Asymptotics 2025/2026 — Solutions for Sheet 9[1]

Multiple scales and averaging

Problem 1: This problem has been treated by a regular perturbation expansion on sheet 5 with the result that $f(t) = e^t + \varepsilon t e^t + \mathcal{O}(\varepsilon^2)$. This expansion is not uniformly valid since the second term on the RHS is bigger than the first term if $t > 1/\varepsilon$. The multiple-scale method can be used to obtain a uniformly valid approximation which in this example is even exact. We assume that f depends on two time-scales, $t_0 = t$ and $t_1 = \varepsilon t$. By the chain rule we have

$$\frac{d}{dt} = \frac{\partial}{\partial t_0} + \varepsilon \frac{\partial}{\partial t_1} \; ,$$

and after insertion into the o.d.e. we obtain

$$\frac{\partial f}{\partial t_0} + \varepsilon \frac{\partial f}{\partial t_1} - f = \varepsilon f^2 e^{-t_0} \; .$$

We expand f in a power series in ε, $f(t_0, t_1) = f(t_0, t_1) + \varepsilon f(t_0, t_1) + \ldots$, insert it into differential equation and initial condition and consider the problem at different orders of ε. At leading order we have

$$\mathcal{O}(\varepsilon^0): \qquad \frac{\partial f_0}{\partial t_0} - f_0 = 0 \; , \qquad f_0(0,0) = 1 \; ,$$

which has the solution $f_0(t_0, t_1) = A(t_1) e^{t_0}$ with $A(0) = 1$. As expected, the solution is not completely determined, we need the further condition that in the next order there should be no secular term. At the next order

$$\mathcal{O}(\varepsilon^1): \qquad \frac{\partial f_1}{\partial t_0} - f_1 = f_0^2 e^{-t_0} - \frac{\partial f_0}{\partial t_1} = A^2 e^{t_0} - \frac{dA}{dt_1} e^{t_0} \; .$$

Both terms on the RHS lead to secular terms since they are solutions of the homogeneous equation (i.e. the LHS). The requirement that they have to vanish leads to

$$A^2 - \frac{dA}{dt_1} = 0 \qquad \Longrightarrow \qquad A(t_1) = \frac{1}{c - t_1} \; ,$$

and the initial condition $A(0) = 1$ yields $c = 1$. The complete first-order approximation is

$$f(t) = \frac{1}{1 - t_1} e^{t_0} = \frac{1}{1 - \varepsilon t} e^t \; .$$

By inserting it into the differential equation and initial condition it can be checked that this is the exact solution.

Problem 2: (a) The exact solution was obtained in the lecture and is given by

$$x(t) = e^{-\varepsilon t} \cos\left(\sqrt{\omega^2 - \varepsilon^2}\, t\right) + \frac{\varepsilon}{\sqrt{\omega^2 - \varepsilon^2}}\, e^{-\varepsilon t} \sin\left(\sqrt{\omega^2 - \varepsilon^2}\, t\right) \; .$$

To approximate it by the multiple scale method we assume that x is a function of $t_0 = t$ and $t_1 = \varepsilon t$. Then the differential equation becomes

$$\frac{\partial^2 x}{\partial t_0^2} + \omega^2 x = -2\varepsilon \left(\frac{\partial x}{\partial t_0} + \frac{\partial^2 x}{\partial t_0 \partial t_1} \right) - \varepsilon^2 \left(2\frac{\partial x}{\partial t_1} + \frac{\partial^2 x}{\partial t_1^2} \right) ,$$

with initial conditions

$$x(0,0) = 1 \qquad \text{and} \qquad \frac{\partial x}{\partial t_0}(0,0) + \varepsilon \frac{\partial x}{\partial t_1}(0,0) = 0 .$$

We expand x in the form $x = x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \dots$, insert it into differential equation and initial conditions and consider the problem at different orders of ε. At leading order

$$\mathcal{O}(\varepsilon^0) : \qquad \frac{\partial^2 x_0}{\partial t_0^2} + \omega^2 x_0 = 0 , \qquad x_0(0,0) = 1 , \qquad \frac{\partial x_0}{\partial t_0}(0,0) = 0 .$$

