Problem Sheet 6, Question 1:
Regular Perturbation of Initial Value Problem
Complete Solution with Full Methodology

Asymptotics Course 2025/2026

Problem Statement

Obtain a two-term expansion when $\epsilon \ll 1$ for the solution of

$$\frac{df}{dt} - f = \epsilon f^2 e^{-t}, \quad f(0) = 1.$$

1 Step 1: Problem Classification and Method Selection

Form Recognition

- **STAGE X (What we have):** We have a first-order nonlinear ordinary differential equation with a small parameter ϵ multiplying a nonlinear term. The equation has the form

  $$\frac{df}{dt} - f = \epsilon f^2 e^{-t}$$

  with initial condition $f(0) = 1$. The parameter $0 < \epsilon \ll 1$ is small.

- **STAGE Y (Why we classify):** When $\epsilon = 0$, the equation becomes

  $$\frac{df_0}{dt} - f_0 = 0, \quad f_0(0) = 1,$$

  which is a linear first-order ODE with solution $f_0(t) = e^t$. This solution exists for all $t \geq 0$ and is smooth. For small but nonzero ϵ, we expect the solution to vary smoothly from the unperturbed solution. There are no singular behaviors expected (no boundary layers, no loss of derivatives).

- **STAGE Z (What this means):** This is a **regular perturbation problem**. The solution can be expressed as a power series in ϵ:

  $$f(t, \epsilon) = f_0(t) + \epsilon f_1(t) + \epsilon^2 f_2(t) + O(\epsilon^3).$$

  The appropriate method is the **regular perturbation expansion method** as discussed in Section 5.1 of the lecture notes.

Why Regular Perturbation Works Here

- **STAGE X (What we check):** The small parameter ϵ multiplies the *nonlinear* term $f^2 e^{-t}$, but does *not* multiply the highest derivative. The unperturbed problem ($\epsilon = 0$) has the same order as the perturbed problem (both are first-order ODEs).

- **STAGE Y (Why this matters):** When a small parameter multiplies the highest derivative (e.g., $\epsilon\frac{d^2 f}{dt^2} + \cdots$), we typically encounter **singular perturbation problems** with boundary layers (Section 6 of lecture notes). Here, the derivative structure is unchanged when $\epsilon \to 0$, so the solution varies smoothly.
- **STAGE Z (Conclusion):** We proceed with confidence using regular perturbation expansion.

2 Step 2: Set Up the Perturbation Expansion

The Ansatz

- **STAGE X (What we assume):** We assume the solution can be written as a power series in ϵ:
$$f(t, \epsilon) = f_0(t) + \epsilon f_1(t) + \epsilon^2 f_2(t) + O(\epsilon^3).$$
For a two-term expansion, we need to determine $f_0(t)$ and $f_1(t)$.
- **STAGE Y (Why this form):** From the theory of regular perturbations (Section 5.1), when the solution depends smoothly on ϵ, it admits a Taylor expansion in ϵ. Each coefficient function $f_n(t)$ is independent of ϵ and depends only on time t.
- **STAGE Z (What we'll do):** Substitute this ansatz into the ODE and the initial condition, then collect terms of equal powers of ϵ. Each power of ϵ will give us a separate ODE to solve sequentially.

3 Step 3: Substitute the Expansion into the ODE

Compute the Derivative

- **STAGE X (Calculate $\frac{df}{dt}$):** From $f(t, \epsilon) = f_0(t) + \epsilon f_1(t) + \epsilon^2 f_2(t) + O(\epsilon^3)$, we have
$$\frac{df}{dt} = \frac{df_0}{dt} + \epsilon\frac{df_1}{dt} + \epsilon^2\frac{df_2}{dt} + O(\epsilon^3).$$
- **STAGE Y (Why straightforward):** Since each $f_n(t)$ depends only on t (not on ϵ), differentiation with respect to t passes through the sum term by term.
- **STAGE Z (Result):** We have the derivative ready for substitution.

