Continuum Mathematics (EMAT 31410)$^{\text{TB1-2}}$
& Eng Math III (EMAT 30012)$^{\text{TB1}}$

Part II: Asymptotics

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Plan of Topics

Week	Topic
7	Algebraic equations & Perturbations
8	Asymptotics – Key concepts
9	Solutions of ODEs
10	Expansion of integrals
11	The Error function
12	Revision

Textbooks

We won't follow any textbook directly, but other useful resources include:

- E. J. Hinch *Perturbation Methods*
  at time of writing this is available here . . .
  `http://inis.jinr.ru/sl/vol2/Mathematics/Hinch,_Perturbation_Methods,1995.pdf`
- J. Heading *An Introduction to Phase-Integral Methods*
- A. Erdelyi *Asymptotic Expansions*
- More advanced:
  Bender & Orszag *Advanced Mathematical Methods for Scientists and Engineers I*
  Dingle *Asymptotic Expansions: Their Derivation and Interpretation*
  (`https://michaelberryphysics.files.wordpress.com/2013/07/dingle.pdf`)

Contents

[Side Notes:] Revision

There are a couple of formulae you'll need to remember how to use:

- The power series in x about a point x_0

$$f(x) = \sum_{n=0}^{\infty} (x - x_0)^n \frac{1}{n!} \frac{d^n}{dx^n} f(x_0) \tag{0.1}$$

  is a Taylor Series of the function $f(x)$ if the series converges.

- Integration by parts

$$\int_a^b uv' dx = [uv]_a^b - \int_a^b u'v \, dx \tag{0.2}$$

- There is also a useful function that generalizes the factorial, called the **Gamma function**

$$\Gamma(x) = (x - 1)! \tag{0.3}$$

  $\Gamma(x)$ is defined for all real x, not just the integers (whereas the factorial is only defined for integers). The Gamma function pops up a lot in asymptotics, so you'll see it occurring below.

1 Algebraic Equations

Asymptotics is all about large or small quantities:

- A **very large** object like the Earth with mass M, is only slightly *perturbed* by a small object like you with mass m. Your state, however, is *dominated* by the Earth.
- The large parameter $K = M/m \gg 1$, or the small parameter $\varepsilon = m/M \ll 1$, can be used to seek approximate or (more powerfully) *asymptotic* description of yours and the Earth's behaviours.

1.1 Regular perturbations

Consider the quadratic equation

$$x^2 + \varepsilon x - 1 = 0 \tag{1.1}$$

- You can solve this easily enough (unlike the problems we'll attack later with the same methods). The solution for x is

$$x = -\tfrac{1}{2}\varepsilon \pm \sqrt{1 + \tfrac{1}{4}\varepsilon^2}\,. \tag{1.2}$$

- You already know how to expand this for small ε (i.e. for ε near zero). For example, use $\sqrt{1+u} = 1 + \frac{1}{2}u - \frac{1}{8}u^2 + \ldots$, then substitute $u = \frac{1}{4}\varepsilon^2$. You get

$$\begin{aligned} x &= -\tfrac{1}{2}\varepsilon \pm \left(1 + \tfrac{1}{8}\varepsilon^2 - \tfrac{1}{128}\varepsilon^4 + \ldots\right) \\ &= \begin{cases} +1 - \tfrac{1}{2}\varepsilon + \tfrac{1}{8}\varepsilon^2 - \tfrac{1}{128}\varepsilon^4 + \mathcal{O}\left(\varepsilon^6\right) \\ -1 - \tfrac{1}{2}\varepsilon - \tfrac{1}{8}\varepsilon^2 + \tfrac{1}{128}\varepsilon^4 + \mathcal{O}\left(\varepsilon^6\right) \end{cases} . \end{aligned} \tag{1.3}$$

  These are polynomial series in the parameter ε. They converge if $|\varepsilon| < 2$.

- Each successive term of these series gives a better approximation for x: if we *truncate* the series at a term with power n, the error in that approximation will be of size or **order** ε^{n+1}.
- Actually in this example it's slightly better than that, the error will be of order ε^{n+2} (written $\mathcal{O}\left(\varepsilon^{n+1}\right)$) because we only have even powers.

Perturbative solution

Let's pretend this was a harder problem that we couldn't solve exactly. How could we have got straight to the approximation?

- We can see easily that for $\varepsilon = 0$ we have $x^2 = 1$, which has two solutions, $x = \pm 1$.

- We can expect for small ε there are two *perturbed* solutions (perturbed just means $\varepsilon \neq 0$), and they should look like $x = \pm 1 +$ powers of ε, so try:

$$x(\varepsilon) = a_0 + a_1\varepsilon + a_2\varepsilon^2 + ... \tag{1.4}$$

- Now just substitute this into the original problem, and compare powers of ε to find the coefficients a_n. So

$$x^2 + \varepsilon x - 1 = 0 \tag{1.5}$$

  becomes

$$\begin{aligned} 0 &= (a_0 + a_1\varepsilon + a_2\varepsilon^2 + ...)^2 + \varepsilon(a_0 + a_1\varepsilon + a_2\varepsilon^2 + ...) - 1 \\ &= (a_0^2 - 1) + \varepsilon(2a_0a_1 + a_0) + \varepsilon^2(a_1^2 + 2a_0a_2 + a_1) + ... \end{aligned} \tag{1.6}$$

  You can see I've expanded out the brackets, and then collected together all terms of successive powers of ε.

- Now the lefthand-side is zero (I have no terms of any power there). So if all terms proportional to ε^0 add to zero, and all terms proportional to ε add to zero, all terms proportional to ε^2 add to zero, etc. I must have

$$\begin{array}{lllll} \text{coeffs. of } \varepsilon^0 & : & 0 = a_0^2 - 1 & \Rightarrow & a_0 = \pm 1 \\ \text{coeffs. of } \varepsilon^1 & : & 0 = 2a_0a_1 + a_0 & \Rightarrow & a_1 = -\frac{1}{2} \\ \text{coeffs. of } \varepsilon^2 & : & 0 = a_1^2 + 2a_0a_2 + a_1 & \Rightarrow & a_2 = \pm\frac{1}{8} \\ \quad ... & : & 0 = ... & & \end{array} \tag{1.7}$$

  As you see, we're successfully reproducing the series expansion we found earlier from the exact solution,

$$x = \pm 1 - \tfrac{1}{2}\varepsilon \pm \tfrac{1}{8}\varepsilon^2 + \mathcal{O}\left(\varepsilon^4\right) \ . \tag{1.8}$$

- Now be careful: I need to have written my ansatz series $x(\varepsilon)$ to a high enough power that adding more terms won't change the answers to the coefficients. I've been careful: going to ε^2 here in my series for $x(\varepsilon)$ is enough to uniquely fix all terms up to ε^2 after substitution.

- The general procedure for such problems is just the same; see side note on *Regular Perturbations* below.

- Try moving the ε to a different place, for example $x^2+x-\varepsilon = 0$ and working through the same: 1. find the exact solution and its expansion, and 2. find the perturbative solutions directly by series approximation. You should find two solutions that now lie near $x = 0$ and $x = -1$ for small ε.

[Side Notes:] Regular Perturbations

An equation

$$F(x, \varepsilon) = 0 \tag{1.9}$$

for small ε is considered to be a perturbation of the problem

$$F(x, 0) = 0 \ . \tag{1.10}$$

- This is a **regular perturbation** if the solutions $x = x(\varepsilon)$ of (1.9) all *converge* to solutions of the *unperturbed* problem (1.10), i.e. if the limits $x(\varepsilon) \to x(0)$ are well behaved.

  E.g. in the lectures we saw a quadratic equation $F(x, \varepsilon) = x^2 + \varepsilon x - 1 = 0$, whose solutions $x(\varepsilon) = -\frac{1}{2}\varepsilon \pm \sqrt{1 + \frac{1}{4}\varepsilon^2}$ converge to $x(0) = \pm 1$.

- To find a (regular) perturbation solution (i.e. for small ε), try the ansatz of a power series in ε:

  $$x(\varepsilon) = \sum_{n=1}^{\infty} a_n \varepsilon^n \ . \tag{1.11}$$

  Substitute this into the problem (1.9) and compare powers of ε to find the coefficients a_n. This is a simple form of **asymptotic balancing**.

- In practice we usually **truncate** the series at some power p, leaving a remainder of size ε^{p+1}. We call this the **order** of the remainder. For example, truncating at order $p = 3$ we write

  $$x(\varepsilon) = a_0 + a_1 \varepsilon + a_2 \varepsilon^2 + a_3 \varepsilon^3 + \mathcal{O}\left(\varepsilon^4\right) \ . \tag{1.12}$$

- The series doesn't have to be in integer powers of ε, it could consist of fractional powers like $x = 1 + \varepsilon^{1/2} + \varepsilon + \varepsilon^{3/2} + ...$ or more complicated functions . . . sometimes you just have to play around to find something that works. Asymptotics is more liberal than the series or approximation methods you may have been used to up to this point.

1.2 Singular perturbations

Things are rather different if the small parameter ε multiplies the highest order term of our equation, i.e. if we consider the quadratic equation

$$\varepsilon x^2 + x - 1 = 0 \tag{1.13}$$

Strangely this only has one solution when $\varepsilon = 0$, namely $x = 1$. But this is a quadratic equation, it should have two roots! Something goes wrong as ε tends to zero. This is a **singular perturbation** problem.

- Let's not lose our heads yet. You can still solve this easily enough. The solution for x is
$$x = \frac{-1 \pm \sqrt{1+4\varepsilon}}{2\varepsilon}\,. \tag{1.14}$$

- Now ε appears in the denominator, so this does not behave well when ε is taken to zero. But again, let's use methods we're familiar with to probe deeper before panicking.

- We can still expand this for small ε. Expanding the square root as $\sqrt{1+u} = 1 + \frac{1}{2}u - \frac{1}{8}u^2 + \frac{1}{16}x^3...$ and substituting $u = 4\varepsilon$, you get
$$\begin{aligned} x &= \frac{-1 \pm \left(1 + 2\varepsilon - 2\varepsilon^2 + 4\varepsilon^3 ...\right)}{2\varepsilon} \\ &= \tfrac{1}{2\varepsilon}\begin{cases} 2\varepsilon - 2\varepsilon^2 + 4\varepsilon^3 + \mathcal{O}\left(\varepsilon^4\right) \\ -2 - 2\varepsilon + 2\varepsilon^2 - 4\varepsilon^3 + \mathcal{O}\left(\varepsilon^4\right) \end{cases} \\ &= \begin{cases} 1 - \varepsilon + 2\varepsilon^2 + \mathcal{O}\left(\varepsilon^3\right) \\ -\frac{1}{\varepsilon} - 1 + \varepsilon - 2\varepsilon^2 + \mathcal{O}\left(\varepsilon^3\right) \end{cases} \quad . \end{aligned} \tag{1.15}$$

  Like before, these are polynomial series in the parameter ε.

- The first solution is perfectly good, it just goes to $x = 1$ at $\varepsilon = 0$. We call this the **regular** solution.

- The second solution 'blows up' or 'diverges' or 'goes to infinity' at $\varepsilon = 0$. We call this the **singular** solution.

- This explains what happen setting $\varepsilon = 0$ in the equation (1.13): it only had one **regular** solution, the singular one whizzes off to infinity when $\varepsilon = 0$!

Perturbative solution

We can find these solutions by series expansion just as we did before.

- We can see that now we're going to need our series to start at order ε^{-1}. (We'll look later at how we know what order to start at more generally).

- So we expect the *perturbed* ($\varepsilon \neq 0$) solution to look like

$$x(\varepsilon) = a_{-1}\varepsilon^{-1} + a_0 + a_1\varepsilon + a_2\varepsilon^2 + ... \tag{1.16}$$

- The procedure is the same as before. Just substitute this into the original problem, and compare powers of ε to find the coefficients a_n. So

$$\varepsilon x^2 + x - 1 = 0 \tag{1.17}$$

  becomes

$$\begin{aligned} 0 &= \varepsilon(a_{-1}\varepsilon^{-1} + a_0 + a_1\varepsilon + ...)^2 + (a_{-1}\varepsilon^{-1} + a_0 + a_1\varepsilon + ...) - 1 \\ &= \varepsilon^{-1}\left(a_{-1}^2 + a_{-1}\right) + (2a_{-1}a_0 + a_0 - 1) + \varepsilon\left(2a_{-1}a_1 + a_0^2 + a_1\right) + ... \end{aligned} \tag{1.18}$$

  As before I've expanded out the brackets, then collected together all terms of successive powers of ε.

- Now the lefthand-side is zero (I have no terms of any power there). So if all terms proportional to ε^{-1} add to zero, and all terms proportional to ε^0 add to zero, all terms proportional to ε add to zero, etc. I must have

$$\begin{array}{lclcl} \text{coeffs. of } \varepsilon^{-1} & : & 0 = a_{-1}^2 + a_{-1} & \Rightarrow & a_{-1} = 0 \quad \text{or } -1 \\ \text{coeffs. of } \varepsilon^{0} & : & 0 = 2a_{-1}a_0 + a_0 - 1 & \Rightarrow & a_0 = +1 \text{ or } -1 \\ \text{coeffs. of } \varepsilon^{1} & : & 0 = 2a_{-1}a_1 + a_0^2 + a_1 & \Rightarrow & a_1 = -1 \text{ or } +1 \\ \quad . \ . \ . & : & 0 = ... & & \end{array} \tag{1.19}$$

  Again we're successfully reproducing the series expansions we found earlier from the exact solution,

$$x = \begin{cases} 1 - \varepsilon + \mathcal{O}\left(\varepsilon^2\right) \\ -\frac{1}{\varepsilon} - 1 + \varepsilon + \mathcal{O}\left(\varepsilon^2\right) \end{cases} . \tag{1.20}$$

- The general procedure for such problems is just the same; see the Side Notes.

