

Exercise Sheet 1, Question 6: Existence and Uniqueness

Methods of Applied Mathematics [SEMT30006]

Problem Statement

Solve the following initial value problems to find a solution $x(t)$ in terms of x_0 :

- (a) $\dot{x} = x^2$ with $x(0) = x_0$
- (b) $\dot{x} = |x|$ with $x(0) = x_0$
- (c) $\dot{x} = |x|^{1/2}$ with $x(0) = x_0$

Then answer the following questions for each system:

- (i) Consider three initial conditions $x_0 = 0$, $x_0 = -1$ and $x_0 = +1$. From each, where does the solution go and how long does it take?
- (ii) Identify the different orbits of each system.
- (iii) Are the orbits uniquely determined by the ODE (for a given x_0 is there only one unique solution)?
- (iv) Show that each system is/isn't Lipschitz continuous, and say how this agrees with your answer to (iii).

Theoretical Framework: Existence and Uniqueness

Explanation 1 (Picard-Lindelöf Theorem). *For the initial value problem $\dot{x} = f(x, t)$ with $x(t_0) = x_0$:*

Existence: *If f is continuous, then a solution exists locally.*

Uniqueness: *If f is Lipschitz continuous in x , then the solution is unique.*

Lipschitz Condition: *A function $f(x)$ is Lipschitz continuous if there exists a constant $L > 0$ such that:*

$$|f(x_1) - f(x_2)| \leq L|x_1 - x_2| \quad \text{for all } x_1, x_2$$

Equivalently, if f is differentiable and $|f'(x)|$ is bounded, then f is Lipschitz.

KEY INSIGHT: When Lipschitz condition fails (e.g., $f'(x)$ unbounded), uniqueness can fail. We may have multiple solutions from the same initial condition!

1 Part (a): $\dot{x} = x^2$ with $x(0) = x_0$

Solving the ODE

Solution 1. Step 1: Separate Variables

The equation $\dot{x} = x^2$ is separable. For $x \neq 0$:

$$\frac{dx}{dt} = x^2 \quad \Rightarrow \quad \frac{dx}{x^2} = dt$$

Step 2: Integrate Both Sides

$$\int \frac{dx}{x^2} = \int dt$$
$$-\frac{1}{x} = t + C$$

Step 3: Apply Initial Condition

At $t = 0$: $x(0) = x_0$

$$-\frac{1}{x_0} = 0 + C \quad \Rightarrow \quad C = -\frac{1}{x_0}$$

Therefore:

$$-\frac{1}{x} = t - \frac{1}{x_0}$$

Step 4: Solve for $x(t)$

$$\frac{1}{x} = \frac{1}{x_0} - t = \frac{1 - x_0 t}{x_0}$$

$$\boxed{x(t) = \frac{x_0}{1 - x_0 t}}$$

Explanation 2 (Special Case: $x_0 = 0$). If $x_0 = 0$, then $\dot{x} = 0^2 = 0$, so:

$$\boxed{x(t) = 0 \quad \text{for all } t}$$

This is the trivial solution (constant at origin).

(i) Behavior for Three Initial Conditions

Solution 2. Case 1: $x_0 = 0$

$$x(t) = 0 \quad \text{for all } t$$

- **Where does it go?** Stays at $x = 0$ forever
- **How long?** Remains there for all time $t \in (-\infty, \infty)$
- **Nature:** Equilibrium solution

Case 2: $x_0 = -1$

$$x(t) = \frac{-1}{1 - (-1)t} = \frac{-1}{1 + t}$$

- At $t = 0$: $x = -1$
- As $t \rightarrow \infty$: $x(t) \rightarrow 0^-$ (approaches zero from below)
- As $t \rightarrow -1^+$: $x(t) \rightarrow -\infty$ (blows up to $-\infty$ as t approaches -1 from right)
- **Where does it go?** Approaches $x = 0$ as $t \rightarrow +\infty$
- **How long?** Takes infinite time to reach $x = 0$

Explanation 3 (Why Approaches Zero). For $x_0 < 0$, we have $x(t) = \frac{x_0}{1 - x_0 t}$ where $1 - x_0 t > 1$ for $t > 0$.

