

Asymptotics Problem 9.3: Complete Pedagogical Solution

Averaging Method for the Duffing Oscillator with Damping

Problem 1. Consider the Duffing oscillator with a damping term

$$\frac{d^2y}{dt^2} + y + k \frac{dy}{dt} + \varepsilon y^3 = 0$$

with initial conditions $y(0) = a$ and $y'(0) = 0$. Determine an approximate solution for $\varepsilon, k \ll 1$ by using the averaging method.

Solution: Step-by-Step Atomic Breakdown

Step 1: Understanding the Physical System

Strategy: *The Duffing oscillator is a fundamental model in nonlinear dynamics. It describes oscillations in a potential with a cubic nonlinearity, arising from Taylor expansion of symmetric potentials around a stable equilibrium. The damping term $k\dot{y}$ causes energy dissipation. For small ε and k , the solution should resemble a simple harmonic oscillator with slowly varying amplitude and phase.*

Justification: *From Lecture Notes §7.2, equations (447)–(449), the Duffing equation arises when expanding any symmetric potential $V(y) = V(-y)$ around a stable equilibrium:*

$$m\ddot{y} = -V''(0)y - \frac{1}{6}V^{(4)}(0)y^3 + \dots$$

Identifying $\omega^2 = V''(0)/m$ and $\alpha = -V^{(4)}(0)/(6m)$ gives $\ddot{y} + \omega^2 y = \alpha y^3$. Our equation has $\omega = 1$ (since the coefficient of y is 1) and includes both the nonlinear restoring force εy^3 and linear damping $k\dot{y}$.

Step 2: Recasting the Equation in Standard Form

Goal: Write the ODE in the form $\ddot{y} + \omega^2 y = \varepsilon f(y, \dot{y}, t)$ suitable for the averaging method. Rearranging our equation:

$$\ddot{y} + y = -k\dot{y} - \varepsilon y^3.$$

This is of the form:

$$\ddot{y} + \omega^2 y = \varepsilon f(y, \dot{y}),$$

where:

$$\begin{aligned}\omega &= 1, \\ \varepsilon f(y, \dot{y}) &= -k\dot{y} - \varepsilon y^3.\end{aligned}$$

Key Concept: *In the averaging method, we treat both perturbations (damping $k\dot{y}$ and nonlinearity εy^3) as small. Formally, both k and ε are $O(\varepsilon)$ in the sense that they are small parameters causing slow modulation of the oscillator's amplitude and phase.*

Step 3: The Averaging Method Framework

Key Concept: *The Averaging Method of Krylov–Bogoliubov* (Lecture Notes §7.2, equations (456)–(464)):

For the perturbed oscillator $\ddot{y} + \omega^2 y = \varepsilon f(y, \dot{y}, t)$, we make the ansatz:

$$\begin{aligned} y(t) &= R(t) \cos \mu, \\ \dot{y}(t) &= -\omega R(t) \sin \mu, \end{aligned}$$

where $\mu = \omega t + \Phi(t)$, and $R(t)$ (amplitude) and $\Phi(t)$ (phase) are slowly varying functions.

The evolution equations for R and Φ are obtained by averaging over one period $\bar{T} = 2\pi/\omega$:

$$\begin{aligned} \dot{R} &= -\frac{\varepsilon}{\omega} \langle f \sin \mu \rangle, \\ \dot{\Phi} &= -\frac{\varepsilon}{\omega R} \langle f \cos \mu \rangle, \end{aligned}$$

where the average is defined as:

$$\langle g \rangle = \frac{1}{\bar{T}} \int_{t_0}^{t_0 + \bar{T}} g(t) dt = \frac{\omega}{2\pi} \int_0^{2\pi/\omega} g(t) dt.$$

Justification: The averaging method replaces the rapidly oscillating dynamics with their time-averaged effect. Since R and Φ vary slowly compared to the oscillation period, they can be treated as approximately constant during the averaging. This yields the same results as the multiple scales method but through a different approach (Lecture Notes §7.2, after equation (464)).

Step 4: Setting Up the Ansatz

With $\omega = 1$, our ansatz becomes:

$$\begin{aligned} y(t) &= R(t) \cos \mu, \\ \dot{y}(t) &= -R(t) \sin \mu, \end{aligned}$$

where $\mu = t + \Phi(t)$.

