Exercise Sheet 3: Bifurcations
Question 8 - Complete Solution

Methods of Applied Mathematics

Problem Statement

Determine what bifurcation happens as μ changes in the system:

$$\frac{dx}{dt} = x - x^3 + \mu$$

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1 Step 1: Analyze System Structure

Rewrite equilibrium condition

For equilibria, set $\dot{x} = 0$:

$$x - x^3 + \mu = 0$$

Rearranging:

$$x^3 - x = \mu$$

or equivalently:

$$x^3 = x + \mu$$

Geometric interpretation

Define $h(x) = x^3 - x$. Then equilibria occur where:

$$h(x) = \mu$$

This means equilibria are intersections of the cubic curve $y = x^3 - x$ with horizontal line $y = \mu$.

XYZ Analysis of Problem Structure

- **STAGE X (What we have):** A 1D system where equilibria are roots of cubic equation. The parameter μ appears additively, shifting the equilibrium equation vertically.
- **STAGE Y (Why this approach):** Unlike previous problems where equilibria had explicit formulas, this cubic generally requires graphical/numerical analysis. The key insight: view the problem as finding where a fixed cubic curve $h(x) = x^3 - x$ intersects a moving horizontal line $y = \mu$. As μ varies:
  - High horizontal line (μ large): may intersect cubic once
  - Medium height: may intersect three times
  - Low horizontal line (μ very negative): may intersect once

The number of intersections (equilibria) changes when the line becomes tangent to the cubic - this signals a fold bifurcation where two equilibria collide and annihilate.

- **STAGE Z (What to find):** We need to:
  1. Find critical points of $h(x)$ (local max/min)
  2. Evaluate h at these critical points to find bifurcation values of μ
  3. Determine stability of equilibria in each parameter regime
  4. Identify the bifurcation type(s)

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2 Step 2: Analyze the Cubic Function

Define the function

$$h(x) = x^3 - x$$

Factor the function

$$h(x) = x(x^2 - 1) = x(x-1)(x+1)$$

Roots of h: $x = -1, 0, 1$

Find critical points

$$h'(x) = 3x^2 - 1$$

Set $h'(x) = 0$:

$$3x^2 = 1 \quad \Rightarrow \quad x^2 = \frac{1}{3} \quad \Rightarrow \quad x = \pm\frac{1}{\sqrt{3}}$$

Classify critical points

Second derivative:

$$h''(x) = 6x$$

At $x = 1/\sqrt{3}$: $h''(1/\sqrt{3}) = 6/\sqrt{3} = 2\sqrt{3} > 0 \rightarrow$ Local minimum
At $x = -1/\sqrt{3}$: $h''(-1/\sqrt{3}) = -6/\sqrt{3} = -2\sqrt{3} < 0 \rightarrow$ Local maximum

Evaluate h at critical points

At local minimum $x = 1/\sqrt{3}$:

$$\begin{aligned} h\left(\frac{1}{\sqrt{3}}\right) &= \left(\frac{1}{\sqrt{3}}\right)^3 - \frac{1}{\sqrt{3}} \\ &= \frac{1}{3\sqrt{3}} - \frac{1}{\sqrt{3}} \\ &= \frac{1}{3\sqrt{3}} - \frac{3}{3\sqrt{3}} \\ &= -\frac{2}{3\sqrt{3}} = -\frac{2\sqrt{3}}{9} \end{aligned}$$

At local maximum $x = -1/\sqrt{3}$:

$$\begin{aligned} h\left(-\frac{1}{\sqrt{3}}\right) &= -\left(\frac{1}{\sqrt{3}}\right)^3 + \frac{1}{\sqrt{3}} \\ &= -\frac{1}{3\sqrt{3}} + \frac{1}{\sqrt{3}} \\ &= -\frac{1}{3\sqrt{3}} + \frac{3}{3\sqrt{3}} \\ &= \frac{2}{3\sqrt{3}} = \frac{2\sqrt{3}}{9} \end{aligned}$$

