Exercise Sheet 3: Bifurcations
Question 5 - Complete Solution

Methods of Applied Mathematics

Problem Statement

Show that a Hopf bifurcation happens in the system:

$$\frac{dx}{dt} = 1 + x^2y - \mu x - x$$
$$\frac{dy}{dt} = \mu x - x^2y$$

as μ varies.

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1 Step 1: Simplify and Understand System

Rewrite first equation

Collect terms in the first equation:

$$\frac{dx}{dt} = 1 - x - \mu x + x^2y = 1 - (1 + \mu)x + x^2y$$

So the system is:

$$\dot{x} = 1 - (1 + \mu)x + x^2y$$
$$\dot{y} = \mu x - x^2y$$

Observe structure

Notice that both equations involve the term x^2y with opposite signs.

XYZ Analysis of System Structure

- **STAGE X (What we have):** A coupled nonlinear system where both variables appear in both equations. The term x^2y appears with opposite signs in the two equations, suggesting a conservation-like structure.
- **STAGE Y (Why this matters):** The symmetry in how x^2y appears (positive in $\dot{x}$, negative in $\dot{y}$) hints at an underlying structure. If we add the equations:

  $$\dot{x} + \dot{y} = 1 - x - \mu x + x^2y + \mu x - x^2y = 1 - x$$

  The x^2y terms cancel and μx terms cancel, leaving a simple relationship. This kind of structure often appears in chemical reaction models (like the Brusselator family) where x^2y represents an autocatalytic reaction rate.

- **STAGE Z (What to expect):** For Hopf bifurcations, we need:
  - An equilibrium that doesn't move much with parameter (or moves in a controlled way)
  - Eigenvalues that are complex for a range of μ
  - Real part of eigenvalues crossing zero at critical μ
  - Imaginary part remaining nonzero at crossing

  The nonlinearity and coupling suggest oscillatory behavior is possible.

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2 Step 2: Find Equilibria

Set up equilibrium conditions

For equilibria, we require $\dot{x} = 0$ and $\dot{y} = 0$:

$$\begin{aligned} 1 - (1+\mu)x + x^2 y &= 0 \quad \cdots (1) \\ \mu x - x^2 y &= 0 \quad \cdots (2) \end{aligned}$$

Solve systematically

From equation (2):

$$x(\mu - xy) = 0$$

This gives either $x = 0$ or $xy = \mu$.
Case 1: $x = 0$
Substitute into equation (1):

$$1 - 0 + 0 = 1 = 0 \quad \text{Contradiction!}$$

So $x = 0$ is not an equilibrium.
Case 2: $xy = \mu$
This means $y = \mu/x$ (assuming $x \neq 0$).
Substitute into equation (1):

$$\begin{aligned} 1 - (1+\mu)x + x^2 \cdot \frac{\mu}{x} &= 0 \\ 1 - (1+\mu)x + x\mu &= 0 \\ 1 - x - \mu x + \mu x &= 0 \\ 1 - x &= 0 \end{aligned}$$

Therefore: $x = 1$
And: $y = \mu/1 = \mu$

Equilibrium

$$\boxed{(x^*, y^*) = (1, \mu)}$$

Verify equilibrium

Check equation (1): $1 - (1+\mu)(1) + (1)^2(\mu) = 1 - 1 - \mu + \mu = 0$
Check equation (2): $\mu(1) - (1)^2(\mu) = \mu - \mu = 0$

XYZ Analysis of Equilibrium

- **STAGE X (What we found):** Unique equilibrium at $(1, \mu)$. The x-coordinate is constant ($x^* = 1$) while the y-coordinate moves linearly with parameter ($y^* = \mu$).
- **STAGE Y (Why this structure):** The equilibrium position in y tracks the parameter μ directly. This is unusual but not unprecedented. The key observation is that from $\mu x = x^2 y$ at equilibrium, we have $y = \mu/x$. Since $x = 1$ was forced by the first equation (independent of μ), we get $y = \mu$ automatically. The first equation $1 - x + x^2 y - \mu x = 0$ becomes $1 - x = 0$ after using $x^2 y = \mu x$ from the second equation, giving $x = 1$ regardless of μ.
- **STAGE Z (What this means):** The equilibrium moves through the phase plane as μ varies, but only in the y-direction. For Hopf bifurcation, we need the stability at this moving equilibrium to change - specifically, eigenvalues must become complex with zero real part at some critical μ^*. Let's compute the Jacobian to investigate stability.