[Side Notes:] Orders & 'Asymptoticness'

When we say x is of order ε, or write

$$x = \mathcal{O}\left(\varepsilon\right)$$

you can crudely think of this as meaning $x \propto \varepsilon$. So:

- if $x = \mathcal{O}\left(\varepsilon\right)$ then x is small when ε is small, or
- if $x = \mathcal{O}\left(\varepsilon^{-1}\right)$ then x is large when ε is small (or vice versa).
- More precisely

  $$x = \mathcal{O}\left(\varepsilon\right) \qquad \text{means that} \qquad \lim_{\varepsilon\to 0} \frac{x}{\varepsilon} = L \tag{1.21}$$

  where L is neither 0 nor ∞.

  Similarly

  $$x = \mathcal{O}\left(\varepsilon^{-1}\right) \qquad \text{means that} \qquad \lim_{\varepsilon\to 0} \frac{x}{\varepsilon^{-1}} = L \tag{1.22}$$

  where L is neither 0 nor ∞.

- If $x = y + \mathcal{O}\left(\varepsilon\right)$ then x is asymptotic to y in the limit of small ε.
- Functions can be asymptotic to each other as well as just variables. So if $x(t) = y(t) + \mathcal{O}\left(\varepsilon\right)$ then the function $x(t)$ is asymptotic to $y(t)$ in the limit of small ε.
- We say two functions $x(t)$ and $y(t)$ are **asymptotically equivalent**, or simply $x(t)$ **is asymptotic to** $y(t)$ as $\varepsilon \to 0$, if

  $$\lim_{\varepsilon\to 0} \frac{x(t)}{y(t)} = 1 \qquad \text{or} \qquad \lim_{\varepsilon\to 0} |x(t) - y(t)| = 0\ . \tag{1.23}$$

1.3 Re-scaling

How did we know what order to start the series expansion? We know because we can argue what **order** in ε x has, and re-scale to a variable in which the solution is a finite non-zero number when $\varepsilon = 0$. (This is a bit like non-dimensionalization if we think of ε as the 'units' we're measuring in).

This is an incredibly powerful tool in many ways. So pay attention. We need to know what *order in* ε dominates x:

- Is x small when ε is small? So as we take ε towards zero, does x shrink to zero like $x = \mathcal{O}(\varepsilon)$ or $x = \mathcal{O}(\sqrt{\varepsilon})$ or $x = \mathcal{O}\left(e^{-1/\varepsilon}\right)$ or some other order?
- Or is x large when ε is small? I.e. as we take ε towards zero, does x grow to infinity like $x = \mathcal{O}\left(\varepsilon^{-1}\right)$ or $x = \mathcal{O}\left(\varepsilon^{-2}\right)$ or $x = \mathcal{O}\left(e^{1/\varepsilon}\right)$ or some other order?
- We can confirm these by re-scaling to a new variable X, for instance if $x = \mathcal{O}(\varepsilon)$ then let $x = \varepsilon X$, or if $x = \mathcal{O}\left(\varepsilon^{-1}\right)$ then let $x = \varepsilon^{-1}X$, then when you substitute in you'll find that X is independent of ε to leading order (i.e. X = number + $\mathcal{O}(\varepsilon)$).
- To work out the order you can try a general form like $x = \mathcal{O}(\varepsilon^p)$, then define $x = \varepsilon^p X$, and work out what value of p would give a **regular** solution for X, i.e. X = number + $\mathcal{O}(\varepsilon)$. Usually only one value of p will do this.
- See Exercise Sheet 1 'Re-scaling' for a worked example on the quadratic equation from the lecture above.
- Of course it isn't usually quadratic equations we're trying to solve. It is when we come to solving differential equations or integrals that these ideas – series expansions and scalings – really save the day.

[Further Reading Only:] Frobenius series

Extending the series methods above, some differential equations can be solved by seeking series solutions of certain forms.
The Frobenius method uses a series

$$y(x) = x^\alpha \sum_{n=0}^{\infty} a_n x^n \tag{1.24}$$

to solve problems of the form

$$y'' + p(x)y' + q(x)y = g(x) \tag{1.25}$$

provided that p, q, g, are analytic in x (at least near $x = 0$). In particular it is useful for

$$x^2 y'' + p(x)xy' + q(x)y = 0 \tag{1.26}$$

provided that p and q are analytic in x (at least near $x = 0$).
In certain circumstances a modified form is needed, such as

$$y(x) = y_0 \ln x + x^\alpha \sum_{n=0}^{\infty} a_n x^n \tag{1.27}$$

There are a great variety of solution methods like this, asymptotics can take a lot of trial-and-error, and is largely a skill in getting a feel for how to balance large and small quantities.

[Further Reading Only:] Iteration

When we solved the quadratic equation

$$x^2 + \varepsilon x - 1 = 0$$

using a series expansion, we used some kind of iterative process: expanding to successively higher orders, balancing orders of ε, and solving for the coefficients.

There is a more brute force way to obtain the series by iteration.

- Try to re-arrange the equation into $x =$ 'something where ε is a small (regular) perturbation', e.g.

$$x = \sqrt{1 - \varepsilon x}$$

- then turn this into an iterative formula

$$x_{n+1} = \sqrt{1 - \varepsilon x_n}\ .$$

- Now this formula must be satisfied for x, and if you make an initial guess, then iterating this formula gives you successively better guesses.
- So iterate: start with x_0 being the solution when $\varepsilon = 0$, so $x_0 = 1$, and off we go, work out x_1, x_2, and so on. . .
- . . . work out successive iterations, and compare them with the exact solution and the series solutions we found in the first lecture.

Similar things are possible for singular perturbation problems and differential equations.
But iteration like this is not a great method. Progressively more iterations are needed to get higher order corrections right, and you don't know a given term is right until you do the next iteration and make sure it no longer changes. Alas in some cases, it may be all that's available, or may be the simplest route to an answer. And of course, a computer may be able to do it for you rather quickly.

2 Asymptotics – key concepts

Asymptotics is much more powerful than having exact solutions to problems. It is about finding out what they 'look like', *qualitatively*, and yet very *precisely*!

Consider the initial value problem

$$y'(x) = 1 - y^2(x) \;, \qquad y(0) = 0 \;. \tag{2.1}$$

We'll use the notation that $y' = \frac{dy}{dx}$, $y'' = \frac{d^2y}{dx^2}$, etc.

Now it just so happens that the solution of this is

$$y(x) = \tanh(x) \;. \tag{2.2}$$

We can use this simple example to demonstrate almost all of the main concepts and methods of asymptotics. Then we'll go on to apply them to tougher functions.

[Side Notes:]

There is a little artistic license in using (2.1) to demonstrate so much of asymptotic theory, but there is nothing actually incorrect in what we'll do, and we are able to introduce key concepts and methods just as you'll apply them to tougher problems. Look in any asymptotics textbook and you'll appreciate — when you see the labyrinth of examples they employ to illustrate all these different methods — how much we've drawn together under this one example.

2.1 Series expansions of functions

- Consider these functions:

$$y(x) \ = \ \frac{x}{1+|x|} \ , \qquad \frac{x}{\sqrt{1+x^2}} \ , \qquad \tfrac{2}{\pi}\arctan(x) \ . \tag{2.3}$$

 How do they behave compared to $\tanh(x)$? Could they be valid approximate solutions to (2.1)?

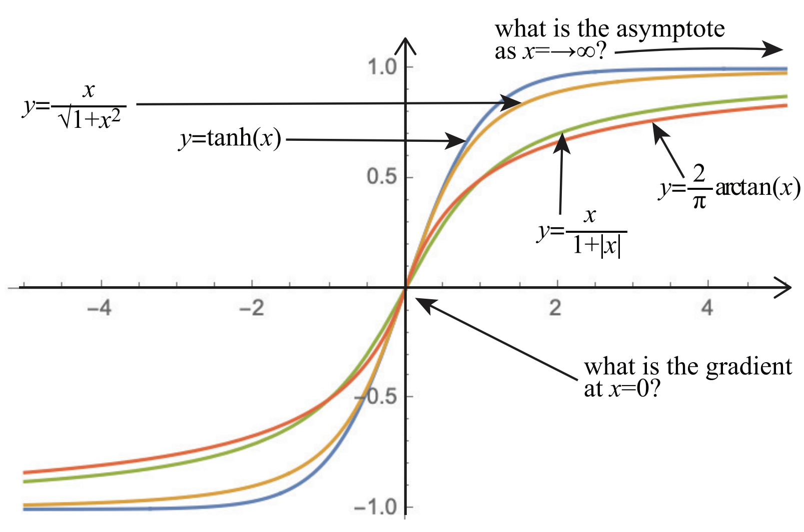
- All of these behave like:

$$\begin{aligned} &y(x) = 0 \quad \text{at} \quad x = 0 \\ &y(x) \to +1 \quad \text{as} \quad x \to +\infty \\ &y(x) \to -1 \quad \text{as} \quad x \to -\infty \end{aligned}$$

 but they approach these values differently.

- They have the same *asymptotes*, i.e they tend towards ± 1 as $x \to \pm\infty$, but they approach them with different *asymptotic behaviour* (or simply different *asymptotics*).

- On a plot they look quite similar:



 but actually they differ in important ways, to do with *how* they approach $x = 0$ and ∞.

- Let's look more closely, starting at $x = 0$. They all behave almost the same there, except ...spot the odd one out!

  The best way to see how they behave near $x = 0$ is to find their series expansions for small x:

$$\begin{aligned}
\frac{x}{1+|x|} &= x - x|x| + x^3 - x^3|x| + x^5 - ... \\
\frac{x}{\sqrt{1+x^2}} &= x - \tfrac{1}{2}x^3 + \tfrac{3}{8}x^5 - ... \\
\tfrac{2}{\pi}\arctan(x) &= \tfrac{2}{\pi}\left(x - \tfrac{1}{3}x^3 + \tfrac{1}{5}x^5 - ...\right) \\
\tanh(x) &= x - \tfrac{1}{3}x^3 + \tfrac{2}{15}x^5 - ...
\end{aligned}$$

  The Exercise Sheet will give you some pointers on comparing these.

- But 'near $x = 0$' only covers a tiny bit of these function's behaviour, and it is what happens in their asymptotes — as x goes off to infinity — that really distinguishes them. This matter a lot because, loosely speaking, x is 'large' over a much bigger range than it is 'small'!

- We can expand these functions for 'large x' or 'x near infinity', which is just a case of expanding for small x^{-1} (see Exercise Sheet):

$$\begin{aligned}
\frac{x}{1+|x|} &= 1 - x^{-1} + x^{-2} - x^{-3} + ... \\
\frac{x}{\sqrt{1+x^2}} &= 1 - \frac{1}{2x^2} + \frac{3}{8x^4} + \mathcal{O}\left(x^{-6}\right) \\
\tfrac{2}{\pi}\arctan(x) &= 1 + \tfrac{2}{\pi}\left(-\frac{1}{x} + \frac{1}{3x^3} - \frac{1}{5x^5} + \mathcal{O}\left(x^{-7}\right)\right) \\
\tanh(x) &= 1 - 2e^{-2x} + 2e^{-4x} + \mathcal{O}\left(e^{-6x}\right)
\end{aligned}$$

  Deriving these is not so different to 'expanding for small x', and you should familiarize yourselves with different ways of finding expansions 'for large x' like these, such as:

  1. expanding by hand – see Exercise Sheet;
  2. using Maple or Wolfram Alpha;
  3. looking them up on Wikipedia, or a proper resource like Abramowitz and Stegun's page-turner *Handbook of Mathematical Functions*.

[Side Notes:] Large variable expansions

You are used to expanding a function $f(x)$ for small x, using a Taylor series

$$f(x) = \sum_{n=0}^{\infty} x^n \frac{1}{n!} \frac{d^n}{dx^n} f(0) \tag{2.4}$$

Much more general series are possible just by simple substitutions.

- E.g. you know how to expand $f(x) = \frac{1}{1-x}$ for small x:

$$f(x) = \frac{1}{1-x} = 1 + x + x^2 + ... = \sum_{n=0}^{\infty} x^n \tag{2.5}$$

- but you can also expand $f(x) = \frac{1}{1-\frac{1}{x}}$ for large x (when $\frac{1}{x}$ is small)

$$f(x) = \frac{1}{1-\frac{1}{x}} = 1 + x^{-1} + x^{-2} + ... = \sum_{n=0}^{\infty} x^{-n} \tag{2.6}$$

- and you can expand $f(x) = \frac{1}{1-e^{-x}}$ for large x (when e^{-x} is small)

$$f(x) = \frac{1}{1-e^{-x}} = 1 + e^{-x} + e^{-2x} + ... = \sum_{n=0}^{\infty} e^{-nx} \tag{2.7}$$

- See Exercise sheet for more. Many complicated functions or solutions of equations can be understood using novel series expansions like these, in terms of any ‘kernel’ you fancy trying, whether x, x^{-1}, e^x, $\ln(x)$, $\sin(x)$, . . . we’ll see methods that result in a few examples.

- Going back to our four series expansions above, and dropping the higher orders, their asymptotic behaviour as $x \to +\infty$ is

$$\begin{aligned} \frac{x}{1+|x|} &\sim 1 - x^{-1} \\ \frac{x}{\sqrt{1+x^2}} &\sim 1 - \tfrac{1}{2}x^{-2} \\ \tfrac{2}{\pi}\arctan(x) &\sim 1 - \tfrac{2}{\pi}x^{-1} \\ \tanh(x) &\sim 1 - 2e^{-2x} \end{aligned}$$

  So two of them approach $f(x) \to 1$ like x^{-1}, the square root function approaches it like x^{-2}, while the tanh function approaches it like e^{-2x}.

  This gives them their quite different rates of approach to the asymptote (see figure above). In applications this can be important, and to study the functions it is vital to know whether the approach to an asymptote like this is polynomial (e.g. x^{-1}) or exponential (e.g. e^{-x}).

- This is their **asymptotic behaviour**. In applications this is what matters more than their *exact* or even *approximate* behaviour.

2.2 Dominant terms and 'Asymptoticness'

The behaviour of an asymptotic series depends strongly on its **dominant** term, i.e. its largest term.