Since $|x_0| < 1 - x_0 t$ for positive t :

$$|x(t)| = \frac{|x_0|}{1 - x_0 t} < |x_0| \quad \text{and decreases as } t \rightarrow \infty$$

Case 3: $x_0 = +1$

$$x(t) = \frac{1}{1 - t}$$

- At $t = 0$: $x = 1$
- As $t \rightarrow 1^-$: $x(t) \rightarrow +\infty$ (blows up to $+\infty$)
- **Where does it go?** Diverges to $+\infty$
- **How long?** Reaches $+\infty$ at time $t = 1$ (finite-time blow-up!)
- Solution only exists on $t \in (-\infty, 1)$

Explanation 4 (Finite-Time Blow-Up). The solution becomes singular at $t = t^* = \frac{1}{x_0}$ when $x_0 > 0$:

$$x(t) = \frac{x_0}{1 - x_0 t} \rightarrow \infty \quad \text{as } t \rightarrow \frac{1}{x_0}$$

For $x_0 = 1$: blow-up time is $t^* = 1$

This is characteristic of super-linear growth ($\dot{x} = x^2$).

Summary:

x_0	Long-time behavior	Time to destination
0	Stays at $x = 0$	All time
-1	$x(t) \rightarrow 0$ as $t \rightarrow \infty$	Infinite time
+1	$x(t) \rightarrow +\infty$ as $t \rightarrow 1$	Finite time ($t^* = 1$)

(ii) Different Orbits

Solution 3. An **orbit** is the trajectory $x(t)$ in phase space (here, the x -axis).

Classification of Orbits:

Type 1: Equilibrium Orbit

$$x(t) = 0 \quad \text{for all } t$$

Starting point: $x_0 = 0$

Type 2: Monotonic Decay to Zero ($x_0 < 0$)

$$x(t) = \frac{x_0}{1 - x_0 t} \quad \text{with } x_0 < 0$$

- Solution exists for all $t > 0$
- $x(t)$ increases (becomes less negative) toward 0
- $x(t) \rightarrow 0$ as $t \rightarrow \infty$
- Example: $x_0 = -1$ gives $x(t) = \frac{-1}{1+t}$

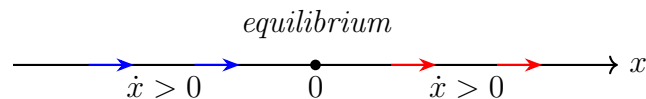
Type 3: Finite-Time Blow-Up ($x_0 > 0$)

$$x(t) = \frac{x_0}{1 - x_0 t} \quad \text{with } x_0 > 0$$

- Solution exists only on $t \in (-\infty, 1/x_0)$
- $x(t)$ increases without bound
- $x(t) \rightarrow +\infty$ as $t \rightarrow (1/x_0)^-$
- Blow-up time: $t^* = \frac{1}{x_0}$
- Example: $x_0 = 1$ gives $x(t) = \frac{1}{1-t}$ with $t^* = 1$

Explanation 5 (Phase Portrait). *On the phase line:*

- *Equilibrium at $x = 0$ (semi-stable: attracts from left, repels to right)*
- *For $x < 0$: flow toward zero ($\dot{x} = x^2 > 0$ means increasing)*
- *For $x > 0$: flow away from zero ($\dot{x} = x^2 > 0$ means increasing)*



Note: Both sides have $\dot{x} > 0$ because x^2 is always positive (except at $x = 0$).

Summary: Three distinct types of orbits based on initial condition.

(iii) Uniqueness of Orbits

Solution 4. Question: For a given x_0 , is there only one solution?

Answer: YES - solutions are unique

Explanation 6 (Why Uniqueness Holds). *From any initial condition x_0 , there is exactly one solution:*

- If $x_0 = 0$: unique solution $x(t) = 0$
- If $x_0 \neq 0$: unique solution $x(t) = \frac{x_0}{1-x_0 t}$

We derived these solutions using separation of variables, which gives a unique solution (up to the domain of existence).

The only subtlety is at $x_0 = 0$, where we cannot divide by x^2 . But directly from the ODE: if $x(0) = 0$, then $\dot{x}(0) = 0^2 = 0$, so the solution must remain at $x = 0$.