Summary of cubic properties

$$\boxed{\begin{array}{lll} \text{Local maximum:} & x = -\dfrac{1}{\sqrt{3}}, & h = \dfrac{2\sqrt{3}}{9} \approx 0.3849 \\ \text{Local minimum:} & x = \dfrac{1}{\sqrt{3}}, & h = -\dfrac{2\sqrt{3}}{9} \approx -0.3849 \end{array}}$$

XYZ Analysis of Cubic Structure

- **STAGE X (What we found):** The cubic $h(x) = x^3 - x$ has two critical points: a local max at $x = -1/\sqrt{3}$ with value $2\sqrt{3}/9$, and a local min at $x = 1/\sqrt{3}$ with value $-2\sqrt{3}/9$.
- **STAGE Y (Why these values matter):** These critical values of h are where horizontal lines become tangent to the cubic curve. They represent threshold values of μ:
  - If $\mu > 2\sqrt{3}/9$: line $y = \mu$ is above the local max, intersecting cubic only once (far right)
  - If $\mu = 2\sqrt{3}/9$: line is tangent at local max (two equilibria touch)
  - If $-2\sqrt{3}/9 < \mu < 2\sqrt{3}/9$: line intersects cubic three times
  - If $\mu = -2\sqrt{3}/9$: line is tangent at local min (two equilibria touch)
  - If $\mu < -2\sqrt{3}/9$: line is below local min, intersecting cubic only once (far left)

  The tangency points are fold bifurcations where pairs of equilibria collide and annihilate.
- **STAGE Z (What this means):** The system undergoes TWO fold bifurcations as μ varies. Between them, three equilibria coexist; outside, only one equilibrium exists. This is richer structure than a single fold bifurcation.

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3 Step 3: Count Equilibria by Parameter Range

Number of real roots

The equation $x^3 - x = \mu$ has:

$$\mu > \frac{2\sqrt{3}}{9} : \quad \boxed{\text{1 equilibrium}}$$

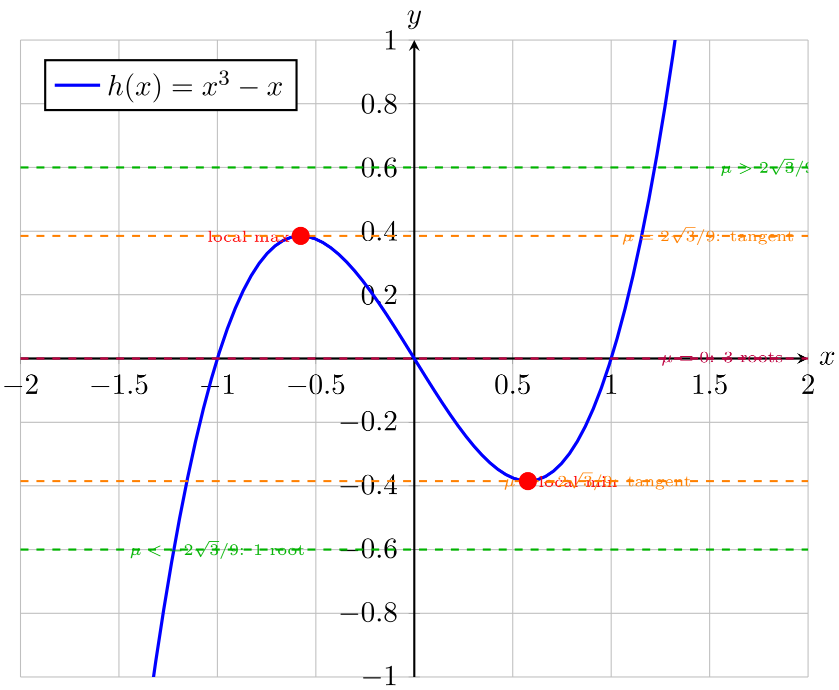
$$\mu = \frac{2\sqrt{3}}{9} : \quad \boxed{\text{2 equilibria (one repeated)}}$$

