Fourier-type integrals and steepest descent

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1. (a) $I(X) = \int_0^1 \tan t \, e^{iXt^4} \, dt$. As $X \to \infty$ the main contribution comes from the stationary point at the origin, even though the tangent vanishes there. We will see that in the following. Let us set $iXt^4 = -s$, or

$$t = e^{i\pi/8} \left(\frac{s}{X}\right)^{1/4} , \qquad dt = \frac{1}{4} e^{i\pi/8} \, s^{-3/4} \, X^{-1/4} \, ds \, .$$

Furthermore we approximate the tangent by its leading term at $t = 0$, i. e. $\tan t \sim t$ as $t \to 0$. We obtain

$$I(X) = \frac{1}{4} e^{i\pi/4} \, X^{-1/2} \int_0^{-iX} s^{-1/2} \, e^{-s} \, ds \, .$$

Finally, we can extend the integral to $-i\infty$, and we are also allowed to shift it onto the real axis (since the contribution from the quarter circle at infinity vanishes). Then with $\int_0^\infty s^{-1/2} \, e^{-s} \, ds = \Gamma(1/2)$ we obtain

$$I(X) \sim \frac{e^{i\pi/4}}{4\sqrt{X}} \; \Gamma\left(\frac{1}{2}\right) = \frac{1}{4}\sqrt{\frac{i\pi}{X}} \quad \text{as} \quad X \to 0 \, .$$

This is indeed the leading order contribution, since, for example, the contribution from the boundary at $t = 1$ is of order X^{-1}.

(b) $I(X) = \int_{1/2}^2 (1+t) \, e^{iX(t^3/3 - t)} \, dt$. The function in the exponent $\Phi(t) = t^3/3 - t$, with $\Phi'(t) = t^2 - 1$ and $\Phi''(t) = 2t$, has stationary points at $t = -1$ and $t = 1$. Only the second one is inside the integration range. Consequently, the leading order asymptotic form of $I(X)$ for $X \to \infty$ is determined by the stationary point at $t = 1$. We expand the exponent around the stationary point up to second order, evaluate the prefactor $(1 + t)$ at $t = 1$, extend the integration to $\pm\infty$, and obtain

$$I(X) \sim \int_{-\infty}^{\infty} 2 \, \exp\left(-\frac{2}{3} iX + iX(t-1)^2\right) \, dt \sim 2 \, e^{-2iX/3} \sqrt{\frac{i\pi}{X}} \quad \text{as} \quad X \to \infty \, .$$

(c) $I(X) = \int_0^\infty (1+t^2)^{-1} \, e^{iXt} \, dt$. There is no stationary point in the interval of integration. Therefore we obtain the leading order contribution for $X \to \infty$ by integration by parts

$$\int_0^\infty \frac{1}{1+t^2} \, e^{iXt} \, dt = \left[\frac{1}{iX} \frac{1}{1+t^2} \, e^{iXt}\right]_0^\infty - \frac{1}{iX} \int_0^\infty \frac{d}{dt} \left(\frac{1}{1+t^2}\right) \, e^{iXt} \, dt \, .$$

The integral on the RHS vanishes as $X \to \infty$ because of the Riemann-Lebesgue lemma. Therefore we obtain

$$I(X) \sim \frac{i}{X} \quad \text{as} \quad X \to \infty\,.$$

(d) $I(X) = \int_0^\infty e^{iX(2t-t^2)} \log(1+t^2)\,dt$. The function in the exponent $\Phi(t) = 2t - t^2$, with $\Phi'(t) = 2 - 2t$ and $\Phi''(t) = -2$, has a stationary point at $t = 1$. We obtain the leading order approximation for $X \to \infty$ by expanding the exponent around the stationary point up to second order, evaluating the prefactor $\log(1+t)$ at $t = 1$, and extending the integration range to $\pm\infty$. This results in

$$I(X) \sim \int_{-\infty}^{\infty} \log 2\, \exp(iX - iX(t-1)^2)\,dt = \log 2 \sqrt{\frac{\pi}{iX}}\, e^{iX} \quad \text{as} \quad X \to \infty\,.$$

