Asymptotics 2025/2026   Solutions for Sheet 1[1]

Algebraic Equations and Asymptotic Expansions

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1. Two-term expansions for the solutions of the following equations are sought for

(a)
$$(x - 1)(x - 2)(x - 3) + \epsilon = 0 \ .$$

The solutions of the unperturbed equation ($\epsilon = 0$) are $x = 1$, 2, and 3, respectively. To obtain the first-order correction for the first solution we insert $x = 1 + x_1\epsilon + O(\epsilon^2)$ into the equation and obtain

$$(x_1\epsilon + \ldots)\,(-1 + x_1\epsilon + \ldots)\,(-2 + x_1\epsilon + \ldots) + \epsilon = 0 \ ,$$
$$(2x_1 + 1)\epsilon + O(\epsilon^2) = 0 \ .$$

The requirement that the coefficient of ϵ has to vanish leads to $x_1 = -\frac{1}{2}$, so the first solution is $x = 1 - \frac{1}{2}\epsilon + O(\epsilon^2)$. Similarly, we obtain the expansions for the other two solutions: $x = 2 + \epsilon + O(\epsilon^2)$ and $x = 3 - \frac{1}{2}\epsilon + O(\epsilon^2)$.

(b)
$$x^3 + x^2 - \epsilon = 0 \ .$$

The solutions of the unperturbed equation ($\epsilon = 0$) are $x = 0$, 0, and -1, respectively. Since 0 is a double solution we can expect the perturbation problem to be singular. Let us, however, first consider the regular solution, i.e. the perturbation of $x = -1$. We substitute

$$x = -1 + x_1\epsilon + x_2\epsilon^2 + \ldots$$

into $x^3 + x^2 - \epsilon$ and obtain

$$(-1 + x_1\epsilon + \ldots)^3 + (-1 + x_1\epsilon + \ldots)^2 - \epsilon = 0 \ ,$$
$$(-1 + 1) + (3x_1 - 2x_1 - 1)\epsilon + O(\epsilon^2) = 0 \ .$$

Therefore $x_1 = 1$ and $x = -1 + \epsilon + O(\epsilon^2)$. For the other solutions try

$$x = x_1\epsilon^\alpha + x_2\epsilon^{2\alpha} + \ldots \ .$$

Then we have

$$\left(x_1\epsilon^\alpha + x_2\epsilon^{2\alpha} + \ldots\right)^3 + \left(x_1\epsilon^\alpha + x_2\epsilon^{2\alpha} + \ldots\right)^2 - \epsilon = 0 \ ,$$
$$(x_1^3\epsilon^{3\alpha} + 3x_1^2 x_2\epsilon^{4\alpha} + \ldots) + (x_1^2\epsilon^{2\alpha} + 2x_1 x_2\epsilon^{3\alpha} + \ldots) - \epsilon = 0 \ .$$

We can balance the ϵ term only if we set $\alpha = 1/2$. Then we have

$$\text{At}\quad O(\epsilon): \qquad x_1^2 - 1 = 0; \qquad\qquad \Longrightarrow \qquad x_1 = \pm 1.$$
$$\text{At}\quad O(\epsilon^{3/2}): \qquad x_1^3 + 2x_1 x_2 = 0; \qquad \Longrightarrow \qquad x_2 = -\frac{1}{2}.$$

Therefore

$$x = \pm \epsilon^{1/2} - \frac{1}{2}\epsilon + O(\epsilon^{3/2}).$$

(c)

$$\epsilon x^3 + x^2 + 2x + 1 = 0 .$$

The solutions of the unperturbed equation ($\epsilon = 0$) are $x = -1$ (twice). Based on our experience with double solutions we try

$$x = -1 + \epsilon^{1/2} x_1 + \epsilon x_2 + \dots .$$

After substitution into the equation we obtain

$$\epsilon(-1 + \dots)^3 + (-1 + \epsilon^{1/2} x_1 + \epsilon x_2 + \dots)^2 + 2(-1 + \epsilon^{1/2} x_1 + \epsilon x_2 + \dots) + 1 = 0 ,$$
$$(1 - 2 + 1) + \epsilon^{1/2}(-2x_1 + 2x_1) + \epsilon(-1 + x_1^2 - 2x_2 + 2x_2) + O(\epsilon^{3/2}) = 0 .$$

We have

$$\text{at}\quad O(\epsilon): \qquad x_1^2 - 1 = 0 \qquad \Longrightarrow \qquad x_1 = \pm 1 .$$

Thus

$$x = -1 \pm \epsilon^{1/2} + O(\epsilon).$$

The third solution is expected to go to infinity as $\epsilon \to 0$. For large x we can neglect $2x$ and 1 in comparison with x^2 in the equation. Thus the dominant balance analysis yields

$$\epsilon x^3 \sim -x^2 \qquad \Longrightarrow \qquad x \sim -\frac{1}{\epsilon} , \qquad \text{as} \quad \epsilon \to 0 .$$

Hence we try

$$x = -\frac{1}{\epsilon} + x_0 + \dots$$

and obtain

$$\epsilon\left(-\frac{1}{\epsilon} + x_0 + \dots\right)^3 + \left(-\frac{1}{\epsilon} + x_0 + \dots\right)^2 + 2\left(-\frac{1}{\epsilon} + \dots\right) + 1 = 0 ,$$
$$\frac{1}{\epsilon^2}(-1 + 1) + \frac{1}{\epsilon}(3x_0 - 2x_0 - 2) + O(1) = 0 .$$

This yields $x_0 = 2$ and thus $x = -\frac{1}{\epsilon} + 2 + \dots$.