- Two functions $f(x)$ and $g(x)$ can have the same asymptotic behaviour in some limit, but be very different, e.g.

$$f(x) = p(x) + a(x) \qquad \& \qquad g(x) = p(x) + b(x) \tag{2.8}$$

  are given asymptotically by $f(x) \approx g(x) \approx p(x)$ for large x, provided

$$a(x) \ll p(x) \qquad \& \qquad b(x) \ll p(x) \tag{2.9}$$

  In each case $p(x)$ is the *dominant* term.

- This may mean

$$\lim_{x\to\infty} a(x) = 0 \qquad \& \qquad \lim_{x\to\infty} b(x) = 0 \tag{2.10}$$

  but does not have to be quite this simple. The important thing is that the behaviour of f and g is *dominated* by the function $p(x)$ as $x \to \infty$, in which case

$$\lim_{x\to\infty} \frac{f(x)}{p(x)} = 1 \qquad \& \qquad \lim_{x\to\infty} \frac{g(x)}{p(x)} = 1 \tag{2.11}$$

- When two functions are asymptotic in some limit, we often write

$$f(x) \sim g(x) \qquad \Leftrightarrow \qquad f(x) \text{ is asymptotic to } g(x)$$

  which is more precise than just

$$f(x) \approx g(x) \qquad \Leftrightarrow \qquad f(x) \text{ is approximately equal to } g(x)$$

- Example 1. The functions

$$f(x) = \frac{1}{x} \qquad \text{and} \qquad g(x) = \frac{1}{x} - \frac{1}{x^3} \tag{2.12}$$

have the same dominant term $\frac{1}{x}$ for large x, because the difference between them is much smaller, i.e. $\frac{1}{x^3} \ll \frac{1}{x}$ for large x.

- Example 2. The functions

$$f(x) = e^{-2x}\,, \qquad g(x) = e^{-2x} + e^{-3x} + x^{-1}e^{-2x}\,, \qquad h(x) = \frac{1 - \tanh(x)}{2}\,,$$

have the same dominant term e^{-2x} for large x, because the difference between these functions is much smaller than this dominant term, i.e.

$$e^{-3x},\ x^{-1}e^{-2x},\ e^{-4x} \ll\ e^{-2x}$$

for large x.

- It is important that you plot these functions (in Maple or Matlab or Wolfram Alpha or whatever), and look at how these asymptotic expressions show these behaviours for large x (strictly as $x \to \infty$). The Exercise Sheet will help you.

- Above we've mainly been thinking about x large and positive. What happens if x is negative?
- We can do all the same analysis, but we have to be careful to consider $x \to +\infty$ and $x \to -\infty$ separately. Going back to the last two examples:
- Example 1. The functions

$$f(x) = \frac{1}{x} \qquad \text{and} \qquad g(x) = \frac{1}{x} - \frac{1}{x^3} \tag{2.13}$$

  have the same asymptotics for $x \to +\infty$ and $x \to -\infty$.

  But . . .
- Example 2. The functions

$$f(x) = e^{-2x}\ , \qquad g(x) = e^{-2x} + e^{-3x} + x^{-1}e^{-2x}\ , \qquad h(x) = \frac{1 - \tanh(x)}{2}\ ,$$

  do not. For negative x the exponential e^{-x} is large instead of small.

  Instead for negative x we have

$$\begin{aligned} e^{-2x} &\sim e^{-2x} \\ e^{-2x} + e^{-3x} + x^{-1}e^{-2x} &\sim e^{-3x} \\ (1 - \tanh(x))/2 &\sim 1\ , \end{aligned}$$

- Again it is important that you plot these functions to see how these asymptotic expressions pick up their behaviour, now for large negative x.
- The key message here is that the dominant term of a function's asymptotics can change. In these examples this happens as x changes sign, i.e. at $x = 0$.
- Change of the dominant term occurs when two candidate terms have the same size,

  e.g. in $e^{-2x} + e^{-3x} + x^{-1}e^{-2x}$ the dominant term changed from e^{-2x} to e^{-3x} when these two terms were equal, i.e. at $x = 0$ when $e^{-2x} = e^{-3x}$.

Asymptotic series have some behaviours that seem strange at first . . .

2.3 Stokes' discontinuities

The asymptotic series of a smooth function does not itself have to be smooth. It can have jumps.

- We can see this happening in the function tanh, the dominant term $+1$ switches to -1 as x changes sign:

  for $x \to +\infty$ we have $\tanh(x) = +1 - 2e^{-2x} + ...$, but
  for $x \to -\infty$ we have $\tanh(x) = -1 + 2e^{2x} - ...$

  So this is discontinuous, despite the function $\tanh(x)$ being continuous.

- When this was first observed in the mid 1800's it caused alarm. Taking time out from measuring the density of the Earth and weighing Jupiter, George Biddell Airy worked out an equation to explain rainbows. His colleague George Gabriel Stokes, trying to study the equation with series expansions, found his series were discontinuous in this way. He was so upset he wrote to his fiancée in 1857:

  > *When the cat's away the mice may play.*
  > *You are the cat and I am the mouse. I have been doing*
  > *what I guess you won't let me do when we are married,*
  > *sitting up till 3 o'clock in the morning*
  > *fighting hard against a mathematical difficulty.*
  > *Some years ago I attacked an integral of Airy's, and*
  > *after a severe trial reduced it to a readily calculable form.*
  > *But there was one difficulty about it which,*
  > *though I tried till I almost made myself ill,*
  > *I could not get over ... the discontinuity of arbitrary constants.*

  despite which she still married him.

- Just note that the discontinuity that worried Stokes, and happens in tanh above, happens at $x = 0$, where the asymptotic series ceases to be a good approximation because each term has the same order. Near there, we need to be very careful how we sum in order to get a continuous function.

- There are ways to handle the series near the jumps, but mostly we are just interested in the leading order behaviour, and noting the location of any such jumps, which these days bare the name of **Stokes' discontinuities**.

2.4 Divergence

Asymptotic series don't have to converge, unlike the usual regular power or power series you're probably more used to.

- In asymptotics we often encounter series that converge for their first few terms, but thereafter diverge, often due to competition between power and factorials something like

$$1 + \frac{1}{x} + \frac{2!}{x^2} + \frac{3!}{x^3} + ... + \frac{n!}{x^n} + ... \tag{2.14}$$

  This is decreasing for $n < x - 1$ and increasing for $n > x - 1$ (show this by assuming two consecutive terms are increasing, $\frac{n!}{x^n} < \frac{(n+1)!}{x^{n+1}}$ which implies $n > x - 1$).

- Just taking the first one (or perhaps two) terms of a series is called **leading order asymptotics**.

- The rest of the series is called it **tail**.

- We won't concern ourselves with the tail and will usually **truncate** to just the leading order terms. But it is useful to know that methods for handling the tail exist:

  - Truncating a divergent series at its least term is called **superasymptotics**.
  - There are then methods to estimate the size of the **remainder** in the divergent *tail* of the series (see Dingle 1973).
  - There are even methods to add up the terms in the divergent *tail*, called **hyperasymptotics**.

[Side Notes:] Careful with divergent series

Because asymptotic series can diverge, and because they involve careful balancing of infinities, they can be incredibly powerful (e.g. the poles in the Complex part of this course), but can also lead to some paradoxical arguments without sufficient care.

- E.g. Take the geometric series

$$f(x) = \sum_{n=0}^{\infty} x^n \ . \tag{2.15}$$

Let's expand this out, and re-arrange it a little, as follows:

$$\begin{aligned} f(x) &= 1 + x + x^2 + x^3 + \ldots \\ &= 1 + x(1 + x + x^2 + \ldots) \\ &= 1 + x \sum_{n=0}^{\infty} x^n \\ &= 1 + x f(x) \end{aligned} \tag{2.16}$$

which seems innocent enough, and now re-arranges to show

$$f(x) = \frac{1}{1-x} \ . \tag{2.17}$$

So this implies that $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$, which you know already, it is just a perfectly valid Taylor series for $|x| < 1$.

But we didn't assume $|x| < 1$ in deriving this identity, and our algebra suggests this formula is true *for any* x. Try it for $x = 2$, the left-hand side gives $\frac{1}{1-2} = -1$, while the righthand-side is $\sum x^n = 1 + 2 + 4 + 8 + \ldots$ which cannot possibly be negative! What has gone wrong?

We'll discuss this if you offer me some arguments!

3 Solutions of ODEs: by asymptotic series

We've seen we can use asymptotics to approximate functions, or approximate the solutions of algebraic equations. Perhaps its most important application, however, is to solve *differential equations*.

So imagine that we didn't know the solution to (2.1)

$$y'(x) = 1 - y^2(x) , \qquad y(0) = 0 ,$$

at all. Imagine it wasn't even possible to solve this equation in closed form. How can we still work out what its solutions look like?

Now you wouldn't really use the methods we're about to learn on this precise example, not least because you can solve it exactly. But using this example I can introduce all of the asymptotic methods that I really need to. Then we'll turn them to much harder problems afterwards.

The fact that we know the solution is $y(x) = \tanh(x)$ helps us understand how and why these methods work. It also gives us surprising insight into why something like tanh is actually a more sophisticated function than it looks.

3.1 Series solutions

Let's try to find a series solution to the problem (2.1).

- To approximate the solution for small x, let's look for a solution of the form

$$y(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + ...$$

  just substitute this into the ODE, and solve for successive orders of x (see Exercise Sheet). You'll find $a_n = 0$ for n even, while $a_1 = 1$, $a_3 = -\frac{1}{3}$, $a_5 = \frac{2}{15}$, ... giving

$$y(x) = x - \tfrac{1}{3}x^3 + \tfrac{2}{15}x^5 - ... \tag{3.1}$$

  This turns out to be just the power series for $\tanh(x)$ expanded about $x = 0$. This is the approximation of the ODE's solutions for small x.

- To approximate the solution for large x, let us try something similar, but using terms that are small when x is large, e.g. powers of $1/x$, so try

$$y(x) = \sum_{n=0}^{\infty} a_n x^{-n} = a_0 + a_1 x^{-1} + a_2 x^{-2} + ...$$

  Now when you substitute this into the ODE and solve for successive orders of x (see Exercise Sheet) you'll find $a_0 = \pm 1$, but all others vanish $a_1 = a_2 = ... = 0$. Why?

- This is the first hint of something called **beyond all orders** asymptotics: quite simply the behaviour of y is **too close** to $y \approx \pm 1$ to be distinguished from it **to any power of** x^n. So a series in powers of x won't work.

[Side Notes:] Beyond all orders asymptotics

"Beyond all orders" terms can result in some exotic phenomena in physics and engineering. But "beyond all orders" really just means that the behaviour you are looking for is too small to capture with the representation you are using, often that you're looking for a polynomial (x^n) approximation when you need an exponential (e^{-x}) one.

- For example, if you try to approximate
$$\frac{1}{1-e^{-1/x}} = a_0 + a_1 x + a_2 x^2 + ... \tag{3.2}$$
you find $a_0 = 0$ or 1, while all other coefficients $a_{n\geq 1}$ must be zero (e.g. evaluate $f(0)$, $f'(0)$, $f''(0)$, ... both sides and solve for the a_n's). This is because the lefthand-side is changing too slowly to equate to the lefthand-side polynomial terms. Any useful description is *beyond all polynomial orders*. For a useful series we can instead write for small $x > 0$
$$\frac{1}{1-e^{-1/x}} = 1 + e^{-1/x} + e^{-2/x} + e^{-3/x} + ... \tag{3.3}$$
- We'll deal more with large x rather than small x, so for example you cannot expand $\frac{1}{1-e^{-x}}$ as a polynomial in x or even x^{-1} for large x, instead you must expand as
$$\frac{1}{1-e^{-x}} = 1 + e^{-x} + e^{-2x} + e^{-3x} + ... \tag{3.4}$$
- In asymptotic series, you can use whatever 'kernel' (small term) you want to develop a power series out of. We'll see ways to find the appropriate terms later.

- Returning to our problem to solve (2.1), we found tanh was too close to ± 1 for large x to distinguish it with polynomial terms like x or x^{-1}. Instead it is exponentially close to ± 1, so try an exponential series in powers of e^{-x}, say

$$y(x) = \sum_{n=0} a_n e^{-nx} = a_0 + a_1 e^{-x} + a_2 e^{-2x} + ... \tag{3.5}$$

Then, as usual, substitute into the ODE and compare orders, $y' = 1 - y^2$ becomes

$$\begin{array}{llll} -a_1 e^{-x} - 2a_2 e^{-2x} - ... = 1 - [a_0 + a_1 e^{-x} + a_2 e^{-2x} ...]^2 & & & \\ \Rightarrow \quad \text{coeffs. of } e^0 : & 1 - a_0^2 = 0 & \Rightarrow & a_0 = 1 \\ \quad \text{coeffs. of } e^{-x} : & -a_1 = -2a_0 a_1 & \Rightarrow & a_1 = 0 \\ \quad \text{coeffs. of } e^{-2x} : & -2a_2 = -a_1^2 - 2a_0 a_2 & \Rightarrow & a_2 =? \\ \quad \text{coeffs. of } e^{-3x} : & -3a_3 = -2a_1 a_2 - 2a_0 a_3 & \Rightarrow & a_3 = 0 \\ \quad \text{coeffs. of } e^{-4x} : & -4a_4 = -a_2^2 - 2a_1 a_3 - 2a_0 a_4 & \Rightarrow & a_4 = \frac{1}{2}a_2^2 \\ \quad \cdots & & & \\ \quad \cdots \quad : & a_{2k+1} = 0 \quad \& \quad a_{2k} = 2(\frac{1}{2}a_2)^k \ , \quad k \in \mathbb{N} & & \end{array}$$

We don't seem to be able to solve for a_2, but all other coefficients are fixed, so we have

$$y(x) = 1 + a_2 e^{-2x} + \tfrac{1}{2}a_2^2 e^{-4x} + ... + 2(\tfrac{1}{2}a_2)^k e^{-2kx} + ... \tag{3.6}$$

(You might also find a solution with $a_0 = -1$, but then all other coefficients vanish, so like the power series this is not a useful solution.)

