6 Linear stability

A lot of what we do is **local dynamics**, which means looking at behaviour only near certain points.

- For the simple population model $\dot{N} = \alpha N$ (with $\alpha = \beta - \delta$) we had solutions

$$N(t) = N_0 e^{\alpha t} \tag{6.1}$$

  For $\alpha < 0$ we saw that this tends towards $N(t) \to 0$ as $t \to \infty$.

- For the cut-off population model $\dot{x} = (\beta - x)x$ (with $N = x/\gamma$) we had solutions

$$x(t) = \frac{\beta x_0 e^{\beta t}}{\beta - x_0 + x_0 e^{\beta t}} \tag{6.2}$$

- Taking (6.2), let's ask what happens when t is large and negative ($t \to -\infty$). Then the term e^t is small. We can expand in a Taylor series (in small $w = e^t$ if you like), giving

$$\begin{aligned} x(t) &\approx \tfrac{\beta x_0}{\beta - x_0} e^{\beta t} \left\{ 1 - \tfrac{x_0 e^{\beta t}}{\beta - x_0} + \tfrac{x_0^2 e^{2\beta t}}{(\beta - x_0)^2} - ... \right\} \\ &\approx \rho e^{\beta t} \qquad \text{with} \;\; \rho = \tfrac{\beta x_0}{\beta - x_0} \end{aligned} \tag{6.3}$$

  which looks rather like the solution (6.1) just with different constants.

  In fact if the initial condition x_0 is small, then

$$\rho = \tfrac{\beta x_0}{\beta - x_0} = x_0 \{ 1 + \tfrac{x_0}{\beta} + \tfrac{x_0^2}{\beta^2} + ... \} \approx x_0$$

  so

$$x(t) \approx x_0 e^{\beta t} \qquad \text{or} \qquad N(t) \approx N_0 e^{\beta t} \tag{6.4}$$

  which behaves exactly like (6.1) when the population is small, just with the expansion rate α replaced by β.

- The function (6.4) is a solution to the **local** problem

$$\dot{x} = \beta x \tag{6.5}$$

namely the local approximation $\dot{x} = (\beta - x)x \approx \beta x$ near $x = 0$.

- We call $\dot{x} = \beta x$ the **linearization** of $\dot{x} = (\beta - x)x$ about $x = 0$.

  It tells us what happens near $x = 0$ when approximated to linear order in x.

- Let's go back to (6.2) and ask what happens when t is large and positive ($t \to +\infty$). Then the term $e^{\beta t}$ is large. But we can re-write

$$x(t) = \frac{\beta x_0}{(\beta - x_0)e^{-\beta t} + x_0} \tag{6.6}$$

then the term $e^{-\beta t}$ is small. Now we can expand in a Taylor series (in small $w = e^{-\beta t}$ if you like), so this looks like

$$\begin{aligned} x(t) &\approx \beta \left\{1 - \tfrac{(\beta - x_0)e^{-\beta t}}{x_0} + \tfrac{(\beta - x_0)^2 e^{-2\beta t}}{x_0^2} - \ldots\right\} \\ &\approx \beta + \tfrac{x_0 - \beta}{x_0}\beta e^{-\beta t} \\ \Rightarrow \qquad \hat{x}(t) &\approx \hat{\rho} e^{-\beta t} \end{aligned} \tag{6.7}$$

where $\hat{x}(t) = x(t) - \beta$ and $\hat{\rho} = \frac{x_0 - \beta}{x_0}\beta$.

So again we have the same kind of exponential behaviour near $\hat{x} = 0$, which means near $x = \beta$ or $N = \beta/\gamma$.

- The function $\hat{x}(t) = \rho e^{-\beta t}$ is a solution to the **local** problem

$$\dot{\hat{x}} = -\beta \hat{x} \tag{6.8}$$

namely the local approximation $\dot{\hat{x}} = (\beta - x)x \approx -\beta\hat{x}$ near $\hat{x} = 0$ (equivalently the local approximation $\dot{x} = (\beta - x)x \approx \beta - x$ near $x = \beta$).

- We call $\dot{x} = \beta - x$ the **linearization** of $\dot{x} = (\beta - x)x$ about $x = \beta$. It tells us what happens near $x = \beta$ when approximated to linear order in x.

- So we are starting to see that we will *always* find this kind of exponential behaviour around an equilibrium, i.e. any point x_* where $f(x_*) = 0$.
- As we look into the distant future ($t \to +\infty$) or past ($t \to -\infty$), solutions tend towards equilibria exponentially, or else they fly off (diverge) to infinity. We say *solutions asymptote towards/away from the equilibria*.

Let's take the two-dimensional system and try the same.

The predator-prey model (4.1) had two equilibria (solutions of $\dot{x} = 0$ and $\dot{y} = 0$), at $(x_{*1}, y_{*1}) = (0, 0)$ and $(x_{*2}, y_{*2}) = (\gamma, \alpha)$.

- Let's approximate about these two equilibria:
  - Linearize about $(x_{*1}, y_{*1}) = (0, 0)$:

$$\begin{aligned} \dot{x} &= (\alpha - y)x \approx \alpha x + ... \\ \dot{y} &= (x - \gamma)y \approx -\gamma y + ... \end{aligned} \tag{6.9}$$

    These look a bit like the linearizations of the 1d population model, an exponential growth in x away from an equilibrium at $x = 0$, and an exponential contraction in y towards an equilibrium at $y = 0$.
  - Linearize about $(x_{*2}, y_{*2}) = (\gamma, \alpha)$:

$$\begin{aligned} \dot{x} &= (\alpha - y)x \approx (\alpha - y)\gamma + ... \\ \dot{y} &= (x - \gamma)y \approx (x - \gamma)\alpha + ... \end{aligned} \tag{6.10}$$

    Again these look a bit like the linearizations of the 1d population model but the x and y terms are mixed up. You might recognize this as a second order ODE that gives oscillations, since differentiating gives $\ddot{x} \approx -\gamma\alpha x + \gamma^2\alpha$ and $\ddot{y} \approx -\alpha\gamma y + \alpha^2\gamma$.