(e) $I(X) = \int_0^\pi \sin(X\,\cos t)\, e^{-t^2}\,dt$. We write this in the form

$$I(X) = \operatorname{Im} \int_0^\pi e^{-t^2}\, e^{iX\cos t}\,dt\,.$$

The function in the exponent $\Phi(t) = \cos t$, with $\Phi'(t) = -\sin t$ and $\Phi''(t) = -\cos t$, has stationary points at the two end points of the integration range. Both contribute. The contribution of a stationary point at an end point is half the contribution of an internal stationary point. Using the formula from the lecture for the contribution of a stationary point at an end point $t = c$

$$\sqrt{\frac{i\,\pi}{2\,X\,\Phi''(c)}}\, f(c)\, e^{i\,X\,\Phi(c)}\,,$$

we find for $X \to \infty$

$$I(X) \sim \operatorname{Im} \sqrt{\frac{\pi}{2\,i\,X}}\, e^{iX} + \operatorname{Im} \sqrt{\frac{i\,\pi}{2\,X}}\, e^{-iX-\pi^2} \sim \sqrt{\frac{\pi}{2\,X}}\, \sin\left(X - \frac{\pi}{4}\right)\left(1 - e^{-\pi^2}\right)\,.$$

2. (a) $I(X) = \int_{-1}^\infty e^{X(t+it-t^2/2)}\,dt$. We rename t by z and look for the path of steepest descent, starting at $z = -1$. The function in the exponent $\Phi(z) = z + iz - z^2/2$ has real and imaginary parts (with $z = x + iy$)

$$u(x,y) = x - y - \frac{x^2}{2} + \frac{y^2}{2} \quad \text{and} \quad v(x,y) = x + y - xy\,.$$

Thus the steepest contour is given by $v(x,y) = v(-1,0)$, or $y = (x+1)/(x-1)$. Looking at the real part of $\Phi(z)$ we find that the part for $-\infty < x \le -1$ corresponds to the contour of steepest descent, which we call C_1.

To deform the original path of integration onto curves of steepest decent, we somehow have to get back to $z = +\infty$. This can only happen along a path through a saddle point. The derivative of $\Phi(z)$ is $\Phi'(z) = 1 + i - z$, and hence there is a saddle point at $z = 1 + i$. Let us look at the steepest contours through this saddle point. They are determined by $v(x,y) = v(1,1) = 1$, or $0 = v(x,y) - 1 = -(y-1)\,(x-1)$, i. e.

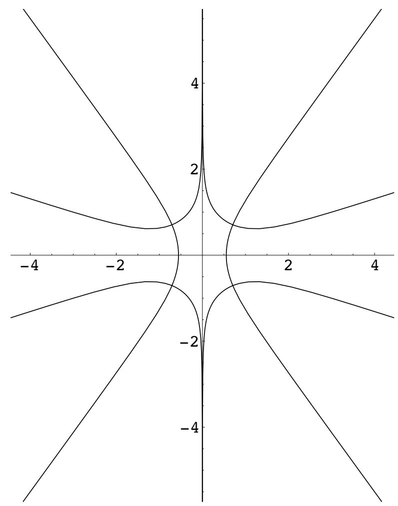

Figure 1: Steepest ascent-descent paths passing through saddle points of $i(t^5/5+t)$ in part 2(b).

by $x = 1$ and $y = 1$. Looking at the real part of $\Phi(z)$ we see that $y = 1$ corresponds to the contour of steepest descent. We call this curve C_2, and it is possible to deform the integration path onto the contours C_1 plus C_2 (the difference at infinity goes to zero).

We conclude that the possible asymptotic contributions to the integral come from small vicinities of the starting point $z = -1$ and the saddle point $z = 1+i$. Comparing the real parts of $\Phi(z)$ at these points, $u(-1, 0) = -3/2$ and $u(1, 1) = 0$, we see that the contribution from the endpoint is subdominant and can be neglected. Hence the full asymptotic contribution comes from the path C_2 through the saddle point. We evaluate this integral by setting $z = x + i$ and find

$$I(X) \sim \int_{-\infty}^{\infty} e^{iX - X(x^2/2 - x + 1/2)}\, dx = e^{iX} \int_{-\infty}^{\infty} e^{-X(x-1)^2/2}\, dx = e^{iX}\sqrt{\frac{2\pi}{X}} \quad \text{as} \quad X \to \infty.$$