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3 Step 3: Compute Jacobian Matrix

Partial derivatives

For $f(x, y) = 1 - (1 + \mu)x + x^2 y$ and $g(x, y) = \mu x - x^2 y$:

$$\frac{\partial f}{\partial x} = -(1 + \mu) + 2xy$$
$$\frac{\partial f}{\partial y} = x^2$$
$$\frac{\partial g}{\partial x} = \mu - 2xy$$
$$\frac{\partial g}{\partial y} = -x^2$$

Jacobian at general point

$$J(x, y) = \begin{pmatrix} -(1 + \mu) + 2xy & x^2 \\ \mu - 2xy & -x^2 \end{pmatrix}$$

Jacobian at equilibrium $(1, \mu)$

Substitute $x = 1, y = \mu$:

$$\left.\frac{\partial f}{\partial x}\right|_{(1,\mu)} = -(1 + \mu) + 2(1)(\mu) = -1 - \mu + 2\mu = \mu - 1$$
$$\left.\frac{\partial f}{\partial y}\right|_{(1,\mu)} = (1)^2 = 1$$
$$\left.\frac{\partial g}{\partial x}\right|_{(1,\mu)} = \mu - 2(1)(\mu) = \mu - 2\mu = -\mu$$
$$\left.\frac{\partial g}{\partial y}\right|_{(1,\mu)} = -(1)^2 = -1$$

Therefore:

$$J(1, \mu) = \begin{pmatrix} \mu - 1 & 1 \\ -\mu & -1 \end{pmatrix}$$

XYZ Analysis of Jacobian

- **STAGE X (What we have):** A parameter-dependent Jacobian with all four entries depending on μ. Unlike previous examples with diagonal or simple structure, this is a fully-populated 2×2 matrix.
- **STAGE Y (Why this structure):** The Jacobian entries show how the linearized dynamics depend on μ:
  - Upper-left $(\mu - 1)$: Self-feedback in x-equation, changes sign at $\mu = 1$
  - Upper-right (1): Constant positive coupling from y to $\dot{x}$
  - Lower-left $(-\mu)$: Negative feedback from x to $\dot{y}$, grows with μ
  - Lower-right (-1): Constant negative self-feedback in y-equation (damping)

  The non-diagonal structure means the eigenvalues will generally be complex. The parameter-dependence in multiple entries suggests eigenvalues will move in the complex plane as μ varies.
- **STAGE Z (What to compute):** To identify a Hopf bifurcation, we need:
  1. Find the eigenvalues as functions of μ
  2. Identify critical value μ^* where real part is zero
  3. Verify imaginary part is nonzero at μ^*
  4. Check that real part crosses zero transversely (nonzero derivative)

  Let's compute the characteristic equation.

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4 Step 4: Find Eigenvalues

Compute trace and determinant

$$\begin{aligned}\text{Trace: } \tau &= (\mu - 1) + (-1) = \mu - 2 \\ \text{Determinant: } \Delta &= (\mu - 1)(-1) - (1)(-\mu) \\ &= -\mu + 1 + \mu = 1\end{aligned}$$

Notice: $\Delta = 1 > 0$ for all μ (constant!)

Characteristic equation

$$\lambda^2 - \tau\lambda + \Delta = 0$$

$$\lambda^2 - (\mu - 2)\lambda + 1 = 0$$

Solve for eigenvalues

Using quadratic formula:

$$\lambda = \frac{(\mu - 2) \pm \sqrt{(\mu - 2)^2 - 4}}{2}$$

Analyze discriminant

Let $\Delta_{disc} = (\mu - 2)^2 - 4$

$$\begin{aligned}
\Delta_{disc} > 0 \quad &\Rightarrow \quad \text{real eigenvalues} \\
\Delta_{disc} = 0 \quad &\Rightarrow \quad \text{repeated real eigenvalue} \\
\Delta_{disc} < 0 \quad &\Rightarrow \quad \text{complex eigenvalues}
\end{aligned}$$