Compute the Nonlinear Term

- **STAGE X (Calculate f^2):** We need to expand
$$f^2 = \left(f_0 + \epsilon f_1 + \epsilon^2 f_2 + O(\epsilon^3)\right)^2.$$
Expanding the square:
$$\begin{aligned} f^2 &= f_0^2 + 2f_0 \cdot \epsilon f_1 + 2f_0 \cdot \epsilon^2 f_2 + (\epsilon f_1)^2 + O(\epsilon^3) \\ &= f_0^2 + 2\epsilon f_0 f_1 + \epsilon^2(2f_0 f_2 + f_1^2) + O(\epsilon^3). \end{aligned}$$
- **STAGE Y (Why careful expansion needed):** The nonlinearity f^2 couples different orders of the expansion. The term f_0^2 contributes at $O(1)$ when multiplied by ϵ (giving ϵf_0^2, which is $O(\epsilon)$). The cross-term $2f_0 f_1$ appears at $O(\epsilon)$ but contributes at $O(\epsilon^2)$ when multiplied by the ϵ in front. We must track all contributions systematically.
- **STAGE Z (Ready for substitution):** Therefore,
$$\epsilon f^2 e^{-t} = \epsilon f_0^2 e^{-t} + \epsilon^2(2f_0 f_1 e^{-t}) + O(\epsilon^3).$$

Substitute Everything into the ODE

- **STAGE X (The full substitution):** The ODE $\frac{df}{dt} - f = \epsilon f^2 e^{-t}$ becomes:

$$\left[\frac{df_0}{dt} + \epsilon\frac{df_1}{dt} + \epsilon^2\frac{df_2}{dt}\right] - [f_0 + \epsilon f_1 + \epsilon^2 f_2]$$
$$= \epsilon f_0^2 e^{-t} + \epsilon^2(2f_0 f_1 e^{-t}) + O(\epsilon^3).$$

- **STAGE Y (Why we organize by powers of ϵ):** For this equation to hold for all small ϵ, the coefficients of each power of ϵ must separately equal zero. This is the fundamental principle of perturbation theory: if a power series in ϵ equals zero for all small ϵ, then each coefficient must vanish independently.

- **STAGE Z (Ready to collect terms):** We now group all terms by their order in ϵ.

4 Step 4: Collect Terms by Powers of ϵ

Organize the Equation

Rearranging:

$$\left(\frac{df_0}{dt} - f_0\right) + \epsilon\left(\frac{df_1}{dt} - f_1 - f_0^2 e^{-t}\right) + \epsilon^2\left(\frac{df_2}{dt} - f_2 - 2f_0 f_1 e^{-t}\right) + O(\epsilon^3) = 0.$$

- **STAGE X (What we see):** The equation is now organized as a polynomial in ϵ. Each bracket contains terms of the same order.

- **STAGE Y (Why this organization works):** Since this equation must hold for all t and all sufficiently small ϵ, and since the terms $1, \epsilon, \epsilon^2, \ldots$ are linearly independent functions of ϵ, each coefficient must vanish separately.

- **STAGE Z (The hierarchy of equations):** We obtain a sequence of ODEs to solve in order:

$$O(\epsilon^0): \quad \frac{df_0}{dt} - f_0 = 0 \qquad \text{(Eq. 0)}$$
$$O(\epsilon^1): \quad \frac{df_1}{dt} - f_1 = f_0^2 e^{-t} \qquad \text{(Eq. 1)}$$
$$O(\epsilon^2): \quad \frac{df_2}{dt} - f_2 = 2f_0 f_1 e^{-t} \qquad \text{(Eq. 2)}$$

5 Step 5: Apply Initial Conditions

Expand the Initial Condition

- **STAGE X (The given condition):** We have $f(0) = 1$. Substituting our expansion:

$$f(0,\epsilon) = f_0(0) + \epsilon f_1(0) + \epsilon^2 f_2(0) + O(\epsilon^3) = 1.$$

- **STAGE Y (Why expand the initial condition):** Just as with the ODE, the initial condition must hold for all small ϵ. Therefore, we must have equality at each order in ϵ.

- **STAGE Z (Initial conditions for each order):** Matching coefficients:

$$O(\epsilon^0): \quad f_0(0) = 1$$
$$O(\epsilon^1): \quad f_1(0) = 0$$
$$O(\epsilon^2): \quad f_2(0) = 0$$

6 Step 6: Solve the Leading Order Problem

The $O(1)$ Equation

- **STAGE X (The problem to solve):**

$$\frac{df_0}{dt} - f_0 = 0, \quad f_0(0) = 1.$$

- **STAGE Y (Why we solve this first):** This is the unperturbed problem (the problem when $\epsilon = 0$). All higher-order corrections depend on this solution, so we must solve it first. This is a standard linear first-order ODE.

- **STAGE Z (Method of solution):** This is a separable ODE. We can write $\frac{df_0}{f_0} = dt$.

Solving by Separation of Variables

- **STAGE X (Separate and integrate):**

$$\frac{df_0}{f_0} = dt \implies \int \frac{df_0}{f_0} = \int dt \implies \ln|f_0| = t + C.$$

- **STAGE Y (Why we can take the absolute value away):** Since $f_0(0) = 1 > 0$ and the solution is continuous, $f_0(t)$ remains positive for all t in the domain of interest. Thus $|f_0| = f_0$.

