Asymptotics Problem 8.6: Complete Pedagogical Solution

Boundary Layers When $p(x)$ Vanishes at Both Boundaries

Problem 1. For the o.d.e. $\varepsilon y'' + \sin(x)y' + \sin(2x)y = 0$, $0 \le x \le \pi$, with $y(0) = \pi$, $y(\pi) = 0$, consider asymptotic expansions for $\varepsilon \to 0$ with a boundary layer at either end of the interval, and find one or more asymptotic expansions for the solution $y(x)$ to leading order.

Solution: Step-by-Step Atomic Breakdown

Step 1: Identify the Problem Structure and Classification

Strategy: *We have a singularly perturbed second-order linear ODE of the general form (Lecture Notes §6.2, Eq. (340)):*

$$\varepsilon y'' + p(x)y' + q(x)y = 0, \quad y(0) = \alpha, \quad y(1) = \beta,$$

where in our case:

$$\begin{aligned} p(x) &= \sin(x) \\ q(x) &= \sin(2x) = 2\sin(x)\cos(x) \\ \textit{Domain:} \quad & [0, \pi] \\ \alpha = \pi, &\quad \beta = 0 \end{aligned}$$

The key question is: ***Where are the boundary layers located?***
Key Concept: *From Lecture Notes §6.2.1, the standard theory tells us:*

- *If $p(x) > 0$ throughout $[a, b]$, the boundary layer is at the* ***left*** *endpoint ($x = a$)*
- *If $p(x) < 0$ throughout $[a, b]$, the boundary layer is at the* ***right*** *endpoint ($x = b$)*
- *If $p(x_0) = 0$ for some x_0, special analysis is required*

Step 2: Analyze Where $p(x) = \sin(x)$ Vanishes

What we observe:

$$p(x) = \sin(x) = 0 \quad \text{at} \quad x = 0 \quad \text{and} \quad x = \pi$$

Warning: *The coefficient $p(x) = \sin(x)$ vanishes at* ***both*** *boundaries of the domain $[0, \pi]$! This means the standard theory (which assumes $p(x) \neq 0$ on $[0, 1]$, see Lecture Notes Eq. (341)) does not directly apply. We cannot immediately determine the boundary layer location from the sign of $p(x)$ at the boundaries.*
Justification: *Why is this problematic? Recall from Lecture Notes §6.2.1 that boundary layer location depends on the sign of $p(x)$:*

- *When $p(x_0) > 0$: The inner solution has form $Y_0(X) = A + Be^{-p_0 X}$, which decays as $X \to +\infty$ only if $p_0 > 0$*
- *When $p(x_0) < 0$: The exponential $e^{-p_0 X}$ grows as $X \to +\infty$, preventing matching*

When $p(x_0) = 0$, the exponential character changes completely, and we need a different dominant balance analysis.
Strategy: *Since we cannot determine the boundary layer location a priori, we must:*

1. *Compute the outer solution (valid away from any boundary layers)*
2. *Try an inner solution at $x = 0$ and check if matching is possible*
3. *Try an inner solution at $x = \pi$ and check if matching is possible*
4. *Based on which matchings succeed, determine the actual structure*

This is the systematic workflow from Lecture Notes §6.2.3.

Step 3: Compute the Outer Solution

What we do: Set $\varepsilon = 0$ in the ODE to find the leading-order outer solution.
Technique: *The outer expansion is $y = y_0 + \varepsilon y_1 + \cdots$. At leading order ($\varepsilon^0$):*

$$\sin(x)y_0' + \sin(2x)y_0 = 0$$

This is a first-order linear ODE (we've lost the highest derivative!).

Step 3a: Solving the Reduced Equation

Simplify using $\sin(2x) = 2\sin(x)\cos(x)$:

$$\sin(x)y_0' + 2\sin(x)\cos(x)y_0 = 0$$

For $x \neq 0, \pi$ (where $\sin(x) \neq 0$), divide by $\sin(x)$:

$$y_0' + 2\cos(x)y_0 = 0$$

Technique: *This is separable:*

$$\frac{dy_0}{y_0} = -2\cos(x)\,dx$$

Integrate both sides:

$$\ln|y_0| = -2\sin(x) + C$$

Therefore:

$$y_0(x) = a\,e^{-2\sin(x)}$$

where a is an arbitrary constant.
Justification: *Why don't we apply boundary conditions yet? We have only **one** integration constant a, but **two** boundary conditions ($y(0) = \pi$ and $y(\pi) = 0$). The outer solution alone cannot satisfy both boundary conditions — this is the hallmark of a singular perturbation problem. We must determine which boundary condition the outer solution should satisfy based on where the boundary layer is located.*

