Exercise Sheet 1, Question 3: Autonomy (Time-Independence) Complete Solution with XYZ Methodology Methods of Applied Mathematics [SEMT30006]

Problem Statement

We've seen systems that either depend on time or don't. A system that does not depend explicitly on its independent variable is called **autonomous**. Which of the following is autonomous? What is the independent variable?

(a) The ODE $\ddot{u} = u + \sin(t)$

(b) The ODE $y'' - y - \sin(x) = 0$

(c) The ODE $\ddot{\theta} + a\dot{\theta} + b = 0$

(d) The map $x_{n+1} = ax_n + x_n^2$

(e) The map $x_{n+1} = nx_n + b$

1 Foundational Concepts

What is Autonomy?

- **STAGE X (Definition):** A dynamical system is **autonomous** if it does not depend explicitly on the independent variable.

  For ODEs: The right-hand side depends only on the state variables, not on time t explicitly.

$$\text{Autonomous:} \quad \dot{x} = f(x) \tag{1}$$
$$\text{Non-autonomous:} \quad \dot{x} = f(x, t) \tag{2}$$

  For maps: The map depends only on the current state x_n, not on the index n explicitly.

$$\text{Autonomous:} \quad x_{n+1} = f(x_n) \tag{3}$$
$$\text{Non-autonomous:} \quad x_{n+1} = f(x_n, n) \tag{4}$$

- **STAGE Y (Why this matters):** Autonomous systems have special properties:

  1. **Time-translation invariance:** If $x(t)$ is a solution, then $x(t + t_0)$ is also a solution for any constant t_0. The system "looks the same" at all times.
  2. **Phase space structure:** Trajectories in phase space never cross (uniqueness). The flow is time-independent.
  3. **Equilibria are stationary:** Fixed points don't move with time.
  4. **Simpler analysis:** We can study the system in phase space without tracking time explicitly.

Non-autonomous systems can have time-varying equilibria, crossing trajectories (at different times), and more complex behavior.

- **STAGE Z (How to identify):**
  1. Identify the independent variable (usually t for ODEs, n for maps)
  2. Check if this variable appears explicitly on the right-hand side
  3. If it appears explicitly $\Rightarrow$ non-autonomous
  4. If it doesn't appear $\Rightarrow$ autonomous

KEY DISTINCTION: The independent variable appearing in derivatives (like $\frac{d}{dt}$) doesn't count—it must appear explicitly in the function itself.

2 Part (a): The ODE $\ddot{u} = u + \sin(t)$

Step 1: Identify the Independent Variable

- **STAGE X (What we have):** The equation is:

$$\ddot{u} = u + \sin(t) \tag{5}$$

  where $\ddot{u} = \frac{d^2u}{dt^2}$.

- **STAGE Y (Finding the independent variable):** The notation $\ddot{u}$ means differentiation with respect to time t. Therefore, the independent variable is:

$$\boxed{\text{Independent variable: } t \text{ (time)}} \tag{6}$$

  The dependent (state) variable is u.

- **STAGE Z (Clear identification):** This is an ODE where we evolve the state u as the independent variable t changes.

Step 2: Check for Explicit Dependence on Independent Variable

- **STAGE X (Rewrite in standard form):** Write as a first-order system. Let $u_1 = u$ and $u_2 = \dot{u}$:

$$\dot{u}_1 = u_2 \tag{7}$$

$$\dot{u}_2 = u_1 + \sin(t) \tag{8}$$

  Or in vector form with $\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$:

$$\dot{\mathbf{u}} = \begin{pmatrix} u_2 \\ u_1 + \sin(t) \end{pmatrix} = f(\mathbf{u}, t) \tag{9}$$

- **STAGE Y (Does t appear explicitly?):** YES! The term $\sin(t)$ contains t explicitly on the right-hand side. The function f depends on both $\mathbf{u}$ and t:

$$f(\mathbf{u}, t) = \begin{pmatrix} u_2 \\ u_1 + \sin(t) \end{pmatrix} \tag{10}$$

  We cannot write this as $f(\mathbf{u})$ alone—the time t must be included.

- **STAGE Z (Conclusion):**

$$\boxed{\text{NOT AUTONOMOUS}} \tag{11}$$

  The system is **non-autonomous** because of the explicit $\sin(t)$ term.