$(\mu - 2)^2 - 4 = 0$ when $|\mu - 2| = 2$, i.e., $\mu = 0$ or $\mu = 4$

- For $\mu < 0$: $\Delta_{disc} > 0$ (real eigenvalues)
- For $0 < \mu < 4$: $\Delta_{disc} < 0$ (complex eigenvalues)
- For $\mu > 4$: $\Delta_{disc} > 0$ (real eigenvalues)

Complex eigenvalues for $0 < \mu < 4$

When eigenvalues are complex:

$$\lambda = \frac{\mu - 2}{2} \pm i\frac{\sqrt{4 - (\mu - 2)^2}}{2}$$

Define:

$$\begin{aligned}
\rho(\mu) = \text{Re}(\lambda) &= \frac{\mu - 2}{2} \\
\omega(\mu) = \text{Im}(\lambda) &= \pm\frac{\sqrt{4 - (\mu - 2)^2}}{2}
\end{aligned}$$

XYZ Analysis of Eigenvalue Structure

- **STAGE X (What we found):** The determinant is constant ($\Delta = 1$) but the trace varies linearly with μ ($\tau = \mu - 2$). For $0 < \mu < 4$, eigenvalues are complex conjugates.
- **STAGE Y (Why complex eigenvalues):** The constant positive determinant $\Delta = 1$ ensures:

  $$\lambda_1 \cdot \lambda_2 = 1 > 0$$

  So eigenvalues have the same sign (both positive, both negative, or complex conjugates). The discriminant:

  $$\tau^2 - 4\Delta = (\mu - 2)^2 - 4$$

  is negative when $|\mu - 2| < 2$, forcing eigenvalues to be complex in this range. The geometry: in the trace-determinant plane, the point $(\tau, \Delta) = (\mu - 2, 1)$ moves horizontally (at fixed $\Delta = 1$) as μ varies. It crosses into the "complex eigenvalue" region when $|\tau| < 2\sqrt{\Delta} = 2$.
- **STAGE Z (What this means):** Complex eigenvalues indicate oscillatory behavior (spirals in phase plane). The real part $\rho(\mu) = (\mu - 2)/2$ determines whether spirals approach equilibrium (stable, $\rho < 0$) or recede (unstable, $\rho > 0$). The critical value $\rho = 0$ occurs at $\mu = 2$. This is where the Hopf bifurcation happens!

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5 Step 5: Identify Hopf Bifurcation Point

Find critical parameter value

Real part is zero when:

$$\rho(\mu) = \frac{\mu - 2}{2} = 0 \quad \Rightarrow \quad \boxed{\mu^* = 2}$$

Check imaginary part at critical value

At $\mu = 2$:

$$\omega(2) = \pm\frac{\sqrt{4 - (2-2)^2}}{2} = \pm\frac{\sqrt{4}}{2} = \pm 1$$

So the eigenvalues at $\mu = 2$ are:

$$\lambda = 0 \pm i(1) = \pm i$$

Verification by substituting into characteristic equation:

$$\lambda^2 - (2-2)\lambda + 1 = \lambda^2 + 1 = 0 \quad \Rightarrow \quad \lambda = \pm i \quad \checkmark$$

Verify imaginary part is nonzero

$$\omega(2) = 1 \neq 0 \quad \checkmark$$

XYZ Analysis of Critical Point

- **STAGE X (What we found):** At $\mu = 2$, the equilibrium $(1, 2)$ has purely imaginary eigenvalues $\lambda = \pm i$.
- **STAGE Y (Why this is special):** Purely imaginary eigenvalues mean the linearized system near the equilibrium has periodic solutions - closed orbits with frequency $\omega = 1$ (period 2π). The zero real part means these orbits neither grow nor decay - they're neutral. This is the borderline between:
  - $\mu < 2$: $\rho < 0$, stable spiral (trajectories approach equilibrium)
  - $\mu = 2$: $\rho = 0$, center-like behavior in linearization
  - $\mu > 2$: $\rho > 0$, unstable spiral (trajectories recede from equilibrium)

  At exactly $\mu = 2$, we're at a critical transition point.
- **STAGE Z (What happens):** This is a necessary condition for Hopf bifurcation, but not sufficient. We still need to verify that:
  1. The real part crosses zero transversely (derivative nonzero)
  2. The system is "generic" (non-degenerate)

  The full Hopf bifurcation theorem tells us that under these conditions, a limit cycle (periodic orbit) emerges for μ near 2. Let's verify the transversality condition.

