# Methods of Applied Mathematics - Part 1
# Exercise Sheet 2: Question 6
# Topological Equivalence

Complete Solution with XYZ Methodology

## Problem Statement

Consider the two linear systems:

**System 1:**

$$\dot{x}_1 = x_1 \tag{1}$$
$$\dot{x}_2 = x_2 \tag{2}$$

**System 2:**

$$\dot{y}_1 = y_1 - y_2 \tag{3}$$
$$\dot{y}_2 = y_1 + y_2 \tag{4}$$

## 1 Question 6(a): Sketch 2D Phase Portraits

### Step 1: Analyze System 1

**Solution 1.** • **STAGE X (System structure):** System 1 can be written in matrix form as:

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = I \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \tag{5}$$

- **STAGE Y (Why analysis is simple):** The matrix $A_1 = I$ is already diagonal. Eigenvalues: $\lambda_1 = \lambda_2 = 1$. Eigenvectors: any direction (identity matrix).
- **STAGE Z (Classification):** From Lecture Notes (Section 8, page 29), with both eigenvalues positive and equal, this is an **unstable star node**.

**Solution for System 1:**

$$\mathbf{x}(t) = e^t \mathbf{x}_0 \quad \Rightarrow \quad \begin{cases} x_1(t) = x_{1,0} e^t \\ x_2(t) = x_{2,0} e^t \end{cases} \tag{6}$$

**Explanation 1** (Phase Portrait Features - System 1)**.**

- *Trajectories are straight rays from the origin*
- *Ratio $x_2/x_1$ constant along each trajectory*
- *All trajectories escape to infinity exponentially*
- *Classical **unstable star node***

## Step 2: Analyze System 2

Matrix form:

$$\begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \tag{7}$$

Let $A_2 = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$.

**Find Eigenvalues:**

$$\begin{aligned}
\det(A_2 - \lambda I) &= \det \begin{pmatrix} 1-\lambda & -1 \\ 1 & 1-\lambda \end{pmatrix} && (8) \\
&= (1-\lambda)^2 + 1 && (9) \\
&= 1 - 2\lambda + \lambda^2 + 1 && (10) \\
&= \lambda^2 - 2\lambda + 2 && (11)
\end{aligned}$$

Using quadratic formula:

$$\begin{aligned}
\lambda &= \frac{2 \pm \sqrt{4-8}}{2} = \frac{2 \pm \sqrt{-4}}{2} = \frac{2 \pm 2i}{2} && (12) \\
&= 1 \pm i && (13)
\end{aligned}$$

Therefore: $\lambda_{1,2} = 1 \pm i$ (complex conjugate pair).

**Explanation 2** (Classification - System 2)**.** *From Lecture Notes (Section 8, page 29):*

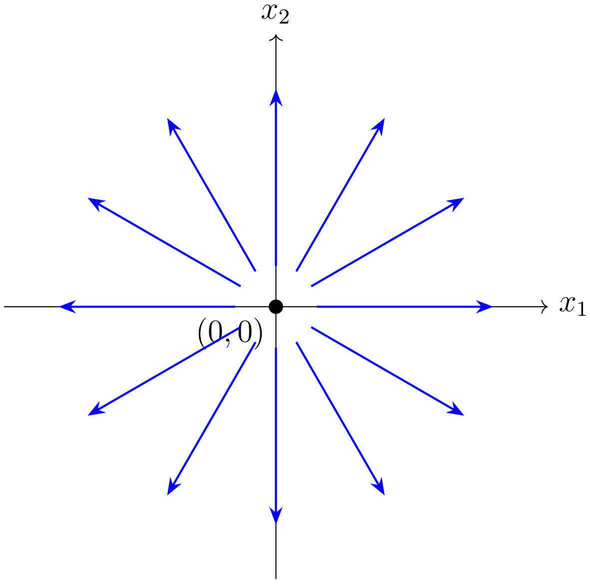
- *Complex eigenvalues* $\Rightarrow$ *rotation (spiraling)*
- *Re*$(\lambda) = 1 > 0 \Rightarrow$ *unstable (outward spiral)*
- *Im*$(\lambda) = \pm 1 \Rightarrow$ *angular frequency* $\omega = 1$