- The oddity here is that we haven't fully solved the problem because we don't know a_2. Why not?

- There's a good reason for this. This solution applies for large x, but for large x we only have the ODE $y'' = 1 - y^2$, we cannot use the boundary condition $y(0) = 0$, so the problem we're solving for large x isn't fully determined.

- So indeed there *should* be an indeterminable constant in our solution that can only be fixed with an appropriate boundary condition, one that we *can* use for large x.

- The initial or boundary condition at $x = 0$ lies separated from the 'large x' domain of our series solution by something called a **boundary layer**. You can think of this like the meniscus in a fluid: the solution of an ODE over the bulk of a fluid undergoes an adjustment near the wall of its container, to satisfy a boundary condition there, and indeed the concepts are related.

- So we have a large x or *outer* solution

$$y(x) = 1 + a_2 e^{-2x} + \tfrac{1}{2} a_2^2 e^{-4x} + ... + 2(\tfrac{1}{2}a_2)^k e^{-2kx} + ... \tag{3.7}$$

  and a small x or *inner* solution

$$y(x) = x - \tfrac{1}{3}x^3 + \tfrac{2}{15}x^5 - ... \tag{3.8}$$

  One of these is fully determined (because the boundary condition lies in its domain), while the other has an undetermined coefficient — this is the typical situation when we have an inner and outer solution.

[Side Notes:] Boundary layers are usually 'small'

Usually the boundary layer refers to a 'small layer' relative to some parameter. If we started instead of (2.1) with

$$\varepsilon y'(x) = 1 - y^2(x)$$

then the analysis above would all proceed in a very similar manner, but with x replaced everywhere by x/ε. Small x would then refer to a boundary layer on the small region $|x| \ll \varepsilon$ where the inner solution applies.

3.2 Matching

To fix the coefficient we need to **match** the inner and outer approximations to each other, forming a **uniform** solution valid over the whole domain.

- Let's imagine that our outer solution remains valid for small x (even though strictly speaking it does not, but in asymptotics it often pays not to be too strict, if we proceed with care).

  Expand the outer solution for small x, in which case we can expand the exponentials as power series, giving

$$\begin{aligned} y(x) &= 1 + a_2 e^{-2x} + \tfrac{1}{2} a_2^2 e^{-4x} + ... \\ &= 1 + a_2(1 - 2x + 2x^2 - ...) + \tfrac{1}{2} a_2^2 (1 - 4x + 8x^2 - ...) + ... \\ &= 1 + a_2 + \tfrac{1}{2} a_2^2 + ... + 2(\tfrac{1}{2} a_2)^k + \mathcal{O}(x) \end{aligned} \tag{3.9}$$

  Looking just at the leading order (constant) terms, notice these just form a geometric series, giving

$$\begin{aligned} y(x) &= 1 + a_2(1 + \tfrac{a_2}{2} + (\tfrac{a_2}{2})^2 + ... + (\tfrac{a_2}{2})^k + ...) + \mathcal{O}(x) \\ &= 1 + \frac{a_2}{1 - \frac{1}{2} a_2} + \mathcal{O}(x) \end{aligned} \tag{3.10}$$

  and now we can apply the boundary condition $y(0) = 0$, giving

$$y(0) = 0 = 1 + \frac{a_2}{1 - \frac{1}{2} a_2} \qquad \Rightarrow \qquad a_2 = -2\ . \tag{3.11}$$

- So finally for large x we have

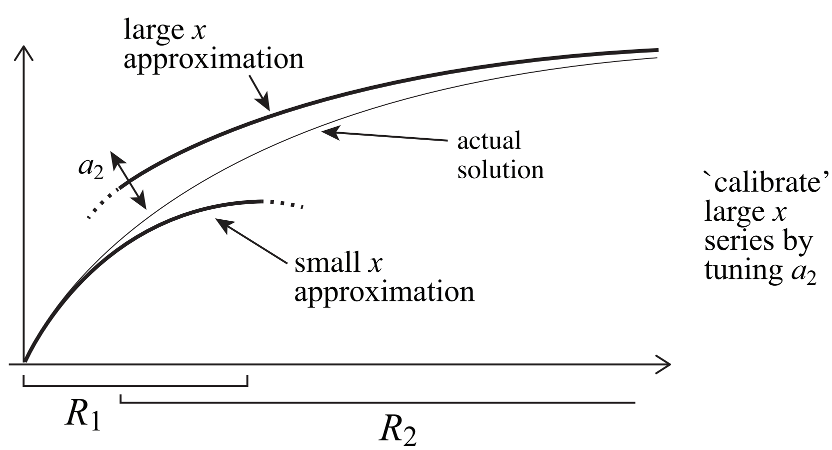
$$y(x) = 1 - 2e^{-2x} + 2e^{-4x} + ... + 2(-1)^k e^{-2kx} + ... \tag{3.12}$$

  Remarkably this worked, and again we have the correct large positive x series for $\tanh(x)$. If we consider large negative x we will obtain the usual counterpart with $y(x) = -1 + 2e^{2x} - ...$

[Side Notes:] Asymptotic Matching

Typically solving a problem with different assumptions of largeness/smallness, we get different solutions.

- If those solutions are valid on regions R_1 (e.g. 'small x') and R_2 (e.g. 'large x'), when we define those regions rigorously we may find they have an overlap, i.e. that both solutions are valid on some mid-range region.
- In such a case there is usually at least one undetermined parameter in one of these solutions, which cannot be determined because it requires information from outside its own region.
- We then expand the R_2 solution under the assumptions of R_1 (i.e. expand the large x solution for small x), or vice verse, and we should find that the problem (the differential equation and any boundary conditions) are enough to fix the undetermined parameter(s).
- E.g. in the ODE example above, the small x (inner) region was $R_1 = \{|x| < 1\}$, and the large x (outer) region we might have expected to be just $\{|x| > 1\}$, but actually the way we obtained the solution it remains valid on $R_2 = \{x > 0\}$. When we expanded the outer solution for small x we were able to fix its undetermined parameter a_2.



3.3 Method of dominant balance

When we cannot guess a series form, we can use a method called dominant balance.

- Guess an anstaz — this will allow us to build the first term in our series, without guessing its whole form. Typically in this method we try assuming

$$y(x) \sim e^{S(x)} \tag{3.13}$$

  if y is close to zero for large x.

- We then substitute this into the ODE, and solve to find $S(x)$.

- When we solve for S we only do it *asymptotically*, not equating terms *exactly* as we did with coefficients in the series methods above.

- Let's try this for (2.1). Now we said e^S should be close to zero for x large, and we know (2.1) has a solution y tending to 1 for large x, so just adapt this slightly to try an ansatz $y(x) \sim 1 + e^{S(x)}$, and substitute into the ODE to get

$$S'e^S \sim 1 - (1 + e^S)^2 \quad \Rightarrow \quad S'e^S \sim -2e^S - e^{2S}$$

  Now divide by e^S to give $S' \sim -2 - e^S$. The last term is small, $e^S \ll 1$, so solve

$$S' \sim -2 \quad \Rightarrow \quad S \sim -2x + k$$

  for some integration constant k, so we have

$$y(x) \sim 1 + e^{-2x+k} = 1 + a_2 e^{-2x}$$

  where $a_2 = e^{-k}$ is the free parameter we had to fix above by matching.

- To improve the approximation and find the next terms in the series, we then assume $y(x) \sim e^{S(x)+T(x)}$ where $|T| \ll |S|$ and solve for T, then $y(x) \sim e^{S(x)+T(x)+W(s)}$ where $|W| \ll |T| \ll |S|$ and solve for W and so on, carefully eliminating terms by balancing dominant terms each time. We'll use this later – it won't work well for our example above for the following reason.

- Usually we use this method when and ODE has a singular point (a coefficient goes to infinity) in the limit we are expanding about. Even though this doesn't apply to (2.1), it still catches the leading order.

3.4 Series solutions for perturbed ODEs

A hybrid of the things we've seen so far is a problem with a small ε which we can approximate as a function of x and ε.

E.g. Consider

$$y'(x) = \varepsilon - y^2(x) \ , \qquad y(0) = c\varepsilon \ , \tag{3.14}$$

for small ε. Note we've made the initial condition also ε dependent, and we'll see how to handle this.

- Let's seek a solution for small ε, making no assumptions on x. We try something like we did for regularly perturbed algebraic solutions, in this case an ansatz

$$y(x,\varepsilon) = y_0(x) + y_1(x)\varepsilon + y_2(x)\varepsilon^2 + ... \tag{3.15}$$

  which is a power series in ε, but now the coefficients are functions $y_n(x)$ (previously they were just constants a_n).

- The initial condition $y(0) = \varepsilon$ tells us that for these we have initial conditions

$$y_0(0) = 0 \ , \quad y_1(0) = c \ , \quad y_2(0) = 0 \ , \quad y_3(0) = 0 \ , \quad ... \tag{3.16}$$

  From hereon the method is the same as in section 1: substitute the ansatz into the equation (the ODE) and solve for the coefficients of ε.

- The ODE (3.14) becomes (writing just y_n instead of $y_n(x)$ everywhere)

$$[y_0' + y_1'\varepsilon + y_2'\varepsilon^2 + ...] = \varepsilon - [y_0 + y_1\varepsilon + y_2\varepsilon^2 + ...]^2 \ , \tag{3.17}$$

  with the initial conditions (3.16). Comparing coefficients of ε we have:

$$\begin{array}{lllll}
\text{coeffs. of } \varepsilon^0 & : & y_0' = -y_0^2 & \Rightarrow & y_0 = \frac{y_0(0)}{1+y_0(0)x} = 0 \\
\text{coeffs. of } \varepsilon^1 & : & y_1' = 1 - 2y_0y_1 & \Rightarrow & y_1 = x + y_1(0) = x + c \\
\text{coeffs. of } \varepsilon^2 & : & y_2' = -y_1^2 - 2y_0y_2 & \Rightarrow & y_2 = -\frac{1}{3}(x+c)^3 + \frac{1}{3} \\
\quad \ldots & : & ... = ... & &
\end{array} \tag{3.18}$$

  so

$$y(x,\varepsilon) = (x+c)\varepsilon + \left(\tfrac{1}{3} - \tfrac{1}{3}(x+c)^3\right)\varepsilon^2 + \mathcal{O}\left(\varepsilon^3\right) \ . \tag{3.19}$$

E.g. Consider

$$y'(x) = 1 - \varepsilon y^2(x) \,, \qquad y(0) = c\varepsilon^2 \,, \tag{3.20}$$

for small ε. We've taken a different initial condition to further illustrate how this works.

- Again seek a solution for small ε, making no assumptions on x, as an ansatz

$$y(x, \varepsilon) = y_0(x) + y_1(x)\varepsilon + y_2(x)\varepsilon^2 + ... \tag{3.21}$$

where again the coefficients are functions $y_n(x)$.

- Here the initial condition $y(0) = \varepsilon^2$ tells us that for these we have initial conditions

$$y_0(0) = 0 \,, \quad y_1(0) = 0 \,, \quad y_2(0) = c \,, \quad y_3(0) = 0 \,, \quad ... \tag{3.22}$$

As usual now: substitute the ansatz into the equation (the ODE) and solve for the coefficients of ε.

- The ODE (3.20) becomes (writing just y_n instead of $y_n(x)$ everywhere)

$$[y_0' + y_1'\varepsilon + y_2'\varepsilon^2 + ...] = 1 - \varepsilon[y_0 + y_1\varepsilon + y_2\varepsilon^2 + ...]^2 \,, \tag{3.23}$$

with the initial conditions (3.22). Comparing coefficients of ε we have:

$$\begin{array}{lcll} \text{coeffs. of } \varepsilon^0 & : & y_0' = 1 & \Rightarrow \quad y_0 = x + y_0(0) = x \\ \text{coeffs. of } \varepsilon^1 & : & y_1' = -y_0^2 & \Rightarrow \quad y_1 = -\frac{1}{3}x^3 + y_1(0) = -\frac{1}{3}x^3 \\ \text{coeffs. of } \varepsilon^2 & : & y_2' = -2y_0y_1 & \Rightarrow \quad y_2 = \frac{2}{15}x^5 + y_2(0) = \frac{2}{15}x^5 + c \\ \quad ... & : & ... = ... & \end{array} \tag{3.24}$$

so

$$y(x, \varepsilon) = x - \tfrac{1}{3}x^3\varepsilon + (\tfrac{2}{15}x^5 + c)\varepsilon^2 + \mathcal{O}\left(\varepsilon^3\right) \,. \tag{3.25}$$

[Further Reading Only:] Remark on these solutions

Again we can actually solve the two problems above exactly. Then let's compare them to $y'(x) = 1 - y^2(x)$, whose solution is $y(x) = \tanh(x)$.

- Let $c = 0$ in both cases for simplicity, then:

  the solution to (3.14) is just

  $$y(x) = \sqrt{\varepsilon}\tanh(x\sqrt{\varepsilon})$$

  and the solution to (3.20) is just

  $$y(x) = \tfrac{1}{\sqrt{\varepsilon}}\tanh(x\sqrt{\varepsilon})\ .$$

- If you expand the tanh for small ε you obtain the solutions we got above using a power series in ε.

- For this problem, that's just the same as expanding for small x, i.e. if you expand the tanh for small x as we did in section 3.1 you'll get the approximations in powers of ε as above.

- This should hint to you that you can turn $y' = \varepsilon - y^2$ and $y' = 1 - \varepsilon y^2$ into the ε-independent ODE $y' = 1 - y^2$ with an appropriate re-scaling.

- You can try to see if you can find that re-scaling. Take $y' = \varepsilon - y^2$ or $y' = 1 - \varepsilon y^2$, and let $y = \varepsilon^a Y$ and $x = \varepsilon^b X$, substitute these into the ODEs and find the values of a and b that give $Y' = 1 - Y^2$, whose solution is $Y = \tanh(X)$.

  [You should find $b = -a = -\frac{1}{2}$ for $y' = \varepsilon - y^2$ and $b = a = -\frac{1}{2}$ for $y' = 1 - \varepsilon y^2$]

4 Integral Methods

Sometimes we can find an integral that solves an ODE, but we cannot solve the integral itself, so we need an asymptotic series for it instead. To illustrate this let's go back to our nice simple equation (2.1).