(b) $I(X) = \int_{-\infty}^{\infty} t\,(1+t^2)^{-1}\, e^{iX(t^5/5+t)}\, dt$. We rename t by z. The function in the exponent $\Phi(z) = i(z^5/5 + z)$, with derivative $\Phi'(z) = i(z^4 + 1)$, has saddle points at

$$e^{i\pi/4}, \qquad e^{3i\pi/4}, \qquad e^{5i\pi/4}, \qquad e^{7i\pi/4}.$$

In polar coordinates, the real and imaginary parts of $\Phi(z)$ are given by

$$\Phi(z) = u(r,\theta) + i\, v(r,\theta) = -\left(\frac{r^5}{5}\sin(5\theta) + r\sin\theta\right) + i\left(\frac{r^5}{5}\cos(5\theta) + r\cos\theta\right).$$

First we determine the steepest contours. At the saddle points $z^4 = -1$ and thus $\Phi(z) = \frac{4}{5}iz$. For example, at the saddle point $z = e^{i\pi/4}$ we have $\Phi(e^{i\pi/4}) = \frac{4}{5}e^{3i\pi/4}$ and therefore its imaginary part is given by $v(1, \pi/4) = \frac{4}{5\sqrt{2}}$. Considering also the other saddle points we find that the steepest contours are determined by the equation

$$v(r,\theta) = \pm\frac{4}{5\sqrt{2}}.$$

All these steepest descent-ascent curves are plotted in figure 1. To find out which of these are steepest descent paths we compare the values of $\operatorname{Re}\Phi(z)$ at the stationary points with values at other points, e. g. on the x-axis where $\theta = 0$ or $\theta = \pi$. The result is that the curves that intersect the x-axis are steepest ascent curves for the saddle points $e^{i\pi/4}$ and $e^{3i\pi/4}$, and steepest descent curves for $e^{5i\pi/4}$ and $e^{7i\pi/4}$. Consequently, the curves that lie entirely above the x-axis are steepest descent curves, and those that lie entirely below the x-axis are steepest ascent curves. For this reason, we shift the path of integration onto the curves that lie entirely above the x-axis and pass through $e^{i\pi/4}$ and $e^{i3\pi/4}$.

We integrate along the steepest descent paths by setting

$$\Phi(z) = \Phi(e^{i\pi/4}) - s^2\,, \qquad \text{and} \qquad \Phi(z) = \Phi(e^{3i\pi/4}) - t^2\,,$$

where s and t are real along these paths. Since we are only interested in the leading order contribution we can find explicit relations between s and z (t and z) by expanding $\Phi(z)$ only up to the quadratic term

$$2\,e^{5i\pi/4}(z - e^{i\pi/4})^2 \approx -s^2\,, \qquad \text{and} \qquad 2\,e^{3i\pi/4}(z - e^{3i\pi/4})^2 \approx -t^2\,.$$

So

$$ds \approx \sqrt{2}\,e^{i\pi/8}\,dz\,, \qquad \text{and} \qquad dt \approx \sqrt{2}\,e^{-i\pi/8}\,dz\,.$$

We then have

$$\begin{aligned}
I(X) &\sim \frac{e^{i\pi/8}}{\sqrt{2}\,(1+e^{i\pi/2})}\int_{-\infty}^{\infty} e^{\frac{4X}{5}e^{3i\pi/4} - X\,s^2}\,ds + \frac{e^{7i\pi/8}}{\sqrt{2}\,(1+e^{3i\pi/2})}\int_{-\infty}^{\infty} e^{\frac{4X}{5}e^{5i\pi/4} - X\,t^2}\,dt \\
&\sim \sqrt{\frac{\pi}{2\,X}}\left(\frac{e^{i\pi/8}}{1+i}e^{-\frac{4}{5\sqrt{2}}(1-i)X} + \frac{e^{7i\pi/8}}{1-i}e^{-\frac{4}{5\sqrt{2}}(1+i)X}\right) \\
&\sim \frac{1}{2}\sqrt{\frac{\pi}{X}}\,e^{-\frac{4}{5\sqrt{2}}X}\left(e^{-i\pi/8 + \frac{4iX}{5\sqrt{2}}} - e^{i\pi/8 - \frac{4iX}{5\sqrt{2}}}\right) \\
&\sim i\sqrt{\frac{\pi}{X}}\,e^{-\frac{4}{5\sqrt{2}}X}\,\sin\left(\frac{4X}{5\sqrt{2}} - \frac{\pi}{8}\right) \qquad \text{as} \qquad X \to \infty\,.
\end{aligned}$$