General solution of the local system

- Because these local systems are linear we can solve them exactly (see Exercise Sheet).

- If a system has an equilibrium at $\mathbf{x} = 0$, nearby it will look in vector form like $\dot{\mathbf{x}} = \underline{\underline{A}}\mathbf{x}$, whose solution is $\mathbf{x}(t) = e^{\underline{\underline{A}}t}\mathbf{x}_0$.

- More generally around an equilibrium at $\mathbf{x} = \mathbf{x}_*$ the system will look like

$$\dot{\mathbf{x}} = \underline{\underline{A}}(\mathbf{x} - \mathbf{x}_*) \qquad \text{with solution} \qquad \mathbf{x}(t) = \mathbf{x}_* + e^{\underline{\underline{A}}t}(\mathbf{x}_0 - \mathbf{x}_*)\ .$$

  - Superficially that just looks the same as the 1d systems above.
  - In a solution to a scalar problem $x(t) = x_0 e^{\alpha t}$, the exponent gives us the rate α of repulsion from ($\alpha > 0$) or attraction to ($\alpha < 0$) an equilibrium.
  - Now we have a 'rate' matrix $\underline{\underline{A}}$. Just like the rate α this tells us whether solutions are repelled to or attracted from the equilibrium, but also whether they circulate around it.

- We must remember these are only local approximations near the equilibria. Let's go back to our predator-prey example.

- For the predator-prey system, re-writing $\mathbf{x} = \binom{x}{y}$,

  - the linearization about $(x_{*1}, y_{*1}) = (0, 0)$ is:

$$\dot{\mathbf{x}} \approx \underline{\underline{A}}.\mathbf{x} \qquad \text{where} \qquad \underline{\underline{A}} = \begin{pmatrix} \alpha & 0 \\ 0 & -\gamma \end{pmatrix} \tag{6.11}$$

    whose solution is

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} \approx e^{\underline{\underline{A}}t}\mathbf{x}_0 = \begin{pmatrix} e^{\alpha t}x_0 \\ e^{-\gamma t}y_0 \end{pmatrix} \tag{6.12}$$

    That's easy enough, the x and y variables just grow or shrink with t, depending whether α and γ or positive or negative.

  - the linearization about $(x_{*2}, y_{*2}) = (\gamma, \alpha)$ is:

$$\dot{\mathbf{x}} \approx \underline{\underline{A}}.(\mathbf{x} - \mathbf{x}_{*2}) \qquad \text{where} \qquad \underline{\underline{A}} = \begin{pmatrix} 0 & -\gamma \\ \alpha & 0 \end{pmatrix}, \quad \mathbf{x}_{*2} = \binom{\gamma}{\alpha}, \tag{6.13}$$

    whose solution is (writing $\mathbf{x}(t) - \mathbf{x}_*$ on the lefthand side)

$$\begin{aligned} \begin{pmatrix} x(t) - \gamma \\ y(t) - \alpha \end{pmatrix} &\approx \mathbf{x}_* + e^{\underline{\underline{A}}t}(\mathbf{x}_0 - \mathbf{x}_*) \qquad (6.14) \\ &= \begin{pmatrix} \cos(t\sqrt{\alpha\gamma}) & -\frac{\gamma}{\sqrt{\alpha\gamma}}\sin(t\sqrt{\alpha\gamma}) \\ \frac{\alpha}{\sqrt{\alpha\gamma}}\sin(t\sqrt{\alpha\gamma}) & \cos(t\sqrt{\alpha\gamma}) \end{pmatrix} \begin{pmatrix} x_0 - \gamma \\ y_0 - \alpha \end{pmatrix} \end{aligned}$$

    (you can find that using ODE methods you've learned, or directly expanding the matrix exponential [see Exercise Sheet]).

- So you can easily see from this what solutions do, yes?

- No? Me neither. We're going to need a slightly smarter way to understand matrices. The Exercise Sheet tries to explore this a bit. (You can get a rough idea that for $\alpha\gamma > 0$ the solution around (x_{*2}, y_{*2}) is going to oscillate, because the cos and sin functions are going to oscillate as t changes, but for $\alpha\gamma < 0$ these become cosh and sinh functions which are like exponentials, giving growth or decay similar to the solution around (x_{*1}, y_{*1}). But we need to do better . . .)

7 Stability & Eigendecomposition

Here's where the eigenvectors and eigenvalues of a matrix come in.