If $y(x) = \tanh(x)$, then its derivative is $y'(x) = \operatorname{sech}^2(x)$. Let's use this to write tanh as an integral

$$\tanh(x) = \int_0^x \operatorname{sech}^2(t)dt \tag{4.1}$$

We'll now use this as another way to find the asymptotics of tanh.

4.1 Series expansion of integrals

- To approximate this for small x, the integral only goes up to small $t = x$, so expand the integrand $\operatorname{sech}^2(t)$ as a power series for small t (use a Taylor or Maclaurin series, or other tricks we've learnt such as $\operatorname{sech}^2(t) = 1/\cosh^2(t) = 1/(1 + \frac{1}{2}t^2 + ...)^2 = ...$), giving

$$\begin{aligned}\tanh(x) &= \int_0^x (1 - t^2 + \tfrac{2}{3}t^4 - \tfrac{17}{45}t^6 + ...)dt \\ &= \left[t - \tfrac{1}{3}t^3 + \tfrac{2}{15}t^5 - \tfrac{17}{315}t^7 + ...\right]_0^x \\ &= x - \tfrac{1}{3}x^3 + \tfrac{2}{15}x^5 - \tfrac{17}{315}x^7 + ... \end{aligned} \tag{4.2}$$

which is just the small x power series for $\tanh(x)$, as we'd expect.

- To find the large x asymptotics, as we've seen before, such a direct polynomial series won't work. Instead, first look at the extreme value or *limit*: when x approaches infinity we have an integral we can do,

$$\tanh(x) = \int_0^\infty \operatorname{sech}^2(t)dt = 1 \qquad \text{for} \quad x \to \infty \tag{4.3}$$

- Let's pull this out of the function we're looking at, by writing

$$\begin{aligned}\tanh(x) &= \int_0^x \operatorname{sech}^2(t)dt \\ &= \left\{\int_0^\infty - \int_x^\infty\right\} \operatorname{sech}^2(t)dt \\ &= \quad 1 \quad - \int_x^\infty \operatorname{sech}^2(t)dt \end{aligned} \tag{4.4}$$

  This '1' is the familiar first term of our asymptotic series. Now for the rest...

- We're left with having to evaluate the integral I'll call "canh" (not a real name)

$$\operatorname{canh}(x) = \int_x^\infty \operatorname{sech}^2(t)dt \tag{4.5}$$

  This is good for a few reasons. For large x the interval we're integrating over to find canh is small, and also the integrand $\operatorname{sech}^2(t)$ is small for large t, and zero in the limit $t \to \infty$. So we just need to find the asymptotics of this small integral canh.

- We can do this by approximating the integrand about the point $t = x$.

- Again we know a polynomial expansion won't work, but we can use the formula $\text{sech}(t) = \frac{2}{e^t+e^{-t}}$, or better, since we're interested in large x, divide by e^{-t} to write this in terms of a small e^{-t} term as $\text{sech}(t) = \frac{2e^{-t}}{1+e^{-2t}}$, then

$$\text{canh}(x) = \int_x^\infty \frac{4e^{-2t}}{(1+e^{-2t})^2} dt \tag{4.6}$$

Since e^{-2t} is small, I can expand this as a power series of e^{-2t} terms,

$$\begin{aligned} \text{canh}(x) &= \int_x^\infty \frac{4e^{-2t}}{(1+e^{-2t})^2} dt \\ &= \int_x^\infty 4e^{-2t} \left\{1 - 2e^{-2t} + ... + (n+1)(-1)^n e^{-2nt} + ...\right\} dt \end{aligned} \tag{4.7}$$

- If I can't integrate this easily, I might need more tricks. A common one is to define a new integration variable $u = t - x$, and do another expansion of the "{...}" part as a power series in u. But this case is easy, I can just integrate term-by-term to get

$$\begin{aligned} \text{canh}(x) &= 4\left[e^{-2t}\left\{-\tfrac{1}{2} - \tfrac{2}{-4}e^{-2t} - ... - (-1)^n e^{-2nt} + ...\right\}\right]_x^\infty \\ &= 2e^{-2x}\left\{1 - e^{-2x} + ... - (-1)^n e^{-2nx} + ...\right\} \end{aligned} \tag{4.8}$$

- Putting this together with the first part of the integral we have

$$\tanh(x) = 1 - 2e^{-2x} + 2e^{-4x} + ... + 2(-1)^n e^{-2nx} + ... \tag{4.9}$$

which is, of course, our by now familiar large x series expansion for $\tanh(x)$.

- What about x large but negative?

  I could start right back from tanh in (4.3), but instead I'm going to just start again from (4.5) to illustrate something important.

- The integral (4.5) is no good for x near $-\infty$, instead I want an integral over $[x, -\infty)$. That's easy to get, just re-write

$$\text{canh}(x) = \left\{ \int_{-\infty}^{\infty} + \int_{x}^{-\infty} \right\} \text{sech}^2(t)dt = 2 + \int_{x}^{-\infty} \text{sech}^2(t)dt \tag{4.10}$$

  as $\int_{-\infty}^{\infty} \text{sech}^2(t)dt = 2$.

- For the last part of this integral just follow steps analogous to those above but for x large and negative. Or take the result

$$\int_{x}^{\infty} \text{sech}^2(t)dt = 2e^{-2x} \left\{ 1 - e^{-2x} + ... - (-1)^n e^{-2nx} + ... \right\} \tag{4.11}$$

  and substitute x with $-x$ and t with $-t$, giving

$$\int_{x}^{-\infty} \text{sech}^2(t)dt = -2e^{2x} \left\{ 1 - e^{2x} + ... - (-1)^n e^{2nx} + ... \right\} \tag{4.12}$$

- Putting this together with the other parts of the integral we have

$$\begin{aligned} \tanh(x) &= 1 - 2 + 2e^{2x} - 2e^{4x} + ... + (-1)^n e^{2nx} + ... \\ &= -1 + 2e^{2x} - 2e^{4x} + ... + (-1)^n e^{2nx} + ... \end{aligned} \tag{4.13}$$

  i.e. our familiar large negative x series expansion for $\tanh(x)$.

The thing I want to point out here is the *Stokes discontinuity*. Because I had to choose a different integration path for large negative x, compared to large positive x, I picked up an extra '2' term that wasn't there before, and this switches the leading order term from $+1$ to -1.

- This is typical of how Stokes discontinuities occur: a change or *bifurcation* in the integration path introduces new terms in the asymptotic series that were not there before.

There are two useful and general methods used to derive series expansions like those above. They are **Watson's Lemma** and **Integration by parts**.

4.2 Watson's Lemma

Watson's Lemma generalizes the approximation we used above for the integral.

- We had

$$\operatorname{canh}(x) = \int_x^\infty \operatorname{sech}^2(t)dt = 4\int_x^\infty e^{-2t}p(t)dt \tag{4.14}$$

  where $p(t) = (1+e^{-2t})^{-2}$, and we expanded $p(t)$ as a series of exponentials. Integrating term-by-term we obtained an asymptotic series for canh, and hence for tanh.

  (Just to remind you, "canh" was our made up name for this function, it isn't a real term.)

[Side Notes:] Watson's Lemma

The proper form of Watson's Lemma provides a general asymptotic expansion

$$I = \int_0^c e^{-kt} f(t)dt \quad \sim \quad \sum_{n=0}^\infty \frac{g^{(n)}(0)\Gamma(\lambda+n+1)}{n!\; k^{\lambda+n+1}} \tag{4.15}$$

This assumes $k > 0$ is large (strictly $k \to \infty$), and $f(t)$ can be written as $f(t) = t^\lambda g(t)$, where $g(0) \neq 0$, $\lambda > -1$, and g is smooth (has infinitely many derivatives) at $t = 0$, moreover $f(t)$ is bounded such that $\int_0^c |f(t)|dt < \infty$.

Given an integral of the more general form

$$I = \int_a^b e^{\Phi(s)}h(s)ds \tag{4.16}$$

if you try to expand about $s = a$ (or similar about $s = b$), just expand the exponent $\Phi(s) = \Phi(a) + (s-a)\Phi'(a) + \frac{1}{2}(s-a)^2 + ...$, let $t = s - a$, $c = b - a$, and $k = -\Phi'(a)$, to get

$$I = \int_0^c e^{-kt} f(t)dt\ , \qquad f(t) = h(t)e^{\Phi(a)+\frac{1}{2}t^2\Phi''(a)+\dots} \tag{4.17}$$

We might only know h or f themselves as a series expansion.

4.3 Integration by parts

Integration by parts can often be used iteratively to develop the asymptotic series.

- We already wrote our integral above in the form

$$\int_x^\infty \operatorname{sech}^2 t\, dt = 4\int_x^\infty p(t)e^{-2t}dt \tag{4.18}$$

with $p(t) = (1+e^{-2t})^{-2}$. Note $p'(t) = 4p^{3/2}(t)e^{-2t}$. Now integrate by parts as follows:

$$\text{let } \begin{cases} u = -\frac{1}{2}p\,, & v' = -2e^{-2t}\,, \\ u' = -\frac{1}{2}p'\,, & v = e^{-2t}\,, \end{cases}$$

$$\Rightarrow \int_x^\infty pe^{-2t}\,dt = -\tfrac{1}{2}\left[pe^{-2t}\right]_x^\infty + 2\int_x^\infty p^{3/2}e^{-4t}dt$$

$$\text{then let } \begin{cases} u = -\frac{1}{4}p^{3/2}\,, & v' = -4e^{-4t}\,, \\ u' = -\frac{3}{8}p^{1/2}p'\,, & v = e^{-4t}\,, \end{cases}$$

$$= -\tfrac{1}{2}\left[pe^{-2t} + p^{3/2}e^{-4t}\right]_x^\infty + 3\int_x^\infty p^2e^{-6t}dt$$

$$\text{then let } \begin{cases} u = -\frac{1}{6}p^2\,, & v' = -6e^{-6t}\,, \\ u' = -\frac{1}{3}pp'\,, & v = e^{-6t}\,, \end{cases}$$

$$\begin{aligned} &= \left[-\tfrac{1}{2}pe^{-2t} - \tfrac{1}{2}p^{3/2}e^{-4t} - \tfrac{1}{2}p^2e^{-6t}\right]_x^\infty + \int_x^\infty pp'e^{-8t}dt \\ &= -\tfrac{1}{2}\left[pe^{-2t} + p^{3/2}e^{-4t} + p^2e^{-6t}\right]_x^\infty + 4\int_x^\infty p^{5/2}e^{-8t}dt \end{aligned}$$

and so on . . . .

Evaluating the terms in square brackets we have

$$\begin{aligned} \int_x^\infty \operatorname{sech}^2 t\, dt &= 2\left(p(x)e^{-2x} + p^{3/2}(x)e^{-4x} + p^2(x)e^{-6x}\right) + ... \\ &= 2e^{-2x} - 2e^{-4x} + 2e^{-6x} - ... - (-1)^n e^{-2nx} + ... \end{aligned} \tag{4.19}$$

as we found before.

- This shows the general procedure. Ironically the algebra (with all these p's and their derivatives) is quite tough for this example, but it often turns out to be rather easy, powerful, and widely applicable to many complicated integrals involving a large variable.

[Side Notes:] Integration by parts

In a more general form we may write the procedure above by taking an integral in the form

$$\int p(t)e^{S(t)}dt \tag{4.20}$$

Let

$$\begin{aligned} & u = p(t)/S'(t)\ , && v' = S'(t)e^{S(t)} \\ \Rightarrow\quad & u' = \dots\ , && v = e^{S(t)} \end{aligned} \tag{4.21}$$

and use the formula for integration by parts

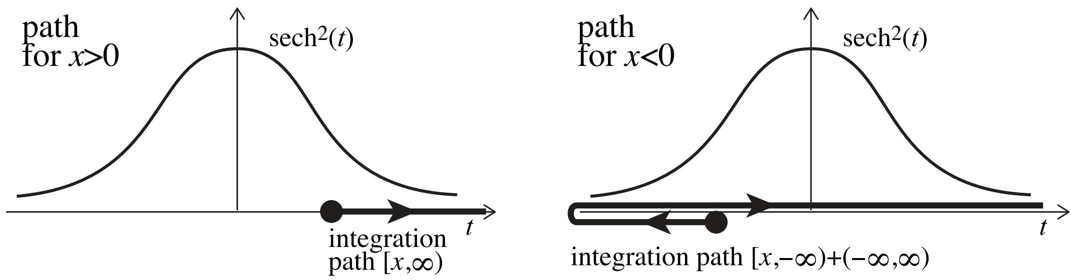
$$\int uv'dt = [uv] - \int u'v\ dt \tag{4.22}$$

The integral $\int u'v\ dt$ will again be in the form (4.20), so we apply the method again, and iterate over and over to generate an asymptotic series.

The great thing about this method is how few assumptions is makes about the functions p and S, but often we just need to try it out to see if the resulting series is useful.

4.4 Blazing saddles: Geometry of the integral

Look again at the different integration paths we used for $x > 0$ and for $x < 0$, and also look at the value of the integrand sech^2 along it:



There were actually just two things that contributed to our integral for tanh, and they were

- the endpoint of the integration path at $t = x$,
- a maximum in the value of the integrand at $t = 0$.

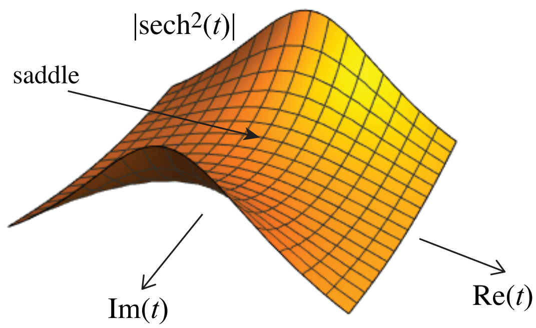
Note the contribution from the maximum switched on or off depending on where x lay in our domain (just $x > 0$ or $x < 0$ here), via a *Stokes discontinuity*.