The period of unperturbed oscillation is $\bar{T} = 2\pi$.

Step 5: Expressing f in Terms of R and μ

We have $\varepsilon f = -k\dot{y} - \varepsilon y^3$.

Substituting the ansatz:

$$\begin{aligned} y &= R \cos \mu, \\ \dot{y} &= -R \sin \mu, \\ y^3 &= R^3 \cos^3 \mu. \end{aligned}$$

Therefore:

$$\varepsilon f = -k(-R \sin \mu) - \varepsilon R^3 \cos^3 \mu = kR \sin \mu - \varepsilon R^3 \cos^3 \mu.$$

Step 6: Computing the Equation for \dot{R}

From the averaging method:

$$\dot{R} = -\frac{1}{\omega} \langle \varepsilon f \sin \mu \rangle = -\langle \varepsilon f \sin \mu \rangle \quad (\text{since } \omega = 1).$$

Substituting εf :

$$\dot{R} = -\langle (kR \sin \mu - \varepsilon R^3 \cos^3 \mu) \sin \mu \rangle.$$

Distributing:

$$\dot{R} = -\langle kR \sin^2 \mu \rangle + \langle \varepsilon R^3 \cos^3 \mu \sin \mu \rangle.$$

Step 6a: Computing $\langle \sin^2 \mu \rangle$

Technique: Using the identity $\sin^2 \mu = \frac{1}{2}(1 - \cos 2\mu)$:

$$\langle \sin^2 \mu \rangle = \frac{1}{2} \langle 1 - \cos 2\mu \rangle = \frac{1}{2} (1 - \langle \cos 2\mu \rangle).$$

From Lecture Notes §7.2, equation (465): $\langle \cos(n\mu) \rangle = 0$ for all integers $n \neq 0$.

Therefore:

$$\langle \sin^2 \mu \rangle = \frac{1}{2}.$$

Step 6b: Computing $\langle \cos^3 \mu \sin \mu \rangle$

Technique: First, use the identity $\cos^2 \mu = \frac{1}{2}(1 + \cos 2\mu)$:

$$\cos^3 \mu = \cos \mu \cdot \cos^2 \mu = \cos \mu \cdot \frac{1}{2}(1 + \cos 2\mu) = \frac{1}{2} \cos \mu + \frac{1}{2} \cos \mu \cos 2\mu.$$

For the second term, use the product-to-sum formula:

$$\cos \mu \cos 2\mu = \frac{1}{2}(\cos 3\mu + \cos \mu).$$

Therefore:

$$\cos^3 \mu = \frac{1}{2} \cos \mu + \frac{1}{4} \cos 3\mu + \frac{1}{4} \cos \mu = \frac{3}{4} \cos \mu + \frac{1}{4} \cos 3\mu.$$

Now multiply by $\sin \mu$:

$$\cos^3 \mu \sin \mu = \frac{3}{4} \cos \mu \sin \mu + \frac{1}{4} \cos 3\mu \sin \mu.$$

Using the product-to-sum formulas:

$$\begin{aligned} \cos \mu \sin \mu &= \frac{1}{2} \sin 2\mu, \\ \cos 3\mu \sin \mu &= \frac{1}{2}(\sin 4\mu - \sin 2\mu). \end{aligned}$$

Therefore:

$$\cos^3 \mu \sin \mu = \frac{3}{8} \sin 2\mu + \frac{1}{8} \sin 4\mu - \frac{1}{8} \sin 2\mu = \frac{2}{8} \sin 2\mu + \frac{1}{8} \sin 4\mu = \frac{1}{4} \sin 2\mu + \frac{1}{8} \sin 4\mu.$$

Taking the average and using $\langle \sin(n\mu) \rangle = 0$ for $n \neq 0$:

$$\langle \cos^3 \mu \sin \mu \rangle = \frac{1}{4} \langle \sin 2\mu \rangle + \frac{1}{8} \langle \sin 4\mu \rangle = 0.$$

