchfork where unstable equilibria would emerge, offering no stable alternative. In applications, supercritical bifurcations are "gentler" - the system finds new stable states rather than diverging.

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6 Step 6: Identify Bifurcation Type

Observed characteristics

1. Number of equilibria changes: $1 \to 3$ as α increases through zero
2. Origin loses stability at $\alpha = 0$
3. Two new stable equilibria emerge symmetrically
4. One eigenvalue crosses zero at bifurcation
5. System has symmetry: $\dot{x}(\alpha x - x^3)$ is odd in x
6. New equilibria are stable (not unstable)

Check for pitchfork symmetry

The $\dot{x}$ equation has reflectional symmetry:

$$f(-x) = \alpha(-x) - (-x)^3 = -\alpha x + x^3 = -(\alpha x - x^3) = -f(x)$$

So $f(x, \alpha)$ is an odd function of x.

Conclusion

$$\boxed{\text{SUPERCRITICAL PITCHFORK BIFURCATION at } \alpha = 0}$$

XYZ Analysis of Bifurcation Classification

- **STAGE X (What identifies this):** All characteristics match pitchfork bifurcation: one equilibrium splits into three, symmetry present, eigenvalue crosses zero.
- **STAGE Y (Why pitchfork specifically):**
  - **Not fold**: No annihilation - equilibria are created, not destroyed
  - **Not transcritical**: No pinned equilibrium being passed through - instead, one equilibrium "branches" into three
  - **Pitchfork**: Requires reflectional symmetry $f(-x) = -f(x)$, which we verified. This symmetry forces new equilibria to appear in symmetric pairs $\pm\sqrt{\alpha}$
  - **Supercritical**: The new equilibria are stable (eigenvalue $-2\alpha < 0$ for $\alpha > 0$). If they were unstable, it would be subcritical

  The bifurcation diagram shape (see next step) resembles a pitchfork tool - one handle (origin for $\alpha < 0$) splitting into three prongs (one center prong at origin, two outer prongs at $\pm\sqrt{\alpha}$ for $\alpha > 0$).
- **STAGE Z (What symmetry requires):** The reflectional symmetry $x \to -x$ is crucial. It ensures:
  - If $(x_0, 0)$ is an equilibrium, so is $(-x_0, 0)$
  - New equilibria must appear in balanced pairs
  - The origin (fixed point of symmetry) is always an equilibrium

  Pitchfork bifurcations appear in systems with natural symmetries - e.g., mechanical systems with symmetric potentials, or equations invariant under coordinate reflections. Without this symmetry, transcritical or fold bifurcations occur instead.

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7 Step 7: Bifurcation Diagram (α vs x)

Equilibrium curves in (α, x) space

Branch 1: Origin

$$x = 0 \quad \text{for all } \alpha$$

- Stable for $\alpha < 0$ (solid line)
- Unstable (saddle) for $\alpha > 0$ (dashed line)