That is the geometry of the integral. And actually these are generally the only *contributions* that give us the terms of our asymptotic series, consisting of:

- endpoints (the integral approximated about one or other endpoints of the integration path).
- saddlepoints (places where the integrand reaches a maximum or minimum along the integration path). There may be any number of these, and they can switch on or off via Stokes discontinuities.

Why saddlepoints (not just maxima)? It's time to bring the two parts of our course together:

- you know from the first part of our course that the only proper way to study an integral is in the complex plane.
- in the complex plane, an *analytic* function (which mostly we're dealing with) doesn't have maxima, it has saddlepoints: its stationary points are maxima along one direction but minima along another.



This plot shows $|\operatorname{sech}^2(t)|$ plotted not just along the real line, but in $\mathbb{C}$.

To find the asymptotics of an integral then:

- you shouldn't look for maxima along $\mathbb{R}$, but saddlepoints in $\mathbb{C}$. These are points in $\mathbb{C}$ where the integrand is stationary.
- you then approximate the contributions from your integral about the endpoints, and any saddlepoints, in complex variables.
- Fortunately this is easy . . .

The last thing we need to know now is whether to include all of the possible endpoint and saddlepoint contributions. To do that we just have to look at the integral's full integration path, in the complex plane.

4.5 Steepest descent and stationary phase

We know, from the first part of this course, that you can change an integrals path provided you keep the endpoints fixed, and avoid any singularities. In the complex this is very powerful, because for an analytic function there is always a direction along which the function is *changing exponentially fast but not oscillating*, i.e. it looks like the figure of our integrand $\operatorname{sech}^2 t$ above, and in that case $\mathbb{R}$ was exactly that path.

- These are called **steepest descent** or **stationary phase** paths. In most literature the two ideas are kept separate, but they're really the same. We'll show why in appendix A, but we are running out of time in this course.
- For an integral
$$\int f(t)dt$$
the steepest descent path is where $|f(t)|$ is changing most quickly, the stationary phase paths are the contours where $\arg(f(t))$ is constant. In fact these are one and the same path for an analytic function.
- To find the asymptotics of an integral properly, you deform your integration contour to like along paths that keep $\arg(f(t))$ constant, while passing through the necessary endpoints and saddlepoints along the way, and adding their contributions to your integral. The path obtained is usually unique.
- This is an incredibly powerful method, but being more geometrical it requires a little more practice and experiment, so alas time demands we leave these ideas to be developed more in appendix A, which will be non-examinable.

4.6 Time to go beyond tanh ...!

It is time to move on to some other ODE problems, the kind of problems these methods were really invented to tackle.

- Most of the problems of interest involve much more complicated functions than tanh, but the methods are much the same, so you can keep revisiting the examples above.
- In fact, I can't find another text where anyone has actually considered that all of these methods could be applied to something as simple as tanh, and they instead use a bewildering array of complicated functions to illustrate each of these methods.
- Learning what method to use where involves some trial and error, so for these exercises we'll usually hint at what method to try, and like above we'll try multiple methods to attack the same problem.

5 The Error Function – a worked example

Now to turn these methods to some tougher ODE problems. We'll run through all these methods for an important thing called the Error function, then attack a variety of other examples like the Airy equation in the Exercise Sheets.

Consider the ordinary differential equation

$$y''(x) + xy'(x) = 0 \tag{5.1}$$

Find an asymptotic expansion of y for large x.

- First try some simple observations. First, for small x this is like

$$y'' = 0 \qquad \Rightarrow \qquad y' = k \qquad \Rightarrow \qquad y \sim kx$$

  for some constant k.

- Similarly, for large x this is like

$$\tfrac{1}{x}y''(x) + y'(x) = 0 \qquad \Rightarrow \qquad y' = 0 \qquad \Rightarrow \qquad y \sim c$$

  for some constant c.

- That's our first sketch of this function's asymptotics. Whatever else we find must fit this.

- We've already seen that some asymptotic series aren't power series (i.e. powers of x). For this problem we'll need something different too. If it's not a power series, it'll involve fractional powers, or logs, or exponentials. You have to use some initiative and intuition, based on asymptotic analysis. In this case we'll start with an exponential, and try to find the terms of an asymptotic series iteratively by balancing dominant terms.

5.1 Dominant balance

The following method is useful when we want to approximate an ODE's solution about a **singular** point, that is, where the coefficients are infinite. We'll look at large x, i.e. where the coefficient of the y' term (simply x here) is infinite.

- Try the general solution

$$y(x) \sim e^{S(x)} \tag{5.2}$$

  for some rational function $S(x)$. Remember this isn't the exact solution, just the asymptotics — keep this in mind.

- We have $y \sim e^S$, so $y' \sim S'e^S$ and $y'' = [S'' + (S')^2]e^S$.

- Substitute it into the ODE $y''(x) + xy'(x) = 0$, and divide by e^S to leave

$$[S'' + (S')^2] + xS' = 0 \tag{5.3}$$

- Er...!? That's no easier to solve than the original problem. But we're not looking to solve things, only solve them *asymptotically!*

- So let's think what S looks like *asymptotically*.
  (Always think: **What can we asymptotically ignore** in this expression?)

- Typically we can assume $S'' \ll S'$, but to be more precise . . .
- say

$$S(x) = ax^{\alpha} \tag{5.4}$$

then $S' = a\alpha x^{\alpha-1}$ and $S'' = a\alpha(\alpha-1)x^{\alpha-2}$, so (thinking of these as exact)

$$\begin{aligned} [a\alpha(\alpha-1)x^{\alpha-2} + a^2\alpha^2 x^{2\alpha-2}] + xa\alpha x^{\alpha-1} &= 0 \\ \Rightarrow \qquad (\alpha-1)x^{-2} + a\alpha x^{\alpha-2} + 1 &= 0 \end{aligned} \tag{5.5}$$

Note α need not be an integer, but we want to find values of α and a that will make (5.5) work or 'balance', at least to its dominant order.

For (5.5) to work, the third term on the lefthand side (the '1') must be balanced by one of the x terms, meaning that term must be a constant — clearly the first term can't do it (it has an x^{-2}), but the second term can if $\alpha = 2$, then $x^{\alpha-2} = 1$, so assume

$$\alpha = 2 \tag{5.6}$$

and (5.5) becomes

$$x^{-2} + 2a + 1 = 0 \ . \tag{5.7}$$

This is allowed because $x^{-2} \ll 1$, so we neglect the smallest term, and the dominant asymptotics reads

$$2a + 1 \sim 0 \qquad \Rightarrow \qquad a = -1/2 \ . \tag{5.8}$$

- So finally

$$S(x) = -\tfrac{1}{2}x^2 \qquad \Rightarrow \qquad y \sim e^{-\frac{1}{2}x^2} \ . \tag{5.9}$$

- This is only the dominant asymptotic term in the exponent — it isn't the exact solution. It is the first term towards building an asymptotic series.

- Now say

$$y(x) \sim e^{S(x)+T(x)} = e^{-\frac{1}{2}x^2+T(x)} \tag{5.10}$$

Clearly T will be quite a different function to S (otherwise it would have shown up as part of S). In particular we should have $|T| \ll |S|$.
We have

$$\begin{aligned} y &\sim e^{-\frac{1}{2}x^2+T} \\ y' &\sim (-x+T')e^{-\frac{1}{2}x^2+T} \\ y'' &\sim [-1+T''+(-x+T')^2]e^{-\frac{1}{2}x^2+T} \end{aligned}$$

- We just drop this back into the ODE (dividing by e^{T+S}):

$$\begin{aligned} 0 &= [-1+T''+(-x+T')^2]+x(-x+T') \\ &= -1+T''-xT'+(T')^2 \end{aligned} \tag{5.11}$$

- We can assume as before that $|T''| \ll |T'| \ll |T|$. But we'll be left with a similar result for T as for S. Assume something a bit different to $S = ax^\alpha$, namely

$$T = b\ln x$$

then

$$T'',\ (T')^2 \ \ll xT' \sim 1 \tag{5.12}$$

leaving

$$0 = -1 - xT' \qquad \Rightarrow \qquad T = -\ln x \tag{5.13}$$

- So

$$y(x) \sim e^{-\frac{1}{2}x^2-\ln x} = x^{-1}e^{-\frac{1}{2}x^2} \tag{5.14}$$

- In principle we can continue, adding another correction term (e.g. $y(x) \sim e^{-\frac{1}{2}x^2-\ln x+U(x)}$), and gradually improving the approximation.

- The next step would be . . .

$$y \sim e^{-\frac{1}{2}x^2-\ln x+U(x)} \tag{5.15}$$

where U can't be like S or T, so assume $U \sim ax^{-\alpha}$, then $(U')^2 \ll x^{-1}U', U'' \ll xU', x^{-2}$ gives

$$0 \sim -xU' + 2x^{-2} \qquad U = -x^{-2} \tag{5.16}$$

so

$$y \sim e^{-\frac{1}{2}x^2-\ln x-x^{-2})} = x^{-1}e^{-\frac{1}{2}x^2}\left(1 - x^{-2} + \mathcal{O}\left(x^{-}4\right)\right) \tag{5.17}$$

- More fully the solution is

$$\begin{aligned} y(x) &\sim k + ce^{-\frac{1}{2}x^2+k-\ln x+\ldots} \\ &= k + ce^{-\frac{1}{2}x^2}\left(x^{-1} + \mathcal{O}\left(x^{-3}\right)\right) \end{aligned} \tag{5.18}$$

Tips:

- If in doubt: try ignoring any term you like, keep track of your assumptions, give up if you encounter contradictions.

- Keep a picture in your head: S is some power of x, say for large x

$$S \sim x^{\alpha} \quad \gg \quad S' \sim x^{\alpha-1} \quad \gg \quad S'' \sim x^{\alpha-2}$$

(e.g. $S \sim x^4$, so $S' \sim x^3$, $S'' \sim x^2$, and obviously $x^4 \gg x^3 \gg x^2$ for large x),

- This is a powerful and very adaptable method. Obviously it's a bit trial-and-error, but each new term tells us a lot about how the function behaves.

- A generalization known as the WKB method in problems with a small parameter ε uses an ansatz $y \sim e^{\varepsilon^{-1}\sum_{n=0}^{\infty}\varepsilon^n S_n(x)}$.

- If you can guess an *integral* rather than *series* form of the solution, you can often do much better.

5.2 Exact integral

In reality the ODE above is another example we can solve, at least as an integral.

- Let $z = y'$, the ODE becomes

$$\begin{aligned}
&z'(x) + xz(x) = 0 \\
&\quad\Rightarrow \qquad \frac{z'}{z} = -x \qquad \Rightarrow \qquad \int \frac{dz}{z} = -\int x\,dx \\
&\Rightarrow \qquad \ln(z) = -x^2/2 + \text{const} \Rightarrow \qquad z(x) = ce^{-x^2/2}
\end{aligned} \tag{5.19}$$

  for some constant c. Then integrating $z = y'$ gives

$$y(x) = \int_0^x z(t)\,dt = c\int_0^x e^{-t^2/2}\,dt \tag{5.20}$$

- You can see that you could have found the solution by *assuming* that y looked something like

$$y(x) = \int_0^x e^{S(t)}dt$$

  for some function $S(t)$ to be found (give this a try). This is another improvement on the series method, and is often useful.

- Unfortunately this often results, as in this case, in an integral that doesn't have an explicit solution. It is one of an arsenal of **standard integrals** that form the bedrock of the study of differential equations. We give them names. This one is the Error Function

$$\text{Erf}\left(\frac{x}{\sqrt{2}}\right) = \sqrt{\tfrac{2}{\pi}}\int_0^x e^{-t^2/2}\,dt \tag{5.21}$$

- But that's not terribly useful in itself. Your computer can calculate this integral and plot it as a graph for you. If using a computer is enough for you, the exits are here . . . here . . and here . . . help yourselves to oxygen masks and parachutes. Everyone else ...

5.3 Integral asymptotics

We want to understand what this function looks like. We need asymptotic expressions, again, for large or small x.

- For small x this is easy. If x is close to 0, then the integral is restricted to t close to 0, so expand inside the integral

$$\begin{aligned}
\operatorname{Erf}\left(\frac{x}{\sqrt{2}}\right) &= \sqrt{\tfrac{2}{\pi}} \int_0^x \left(1 - \tfrac{1}{2}t^2 + \tfrac{1}{8}t^4 - ...\right) dt \\
&= \sqrt{\tfrac{2}{\pi}} \left(x - \tfrac{1}{6}x^2 + \tfrac{1}{40}x^4\right) + \mathcal{O}\left(x^6\right)
\end{aligned} \tag{5.22}$$

- For large x none of our tricks so far will work. But first look at the extreme value or *limit*: if x is *so large* that it tends to infinity, we have an integral we can do,

$$\sqrt{\tfrac{2}{\pi}} \int_0^\infty e^{-t^2/2}\, dt = 1\ . \tag{5.23}$$

  Now, like we did for the tanh example earlier. . .