Step 3: Physical Interpretation

- **STAGE X (What this equation represents):** This could model a forced oscillator:

$$\ddot{u} - u = \sin(t) \tag{12}$$

  The $\sin(t)$ term is an external periodic forcing that varies with time.

- **STAGE Y (Why non-autonomy matters):** Because the forcing varies with time:
  1. The "rules" of the system change with t
  2. Different solutions starting at the same state u but different times t will evolve differently
  3. Phase portraits change with time
  4. Equilibria (if any) may move with time

- **STAGE Z (Example):** If we evaluate at $t = 0$: $\ddot{u} = u + \sin(0) = u$

  If we evaluate at $t = \pi/2$: $\ddot{u} = u + \sin(\pi/2) = u + 1$

  The system dynamics differ at different times, confirming non-autonomy.

KEY INSIGHT: The presence of an explicit time-dependent forcing term $\sin(t)$ makes the system non-autonomous. This is common in physics with external driving forces.

3 Part (b): The ODE $y'' - y - \sin(x) = 0$

Step 1: Identify the Independent Variable

- **STAGE X (What we have):** The equation is:

$$y'' - y - \sin(x) = 0 \tag{13}$$

or equivalently:

$$y'' = y + \sin(x) \tag{14}$$

- **STAGE Y (Interpreting the notation):** The prime notation y'' means:

$$y'' = \frac{d^2y}{dx^2} \tag{15}$$

We are differentiating y with respect to x. Therefore:

$$\boxed{\text{Independent variable: } x} \tag{16}$$

The dependent (state) variable is y.

- **STAGE Z (Important distinction):** Here, x is the independent variable (like time), not the state variable! The state variable is y, which depends on x.

Step 2: Check for Explicit Dependence on Independent Variable

- **STAGE X (Rewrite as first-order system):** Let $y_1 = y$ and $y_2 = y'$ where primes denote derivatives with respect to x:

$$y_1' = y_2 \tag{17}$$
$$y_2' = y_1 + \sin(x) \tag{18}$$

In vector form with $\mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$:

$$\mathbf{y}' = \begin{pmatrix} y_2 \\ y_1 + \sin(x) \end{pmatrix} = f(\mathbf{y}, x) \tag{19}$$

- **STAGE Y (Does x appear explicitly?):** YES! The term $\sin(x)$ contains the independent variable x explicitly on the right-hand side. The function depends on both $\mathbf{y}$ and x:

$$f(\mathbf{y}, x) = \begin{pmatrix} y_2 \\ y_1 + \sin(x) \end{pmatrix} \tag{20}$$

We cannot write this purely as a function of $\mathbf{y}$.

- **STAGE Z (Conclusion):**

$$\boxed{\text{NOT AUTONOMOUS}} \tag{21}$$

The system is **non-autonomous** because the independent variable x appears explicitly in $\sin(x)$.

Step 3: Comparison with Part (a)

- **STAGE X (Structural similarity):** Compare parts (a) and (b):

$$\text{Part (a):} \quad \ddot{u} = u + \sin(t) \quad \text{(independent var: } t\text{)} \tag{22}$$
$$\text{Part (b):} \quad y'' = y + \sin(x) \quad \text{(independent var: } x\text{)} \tag{23}$$

  These have **identical mathematical structure**!

- **STAGE Y (Why both are non-autonomous):** In both cases:
  1. The independent variable appears explicitly in a $\sin(\cdot)$ term
  2. The "forcing" changes as we move along the independent variable
  3. Different starting positions along the independent variable lead to different dynamics

- **STAGE Z (The confusion):** Students often think (b) might be different because x "looks like" a state variable. But:
  - In part (a): t is independent, u is dependent (state)
  - In part (b): x is independent, y is dependent (state)

  Both have the same non-autonomous structure: independent variable appears explicitly in sin(independent var).

COMMON MISTAKE: Don't assume x is always a state variable! Here, x is the independent variable (like time), and y is evolving along x.

4 Part (c): The ODE $\ddot{\theta} + a\dot{\theta} + b = 0$

Step 1: Identify the Independent Variable

- **STAGE X (What we have):** The equation is:

$$\ddot{\theta} + a\dot{\theta} + b = 0 \tag{24}$$

- **STAGE Y (Interpreting the notation):** The dot notation means differentiation with respect to time:

$$\dot{\theta} = \frac{d\theta}{dt}, \quad \ddot{\theta} = \frac{d^2\theta}{dt^2} \tag{25}$$

  Therefore:

$$\boxed{\text{Independent variable: } t \text{ (time)}} \tag{26}$$

  The dependent (state) variable is θ.

