Exercise Sheet 4: Maps
Question 5 - Complete Solution

Methods of Applied Mathematics

Problem Statement

Solve the map:

$$x_{n+1} = 2x_n - y_n$$
$$y_{n+1} = 2y_n - x_n$$

with initial condition $x_0 = 1$, $y_0 = 0$.
Part (a): Iterate the map repeatedly until a pattern emerges.
Part (b): Use eigenvalue decomposition as for ODEs, but with λ^n powers instead of $e^{\lambda t}$ terms.

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1 Part (a): Direct Iteration Method

Set up iteration procedure

The map is a discrete-time dynamical system. Starting from $(x_0, y_0) = (1, 0)$, we repeatedly apply:

$$x_{n+1} = 2x_n - y_n$$
$$y_{n+1} = 2y_n - x_n$$

Compute successive iterates

Iteration 0 → 1:

$$x_1 = 2x_0 - y_0 = 2(1) - 0 = 2$$
$$y_1 = 2y_0 - x_0 = 2(0) - 1 = -1$$

Iteration 1 → 2:

$$x_2 = 2x_1 - y_1 = 2(2) - (-1) = 5$$
$$y_2 = 2y_1 - x_1 = 2(-1) - 2 = -4$$

Iteration 2 → 3:

$$x_3 = 2x_2 - y_2 = 2(5) - (-4) = 14$$
$$y_3 = 2y_2 - x_2 = 2(-4) - 5 = -13$$

Iteration 3 → 4:

$$x_4 = 2x_3 - y_3 = 2(14) - (-13) = 41$$
$$y_4 = 2y_3 - x_3 = 2(-13) - 14 = -40$$

Iteration 4 → 5:

$$x_5 = 2x_4 - y_4 = 2(41) - (-40) = 122$$
$$y_5 = 2y_4 - x_4 = 2(-40) - 41 = -121$$

Tabulate results

n	x_n	y_n
0	1	0
1	2	-1
2	5	-4
3	14	-13
4	41	-40
5	122	-121

XYZ Analysis of Iteration Pattern

- **STAGE X (What we observe):** The values grow rapidly. The x_n sequence is $1, 2, 5, 14, 41, 122, \ldots$ and the y_n sequence is $0, -1, -4, -13, -40, -121, \ldots$. Note that x_n and y_n are always close: $x_n - y_n = 1, 3, 9, 27, 81, 243, \ldots = 3^n$.
- **STAGE Y (Why this pattern):** Looking at combinations:
  - $x_n + y_n$: $1, 1, 1, 1, 1, 1, \ldots$ (constant!)
  - $x_n - y_n$: $1, 3, 9, 27, 81, 243, \ldots$ (powers of 3: 3^n)

  The map preserves $x_n + y_n = 1$ but amplifies the difference $x_n - y_n$ by factor of 3 each iteration. This suggests the system has two eigenvalues: $\lambda_1 = 3$ (exponential growth in the difference direction) and $\lambda_2 = 1$ (constant in the sum direction).
- **STAGE Z (What this means):** From the pattern $x_n - y_n = 3^n$ and $x_n + y_n = 1$, we can solve:
  $$x_n = \frac{(x_n + y_n) + (x_n - y_n)}{2} = \frac{1 + 3^n}{2}$$
  $$y_n = \frac{(x_n + y_n) - (x_n - y_n)}{2} = \frac{1 - 3^n}{2}$$
  This gives explicit formulas, which we'll verify rigorously in part (b).

Conjectured solution from iteration

$$\boxed{x_n = \frac{3^n + 1}{2}, \quad y_n = \frac{1 - 3^n}{2}}$$

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2 Part (b): Eigenvalue Decomposition Method

Step 1: Write system in matrix form

The map can be written as:

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} x_n \\ y_n \end{pmatrix}$$

Define the matrix:

$$A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

XYZ Analysis of Matrix Structure

- **STAGE X (What we have):** A 2×2 symmetric matrix with 2's on the diagonal and -1's off-diagonal.
- **STAGE Y (Why this form matters):** Symmetry guarantees real eigenvalues and orthogonal eigenvectors. The structure $A = 2I - J$ where J is the all-ones off-diagonal suggests eigenvalues related to sum and difference coordinates.
- **STAGE Z (What to compute):** Find eigenvalues λ and eigenvectors $\mathbf{v}$ to decompose the solution as $\mathbf{x}_n = \alpha_1 \lambda_1^n \mathbf{v}_1 + \alpha_2 \lambda_2^n \mathbf{v}_2$.