NO NON-UNIQUENESS: *Unlike part (c), this system has unique solutions from every initial condition.*

(iv) Lipschitz Continuity

Solution 5. Question: Is $f(x) = x^2$ Lipschitz continuous?

Step 1: Check Lipschitz Condition

For $f(x) = x^2$, compute the derivative:

$$f'(x) = 2x$$

Step 2: Is $|f'(x)|$ Bounded?

$$|f'(x)| = |2x| = 2|x|$$

This is **NOT** bounded as $x \rightarrow \pm\infty$.

Therefore, $f(x) = x^2$ is **NOT globally Lipschitz continuous**.

Step 3: Local Lipschitz Continuity

However, on any bounded interval $|x| \leq M$:

$$|f'(x)| = 2|x| \leq 2M$$

So f is Lipschitz continuous on any compact set with Lipschitz constant $L = 2M$.

Locally Lipschitz, but NOT globally Lipschitz

Explanation 7 (Lipschitz vs. Uniqueness). *Despite $f(x) = x^2$ not being globally Lipschitz, we still have uniqueness!*

Why? The Picard-Lindelöf theorem says:

- Lipschitz \Rightarrow Uniqueness (sufficient condition)
- But uniqueness can hold even without global Lipschitz (not necessary)

Local Lipschitz is enough: Since $f(x) = x^2$ is locally Lipschitz, and solutions remain in compact regions over finite time, uniqueness holds locally. As long as we haven't reached blow-up, uniqueness is guaranteed.

What about blow-up? The finite-time blow-up for $x_0 > 0$ doesn't violate uniqueness—the solution simply ceases to exist beyond $t^* = 1/x_0$. There's still only one solution on $[0, t^*)$.

Agreement with (iii):

- Uniqueness holds ✓
- Local Lipschitz condition guarantees this ✓
- Lack of global Lipschitz doesn't prevent uniqueness ✓
- Finite-time blow-up is consistent with local uniqueness ✓

2 Part (b): $\dot{x} = |x|$ with $x(0) = x_0$

Solving the ODE

Solution 6. Step 1: Case Analysis

Since $|x|$ has different forms depending on sign of x , we solve separately:

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

Case 1: $x > 0$

$$\frac{dx}{dt} = x \quad \Rightarrow \quad \frac{dx}{x} = dt$$

Integrating:

$$\ln |x| = t + C \quad \Rightarrow \quad x = Ae^t$$

With $x(0) = x_0 > 0$: $A = x_0$

$$\boxed{x(t) = x_0 e^t \quad \text{for } x_0 > 0}$$

Case 2: $x < 0$

$$\frac{dx}{dt} = -x \quad \Rightarrow \quad \frac{dx}{x} = -dt$$

Integrating:

$$\ln |x| = -t + C \quad \Rightarrow \quad x = -Be^{-t}$$

With $x(0) = x_0 < 0$: $-B = x_0$, so $B = -x_0 > 0$

$$\boxed{x(t) = x_0 e^{-t} \quad \text{for } x_0 < 0}$$

Case 3: $x = 0$

If $x(0) = 0$, then $\dot{x} = |0| = 0$, so:

$$\boxed{x(t) = 0 \quad \text{for all } t}$$

Complete Solution:

$$\boxed{x(t) = \begin{cases} x_0 e^t & \text{if } x_0 > 0 \\ 0 & \text{if } x_0 = 0 \\ x_0 e^{-t} & \text{if } x_0 < 0 \end{cases}}$$

Explanation 8 (Verification). *Check each case satisfies $\dot{x} = |x|$:*

- If $x_0 > 0$: $x = x_0 e^t > 0$, so $\dot{x} = x_0 e^t = x = |x|$
- If $x_0 = 0$: $x = 0$, so $\dot{x} = 0 = |0|$
- If $x_0 < 0$: $x = x_0 e^{-t} < 0$, so $\dot{x} = -x_0 e^{-t} = -x = |x|$

(i) Behavior for Three Initial Conditions

Solution 7. Case 1: $x_0 = 0$

$$x(t) = 0 \quad \text{for all } t$$

- **Where does it go?** Stays at $x = 0$ forever
- **How long?** Remains there for all time
- **Nature:** Equilibrium solution