Step 6c: Assembling the Equation for \dot{R}

Substituting back:

$$\dot{R} = -kR \cdot \frac{1}{2} + \varepsilon R^3 \cdot 0 = -\frac{k}{2}R.$$

Amplitude Equation:

$$\dot{R} = -\frac{k}{2}R$$

Step 7: Computing the Equation for $\dot{\Phi}$

From the averaging method:

$$\dot{\Phi} = -\frac{1}{\omega R} \langle \varepsilon f \cos \mu \rangle = -\frac{1}{R} \langle \varepsilon f \cos \mu \rangle.$$

Substituting εf :

$$\dot{\Phi} = -\frac{1}{R} \langle (kR \sin \mu - \varepsilon R^3 \cos^3 \mu) \cos \mu \rangle.$$

Distributing:

$$\dot{\Phi} = -\langle k \sin \mu \cos \mu \rangle + \langle \varepsilon R^2 \cos^4 \mu \rangle.$$

Step 7a: Computing $\langle \sin \mu \cos \mu \rangle$

Technique: Using the identity $\sin \mu \cos \mu = \frac{1}{2} \sin 2\mu$:

$$\langle \sin \mu \cos \mu \rangle = \frac{1}{2} \langle \sin 2\mu \rangle = 0.$$

Step 7b: Computing $\langle \cos^4 \mu \rangle$

Technique: Using $\cos^2 \mu = \frac{1}{2}(1 + \cos 2\mu)$:

$$\cos^4 \mu = (\cos^2 \mu)^2 = \frac{1}{4}(1 + \cos 2\mu)^2 = \frac{1}{4}(1 + 2 \cos 2\mu + \cos^2 2\mu).$$

Using $\cos^2 2\mu = \frac{1}{2}(1 + \cos 4\mu)$:

$$\cos^4 \mu = \frac{1}{4} \left(1 + 2 \cos 2\mu + \frac{1}{2}(1 + \cos 4\mu) \right) = \frac{1}{4} \left(\frac{3}{2} + 2 \cos 2\mu + \frac{1}{2} \cos 4\mu \right).$$

Simplifying:

$$\cos^4 \mu = \frac{3}{8} + \frac{1}{2} \cos 2\mu + \frac{1}{8} \cos 4\mu.$$

Taking the average:

$$\langle \cos^4 \mu \rangle = \frac{3}{8} + \frac{1}{2} \langle \cos 2\mu \rangle + \frac{1}{8} \langle \cos 4\mu \rangle = \frac{3}{8}.$$

Step 7c: Assembling the Equation for $\dot{\Phi}$

Substituting back:

$$\dot{\Phi} = -k \cdot 0 + \varepsilon R^2 \cdot \frac{3}{8} = \frac{3\varepsilon R^2}{8}.$$

Phase Equation:

$$\dot{\Phi} = \frac{3\varepsilon R^2}{8}$$

Step 8: Solving the Amplitude Equation

The ODE: $\dot{R} = -\frac{k}{2}R$ with initial condition $R(0) = ?$

Step 8a: Determining the Initial Condition for R

From the ansatz at $t = 0$:

$$\begin{aligned}y(0) &= R(0) \cos \Phi(0) = a, \\ \dot{y}(0) &= -R(0) \sin \Phi(0) = 0.\end{aligned}$$

From $\dot{y}(0) = 0$: Either $R(0) = 0$ (trivial solution) or $\sin \Phi(0) = 0$.

Since we want a non-trivial solution with $y(0) = a \neq 0$, we need $\sin \Phi(0) = 0$, which means $\Phi(0) = 0$ (or $\Phi(0) = n\pi$ for integer n ; we choose $\Phi(0) = 0$ for simplicity).

From $y(0) = R(0) \cos(0) = R(0) = a$:

$$R(0) = a, \quad \Phi(0) = 0.$$

Step 8b: Solving for $R(t)$

The equation $\dot{R} = -\frac{k}{2}R$ is a first-order linear ODE.

Technique: *Separating variables:*

$$\frac{dR}{R} = -\frac{k}{2}dt.$$

Integrating:

$$\ln R = -\frac{k}{2}t + C.$$

Exponentiating:

$$R(t) = R(0)e^{-kt/2}.$$

With $R(0) = a$:

$R(t) = ae^{-kt/2}$

Reflection: *The amplitude decays exponentially with decay rate $k/2$. This is the expected behavior for a damped oscillator: the damping constant k determines how quickly energy is dissipated. The factor of $1/2$ arises because energy goes as R^2 , and $R^2 \propto e^{-kt}$ corresponds to amplitude $R \propto e^{-kt/2}$.*

Step 9: Solving the Phase Equation

The ODE: $\dot{\Phi} = \frac{3\varepsilon R^2}{8}$ with $\Phi(0) = 0$.