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6 Step 6: Verify Hopf Bifurcation Conditions

Condition (B1): Equilibrium exists

At $\mu = 2$, the equilibrium is $(1, 2)$.

Condition (B2): Eigenvalues purely imaginary

At $\mu = 2$: $\lambda = \pm i$ (pure imaginary).

Condition (G1): Imaginary part nonzero

$$\omega(2) = 1 \neq 0 \quad \checkmark$$

Condition (G2): Real part crosses zero transversely

Compute the derivative of the real part with respect to μ:

$$\frac{d\rho}{d\mu} = \frac{d}{d\mu}\left[\frac{\mu - 2}{2}\right] = \frac{1}{2}$$

At $\mu = 2$:

$$\left.\frac{d\rho}{d\mu}\right|_{\mu=2} = \frac{1}{2} \neq 0 \quad \checkmark$$

Since the derivative is positive, the real part crosses from negative to positive as μ increases through 2.

Conclusion

All Hopf bifurcation conditions are satisfied:

$$\boxed{\text{HOPF BIFURCATION at } \mu = 2, \text{ equilibrium } (1, 2)}$$

XYZ Analysis of Verification

- **STAGE X (What we verified):** All four conditions (B1, B2, G1, G2) for a Hopf bifurcation hold at $\mu = 2$.
- **STAGE Y (Why each condition matters):**
  - **(B1) Equilibrium exists**: Fundamental - need a reference point for bifurcation
  - **(B2) Pure imaginary eigenvalues**: Indicates neutrally stable oscillations in linearization at critical point
  - **(G1) Imaginary part nonzero**: Ensures genuine oscillation (frequency $\omega = 1$, not degenerate case $\omega = 0$). Guarantees the periodic behavior is robust
  - **(G2) Transverse crossing**: Ensures the stability change is non-degenerate. The derivative $d\rho/d\mu = 1/2 > 0$ means:
    - For μ slightly less than 2: $\rho < 0$ (stable spiral)
    - For μ slightly greater than 2: $\rho > 0$ (unstable spiral)

    The crossing is transverse (not tangential), guaranteeing a limit cycle emerges rather than higher-order degeneracy
- **STAGE Z (What this guarantees):** By the Hopf Bifurcation Theorem (Section 14 of lecture notes, pages 53-55), the system exhibits:
  - For $\mu < 2$: Stable equilibrium at $(1, \mu)$, no periodic orbits nearby
  - For μ near 2: A periodic orbit (limit cycle) emerges
  - The amplitude of the limit cycle grows like $\sqrt{|\mu - 2|}$ near the bifurcation

– Whether the bifurcation is supercritical (stable limit cycle for $\mu > 2$) or subcritical (unstable limit cycle for $\mu < 2$) depends on the sign of the first Lyapunov coefficient ℓ_1 (not computed here)

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7 Step 7: Characterize the Bifurcation

Eigenvalue paths in complex plane

As μ varies, the eigenvalues trace paths:

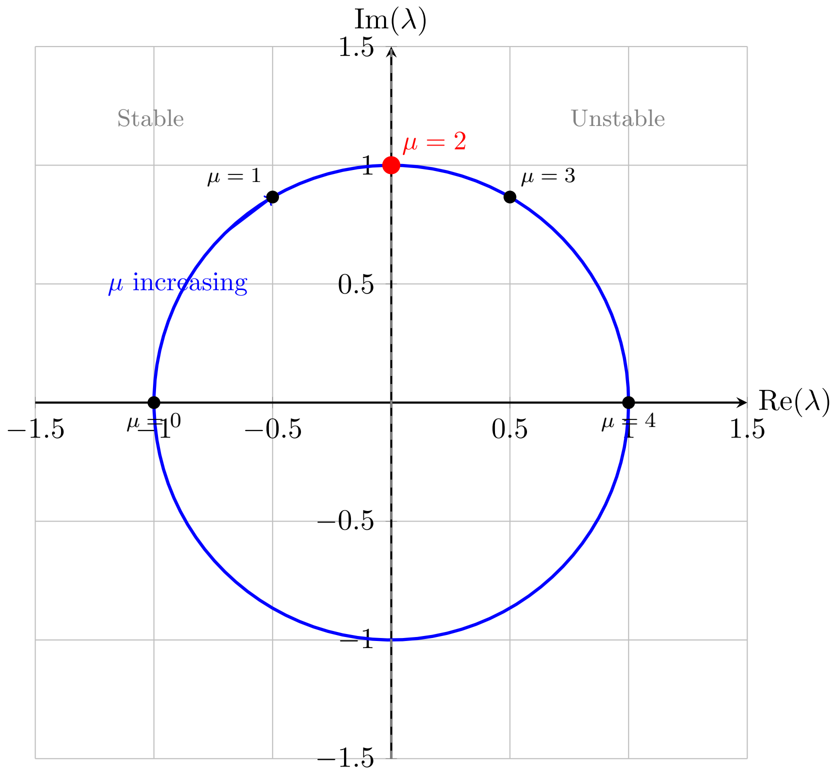
For $0 < \mu < 4$ (complex eigenvalues):