Step 3b: Evaluating the Outer Solution at Boundaries

At $x = 0$:

$$y_0(0) = a\,e^{-2\sin(0)} = a\,e^0 = a$$

At $x = \pi$:

$$y_0(\pi) = a\,e^{-2\sin(\pi)} = a\,e^0 = a$$

Key Concept: *Interestingly, $y_0(0) = y_0(\pi) = a$ because $\sin(0) = \sin(\pi) = 0$. This means the outer solution approaches the **same** constant a at both endpoints. This will be crucial for the matching analysis.*

Step 4: Try Inner Solution at $x = 0$

Strategy: *Assume there is a boundary layer at $x = 0$. Define the inner variable $X = x/\delta$ where $\delta(\varepsilon) \to 0$ as $\varepsilon \to 0$. The boundary layer width δ will be determined by dominant balance.*

Step 4a: Transform the ODE

Set $x = \delta X$ and $Y(X) = y(x)$. Then:

$$\frac{dy}{dx} = \frac{1}{\delta}\frac{dY}{dX} = \frac{Y'}{\delta}$$
$$\frac{d^2y}{dx^2} = \frac{1}{\delta^2}\frac{d^2Y}{dX^2} = \frac{Y''}{\delta^2}$$

Substitute into the ODE:

$$\varepsilon \cdot \frac{Y''}{\delta^2} + \sin(\delta X) \cdot \frac{Y'}{\delta} + \sin(2\delta X) \cdot Y = 0$$

Step 4b: Taylor Expand the Coefficients

For small δX:

$$\sin(\delta X) = \delta X - \frac{(\delta X)^3}{6} + \cdots \approx \delta X$$
$$\sin(2\delta X) = 2\delta X - \frac{(2\delta X)^3}{6} + \cdots \approx 2\delta X$$

The ODE becomes:

$$\frac{\varepsilon}{\delta^2} Y'' + \frac{\delta X}{\delta} Y' + 2\delta X \cdot Y = 0$$
$$\frac{\varepsilon}{\delta^2} Y'' + XY' + 2\delta XY = 0$$

Justification: *Why can we use Taylor expansion? Near $x = 0$, the inner variable $X = O(1)$ corresponds to $x = \delta X = O(\delta)$, which is small. Therefore $\sin(\delta X) \approx \delta X$ is an excellent approximation within the boundary layer.*

Step 4c: Dominant Balance Analysis

Technique: *We need to determine $\delta(\varepsilon)$ such that the two leading terms balance. Consider the relative sizes:*

- *First term: $\frac{\varepsilon}{\delta^2} Y''$*
- *Second term: XY' (coefficient is $O(1)$)*
- *Third term: $2\delta XY$ (coefficient is $O(\delta)$, subdominant)*

For a non-trivial boundary layer, the first two terms must balance:

$$\frac{\varepsilon}{\delta^2} = O(1) \quad \Longrightarrow \quad \delta^2 = \varepsilon \quad \Longrightarrow \quad \boxed{\delta = \sqrt{\varepsilon}}$$

Justification: *Why this balance? If $\delta \ll \sqrt{\varepsilon}$, then $\varepsilon/\delta^2 \gg 1$ and the Y'' term would dominate alone (no balance). If $\delta \gg \sqrt{\varepsilon}$, then $\varepsilon/\delta^2 \ll 1$ and the Y'' term would be negligible (we'd get the outer equation). Only $\delta = \sqrt{\varepsilon}$ gives a proper balance.*

This is consistent with Lecture Notes §6.2.2, Eq. (356): when $p(x_0) = 0$ but $p'(x_0) \neq 0$, the boundary layer width is $\delta = \sqrt{\varepsilon}$, not $\delta = \varepsilon$.