Step 2: Find eigenvalues

Solve the characteristic equation $\det(A - \lambda I) = 0$:

$$\det \begin{pmatrix} 2-\lambda & -1 \\ -1 & 2-\lambda \end{pmatrix} = 0$$

Expand:

$$\begin{aligned} (2-\lambda)^2 - (-1)(-1) &= 0 \\ (2-\lambda)^2 - 1 &= 0 \\ 4 - 4\lambda + \lambda^2 - 1 &= 0 \\ \lambda^2 - 4\lambda + 3 &= 0 \end{aligned}$$

Factor:

$$(\lambda - 3)(\lambda - 1) = 0$$

Therefore:

$$\boxed{\lambda_1 = 3, \quad \lambda_2 = 1}$$

XYZ Analysis of Eigenvalues

- **STAGE X (What we found):** Two positive real eigenvalues: $\lambda_1 = 3 > 1$ and $\lambda_2 = 1$.
- **STAGE Y (Why these values):**
  - $\lambda_1 = 3 > 1$: This eigenvalue causes exponential growth. Points in this eigendirection grow by factor 3 each iteration.
  - $\lambda_2 = 1$: This eigenvalue preserves magnitude. Points in this eigendirection remain at constant distance from origin.

  For stability analysis: $|\lambda_1| = 3 > 1$ means unstable (exponential growth), $|\lambda_2| = 1$ means marginally stable (neutral). Any initial condition with nonzero component in the λ_1 eigendirection will grow to infinity.
- **STAGE Z (What this means dynamically):** The system has one unstable direction (growing like 3^n) and one neutral direction (constant). Our initial condition $(1, 0)$ must have components in both directions, explaining why the iterations showed both growth and a constant pattern.

Step 3: Find eigenvectors

For $\lambda_1 = 3$:
Solve $(A - 3I)\mathbf{v}_1 = \mathbf{0}$:

$$\begin{pmatrix} 2-3 & -1 \\ -1 & 2-3 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

From first row: $-v_1 - v_2 = 0 \Rightarrow v_2 = -v_1$
Choose $v_1 = 1$, then $v_2 = -1$:

$$\boxed{\mathbf{v}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}}$$

For $\lambda_2 = 1$:
Solve $(A - I)\mathbf{v}_2 = \mathbf{0}$:

$$\begin{pmatrix} 2-1 & -1 \\ -1 & 2-1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

From first row: $v_1 - v_2 = 0 \Rightarrow v_2 = v_1$
Choose $v_1 = 1$, then $v_2 = 1$:

$$\boxed{\mathbf{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}}$$

Verify orthogonality

$$\mathbf{v}_1 \cdot \mathbf{v}_2 = (1)(1) + (-1)(1) = 1 - 1 = 0 \quad \checkmark$$

The eigenvectors are orthogonal, as expected for a symmetric matrix.

XYZ Analysis of Eigenvectors

- **STAGE X (What we found):**
  - $\mathbf{v}_1 = (1, -1)^T$: the "difference" direction
  - $\mathbf{v}_2 = (1, 1)^T$: the "sum" direction
- **STAGE Y (Why these directions):**
  - $\mathbf{v}_1 = (1, -1)$: Points in this direction have $x = -y$ (opposite signs). This is the direction where $x - y$ is maximized. Along this direction, the map scales by $\lambda_1 = 3$.
  - $\mathbf{v}_2 = (1, 1)$: Points in this direction have $x = y$ (same values). This is the direction where $x + y$ is constant. Along this direction, the map scales by $\lambda_2 = 1$ (unchanged).

  These match the observed pattern from part (a): $x_n - y_n$ grows like 3^n, while $x_n + y_n$ stays constant at 1.
- **STAGE Z (What this geometric picture means):** Any initial point can be decomposed into components along these two perpendicular axes. The component along $\mathbf{v}_1$ grows exponentially, while the component along $\mathbf{v}_2$ remains constant. This explains the long-term behavior: trajectories move along lines parallel to $\mathbf{v}_1$ while maintaining their projection onto $\mathbf{v}_2$.