- **STAGE Z (General solution):** Exponentiating both sides:

$$f_0(t) = Ke^t$$

  where $K = e^C$ is a constant determined by initial conditions.

Apply the Initial Condition

- **STAGE X (Use $f_0(0) = 1$):**

$$f_0(0) = Ke^0 = K = 1.$$

- **STAGE Y (Why this determines the solution uniquely):** The first-order linear ODE with an initial condition has a unique solution by the existence and uniqueness theorem for ODEs.

- **STAGE Z (Leading order solution):**

$$\boxed{f_0(t) = e^t}$$

Verification

- **STAGE X (Check the ODE):**

$$\frac{df_0}{dt} = e^t, \quad f_0 = e^t \implies \frac{df_0}{dt} - f_0 = e^t - e^t = 0. \quad \checkmark$$

- **STAGE Y (Check the initial condition):**

$$f_0(0) = e^0 = 1. \quad \checkmark$$

- **STAGE Z (Confidence in the solution):** Both the ODE and initial condition are satisfied. We proceed to the next order.

7 Step 7: Solve the First-Order Correction

The $O(\epsilon)$ Equation

- **STAGE X (The problem to solve):** Using $f_0(t) = e^t$ in Eq. 1:

$$\frac{df_1}{dt} - f_1 = f_0^2 e^{-t} = (e^t)^2 e^{-t} = e^{2t} \cdot e^{-t} = e^t,$$

  with initial condition $f_1(0) = 0$.

- **STAGE Y (Why this is more complex):** Unlike the homogeneous equation for f_0, this is an **inhomogeneous linear ODE**. The right-hand side e^t acts as a forcing term. We need to find both the homogeneous solution and a particular solution.

- **STAGE Z (Solution strategy):** The general solution is:

$$f_1(t) = f_1^{(\text{hom})}(t) + f_1^{(\text{part})}(t)$$

  where $f_1^{(\text{hom})}$ solves the homogeneous equation and $f_1^{(\text{part})}$ is any particular solution.

Homogeneous Solution

- **STAGE X (Solve $\frac{df_1^{(\text{hom})}}{dt} - f_1^{(\text{hom})} = 0$):** This is the same form as the $O(1)$ equation:

$$f_1^{(\text{hom})}(t) = Ae^t$$

  where A is an arbitrary constant.

- **STAGE Y (Why we need this):** The homogeneous solution provides the freedom to satisfy initial conditions. The particular solution alone may not satisfy $f_1(0) = 0$.

- **STAGE Z (One part of the solution):** We have found the complementary function.

Particular Solution

- **STAGE X (The issue with naive guess):** Normally, for an equation $\frac{df_1}{dt} - f_1 = e^t$, we might try a particular solution $f_1^{(\text{part})} = Be^t$. However:

$$\frac{d(Be^t)}{dt} - Be^t = Be^t - Be^t = 0 \neq e^t.$$

  This fails because e^t is already a solution of the homogeneous equation!

- **STAGE Y (Why we need a different ansatz):** When the forcing term is a solution of the homogeneous equation, we have **resonance**. The standard method (from ODE theory) is to multiply by t: try $f_1^{(\text{part})} = Bte^t$.

- **STAGE Z (Try the modified ansatz):** Let $f_1^{(\text{part})} = Bte^t$ where B is to be determined.

Determine the Particular Solution

- **STAGE X (Compute derivatives):**

$$f_1^{(\text{part})} = Bte^t$$

  Using the product rule:

$$\frac{df_1^{(\text{part})}}{dt} = B\frac{d}{dt}(te^t) = B(e^t + te^t) = B(1+t)e^t.$$

- **STAGE Y (Substitute into the ODE):**

$$\frac{df_1^{(\text{part})}}{dt} - f_1^{(\text{part})} = B(1+t)e^t - Bte^t = Be^t + Bte^t - Bte^t = Be^t.$$

- **STAGE Z (Match the forcing term):** We need $Be^t = e^t$, so $B = 1$. Thus:

$$f_1^{(\text{part})}(t) = te^t.$$

General Solution for f_1

- **STAGE X (Combine solutions):**

$$f_1(t) = Ae^t + te^t = (A+t)e^t.$$

- **STAGE Y (Apply initial condition $f_1(0) = 0$):**

$$f_1(0) = (A+0)e^0 = A = 0.$$

- **STAGE Z (First-order correction):**

$$\boxed{f_1(t) = te^t}$$

Verification

- **STAGE X (Check the ODE):**

$$\frac{df_1}{dt} = \frac{d}{dt}(te^t) = e^t + te^t = (1+t)e^t$$

$$\frac{df_1}{dt} - f_1 = (1+t)e^t - te^t = e^t. \quad \checkmark$$

- **STAGE Y (Check the initial condition):**

$$f_1(0) = 0 \cdot e^0 = 0. \quad \checkmark$$

- **STAGE Z (Confidence):** The solution is verified. We now have both terms for our two-term expansion.