Step 4d: Leading-Order Inner Equation at $x = 0$

With $\delta = \sqrt{\varepsilon}$, the inner equation becomes:

$$Y'' + XY' + 2\sqrt{\varepsilon}XY = 0$$

At leading order (neglecting the $O(\sqrt{\varepsilon})$ term):

$$\boxed{Y_0'' + XY_0' = 0}$$

This is a second-order ODE with non-constant coefficients.

Step 4e: Solve the Leading-Order Inner Equation

Technique: *The equation $Y_0'' + XY_0' = 0$ can be reduced in order. Let $W = Y_0'$:*

$$W' + XW = 0 \quad \Longrightarrow \quad \frac{dW}{W} = -X\,dX$$

Integrate:

$$\ln|W| = -\frac{X^2}{2} + C_1 \quad \Longrightarrow \quad W = Y_0' = A\exp\left(-\frac{X^2}{2}\right)$$

Integrate again to find Y_0:

$$Y_0(X) = A\int_0^X \exp\left(-\frac{s^2}{2}\right) ds + B$$

Justification: *Why integrate from 0 to X? This choice makes the integral vanish at $X = 0$, which simplifies applying the boundary condition. The constant of integration is absorbed into B.*

Step 4f: Apply Boundary Condition at $x = 0$

At $x = 0$, we have $X = 0$ and $y(0) = \pi$. Therefore:

$$Y_0(0) = A\int_0^0 \exp\left(-\frac{s^2}{2}\right) ds + B = 0 + B = B$$

The boundary condition $Y_0(0) = \pi$ gives:

$$\boxed{B = \pi}$$

So the inner solution at $x = 0$ is:

$$Y_{0,a}(X) = A\int_0^X \exp\left(-\frac{s^2}{2}\right) ds + \pi$$

Step 4g: Check Matching at $x = 0$

Technique: *For matching, we need to check the behavior as $X \to +\infty$ (moving from the inner region toward the outer region). The Gaussian integral:*

$$\int_0^\infty \exp\left(-\frac{s^2}{2}\right) ds = \sqrt{\frac{\pi}{2}}$$

Therefore:

$$\lim_{X\to+\infty} Y_{0,a}(X) = A\sqrt{\frac{\pi}{2}} + \pi$$

*This is a **finite constant!***

Key Concept: *The inner solution approaches a constant as $X \to +\infty$. This means matching with the outer solution is **possible** in principle. The outer solution also approaches a constant (a) as $x \to 0$. We will determine both A and a through the matching condition.*

Step 5: Try Inner Solution at $x = \pi$

Strategy: *Now assume there is a boundary layer at $x = \pi$. Define the inner variable by $x - \pi = \delta X$, or equivalently $x = \pi + \delta X$.*

Step 5a: Transform the ODE

Set $x = \pi + \delta X$ and $Y(X) = y(x)$. The derivatives transform the same way:

$$\frac{dy}{dx} = \frac{Y'}{\delta}, \quad \frac{d^2y}{dx^2} = \frac{Y''}{\delta^2}$$

Substitute into the ODE:

$$\frac{\varepsilon}{\delta^2} Y'' + \sin(\pi + \delta X) \cdot \frac{Y'}{\delta} + \sin(2\pi + 2\delta X) \cdot Y = 0$$

Step 5b: Taylor Expand Around $x = \pi$

Use trigonometric identities and Taylor expansion:

$$\sin(\pi + \delta X) = -\sin(\delta X) \approx -\delta X$$
$$\sin(2\pi + 2\delta X) = \sin(2\delta X) \approx 2\delta X$$

The ODE becomes:

$$\frac{\varepsilon}{\delta^2} Y'' + \frac{-\delta X}{\delta} Y' + 2\delta X \cdot Y = 0$$
$$\frac{\varepsilon}{\delta^2} Y'' - XY' + 2\delta XY = 0$$

Key Concept: *Notice the crucial difference: the coefficient of Y' is now $-X$ instead of $+X$! This sign change will have dramatic consequences for whether matching is possible.*

Step 5c: Dominant Balance

By the same argument as before, $\delta = \sqrt{\varepsilon}$.

Step 5d: Leading-Order Inner Equation at $x = \pi$

At leading order:

$$\boxed{Y_0'' - XY_0' = 0}$$

Note the **minus sign** compared to the equation at $x = 0$.

