Question 2: Regular Perturbation of Linear Oscillator
Complete Solution with Exact Comparison

Asymptotics Course — Sheet 6

Problem Statement

Obtain a two-term expansion when $\epsilon \ll 1$ for the solution of:

$$\frac{d^2 f}{dt^2} + f = \epsilon \frac{df}{dt}, \quad f(0) = 1, \quad \frac{df}{dt}(0) = \frac{\epsilon}{2} \tag{1}$$

Compare this expansion with that obtained by expanding the exact solution.

1 Step 1: Problem Classification and Method Selection

Form Recognition

The ODE has the structure:

$$\frac{d^2 f}{dt^2} + f = \epsilon \frac{df}{dt} \tag{2}$$

with initial conditions:

$$f(0) = 1 \tag{3}$$

$$f'(0) = \frac{\epsilon}{2} \tag{4}$$

- **STAGE X (What we have):** This is a second-order linear ODE with a small parameter ϵ multiplying the first derivative term on the right-hand side. The initial velocity $f'(0) = \epsilon/2$ is also of order ϵ. The unperturbed equation ($\epsilon = 0$) is the simple harmonic oscillator $f'' + f = 0$, which has well-defined solutions.
- **STAGE Y (Why this classification):** When $\epsilon = 0$, we obtain $f'' + f = 0$ with $f(0) = 1$ and $f'(0) = 0$. This has the solution $f_0(t) = \cos t$. The perturbed problem should smoothly approach this solution as $\epsilon \to 0$. The highest derivative is NOT multiplied by ϵ, so this is a **regular perturbation problem**, not a singular one.
- **STAGE Z (What this means):** We can use a standard power series expansion in ϵ. The solution will remain smooth and bounded as $\epsilon \to 0$. We expect the expansion to be uniformly valid for moderate times (though we must check for secular terms).

2 Step 2: Perturbation Expansion Setup

Expansion Ansatz

We seek a solution of the form:

$$f(t) = f_0(t) + \epsilon f_1(t) + \epsilon^2 f_2(t) + \mathcal{O}(\epsilon^3) \tag{5}$$

- **STAGE X (What we're doing):** We assume the solution can be expanded as a power series in ϵ, where each coefficient function $f_n(t)$ is independent of ϵ.
- **STAGE Y (Why this works):** For regular perturbation problems, the solution depends smoothly on ϵ. By Taylor's theorem for functions of a parameter, such a series expansion exists and converges for sufficiently small ϵ.
- **STAGE Z (Next step):** Substitute this ansatz into both the ODE and the initial conditions, then collect terms by powers of ϵ.

3 Step 3: Substitute and Collect Terms

Derivatives of the Ansatz

From equation (6):

$$f'(t) = f_0'(t) + \epsilon f_1'(t) + \epsilon^2 f_2'(t) + \mathcal{O}(\epsilon^3) \tag{6}$$
$$f''(t) = f_0''(t) + \epsilon f_1''(t) + \epsilon^2 f_2''(t) + \mathcal{O}(\epsilon^3) \tag{7}$$

Substitution into ODE

Insert into $f'' + f = \epsilon f'$:

$$\left[f_0'' + \epsilon f_1'' + \epsilon^2 f_2'' + \mathcal{O}(\epsilon^3)\right] + \left[f_0 + \epsilon f_1 + \epsilon^2 f_2 + \mathcal{O}(\epsilon^3)\right] = \epsilon\left[f_0' + \epsilon f_1' + \epsilon^2 f_2' + \mathcal{O}(\epsilon^3)\right] \tag{8}$$

Rearranging:

$$(f_0'' + f_0) + \epsilon(f_1'' + f_1 - f_0') + \epsilon^2(f_2'' + f_2 - f_1') + \mathcal{O}(\epsilon^3) = 0 \tag{9}$$

- **STAGE X (What we observe):** After substitution, we have a polynomial in ϵ equal to zero.
- **STAGE Y (Why we can equate coefficients):** For the equation to hold for all small ϵ, each power of ϵ must independently vanish. This is because polynomials can only be identically zero if all coefficients are zero.
- **STAGE Z (Result):** We obtain a hierarchy of ODEs, one for each order of ϵ.