- Let's pull this out of the function we're looking at, by writing

$$\begin{aligned}
\operatorname{Erf}\left(\frac{x}{\sqrt{2}}\right) &= \sqrt{\tfrac{2}{\pi}} \int_0^x e^{-t^2/2}\, dt \\
&= \sqrt{\tfrac{2}{\pi}} \left\{\int_0^\infty - \int_x^\infty\right\} e^{-t^2/2}\, dt
\end{aligned} \tag{5.24}$$

  The first part just evaluate as one, the second part is called the Complementary Error Function,

$$\operatorname{Erfc}\left(\frac{x}{\sqrt{2}}\right) = \sqrt{\tfrac{2}{\pi}} \int_x^\infty e^{-t^2/2}\, dt \tag{5.25}$$

- So we just want the large x behaviour of Erfc.
- We can't expand $e^{-t^2/2}$ as any sensible series for large t, but we can expand the exponent around $t = x$,

$$\begin{aligned}
\mathrm{Erfc}\left(\frac{x}{\sqrt{2}}\right) &= \sqrt{\tfrac{2}{\pi}} \int_x^\infty e^{-t^2/2}\, dt \\
&= \sqrt{\tfrac{2}{\pi}} \int_x^\infty e^{-x^2/2 - x(t-x) - (t-x)^2/2}\, dt \\
&\qquad \text{group as follows ...} \\
&= \sqrt{\tfrac{2}{\pi}} e^{-x^2/2} \int_x^\infty e^{-x(t-x)} e^{-(t-x)^2/2}\, dt \\
&\qquad \text{now just expand the last term ...} \\
&= \sqrt{\tfrac{2}{\pi}} e^{-x^2/2} \int_x^\infty e^{-x(t-x)} \left(1 - \tfrac{(t-x)^2}{2} + ...\right)\, dt \\
&= \sqrt{\tfrac{2}{\pi}} e^{-x^2/2} \left[e^{-x(t-x)} \left(-\tfrac{1}{x} + \tfrac{1}{x^3} + \tfrac{(t-x)}{x^2} + \tfrac{(t-x)^2}{2x} + ...\right)\right]_x^\infty \\
&= \sqrt{\tfrac{2}{\pi}} e^{-x^2/2} \left(x^{-1} - x^{-3} + ...\right) \qquad (5.26)
\end{aligned}$$

  and of course we can expand as far as we want.

  (The integration in the second-to-last step there is actually a little tricky, it comes from integrating the two terms directly, i.e.

$$\begin{aligned}
1^{st}\text{ term:} &\quad \int_x^\infty e^{-x(t-x)}\,(1)\,dt = \left[e^{-x(t-x)}(-\tfrac{1}{x})\right]_x^\infty \\
2^{nd}\text{ term:} &\quad \int_x^\infty e^{-x(t-x)} \left(-\tfrac{(t-x)^2}{2}\right) dt = \left[e^{-x(t-x)} \left(\tfrac{1}{x^3} + \tfrac{(t-x)}{x^2} + \tfrac{(t-x)^2}{2x}\right)\right]_x^\infty
\end{aligned}$$

  ... the 2^{nd} term is the tricky bit, but to check its right, differentiate the term in square brackets on the right, to verify that it gives the integrand on the left. We don't bother finding further terms of the series as we spot they'll all be higher order (smaller) in x^{-1}.)

- More importantly, notice that we had to expand the exponent first, taking out the term $e^{-x(t-x)}$, and then only expanding the term $e^{-(t-x)^2/2}$ as a polynomial. Without keeping the $e^{-x(t-x)}$ term we wouldn't have got a convergent integral. This is similar to our problems with tanh earlier, we had to make sure we kept enough exponential terms for the result to work.

Here our other methods to obtain further terms of the series — Watson's Lemma, integration by parts, or geometric methods — are better, so let's try them . . .

5.4 Integration by parts

Another way that we saw to analyze a function from its integral was to use integration by parts. We can apply the method exactly as described earlier.

We start with

$$\operatorname{Erf}\left(\frac{x}{\sqrt{2}}\right) = \sqrt{\tfrac{2}{\pi}} \int_0^x e^{-t^2/2}\, dt$$

- According to our method we expand as follows,

$$\text{let } \begin{cases} u = t^{-1}\,, & v' = te^{-t^2/2}\,, \\ u' = -t^{-2}\,, & v = e^{-t^2/2}\,, \end{cases}$$

$$\int_\infty^x e^{-t^2/2}\, dt = \left[-t^{-1}e^{-t^2/2}\right]_\infty^x - \int_\infty^x t^{-2}e^{-t^2/2}\, dt$$

$$\text{then let } \begin{cases} u = t^{-3}\,, & v' = te^{-t^2/2}\,, \\ u' = -3t^{-4}\,, & v = e^{-t^2/2}\,, \end{cases}$$

$$= \left[(-t^{-1} + t^{-3})e^{-t^2/2}\right]_\infty^x + 3\int_\infty^x t^{-4}e^{-t^2/2}\, dt$$

$$\text{then let } \begin{cases} u = t^{-5}\,, & v' = te^{-t^2/2}\,, \\ u' = -5t^{-6}\,, & v = e^{-t^2/2}\,, \end{cases}$$

$$= \left[(-t^{-1} + t^{-3} - 3t^{-5})e^{-t^2/2}\right]_\infty^x - 3.5\int_\infty^x t^{-6}e^{-t^2/2}\, dt$$

and so on. Evaluating the square brackets we have

$$\int_\infty^x e^{-t^2/2}\, dt = e^{-x^2/2}\left(-\frac{1}{x} + \frac{1}{x^3} - \frac{3}{x^5} + \mathcal{O}\left(x^{-7}\right)\right) \tag{5.27}$$

and hence

$$\operatorname{Erf}\left(\frac{x}{\sqrt{2}}\right) = 1 + \sqrt{\tfrac{2}{\pi}}e^{-x^2/2}\left(-\frac{1}{x} + \frac{1}{x^3} - \frac{3}{x^5} + \mathcal{O}\left(x^{-7}\right)\right) \tag{5.28}$$

- As before, we've only assumed x large and positive here. For x large but negative we should have looked at the integral to $-\infty$ rather than to ∞. We'd have found the first term had the opposite sign, and putting them together,

$$\mathrm{Erf}\left(\frac{x}{\sqrt{2}}\right) = \pm 1 + \sqrt{\tfrac{2}{\pi}} e^{-x^2/2} \left(-\frac{1}{x} + \frac{1}{x^3} - \frac{3}{x^5} + \mathcal{O}\left(x^{-7}\right)\right) \tag{5.29}$$

  where $\pm$ is the sign of x.

5.5 Watson's Lemma

Let's see how we could have found this result using Watson's Lemma, which gives the general result

$$\int_0^c e^{-kt} f(t) dt \ \sim \ \sum_{n=0}^{\infty} \frac{g^{(n)}(0)\Gamma(\lambda+n+1)}{n!\ k^{\lambda+n+1}}$$

where $f(t) = t^\lambda g(t)$ with $g(0) \neq 0$ for some $\lambda > -1$, and large $k > 0$.

- We first need something like the e^{-kt} term. So we expand about $t = x$, or let $s = t - x$, giving

$$\begin{aligned}
\mathrm{Erfc}\left(\frac{x}{\sqrt{2}}\right) &= \sqrt{\tfrac{2}{\pi}} \int_x^\infty e^{-t^2/2}\, dt \\
&= \sqrt{\tfrac{2}{\pi}} \int_0^\infty e^{-x^2/2 - xs - s^2/2}\, ds \\
&= \int_0^\infty e^{-xs} f(s)\, ds \qquad (5.30)
\end{aligned}$$

  where $f(s) = \sqrt{\frac{2}{\pi}} e^{-x^2/2 - s^2/2}$.

- So Watson's Lemma, for $x > 0$ large gives

$$\begin{aligned}
\mathrm{Erfc}\left(\frac{x}{\sqrt{2}}\right) &= \int_0^\infty e^{-xs} f(s)\, ds \\
&\sim \ \sum_{n=0}^{\infty} \frac{g^{(n)}(0)\Gamma(\lambda+n+1)}{n!\ x^{\lambda+n+1}} \qquad (5.31)
\end{aligned}$$

  where $g(s) = s^{-\lambda} \sqrt{\frac{2}{\pi}} e^{-x^2/2 - s^2/2}$.

- If $\lambda > 0$ then g won't be finite at $s = 0$. We need $\lambda > -1$, to try $\lambda = 0$. Then $g(0) = \sqrt{\frac{2}{\pi}} e^{-x^2/2}$, $g''(0) = -\sqrt{\frac{2}{\pi}} e^{-x^2/2} s$, $g^{iv}(0) = 3\sqrt{\frac{2}{\pi}} e^{-x^2/2}$, . . . and all odd derivatives vanish $g'(0) = g'''(0) = g^v(0) = ... = 0$. So we get

$$\operatorname{Erfc}\left(\frac{x}{\sqrt{2}}\right) = \int_0^\infty e^{-xs} f(s)\, ds$$
$$\sim \sqrt{\tfrac{2}{\pi}} e^{-x^2/2} \sum_{n=0}^{\infty} \frac{h^{(n)}(0)\Gamma(n+1)}{n!\, x^{n+1}} \tag{5.32}$$

where $h(s) = e^{-s^2/2}$.

Don't worry. A lot of asymptotics is less about making long expansions like these, and instead either about just finding the first term or two, or using standard results to solve harder applied problems. But in some cases knowing at least that these things are possible can be very useful. Often we use results like these because computers cannot evaluate them correctly, if at all, and they need a little help by deriving formulae like those above.

We'll now turn this over to the Exercise Sheets to go through some further examples. You've learnt a lot of powerful methods here, which with a little adaptation can solve (asymptotically) a huge range of unsolvable problems.

I will drop one last section in, developing our 'Geometry of the integral' section a little further, to fully bring the two part of our course — Complex Variables and Asymptotics — together. The methods so far are pretty standard fare in asymptotics, while the methods in appendix A below are rather less widely known, at least less widely used well. But alas . . .

WHAT FOLLOWS IS NON-EXAMINABLE. . .

THESE LAST BITS ARE FOR FURTHER READING ONLY
BUT YOU MAY FIND THEM USEFUL
IN THIS AND OTHER COURSES

Appendix

The last few pages below contain important notions you need to be familiar with if you're going to model or study nonlinear dynamical systems. But we have run out of time in our course, so please familiarize yourselves with these topics for future reference, but you won't be tested on them in this course.

A Geometric Integral Methods

Most of the methods above are rather brute force . . . with a search engine, a computer, and enough time, anyone can apply them. Some computer packages (like Wolfram's Mathematica) have them programmed in when you just instruct it to 'Integrate' (not that they always do it well or even correctly, which is why you need to understand them).

But geometric methods require more human intuition and insight, geometrical methods which (to my knowledge at least) have not been programmed into any computer.

So let's revisit section 4.4 and put a little more flesh on those ideas, why only endpoints and saddlepoints contribute to an integral, why we have saddlepoints and not just maxima, and what the paths are that we follow in the complex plane and why.

To summarize our method . . .

Consider an integral

$$I = \int_a^b e^{\Phi(t)} dt \tag{A.1}$$

To find this integral's asymptotics:

- Look for behaviour near distinguished points, which dominate a function's behaviour.
- The places that might dominate this integral are:
  - the endpoints, at $t = a$ and/or $t = b$,
  - the saddlepoints, where $e^{\Phi(t)}$ is stationary, i.e. where $\Phi'(t) = 0$.
- Near any special point $t = t_*$, we proceed by expanding $\Phi(t)$ as a power series

  $$\Phi(t) = \Phi(t_s) + (t - t_*)\Phi'(t_*) + \tfrac{1}{2}(t - t_*)^2\Phi''(t_*) + ... \tag{A.2}$$

  Because we're expanding the exponent only (not the whole e^{Φ}) this is better than just a power series approximation of the integral . . . loosely speaking it will give an 'exponentially good' approximation.
- Fortunately we can easily work out the *contributions* these would give to the integral.

A.1 . . . The Endpoint

From our integral (4.1) we had a contribution near $t = x$ that tailed off towards either $-\infty$ or $+\infty$, depending on the sign of x.

- For $x > 0$ we had an integral along the path $[x, \infty)$. The integrand $\operatorname{sech}^2 t$ decreased exponentially fast from $t = x$ to ∞. So when we approximated this contribution for t near x, we obtained an exponentially good approximation that we can write concisely as

$$\begin{aligned}\int_x^\infty \operatorname{sech}^2 t \, dt &= 4\int_x^\infty e^{-2t} p(t) dt \\ &= 4p(x)\int_x^\infty e^{-2t} dt + \ldots \sim -2p(x)e^{-2x}\end{aligned} \tag{A.3}$$

in terms of a function $p(x) = (1+e^{-2x})^{-2}$ you can find from the steps above.

[Side Notes:] Endpoints

To derive general formulae it is useful to think about integrals written in the form

$$I = \int_a^b e^{\Phi(t)} dt \, . \tag{A.4}$$

I want to approximate the contribution from the endpoint of this integral at $t = a$.

I'll assume the integral is decreasing and I can approximate by pushing the other limit b out to ∞, but there's some ambiguity in this so for now I'll write the upper limit as '∞'. Then:

- Let's expand $\Phi(t)$ as a power series about $t = a$,

$$\begin{aligned}I_a &= \int_a^{`\infty'} e^{\Phi(t)} dt \\ &= e^{\Phi(a)} \int_a^{`\infty'} e^{(t-a)\Phi'(a) + \mathcal{O}\left((t-a)^2\right)} dt\end{aligned} \tag{A.5}$$

then substitute a new integration variable $s = -(t-a)\Phi'(a)$ and just evaluate, ignoring higher order terms,

$$I_a = -\frac{e^{\Phi(a)}}{\Phi'(a)} \int_0^{\infty} e^{-s+\mathcal{O}\left(s^2\right)} ds \quad = \quad -\frac{e^{\Phi(a)}}{\Phi'(a)} + \ldots \tag{A.6}$$

Now initially I replaced the upper endpoint b with '∞' because I didn't know where my integration path should end, but after the transformation to the variable s the sign of Φ' conspires to ensure I can replace it with ∞ exactly. It ensures my integration path is chosen in such a way that the integrand decreases away from a and the integral I_a converges.

- The '...' terms give higher order terms as usual. This provides a very useful and general formula for the **endpoint contribution** to an integral in the form (A.4).
- This wouldn't have worked if a had been a saddlepoint, because then $\Phi'(a) = 0$, and this formula would give infinity.
- An extra trick: to get the form (A.9) from an integral $\int f dt$ just define $\Phi = \ln f$.

A.2 . . . The Saddlepoint

The easiest parts of the integrals above were the terms

$$\int_{-\infty}^{\infty} \mathrm{sech}^2\, t\; dt = 2 \qquad \text{or} \qquad \int_{0}^{\infty} \mathrm{sech}^2\, t\; dt = 1 \tag{A.7}$$

This produced the largest — dominant — term in the asymptotics.