Case 2: $x_0 = -1$

$$x(t) = -e^{-t}$$

- At $t = 0$: $x = -1$
- As $t \rightarrow \infty$: $x(t) = -e^{-t} \rightarrow 0^-$ (approaches zero from below)
- Exponential decay toward zero
- **Where does it go?** Approaches $x = 0$
- **How long?** Infinite time (asymptotic approach)

Case 3: $x_0 = +1$

$$x(t) = e^t$$

- At $t = 0$: $x = 1$
- As $t \rightarrow \infty$: $x(t) = e^t \rightarrow +\infty$
- Exponential growth
- **Where does it go?** Diverges to $+\infty$
- **How long?** Infinite time (but grows unboundedly)
- Solution exists for all $t \in (-\infty, \infty)$

Explanation 9 (Comparison to Part (a)). *Unlike part (a):*

- *NO finite-time blow-up (exponential growth is slower than x^2 growth)*
- *Solutions from $x_0 < 0$ decay exponentially (not algebraically)*
- *Solutions from $x_0 > 0$ grow exponentially (no singularity)*

Summary:

x_0	Long-time behavior	Time to destination
0	Stays at $x = 0$	All time
-1	$x(t) \rightarrow 0$ as $t \rightarrow \infty$	Infinite time
+1	$x(t) \rightarrow +\infty$ as $t \rightarrow \infty$	Infinite time

(ii) Different Orbits

Solution 8. Classification of Orbits:

Type 1: Equilibrium Orbit

$$x(t) = 0 \quad \text{for all } t$$

Starting point: $x_0 = 0$

Type 2: Exponential Decay to Zero ($x_0 < 0$)

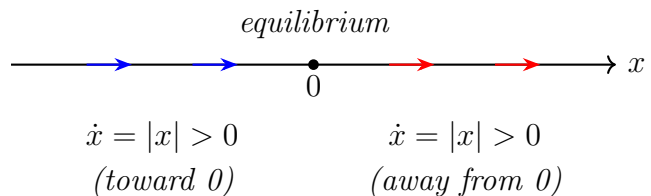
$$x(t) = x_0 e^{-t} \quad \text{with } x_0 < 0$$

- Solution exists for all $t \in \mathbb{R}$
- $x(t)$ increases (becomes less negative)
- $x(t) \rightarrow 0^-$ as $t \rightarrow \infty$
- Decay rate: e^{-t} (time constant $\tau = 1$)

Type 3: Exponential Growth to Infinity ($x_0 > 0$)

$$x(t) = x_0 e^t \quad \text{with } x_0 > 0$$

- Solution exists for all $t \in \mathbb{R}$
- $x(t)$ increases exponentially
- $x(t) \rightarrow +\infty$ as $t \rightarrow \infty$
- Growth rate: e^t (time constant $\tau = 1$)



Explanation 10 (Phase Portrait).

The equilibrium at $x = 0$ is semi-stable:

- *Stable from the left (attracts $x < 0$)*
- *Unstable to the right (repels $x > 0$)*

(iii) Uniqueness of Orbits

Solution 9. Answer: YES - solutions are unique

Explanation 11 (Why Uniqueness Holds). *For any initial condition x_0 :*

- *If $x_0 > 0$: unique solution $x(t) = x_0 e^t$*
- *If $x_0 = 0$: unique solution $x(t) = 0$*

- If $x_0 < 0$: unique solution $x(t) = x_0 e^{-t}$

Each solution was obtained by separation of variables in the appropriate region.

At $x = 0$, the solution must stay at zero because $\dot{x} = |0| = 0$.

No ambiguity: The piecewise nature of $|x|$ doesn't create non-uniqueness because solutions starting in one region ($x > 0$ or $x < 0$) never cross to the other region (except by approaching $x = 0$ asymptotically).

(iv) Lipschitz Continuity

Solution 10. Question: Is $f(x) = |x|$ Lipschitz continuous?

Step 1: Compute the Derivative

$$f'(x) = \frac{d}{dx}|x| = \begin{cases} +1 & \text{if } x > 0 \\ \text{undefined} & \text{if } x = 0 \\ -1 & \text{if } x < 0 \end{cases}$$

Step 2: Check Boundedness of $|f'(x)|$

For $x \neq 0$: $|f'(x)| = 1$ (bounded!)