- On the whole the behaviour created by a 'rate' matrix like we saw above is complicated. But along certain directions in space it is very simple . . . the matrix's eigenvectors.
- If we take an initial condition $\mathbf{x}_0$ that happens to lie along the direction $\mathbf{v}$ from an equilibrium, where $\mathbf{v}$ is an eigenvector of a matrix $\underline{\underline{A}}$ with an eigenvalue λ, then

$$\underline{\underline{A}}.\mathbf{v} = \lambda \mathbf{v} \tag{7.1}$$

then it follows that

$$e^{\underline{\underline{A}}t}.\mathbf{v} = e^{\lambda t}\mathbf{v} \tag{7.2}$$

[Side Notes:] Proof

$$\begin{aligned} \underline{\underline{A}}.\mathbf{v} = \lambda \mathbf{v} \quad &\Rightarrow \quad \underline{\underline{A}}^2.\mathbf{v} = \lambda \underline{\underline{A}}\mathbf{v} = \lambda^2 \mathbf{v} \quad \Rightarrow \quad \ldots \\ &\Rightarrow \quad \underline{\underline{A}}^n.\mathbf{v} = \lambda^n \mathbf{v} \end{aligned} \tag{7.3}$$

so

$$e^{\underline{\underline{A}}t}.\mathbf{v} \;=\; \sum_{n=0}^{\infty} \tfrac{1}{n!} t^n \underline{\underline{A}}^n.\mathbf{v} \;=\; \sum_{n=0}^{\infty} \tfrac{1}{n!} t^n \lambda^n \mathbf{v} \;=\; e^{\lambda t}\mathbf{v} \tag{7.4}$$

In the predator-prey model near (x_{*1}, y_{*1}) the 'rate' matrix was $\underline{\underline{A}} = \begin{pmatrix} \alpha & 0 \\ 0 & -\gamma \end{pmatrix}$. This has :

- eigenvalue α along eigenvector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$,
  so if $\mathbf{x}_0 - \mathbf{x}_* = \begin{pmatrix} x_0 \\ 0 \end{pmatrix} = x_0 \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ then

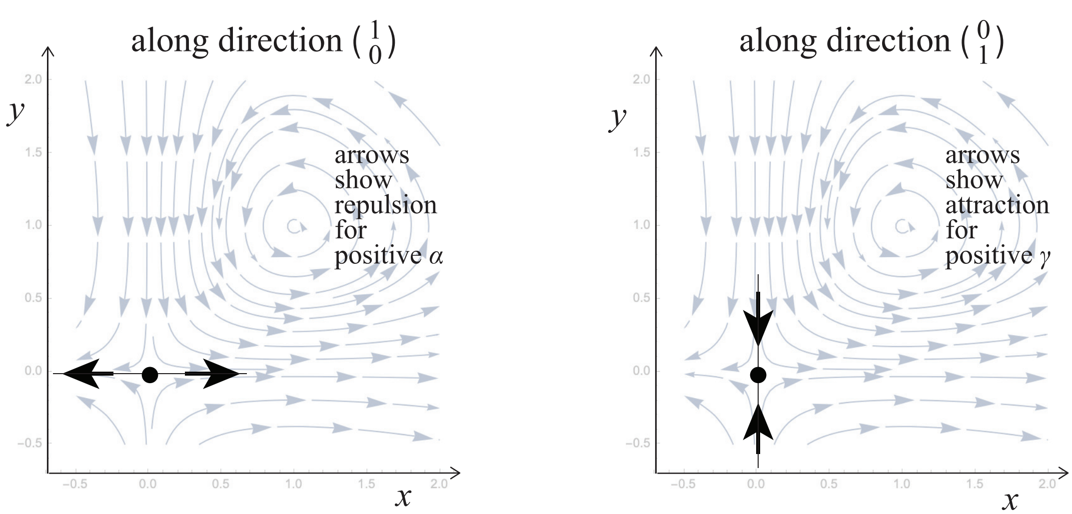
$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} \approx e^{\underline{\underline{A}}t}\mathbf{x}_0 = x_0 e^{\alpha t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{7.5}$$

  i.e. simple 1d attraction/repulsion along direction $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ with rate α.

- eigenvalue $-\gamma$ along eigenvector $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$,
  so if $\mathbf{x}_0 - \mathbf{x}_* = \begin{pmatrix} 0 \\ y_0 \end{pmatrix} = y_0 \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ then

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} \approx e^{\underline{\underline{A}}t}\mathbf{x}_0 = y_0 e^{-\gamma t} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{7.6}$$

  i.e. simple 1d attraction/repulsion along direction $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ with rate $-\gamma$.



- That's too easy, now look at the other equilibrium. . .

Near (x_{*2}, y_{*2}) the 'rate' matrix was $\underline{\underline{A}} = \begin{pmatrix} 0 & -\gamma \\ \alpha & 0 \end{pmatrix}$. This has :

- eigenvalues $\pm\sqrt{-\alpha\gamma}$ along eigenvectors $\begin{pmatrix} \pm\sqrt{-\gamma} \\ \sqrt{\alpha} \end{pmatrix}$,

  so if $\mathbf{x}_0 - \mathbf{x}_* = r\begin{pmatrix} \pm\sqrt{-\gamma} \\ \sqrt{\alpha} \end{pmatrix}$ then

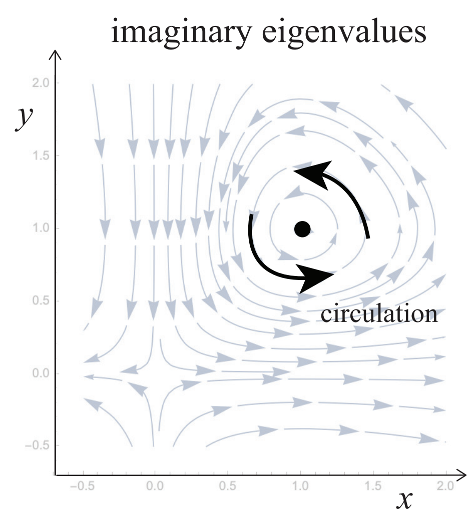
$$\begin{pmatrix} x(t) - \gamma \\ y(t) - \alpha \end{pmatrix} \approx e^{\underline{\underline{A}}t}\mathbf{x}_0 = re^{\pm\sqrt{-\alpha\gamma}t}\begin{pmatrix} \pm\sqrt{-\gamma} \\ \sqrt{\alpha} \end{pmatrix} \tag{7.7}$$