*This is an **unstable focus** (outward spiral).*

## Step 3: Sketch Phase Portraits

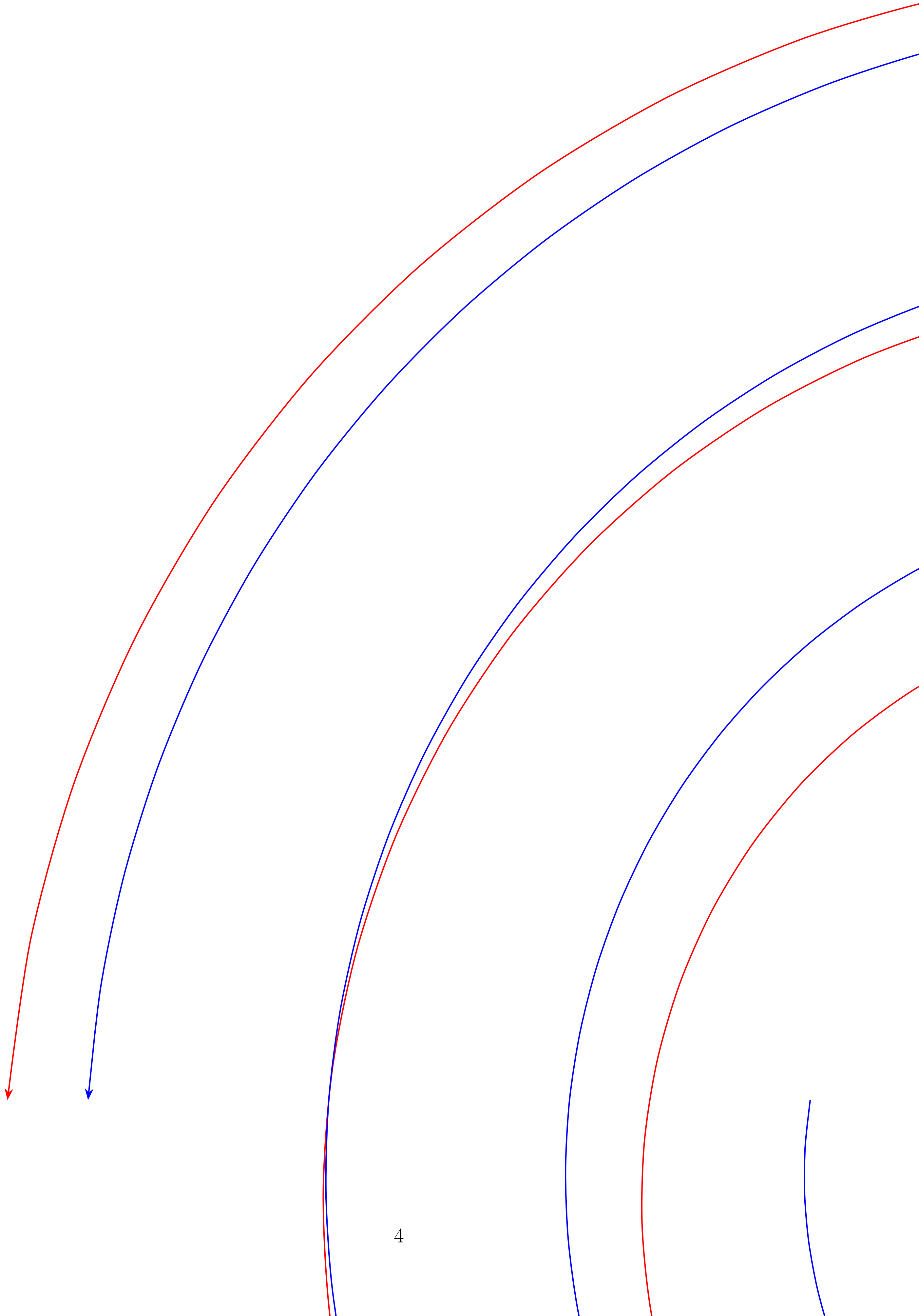
**System 1: Unstable Star Node**

## System 1




Unstable star node
$\lambda = 1, 1$ (real, positive)

## System 2: Unstable Focus



### Final Answer for Part (a)

$$\boxed{\begin{array}{l}\textbf{System 1: } \text{Unstable star node with radial trajectories} \\ \textbf{System 2: } \text{Unstable focus with outward spiraling trajectories}\end{array}} \tag{14}$$

Both systems have unstable equilibria at the origin, but with qualitatively different trajectory structures.