(d)

$$\sqrt{2}\sin(x + \pi/4) - 1 - x + \frac{1}{2}x^2 = -\frac{1}{6}\epsilon .$$

We look for the solution that approaches zero as $\epsilon \to 0$. A Taylor expansion of the left-hand side (LHS) for small values of x yields

$$-\frac{1}{6}x^3 + \frac{1}{24}x^4 + O(x^5) = -\frac{1}{6}\epsilon .$$

To obtain the leading order approximation for x we have to consider only the first term on the LHS. Comparison with the RHS yields $x^3 = \epsilon$, or $x = \epsilon^{1/3}$. The next term is obtained by inserting $x = \epsilon^{1/3} + \alpha\epsilon^\beta + \ldots$, where $\beta > 1/3$, into the equation for x

$$-\frac{1}{6}(\epsilon^{1/3} + \alpha\epsilon^\beta + \ldots)^3 + \frac{1}{24}(\epsilon^{1/3} + \alpha\epsilon^\beta + \ldots)^4 + \ldots = -\frac{1}{6}\epsilon ,$$

$$-\frac{1}{6}\epsilon - \frac{\alpha}{2}\epsilon^{\beta+2/3} + \ldots + \frac{1}{24}\epsilon^{4/3} + \ldots = -\frac{1}{6}\epsilon .$$

From this we conclude that $\beta = 2/3$ and $\alpha = 1/12$, and so $x = \epsilon^{1/3} + \frac{1}{12}\epsilon^{2/3} + o(\epsilon^{2/3})$ as $\epsilon \to 0$.

2. Two-term expansions for the solutions of the following equation are sought for

$$\epsilon^2 x^3 + x^2 + 2x + \epsilon = 0$$

The solutions of the unperturbed equation ($\epsilon = 0$) are $x = 0$ and $x = -2$, respectively. To find the solution of the perturbed equation near zero we try $x = x_1\epsilon + x_2\epsilon^2 + \ldots$. We insert this expression into the equation for x and obtain

$$\epsilon^2(x_1\epsilon + \ldots)^3 + (x_1\epsilon + \ldots)^2 + 2(x_1\epsilon + x_2\epsilon^2 + \ldots) + \epsilon = 0 ,$$

$$\epsilon(2x_1 + 1) + \epsilon^2(x_1^2 + 2x_2) + O(\epsilon^3) = 0 .$$

We conclude that $x_1 = -1/2$, $x_2 = -1/8$, and $x = -\frac{1}{2}\epsilon - \frac{1}{8}\epsilon^2 + O(\epsilon^3)$.

Similarly, we set $x = -2 + x_1\epsilon + \ldots$ to find the solution near $x = -2$. Insertion into the equation for x yields

$$\epsilon^2(-2 + \ldots)^3 + (-2 + x_1\epsilon + \ldots)^2 + 2(-2 + x_1\epsilon + \ldots) + \epsilon = 0 ,$$

$$(4 - 4) + \epsilon(-4x_1 + 2x_1 + 1) + O(\epsilon^2) = 0 .$$

We conclude that $x_1 = 1/2$ and $x = -2 + \frac{1}{2}\epsilon + O(\epsilon^2)$.

Finally, the remaining third solution is expected to occur at large values of x. Then we can neglect $2x$ and ϵ in comparison with x^2 in the equation for x. From the remaining two terms in the equation we conclude that

$$\epsilon^2 x^3 \sim -x^2 \quad\Longrightarrow\quad x \sim -\frac{1}{\epsilon^2} , \quad \text{as} \quad \epsilon \to 0 .$$

Therefore, we look for an expansion of the form

$$x = -\frac{1}{\epsilon^2} + \frac{x_0}{\epsilon} + x_1 + \ldots .$$

We insert this expression into the equation for x and obtain

$$\epsilon^2\left(-\frac{1}{\epsilon^2} + \frac{x_0}{\epsilon} + x_1 + \ldots\right)^3 + \left(-\frac{1}{\epsilon^2} + \frac{x_0}{\epsilon} + x_1 + \ldots\right)^2 + 2\left(-\frac{1}{\epsilon^2} + \ldots\right) + \epsilon = 0 ,$$

$$\frac{1}{\epsilon^4}(-1 + 1) + \frac{1}{\epsilon^3}(3x_0 - 2x_0) + \frac{1}{\epsilon^2}(3x_1 - 3x_0^2 - 2x_1 + x_0^2 - 2) + O(\epsilon^{-1}) = 0 .$$

It follows that $x_0 = 0$, $x_1 = 2$, and $x = -\epsilon^{-2} + 2 + O(\epsilon)$.

