

Question 2: Regular Perturbation of Linear Oscillator

Complete Solution with Exact Comparison

Asymptotics Course — Sheet 6

Problem Statement

Obtain a two-term expansion when $\epsilon \ll 1$ for the solution of:

$$\frac{d^2 f}{dt^2} + f = \epsilon \frac{df}{dt}, \quad f(0) = 1, \quad \frac{df}{dt}(0) = \epsilon^2 \quad (1)$$

Compare this expansion with that obtained by expanding the exact solution.

1 Step 1: Problem Classification and Method Selection

Form Recognition

The ODE has the structure:

$$\frac{d^2 f}{dt^2} + f = \epsilon \frac{df}{dt} \quad (2)$$

with initial conditions:

$$f(0) = 1 \quad (3)$$

$$f'(0) = \epsilon^2 \quad (4)$$

- **STAGE X (What we have):** This is a second-order linear ODE with a small parameter ϵ multiplying the first derivative term. The unperturbed equation ($\epsilon = 0$) is the simple harmonic oscillator $f'' + f = 0$, which has well-defined solutions.
- **STAGE Y (Why this classification):** When $\epsilon = 0$, we obtain $f'' + f = 0$ with $f(0) = 1$ and $f'(0) = 0$. This has the solution $f_0(t) = \cos t$. The perturbed problem should smoothly approach this solution as $\epsilon \rightarrow 0$. The highest derivative is NOT multiplied by ϵ , so this is a **regular perturbation problem**, not a singular one.
- **STAGE Z (What this means):** We can use a standard power series expansion in ϵ . The solution will remain smooth and bounded as $\epsilon \rightarrow 0$. We expect the expansion to be uniformly valid for moderate times (though we must check for secular terms).

2 Step 2: Perturbation Expansion Setup

Expansion Ansatz

We seek a solution of the form:

$$f(t) = f_0(t) + \epsilon f_1(t) + \epsilon^2 f_2(t) + \mathcal{O}(\epsilon^3) \quad (5)$$

- **STAGE X (What we're doing):** We assume the solution can be expanded as a power series in ϵ , where each coefficient function $f_n(t)$ is independent of ϵ .

- **STAGE Y (Why this works):** For regular perturbation problems, the solution depends smoothly on ϵ . By Taylor's theorem for functions of a parameter, such a series expansion exists and converges for sufficiently small ϵ .
- **STAGE Z (Next step):** Substitute this ansatz into both the ODE and the initial conditions, then collect terms by powers of ϵ .

3 Step 3: Substitute and Collect Terms

Derivatives of the Ansatz

From equation (6):

$$f'(t) = f'_0(t) + \epsilon f'_1(t) + \epsilon^2 f'_2(t) + \mathcal{O}(\epsilon^3) \quad (6)$$

$$f''(t) = f''_0(t) + \epsilon f''_1(t) + \epsilon^2 f''_2(t) + \mathcal{O}(\epsilon^3) \quad (7)$$

Substitution into ODE

Insert into $f'' + f = \epsilon f'$:

$$\begin{aligned} [f''_0 + \epsilon f''_1 + \epsilon^2 f''_2 + \mathcal{O}(\epsilon^3)] + [f_0 + \epsilon f_1 + \epsilon^2 f_2 + \mathcal{O}(\epsilon^3)] \\ = \epsilon [f'_0 + \epsilon f'_1 + \epsilon^2 f'_2 + \mathcal{O}(\epsilon^3)] \end{aligned} \quad (8)$$

Rearranging:

$$(f''_0 + f_0) + \epsilon(f''_1 + f_1 - f'_0) + \epsilon^2(f''_2 + f_2 - f'_1) + \mathcal{O}(\epsilon^3) = 0 \quad (9)$$

- **STAGE X (What we observe):** After substitution, we have a polynomial in ϵ equal to zero.
- **STAGE Y (Why we can equate coefficients):** For the equation to hold for all small ϵ , each power of ϵ must independently vanish. This is because polynomials can only be identically zero if all coefficients are zero.
- **STAGE Z (Result):** We obtain a hierarchy of ODEs, one for each order of ϵ .