But $f'(0)$ does not exist (derivative undefined at origin).

Step 3: Direct Lipschitz Test

For any x_1, x_2 :

$$|f(x_1) - f(x_2)| = ||x_1| - |x_2||$$

By the reverse triangle inequality:

$$||x_1| - |x_2|| \leq |x_1 - x_2|$$

Therefore:

$$|f(x_1) - f(x_2)| \leq |x_1 - x_2|$$

This holds with Lipschitz constant $L = 1$.

YES - globally Lipschitz continuous with $L = 1$

Explanation 12 (Lipschitz Despite Non-Differentiability). *Even though $f(x) = |x|$ is not differentiable at $x = 0$, it is still Lipschitz continuous!*

The key insight:

- Lipschitz continuity is about bounding $|f(x_1) - f(x_2)|$
- It doesn't require differentiability everywhere
- The "corner" at $x = 0$ is "mild" enough to satisfy Lipschitz condition

IMPORTANT DISTINCTION:

- Differentiable + bounded derivative \Rightarrow Lipschitz
- But Lipschitz \nRightarrow Differentiable everywhere
- $|x|$ is Lipschitz but not differentiable at 0

Agreement with (iii):

- $f(x) = |x|$ is Lipschitz continuous ✓
- Picard-Lindelöf theorem guarantees uniqueness ✓
- Solutions are indeed unique ✓
- Perfect agreement! ✓

3 Part (c): $\dot{x} = |x|^{1/2}$ with $x(0) = x_0$

Solving the ODE

Solution 11. Note: This problem is provided with the solution in the problem statement. We verify and analyze it.

Given Solution:

$$x(t) = \begin{cases} +(|x_0|^{1/2} + \frac{1}{2}t)^2 & \text{if } x_0 \geq 0 \\ 0 & \text{if } x_0 = 0 \\ -(|x_0|^{1/2} - \frac{1}{2}t)^2 & \text{if } x_0 \leq 0 \end{cases}$$

Step 1: Verify the Solution for $x_0 > 0$

Let $x(t) = \left(\sqrt{x_0} + \frac{t}{2}\right)^2$ with $x_0 > 0$

Check initial condition:

$$x(0) = (\sqrt{x_0})^2 = x_0 \quad \checkmark$$

Compute derivative:

$$\frac{dx}{dt} = 2 \left(\sqrt{x_0} + \frac{t}{2} \right) \cdot \frac{1}{2} = \sqrt{x_0} + \frac{t}{2}$$

Since $x(t) > 0$ for all $t \geq 0$:

$$|x|^{1/2} = \sqrt{x} = \sqrt{\left(\sqrt{x_0} + \frac{t}{2}\right)^2} = \sqrt{x_0} + \frac{t}{2}$$

Therefore $\dot{x} = |x|^{1/2}$

Step 2: Verify the Solution for $x_0 < 0$

Let $x(t) = -\left(\sqrt{|x_0|} - \frac{t}{2}\right)^2$ with $x_0 < 0$

For $0 \leq t < 2\sqrt{|x_0|}$, we have $x(t) < 0$, so:

$$|x|^{1/2} = \sqrt{-x} = \sqrt{\left(\sqrt{|x_0|} - \frac{t}{2}\right)^2} = \sqrt{|x_0|} - \frac{t}{2}$$

Compute derivative:

$$\frac{dx}{dt} = -2 \left(\sqrt{|x_0|} - \frac{t}{2} \right) \cdot \left(-\frac{1}{2} \right) = \sqrt{|x_0|} - \frac{t}{2}$$

Therefore $\dot{x} = |x|^{1/2}$

Explanation 13 (Key Observation). For $x_0 < 0$, the solution $x(t) = -\left(\sqrt{|x_0|} - \frac{t}{2}\right)^2$ reaches zero at time:

$$t^* = 2\sqrt{|x_0|}$$

At this point, the solution becomes $x(t) = 0$ for all $t \geq t^*$.

This is because once $x = 0$, we have $\dot{x} = |0|^{1/2} = 0$, so the solution stays at zero.