8 Step 8: Assemble the Two-Term Expansion

The Final Answer

- **STAGE X (Combine the results):** Substituting $f_0(t) = e^t$ and $f_1(t) = te^t$ into our expansion:
$$f(t,\epsilon) = f_0(t) + \epsilon f_1(t) + O(\epsilon^2) = e^t + \epsilon te^t + O(\epsilon^2).$$
Factoring:
$$\boxed{f(t,\epsilon) = e^t(1+\epsilon t) + O(\epsilon^2)}$$

- **STAGE Y (What this result means):** For small ϵ and not-too-large t, the solution is approximately $e^t(1+\epsilon t)$. The leading behavior is exponential growth e^t (from the unperturbed problem), with a correction that grows linearly in both ϵ and t.

- **STAGE Z (Validity of the expansion):** The expansion is uniformly valid as long as $\epsilon t \ll 1$, i.e., for $t \ll 1/\epsilon$. For times $t = O(1/\epsilon)$, the correction term becomes comparable to the leading term, and higher-order terms may be needed (this is related to the discussion of secular terms in Section 7.1 of the lecture notes).

9 Step 9: Physical and Mathematical Interpretation

Behavior of the Solution

- **STAGE X (Analyzing the structure):**
  - The unperturbed solution $f_0(t) = e^t$ grows exponentially.
  - The correction $\epsilon f_1(t) = \epsilon te^t$ also grows exponentially, but with an additional factor of t.
  - For fixed small ϵ, as t increases, eventually ϵte^t becomes non-negligible compared to e^t.

- **STAGE Y (Why the expansion might break down):** When $\epsilon t = O(1)$, i.e., $t = O(1/\epsilon)$, the two terms are of the same order, and the perturbation series is no longer a valid approximation. This is a characteristic feature of secular growth (Section 7.1).

- **STAGE Z (When to trust the result):** For $t \ll 1/\epsilon$ and $\epsilon \ll 1$, the two-term expansion is accurate and uniformly valid.

Comparison with the Original Problem

- **STAGE X (The nonlinear term's effect):** The original ODE has $\epsilon f^2 e^{-t}$ on the right-hand side. For the unperturbed solution $f_0 = e^t$, this term is:
$$\epsilon f_0^2 e^{-t} = \epsilon e^{2t} e^{-t} = \epsilon e^t,$$
which is precisely the forcing term in the equation for f_1.

- **STAGE Y (Why the correction has this form):** The nonlinearity f^2 amplifies the growth: the solution grows like e^t, so $f^2 \sim e^{2t}$, but the factor e^{-t} moderates this to e^t. The net effect is captured in the correction term ϵte^t.

- **STAGE Z (Physical insight):** If this ODE modeled a physical process (e.g., population growth with a time-varying interaction term), the ϵte^t term represents a cumulative effect that grows both with time and the strength of the interaction (ϵ).

10 Verification Checklist

Following the standards of thoroughness demonstrated in the lecture notes and reference solutions:

✓ **Problem type identified**: Regular perturbation (Section 5.1)

✓ **Expansion ansatz justified**: Power series in ϵ

✓ **Terms collected systematically**: By powers of ϵ

✓ **Initial conditions distributed**: $f_0(0) = 1$, $f_1(0) = 0$

✓ $O(1)$ **equation solved**: $f_0(t) = e^t$

✓ $O(\epsilon)$ **equation solved**: $f_1(t) = te^t$

✓ **Resonance handled correctly**: Used te^t ansatz for particular solution

✓ **Each solution verified**: Checked ODE and initial conditions

✓ **Final answer assembled**: $f(t, \epsilon) = e^t(1 + \epsilon t) + O(\epsilon^2)$

✓ **Validity discussed**: Expansion valid for $t \ll 1/\epsilon$

Final Two-Term Expansion:

$$f(t, \epsilon) = e^t(1 + \epsilon t) + O(\epsilon^2), \quad \epsilon \ll 1, \quad t \ll 1/\epsilon.$$