Step 4: General solution

The general solution has the form:

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = \alpha_1 \lambda_1^n \mathbf{v}_1 + \alpha_2 \lambda_2^n \mathbf{v}_2$$

Substituting our eigenvalues and eigenvectors:

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = \alpha_1 (3)^n \begin{pmatrix} 1 \\ -1 \end{pmatrix} + \alpha_2 (1)^n \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Simplify:

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = \alpha_1 \cdot 3^n \begin{pmatrix} 1 \\ -1 \end{pmatrix} + \alpha_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

In component form:

$$x_n = \alpha_1 \cdot 3^n + \alpha_2$$
$$y_n = -\alpha_1 \cdot 3^n + \alpha_2$$

XYZ Analysis of General Solution Form

- **STAGE X (What the formula shows):** The solution is a linear combination of two modes: one that grows exponentially (3^n term) and one that is constant (the α_2 term).
- **STAGE Y (Why this structure):** Unlike ODEs where solutions involve $e^{\lambda t}$ (continuous exponential growth), maps have discrete time steps, so solutions involve λ^n (discrete exponential growth). Each iteration multiplies by λ rather than adding $\lambda\, dt$. The constants α_1, α_2 weight how much of each eigenmode is present, determined by projecting the initial condition onto the eigenvectors.
- **STAGE Z (What remains):** We need two equations (the initial conditions) to determine two unknowns (α_1, α_2). Once found, we have the complete explicit solution for all time steps n.

Step 5: Apply initial conditions

At $n = 0$: $(x_0, y_0) = (1, 0)$

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \alpha_1 (3)^0 \begin{pmatrix} 1 \\ -1 \end{pmatrix} + \alpha_2 (1)^0 \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \alpha_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} + \alpha_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

This gives two equations:

$$1 = \alpha_1 + \alpha_2 \tag{1}$$
$$0 = -\alpha_1 + \alpha_2 \tag{2}$$

From equation (2):

$$\alpha_2 = \alpha_1$$

Substitute into equation (1):

$$1 = \alpha_1 + \alpha_1 = 2\alpha_1 \quad \Rightarrow \quad \alpha_1 = \frac{1}{2}$$

Therefore:

$$\boxed{\alpha_1 = \frac{1}{2}, \quad \alpha_2 = \frac{1}{2}}$$

XYZ Analysis of Initial Condition Decomposition

- **STAGE X (What we found):** The initial condition $(1, 0)$ has equal weight $(1/2)$ in both eigendirections.
- **STAGE Y (Why equal weights):** The initial point $(1, 0)$ lies exactly halfway between the two eigenvector directions:

  $$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

  Geometrically, $(1, 0)$ is the average of $(1, -1)$ and $(1, 1)$. Since both components are present, the trajectory will show both exponential growth (from the $\lambda_1 = 3$ mode) and a constant background (from the $\lambda_2 = 1$ mode).
- **STAGE Z (What this predicts):** With equal weights, the solution will be $x_n = \frac{1}{2}3^n + \frac{1}{2}$ and $y_n = -\frac{1}{2}3^n + \frac{1}{2}$. For large n, the 3^n terms dominate, and the trajectory approaches the unstable eigendirection $(1, -1)$ (moving along the line $y = -x + 1$).

Step 6: Write explicit solution

Substitute $\alpha_1 = \alpha_2 = 1/2$ into the general solution:

$$x_n = \frac{1}{2} \cdot 3^n + \frac{1}{2} = \frac{3^n + 1}{2}$$
$$y_n = -\frac{1}{2} \cdot 3^n + \frac{1}{2} = \frac{1 - 3^n}{2}$$

$$\boxed{x_n = \frac{3^n + 1}{2}, \quad y_n = \frac{1 - 3^n}{2}}$$

Step 7: Verify solution

Check initial condition at $n = 0$:

$$x_0 = \frac{3^0 + 1}{2} = \frac{1 + 1}{2} = 1 \quad \checkmark$$
$$y_0 = \frac{1 - 3^0}{2} = \frac{1 - 1}{2} = 0 \quad \checkmark$$

Check map is satisfied at $n = 0 \to 1$:

$$x_1 = \frac{3^1 + 1}{2} = \frac{4}{2} = 2$$
$$\text{From map: } x_1 = 2x_0 - y_0 = 2(1) - 0 = 2 \quad \checkmark$$

$$y_1 = \frac{1 - 3^1}{2} = \frac{-2}{2} = -1$$
$$\text{From map: } y_1 = 2y_0 - x_0 = 2(0) - 1 = -1 \quad \checkmark$$