(i) Behavior for Three Initial Conditions

Solution 12. Case 1: $x_0 = 0$

$$x(t) = 0 \quad \text{for all } t$$

- **Where does it go?** Stays at $x = 0$ forever
- **How long?** Remains there for all time
- **Nature:** Equilibrium solution

Case 2: $x_0 = -1$

$$x(t) = -\left(1 - \frac{t}{2}\right)^2 \quad \text{for } 0 \leq t \leq 2$$

After $t = 2$: $x(t) = 0$

- At $t = 0$: $x = -1$
- At $t = 1$: $x = -\left(\frac{1}{2}\right)^2 = -\frac{1}{4}$
- At $t = 2$: $x = 0$
- For $t \geq 2$: $x = 0$
- **Where does it go?** Reaches $x = 0$ and stays there
- **How long?** Reaches zero in finite time $t^* = 2$

Explanation 14 (Finite-Time Arrival at Zero). *Unlike parts (a) and (b), where solutions from $x_0 < 0$ approach zero asymptotically (taking infinite time), here the solution reaches zero in finite time!*

This is because $\dot{x} = |x|^{1/2}$ is a sub-linear growth rate. Near $x = 0$:

$$\frac{dx}{dt} \sim \sqrt{|x|} \rightarrow 0 \quad \text{slowly enough that} \quad \int_0^{x_0} \frac{dx}{\sqrt{|x|}} < \infty$$

Case 3: $x_0 = +1$

$$x(t) = \left(1 + \frac{t}{2}\right)^2$$

- At $t = 0$: $x = 1$
- At $t = 2$: $x = (2)^2 = 4$
- At $t = 4$: $x = (3)^2 = 9$
- As $t \rightarrow \infty$: $x(t) \sim \frac{t^2}{4} \rightarrow +\infty$
- **Where does it go?** Diverges to $+\infty$
- **How long?** Infinite time (algebraic growth)

- Growth rate: $\sim t^2$ (slower than exponential)

Summary:

x_0	Long-time behavior	Time to destination
0	Stays at $x = 0$	All time
-1	Reaches $x = 0$	Finite time ($t^* = 2$)
+1	$x(t) \rightarrow +\infty$	Infinite time

(ii) Different Orbits

Solution 13. Classification of Orbits:

Type 1: Equilibrium Orbit

$$x(t) = 0 \quad \text{for all } t$$

Starting point: $x_0 = 0$

Type 2: Finite-Time Arrival at Zero ($x_0 < 0$)

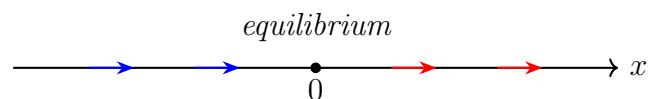
$$x(t) = \begin{cases} -\left(\sqrt{|x_0|} - \frac{t}{2}\right)^2 & \text{for } 0 \leq t \leq 2\sqrt{|x_0|} \\ 0 & \text{for } t > 2\sqrt{|x_0|} \end{cases}$$

- Solution increases from x_0 to 0
- Reaches zero at time $t^* = 2\sqrt{|x_0|}$
- Stays at zero thereafter

Type 3: Algebraic Growth to Infinity ($x_0 > 0$)

$$x(t) = \left(\sqrt{x_0} + \frac{t}{2}\right)^2$$

- Solution exists for all $t \geq 0$
- Grows like $\sim t^2$ for large t
- $x(t) \rightarrow +\infty$ as $t \rightarrow \infty$



$$\dot{x} = |x|^{1/2} > 0 \quad \dot{x} = |x|^{1/2} > 0$$

Explanation 15 (Phase Portrait). *(toward 0, finite time) (away from 0)*

Equilibrium at $x = 0$ is semi-stable, but with a crucial difference from parts (a) and (b): trajectories from $x < 0$ reach the equilibrium in finite time.

(iii) Uniqueness of Orbits

Solution 14. Answer: NO - solutions are NOT unique from $x_0 = 0$

Explanation 16 (Non-Uniqueness at Origin). *From the initial condition $x(0) = 0$, we have **infinitely many solutions!***

Solution 1: Stay at zero

$$x(t) = 0 \quad \text{for all } t$$

This clearly satisfies $\dot{x} = |0|^{1/2} = 0$ and $x(0) = 0$.