$$\lambda(\mu) = \frac{\mu - 2}{2} \pm i\frac{\sqrt{4 - (\mu - 2)^2}}{2}$$

Key values:

$$\begin{aligned}
\mu = 0: &\quad \lambda = -1 \pm i \cdot 0 = -1 \quad \text{(repeated)} \\
\mu = 1: &\quad \lambda = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2} \\
\mu = 2: &\quad \lambda = 0 \pm i \\
\mu = 3: &\quad \lambda = +\frac{1}{2} \pm i\frac{\sqrt{3}}{2} \\
\mu = 4: &\quad \lambda = +1 \pm i \cdot 0 = +1 \quad \text{(repeated)}
\end{aligned}$$

Eigenvalue trajectory



Stability summary

Parameter Range	Eigenvalues	Stability
$\mu < 2$	$\text{Re}(\lambda) < 0$	Stable spiral
$\mu = 2$	$\lambda = \pm i$	Neutral (Hopf point)
$\mu > 2$	$\text{Re}(\lambda) > 0$	Unstable spiral

XYZ Analysis of Eigenvalue Evolution

- **STAGE X (What we see):** Eigenvalues move counterclockwise around a circle of radius 1 in the complex plane as μ increases from 0 to 4. They cross the imaginary axis (Hopf bifurcation) at $\mu = 2$.

- **STAGE Y (Why circular path):** The constant determinant $\Delta = 1$ means:

$$|\lambda_1| \cdot |\lambda_2| = 1$$

  For complex conjugate eigenvalues: $\lambda = \rho \pm i\omega$, we have $|\lambda|^2 = \rho^2 + \omega^2$. So: $|\lambda|^2 = \Delta = 1$, meaning eigenvalues lie on the unit circle in the complex plane.

  As μ varies from 0 to 4, the trace $\tau = \mu - 2$ varies from -2 to $+2$. The eigenvalue formula:

$$\lambda = \frac{\tau}{2} \pm i\frac{\sqrt{4\Delta - \tau^2}}{2} = \frac{\tau}{2} \pm i\frac{\sqrt{4 - \tau^2}}{2}$$

  with $\Delta = 1$ gives a parametric equation for a semicircle. Setting $\tau = 2\cos\theta$ (varies from -2 to $+2$ as θ goes from π to 0), we get:

$$\lambda = \cos\theta \pm i\sin\theta = e^{\pm i\theta}$$

  This is precisely the unit circle! The angle θ relates to μ by $\cos\theta = (\mu - 2)/2$.

- **STAGE Z (What this means physically):** The circular trajectory is special to this system. As μ increases:

  - Eigenvalues start at $\lambda = -1$ (repeated, $\mu = 0$)
  - Split into complex conjugates and move through left half-plane (stable spirals)
  - Cross imaginary axis at $\mu = 2$ (Hopf bifurcation) with frequency $\omega = 1$
  - Continue into right half-plane (unstable spirals)
  - Reconverge to $\lambda = +1$ (repeated, $\mu = 4$)

  The Hopf bifurcation occurs exactly when the circular path crosses the imaginary axis. The frequency of emerging oscillations matches the imaginary part at crossing: $\omega(2) = 1$, giving period $T = 2\pi/\omega = 2\pi$.

