Asymptotics 2025/2026 — Sheet 5[1]

Fourier-type integrals and steepest descent

Please hand in questions 1(a), 1(b) and 2(a) on Thursday 6 November 2025 at 12pm

In all of the below, the relevant asymptotic limit is $X \to \infty$.

1. Use the method of stationary phase to obtain the leading asymptotic behaviour of the following integrals
   (a) $\int_0^1 \tan(t) e^{iXt^4}\, dt$.
   (b) $\int_{1/2}^2 (1+t) e^{iX(-t+t^3/3)}\, dt$.
   (c) $\int_0^\infty \frac{1}{1+t^2} e^{iXt}\, dt$.
   (d) $\int_0^\infty e^{iX(2t-t^2)} \ln(1+t^2)\, dt$.
   (e) $\int_0^\pi \sin(X\cos(t)) e^{-t^2}\, dt$.

2. Use the method of steepest descent to find the leading asymptotic behaviour of the following integrals
   (a) $\int_{-1}^\infty e^{X(t+it-t^2/2)}\, dt$.
   (b) $\int_{-\infty}^\infty \frac{t e^{iX(t^5/5+t)}}{1+t^2}\, dt$.
   (c) $\int_0^\infty e^{iX(t^4/4+t^3/3)} e^{-t}\, dt$.
   (d) $\int_0^1 \frac{e^{iXt^2}}{t^2-t+5/16}\, dt$.
   (e) $\int_{-1}^\infty \sqrt{1+t}\, \cos(Xt^2) e^{X(t-t^3/3)}\, dt$.

3. Find the leading asymptotic behaviour of the Gamma function $\Gamma(X)$ using the integral representation
$$\frac{1}{\Gamma(X)} = \frac{1}{2\pi i}\int_\gamma e^t t^{-X}\, dt$$
where the path γ starts at $-\infty$ below the branch cut along the negative x-axis, encircles the origin and then goes off to $-\infty$ above the branch cut.