Solution 2: Leave zero at any time $t_0 \geq 0$

For any $t_0 \geq 0$, define:

$$x(t) = \begin{cases} 0 & \text{for } 0 \leq t \leq t_0 \\ \left(\frac{t-t_0}{2}\right)^2 & \text{for } t > t_0 \end{cases}$$

Verification:

- Initial condition: $x(0) = 0$
- For $0 \leq t \leq t_0$: $\dot{x} = 0 = |0|^{1/2}$
- For $t > t_0$: $\dot{x} = 2 \cdot \frac{t-t_0}{2} \cdot \frac{1}{2} = \frac{t-t_0}{2}$ and $|x|^{1/2} = \frac{t-t_0}{2}$
- Continuity at $t = t_0$: $\lim_{t \rightarrow t_0^-} x(t) = 0 = \lim_{t \rightarrow t_0^+} x(t)$
- Derivative at $t = t_0$: Both sides give $\dot{x} = 0$

So for EVERY choice of $t_0 \geq 0$, we get a different valid solution!

INFINITELY MANY SOLUTIONS FROM $x_0 = 0$

This is a fundamental failure of uniqueness.

From other initial conditions ($x_0 \neq 0$):

Solutions ARE unique:

- If $x_0 > 0$: unique solution $x(t) = \left(\sqrt{x_0} + \frac{t}{2}\right)^2$
- If $x_0 < 0$: unique solution given by the piecewise formula

The non-uniqueness is localized to $x = 0$.

(iv) Lipschitz Continuity

Solution 15. Question: Is $f(x) = |x|^{1/2}$ Lipschitz continuous?

Step 1: Compute the Derivative (where it exists)

For $x > 0$:

$$f(x) = x^{1/2} \quad \Rightarrow \quad f'(x) = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$$

For $x < 0$:

$$f(x) = (-x)^{1/2} \quad \Rightarrow \quad f'(x) = -\frac{1}{2}(-x)^{-1/2} = \frac{1}{2\sqrt{-x}}$$

At $x = 0$: derivative does not exist.

Step 2: Check Boundedness

$$\lim_{x \rightarrow 0^+} f'(x) = \lim_{x \rightarrow 0^+} \frac{1}{2\sqrt{x}} = +\infty$$

$$\lim_{x \rightarrow 0^-} f'(x) = \lim_{x \rightarrow 0^-} \frac{1}{2\sqrt{-x}} = +\infty$$

The derivative is **UNBOUNDED** near $x = 0$!

Step 3: Direct Lipschitz Test

For Lipschitz continuity, we need:

$$|f(x_1) - f(x_2)| \leq L|x_1 - x_2| \quad \text{for some constant } L$$

Take $x_1 = h > 0$ and $x_2 = 0$ with small h :

$$|f(h) - f(0)| = |\sqrt{h} - 0| = \sqrt{h}$$

$$|x_1 - x_2| = |h - 0| = h$$

For Lipschitz: $\sqrt{h} \leq L \cdot h$, which gives $L \geq \frac{\sqrt{h}}{h} = \frac{1}{\sqrt{h}}$

As $h \rightarrow 0$: $\frac{1}{\sqrt{h}} \rightarrow \infty$

Therefore, no finite L exists!

NOT Lipschitz continuous at $x = 0$

Explanation 17 (Why Not Lipschitz). *The function $f(x) = |x|^{1/2}$ has a "vertical tangent" at $x = 0$:*

- The slope approaches infinity as we approach the origin
- This violates the Lipschitz condition, which requires bounded slopes
- The "corner" at $x = 0$ is too sharp

Compare to part (b): $|x|$ has a "corner" but finite slope (± 1), so it's Lipschitz.
Here: $|x|^{1/2}$ has infinite slope at 0, so NOT Lipschitz.

Agreement with (iii):

- $f(x) = |x|^{1/2}$ is NOT Lipschitz continuous at $x = 0$ ✓
- Picard-Lindelöf theorem does NOT guarantee uniqueness ✓
- Solutions from $x_0 = 0$ are NOT unique ✓
- Perfect agreement! ✓

THE LIPSCHITZ CONDITION IS CRUCIAL FOR UNIQUENESS

When the Lipschitz condition fails, uniqueness can fail. This example demonstrates why the Lipschitz condition in the Picard-Lindelöf theorem is not just a technical requirement—it's essential!