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## 2 Question 6(b): Prove Topological Equivalence

### Overview of Proof Strategy

**Solution 2.**

- **STAGE X (Goal):** Show there exists a continuous, invertible map (homeomorphism) transforming trajectories of System 1 to System 2 while preserving time direction.
- **STAGE Y (Method):** From Lecture Notes (Section 11, page 37), topological equivalence means the systems have the same qualitative behavior. We'll use polar coordinates to reveal the underlying structure and construct an explicit homeomorphism.
- **STAGE Z (Steps):** Convert to polar, solve, construct map $h : (r, \theta) \mapsto (\rho, \phi)$, verify continuity and invertibility.

### Part (i): System 1 in Polar Coordinates

**Coordinate Transformation:**

$$x_1 = r\cos\theta, \quad x_2 = r\sin\theta \tag{15}$$

**Time Derivatives:**

$$\dot{x}_1 = \dot{r}\cos\theta - r\dot{\theta}\sin\theta \tag{16}$$

$$\dot{x}_2 = \dot{r}\sin\theta + r\dot{\theta}\cos\theta \tag{17}$$

**Substitute Original Equations** ($\dot{x}_1 = x_1$, $\dot{x}_2 = x_2$):

$$r\cos\theta = \dot{r}\cos\theta - r\dot{\theta}\sin\theta \tag{18}$$

$$r\sin\theta = \dot{r}\sin\theta + r\dot{\theta}\cos\theta \tag{19}$$

**Find $\dot{r}$:** Multiply (18) by $\cos\theta$, (19) by $\sin\theta$, and add:

$$r\cos^2\theta + r\sin^2\theta = \dot{r}\cos^2\theta + \dot{r}\sin^2\theta \tag{20}$$

$$r = \dot{r} \tag{21}$$

**Find $\dot{\theta}$:** Multiply (18) by $(-\sin\theta)$, (19) by $\cos\theta$, and add:

$$-r\sin\theta\cos\theta + r\sin\theta\cos\theta = -\dot{r}\sin\theta\cos\theta + r\dot{\theta}\sin^2\theta \tag{22}$$

$$+\dot{r}\sin\theta\cos\theta + r\dot{\theta}\cos^2\theta \tag{23}$$

$$0 = r\dot{\theta}(\sin^2\theta + \cos^2\theta) = r\dot{\theta} \tag{24}$$

For $r \neq 0$: $\dot{\theta} = 0$.

## Result for System 1:

$$\boxed{\dot{r} = r, \quad \dot{\theta} = 0} \tag{25}$$

**Explanation 3** (Interpretation).

- *$\dot{r} = r$: Exponential radial growth*
- *$\dot{\theta} = 0$: No angular motion*
- *Confirms straight-line trajectories from origin*

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**Solution 3** (continued). **Part (iii): System 2 in Polar Coordinates**

**Coordinate Transformation:**

$$y_1 = \rho\cos\phi, \quad y_2 = \rho\sin\phi \tag{26}$$

**Time Derivatives:**

$$\dot{y}_1 = \dot{\rho}\cos\phi - \rho\dot{\phi}\sin\phi \tag{27}$$

$$\dot{y}_2 = \dot{\rho}\sin\phi + \rho\dot{\phi}\cos\phi \tag{28}$$

**Substitute Original Equations** ($\dot{y}_1 = y_1 - y_2$, $\dot{y}_2 = y_1 + y_2$):

$$\rho\cos\phi - \rho\sin\phi = \dot{\rho}\cos\phi - \rho\dot{\phi}\sin\phi \tag{29}$$

$$\rho\cos\phi + \rho\sin\phi = \dot{\rho}\sin\phi + \rho\dot{\phi}\cos\phi \tag{30}$$