Initial Conditions Hierarchy

Expand $f(0) = 1$:

$$f_0(0) + \epsilon f_1(0) + \epsilon^2 f_2(0) + \cdots = 1 \tag{10}$$

Expand $f'(0) = \frac{\epsilon}{2}$:

$$f_0'(0) + \epsilon f_1'(0) + \epsilon^2 f_2'(0) + \cdots = \frac{\epsilon}{2} \tag{11}$$

- **STAGE X (Initial conditions by order):** Equating powers of ϵ in the initial conditions:

$$\begin{aligned}
\mathcal{O}(1):&\quad f_0(0) = 1, \quad f_0'(0) = 0\\
\mathcal{O}(\epsilon):&\quad f_1(0) = 0, \quad f_1'(0) = \frac{1}{2}\\
\mathcal{O}(\epsilon^2):&\quad f_2(0) = 0, \quad f_2'(0) = 0
\end{aligned}$$

- **STAGE Y (Why this structure):** The condition $f'(0) = \epsilon/2$ means the initial velocity is of order ϵ. When we expand this in powers of ϵ, the coefficient $1/2$ appears at order $\mathcal{O}(\epsilon)$, giving $f_1'(0) = 1/2$. This is crucial for determining the correct initial conditions at each order.
- **STAGE Z (Ready to solve):** We now have complete ODE problems at each order.

4 Step 4: Solve Order by Order

Order $\mathcal{O}(1)$: Leading Order Problem

ODE and Initial Conditions:

$$f_0'' + f_0 = 0, \quad f_0(0) = 1, \quad f_0'(0) = 0 \tag{12}$$

General Solution: The characteristic equation is $r^2 + 1 = 0 \Rightarrow r = \pm i$, giving:

$$f_0(t) = A\cos t + B\sin t \tag{13}$$

Apply Initial Conditions:

$$f_0(0) = A = 1 \Rightarrow A = 1 \tag{14}$$

$$f_0'(t) = -A\sin t + B\cos t \Rightarrow f_0'(0) = B = 0 \Rightarrow B = 0 \tag{15}$$

Leading Order Solution:

$$\boxed{f_0(t) = \cos t} \tag{16}$$

- **STAGE X (What we found):** The unperturbed solution is pure harmonic oscillation with unit amplitude and zero initial velocity.
- **STAGE Y (Physical meaning):** This represents an undamped oscillator starting from maximum displacement. The perturbation term $\epsilon f'$ represents weak forcing proportional to velocity.
- **STAGE Z (Next order):** This solution becomes the forcing term for the next order.

Order $\mathcal{O}(\epsilon)$: First Correction

ODE and Initial Conditions:

$$f_1'' + f_1 = f_0' = -\sin t, \quad f_1(0) = 0, \quad f_1'(0) = \frac{1}{2} \tag{17}$$

- **STAGE X (What we have):** An inhomogeneous ODE where the forcing term is $-\sin t$, which is a solution of the homogeneous equation.
- **STAGE Y (Why this matters - Resonance):** The forcing frequency equals the natural frequency. This is a resonance condition that will produce a **secular term** (a term growing linearly with time).
- **STAGE Z (Expect secular growth):** The particular solution will contain a term proportional to $t\cos t$.

Solution Method:
The homogeneous solution is:

$$f_{1,h}(t) = C\cos t + D\sin t \tag{18}$$

For the particular solution, since $\sin t$ is a homogeneous solution, we use the ansatz:

$$f_{1,p}(t) = t(E\cos t + F\sin t) \tag{19}$$

Compute derivatives:

$$\begin{aligned} f'_{1,p}(t) &= (E\cos t + F\sin t) + t(-E\sin t + F\cos t) && (20) \\ &= E\cos t + F\sin t - Et\sin t + Ft\cos t && (21) \end{aligned}$$

$$\begin{aligned} f''_{1,p}(t) &= -E\sin t + F\cos t + [-E\sin t - Et\cos t + F\cos t - Ft\sin t] && (22) \\ &= -2E\sin t + 2F\cos t - Et\cos t - Ft\sin t && (23) \end{aligned}$$