(c) $I(X) = \int_0^\infty e^{-t}\,e^{iX(t^4/4+t^3/3)}\,dt$. We rename t by z. The function in the exponent $\Phi(z) = i(z^4/4 + z^3/3)$, with derivative $\Phi'(z) = i(z^3 + z^2)$, has a saddle point at $z = -1$, and one of order two at $z = 0$. We are only interested in the one at $z = 0$. In polar coordinates, the real and imaginary parts of $\Phi(z)$ are given by

$$\Phi(z) = -\left(\frac{r^4}{4}\sin(4\theta) + \frac{r^3}{3}\sin(3\theta)\right) + i\left(\frac{r^4}{4}\cos(4\theta) + \frac{r^3}{3}\cos(3\theta)\right)\,,$$

and the steepest descent-ascent curves through the saddle point $z = 0$ are determined by $\operatorname{Im}\Phi(z) = 0$ or

$$r = -\frac{4\,\cos(3\theta)}{3\,\cos(4\theta)}\,.$$

This condition describes three curves that intersect at equal angles at the saddle point $z = 0$. Let us consider them as six half lines. They approach the origin from angular directions that are determined by $\cos(3\theta) = 0$. If one goes around the saddle point then these six half lines are alternatingly steepest descent and steepest ascent curves. The ones closest to the real axis have angular directions $\theta = \pm\pi/6$, and a closer examination of the real part of $\Phi(z)$ shows that the one above the x-axis is a contour of steepest descent. For large values of r it has an asymptote given by $\theta = \pi/8$. We can shift the integration contour onto this curve.

Since the steepest contour condition implies that $\Phi(z)$ is purely real along this contour we set $\Phi(z) = -s$ where s varies between 0 and infinity. We can simplify the relation between s and z (since we are only interested in the leading order approximation) by replacing $\Phi(z)$ by its leading order term near $z = 0$

$$s = -\Phi(z) \approx -i\frac{z^3}{3} \qquad \text{and} \qquad dz \approx 3^{-2/3}\, e^{i\pi/6}\, s^{-2/3}\, ds\ .$$

Finally, we obtain

$$I(X) \sim 3^{-2/3}\, e^{i\pi/6} \int_0^\infty e^{-Xs}\, s^{-2/3}\, ds \sim \frac{e^{i\pi/6}\,\Gamma\left(\frac{1}{3}\right)}{3^{2/3}\, X^{1/3}} \qquad \text{as} \qquad X \to \infty\ .$$

(d) $I(X) = \int_0^1 \frac{e^{iXt^2}}{t^2-t+\frac{5}{16}}\, dt$. This example is similar to an example discussed in class. Denote by C_4 the interval $[0, 1]$ oriented in the positive direction of the real axis. The difference with the example discussed in class is that now the integrand has a simple pole inside the closed contour $\Gamma = C_1 \cup C_2 \cup (-C_3) \cup (-C_4)$. (Here we have used the same notation as in the notes.) Indeed

$$t^2 - t + \frac{5}{16} = \left(t - \frac{1}{2} - \frac{i}{4}\right)\left(t - \frac{1}{2} + \frac{i}{4}\right).$$

Recall that the residue theorem states that if Γ is a closed contour in the complex plane $f(z)$ is analytic inside Γ except at a finite number of isolated points $a_1, \ldots, a_n$, then

$$\oint_\Gamma f(z)dz = 2\pi i \sum \operatorname*{Res}_{z=a_k} f(z). \tag{1}$$

Here the notation $\operatorname*{Res}_{z=a_k} f(z)$ denotes the residue of f at a_k. If a_k is a simple pole of f, then

$$\operatorname*{Res}_{z=a_k} f(z) = \lim_{z\to a_k} (z - a_k) f(z).$$

Therefore, we have

$$\operatorname*{Res}_{z=\left(\frac{1}{2}+\frac{i}{4}\right)} \frac{e^{iXt^2}}{t^2 - t + \frac{5}{16}} = -2ie^{-\frac{X}{4}\left(1-\frac{3i}{4}\right)}.$$