**Find $\dot{\rho}$:** Multiply (29) by $\cos\phi$, (30) by $\sin\phi$, and add:

$$\rho\cos^2\phi - \rho\sin\phi\cos\phi + \rho\sin\phi\cos\phi + \rho\sin^2\phi \tag{31}$$

$$= \dot{\rho}\cos^2\phi - \rho\dot{\phi}\sin\phi\cos\phi + \dot{\rho}\sin^2\phi + \rho\dot{\phi}\sin\phi\cos\phi \tag{32}$$

$$\rho(\cos^2\phi + \sin^2\phi) = \dot{\rho}(\cos^2\phi + \sin^2\phi) \tag{33}$$

$$\rho = \dot{\rho} \tag{34}$$

**Find $\dot{\phi}$:** Multiply (29) by $(-\sin\phi)$, (30) by $\cos\phi$, and add:

$$-\rho\sin\phi\cos\phi + \rho\sin^2\phi + \rho\cos^2\phi + \rho\sin\phi\cos\phi \tag{35}$$

$$= -\dot{\rho}\sin\phi\cos\phi + \rho\dot{\phi}\sin^2\phi + \dot{\rho}\sin\phi\cos\phi + \rho\dot{\phi}\cos^2\phi \tag{36}$$

$$\rho(\sin^2\phi + \cos^2\phi) = \rho\dot{\phi}(\sin^2\phi + \cos^2\phi) \tag{37}$$

$$\rho = \rho\dot{\phi} \tag{38}$$

For $\rho \neq 0$: $\dot{\phi} = 1$.

## Result for System 2:

$$\boxed{\dot{\rho} = \rho, \quad \dot{\phi} = 1} \tag{39}$$

**Explanation 4** (Interpretation).

- *$\dot{\rho} = \rho$: Same exponential radial growth as System 1*
- *$\dot{\phi} = 1$: Constant angular velocity (rotation)*
- *Produces spiral trajectories: exponential growth + rotation*

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**Solution 4** (continued). **Part (iv): Solve the Polar Systems**

**System 1:** $\dot{r} = r$, $\dot{\theta} = 0$
These are decoupled ODEs:
*For $r(t)$:*

$$\frac{dr}{dt} = r \tag{40}$$
$$\frac{dr}{r} = dt \tag{41}$$
$$\ln|r| = t + C_1 \tag{42}$$
$$r(t) = Ae^t \tag{43}$$

With initial condition $r(0) = r_0$: $A = r_0$, so:

$$r(t) = r_0 e^t \tag{44}$$

*For $\theta(t)$:*

$$\frac{d\theta}{dt} = 0 \tag{45}$$
$$\theta(t) = \text{constant} = \theta_0 \tag{46}$$

**Solution for System 1:**

$$\boxed{r(t) = r_0 e^t, \quad \theta(t) = \theta_0} \tag{47}$$

**System 2:** $\dot{\rho} = \rho$, $\dot{\phi} = 1$
Again, decoupled ODEs:
*For $\rho(t)$:*

$$\frac{d\rho}{dt} = \rho \tag{48}$$
$$\rho(t) = \rho_0 e^t \tag{49}$$

*For $\phi(t)$:*

$$\frac{d\phi}{dt} = 1 \tag{50}$$
$$\phi(t) = t + \phi_0 \tag{51}$$

**Solution for System 2:**

$$\boxed{\rho(t) = \rho_0 e^t, \quad \phi(t) = \phi_0 + t} \tag{52}$$

**Explanation 5** (Comparing Solutions). ***System 1:***

- *Radius grows exponentially: $r = r_0 e^t$*
- *Angle fixed: $\theta = \theta_0$*
- *Straight radial motion*

***System 2:***

- *Radius grows exponentially:* $\rho = \rho_0 e^t$ *(same rate!)*
- *Angle increases linearly:* $\phi = \phi_0 + t$
- *Spiral motion (exponential growth + rotation)*

*The radial growth is identical - only the angular behavior differs!*

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**Solution 5** (continued)**. Part (v): Construct the Homeomorphism**

We need to find a map that transforms solutions of System 1 into solutions of System 2.