Substitute into ODE:

$$\begin{aligned} f''_{1,p} + f_{1,p} &= -2E\sin t + 2F\cos t - Et\cos t - Ft\sin t + Et\cos t + Ft\sin t && (24) \\ &= -2E\sin t + 2F\cos t && (25) \end{aligned}$$

Setting this equal to $-\sin t$:

$$\begin{aligned} -2E\sin t + 2F\cos t &= -\sin t && (26) \\ \Rightarrow \quad E = \frac{1}{2}, &\quad F = 0 && (27) \end{aligned}$$

Particular solution:

$$f_{1,p}(t) = \frac{t}{2}\cos t \tag{28}$$

General solution for f_1:

$$f_1(t) = C\cos t + D\sin t + \frac{t}{2}\cos t \tag{29}$$

Apply Initial Conditions:

$$\begin{aligned} f_1(0) &= C = 0 \Rightarrow C = 0 && (30) \\ f'_1(t) &= -C\sin t + D\cos t + \frac{1}{2}\cos t - \frac{t}{2}\sin t && (31) \\ f'_1(0) &= D + \frac{1}{2} = \frac{1}{2} \Rightarrow D = 0 && (32) \end{aligned}$$

First Order Correction:

$$\boxed{f_1(t) = \frac{t}{2}\cos t} \tag{33}$$

- **STAGE X (What we found):** The first correction is purely a secular term—a resonant response that grows linearly with time.
- **STAGE Y (Why $D = 0$):** The non-zero initial velocity $f'_1(0) = 1/2$ exactly cancels with the $1/2$ contribution from the particular solution's derivative at $t = 0$, leaving $D = 0$.
- **STAGE Z (Simplification):** The solution is remarkably clean: no oscillatory correction at order ϵ, only secular growth.

5 Step 5: Combine the Two-Term Expansion

Two-term perturbation expansion:

$$f(t) = f_0(t) + \epsilon f_1(t) + \mathcal{O}(\epsilon^2) \tag{34}$$
$$= \cos t + \epsilon \cdot \frac{t}{2} \cos t + \mathcal{O}(\epsilon^2) \tag{35}$$

$$\boxed{f(t) = \cos t + \frac{\epsilon t}{2} \cos t + \mathcal{O}(\epsilon^2)} \tag{36}$$

This can also be written as:

$$f(t) = \left(1 + \frac{\epsilon t}{2}\right) \cos t + \mathcal{O}(\epsilon^2) \tag{37}$$

- **STAGE X (What we have):** A two-term expansion with a secular term that represents amplitude growth.
- **STAGE Y (Secular term meaning):** The factor $(1 + \epsilon t/2)$ suggests exponential growth $e^{\epsilon t/2} \approx 1 + \epsilon t/2$ for small ϵt.
- **STAGE Z (Validity):** This expansion is valid for $\epsilon t \ll 1$, i.e., $t \ll 1/\epsilon$.

6 Step 6: Find the Exact Solution

Characteristic Equation

Rewrite the ODE:

$$f'' - \epsilon f' + f = 0 \tag{38}$$

The characteristic equation is:

$$m^2 - \epsilon m + 1 = 0 \tag{39}$$

Using the quadratic formula:

$$m = \frac{\epsilon \pm \sqrt{\epsilon^2 - 4}}{2} = \frac{\epsilon}{2} \pm \frac{1}{2}\sqrt{\epsilon^2 - 4} \tag{40}$$

For small ϵ where $\epsilon^2 < 4$:

$$m = \frac{\epsilon}{2} \pm \frac{i}{2}\sqrt{4 - \epsilon^2} \tag{41}$$

- **STAGE X (Complex roots):** The roots are complex conjugates for $|\epsilon| < 2$.
- **STAGE Y (Oscillatory solution):** Complex roots indicate oscillatory behavior with exponential amplitude modulation.
- **STAGE Z (Exact form):** We can write the solution in terms of exponentials or trigonometric functions.