  – If $\gamma\alpha < 0$ this gives simple 1d attraction along direction $\begin{pmatrix} -\sqrt{-\gamma} \\ \sqrt{\alpha} \end{pmatrix}$ and repulsion along direction $\begin{pmatrix} +\sqrt{-\gamma} \\ \sqrt{\alpha} \end{pmatrix}$.
  (Note I didn't say r had to be real).

  – If $\gamma\alpha > 0$ this can't give a real solution, as $\sqrt{-\alpha\gamma} = i\sqrt{\alpha\gamma}$ so this looks like the complex valued

$$\begin{pmatrix} x(t) - \gamma \\ y(t) - \alpha \end{pmatrix} \approx e^{\underline{\underline{A}}t}\mathbf{x}_0 = re^{\pm i\sqrt{\alpha\gamma}t}\begin{pmatrix} \pm\sqrt{-\gamma} \\ \sqrt{\alpha} \end{pmatrix} \tag{7.8}$$

  but it is easy to enough to play around and turn this into something real-valued [see sidenote below].



  More importantly note what this means. You know
  $$e^{\pm i\sqrt{\alpha\gamma}t} = \cos(\sqrt{\alpha\gamma}t) + i\sin(\sqrt{\alpha\gamma}t)$$
  so solutions with this term oscillate with frequency $\sqrt{\gamma\alpha}$, and in a manner shaped by the eigenvectors.

[Side Notes:] To find the real-plane oscillations

Consider an initial condition that is a linear combination of the two eigenvectors

$$\mathbf{x}_0 - \mathbf{x}_* = r\begin{pmatrix} +\sqrt{-\gamma} \\ \sqrt{\alpha} \end{pmatrix} + r^*\begin{pmatrix} -\sqrt{-\gamma} \\ \sqrt{\alpha} \end{pmatrix}$$

for $r \in \mathbb{C}$ and $\alpha, \gamma > 0$. Despite appearances this, and the solutions $\mathbf{x}(t) = ((x(t), y(t))$ that flow out of it, must be real.

With a little work we can re-arrange these into something more useful, in particular to see they are indeed real. Write

$$\begin{aligned}
\begin{pmatrix} x(t) - x_* \\ y(t) - y_* \end{pmatrix} &\approx e^{\underline{\underline{A}}t}(\mathbf{x}_0 - \mathbf{x}_*) \qquad (7.9) \\
&= re^{+i\sqrt{\alpha\gamma}t}\begin{pmatrix} +\sqrt{-\gamma} \\ \sqrt{\alpha} \end{pmatrix} + r^* e^{-i\sqrt{\alpha\gamma}t}\begin{pmatrix} -\sqrt{-\gamma} \\ \sqrt{\alpha} \end{pmatrix} \\
&= r[C + iS]\begin{pmatrix} +\sqrt{-\gamma} \\ \sqrt{\alpha} \end{pmatrix} + r^*[C - iS]\begin{pmatrix} -\sqrt{-\gamma} \\ \sqrt{\alpha} \end{pmatrix} \\
&\qquad \text{using shorthand } C = \cos\left(\sqrt{\alpha\gamma}t\right),\ S = \sin\left(\sqrt{\alpha\gamma}t\right) \\
&= \begin{pmatrix} (r - r^*)\sqrt{-\gamma} \\ (r + r^*)\sqrt{\alpha} \end{pmatrix}\cos\left(\sqrt{\alpha\gamma}t\right) + i\begin{pmatrix} (r + r^*)\sqrt{-\gamma} \\ (r - r^*)\sqrt{\alpha} \end{pmatrix}\sin\left(\sqrt{\alpha\gamma}t\right) \\
&\qquad \text{let } r = (a + ib)/2 \\
&= \begin{pmatrix} \sqrt{\gamma} & 0 \\ 0 & \sqrt{\alpha} \end{pmatrix}\left\{\begin{pmatrix} -b \\ a \end{pmatrix}\cos\left(\sqrt{\alpha\gamma}t\right) - \begin{pmatrix} a \\ b \end{pmatrix}\sin\left(\sqrt{\alpha\gamma}t\right)\right\}
\end{aligned}$$

So we see that a solution oscillates as cos and sin along two orthogonal directions, creating (uneven) circulation around the equilibrium. The factors $\sqrt{\gamma}$ and $\sqrt{\alpha}$ just skew the whole picture by a constant.

[Side Notes:] Eigendecomposition

You can write any ODE or its solutions in terms of the eigenvectors associated with a given equilibrium.

- A non-singular $n \times n$ matrix $\underline{\underline{A}}$ has n linearly independent eigenvectors $\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_n$. We can therefore express any vector as a linear combination of them, say

$$\mathbf{x} - \mathbf{x}_* = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + ... + c_n\mathbf{v}_n \tag{7.10}$$

  for some coefficients $c_1, ..., c_n$.