The solution is

$$x_0(t_0, t_1) = A(t_1) \cos(\omega t_0) + B(t_1) \sin(\omega t_0) , \qquad A(0) = 1 , \qquad B(0) = 0 .$$

At next order we have

$$\mathcal{O}(\varepsilon^1) : \ \frac{\partial^2 x_1}{\partial t_0^2} + \omega^2 x_1 = -2\frac{\partial x_0}{\partial t_0} - 2\frac{\partial^2 x_0}{\partial t_0 \partial t_1} = 2\omega \sin(\omega t_0) \left(A + \frac{dA}{dt_1} \right) - 2\omega \cos(\omega t_0) \left(B + \frac{dB}{dt_1} \right)$$

with initial conditions

$$x_1(0,0) = 0 , \qquad \frac{\partial x_1}{\partial t_0}(0,0) = -\frac{\partial x_0}{\partial t_1}(0,0) .$$

All terms on the RHS of the differential equation lead to secular terms since they are solutions of the homogeneous equation. The condition that they have to vanish leads to equations for A and B. Taking into regard the initial conditions for A and B, the solutions are

$$A(t_1) = e^{-t_1} \qquad \text{and} \qquad B(t_1) = 0 .$$

Now we solve the equation for x_1. It is

$$x_1(t_0, t_1) = C(t_1) \cos(\omega t_0) + D(t_1) \sin(\omega t_0) , \qquad C(0) = 0 , \qquad D(0) = -\frac{1}{\omega}\frac{dA}{dt_1}(0) = \frac{1}{\omega} .$$

To determine C and D we have to go to the next order

$$\begin{aligned} \mathcal{O}(\varepsilon^2) : \frac{\partial^2 x_0}{\partial t_0^2} + \omega^2 x &= -2\frac{\partial x_1}{\partial t_0} - 2\frac{\partial^2 x_1}{\partial t_0 \partial t_1} - 2\frac{\partial x_0}{\partial t_1} - \frac{\partial^2 x_0}{\partial t_1^2} \\ &= 2\omega \sin(\omega t_0) \left(C + \frac{dC}{dt_1} \right) - \cos(\omega t_0) \left(2\omega D + 2\omega \frac{dD}{dt_1} - e^{-t_1} \right) . \end{aligned}$$

Again the RHS has to vanish, and with the initial conditions we obtain

$$C(t_1) = 0 , \qquad D(t_1) = \frac{1}{\omega} e^{-t_1} + \frac{t_1}{2\omega} e^{-t_1} .$$

The complete approximation is

$$x(t) \approx e^{-\varepsilon t}\cos(\omega t) + \left(\frac{\varepsilon}{\omega} + \frac{\varepsilon^2 t}{2\omega}\right) e^{-\varepsilon t}\sin(\omega t)\ .$$

(b) The same result is obtained by expanding the terms multiplying the exponential functions in the exact solution up to order ε^2. We see that here a secular term arises which cannot be avoided with our choice of t_0 and t_1. Its origin is the expansion of the cosine in the exact solution. To avoid it we would have to choose a different slow time t_0 of the form $t_0 = \delta(\varepsilon)t$. Then the requirement that the expression for $D(t_1)$ should have no secular term would give a condition for the expansion of δ in ε. The intention is that the slow time t_0 should finally correspond to the arguments of sine and cosine so that these functions do not need to be expanded.

Problem 3: According to the lecture, the approximation to the solution of $y'' + \omega^2 y = \varepsilon f(y', y)$ has the form

$$\begin{aligned} y(t) &= R(t)\cos\mu \\ y'(t) &= -\omega R(t)\sin\mu \end{aligned}$$

where $\mu = \omega t + \Phi(t)$, and $R(t)$ and $\Phi(t)$ are determined by the equations

$$\begin{aligned} R' &= -\frac{1}{\omega}\langle \varepsilon f \sin\mu\rangle \\ \Phi' &= -\frac{1}{\omega R}\langle \varepsilon f \cos\mu\rangle \end{aligned}$$