Initial Conditions Hierarchy

Expand $f(0) = 1$:

$$f_0(0) + \epsilon f_1(0) + \epsilon^2 f_2(0) + \dots = 1 \quad (10)$$

Expand $f'(0) = \epsilon^2$:

$$f'_0(0) + \epsilon f'_1(0) + \epsilon^2 f'_2(0) + \dots = \epsilon^2 \quad (11)$$

- **STAGE X (Initial conditions by order):** Equating powers of ϵ in the initial conditions:

$$\begin{aligned} \mathcal{O}(1) : \quad f_0(0) &= 1, & f'_0(0) &= 0 \\ \mathcal{O}(\epsilon) : \quad f_1(0) &= 0, & f'_1(0) &= 0 \\ \mathcal{O}(\epsilon^2) : \quad f_2(0) &= 0, & f'_2(0) &= 1 \end{aligned}$$

- **STAGE Y (Why this structure):** The condition $f'(0) = \epsilon^2$ means the initial velocity is of order ϵ^2 , so it doesn't appear until the second order. This is crucial for determining the correct initial conditions at each order.
- **STAGE Z (Ready to solve):** We now have complete ODE problems at each order.

4 Step 4: Solve Order by Order

Order $\mathcal{O}(1)$: Leading Order Problem

ODE and Initial Conditions:

$$f_0'' + f_0 = 0, \quad f_0(0) = 1, \quad f_0'(0) = 0 \quad (12)$$

General Solution: The characteristic equation is $r^2 + 1 = 0 \Rightarrow r = \pm i$, giving:

$$f_0(t) = A \cos t + B \sin t \quad (13)$$

Apply Initial Conditions:

$$f_0(0) = A = 1 \Rightarrow A = 1 \quad (14)$$

$$f_0'(t) = -A \sin t + B \cos t \Rightarrow f_0'(0) = B = 0 \Rightarrow B = 0 \quad (15)$$

Leading Order Solution:

$$\boxed{f_0(t) = \cos t} \quad (16)$$

- **STAGE X (What we found):** The unperturbed solution is pure harmonic oscillation with unit amplitude and zero initial velocity.
- **STAGE Y (Physical meaning):** This represents an undamped oscillator starting from maximum displacement. The perturbation term $\epsilon f'$ represents weak damping.
- **STAGE Z (Next order):** This solution becomes the forcing term for the next order.

Order $\mathcal{O}(\epsilon)$: First Correction

ODE and Initial Conditions:

$$f_1'' + f_1 = f_0' = -\sin t, \quad f_1(0) = 0, \quad f_1'(0) = 0 \quad (17)$$

- **STAGE X (What we have):** An inhomogeneous ODE where the forcing term is $-\sin t$, which is a solution of the homogeneous equation.
- **STAGE Y (Why this matters - Resonance):** The forcing frequency equals the natural frequency. This is a resonance condition that will produce a **secular term** (a term growing linearly with time).
- **STAGE Z (Expect secular growth):** The particular solution will contain a term proportional to $t \cos t$.

Solution Method:

The homogeneous solution is:

$$f_{1,h}(t) = C \cos t + D \sin t \quad (18)$$

For the particular solution, since $\sin t$ is a homogeneous solution, we use the ansatz:

$$f_{1,p}(t) = t(E \cos t + F \sin t) \quad (19)$$

Compute derivatives:

$$f_{1,p}'(t) = (E \cos t + F \sin t) + t(-E \sin t + F \cos t) \quad (20)$$

$$= E \cos t + F \sin t - Et \sin t + Ft \cos t \quad (21)$$

$$f_{1,p}''(t) = -E \sin t + F \cos t + [-E \sin t - Et \cos t + F \cos t - Ft \sin t] \quad (22)$$

$$= -2E \sin t + 2F \cos t - Et \cos t - Ft \sin t \quad (23)$$

Substitute into ODE:

$$\begin{aligned} f_{1,p}'' + f_{1,p} &= [-2E \sin t + 2F \cos t - Et \cos t - Ft \sin t] \\ &\quad + [Et \cos t + Ft \sin t] = -\sin t \end{aligned} \quad (24)$$