Substituting $R(t) = ae^{-kt/2}$:

$$\dot{\Phi} = \frac{3\varepsilon a^2 e^{-kt}}{8}.$$

Step 9a: Integrating

Technique: *Integrating from 0 to t :*

$$\Phi(t) - \Phi(0) = \frac{3\varepsilon a^2}{8} \int_0^t e^{-ks} ds.$$

Computing the integral:

$$\int_0^t e^{-ks} ds = \left[-\frac{1}{k} e^{-ks} \right]_0^t = -\frac{1}{k} e^{-kt} + \frac{1}{k} = \frac{1}{k} (1 - e^{-kt}).$$

Therefore:

$$\Phi(t) = \frac{3\varepsilon a^2}{8k} (1 - e^{-kt}).$$

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Reflection: The phase shift $\Phi(t)$ increases from zero and asymptotically approaches a finite limit:

$$\lim_{t \rightarrow \infty} \Phi(t) = \frac{3\varepsilon a^2}{8k}.$$

This means the nonlinearity causes a cumulative phase shift, but the total shift remains bounded because the amplitude (and hence the nonlinear effect) decays due to damping. As the oscillation dies out, the phase approaches a constant offset.

Step 10: Assembling the Complete Solution

The solution is given by $y(t) = R(t) \cos \mu$ where $\mu = t + \Phi(t)$. Substituting our results:

$$\begin{aligned} R(t) &= ae^{-kt/2}, \\ \Phi(t) &= \frac{3\varepsilon a^2}{8k} (1 - e^{-kt}), \\ \mu &= t + \Phi(t) = t + \frac{3\varepsilon a^2}{8k} (1 - e^{-kt}). \end{aligned}$$

Complete Approximate Solution:

$$y(t) \approx a \exp\left(-\frac{k}{2}t\right) \cos\left[t + \frac{3\varepsilon a^2}{8k} (1 - e^{-kt})\right]$$

Step 11: Verification of Initial Conditions

Step 11a: Checking $y(0) = a$

At $t = 0$:

$$y(0) = ae^0 \cos\left[0 + \frac{3\varepsilon a^2}{8k} (1 - 1)\right] = a \cos(0) = a. \quad \checkmark$$

Step 11b: Checking $\dot{y}(0) = 0$

From the ansatz $\dot{y} = -R \sin \mu$:

$$\dot{y}(0) = -a \sin(0) = 0. \quad \checkmark$$

Step 12: Physical Interpretation

Reflection: The solution exhibits three distinct physical effects:

1. **Amplitude Decay:** The factor $ae^{-kt/2}$ describes exponential amplitude decay due to the damping term $k\dot{y}$. The decay rate is $k/2$, so the amplitude halves every $t = (2 \ln 2)/k$ time units.
2. **Base Oscillation:** The $\cos(t + \dots)$ term describes oscillations with natural frequency $\omega = 1$. Without perturbations, this would simply be a $\cos t$.
3. **Nonlinear Phase Shift:** The term $\frac{3\varepsilon a^2}{8k} (1 - e^{-kt})$ is the cumulative phase shift caused by the cubic nonlinearity εy^3 . This shift:

- Starts at zero when $t = 0$
- Grows as the nonlinearity “bends” the oscillation frequency
- Saturates at $\frac{3\varepsilon a^2}{8k}$ as $t \rightarrow \infty$ because the decaying amplitude reduces the nonlinear effect

The sign of ε determines the direction of the phase shift:

- $\varepsilon > 0$ (hardening spring): phase increases, effective frequency slightly higher than 1
- $\varepsilon < 0$ (softening spring): phase decreases, effective frequency slightly lower than 1

Step 13: Limiting Cases

Case 1: Pure Damping ($\varepsilon = 0$)

When $\varepsilon = 0$:

$$y(t) = ae^{-kt/2} \cos t.$$

This is the standard damped harmonic oscillator solution, matching the exact solution for the linear damped oscillator.