- Take the linearization of the ODE

$$\begin{aligned} \dot{\mathbf{x}} = \underline{\underline{A}}.(\mathbf{x} - \mathbf{x}_*) &= \underline{\underline{A}}.(c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + ... + c_n\mathbf{v}_n) \\ &= c_1\lambda_1\mathbf{v}_1 + c_2\lambda_2\mathbf{v}_2 + ... + c_n\lambda_n\mathbf{v}_n \end{aligned} \tag{7.11}$$

  giving us equations of motion along these different eigenvectors.

- Similarly for the solutions of the ODE

$$\begin{aligned} \mathbf{x}(t) = \mathbf{x}_* + e^{\underline{\underline{A}}t}.(\mathbf{x}_0 - \mathbf{x}_*) &= \mathbf{x}_* + e^{\underline{\underline{A}}t}.(c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + ... + c_n\mathbf{v}_n) \\ &= \mathbf{x}_* + c_1e^{\lambda_1 t}\mathbf{v}_1 + c_2e^{\lambda_2 t}\mathbf{v}_2 + ... + c_ne^{\lambda_n t}\mathbf{v}_n \end{aligned} \tag{7.12}$$

  giving us rates of attraction to / repulsion from $\mathbf{x}_*$ along the eigenvector directions (and rotation if the λ_rs are complex).

8 Saddle, Node, Focus, or Center?

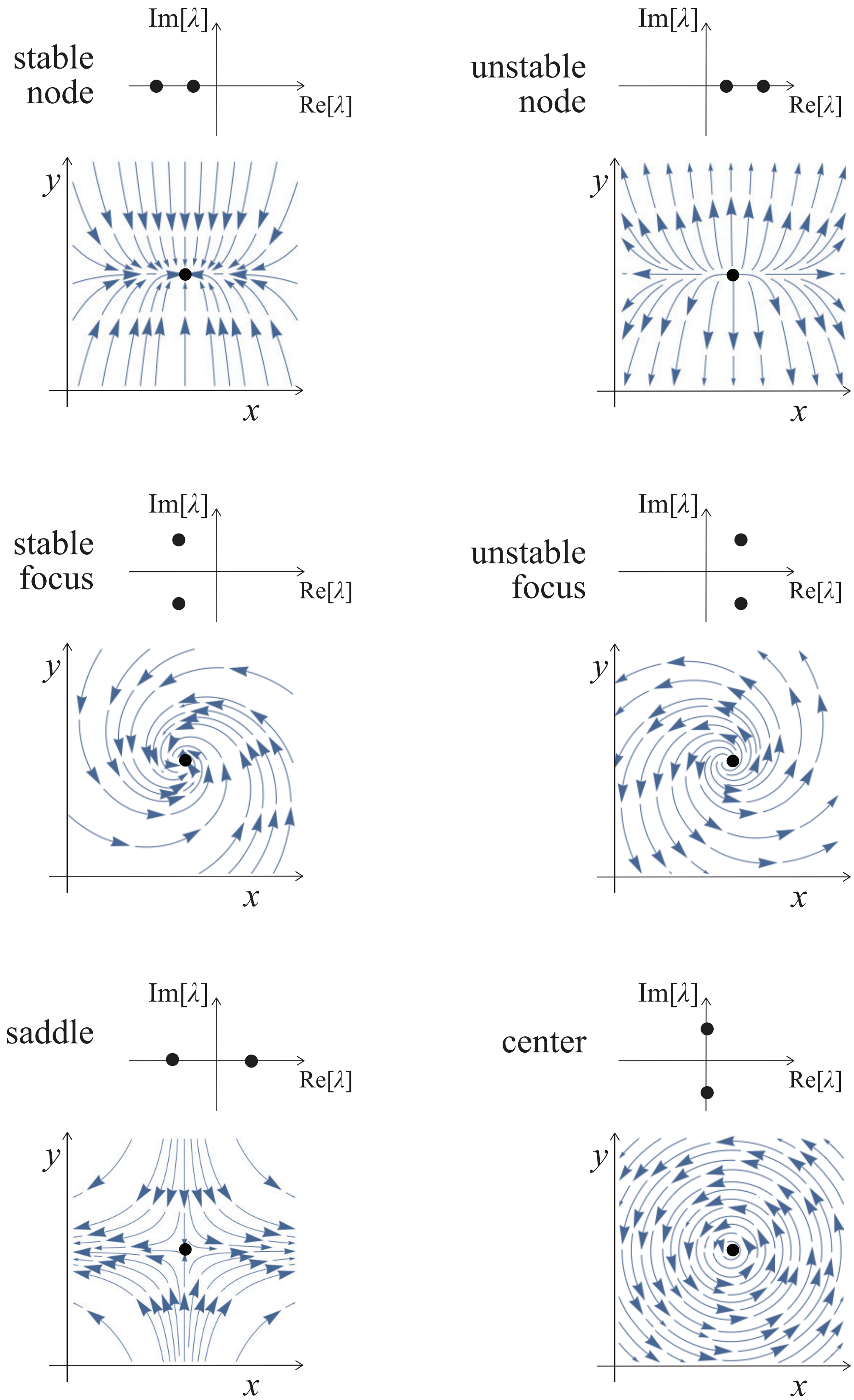
You've seen now pretty much everything that can happen around an equilibrium: as t increases you have either attraction ($x \sim e^{-t}$) or repulsion ($x \sim e^{+t}$), with or without circulation ($x \sim e^{it}$).

The eigenvalues and eigenvectors tell us everything we need about this:

- the eigenvalues tell us the rates of attraction/repulsion/circulation
- the eigenvectors tell us the directions along which these rates apply
- (along other directions the behaviour is just a mixture of these).