$$-\frac{2\sqrt{3}}{9} < \mu < \frac{2\sqrt{3}}{9} : \quad \boxed{\text{3 equilibria}}$$

$$\mu = -\frac{2\sqrt{3}}{9} : \quad \boxed{\text{2 equilibria (one repeated)}}$$

$$\mu < -\frac{2\sqrt{3}}{9} : \quad \boxed{\text{1 equilibrium}}$$

Sketch of cubic and horizontal lines



XYZ Analysis of Equilibrium Count

- **STAGE X (What the diagram shows):** The cubic curve and various horizontal lines. Lines above the local max or below the local min intersect once; lines between the extrema intersect three times.

- **STAGE Y (Why count changes):** The fundamental theorem of algebra guarantees a cubic has three roots (counting multiplicities, including complex). For our equation $x^3 - x - \mu = 0$:

  - When μ is extreme: one real root, two complex conjugate roots
  - When μ is intermediate: three distinct real roots
  - At transition ($\mu = \pm 2\sqrt{3}/9$): three real roots, but two coincide (repeated root)

  The complex roots become real as μ enters the interval $(-2\sqrt{3}/9, 2\sqrt{3}/9)$, and the transition happens via fold bifurcation where a pair of real roots emerges from (or disappears into) the complex plane.

- **STAGE Z (What this predicts):** Two fold bifurcations occur:
  1. At $\mu = 2\sqrt{3}/9$: fold at $x = -1/\sqrt{3}$ (left side of curve)
  2. At $\mu = -2\sqrt{3}/9$: fold at $x = 1/\sqrt{3}$ (right side of curve)

  Between bifurcations, the system has three equilibria. We now need to determine their stabilities.

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4 Step 4: Determine Stability

Compute derivative

For $f(x) = x - x^3 + \mu$:

$$f'(x) = 1 - 3x^2$$

Stability criterion

- $f'(x) < 0$: stable (flow toward equilibrium)
- $f'(x) > 0$: unstable (flow away from equilibrium)
- $f'(x) = 0$: neutral (bifurcation point)

Analyze sign of $f'(x)$

$$f'(x) = 1 - 3x^2 < 0 \quad \Leftrightarrow \quad x^2 > \frac{1}{3} \quad \Leftrightarrow \quad |x| > \frac{1}{\sqrt{3}}$$

So:

$$\begin{aligned}
|x| > \frac{1}{\sqrt{3}}: &\quad f'(x) < 0 \quad \Rightarrow \quad \text{STABLE} \\
|x| < \frac{1}{\sqrt{3}}: &\quad f'(x) > 0 \quad \Rightarrow \quad \text{UNSTABLE} \\
|x| = \frac{1}{\sqrt{3}}: &\quad f'(x) = 0 \quad \Rightarrow \quad \text{NEUTRAL (bifurcation)}
\end{aligned}$$

Stability pattern for three equilibria

When three equilibria exist (for $-2\sqrt{3}/9 < \mu < 2\sqrt{3}/9$), denote them as $x_L < x_M < x_R$ (left, middle, right):

- x_L is far left: $x_L < -1/\sqrt{3}$, so $|x_L| > 1/\sqrt{3} \to$ **Stable**
- x_M is in middle: $|x_M| < 1/\sqrt{3} \to$ **Unstable**
- x_R is far right: $x_R > 1/\sqrt{3}$, so $|x_R| > 1/\sqrt{3} \to$ **Stable**

Stability at bifurcation points

At $\mu = 2\sqrt{3}/9$: Two equilibria at/near $x = -1/\sqrt{3}$

- One at $x = -1/\sqrt{3}$ exactly: $f'(-1/\sqrt{3}) = 0$ (neutral)
- One slightly left: $f' < 0$ (stable)

At $\mu = -2\sqrt{3}/9$: Two equilibria at/near $x = 1/\sqrt{3}$

- One at $x = 1/\sqrt{3}$ exactly: $f'(1/\sqrt{3}) = 0$ (neutral)
- One slightly right: $f' < 0$ (stable)