Case 2: Pure Nonlinearity ($k = 0$)

When $k \rightarrow 0$, we need to carefully take the limit. Using L'Hôpital's rule or Taylor expansion:

$$\frac{1 - e^{-kt}}{k} \rightarrow t \quad \text{as } k \rightarrow 0.$$

Therefore:

$$\Phi(t) \rightarrow \frac{3\varepsilon a^2}{8}t,$$

and the solution becomes:

$$y(t) = a \cos \left(t + \frac{3\varepsilon a^2}{8}t \right) = a \cos \left[\left(1 + \frac{3\varepsilon a^2}{8} \right) t \right].$$

Justification: This is the well-known result for the undamped Duffing oscillator: the nonlinearity causes a frequency shift proportional to the amplitude squared. The effective frequency is:

$$\omega_{\text{eff}} = 1 + \frac{3\varepsilon a^2}{8}.$$

For a hardening spring ($\varepsilon > 0$), the frequency increases with amplitude.

Case 3: Long-Time Behavior ($t \rightarrow \infty$)

As $t \rightarrow \infty$:

- $R(t) = ae^{-kt/2} \rightarrow 0$: the oscillation amplitude decays to zero.
- $\Phi(t) \rightarrow \frac{3\varepsilon a^2}{8k}$: the phase approaches a constant limit.
- $y(t) \rightarrow 0$: the solution decays to the stable equilibrium.

Final Summary

Complete Solution for Problem 9.3:

Given: $\ddot{y} + y + k\dot{y} + \varepsilon y^3 = 0$ with $y(0) = a$, $\dot{y}(0) = 0$, and $\varepsilon, k \ll 1$.

Standard form: $\ddot{y} + \omega^2 y = \varepsilon f$ with $\omega = 1$ and $\varepsilon f = -k\dot{y} - \varepsilon y^3$.

Averaging ansatz: $y = R \cos \mu$, $\dot{y} = -R \sin \mu$, $\mu = t + \Phi(t)$.

Averaged equations:

$$\begin{aligned}\dot{R} &= -\frac{k}{2}R \\ \dot{\Phi} &= \frac{3\varepsilon R^2}{8}\end{aligned}$$

Initial conditions: $R(0) = a$, $\Phi(0) = 0$.

Solutions for slow variables:

$$\begin{aligned}R(t) &= ae^{-kt/2} \\ \Phi(t) &= \frac{3\varepsilon a^2}{8k}(1 - e^{-kt})\end{aligned}$$

Approximate solution:

$$y(t) \approx ae^{-kt/2} \cos \left[t + \frac{3\varepsilon a^2}{8k} (1 - e^{-kt}) \right]$$

Physical interpretation:

- Exponential amplitude decay with rate $k/2$
- Frequency remains approximately $\omega = 1$
- Cumulative phase shift due to nonlinearity, bounded by $\frac{3\varepsilon a^2}{8k}$

Connection to Lecture Notes

Reflection: This problem demonstrates the averaging method from Lecture Notes §7.2, equations (462)–(464). The key steps are:

1. **Transform to amplitude-phase variables:** Express $y = R \cos \mu$ and $\dot{y} = -\omega R \sin \mu$ (equations (456)–(457)).
2. **Derive evolution equations:** The exact equations (460)–(461) become approximate averaged equations (462)–(463).
3. **Compute averages:** Use trigonometric identities and the key property $\langle \cos(n\mu) \rangle = \langle \sin(n\mu) \rangle = 0$ for $n \neq 0$ (equation (465)).
4. **Solve the slow dynamics:** The averaged equations are simpler ODEs that capture the slow modulation.

The averaging method gives identical results to the multiple scales method (as noted after equation (464)), but through a more physically intuitive averaging process rather than explicit introduction of multiple time scales.

This problem combines two perturbative effects:

- **Damping** (ky): *Contributes only to \dot{R} (amplitude decay), not to $\dot{\Phi}$.*
- **Cubic nonlinearity** (εy^3): *Contributes only to $\dot{\Phi}$ (phase shift), not to \dot{R} .*

This separation is a special feature of this particular combination of perturbations.