We get three types of behaviour:

- A **node** has two real eigenvalues λ_1, λ_2, with the same signs:
  - $\lambda_1, \lambda_2 > 0$ means an unstable node (repeller)
  - $\lambda_1, \lambda_2 < 0$ means a stable node (attractor)
  - Order them as $|\lambda_1| < |\lambda_2|$, then λ_2 gives the **strong** eigendirection and λ_1 gives the **weak** eigendirection (telling us the fastest and slowest directions of motion).
  - Note $\det \underline{\underline{A}} = \lambda_1\lambda_2 > 0$.
- A **saddle** has two real eigenvalues λ_1, λ_2, with different signs:
  - There is attraction along one direction and repulsion along the other.
  - Order them as $\lambda_1 < 0 < \lambda_2$, then λ_2 gives the **unstable** (repelling) eigendirection and λ_1 gives the **stable** (attracting) eigendirection.
  - Note $\det \underline{\underline{A}} = \lambda_1\lambda_2 < 0$.
- A **focus** has two complex eigenvalues $\lambda_1, \lambda_2 \in \mathbb{C}$:
  - they are complex conjugates $\lambda_1 = \lambda_2^*$
  - the real part $\mathsf{Re}[\lambda_1] = \mathsf{Re}[\lambda_2]$ tells us the rate of attraction/repulsion just like the real eigenvalues of a node (so $\mathsf{Re}[\lambda_1] > 0$ mean unstable, $\mathsf{Re}[\lambda_1] < 0$ means stable).
  - the imaginary part $\mathsf{Im}[\lambda_1] = -\mathsf{Im}[\lambda_2]$ gives the frequency of circulation.


stable node
Im[λ]
Re[λ]
y
x
unstable node
Im[λ]
Re[λ]
y
x
stable focus
Im[λ]
Re[λ]
y
x
unstable focus
Im[λ]
Re[λ]
y
x
saddle
Im[λ]
Re[λ]
y
x
center
Im[λ]
Re[λ]
y
x

- A **centre** is the special case of a focus with no attraction/repulsion:
  - $\lambda_1, \lambda_2 \in \mathbb{I}$ so $\mathsf{Re}[\lambda_1] = \mathsf{Re}[\lambda_2] = 0$

- The predator-prey model equilibrium (x_{*1}, y_{*1}) classifies as:

	$\gamma < 0$	$\gamma > 0$
$\alpha > 0$	unstable node	saddle
$\alpha < 0$	saddle	stable node

  while the equilibrium (x_{*2}, y_{*2}) classifies as:

	$\gamma < 0$	$\gamma > 0$
$\alpha > 0$	saddle	center
$\alpha < 0$	center	saddle

- In higher dimensions things are just an extension of these:
  - in n dimensions, there will be n eigenvalues. Some might be real, others complex, some positive and others negative, so an equilibrium can be stable and unstable in different directions, and can be like a node, focus, saddle, or centre along different directions. But always:
  - if there are complex eigenvalues they always occur in conjugate pairs,
  - a node has at least two real eigenvalues with the same sign,
  - a saddle has at least two real eigenvalues with different signs,
  - a focus has at least two eigenvalues that are a complex conjugate pair.
  - An equilibrium is typically a composite of these, e.g. a saddle-focus has at least two eigenvalues that are a complex conjugate pair, but not all real parts or real eigenvalues have the same sign.

9 Local stability

Linear systems are actually the only ones we can solve in general, and this is certainly true when we go to multiple dimensions (i.e. multiple variables, like the two population model above).

- But the **linear theory** above would be rather useless if it was only good for solving linear systems.
- As we have seen, linearization provides a local approximation to equilibria more generally, so in a system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$, if $\mathbf{x}_*$ is any point where $\mathbf{f}(\mathbf{x}_*) = 0$, then locally the system looks like

  $$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) \approx \underline{\underline{A}}.(\mathbf{x} - \mathbf{x}_*)$$

  which has solutions

  $$\mathbf{x}(t) \approx \mathbf{x}_* + e^{\underline{\underline{A}}t}.(\mathbf{x}_0 - \mathbf{x}_*)$$

- We rarely use this solution, we just infer the behaviour from the eigenvalues and eigenvectors of $\underline{\underline{A}}$.
- The matrix $\underline{\underline{A}}$ is found from the derivatives of the ODE at the equilibrium:

  $$\text{in 1d} \quad \text{for} \quad \dot{x} = f(x) \quad : \quad A \equiv \lambda = \left.\frac{df(x)}{dx}\right|_{x=x_*} \tag{9.1}$$

  $$\text{in } n\text{d} \quad \text{for} \quad \dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) \quad : \quad \underline{\underline{A}} = \left.\frac{d\mathbf{f}(\mathbf{x})}{d\mathbf{x}}\right|_{\mathbf{x}=\mathbf{x}_*}$$

  $$\& \quad ||\underline{\underline{A}} - \lambda \underline{\underline{I}}|| = 0 \tag{9.2}$$

  where generally $\underline{\underline{A}}$ is the Jacobian matrix

  $$\frac{d\mathbf{f}}{d\mathbf{x}} = \frac{\partial(f, g, ...)}{\partial(x, y, ...)} = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \cdots \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix} \tag{9.3}$$

  with $\mathbf{x} = (x, y, ...)$, $\mathbf{f} = (f, g, ...)$.