Branch 2: Upper equilibrium

$$x = +\sqrt{\alpha} \quad \text{for } \alpha \geq 0$$

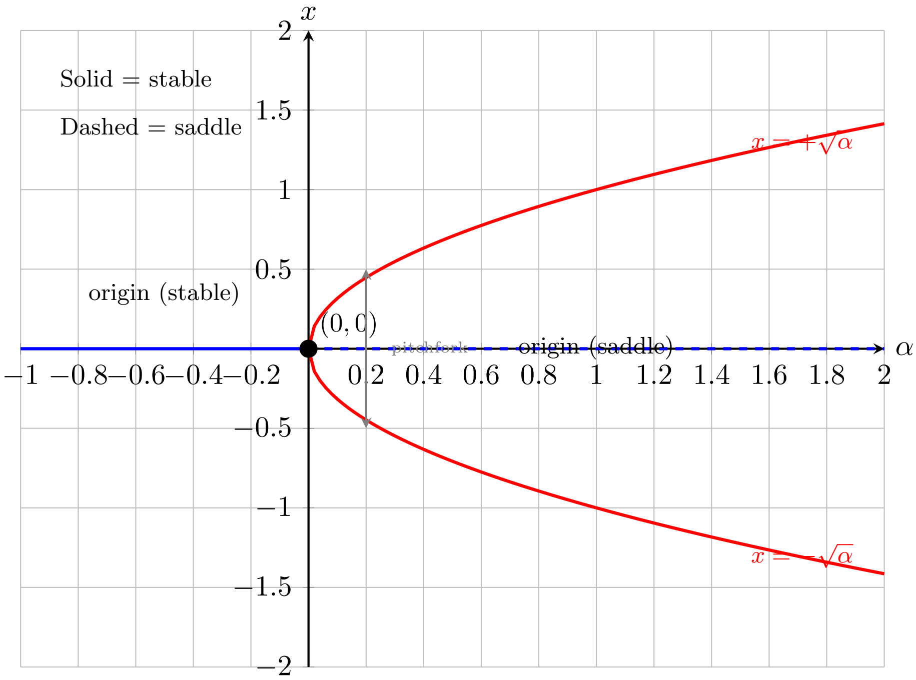
- Emerges from origin at $\alpha = 0$
- Stable for $\alpha > 0$ (solid line)

Branch 3: Lower equilibrium

$$x = -\sqrt{\alpha} \quad \text{for } \alpha \geq 0$$

- Emerges from origin at $\alpha = 0$
- Stable for $\alpha > 0$ (solid line)

Bifurcation diagram: α vs x



XYZ Analysis of Bifurcation Diagram

- **STAGE X (What the diagram shows):** One line for $\alpha < 0$ splitting into three lines for $\alpha > 0$, with the shape of a pitchfork. The middle prong (origin) changes from solid to dashed; the outer prongs are solid.
- **STAGE Y (Why this shape):** The curves $x = \pm\sqrt{\alpha}$ are parabolas on their side in (α, x) space:
  - Squaring: $x^2 = \alpha$ describes a parabola opening to the right
  - This splits into two branches: $x = +\sqrt{\alpha}$ (upper) and $x = -\sqrt{\alpha}$ (lower)
  - Both branches meet at the origin when $\alpha = 0$ (vertex of parabola)
  - The square-root dependence $x \sim \sqrt{\alpha}$ means the branches grow slowly near $\alpha = 0$ - they emerge tangent to the horizontal line at $x = 0$

  For $\alpha < 0$, only the origin branch exists (no real square roots of negative numbers). The pitchfork shape - one handle splitting into three prongs - gives the bifurcation its name.
- **STAGE Z (What this means dynamically):** Reading left to right as α increases:
  - Far left ($\alpha \ll 0$): One stable equilibrium at origin; all trajectories attracted to $(0, 0)$
  - Approaching zero: Still one attractor at origin
  - At zero: Critical point; one equilibrium with zero eigenvalue
  - Just past zero: Three equilibria appear; origin becomes saddle, two stable nodes emerge nearby
  - Far right ($\alpha \gg 0$): Two stable attractors at $x = \pm\sqrt{\alpha}$ far from origin; saddle at origin acts as separatrix

  The system's attractor "bifurcates" - where there was one stable state, there are now two stable states symmetrically placed. This is spontaneous symmetry breaking: for $\alpha > 0$, the system must "choose" between $x > 0$ or $x < 0$ basins.