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8 Step 8: Physical Interpretation

System behavior across bifurcation

For $\mu < 2$:

- Equilibrium $(1, \mu)$ is stable spiral
- Trajectories spiral inward toward equilibrium

- System reaches steady state (no oscillations in long term)

At $\mu = 2$:

- Equilibrium $(1, 2)$ has purely imaginary eigenvalues
- Linearization shows neutral cycles (neither growing nor decaying)
- Critical transition point

For $\mu > 2$ (just above bifurcation):

- Equilibrium $(1, \mu)$ is unstable spiral
- A limit cycle emerges (periodic orbit)
- System exhibits sustained oscillations

Expected dynamics near bifurcation

If supercritical (most common):

- For $\mu \lesssim 2$: Stable equilibrium, small perturbations decay
- For $\mu \gtrsim 2$: Small stable limit cycle emerges, radius $\propto \sqrt{\mu - 2}$
- Trajectories approach limit cycle from inside and outside

If subcritical (less common):

- For $\mu \lesssim 2$: Small unstable limit cycle exists below bifurcation
- For $\mu \gtrsim 2$: Equilibrium unstable, trajectories diverge or approach distant attractor

XYZ Analysis of Physical Behavior

- **STAGE X (What happens):** The system transitions from steady state to oscillatory behavior as μ crosses 2. Sustained periodic oscillations emerge where there were none before.
- **STAGE Y (Why oscillations emerge):** Near the Hopf bifurcation, the linearized dynamics exhibit rotation (from imaginary part of eigenvalues) and growth/decay (from real part). When real part is negative ($\mu < 2$), the rotation is damped and trajectories spiral inward. When real part becomes positive ($\mu > 2$), the rotation is amplified and trajectories spiral outward from equilibrium.

  But the system is nonlinear - the terms x^2y and $-x^2y$ provide nonlinear feedback. As amplitude grows beyond equilibrium, these nonlinear terms eventually balance the linear instability, stabilizing the trajectory into a limit cycle. The Hopf Bifurcation Theorem guarantees this balance creates a periodic orbit for μ near the critical value.
- **STAGE Z (What this represents):** Hopf bifurcations model the onset of oscillations in many physical systems:
  - **Chemical reactions**: Oscillating concentrations (e.g., Belousov-Zhabotinsky reaction)
  - **Biological systems**: Circadian rhythms, neural firing patterns, population cycles
  - **Mechanical systems**: Flutter instability in aircraft wings
  - **Electrical circuits**: Oscillator circuits, laser dynamics
  - **Climate models**: Periodic climate patterns (e.g., El Niño)

The parameter μ represents a control parameter (temperature, reaction rate, feedback gain, etc.) that, when varied, causes the system to spontaneously begin oscillating. This is a qualitative change in dynamics - from equilibrium to periodic motion.

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9 Summary

System

$$\begin{aligned}\dot{x} &= 1 - (1+\mu)x + x^2 y \\ \dot{y} &= \mu x - x^2 y\end{aligned}$$

Equilibrium

$$(x^*, y^*) = (1, \mu) \quad \text{for all } \mu$$

Jacobian at equilibrium

$$J(1, \mu) = \begin{pmatrix} \mu - 1 & 1 \\ -\mu & -1 \end{pmatrix}$$

Eigenvalues

$$\begin{aligned}\tau &= \mu - 2 \\ \Delta &= 1 \\ \lambda &= \frac{\mu - 2}{2} \pm i\frac{\sqrt{4 - (\mu-2)^2}}{2} \quad \text{for } 0 < \mu < 4\end{aligned}$$

Hopf Bifurcation

$$\boxed{\text{Occurs at } \mu^* = 2, \text{ equilibrium } (1, 2)}$$

Verification:

(B1) Equilibrium $(1, 2)$ exists

(B2) Eigenvalues $\lambda = \pm i$ (purely imaginary)

(G1) Frequency $\omega(2) = 1 \neq 0$

(G2) Transversality: $d\rho/d\mu = 1/2 \neq 0$

Dynamics:

- $\mu < 2$: Stable spiral $\rightarrow$ steady state
- $\mu = 2$: Purely imaginary eigenvalues $\rightarrow$ Hopf bifurcation
- $\mu > 2$: Unstable spiral $\rightarrow$ limit cycle emerges

Key insight: The constant determinant ($\Delta = 1$) constrains eigenvalues to unit circle in complex plane. As trace varies linearly with μ, eigenvalues sweep around circle, crossing imaginary axis at $\mu = 2$ - the Hopf bifurcation point where oscillatory behavior emerges.