4 Summary and Comparison

Comparison of All Three Parts

Property	(a) $\dot{x} = x^2$	(b) $\dot{x} = x $	(c) $\dot{x} = x ^{1/2}$
Solutions from Different Initial Conditions			
$x_0 = 0$	$x(t) = 0$	$x(t) = 0$	$x(t) = 0$ (or others!)
$x_0 = -1$	$\frac{-1}{1+t} \rightarrow 0$	$-e^{-t} \rightarrow 0$	Reaches 0 at $t = 2$
$x_0 = +1$	$\frac{1}{1-t}$ blows up	$e^t \rightarrow \infty$	$(1 + \frac{t}{2})^2 \rightarrow \infty$
Time to Reach Destination			
From $x_0 = -1$ to 0	Infinite	Infinite	Finite ($t = 2$)
From $x_0 = +1$ to ∞	Finite ($t = 1$)	Infinite	Infinite
Orbit Types			
Number of orbit types	3	3	3
Equilibrium	Yes (at 0)	Yes (at 0)	Yes (at 0)
Finite-time blow-up	Yes ($x_0 > 0$)	No	No
Finite-time arrival	No	No	Yes ($x_0 < 0$)
Uniqueness and Lipschitz			
Unique solutions?	YES	YES	NO (at $x_0 = 0$)
Globally Lipschitz?	NO	YES	NO
Locally Lipschitz?	YES	YES	NO (at $x = 0$)
$f'(x)$ at $x = 0$	$f'(0) = 0$	Undefined	$+\infty$
Growth Rates			
Type	Superlinear	Linear (exp)	Sublinear
\dot{x} for large x	$\sim x^2$	$\sim x$	$\sim \sqrt{x}$

Key Insights from This Exercise

Explanation 18 (The Role of Lipschitz Continuity). *What we learned:*

- Sufficient but not necessary:** Lipschitz \Rightarrow uniqueness, but uniqueness doesn't require Lipschitz (see part (a))
- Local vs. Global:** Local Lipschitz often sufficient for local uniqueness
- When it fails:**
 - Part (c): $f(x) = |x|^{1/2}$ has $f'(0) = \infty$
 - Result: infinitely many solutions from $x_0 = 0$
 - The vertical tangent allows "branching" of solutions
- Differentiability:**
 - Part (b): $f(x) = |x|$ not differentiable at 0, but still Lipschitz
 - Lipschitz is a weaker condition than differentiability

Finite-Time Phenomena

Explanation 19 (Finite-Time Blow-Up vs. Finite-Time Arrival). ***Finite-Time Blow-Up (Part a, $x_0 > 0$):***

- Growth rate $\dot{x} = x^2$ is superlinear
- Solution reaches $+\infty$ in finite time
- Example: $x(t) = \frac{1}{1-t}$ blows up at $t = 1$

Finite-Time Arrival (Part c, $x_0 < 0$):

- Growth rate $\dot{x} = |x|^{1/2}$ is sublinear
- Solution reaches 0 in finite time (from below)
- Example: $x(t) = -(1 - \frac{t}{2})^2$ reaches 0 at $t = 2$
- After arrival, stays at 0 (or can leave—non-uniqueness!)

Infinite-Time Asymptotic (Part b):

- Growth rate $\dot{x} = |x|$ is linear
- Exponential approach/departure
- Never reaches destination in finite time

Mathematical Rigor: What This Exercise Teaches

1. **Always check assumptions:** Picard-Lindelöf requires Lipschitz—verify it!
2. **Non-uniqueness is possible:** Part (c) shows multiple solutions from same IC
3. **Finite-time events:** Both blow-up and arrival can occur in finite time
4. **Careful case analysis:** Functions like $|x|$ require splitting into cases
5. **Verification matters:** Always check solutions satisfy the ODE and IC

PROFOUND LESSON: The seemingly small difference between $|x|$ and $|x|^{1/2}$ (both non-smooth at origin) leads to dramatically different behavior regarding uniqueness. The rate of growth near the equilibrium determines whether uniqueness holds!

END OF QUESTION 6