[Side Notes:] **Linearization**

Look back at the revision notes:

- Approximating a 1d system about a point x_* we have

$$\dot{x} = f(x) = f(x_*) + f'(x_*)(x - x_*) + ... \tag{9.4}$$

and $f(x_*) = 0$ at an equilibrium, so removing that and dropping higher orders we have

$$\dot{x} = f(x) = A(x - x_*) \tag{9.5}$$

called the *linearization* of the system, where $A = f'(x_*)$ where $f'(x_*) = \frac{d}{dx} f(x)\big|_{x=x_*}$.

- Similarly for an nd system about a point $\mathbf{x}_* = (x_*, y_*, ...)$ we have

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) = \mathbf{f}(\mathbf{x}_*) + \underline{\underline{A}}.(\mathbf{x} - \mathbf{x}_*) + ... \tag{9.6}$$

and $\mathbf{f}(\mathbf{x}_*) = 0$ at an equilibrium, so again removing that and dropping higher orders we have

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) = \underline{\underline{A}}.(\mathbf{x} - \mathbf{x}_*) \tag{9.7}$$

called the *linearization* of the system, where $\underline{\underline{A}}$ is the Jacobian of $\mathbf{f}$ at $\mathbf{x}_*$.

But actually we can say something stronger. The linearization isn't just an approximation, *close enough* to the equilibrium it captures the local behaviour *exactly* (the way the derivative of a graph captures its gradient) . . .

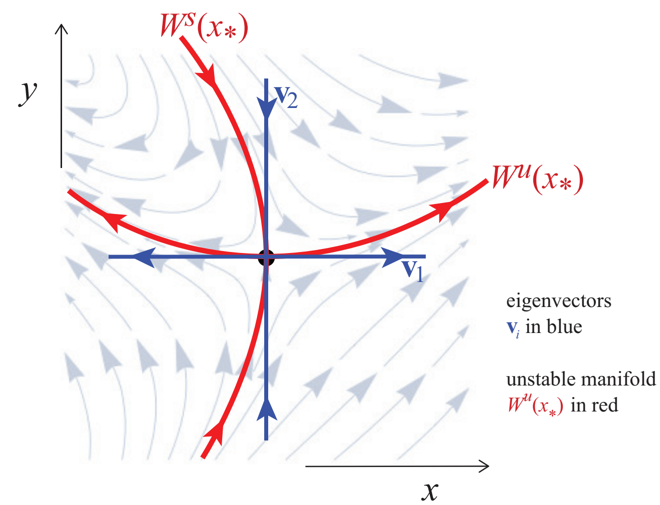
10 Stable and unstable manifolds

The linearization, and the eigenvectors, only tell us the *local* picture.
What happens as you look further away from an equilibrium?

- As we follow the eigenvector direction out away from the equilibrium, we follow an orbit that begins to bend (typically, if there are nonlinear terms in the ODE).
- E.g. take a simple saddle $\dot{x} = x$, $\dot{y} = -y$, and add some nonlinear terms, say

$$\dot{x} = x + y^2 \; ,$$
$$\dot{y} = -y + x^2$$

The figure shows the eigenvectors pointing along the axes from the saddle at the origin. There is an orbit that runs along each eigenvector at the saddle itself, but further away becomes curved.



- Let x_* be a saddle point of the vector field $\dot{x} = f(x)$. The set of all points "ending up at x_*" under the flow of Φ_t of the vector field

$$W^s(x_*) = \{x \in \mathbb{R}^n \mid \Phi_t(x) \to x_* \;\text{ as }\; t \to +\infty\} \; , \tag{10.1}$$

  is called the *stable manifold* of x_*.

- Similarly, the set of all points "coming from x_*"

$$W^u(x_*) = \{x \in \mathbb{R}^n \mid \Phi_t(x) \to x_* \;\text{ as }\; t \to -\infty\} \; , \tag{10.2}$$

  is called the *unstable manifold* of x_*.

- The dimensions of the stable manifolds, plus the dimensions of the unstable manifolds, of an equilibrium, must add up to the total dimension of the system.

Invariant manifolds are hard to find in general but ...

[Side Notes:] **Theorem**

Let x_* be a saddle point of $\dot{x} = f(x)$. Then x_* is also a saddle point of the linearised system

$$\dot{x} = A(x - x_*).$$

Let $E^s(x_*)$ and $E^u(x_*)$ denote the stable and unstable eigenspaces of the linearised system (the directions the eigenvectors point along). In a small enough neighbourhood U of x_*, there exists a local piece of $W^s(x_*)$, that is a smooth manifold that is a graph of some function $h : E^s(x_*) \to W^s(x_*)$. Furthermore, $W^s(x_*)$ is tangent to $E^s(x_*)$ at x_*. The same is true for the local unstable manifold $W^u(x_*)$ which is defined in the same way.

So this says the stable and unstable manifolds, whatever shape they have, are tangent to the eigenvectors at an equilibrium, with the appropriate stability (a stable manifold if the eigenvalue along that direction has negative real part, an unstable manifold if the eigenvalue along that direction has positive real part).

Basin of attraction

In a nonlinear system there might be many equilibria, some stable and some unstable.

