Asymptotics 2025/2026 Sheet 9[1]

Multiple Scales

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1. For the first-order nonlinear differential equation
$$\frac{df}{dt} - f = \varepsilon f^2 e^{-t}$$
with $\varepsilon \ll 1$ and initial condition $f(0) = 1$ determine an approximation by using the multiple-scale method. Show that the resulting expression is the exact solution.

2. For the lightly damped linear oscillator
$$\ddot{x} + 2\varepsilon\dot{x} + \omega^2 x = 0$$
(a) use multiple-scale analysis to the find the solution in second order, i.e. go one order beyond what was done in the lecture.
(b) compare your result with the exact solution.

3. Consider the Duffing oscillator with a damping term
$$\frac{d^2y}{dt^2} + y + k\frac{dy}{dt} + \varepsilon y^3 = 0$$
with initial conditions $y(0) = a$ and $y'(0) = 0$. Determine an approximate solution for $\varepsilon, k \ll 1$ by using the averaging method.

4. For the differential equation
$$\frac{d^2y}{dt^2} + y + \varepsilon|y'|y' = 0$$
with initial conditions $y(0) = 1$, $y'(0) = 0$, and $\varepsilon \ll 1$ find a uniformly valid expansion by using the averaging method.

5. By using the multiple-scale analysis, investigate the behaviour of solutions of the Mathieu equation:
$$\ddot{x} + (a - 2q\cos 2t)x = 0\ .$$
In particular, discuss the regions where the solutions are stable. You are adviced to treat this as partly a literature study. (It is discussed for example in Bender/Orszag). It is an example of the class of o.d.e.s with periodic coefficients.