Check map is satisfied at $n = 1 \to 2$:

$$x_2 = \frac{3^2 + 1}{2} = \frac{10}{2} = 5$$
$$\text{From map: } x_2 = 2x_1 - y_1 = 2(2) - (-1) = 5 \quad \checkmark$$

$$y_2 = \frac{1 - 3^2}{2} = \frac{-8}{2} = -4$$
$$\text{From map: } y_2 = 2y_1 - x_1 = 2(-1) - 2 = -4 \quad \checkmark$$

General verification:
We verify that x_n, y_n satisfy the original map equations. From our solution:

$$\begin{aligned}
2x_n - y_n &= 2 \cdot \frac{3^n + 1}{2} - \frac{1 - 3^n}{2} \\
&= \frac{2(3^n + 1) - (1 - 3^n)}{2} \\
&= \frac{2 \cdot 3^n + 2 - 1 + 3^n}{2} \\
&= \frac{3 \cdot 3^n + 1}{2} \\
&= \frac{3^{n+1} + 1}{2} = x_{n+1} \quad \checkmark
\end{aligned}$$

$$\begin{aligned}
2y_n - x_n &= 2 \cdot \frac{1 - 3^n}{2} - \frac{3^n + 1}{2} \\
&= \frac{2(1 - 3^n) - (3^n + 1)}{2} \\
&= \frac{2 - 2 \cdot 3^n - 3^n - 1}{2} \\
&= \frac{1 - 3 \cdot 3^n}{2} \\
&= \frac{1 - 3^{n+1}}{2} = y_{n+1} \quad \checkmark
\end{aligned}$$

The solution is verified!

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3 Summary and Comparison

Both methods yield the same result

$$\boxed{x_n = \frac{3^n + 1}{2}, \quad y_n = \frac{1 - 3^n}{2}}$$

Method comparison

Method (a): Direct Iteration	Method (b): Eigenvalue Decomposition
Pros: - Simple to implement - No linear algebra required - Easy to program - Good for short-term behavior	**Pros:** - Gives explicit closed-form solution - Reveals system structure (eigenmodes) - Efficient for computing x_n for large n - Explains long-term behavior analytically
Cons: - Must compute every step - Impractical for large n - Pattern recognition needed - No insight into system structure	**Cons:** - Requires eigenvalue computation - More complex setup - Requires understanding of linear algebra - Can be difficult for large systems

XYZ Analysis of Solution Structure

- **STAGE X (What the solution tells us):**
  - x_n grows like $3^n/2$ for large n
  - y_n grows like $-3^n/2$ for large n (same magnitude, opposite sign)
  - The ratio $y_n/x_n \to -1$ as $n \to \infty$
- **STAGE Y (Why this behavior):** The dominant eigenvalue $\lambda_1 = 3$ with eigenvector $(1, -1)$ controls long-term dynamics. The system is unstable: trajectories escape to infinity along the unstable eigendirection. The $\lambda_2 = 1$ mode contributes a constant background $(1/2, 1/2)$, but becomes negligible relative to the growing 3^n terms. This is characteristic of linear maps where $|\lambda_{\max}| > 1$.
- **STAGE Z (What this means):**
  - **Asymptotic behavior:** $(x_n, y_n) \approx (3^n/2, -3^n/2)$ as $n \to \infty$
  - **Trajectory shape:** Points move along lines parallel to $(1, -1)$
  - **Growth rate:** Distance from origin grows like $\sqrt{x_n^2 + y_n^2} \sim 3^n/\sqrt{2}$
  - **Doubling time:** Since $3^n = e^{n \ln 3}$, the system grows by factor e every $1/\ln 3 \approx 0.91$ iterations

Connection to ODEs

The key difference between maps and ODEs:

ODEs	Maps
$\mathbf{x}(t) = \alpha_1 e^{\lambda_1 t}\mathbf{v}_1 + \alpha_2 e^{\lambda_2 t}\mathbf{v}_2$	$\mathbf{x}_n = \alpha_1 \lambda_1^n \mathbf{v}_1 + \alpha_2 \lambda_2^n \mathbf{v}_2$
Continuous time	Discrete time
Stability: $\mathrm{Re}(\lambda) < 0$	Stability: $|\lambda| < 1$
$e^{\lambda t}$ terms	λ^n terms

For our system: $\lambda_1 = 3 > 1$ (unstable), $\lambda_2 = 1$ (marginally stable). The origin is an unstable fixed point.