- **STAGE Z (Parameters):** Note that a and b are constants (parameters), not variables.

Step 2: Rewrite as First-Order System

- **STAGE X (Standard form):** Rewrite the equation:

$$\ddot{\theta} = -a\dot{\theta} - b \tag{27}$$

  Let $\theta_1 = \theta$ and $\theta_2 = \dot{\theta}$:

$$\dot{\theta}_1 = \theta_2 \tag{28}$$

$$\dot{\theta}_2 = -a\theta_2 - b \tag{29}$$

- **STAGE Y (Vector form):** With $\boldsymbol{\theta} = \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix}$:

$$\dot{\boldsymbol{\theta}} = \begin{pmatrix} \theta_2 \\ -a\theta_2 - b \end{pmatrix} = f(\boldsymbol{\theta}) \tag{30}$$

- **STAGE Z (Key observation):** The right-hand side depends only on $\boldsymbol{\theta}$ (the state variables θ_1 and θ_2), not on t explicitly.

Step 3: Check for Explicit Time Dependence

- **STAGE X (Examining the right-hand side):** The function is:

$$f(\boldsymbol{\theta}) = \begin{pmatrix} \theta_2 \\ -a\theta_2 - b \end{pmatrix} \tag{31}$$

- **STAGE Y (Does t appear?):** NO! The independent variable t does not appear anywhere on the right-hand side:
  - θ_2 is a state variable (not t)
  - a and b are constants (not t)

- No terms like $\sin(t)$, e^t, t^2, etc.

The function depends only on the state $\boldsymbol{\theta}$, not on t.

- **STAGE Z (Conclusion):**

$$\boxed{\text{AUTONOMOUS}} \tag{32}$$

The system is **autonomous**.

Step 4: Physical Interpretation

- **STAGE X (What this equation models):** This is a damped system with constant forcing:

$$\ddot{\theta} + a\dot{\theta} = -b \tag{33}$$

Example: A pendulum with friction (damping coefficient a) and a constant external torque ($-b$).

- **STAGE Y (Why it's autonomous):** The damping coefficient a and forcing b are constant—they don't change with time. The system follows the same rules at all times t.

Key properties:

1. Time-translation invariant: shifting a solution in time gives another solution
2. Phase portrait is time-independent
3. Equilibria are stationary (don't move with t)

- **STAGE Z (Finding equilibrium):** At equilibrium: $\dot{\theta}_1 = 0$ and $\dot{\theta}_2 = 0$

From the first equation: $\theta_2 = 0$

From the second: $0 = -a(0) - b = -b$

If $b \neq 0$, there's no equilibrium in this system! This represents constant acceleration (if $b \neq 0$).

If $b = 0$: equilibrium at $(\theta_1, \theta_2) = (\theta^*, 0)$ for any θ^* (line of equilibria).

KEY INSIGHT: Constant parameters (like a and b) don't make a system non-autonomous. Only explicit dependence on the independent variable matters.

5 Part (d): The map $x_{n+1} = ax_n + x_n^2$

Step 1: Understand Discrete Maps

- **STAGE X (What is a map?):** A discrete dynamical system (or map) evolves in discrete steps rather than continuously:

$$x_{n+1} = f(x_n, n) \tag{34}$$

  where:

  - n is the discrete "time" index ($n = 0, 1, 2, 3, \ldots$)
  - x_n is the state at step n
  - x_{n+1} is the state at step $n + 1$

- **STAGE Y (Independent variable for maps):** For discrete maps, the independent variable is n (the iteration index), analogous to time t in ODEs.

- **STAGE Z (Autonomy for maps):** A map is autonomous if f depends only on x_n, not on n explicitly:

$$\text{Autonomous:} \quad x_{n+1} = f(x_n) \tag{35}$$
$$\text{Non-autonomous:} \quad x_{n+1} = f(x_n, n) \tag{36}$$

Step 2: Identify the Independent Variable

- **STAGE X (Our map):**

$$x_{n+1} = ax_n + x_n^2 \tag{37}$$

- **STAGE Y (Independent variable):** The subscript n indicates the iteration number. We iterate forward: $n \to n + 1$.

$$\boxed{\text{Independent variable: } n \text{ (iteration index)}} \tag{38}$$

- **STAGE Z (State variable):** The state variable is x (which takes value x_n at iteration n).