Step 5e: Solve the Leading-Order Inner Equation

Technique: *Let $W = Y_0'$:*

$$W' - XW = 0 \quad \Longrightarrow \quad \frac{dW}{W} = X\,dX$$

Integrate:

$$\ln|W| = \frac{X^2}{2} + C_1 \quad \Longrightarrow \quad W = Y_0' = C \exp\left(\frac{X^2}{2}\right)$$

Integrate again:

$$Y_0(X) = C \int_0^X \exp\left(\frac{s^2}{2}\right) ds + D$$

Step 5f: Apply Boundary Condition at $x = \pi$

At $x = \pi$, we have $X = 0$ and $y(\pi) = 0$. Therefore:

$$Y_0(0) = C \cdot 0 + D = D = 0 \quad \Longrightarrow \quad \boxed{D = 0}$$

So the inner solution at $x = \pi$ is:

$$Y_{0,b}(X) = C \int_0^X \exp\left(\frac{s^2}{2}\right) ds$$

Step 5g: Check Matching at $x = \pi$

Technique: *For matching, we need to check the behavior as $X \to -\infty$ (moving from the boundary layer at $x = \pi$ toward the interior). Consider:*

$$\int_0^X \exp\left(\frac{s^2}{2}\right) ds \quad \text{as } X \to -\infty$$

Since $\exp(s^2/2) > 0$ *for all s, the integral from* 0 *to* $X < 0$ *is:*

$$\int_0^X = -\int_X^0 \exp\left(\frac{s^2}{2}\right) ds$$

As $X \to -\infty$, the integrand $\exp(s^2/2) \to \infty$*, so:*

$$\left| \int_X^0 \exp\left(\frac{s^2}{2}\right) ds \right| \to \infty$$

Warning: *The inner solution at $x = \pi$* ***diverges exponentially*** *as $X \to -\infty$:*

$$Y_{0,b}(X) \sim C \cdot (\textit{exponentially large}) \quad \text{as } X \to -\infty$$

Unless $C = 0$, this cannot be matched to the outer solution, which is finite.
Justification: *Why does this happen? The sign difference in the inner equation changes the Gaussian from $e^{-X^2/2}$ (decaying) to $e^{+X^2/2}$ (growing). The growing Gaussian cannot be integrated to give a finite limit. This is the mathematical manifestation of the physical fact: boundary layers occur where the flow of information (determined by the sign of $p(x)$) is into the boundary, not out of it.*

Step 6: Determine the Boundary Layer Structure

Key Concept: *Our analysis reveals:*

1. ***At*** *$x = 0$: Inner solution approaches a finite limit as $X \to +\infty$.* ***Matching IS possible.***
2. ***At*** *$x = \pi$: Inner solution diverges as $X \to -\infty$.* ***Matching is NOT possible.***

Conclusion: *There is a boundary layer* ***only at*** *$x = 0$, and* ***no boundary layer at*** *$x = \pi$.*
Justification: *Why no boundary layer at $x = \pi$? Near $x = \pi$, we have $p(x) = \sin(x) \approx -(\pi - x) < 0$ for x slightly less than π. According to the general theory (Lecture Notes §6.2.1), when $p < 0$ near a boundary point, matching fails because the inner solution grows exponentially in the wrong direction. Even though $p(\pi) = 0$ exactly, the* ***sign of*** *$p'(\pi) = \cos(\pi) = -1 < 0$ determines that the effective behavior is as if $p < 0$.*

Step 7: Apply Boundary Condition to Outer Solution

Since there is no boundary layer at $x = \pi$, the outer solution must satisfy the boundary condition there:

$$y_0(\pi) = 0$$

From Step 3b, $y_0(\pi) = a$. Therefore:

$$\boxed{a = 0}$$

The outer solution is:

$$\boxed{y_0(x) = 0 \quad \text{for all } x \in (0, \pi]}$$

Reflection: *This is a remarkable result! The outer solution vanishes identically. The entire non-trivial behavior of the solution is concentrated in the boundary layer at $x = 0$. This happens because the boundary condition at $x = \pi$ forces $a = 0$, and the function $e^{-2\sin(x)}$ (while varying between e^{-2} and 1) is multiplied by zero.*

Step 8: Perform Matching at $x = 0$

Now we match the inner solution at $x = 0$ with the (now known) outer solution $y_0(x) = 0$.