The residue theorem gives us

$$\begin{aligned} \int_0^1 \frac{e^{iXt^2}}{t^2 - t + \frac{5}{16}} dt = \int_{C_1} \frac{e^{iXt^2}}{t^2 - t + \frac{5}{16}} dt + \int_{C_2} \frac{e^{iXt^2}}{t^2 - t + \frac{5}{16}} dt \\ - \int_{C_3} \frac{e^{iXt^2}}{t^2 - t + \frac{5}{16}} dt - 4\pi e^{-\frac{X}{4}\left(1-\frac{3i}{4}\right)}. \end{aligned} \tag{2}$$

Following the same steps of the notes and using (2) we arrive at

$$I(X) \sim \frac{8}{5}\sqrt{\frac{i\pi}{X}} - 4\pi e^{-\frac{X}{4}\left(1-\frac{3i}{4}\right)}. \quad \text{as} \quad X \to \infty\ .$$

(e) $I(X) = \int_{-1}^{\infty} \sqrt{1+t}\, \cos(Xt^2)\, e^{X(t-t^3/3)}\, dt$. The integral can be written in the form

$$I(X) = \mathrm{Re} \int_{-1}^{\infty} \sqrt{1+z}\, e^{X(z+iz^2-z^3/3)}\, dz \ .$$

The function in the exponent, $\Phi(z) = z + iz^2 - z^3/3$, has derivative $\Phi'(z) = -(z-i)^2$ and thus a saddle point at $z = i$. Real and imaginary parts of $\Phi(z)$ are given by

$$\Phi(x+iy) = \left(-\frac{x^3}{3} + xy^2 - 2xy + x \right) + i \left(\frac{1}{3}(y-1)^3 - x^2\,(y-1) + \frac{1}{3} \right) \ .$$

The steepest contours through the saddle point are determined by $\mathrm{Im}\,\Phi(z) = 1/3$, leading to $y = 1$ and $y = \pm\sqrt{3}x + 1$. Since it is a saddle point of order two, these are *three* lines intersecting at equal angles at the saddle point. If one goes around the saddle point then they represent six rays which alternatingly are contours of steepest ascent and steepest descent. The line $y = 1$, $x > 0$, is a steepest descent curve. If possible, we would like to deform the integration path onto curves of steepest descent.

Let us look at the steepest descent path that starts at the starting point of the integral $z = -1$. It is determined by $\mathrm{Im}\,\Phi(z) = 1$, runs in the third quadrant, and has $y = \sqrt{3}x + 1$ as an asymptote. This asymptote, however, is a steepest descent path of the saddle point. For this reason, we can deform the integration path in the following way. Starting from $z = -1$ we follow the steepest descent path $\mathrm{Im}\,\Phi(z) = 1$ in the third quadrant until infinity. We then come back along $y = \sqrt{3}x + 1$ to the saddle point $z = i$. Finally, we follow the steepest descent path $y = 1$ in the first quadrant until infinity.

The leading order asymptotic contribution comes from the vicinity of the stationary point. First we consider the line $y = 1$, $x > 0$. To integrate along this line we set $(z - i) = s$, $s \in [0, \infty)$. Furthermore, we can approximate $\Phi(z)$ by its Taylor expansion up to the first non-trivial term, since we are only interested in the leading order behaviour as $X \to \infty$,

$$\Phi(z) \approx \Phi(i) - \frac{1}{3}(z-i)^3 = \frac{i}{3} - \frac{s^3}{3} \ .$$

We thus obtain the following asymptotic contribution, as $X \to \infty$, from the path along $y = 1$, $x > 0$,

$$\begin{aligned} I_1(X) &\sim \mathrm{Re}\ \sqrt{1+i} \int_0^{\infty} e^{iX/3 - Xs^3/3}\, ds \\ &\sim \mathrm{Re}\ 2^{1/4}\, e^{i\pi/8} \int_0^{\infty} e^{iX/3}\, 3^{-2/3}\, X^{-1/3}\, t^{-2/3}\, e^{-t}\, dt \\ &\sim \mathrm{Re}\ \frac{2^{1/4}\,\Gamma(1/3)}{3^{2/3}\, X^{1/3}} e^{iX/3 + i\pi/8} \ , \end{aligned}$$

where $t = Xs^3/3$.