**Proposed Homeomorphism:**

$$h : (r, \theta) \mapsto (\rho, \phi) \quad \text{where} \quad \begin{cases} \rho = r \\ \phi = \theta - \ln(r) \end{cases} \tag{53}$$

## Step 1: Verify the Map Transforms Solutions

**Start with System 1 solution:** $r(t) = r_0 e^t$, $\theta(t) = \theta_0$

**Apply the map:**

$$\rho(t) = r(t) = r_0 e^t \tag{54}$$

$$\phi(t) = \theta(t) - \ln(r(t)) = \theta_0 - \ln(r_0 e^t) = \theta_0 - \ln(r_0) - t \tag{55}$$

**Verify this satisfies System 2:**

Check $\dot{\rho} = \rho$:

$$\dot{\rho} = \frac{d}{dt}(r_0 e^t) = r_0 e^t = \rho \quad \checkmark \tag{56}$$

Check $\dot{\phi} = 1$:

$$\dot{\phi} = \frac{d}{dt}(\theta_0 - \ln(r_0) - t) = 0 - 0 - 1 = -1 \quad ??? \tag{57}$$

**Issue:** We get $\dot{\phi} = -1$, not $+1$. Let me reconsider the map.

**Corrected Homeomorphism:**

$$h : (r, \theta) \mapsto (\rho, \phi) \quad \text{where} \quad \begin{cases} \rho = r \\ \phi = \theta + \ln(r) \end{cases} \tag{58}$$

**Apply the corrected map:**

$$\rho(t) = r(t) = r_0 e^t \tag{59}$$

$$\phi(t) = \theta(t) + \ln(r(t)) = \theta_0 + \ln(r_0 e^t) = \theta_0 + \ln(r_0) + t \tag{60}$$

**Verify System 2:**

Check $\dot{\rho} = \rho$:

$$\dot{\rho} = r_0 e^t = \rho \quad \checkmark \tag{61}$$

Check $\dot{\phi} = 1$:

$$\dot{\phi} = \frac{d}{dt}(\theta_0 + \ln(r_0) + t) = 0 + 0 + 1 = 1 \quad \checkmark \tag{62}$$

Perfect! The corrected map works.

## Step 2: Verify Homeomorphism Properties

A homeomorphism must be:

1. Continuous
2. Bijective (one-to-one and onto)
3. Have continuous inverse

**Forward Map:**

$$h(r, \theta) = (r, \theta + \ln r) = (\rho, \phi) \tag{63}$$

**Inverse Map:**
From $\rho = r$ and $\phi = \theta + \ln r$:

$$r = \rho \tag{64}$$
$$\theta = \phi - \ln \rho \tag{65}$$

So:

$$h^{-1}(\rho, \phi) = (\rho, \phi - \ln \rho) = (r, \theta) \tag{66}$$

**Continuity Check:**

- $h$: The function $(r, \theta) \mapsto (r, \theta + \ln r)$ is continuous for $r > 0$ (logarithm continuous on positive reals).
- $h^{-1}$: The function $(\rho, \phi) \mapsto (\rho, \phi - \ln \rho)$ is continuous for $\rho > 0$.

**Bijectivity:**

- *Injective:* If $h(r_1, \theta_1) = h(r_2, \theta_2)$, then $(r_1, \theta_1 + \ln r_1) = (r_2, \theta_2 + \ln r_2)$. This implies $r_1 = r_2$ and $\theta_1 + \ln r_1 = \theta_2 + \ln r_2$, hence $\theta_1 = \theta_2$. So $h$ is injective.
- *Surjective:* For any $(\rho, \phi)$ with $\rho > 0$, we can find $(r, \theta) = (\rho, \phi - \ln \rho)$ such that $h(r, \theta) = (\rho, \phi)$. So $h$ is surjective.