Step 8a: Van Dyke Matching

Technique: *Apply Van Dyke's matching rule (Lecture Notes §6.1.3): The outer expansion of the inner solution must equal the inner expansion of the outer solution.*

Inner expansion of outer solution:

The outer solution is $y_0(x) = 0$. Express in inner variable $x = \sqrt{\varepsilon}X$:

$$y_0(\sqrt{\varepsilon}X) = 0$$

This is already 0 for all X.

Outer expansion of inner solution:

The inner solution is:

$$Y_{0,a}(X) = A\int_0^X \exp\left(-\frac{s^2}{2}\right) ds + \pi$$

Express in outer variable $X = x/\sqrt{\varepsilon}$ and take $\varepsilon \to 0$ (which means $X \to \infty$ for fixed $x > 0$):

$$\lim_{X\to\infty} Y_{0,a}(X) = A\sqrt{\frac{\pi}{2}} + \pi$$

Step 8b: Matching Condition

For matching:

$$\lim_{X\to\infty} Y_{0,a}(X) = \lim_{x\to 0^+} y_0(x)$$

$$A\sqrt{\frac{\pi}{2}} + \pi = 0$$

Solve for A:

$$A = -\frac{\pi}{\sqrt{\pi/2}} = -\pi\sqrt{\frac{2}{\pi}} = -\sqrt{2\pi}$$

$$\boxed{A = -\sqrt{2\pi}}$$

Step 9: Write the Complete Inner Solution

The matched inner solution at $x = 0$ is:

$$Y_0(X) = -\sqrt{2\pi}\int_0^X \exp\left(-\frac{s^2}{2}\right) ds + \pi$$

We can express this using the error function. Recall (Lecture Notes §2.6.2, Eq. (71)):

$$\mathrm{erf}(z) = \frac{2}{\sqrt{\pi}}\int_0^z e^{-t^2} dt$$

With substitution $t = s/\sqrt{2}$:

$$\int_0^X e^{-s^2/2} ds = \sqrt{2}\int_0^{X/\sqrt{2}} e^{-t^2} dt = \sqrt{2}\cdot\frac{\sqrt{\pi}}{2}\,\mathrm{erf}\left(\frac{X}{\sqrt{2}}\right) = \sqrt{\frac{\pi}{2}}\,\mathrm{erf}\left(\frac{X}{\sqrt{2}}\right)$$

Therefore:

$$\begin{aligned}
Y_0(X) &= -\sqrt{2\pi}\cdot\sqrt{\frac{\pi}{2}}\,\mathrm{erf}\left(\frac{X}{\sqrt{2}}\right) + \pi \\
&= -\pi\,\mathrm{erf}\left(\frac{X}{\sqrt{2}}\right) + \pi \\
&= \pi\left[1 - \mathrm{erf}\left(\frac{X}{\sqrt{2}}\right)\right]
\end{aligned}$$

Step 10: Write the Composite Solution

Since the outer solution vanishes ($y_0 = 0$), the composite solution equals the inner solution expressed in the original variable x.
Recall $X = x/\sqrt{\varepsilon}$, so:

$$\frac{X}{\sqrt{2}} = \frac{x}{\sqrt{2\varepsilon}}$$

Final Leading-Order Composite Solution:

$$\boxed{y_c(x) = \pi\left[1 - \mathrm{erf}\left(\frac{x}{\sqrt{2\varepsilon}}\right)\right] = \pi\,\mathrm{erfc}\left(\frac{x}{\sqrt{2\varepsilon}}\right)}$$

where $\mathrm{erfc}(z) = 1 - \mathrm{erf}(z)$ is the complementary error function.