Simplifying:

$$-2E \sin t + 2F \cos t = -\sin t \quad (25)$$

Coefficient matching:

$$\cos t : 2F = 0 \Rightarrow F = 0 \quad (26)$$

$$\sin t : -2E = -1 \Rightarrow E = \frac{1}{2} \quad (27)$$

Particular solution:

$$f_{1,p}(t) = \frac{t}{2} \cos t \quad (28)$$

General solution:

$$f_1(t) = C \cos t + D \sin t + \frac{t}{2} \cos t \quad (29)$$

Apply initial conditions:

$$f_1(0) = C = 0 \Rightarrow C = 0 \quad (30)$$

$$f_1'(t) = -C \sin t + D \cos t + \frac{1}{2} \cos t - \frac{t}{2} \sin t \quad (31)$$

$$f_1'(0) = D + \frac{1}{2} = 0 \Rightarrow D = -\frac{1}{2} \quad (32)$$

First Order Correction:

$$f_1(t) = -\frac{1}{2} \sin t + \frac{t}{2} \cos t \quad (33)$$

- **STAGE X (What we found):** The first correction contains both a bounded oscillatory part $-\frac{1}{2} \sin t$ and a secular term $\frac{t}{2} \cos t$ that grows without bound.
- **STAGE Y (Why secular terms appear):** The forcing term $-\sin t$ is in resonance with the natural frequency. The secular term represents the cumulative effect of resonant forcing.
- **STAGE Z (Validity limitation):** This expansion breaks down when $t \sim \mathcal{O}(1/\epsilon)$ because then the "correction" term $\epsilon f_1 \sim \epsilon t \sim 1$ becomes comparable to the leading term. For moderate times $t = \mathcal{O}(1)$, the expansion remains valid.

5 Step 5: Two-Term Perturbation Result

Combining the results:

$$f(t) = \cos t + \epsilon \left(-\frac{1}{2} \sin t + \frac{t}{2} \cos t \right) + \mathcal{O}(\epsilon^2) \quad (34)$$

This can be rewritten as:

$$f(t) = \cos t - \frac{\epsilon}{2} \sin t + \frac{\epsilon t}{2} \cos t + \mathcal{O}(\epsilon^2) \quad (35)$$

- **STAGE X (Two-term expansion complete):** We have the leading order and first correction.
- **STAGE Y (Domain of validity):** This expansion is uniformly valid for $t \ll 1/\epsilon$. For longer times, the secular term dominates and the expansion fails.
- **STAGE Z (Next task):** Compare with the exact solution.

6 Step 6: Exact Solution

Solve the Characteristic Equation

The ODE is:

$$f'' - \epsilon f' + f = 0 \quad (36)$$

Try $f = e^{rt}$:

$$r^2 - \epsilon r + 1 = 0 \quad (37)$$

Using the quadratic formula:

$$r = \frac{\epsilon \pm \sqrt{\epsilon^2 - 4}}{2} = \frac{\epsilon \pm \sqrt{-(4 - \epsilon^2)}}{2} = \frac{\epsilon \pm i\sqrt{4 - \epsilon^2}}{2} \quad (38)$$

- **STAGE X (Complex roots):** The discriminant is negative for small ϵ , giving complex conjugate roots.
- **STAGE Y (Oscillatory solution):** Complex roots indicate oscillatory behavior with exponential amplitude modulation.
- **STAGE Z (Exact form):** We can write the solution in terms of exponentials or trigonometric functions.

Let:

$$\alpha = \frac{\epsilon}{2} \quad (39)$$

$$\omega = \frac{\sqrt{4 - \epsilon^2}}{2} = \frac{1}{2}\sqrt{4 - \epsilon^2} \quad (40)$$

General solution:

$$f(t) = e^{\alpha t}(A \cos(\omega t) + B \sin(\omega t)) \quad (41)$$

Apply Initial Conditions

Condition 1: $f(0) = 1$

$$f(0) = e^0(A \cos 0 + B \sin 0) = A = 1 \Rightarrow A = 1 \quad (42)$$

Condition 2: $f'(0) = \epsilon^2$

First compute $f'(t)$:

$$f'(t) = \alpha e^{\alpha t}(A \cos(\omega t) + B \sin(\omega t)) \quad (43)$$

$$+ e^{\alpha t}(-A\omega \sin(\omega t) + B\omega \cos(\omega t)) \quad (44)$$

At $t = 0$:

$$f'(0) = \alpha A + B\omega = \frac{\epsilon}{2} + B\omega = \epsilon^2 \quad (45)$$

Solving for B :

$$B\omega = \epsilon^2 - \frac{\epsilon}{2} \Rightarrow B = \frac{\epsilon^2 - \frac{\epsilon}{2}}{\omega} = \frac{2\epsilon^2 - \epsilon}{2\omega} \quad (46)$$

Exact solution:

$$f_{\text{exact}}(t) = e^{\frac{\epsilon t}{2}} \left[\cos \left(\frac{t}{2} \sqrt{4 - \epsilon^2} \right) + \frac{2\epsilon^2 - \epsilon}{2\omega} \sin \left(\frac{t}{2} \sqrt{4 - \epsilon^2} \right) \right] \quad (47)$$

where $\omega = \frac{1}{2}\sqrt{4 - \epsilon^2}$.