XYZ Analysis of Stability

- **STAGE X (What we found):** The stability depends only on position: equilibria with $|x| > 1/\sqrt{3}$ are stable, those with $|x| < 1/\sqrt{3}$ are unstable. When three equilibria exist, the pattern is stable-unstable-stable.
- **STAGE Y (Why this pattern):** The derivative $f'(x) = 1 - 3x^2$ is a downward-opening parabola in x:
  - Positive near $x = 0$ (unstable equilibria)
  - Negative for $|x|$ large (stable equilibria)
  - Zero at $x = \pm 1/\sqrt{3}$ (bifurcation points)

  The physical interpretation: for $|x|$ small, the linear term x dominates (positive coefficient $\rightarrow$ unstable). For $|x|$ large, the cubic term $-x^3$ dominates (negative coefficient for large $|x|$ $\rightarrow$ stable). The balance point is at $|x| = 1/\sqrt{3}$.

  This means equilibria born in fold bifurcations inherit predictable stabilities based on their positions. At the right fold ($x = 1/\sqrt{3}$), a stable equilibrium (with $x > 1/\sqrt{3}$) collides with an unstable one (with $x < 1/\sqrt{3}$). Similarly at the left fold.
- **STAGE Z (What this means for dynamics):** In the three-equilibria regime, the middle equilibrium is a separatrix:
  - Initial conditions $x < x_M$: flow toward x_L (left stable equilibrium)
  - Initial conditions $x > x_M$: flow toward x_R (right stable equilibrium)

  The unstable middle equilibrium divides the phase space into two basins of attraction. At fold bifurcations, these basins merge or separate as equilibria are created/destroyed.

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5 Step 5: Verify Fold Bifurcation Conditions

Left fold at $\mu = 2\sqrt{3}/9$, $x = -1/\sqrt{3}$

For $f(x, \mu) = x - x^3 + \mu$:

(B1) Equilibrium exists:

$$f\left(-\frac{1}{\sqrt{3}}, \frac{2\sqrt{3}}{9}\right) = -\frac{1}{\sqrt{3}} - \left(-\frac{1}{\sqrt{3}}\right)^3 + \frac{2\sqrt{3}}{9} = -\frac{1}{\sqrt{3}} + \frac{1}{3\sqrt{3}} + \frac{2\sqrt{3}}{9}$$

$$= -\frac{3}{3\sqrt{3}} + \frac{1}{3\sqrt{3}} + \frac{2\sqrt{3}}{9} = -\frac{2}{3\sqrt{3}} + \frac{2\sqrt{3}}{9} = -\frac{2\sqrt{3}}{9} + \frac{2\sqrt{3}}{9} = 0 \quad \checkmark$$

(B2) Zero eigenvalue:

$$\left.\frac{\partial f}{\partial x}\right|_{x=-1/\sqrt{3}} = 1 - 3\left(\frac{1}{3}\right) = 1 - 1 = 0 \quad \checkmark$$

(G1) Second derivative nonzero:

$$\frac{\partial^2 f}{\partial x^2} = -6x \quad \Rightarrow \quad \left.\frac{\partial^2 f}{\partial x^2}\right|_{x=-1/\sqrt{3}} = -6 \cdot \left(-\frac{1}{\sqrt{3}}\right) = \frac{6}{\sqrt{3}} = 2\sqrt{3} \neq 0 \quad \checkmark$$

(G2) Parameter derivative nonzero:

$$\frac{\partial f}{\partial \mu} = 1 \quad \Rightarrow \quad \left.\frac{\partial f}{\partial \mu}\right|_{\text{any point}} = 1 \neq 0 \quad \checkmark$$

Right fold at $\mu = -2\sqrt{3}/9$, $x = 1/\sqrt{3}$

For $f(x,\mu) = x - x^3 + \mu$:

(B1) Equilibrium exists:

$$f\left(\frac{1}{\sqrt{3}}, -\frac{2\sqrt{3}}{9}\right) = \frac{1}{\sqrt{3}} - \left(\frac{1}{\sqrt{3}}\right)^3 - \frac{2\sqrt{3}}{9} = \frac{1}{\sqrt{3}} - \frac{1}{3\sqrt{3}} - \frac{2\sqrt{3}}{9}$$

$$= \frac{3}{3\sqrt{3}} - \frac{1}{3\sqrt{3}} - \frac{2\sqrt{3}}{9} = \frac{2}{3\sqrt{3}} - \frac{2\sqrt{3}}{9} = \frac{2\sqrt{3}}{9} - \frac{2\sqrt{3}}{9} = 0 \quad \checkmark$$

(B2) Zero eigenvalue:

$$\left.\frac{\partial f}{\partial x}\right|_{x=1/\sqrt{3}} = 1 - 3\left(\frac{1}{3}\right) = 0 \quad \checkmark$$

(G1) Second derivative nonzero:

$$\left.\frac{\partial^2 f}{\partial x^2}\right|_{x=1/\sqrt{3}} = -6 \cdot \frac{1}{\sqrt{3}} = -\frac{6}{\sqrt{3}} = -2\sqrt{3} \neq 0 \quad \checkmark$$

(G2) Parameter derivative nonzero:

$$\frac{\partial f}{\partial \mu} = 1 \neq 0 \quad \checkmark$$

Conclusion

$$\boxed{\begin{array}{l} \text{FOLD BIFURCATION at } \mu = \frac{2\sqrt{3}}{9},\ x = -\frac{1}{\sqrt{3}} \\ \text{FOLD BIFURCATION at } \mu = -\frac{2\sqrt{3}}{9},\ x = \frac{1}{\sqrt{3}} \end{array}}$$

XYZ Analysis of Verification

- **STAGE X (What we verified):** Both bifurcation points satisfy all four conditions (B1, B2, G1, G2) for fold bifurcations.
- **STAGE Y (Why two folds):** The cubic equation can have up to two pairs of equilibria that collide. Each collision is independent:
  - Left fold: Occurs at local maximum of $h(x)$. As μ decreases through $2\sqrt{3}/9$, two equilibria emerge on the left branch
  - Right fold: Occurs at local minimum of $h(x)$. As μ increases through $-2\sqrt{3}/9$, two equilibria emerge on the right branch

  The sign difference in $\partial^2 f/\partial x^2$ ($+2\sqrt{3}$ vs $-2\sqrt{3}$) reflects the different curvatures at left (max) and right (min) critical points. But both are fold bifurcations - the sign of second derivative just indicates which branch is stable.
- **STAGE Z (What this represents):** Systems with multiple fold bifurcations exhibit hysteresis and bistability:
  - For intermediate μ: two stable states coexist with one unstable separatrix
  - Slowly increasing μ from $\mu \ll 0$: system stays on right stable branch until right fold at $\mu = -2\sqrt{3}/9$, then jumps to left stable branch
  - Slowly decreasing μ from $\mu \gg 0$: system stays on left stable branch until left fold at $\mu = 2\sqrt{3}/9$, then jumps to right stable branch

  The path taken depends on history - this is hysteresis, common in mechanical buckling, optical bistability, and ecological regime shifts.

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6 Step 6: Bifurcation Diagram