8 Step 8: Bifurcation Diagram (α vs y)

Equilibrium curves in (α, y) space

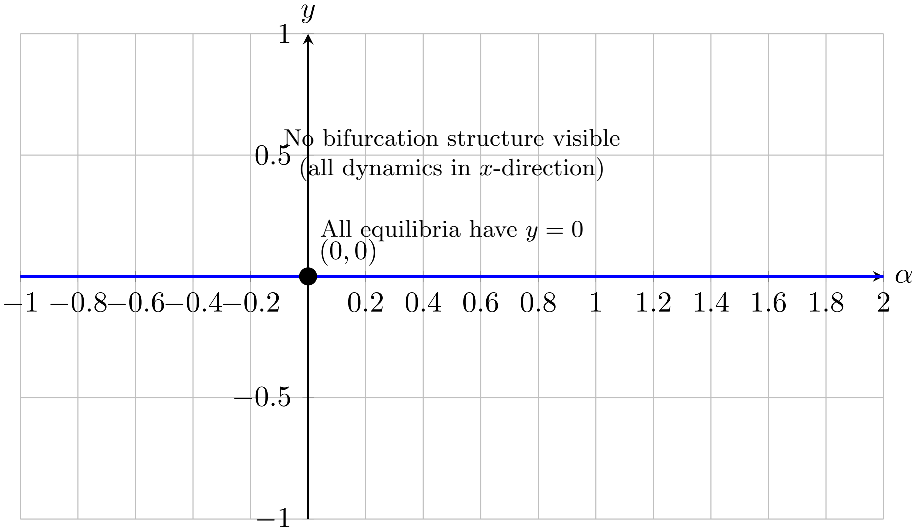
All equilibria have $y = 0$ (from $\dot{y} = -y = 0$).

Therefore, regardless of α or which equilibrium we consider:

All branches:

$$y = 0 \quad \text{for all } \alpha$$

Bifurcation diagram: α vs y



XYZ Analysis of (α, y) Diagram

- **STAGE X (What we see):** A completely flat diagram - just a horizontal line at $y = 0$. No branches separate, no structure visible.
- **STAGE Y (Why this triviality):** The y-component of all equilibria is zero because:
  - The equilibrium condition $\dot{y} = 0$ gives $-y = 0$, hence $y = 0$
  - This constraint is independent of α and independent of x
  - Whether we're at origin, $+\sqrt{\alpha}$, or $-\sqrt{\alpha}$, the y-coordinate is always zero
  - The decoupling means y-dynamics don't "know about" the bifurcation in x

  In the (α, y) projection, all three equilibrium branches collapse onto the same line $y = 0$. We lose all information about the pitchfork structure because it occurs in the orthogonal (x) direction.
- **STAGE Z (What this reveals):** The bifurcation is fundamentally 1-dimensional - it happens in the x-direction. The y-direction is a "spectator":
  - Always stable (exponential decay)
  - Always returns to $y = 0$
  - Doesn't participate in bifurcation

This demonstrates that bifurcation diagrams must be plotted against the relevant coordinate. Plotting against y (the stable, non-bifurcating direction) reveals nothing. The choice of which coordinate to plot matters - we need the coordinate where the eigenvalue crosses zero.

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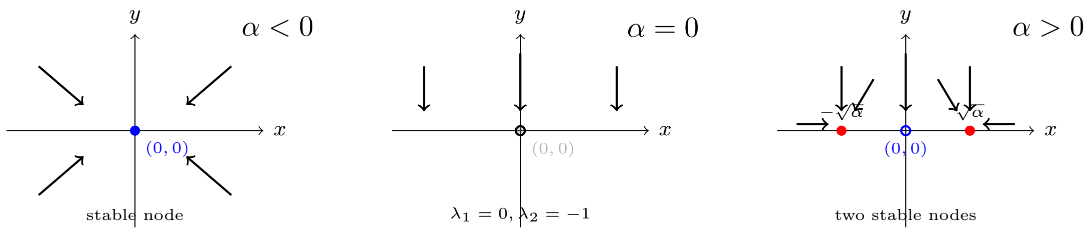
9 Step 9: Phase Portraits

Three scenarios



Notation: Filled circle = stable, hollow circle = saddle

XYZ Analysis of Phase Portrait Evolution

- **STAGE X (What we see):** The single attractor at the origin splits into two symmetric attractors on the x-axis. All trajectories decay toward $y = 0$, then flow along the x-axis toward whichever stable equilibrium is nearest.
- **STAGE Y (Why these flows):** The decoupled structure creates two-stage dynamics:
  - **Stage 1 (fast)**: $\dot{y} = -y$ causes rapid exponential decay toward $y = 0$. This happens on timescale $\tau \sim 1/|\lambda_2| = 1$.
  - **Stage 2 (slower near bifurcation)**: Once $y \approx 0$, motion is along x-axis governed by $\dot{x} = \alpha x - x^3$. Near the bifurcation ($\alpha \approx 0$), this is slow because the driving eigenvalue is small.