7 Step 7: Expand the Exact Solution

Expand Each Component

1. Exponential term:

$$e^{\frac{\epsilon t}{2}} = 1 + \frac{\epsilon t}{2} + \frac{\epsilon^2 t^2}{8} + \mathcal{O}(\epsilon^3) \quad (48)$$

2. Frequency term:

$$\omega = \frac{1}{2}\sqrt{4 - \epsilon^2} = \frac{1}{2} \cdot 2\sqrt{1 - \frac{\epsilon^2}{4}} \quad (49)$$

$$= 1 \cdot \left(1 - \frac{\epsilon^2}{8} + \mathcal{O}(\epsilon^4) \right) = 1 - \frac{\epsilon^2}{8} + \mathcal{O}(\epsilon^4) \quad (50)$$

- **STAGE X (Binomial expansion):** Used $(1+x)^{1/2} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \dots$ with $x = -\epsilon^2/4$.
- **STAGE Y (Why to order ϵ^2):** We need ω accurate to $\mathcal{O}(\epsilon^2)$ to get f correct to $\mathcal{O}(\epsilon)$.
- **STAGE Z (Apply to trig functions):** The argument of sine and cosine becomes $t(1 - \epsilon^2/8) = t - \epsilon^2 t/8$.

3. Cosine term:

$$\cos(\omega t) = \cos \left(t \left[1 - \frac{\epsilon^2}{8} \right] \right) = \cos \left(t - \frac{\epsilon^2 t}{8} \right) \quad (51)$$

$$= \cos t \cos \left(\frac{\epsilon^2 t}{8} \right) + \sin t \sin \left(\frac{\epsilon^2 t}{8} \right) \quad (52)$$

$$= \cos t \left[1 - \frac{1}{2} \left(\frac{\epsilon^2 t}{8} \right)^2 + \dots \right] + \sin t \left[\frac{\epsilon^2 t}{8} + \dots \right] \quad (53)$$

$$= \cos t + \frac{\epsilon^2 t}{8} \sin t + \mathcal{O}(\epsilon^4) \quad (54)$$

4. Coefficient of sine:

$$\frac{2\epsilon^2 - \epsilon}{2\omega} = \frac{2\epsilon^2 - \epsilon}{2(1 - \epsilon^2/8 + \mathcal{O}(\epsilon^4))} \quad (55)$$

$$= \frac{2\epsilon^2 - \epsilon}{2} (1 + \epsilon^2/8 + \mathcal{O}(\epsilon^4)) \quad (56)$$

$$= (\epsilon^2 - \frac{\epsilon}{2})(1 + \epsilon^2/8) + \mathcal{O}(\epsilon^3) \quad (57)$$

$$= -\frac{\epsilon}{2} + \epsilon^2 + \mathcal{O}(\epsilon^3) \quad (58)$$

5. Sine term:

$$\sin(\omega t) = \sin\left(t - \frac{\epsilon^2 t}{8}\right) \quad (59)$$

$$= \sin t \cos\left(\frac{\epsilon^2 t}{8}\right) - \cos t \sin\left(\frac{\epsilon^2 t}{8}\right) \quad (60)$$

$$= \sin t + \mathcal{O}(\epsilon^4) \quad (61)$$

Combine All Terms

$$f_{\text{exact}}(t) = \left(1 + \frac{\epsilon t}{2} + \mathcal{O}(\epsilon^2)\right) \times \left[\cos t + \frac{\epsilon^2 t}{8} \sin t + \left(-\frac{\epsilon}{2} + \mathcal{O}(\epsilon^2)\right) \sin t + \mathcal{O}(\epsilon^2)\right] \quad (62)$$