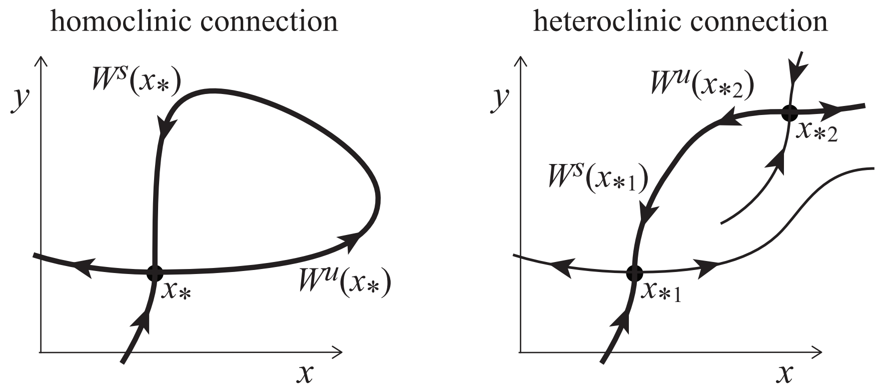
- Around each stable equilibrium there will be a region of orbits that are all attracted into that equilibrium. This is called its **basin of attraction**.
- The size and shape of the basin of attraction is typically determined by the positions of the stable and unstable manifolds from the various equilibria. These can be hard to find exactly.

Connections

The stable and unstable manifolds of two equilibria can connect to each other.

- A **homoclinic connection** occurs when the unstable manifold of an equilibrium connects back to the stable manifold of the same equilibrium.

- A **heteroclinic connection** occurs when the unstable manifold of one equilibrium connects to the stable manifold of another equilibrium.

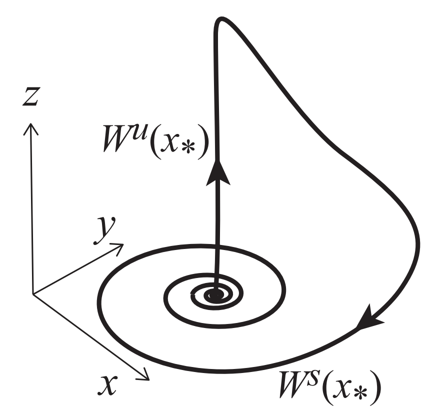


- Note in each case the dimension of the stable and unstable manifolds do not have to be the same.

- E.g. The 'Shilnikov bifurcation' is a homoclinic connection where a one-dimensional unstable manifold connects back to the two-dimensional stable manifold of a saddle-focus (two complex eigenvalues with negative real part, and a positive real eigenvalue), for example in the Rössler system

$$\begin{aligned}\dot{x} &= -y - z\\ \dot{y} &= x + ay\\ \dot{z} &= bx - cz + xz\end{aligned}$$

with $a = 0.38$, $b = 0.3$, $c = 4.820$.



11 Similar systems

We have seen from the cut-off population model, and the predatory-prey model, that:

- the linearization gives a local approximation around the equilibrium, or put another way,
- the system looks *similar to* the linearization near the equilibrium.

This idea of two systems looking *similar* is extremely powerful.
We call it **topological equivalence**.

- Essentially two systems are topologically equivalent if we can find a homeomorphism between them — a map that continuously (and invertibly) warps the orbits of one system into those of another.

> **[Side Notes:] Topological equivalence**
>
> Suppose we have two vector fields
>
> $$\dot{x} = f(x) \qquad \text{and} \qquad \dot{y} = g(y) \tag{11.1}$$
>
> where $x \in U$ and $y \in V$, on domains $U, V \subset \mathbb{R}^n$. Then these are *topologically equivalent* if we can find a continuous and invertible map (a *homeomorphism*) $h : U \to V$ that maps orbits of one system to the other, respecting the direction of time.

- An important concept in topological equivalence is hyperbolicity. An equilibrium is **hyperbolic** if none of its eigenvalues lie on the imaginary axis.

[Side Notes:] Theorem: on Topological Equivalence for linear flows

Consider the two linear vector fields

$$\dot{x} = Ax \qquad \text{and} \qquad \dot{x} = Bx \tag{11.2}$$

where $x \in \mathbb{R}^n$, and A and B are $n \times n$ matrices. Let $n_{\pm}(A)$ denote the number of eigenvalues of a matrix A that have a $\pm$ve real part.
Then these systems are topologically equivalent if and only if

$$n_+(A) = n_+(B) \neq 0 \qquad \text{and} \qquad n_-(A) = n_-(B) \neq 0\,.$$

- Now that isn't much use if we can't extend it beyond linear systems. So we need to be able to describe topological equivalence, between nonlinear systems, or between a nonlinear system and its linearization. The next theorem does this.

[Side Notes:] **Theorem of Hartman & Grobman**

If the system

$$\dot{x} = f(x) \tag{11.3}$$

has a hyperbolic equilibrium at $x = x_*$, then there exists a neighbourhood U of x_* such that the system on U is topologically equivalent to the linearised system

$$\dot{y} = Ay \tag{11.4}$$

on an (arbitrary) neighbourhood V of the origin, where A is the Jacobian of the first system at x_*, and $y = x - x_*$.

12 Stability and genericity

- If a small change in a system (e.g. a small change in its parameter values) results in a topologically equivalent system, we say it is **structurally stable**.

- We say a system is **generic** if it occurs '*typically*', i.e. has a nonzero probability of occurring in a given system (or its occurrence is 'not measure zero').

- Genericity therefore depends on the *class* of systems we are talking about — the defining conditions. E.g. is the system in $\mathbb{R}$ or $\mathbb{R}^n$ or only a limited domain $x > 0$, is the system conservative, reversible, does it have a symmetry, and so on? If so, we can talk about genericity within a given class of system.

- Usually a generic system is structurally stable, otherwise any small change will result in a different system, so the system has a zero-measure chance of occurring.

- If a major structural (qualtitative/topological) change does take place we'll call it a *bifurcation*.