where $< \cdots >$ denotes the averaging over one period $T = 2\pi/\omega$. Here we have $\omega = 1$ and $\varepsilon f = -ky' - \varepsilon y^3$. Thus the equations for R and Φ are

$$\begin{aligned} R' &= \langle (ky' + \varepsilon y^3)\sin\mu\rangle \\ &= \langle -kR\sin^2\mu + \varepsilon R^3\cos^3\mu\sin\mu\rangle \\ &= \langle -\frac{kR}{2}(1-\cos 2\mu) + \varepsilon R^3\frac{1}{2}(1+\cos 2\mu)\frac{1}{2}\sin 2\mu\rangle \\ &= \langle -\frac{kR}{2}(1-\cos 2\mu) + \frac{\varepsilon R^3}{4}(\sin 2\mu + \frac{1}{2}\sin 4\mu)\rangle \\ &= -\frac{1}{2}kR \end{aligned}$$

and

$$\begin{aligned} \Phi' &= \frac{1}{R}\langle (ky' + \varepsilon y^3)\cos\mu\rangle \\ &= \langle -k\sin\mu\cos\mu + \varepsilon R^2\cos^4\mu\rangle \\ &= \langle -\frac{k}{2}\sin 2\mu + \varepsilon R^2\frac{1}{4}(1+\cos 2\mu)^2\rangle \\ &= \langle -\frac{k}{2}\sin 2\mu + \frac{\varepsilon R^2}{4}(1 + 2\cos 2\mu + \frac{1}{2}(1+\cos 4\mu))\rangle \\ &= \frac{3}{8}\varepsilon R^2 \end{aligned}$$

where we used $\langle \cos n\mu \rangle = \langle \sin n\mu \rangle = 0$ for integer n. From the initial conditions follows that $R(0) = a$ and $\Phi(0) = 0$. The solution for $R(t)$ is

$$R(t) = a \exp\left(-\frac{1}{2}kt\right) ,$$

and that for $\Phi(t)$

$$\Phi(t) = \frac{3\varepsilon a^2}{8k}\left(1 - e^{-kt}\right) .$$

The complete result is

$$y(t) \approx a \exp\left(-\frac{1}{2}kt\right) \cos\left[t + \frac{3\varepsilon a^2}{8k}\left(1 - e^{-kt}\right)\right] .$$

Problem 4: Again we use the averaging method (see problem 3) now with $\omega = 1$ and $\varepsilon f = -\varepsilon|y'|y'$. The equations for $R(t)$ and $\Phi(t)$ are

$$R'(t) = \langle \varepsilon\, |y'|\, y' \sin\mu \rangle = -\langle \varepsilon R^2 |\sin\mu| \sin^2\mu \rangle = -2\frac{\varepsilon R^2}{2\pi}\int_0^\pi \sin^3 t\, dt = -\frac{4\varepsilon R^2}{3\pi}$$

and

$$\Phi'(t) = \frac{1}{R}\langle \varepsilon\, |y'|\, y' \cos\mu \rangle = -\langle \varepsilon R |\sin\mu| \sin\mu\cos\mu \rangle = -2\frac{\varepsilon R}{2\pi}\int_0^\pi \sin t\, \frac{1}{2}\sin 2t\, dt = 0$$

where we used the fact that the integral from π to 2π gives the same result as that from 0 to π. From the initial conditions follows that $R(0) = 1$ and $\Phi(0) = 0$. The solution for $R(t)$ is

$$R(t) = \frac{3\pi}{4\varepsilon t + 3\pi}$$

and for Φ we have $\Phi(t) = 0$, so that the complete approximation is given by

$$y(t) \approx \frac{3\pi}{4\varepsilon t + 3\pi}\cos t .$$

Problem 5: See Bender/Orszag p. 560 onwards; or N.W. MacLachlan, *Theory and application of Mathieu functions* [Queens Library QA405 MAC]; or F.M. Arscott, *Periodic differential equations: An introduction to Mathieu, Lamé and allied functions* [Queens Library QA371 ARS].