Expanding to $\mathcal{O}(\epsilon)$:

$$f_{\text{exact}}(t) = \left(1 + \frac{\epsilon t}{2}\right) \left[\cos t - \frac{\epsilon}{2} \sin t\right] + \mathcal{O}(\epsilon^2) \quad (63)$$

$$= \cos t - \frac{\epsilon}{2} \sin t + \frac{\epsilon t}{2} \cos t - \frac{\epsilon^2 t}{4} \sin t + \mathcal{O}(\epsilon^2) \quad (64)$$

$$= \cos t - \frac{\epsilon}{2} \sin t + \frac{\epsilon t}{2} \cos t + \mathcal{O}(\epsilon^2) \quad (65)$$

Expanded exact solution:

$$f_{\text{exact}}(t) = \cos t - \frac{\epsilon}{2} \sin t + \frac{\epsilon t}{2} \cos t + \mathcal{O}(\epsilon^2) \quad (66)$$

8 Step 8: Comparison and Verification

Direct Comparison

Perturbation expansion result (from Step 5):

$$f_{\text{pert}}(t) = \cos t - \frac{\epsilon}{2} \sin t + \frac{\epsilon t}{2} \cos t + \mathcal{O}(\epsilon^2) \quad (67)$$

Expanded exact solution (from Step 7):

$$f_{\text{exact}}(t) = \cos t - \frac{\epsilon}{2} \sin t + \frac{\epsilon t}{2} \cos t + \mathcal{O}(\epsilon^2) \quad (68)$$

PERFECT AGREEMENT: The two-term perturbation expansion matches the expansion of the exact solution identically to $\mathcal{O}(\epsilon)$.

- **STAGE X (What we verified):** Both methods give identical results through order ϵ .
- **STAGE Y (Why this confirms validity):** The perturbation method is correct for this regular problem. The secular term $\frac{\epsilon t}{2} \cos t$ is a genuine feature of the exact solution, not an artifact of the method.
- **STAGE Z (Physical interpretation):** The damping term $\epsilon f'$ causes both:
 1. A phase shift (the $-\frac{\epsilon}{2} \sin t$ term)
 2. A slow amplitude growth that appears as secular growth at this order but is actually bounded by the exponential $e^{\epsilon t/2} \approx 1 + \epsilon t/2$ for small t .

Key Observations

1. Secular Term Resolution:

The secular term $\frac{\epsilon t}{2} \cos t$ appears unbounded as $t \rightarrow \infty$, but in the exact solution it's part of:

$$e^{\frac{\epsilon t}{2}} \cos(\omega t) \approx \left(1 + \frac{\epsilon t}{2}\right) \cos t \quad (69)$$

The exponential growth is actually bounded and represents weak growth due to the negative damping coefficient (the term enters as $-\epsilon f'$ on the RHS, which is anti-damping for $f' < 0$).

2. Validity Domain:

The expansion is uniformly valid for:

$$t \ll \frac{1}{\epsilon} \quad (70)$$

For $t = \mathcal{O}(1/\epsilon)$, the $\mathcal{O}(\epsilon)$ term becomes $\mathcal{O}(1)$ and the expansion breaks down. The multiple scales method would be needed for longer times.

3. Initial Condition Impact:

The unusual initial condition $f'(0) = \epsilon^2$ means:

- No contribution to $f'_0(0)$ or $f'_1(0)$
- The initial velocity is extremely small, entering only at $\mathcal{O}(\epsilon^2)$
- This affects the coefficient B in the exact solution

9 Verification Checklist

- ✓ **Problem classified:** Regular perturbation (highest derivative not multiplied by ϵ)
- ✓ **Expansion ansatz:** $f = f_0 + \epsilon f_1 + \mathcal{O}(\epsilon^2)$
- ✓ **Order $\mathcal{O}(1)$:** $f_0 = \cos t$
- ✓ **Order $\mathcal{O}(\epsilon)$:** $f_1 = -\frac{1}{2} \sin t + \frac{t}{2} \cos t$ (with secular term)
- ✓ **Exact solution found:** Using characteristic equation method
- ✓ **Exact solution expanded:** Careful Taylor expansion to $\mathcal{O}(\epsilon)$
- ✓ **Perfect agreement:** Perturbation and exact expansion match
- ✓ **Secular term explained:** Genuine feature from exponential growth $e^{\epsilon t/2}$
- ✓ **Validity domain identified:** $t \ll 1/\epsilon$

This solution demonstrates that regular perturbation theory correctly captures the behavior of damped oscillators for moderate times, including secular terms that represent physical exponential growth.