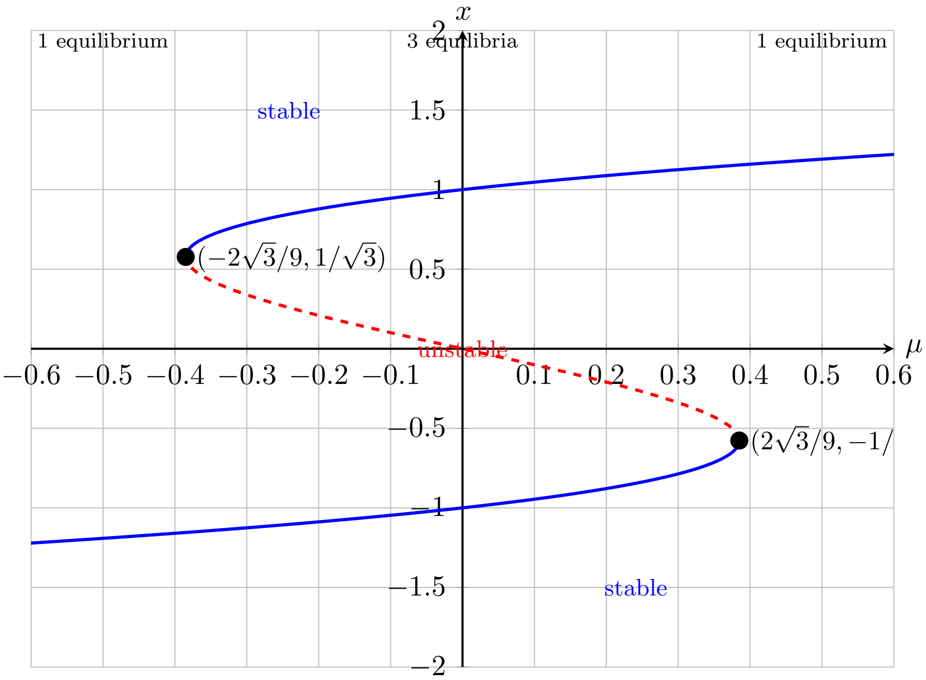
Equilibrium curves

From $x^3 - x = \mu$, we can plot x vs μ.

Alternatively, parametrize by x and compute $\mu(x) = x^3 - x$:

- For $x < -1/\sqrt{3}$: upper-left branch (stable)
- At $x = -1/\sqrt{3}$: fold point, $\mu = 2\sqrt{3}/9$
- For $-1/\sqrt{3} < x < 1/\sqrt{3}$: middle branch (unstable)
- At $x = 1/\sqrt{3}$: fold point, $\mu = -2\sqrt{3}/9$
- For $x > 1/\sqrt{3}$: lower-right branch (stable)

Bifurcation diagram: μ vs x



XYZ Analysis of Bifurcation Diagram

- **STAGE X (What the diagram shows):** An "S-shaped" or "N-shaped" curve (depending on orientation). Two fold points where the curve turns back on itself. Solid lines (stable) on outer branches, dashed line (unstable) on middle branch.

- **STAGE Y (Why this shape):** The curve is simply the graph of $\mu = x^3 - x$ rotated 90° (plotted with axes swapped). The S-shape comes from the cubic function:

  - For μ far left: line $y = \mu$ intersects cubic once (upper left) $\rightarrow$ single equilibrium at large negative x
  - As μ increases to $-2\sqrt{3}/9$: line rises to tangency point $\rightarrow$ fold bifurcation, two new equilibria born
  - For intermediate μ: line intersects cubic three times $\rightarrow$ three coexisting equilibria
  - As μ increases to $2\sqrt{3}/9$: line reaches upper tangency $\rightarrow$ second fold bifurcation, two equilibria annihilate
  - For μ far right: line intersects once (lower right) $\rightarrow$ single equilibrium at large positive x

  The middle branch "folds back" because $\mu(x) = x^3 - x$ is not monotonic - it decreases then increases.

- **STAGE Z (What this means for control):** Reading vertically at fixed μ:

  - Outside folds: one equilibrium (unique stable state)
  - Between folds: three equilibria (bistability - two stable attractors separated by unstable saddle)

  Reading horizontally shows hysteresis: slowly varying μ causes system to jump discontinuously at fold points. The system "remembers" which branch it's on. Applications include:

- Mechanical systems: beam buckling under load
- Optical systems: bistable lasers
- Climate models: ice-albedo feedback leading to abrupt transitions
- Ecological systems: lake eutrophication with multiple stable states