- Importantly these integrals go through the point $t = 0$, and since

$$\frac{d}{dt}\,\mathrm{sech}^2\, t = 0 \qquad \text{at} \qquad t = 0\,, \tag{A.8}$$

  this point is the unique maximum of the integrand $\mathrm{sech}^2\, t$. For this function, this is indeed the unique stationary point of $\mathrm{sech}^2\, t$ in the complex plane.

- The integral $\int_{-\infty}^{\infty} \mathrm{sech}^2\, t\; dt$ passes through the whole saddlepoint, and picks up its full value of '2', while the integral $\int_0^{\infty} \mathrm{sech}^2\, t\; dt$ only runs over half of the saddlepoint and picks up half its full value of '1'.

- We often find that we can integrate the saddlepoint contributions exactly, as we could here. When we can't, there's a general formula for them . . .

[Side Notes:] **Saddlepoints**

As in the last sidenote, consider in general an integral written in the form

$$I = \int_a^b e^{\Phi(t)} dt\,. \tag{A.9}$$

If e^{Φ} reaches a maximum, it'll be where Φ reaches a maximum, i.e. $\Phi' = 0$.

- Say a maximum of Φ exists at some $t = t_s$. Since $\Phi'(t_s) = 0$, when we expand Φ about $t = t_s$ we are left with just

$$\Phi(t) = \Phi(t_s) + \tfrac{1}{2}(t - t_s)^2 \Phi''(t_s) + \ldots \tag{A.10}$$

- Let's substitute a new integration variable $u = (t - t_s)\sqrt{-\Phi''(t_s)}$ to simplify things, so

$$\Phi(t) = \Phi(t_s) - \tfrac{1}{2}u^2 + \ldots \tag{A.11}$$

and substituting $du = dt\sqrt{-\Phi''(t_s)}$ in the integral we have

$$I_s = \frac{e^{\Phi(t_s)}}{\sqrt{-\Phi''(t_s)}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}u^2 + \cdots} du \tag{A.12}$$

with the $e^{\Phi(t_s)}$ term being taken outside the integral as it is just a constant.

- For the integral that's left we can just evaluate $\displaystyle\int_{-\infty}^{\infty} e^{-\frac{1}{2}u^2} du = \sqrt{2\pi}$, giving

$$I_s = \sqrt{\frac{2\pi}{-\Phi''(t_s)}} e^{\Phi(t_s)} + \ldots \tag{A.13}$$

- The '...' terms give higher order terms as usual. This provides a very useful and general formula for the **saddlepoint contribution** to an integral in the form (A.9).

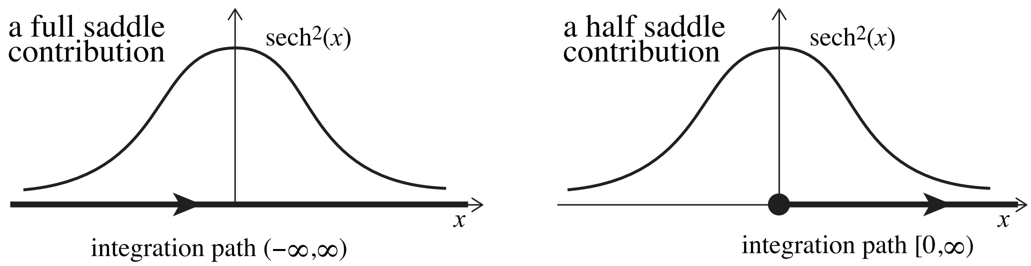
- Since this is only an approximation near t_s, I don't care about the endpoints a and b, so I've replaced (a, b) with $(-\infty, \infty)$. This is okay because $e^{-\frac{1}{2}u^2}$ is shrinking exponentially fast as we move away from the saddlepoint at $u = 0$. We've already seen how to deal with the endpoints separately.

Two things to be aware of:

[Side Notes:] An endpoint-saddlepoint

What if the endpoint of an integral is also a saddlepoint?

- If an endpoint lies on the saddle, then the integration path only runs over half of the maximum there.



- So you simply take half of the saddle contribution.

  Taking our general form

$$I = \int_a^b e^{\Phi(t)} dt \; , \tag{A.14}$$

if there is a saddlepoint at $t = a$ (so $\Phi'(a) = 0$) then we just have

$$I_s = \frac{1}{2}\sqrt{\frac{2\pi}{-\Phi''(a)}} e^{\Phi(a)} + \ldots \tag{A.15}$$

[Side Notes:] A double-saddlepoint

- We saw that if we tried to evaluate an endpoint contribution, an alarm bell would ring if $\Phi' = 0$ that indicated the endpoint coincided with a saddlepoint.
- If you evaluate a saddlepoint contribution and find $\Phi'' = 0$, that's another alarm bell., and the saddlepoint formula above will be infinite. This one tells you that two saddles have collided.
- Fortunately there are also general methods to deal with this situation, but they're just extensions of the ideas above. We encounter them when we study the Airy function. They're responsible for a lot of the bright fringes or *caustics* of light you see around you, from reflections on swimming pools, drinking cups or glasses, and rainbows.

Like the boundary layer corresponded to a meniscus in fluids, these endpoints and saddlepoints can have direct physical meaning. E.g. in a wave (optics, sound, or water) problem, the maxima of e^{Φ} often correspond to *rays* or *characteristics* of the wave field, endpoints correspond to waves scattered off particular features in whatever medium the wave is traveling through.

A.3 Contributions – Which to include?

Our formulae for the endpoint and saddlepoint contributions are beautiful things, but we need to know how to use them.

- These are going to provide the dominant terms in the asymptotics of our integral, something like

$$\begin{aligned} I = \int_a^b e^{\Phi(t)} dt \ &\sim \ I_a + I_b + I_s \\ &= -\frac{e^{\Phi(a)}}{\Phi'(a)} + \frac{e^{\Phi(b)}}{\Phi'(b)} + e^{\Phi(t_s)} \sqrt{\frac{2\pi}{-\Phi''(t_s)}} \end{aligned} \tag{A.16}$$

- ... but not quite. We don't always want both endpoints, and there can be any number of saddlepoints t_s depending on how many turning points Φ has.

- To find out which parts to keep requires a little geometric game: a maze in the complex plane, and this takes a human brain!

- The integral I started out as a sum along the real line from a to b. That's a pretty dumb way to get from a to b, and if you look in the complex plane you can find something much better.

- We learnt in the *Complex Variables* part of this course that, when integrating an analytic function, you can change the integration path however you like provided you: i) don't pass through any singularities, and ii) don't change the endpoints.

- So we can change the path taken to get from a to b in the integral. If that path takes in the contributions I_a, I_b, and I_s that we worked out above, we add them to our integral.

A.4 What path?

In none of the analysis above did we assume that the functions we were dealing with were real, in fact, all of this works just as well, or better, if they are complex.

- The function we're integrating looks like this:

$$\begin{aligned} e^{\Phi(t)} &= \left|e^{\Phi(t)}\right| \times e^{i\arg[e^{\Phi(t)}]} \\ &= e^{\mathsf{Re}\Phi(t)} \times e^{i\mathsf{Im}\Phi(t)} \\ &= e^{\mathsf{Re}\Phi(t)} \times \{\cos\mathsf{Im}\Phi(t) + i\sin\mathsf{Im}\Phi(t)\} \\ &= \text{magnitude} \times \text{oscillation} \end{aligned} \tag{A.17}$$

  so $\mathsf{Re}\Phi$ tells us the size of e^{Φ}, and $\mathsf{Im}\Phi$ tells us about its oscillations or *phase*.

- Typically as we change t and travel along the integration contour, we'll see the magnitude of the function change, and we'll see it oscillating.

- But we can choose the path taken in the complex plane to get from a to b.

- This becomes a very simple function if we choose a path on which

$$\mathsf{Im}\Phi(t) = \text{constant} \tag{A.18}$$

  along which there is no oscillation — the function just tends to increase or decrease with t.

- In fact, these are also the paths where e^{Φ} is fastest varying (see the next Side Note).

- This is called a **steepest descent** path.

- When doing this kind of asymptotics on integrals, we *only* follow integration paths along directions of steepest descent.

- The expressions we derived for the endpoint or saddlepoint contributions automatically chose (when we changed the integration variable to $s = -(t-b)\Phi'(b)$ or $u = (t-t_s)\sqrt{-\Phi''(t_s)}$) a steepest descent path through the saddle or end point.

[Side Notes:] Steepest descent = Stationary phase

The directions of steepest descent, where the integrand is most quickly changing, are also the directions where it is non-oscillating or has 'stationary phase'. This is a consequence of the Cauchy-Riemann equations.

If we write the integral in the form

$$\int e^{\Phi(t)} dt$$

then

- steepest descent means $|e^{\Phi}| = e^{\mathsf{Re}\Phi}$ is fastest varying,
- stationary phase means $\arg(e^{\Phi}) = e^{\mathsf{Im}\Phi}$ is constant,

If Φ is differentiable with respect to complex t (hence analytic in t), then we can show that the steepest descent direction of $\mathsf{Re}\Phi$ is also a contour of $\mathsf{Im}\Phi$.

We show this as follows.

- If e^{Φ} is differentiable, then Φ is differentiable, and so analytic and satisfies the Cauchy-Riemann equations. What is the angle between the steepest descent directions (the gradient vectors) of $\mathsf{Re}\Phi$ and $\mathsf{Im}\Phi$?
- Letting $t = x + iy$, calculate:

$$\begin{aligned} \nabla \mathsf{Im}\Phi \cdot \nabla \mathsf{Re}\Phi &= \begin{pmatrix} \frac{\partial \mathsf{Im}\Phi}{\partial x} \\ \frac{\partial \mathsf{Im}\Phi}{\partial y} \end{pmatrix} \cdot \begin{pmatrix} \frac{\partial \mathsf{Re}\Phi}{\partial x} \\ \frac{\partial \mathsf{Re}\Phi}{\partial y} \end{pmatrix} \\ &= \frac{\partial \mathsf{Im}\Phi}{\partial x}\frac{\partial \mathsf{Re}\Phi}{\partial x} + \frac{\partial \mathsf{Im}\Phi}{\partial y}\frac{\partial \mathsf{Re}\Phi}{\partial y} \\ &= \frac{\partial \mathsf{Im}\Phi}{\partial x}\frac{\partial \mathsf{Im}\Phi}{\partial y} - \frac{\partial \mathsf{Im}\Phi}{\partial y}\frac{\partial \mathsf{Im}\Phi}{\partial x} \;=\; 0 \end{aligned} \tag{A.19}$$

  since $\frac{\partial \mathsf{Re}\Phi}{\partial x} = \frac{\partial \mathsf{Im}\Phi}{\partial y}$ and $\frac{\partial \mathsf{Re}\Phi}{\partial y} = \frac{\partial \mathsf{Im}\Phi}{\partial x}$ by the Cauchy-Riemann equations.
- So $\nabla \mathsf{Im}\Phi$ and $\nabla \mathsf{Re}\Phi$ are perpendicular.
- Recall that given a function $f(x, y)$, the gradient vector $\nabla f(x, y)$ points:

1. along the steepest descent direction of f
   (as the rate of variation $\mathbf{n} \cdot \nabla f$ along a direction $\mathbf{n}$ is greatest when $\mathbf{n} \parallel \nabla f$)
2. perpendicular to the contours f =constant
   (as if $\mathbf{n}$ points along a contour, f is non-varying along that direction, so $\mathbf{n} \cdot \nabla f = 0$, which implies $\mathbf{n} \perp \nabla f$)

- Hence wherever the Cauchy-Riemann equations hold, they imply that the fastest change (*steepest descent*) of $\mathsf{Re}\Phi$ occurs long the contours of $\mathsf{Im}\Phi$.

A.5 Stokes Lines

The integration contour can change whether or not it passes through a given saddlepoint t_s.

- This turns the saddlepoint contribution from t_s on (if the path goes through t_s) or off (if it doesn't).

- With a little thought, you can see that this change can only happen when the saddlepoint t_s is connected by a steepest descent path to either another saddle t_w, or an endpoint a or b, that is, assuming an integral

$$\int_a^b e^{\Phi(t)} dt \tag{A.20}$$

  with saddles at t_s and t_w, these occur when

$$\mathsf{Im}\left(\Phi(t_s)\right) = \mathsf{Im}\left(\Phi(t_w)\right) \quad \text{or} \quad \mathsf{Im}\left(\Phi(t_s)\right) = \mathsf{Im}\left(\Phi(a \text{ or } b)\right) \tag{A.21}$$

- These equations define the **Stokes lines** where **Stokes discontinuities** turn saddlepoints on or off. It occurs when the endpoint and saddlepoint are connected by a steepest descent line.

B False exponents

Consider the more general integral

$$I = \int_a^b q(t) e^{\Phi(t)} dt \tag{B.1}$$

1. If $q(t)$ slow varying compared to $e^{\Phi(t)}$ we approximate $q(t) \approx q(t_s)$.

   E.g. $q(t)$ is a polynomial or rational function of x.

2. If $q(t)$ is fast varying despite not explicitly being an exponential, we can write

   $$q(t)e^{\Phi(t)} = e^{\Psi(t)} \qquad \text{where} \quad \Psi = \Phi + \ln(q)$$

   and use the same methods as above for Ψ instead of Φ .

   E.g. $q(t) = t^n$ where n is a large power.

   In this case we might call $\ln(q)$ a 'false exponent' or 'slow exponent' — q is really a polynomial, but has a high enough power that it varies fast enough that treating it as an exponential provides a good approximation.

C Further reading

These are the main techniques of asymptotic theory for use in modeling, where we mostly care about leading order behaviour. There is some much more complicated theory that deals with working out the best place to truncate a series, estimating the size of the remainder, or even re-summing the tail of the series itself, for which topics you might like to look up include: optimal trauncation of divergent series, Borel summation, Mellin transforms, Euler-Maclaurin series, resurgence, uniformization, and hyperasymptotics.

For other asymptotic methods extending the ideas here try also WKB methods, phase-integral methods, methods of multiple scales, methods of strained coordinates.