Step 3: Check for Explicit Dependence on n

- **STAGE X (Examine the right-hand side):**

$$x_{n+1} = ax_n + x_n^2 \tag{39}$$

  The right-hand side contains:

  - a: a constant parameter
  - x_n: the state at iteration n
  - x_n^2: a function of the state

- **STAGE Y (Does n appear explicitly?):** NO! The iteration index n only appears as a subscript to indicate "which iteration," not as an explicit variable in the function.

  The function is:

$$f(x_n) = ax_n + x_n^2 \tag{40}$$

  This depends only on the current state x_n, not on n itself.

- **STAGE Z (Conclusion):**

$$\boxed{\text{AUTONOMOUS}} \tag{41}$$

The map is **autonomous**.

Step 4: Understanding Map Dynamics

- **STAGE X (Iterative evolution):** Starting from x_0:

$$x_1 = ax_0 + x_0^2 \tag{42}$$
$$x_2 = ax_1 + x_1^2 = a(ax_0 + x_0^2) + (ax_0 + x_0^2)^2 \tag{43}$$
$$x_3 = ax_2 + x_2^2 \tag{44}$$
$$\vdots \tag{45}$$

- **STAGE Y (Why it's autonomous):** The rule for computing x_{n+1} from x_n is the same at every iteration:
  - The map at $n = 0$: $x_1 = ax_0 + x_0^2$
  - The map at $n = 5$: $x_6 = ax_5 + x_5^2$
  - The map at $n = 100$: $x_{101} = ax_{100} + x_{100}^2$

  All use the same rule! The iteration number n doesn't affect the dynamics.

- **STAGE Z (Fixed points):** Fixed points satisfy $x^* = ax^* + (x^*)^2$:

$$(x^*)^2 + (a-1)x^* = 0 \tag{46}$$
$$x^*(x^* + a - 1) = 0 \tag{47}$$

  Fixed points: $x^* = 0$ and $x^* = 1 - a$.

  These don't change with n (time-independent), confirming autonomy.

COMMON CONFUSION: The subscript n in x_n is just notation for "value at iteration n." It doesn't make the system non-autonomous unless n appears explicitly in the function (like nx_n or $\sin(n)$).

6 Part (e): The map $x_{n+1} = nx_n + b$

Step 1: Identify the Independent Variable

- **STAGE X (Our map):**

$$x_{n+1} = nx_n + b \tag{48}$$

- **STAGE Y (Independent variable):** As with part (d), the iteration index n is the independent variable:

$$\boxed{\text{Independent variable: } n \text{ (iteration index)}} \tag{49}$$

  The state variable is x.

- **STAGE Z (Parameters):** b is a constant parameter.

Step 2: Check for Explicit Dependence on n

- **STAGE X (Examine the right-hand side):**

$$x_{n+1} = nx_n + b \tag{50}$$

  The right-hand side contains:

  – n: the iteration index (independent variable!)
  – x_n: the state at iteration n
  – b: a constant

- **STAGE Y (Does n appear explicitly?):** YES! The iteration number n appears explicitly as a coefficient multiplying x_n.

  The function is:

$$f(x_n, n) = nx_n + b \tag{51}$$

  This depends on **both** the current state x_n **and** the iteration number n.

- **STAGE Z (Conclusion):**

$$\boxed{\text{NOT AUTONOMOUS}} \tag{52}$$

  The map is **non-autonomous**.

Step 3: Understanding the Non-Autonomous Dynamics

- **STAGE X (How the map changes):** Starting from x_0:

$$x_1 = 0 \cdot x_0 + b = b \tag{53}$$
$$x_2 = 1 \cdot x_1 + b = 1 \cdot b + b = 2b \tag{54}$$
$$x_3 = 2 \cdot x_2 + b = 2(2b) + b = 5b \tag{55}$$
$$x_4 = 3 \cdot x_3 + b = 3(5b) + b = 16b \tag{56}$$
$$\vdots \tag{57}$$

- **STAGE Y (Why it's non-autonomous):** The "rule" changes at each iteration:
  - At $n = 0$: multiply by 0 and add b
  - At $n = 1$: multiply by 1 and add b
  - At $n = 2$: multiply by 2 and add b
  - At $n = 10$: multiply by 10 and add b

  The dynamics are **different at each iteration** because the coefficient n changes. This is the hallmark of non-autonomy.