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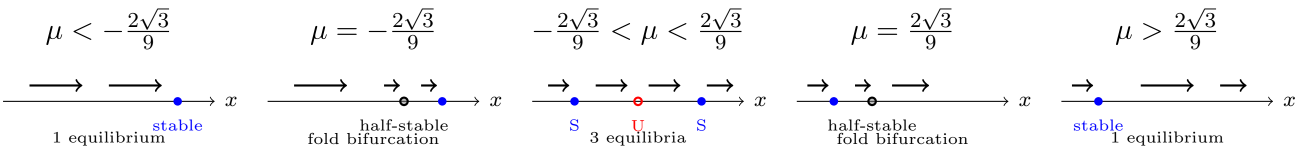
7 Step 7: Phase Portraits

Five representative scenarios



XYZ Analysis of Phase Portraits

- **STAGE X (What we see):** The number and type of equilibria changing dramatically as μ varies. Single stable state → fold → three states (bistability) → fold → single stable state.
- **STAGE Y (Why these transitions):** The 1D flow $\dot{x} = x - x^3 + \mu$ has:
  - **Far left regime**: μ very negative makes $\dot{x}$ negative for most x, except far right where x term dominates → single stable equilibrium at large positive x
  - **Right fold** ($\mu = -2\sqrt{3}/9$): Two equilibria merge at $x = 1/\sqrt{3}$. The derivative is zero there (half-stable point)
  - **Central regime**: Three equilibria coexist. The outer two are stable (large $|x|$ where $-x^3$ dominates), middle is unstable (small $|x|$ where x dominates)
  - **Left fold** ($\mu = 2\sqrt{3}/9$): Two equilibria merge at $x = -1/\sqrt{3}$
  - **Far right regime**: μ very positive makes $\dot{x}$ positive for most x, except far left where $-x^3$ term dominates → single stable equilibrium at large negative x
- **STAGE Z (What this means globally):** The system exhibits path dependence (hysteresis):
  - Start with $\mu \ll 0$, system at stable equilibrium (far right)
  - Slowly increase μ: system stays on right stable branch, passing through bistable region
  - At $\mu = 2\sqrt{3}/9$: right stable branch disappears in fold → system must jump to left stable branch
  - Continue increasing μ: system remains on left stable branch
  - Now decrease μ: system stays on left stable branch, retracing through bistable region
  - At $\mu = -2\sqrt{3}/9$: left stable branch disappears in fold → system must jump to right stable branch

  The path followed going up differs from the path going down - this creates a hysteresis loop. The system "remembers" where it came from via which stable branch it occupies.

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8 Summary

System

$$\dot{x} = x - x^3 + \mu$$

Bifurcations

$$\boxed{\begin{array}{lll} \text{Fold at:} & \mu = \dfrac{2\sqrt{3}}{9} \approx 0.3849, & x = -\dfrac{1}{\sqrt{3}} \approx -0.5774 \\ \text{Fold at:} & \mu = -\dfrac{2\sqrt{3}}{9} \approx -0.3849, & x = \dfrac{1}{\sqrt{3}} \approx 0.5774 \end{array}}$$

Equilibrium structure by parameter regime

Parameter Range	**Number**	**Stability Pattern**
$\mu < -2\sqrt{3}/9$	1	Stable (far right)
$\mu = -2\sqrt{3}/9$	2	Half-stable + stable
$-2\sqrt{3}/9 < \mu < 2\sqrt{3}/9$	3	Stable–Unstable–Stable
$\mu = 2\sqrt{3}/9$	2	Stable + half-stable
$\mu > 2\sqrt{3}/9$	1	Stable (far left)

Key phenomena

- **Bistability**: For $|\mu| < 2\sqrt{3}/9$, two stable attractors coexist
- **Hysteresis**: System path depends on direction of parameter variation
- **Catastrophic jumps**: At fold points, sudden transitions between distant stable states
- **S-shaped bifurcation diagram**: Characteristic of systems with cubic nonlinearity

Physical interpretation

This structure appears in:

- Buckled beams (displacement vs load)
- Optical bistability (intensity vs detuning)
- Climate tipping points (temperature vs forcing)
- Ecological regime shifts (biomass vs nutrient input)

The two fold bifurcations create a parameter window of bistability between catastrophic transition thresholds.