3. We are asked to verify the following statements

   (a) $\sin x^{1/3} = O(x^{1/3})$, $x \to 0+$. This is correct, since the following limit is finite

   $$\lim_{x \to 0+} \frac{\sin x^{1/3}}{x^{1/3}} = \lim_{x \to 0+} \frac{x^{1/3} - \frac{1}{6}x + \dots}{x^{1/3}} = 1 \ .$$

   (b) $\cos(x) = O(1)$, $x \to \infty$. This is correct since the following quotient stays bounded as $x \to \infty$

   $$\left| \frac{\cos x}{1} \right| \leq 1 \ , \qquad \text{all } x \ .$$

   (c) $\sin x = O(x \cos x)$, $x \to 0$. This is correct, since the following limit is finite

   $$\lim_{x \to 0} \frac{\sin x}{x \cos x} = \lim_{x \to 0} \frac{x - \frac{1}{6}x^3 + \dots}{x - \frac{1}{2}x^3 + \dots} = 1 \ .$$

   (d) $\log(\log \frac{1}{x}) = o(\log(x))$, $x \to 0+$. This is correct, since the following limit vanishes (as is shown by using the rule of de l'Hospital)

   $$\lim_{x \to 0+} \frac{\log(\log \frac{1}{x})}{\log x} = \lim_{x \to 0+} \frac{(\log \frac{1}{x})^{-1} x \left(-x^{-2} \right)}{x^{-1}} = \lim_{x \to 0+} \frac{1}{\log x} = 0 \ .$$

4. The sequence $\{\phi_n(x) = x^{-n} \cos(nx)\}$, $n = 0, 1, \dots$, would be an asymptotic sequence as $x \to \infty$, if for all n

   $$\lim_{x \to \infty} \frac{\phi_{n+1}(x)}{\phi_n(x)} = 0 \ .$$

   For this reason, we consider

   $$\frac{\phi_{n+1}(x)}{\phi_n(x)} = \frac{\cos((n+1)x)}{x \cos(nx)} = \frac{\cos(nx)\cos(x) - \sin(nx)\sin(x)}{x \cos(nx)} = \frac{\cos(x)}{x} - \frac{\tan(nx)\sin(x)}{x} \ .$$

   However, the limit of this expression as $x \to \infty$ does not exist, since it diverges for all $x = \frac{\pi}{n}(\frac{1}{2} + m)$, $m = 0, 1, 2, \dots$. Therefore the sequence $\{\phi_n(x)\}$ is not an asymptotic sequence.

5. To prove that $\sum_{n=1}^{\infty} \frac{1}{z^n}$ is an asymptotic expansion of $\frac{1}{z-1}$ as $z \to \infty$, we may use the formula for the coefficients of an asymptotic expansion

   $$f(z) \sim \sum_{n=0}^{\infty} a_n \phi_n(z), \quad z \to \infty, \quad \text{where} \quad a_{n+1} = \lim_{z \to \infty} \frac{f(z) - \sum_{k=0}^{n} a_k \phi_k(z)}{\phi_{n+1}(z)}.$$

   We have $\phi_n(z) = z^{-n}$ and $a_0 = 0$. Using $\sum_{k=1}^{n} c^k = (c^{n+1} - c)/(c-1)$ we find

   $$a_{n+1} = \lim_{z \to \infty} \left[z^{n+1} \left(\frac{1}{z-1} - \sum_{k=1}^{n} \frac{1}{z^k} \right) \right] = \lim_{z \to \infty} \left[z^{n+1} \left(\frac{1}{z^n(z-1)} \right) \right] = 1 \ ,$$

   and thus the assertion is proved. Alternatively, we might have used the formula for a geometric series (valid for $z > 1$)

   $$\sum_{n=1}^{\infty} \frac{1}{z^n} = \sum_{n=0}^{\infty} \frac{1}{z^n} - 1 = \frac{1}{1 - z^{-1}} - 1 = \frac{1}{z-1} \ .$$

6. Choose for example $f(x) = x^2 + x$ and $g(x) = x^2$. Then $f(x) \sim g(x)$ as $x \to \infty$, because

$$\lim_{x \to \infty} \frac{f(x)}{g(x)} = \lim_{x \to \infty} \frac{x^2 + x}{x^2} = 1 \ .$$

But $\exp(f(x))$ is not asymptotic to $\exp(g(x))$ as $x \to \infty$, because

$$\frac{\exp(f(x))}{\exp(g(x))} = \frac{\exp(x^2 + x)}{\exp(x^2)} = \exp(x) \to \infty \quad \text{as} \quad x \to \infty \ .$$