- **STAGE Z (Growth behavior):** As n increases, the coefficient grows, causing increasingly rapid changes in x_n. The system "accelerates" over iterations.

  For $b > 0$ and typical initial conditions, this map exhibits explosive growth as $n \to \infty$.

Step 4: Comparison with Part (d)

- **STAGE X (Side-by-side comparison):**

$$\text{Part (d):} \quad x_{n+1} = ax_n + x_n^2 \quad \text{(autonomous)} \tag{58}$$

$$\text{Part (e):} \quad x_{n+1} = nx_n + b \quad \text{(non-autonomous)} \tag{59}$$

- **STAGE Y (The crucial difference):**
  - In (d): a is a **constant**—same at all iterations
  - In (e): n is the **iteration index**—changes with each step

  Part (d) uses a constant parameter a, while part (e) uses the variable n that increases with each iteration.

- **STAGE Z (Physical analogy):**
  - Part (d): Like a system with constant friction coefficient
  - Part (e): Like a system where friction increases with each time step

  The changing "rules" in (e) make it non-autonomous.

KEY DISTINCTION:

- $x_{n+1} = ax_n + b$ with constant $a \Rightarrow$ AUTONOMOUS
- $x_{n+1} = nx_n + b$ with variable $n \Rightarrow$ NON-AUTONOMOUS

The explicit appearance of n (not just as a subscript) makes all the difference!

7 Summary Table and Final Comparison

Complete Results

Part	System	Independent Var	Autonomous?	Reason
(a)	$\ddot{u} = u + \sin(t)$	t	NO	t in $\sin(t)$
(b)	$y'' - y - \sin(x) = 0$	x	NO	x in $\sin(x)$
(c)	$\ddot{\theta} + a\dot{\theta} + b = 0$	t	YES	No explicit t
(d)	$x_{n+1} = ax_n + x_n^2$	n	YES	No explicit n
(e)	$x_{n+1} = nx_n + b$	n	NO	n multiplies x_n

Pattern Recognition

- **STAGE X (Common patterns for NON-autonomous):**
  1. Time-varying forcing: $f(x, t) = \text{function}(x) + g(t)$
     Examples: $\sin(t)$, e^t, t^2, etc.
  2. Time-varying coefficients: $f(x, t) = h(t) \cdot x$
     Examples: tx, $\sin(t) \cdot x$, etc.
  3. For maps: Iteration-dependent terms
     Examples: nx_n, $\sin(n)$, n^2, etc.
- **STAGE Y (Common patterns for AUTONOMOUS):**
  1. Constant coefficients: $f(x) = ax + bx^2 + c$
  2. State-dependent only: $f(x) = \sin(x)$, x^3, e^x, etc.
  3. No explicit time/iteration dependence
- **STAGE Z (How to avoid confusion):**
  1. First, identify the independent variable clearly
  2. Then scan the right-hand side for explicit occurrences
  3. Remember: subscripts are just notation, not explicit dependence
  4. Constants/parameters don't count as variables

Connection to Course Material

- **STAGE X (Lecture notes context):** The lecture notes primarily focus on **autonomous systems** because:
  1. They are simpler to analyze
  2. Phase portraits are time-independent
  3. Equilibrium analysis is straightforward
  4. They capture essential dynamics of many physical systems
- **STAGE Y (When non-autonomy arises):** Non-autonomous systems appear when:
  1. External forcing varies with time (e.g., seasonal effects, periodic driving)
  2. Parameters change with time (e.g., aging, growth)
  3. Boundary conditions move (e.g., moving walls)
  4. Control inputs are time-dependent

- **STAGE Z (Converting non-autonomous to autonomous):** A non-autonomous ODE can sometimes be made autonomous by introducing time as a state variable:

$$\dot{x} = f(x, t) \quad \text{(non-autonomous)} \tag{60}$$

Introduce $y = t$ with $\dot{y} = 1$:

$$\dot{x} = f(x, y) \tag{61}$$

$$\dot{y} = 1 \quad \text{(autonomous in } (x, y) \text{ space!)} \tag{62}$$

This trick increases the dimension but makes the system autonomous.

EXAM STRATEGY:

1. Always identify the independent variable first
2. Look for explicit appearances in the function
3. Constants and parameters don't make systems non-autonomous
4. Subscripts in maps (x_n) are just notation

END OF QUESTION 3