Let:

$$\alpha = \frac{\epsilon}{2} \tag{42}$$
$$\omega = \frac{\sqrt{4 - \epsilon^2}}{2} = \frac{1}{2}\sqrt{4 - \epsilon^2} \tag{43}$$

General solution:

$$f(t) = e^{\alpha t}(A \cos(\omega t) + B \sin(\omega t)) \tag{44}$$

Apply Initial Conditions

Condition 1: $f(0) = 1$

$$f(0) = e^0(A\cos 0 + B\sin 0) = A = 1 \Rightarrow A = 1 \tag{45}$$

Condition 2: $f'(0) = \frac{\epsilon}{2}$
First compute $f'(t)$:

$$f'(t) = \alpha e^{\alpha t}(A\cos(\omega t) + B\sin(\omega t)) \tag{46}$$
$$+ e^{\alpha t}(-A\omega\sin(\omega t) + B\omega\cos(\omega t)) \tag{47}$$

At $t = 0$:

$$f'(0) = \alpha A + B\omega = \frac{\epsilon}{2} \cdot 1 + B\omega = \frac{\epsilon}{2} \tag{48}$$

Solving for B:

$$B\omega = \frac{\epsilon}{2} - \frac{\epsilon}{2} = 0 \Rightarrow B = 0 \tag{49}$$

Exact solution:

$$\boxed{f_{\text{exact}}(t) = e^{\frac{\epsilon t}{2}} \cos\left(\frac{t}{2}\sqrt{4-\epsilon^2}\right)} \tag{50}$$

- **STAGE X (Simple form):** The exact solution has a remarkably simple form because $B = 0$.
- **STAGE Y (Why $B = 0$):** The initial condition $f'(0) = \epsilon/2$ was chosen precisely so that the sine coefficient vanishes. This is what makes the comparison with the perturbation expansion particularly clean.
- **STAGE Z (Structure):** The solution is an exponentially growing cosine with a slightly modified frequency.

7 Step 7: Expand the Exact Solution

Expand Each Component

1. Exponential term:

$$e^{\frac{\epsilon t}{2}} = 1 + \frac{\epsilon t}{2} + \frac{\epsilon^2 t^2}{8} + \mathcal{O}(\epsilon^3) \tag{51}$$

2. Frequency term:

$$\omega = \frac{1}{2}\sqrt{4-\epsilon^2} = \frac{1}{2} \cdot 2\sqrt{1-\frac{\epsilon^2}{4}} \tag{52}$$
$$= 1 \cdot \left(1 - \frac{\epsilon^2}{8} + \mathcal{O}(\epsilon^4)\right) = 1 - \frac{\epsilon^2}{8} + \mathcal{O}(\epsilon^4) \tag{53}$$

- **STAGE X (Binomial expansion):** Used $(1+x)^{1/2} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \cdots$ with $x = -\epsilon^2/4$.
- **STAGE Y (Why to order ϵ^2):** We need ω accurate to $\mathcal{O}(\epsilon^2)$ to get f correct to $\mathcal{O}(\epsilon)$.
- **STAGE Z (Apply to trig functions):** The argument of cosine becomes $t(1 - \epsilon^2/8) = t - \epsilon^2 t/8$.

3. Cosine term:

$$\cos(\omega t) = \cos\left(t\left[1 - \frac{\epsilon^2}{8}\right]\right) = \cos\left(t - \frac{\epsilon^2 t}{8}\right) \tag{54}$$

$$= \cos t \cos\left(\frac{\epsilon^2 t}{8}\right) + \sin t \sin\left(\frac{\epsilon^2 t}{8}\right) \tag{55}$$

$$= \cos t \left[1 - \frac{1}{2}\left(\frac{\epsilon^2 t}{8}\right)^2 + \cdots\right] + \sin t\left[\frac{\epsilon^2 t}{8} + \cdots\right] \tag{56}$$

$$= \cos t + \mathcal{O}(\epsilon^2) \tag{57}$$

Combine All Terms

$$f_{\text{exact}}(t) = e^{\frac{\epsilon t}{2}} \cos(\omega t) \tag{58}$$

$$= \left(1 + \frac{\epsilon t}{2} + \mathcal{O}(\epsilon^2)\right)\left(\cos t + \mathcal{O}(\epsilon^2)\right) \tag{59}$$

$$= \cos t + \frac{\epsilon t}{2}\cos t + \mathcal{O}(\epsilon^2) \tag{60}$$

Expanded exact solution:

$$\boxed{f_{\text{exact}}(t) = \cos t + \frac{\epsilon t}{2}\cos t + \mathcal{O}(\epsilon^2)} \tag{61}$$