  For $\alpha < 0$: The 1D flow $\dot{x} = \alpha x - x^3$ has $\dot{x} < 0$ for $x > 0$ (flow leftward) and $\dot{x} > 0$ for $x < 0$ (flow rightward), both toward origin.

  For $\alpha > 0$: The 1D flow has three regions:

  - $x < -\sqrt{\alpha}$: $\dot{x} < 0$ (flow left, away from basin)
  - $-\sqrt{\alpha} < x < 0$: $\dot{x} > 0$ (flow right, toward $-\sqrt{\alpha}$)
  - $0 < x < \sqrt{\alpha}$: $\dot{x} > 0$ (flow right, toward $\sqrt{\alpha}$)
  - $x > \sqrt{\alpha}$: $\dot{x} < 0$ (flow left, toward $\sqrt{\alpha}$)
- **STAGE Z (What this means globally):** Post-bifurcation, the phase plane splits into two basins of attraction:
  - **Right basin** ($x > 0$): trajectories flow toward $(\sqrt{\alpha}, 0)$
  - **Left basin** ($x < 0$): trajectories flow toward $(-\sqrt{\alpha}, 0)$
  - **Separatrix**: the stable manifold of the saddle at origin (the y-axis) divides the basins

  This is spontaneous symmetry breaking: the system is symmetric under $x \to -x$, but any trajectory starting away from $x = 0$ must "choose" one basin. The final state breaks the symmetry even though the equations preserve it. This phenomenon appears in physics (ferromagnets below Curie temperature), biology (left/right asymmetry in embryos), and many other contexts.

10 Summary

Part (a): Equilibria and Stability

For all α: all equilibria have $y = 0$ (from $\dot{y} = -y = 0$)

Equilibrium 1: $(0, 0)$

- Always exists
- Stable node for $\alpha < 0$ (eigenvalues: $\alpha < 0, -1$)
- Neutral for $\alpha = 0$ (eigenvalues: $0, -1$)
- Saddle for $\alpha > 0$ (eigenvalues: $\alpha > 0, -1$)

Equilibria 2 and 3: $(\pm\sqrt{\alpha}, 0)$

- Exist only for $\alpha > 0$
- Both are stable nodes (eigenvalues: $-2\alpha < 0, -1$)
- Emerge from origin at $\alpha = 0$
- Move symmetrically along x-axis as $\pm\sqrt{\alpha}$

Part (b): Bifurcation at $\alpha = 0$

$$\boxed{\text{SUPERCRITICAL PITCHFORK BIFURCATION}}$$

Characteristics:

- One equilibrium splits into three
- System has reflectional symmetry: $f(-x) = -f(x)$
- New equilibria are stable (supercritical)
- Origin changes from stable node to saddle
- One eigenvalue crosses zero

Part (c): Bifurcation Diagrams

α vs x diagram:

- Shows pitchfork structure
- One branch ($x = 0$) for $\alpha < 0$ (solid)
- Three branches for $\alpha > 0$:
  - Center: $x = 0$ (dashed - saddle)
  - Upper: $x = +\sqrt{\alpha}$ (solid - stable)
  - Lower: $x = -\sqrt{\alpha}$ (solid - stable)
- Branches meet tangentially at $(\alpha, x) = (0, 0)$

α vs y diagram:

- Completely trivial: $y = 0$ for all α
- No bifurcation structure visible
- All equilibria collapse onto single horizontal line
- Demonstrates bifurcation is 1D (in x-direction only)

Key insight: The bifurcation is confined to the x-direction where the eigenvalue crosses zero. The y-direction is always stable (spectator mode) and shows no bifurcation structure. Plotting against y reveals nothing; plotting against x reveals the pitchfork.