Similarly, we integrate along the steepest descent path $y = \sqrt{3}x + 1$, $x < 0$, by setting $(z - i) = e^{-2\pi i/3} s$, $s \in [0, \infty)$. Then $dz = e^{-2\pi i/3}\, ds$. We also approximate $\Phi(z)$ again by

$$\Phi(z) \approx \Phi(i) - \frac{1}{3}(z-i)^3 = \frac{i}{3} - \frac{s^3}{3} \ .$$

The contribution from the path along $y = \sqrt{3}x + 1$, $x < 0$, is thus given by

$$
\begin{aligned}
I_2(X) &\sim \mathrm{Re}\,\sqrt{1+i}\int_\infty^0 e^{iX/3 - Xs^3/3}\, e^{-2\pi i/3}\, ds \\
&\sim -\,\mathrm{Re}\,2^{1/4}\, e^{i\pi/8}\, e^{-2\pi i/3} \int_0^\infty e^{iX/3}\, 3^{-2/3}\, X^{-1/3}\, t^{-2/3}\, e^{-t}\, dt \\
&\sim -\,\mathrm{Re}\,\frac{2^{1/4}\,\Gamma(1/3)}{3^{2/3}\, X^{1/3}} e^{iX/3 + i\pi/8 - 2\pi i/3} \,,
\end{aligned}
$$

as $X \to \infty$.

The final result for the asymptotic behaviour of $I(X)$ as $X \to \infty$ has the form

$$
I(X) = I_1(X) + I_2(X) \sim \frac{2^{1/4}\,\Gamma\left(\frac{1}{3}\right)}{3^{1/6}\, X^{1/3}} \cos\left(\frac{X}{3} + \frac{7\pi}{24}\right) .
$$

3. $[\Gamma(X)]^{-1} = (2\pi i)^{-1}\int_\gamma e^{t - X\log t}\, dt$. The calculation is similar to that in the lecture for the integral representation of $\Gamma(X)$. First we set $t = Xz$ in order that the function multiplying X in the exponent has a saddle point

$$
\frac{1}{\Gamma(X)} = \frac{X}{2\pi i}\int_\gamma e^{X(z - \log z) - X\log X}\, dz \,.
$$

Now the function $\Phi(z) = z - \log z$, with $\Phi'(z) = 1 - z^{-1}$ and $\Phi''(z) = z^{-2}$, has a saddle point at $z = 1$. The steepest contours through this saddle point are determined by the condition $0 = \mathrm{Im}\,\Phi(z) = r\sin\theta - \theta$, or

$$
r = \frac{\theta}{\sin\theta} \,.
$$

The solution $\theta = 0$ corresponds to the steepest ascent curve, and the other solution has the property that for $r \to \infty$ the angular coordinate behaves as $\theta \to \pm\pi$. Furthermore, $y = r\sin\theta = \theta \to \pm\pi$. Consequently $y = \pm\pi$ are the two asymptotes of the curve of steepest descent as $x \to -\infty$. We can shift the integration contour onto this steepest descent curve.

We perform a change of parameter $\Phi(z) = \Phi(0) - s^2$ where s is real along the steepest descent curve and varies between $-\infty$ and ∞. Since we are interested only in the leading order behaviour of $I(X)$ as $X \to \infty$, we can simplify the relation between z and s by expanding $\Phi(z)$ near the saddle point up to second order

$$
\Phi(z) \approx 1 + \frac{1}{2}(z-1)^2 \qquad \Longrightarrow \qquad s \approx \frac{-i}{\sqrt{2}}(z-1) \,.
$$

We chose $\sqrt{-1} = -i$ here in order that the transformation from z to s does not change the sense of integration. (The steepest descent curve has a vertical tangent at $z = 1$ so that locally $z \approx 1 + iy$.) Finally, we find

$$
\frac{1}{\Gamma(X)} \sim \frac{i\,X\,e^{-X\log X}}{2^{1/2}\,\pi\, i}\int_{-\infty}^{\infty} e^{X - Xs^2}\, ds \sim \frac{e^X}{X^X}\sqrt{\frac{X}{2\pi}} \qquad \text{as} \qquad X \to \infty \,.
$$