Step 11: Verify the Solution Properties

Technique: *Let's verify that our solution satisfies the required properties:*

Boundary condition at $x = 0$***:***

$$y_c(0) = \pi\left[1 - \mathit{erf}(0)\right] = \pi[1 - 0] = \pi \quad \checkmark$$

Boundary condition at $x = \pi$***:***
As $\varepsilon \to 0$ *with* $x = \pi$ *fixed:*

$$\frac{\pi}{\sqrt{2\varepsilon}} \to \infty$$

$$\mathit{erf}\left(\frac{\pi}{\sqrt{2\varepsilon}}\right) \to 1$$

$$y_c(\pi) \to \pi[1 - 1] = 0 \quad \checkmark$$

Behavior in the outer region *($x = O(1)$, $x > 0$):*
For fixed $x > 0$, as $\varepsilon \to 0$:

$$\frac{x}{\sqrt{2\varepsilon}} \to \infty, \quad erf \to 1, \quad y_c(x) \to 0 = y_0(x) \quad \checkmark$$

Boundary layer width:
The transition from $y \approx \pi$ to $y \approx 0$ occurs when $x/\sqrt{2\varepsilon} = O(1)$, i.e., when $x = O(\sqrt{\varepsilon})$. This confirms the boundary layer width is $\delta = \sqrt{\varepsilon}$.

Step 12: Physical and Mathematical Interpretation

Reflection: *The solution structure reveals several important features:*

1. ***Boundary layer only at*** *$x = 0$: Although $p(x) = \sin(x)$ vanishes at both endpoints, a boundary layer exists only at $x = 0$. This is determined by the* ***sign of*** *$p'(x)$:*
   - *At $x = 0$: $p'(0) = \cos(0) = +1 > 0 \Rightarrow$ boundary layer exists*
   - *At $x = \pi$: $p'(\pi) = \cos(\pi) = -1 < 0 \Rightarrow$ no boundary layer*
2. ***Boundary layer width*** *$\sim \sqrt{\varepsilon}$: When $p(x_0) = 0$ but $p'(x_0) \neq 0$, the boundary layer is* ***thicker*** *than the standard $O(\varepsilon)$ width. This is because the coefficient of y' is $O(\delta X) = O(\sqrt{\varepsilon})$ near the boundary, requiring a larger region for the $\varepsilon y''$ term to be important.*
3. ***Gaussian profile****: The inner solution involves the error function, which arises from integrating a Gaussian. This is characteristic of boundary layers at points where $p(x)$ has a simple zero.*
4. ***Outer solution vanishes****: The solution is essentially concentrated entirely in the boundary layer. Outside the layer, the solution is asymptotically zero. This happens because the boundary condition $y(\pi) = 0$, combined with the fact that y_0 is constant at both endpoints, forces $y_0 \equiv 0$.*
5. ***Connection to parabolic cylinder functions****: The inner equation $Y'' + XY' = 0$ is related to parabolic cylinder equations (Lecture Notes §6.2.2). For this specific case, the solution reduces to error functions, but more general cases might require parabolic cylinder functions.*

Summary Table

Component	**Result**
ODE	$\varepsilon y'' + \sin(x)y' + \sin(2x)y = 0$
Domain	$[0, \pi]$
Boundary conditions	$y(0) = \pi$, $y(\pi) = 0$
Outer solution	$y_0(x) = 0$
Boundary layer location	$x = 0$ only
Boundary layer width	$\delta = \sqrt{\varepsilon}$
Inner variable	$X = x/\sqrt{\varepsilon}$
Inner equation	$Y_0'' + XY_0' = 0$
Inner solution	$Y_0(X) = \pi \left[1 - \operatorname{erf}\left(\frac{X}{\sqrt{2}}\right)\right]$
Composite solution	$y_c(x) = \pi \operatorname{erfc}\left(\dfrac{x}{\sqrt{2\varepsilon}}\right)$

Connection to Lecture Material

Reflection: *This problem illustrates several key concepts from the course:*

- ***Workflow for boundary layers*** *(Lecture Notes §6.2.3): We followed the systematic approach of identifying candidate locations, computing inner and outer solutions, and checking matching.*
- ***Boundary layers of non-standard width*** *(Lecture Notes §6.2.2): When $p(x_0) = 0$, the dominant balance gives $\delta = \sqrt{\varepsilon}$ rather than $\delta = \varepsilon$.*
- ***Sign of $p'(x_0)$ determines layer existence****: Even when $p(x_0) = 0$ at both boundaries, only one may have a boundary layer, determined by whether the exponential in the inner solution decays or grows.*
- ***Error function solutions****: The Gaussian integrals appearing in boundary layer problems (Lecture Notes §2.6.2) lead naturally to error functions in the final solution.*
- ***Matching determines integration constants****: The constants A in the inner solution and a in the outer solution were both determined by matching conditions and boundary conditions, not independently.*