8 Step 8: Comparison and Verification

Direct Comparison

Perturbation expansion result (from Step 5):

$$f_{\text{pert}}(t) = \cos t + \frac{\epsilon t}{2}\cos t + \mathcal{O}(\epsilon^2) \tag{62}$$

Expanded exact solution (from Step 7):

$$f_{\text{exact}}(t) = \cos t + \frac{\epsilon t}{2}\cos t + \mathcal{O}(\epsilon^2) \tag{63}$$

PERFECT AGREEMENT: The two-term perturbation expansion matches the expansion of the exact solution identically to $\mathcal{O}(\epsilon)$.

- **STAGE X (What we verified):** Both methods give identical results through order ϵ.
- **STAGE Y (Why this confirms validity):** The perturbation method is correct for this regular problem. The secular term $\frac{\epsilon t}{2}\cos t$ is a genuine feature of the exact solution, not an artifact of the method.
- **STAGE Z (Physical interpretation):** The term $\epsilon f'$ on the right-hand side acts as anti-damping (energy input), causing:
  1. Exponential amplitude growth captured by $e^{\epsilon t/2} \approx 1 + \epsilon t/2$
  2. A slight frequency shift of order $\mathcal{O}(\epsilon^2)$, not visible at this order

Key Observations

1. Secular Term Resolution:

The secular term $\frac{\epsilon t}{2}\cos t$ appears unbounded as $t \to \infty$, but in the exact solution it's part of:

$$e^{\frac{\epsilon t}{2}}\cos(\omega t) \approx \left(1 + \frac{\epsilon t}{2}\right)\cos t \tag{64}$$

The linear growth $1+\epsilon t/2$ is actually the leading-order expansion of the bounded exponential $e^{\epsilon t/2}$, representing physical amplitude growth due to the anti-damping effect.

2. Validity Domain:

The expansion is uniformly valid for:

$$t \ll \frac{1}{\epsilon} \tag{65}$$

For $t = \mathcal{O}(1/\epsilon)$, the $\mathcal{O}(\epsilon)$ term becomes $\mathcal{O}(1)$ and the expansion breaks down. The multiple scales method would be needed for longer times.

3. Initial Condition Impact:

The initial condition $f'(0) = \epsilon/2$ was specifically chosen so that:

- The leading order has zero initial velocity: $f_0'(0) = 0$
- The first correction has initial velocity $f_1'(0) = 1/2$
- This exactly cancels the contribution from the particular solution, giving $D = 0$
- In the exact solution, this makes $B = 0$, yielding a pure growing cosine

9 Verification Checklist

✓ **Problem classified:** Regular perturbation (highest derivative not multiplied by ϵ)

✓ **Expansion ansatz:** $f = f_0 + \epsilon f_1 + \mathcal{O}(\epsilon^2)$

✓ **Initial conditions distributed:** $f_0(0) = 1$, $f_0'(0) = 0$; $f_1(0) = 0$, $f_1'(0) = 1/2$

✓ **Order $\mathcal{O}(1)$:** $f_0 = \cos t$

✓ **Order $\mathcal{O}(\epsilon)$:** $f_1 = \frac{t}{2}\cos t$ (secular term only)

✓ **Exact solution found:** $f = e^{\epsilon t/2}\cos(\omega t)$ with $B = 0$

✓ **Exact solution expanded:** Careful Taylor expansion to $\mathcal{O}(\epsilon)$

✓ **Perfect agreement:** Perturbation and exact expansion match

✓ **Secular term explained:** Genuine feature from exponential growth $e^{\epsilon t/2}$

✓ **Validity domain identified:** $t \ll 1/\epsilon$

This solution demonstrates that regular perturbation theory correctly captures the behavior of weakly anti-damped oscillators for moderate times, including secular terms that represent physical exponential growth.