Therefore, $h$ is a valid homeomorphism.

## Step 3: Geometric Interpretation

**Explanation 6** (What the Homeomorphism Does)**.** *The map $h(r, \theta) = (r, \theta + \ln r)$ can be understood as:*

- ***Radial component unchanged:*** *$\rho = r$ (same distance from origin)*
- ***Angular component modified:*** *$\phi = \theta + \ln r$ (angle adjusted by logarithm of radius)*
- ***Effect on trajectories:***
  - *System 1: Straight line at fixed angle $\theta_0$, growing exponentially*
  - *After mapping: The angle becomes $\phi = \theta_0 + \ln(r_0 e^t) = \theta_0 + \ln r_0 + t$*

- *This increases linearly with time - exactly what System 2 does!*

- ***Physical picture:*** *The homeomorphism "unwinds" the radial motion of System 1 into the spiral motion of System 2 by adding an angle that depends logarithmically on the radius.*

*From Lecture Notes (Section 11, page 37): Topological equivalence preserves the qualitative structure. Both systems:*

- *Have unstable equilibrium at origin*
- *All trajectories escape to infinity*
- *No limit cycles or other attractors*

*The homeomorphism shows these systems are fundamentally "the same" topologically, even though one has straight trajectories and the other has spirals.*

## Final Answer for Part (b)

$$\boxed{\begin{array}{l}\text{The homeomorphism } h : \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}^2 \setminus \{0\} \text{ defined by:} \\ \qquad h(r,\theta) = (r, \theta + \ln r) = (\rho, \phi) \\ \text{transforms solutions of System 1 into solutions of System 2.} \\ \text{This proves the systems are topologically equivalent.}\end{array}} \tag{67}$$

**Summary of verification:**

- System 1 solution: $r(t) = r_0 e^t$, $\theta(t) = \theta_0$
- After mapping: $\rho(t) = r_0 e^t$, $\phi(t) = \theta_0 + \ln r_0 + t$
- This satisfies System 2: $\dot{\rho} = \rho$, $\dot{\phi} = 1$ ✓
- $h$ is continuous, bijective, with continuous inverse ✓
- Therefore: Systems are topologically equivalent ✓

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# Summary: Topological Equivalence and its Significance

## Key Results

1. **System 1** ($\dot{x}_1 = x_1$, $\dot{x}_2 = x_2$): Unstable star node with straight radial trajectories
2. **System 2** ($\dot{y}_1 = y_1 - y_2$, $\dot{y}_2 = y_1 + y_2$): Unstable focus with outward spiral trajectories
3. **Polar forms reveal similarity:**
   - System 1: $\dot{r} = r$, $\dot{\theta} = 0$
   - System 2: $\dot{\rho} = \rho$, $\dot{\phi} = 1$

- Same radial dynamics, different angular dynamics

4. **Homeomorphism:** $h(r, \theta) = (r, \theta + \ln r)$ continuously deforms one phase portrait into the other

5. **Topological equivalence:** Despite different geometric appearance, the systems have identical topological structure

## Connection to Lecture Notes

From Section 11 (pages 37-39):

- **Hartman-Grobman Theorem:** For hyperbolic equilibria, nonlinear systems are topologically equivalent to their linearizations

- **Hyperbolicity matters:** Both systems have eigenvalues with positive real part (hyperbolic), enabling the topological equivalence

- **Topological classification:** Systems are grouped by the number and type of eigenvalues:
  - System 1: Two positive real eigenvalues $\Rightarrow$ unstable node
  - System 2: Complex pair with positive real part $\Rightarrow$ unstable focus
  - Both are "unstable" topologically (trajectories escape)

- **Importance:** Topological equivalence is coarser than similarity but more robust - it persists under continuous deformations of the vector field

## Physical Intuition

The homeomorphism demonstrates that:

- Adding rotation to radial growth changes geometry but not topology

- Continuous transformations can relate seemingly different dynamical behaviors

- Qualitative features (stability, number of equilibria, existence of cycles) are topologically invariant