1 ODEs and Dynamical Systems

This part of the course is about the **qualitative** approach to studying dynamical systems.

- 'Qualitative' means we are interested in finding geometrical descriptions of a system's behaviour, in describing its general features rather than solving it exactly.

- Very few systems can be solved exactly. Even computational solutions can be difficult to interpret, and sometimes misleading. A qualitative understanding of a system is vital before rushing into computer simulations. And it can tell you everything you need to understand without solving its equations at all.

A dynamical system is any system of equations that tells you how something evolves in time.

- An equation $\dot{x} = f(x)$ defines an *ordinary differential equation* (ODE) for the dependent variable x, in terms of the independent variable t, where $\dot{x}$ is shorthand for $\frac{dx}{dt}$.

- An equation $x_{n+1} = f(x_n)$ defines a *difference equation* or *map* in the discrete variable x_n.

- Both of these are types of dynamical system. Their solutions are functions $x(t)$ (for the ODE) or sequences $x_0, x_1, x_2, ...$ (for the map) that tell us how a system behaves over time, from an initial condition $x(0)$ or x_0.

- We can have sets of such equations defining an n-dimensional system, e.g. a set of ODEs in continuous variables $x, y, ...$

  $$(\dot{x}, \dot{y}, ...) = \{f(x, y, ...), g(x, y, ...), ...\}$$

  or a set of equations in discrete variables $x_n, y_n, ...$

  $$(x_{n+1}, y_{n+1}, ...) = \{f(x_n, y_n, ...), g(x_n, y_n, ...), ...\}$$

- There are other kinds of dynamical systems such as partial differential equations, cellular automata, delay differential equations, integral equations, renewal equations, stochastic differential equations, hybrid systems, piecewise smooth dynamical systems, . . . A lot of what we will study here provide ideas that can be extended to these.

2 An example of population growth

Take a population that has N individuals at time t, evolving as

$$\dot{N} = B - D + M \tag{2.1}$$

where B = births, D = deaths, M = migrations, per unit time.

A simple model is to say these changes are proportional to the number of individuals, so define a birth rate β such that $B = \beta N$, and death rate δ such that $D = \delta N$. These are defined per individual per unit time, with $\beta > 0$ and $\delta > 0$. For now say $M = 0$ (no migration, i.e. closed borders).

Then we have

$$\dot{N} = (\beta - \delta)N . \tag{2.2}$$

- This is easy to solve, e.g. by separation of variables

  $$\frac{dN}{N} = (\beta - \delta)dt \quad \Rightarrow \quad \ln \tfrac{N(t)}{N_0} = (\beta - \delta)t \quad \Rightarrow \quad N(t) = N_0 e^{(\beta-\delta)t} \tag{2.3}$$

  with initial condition $N(0) = N_0$.

- If $N_0 = 0$ then $N(t) = 0$ for all times t. We call this an *equilibrium*, i.e. a state where $\frac{dN}{dt} = 0$ so the system feels no impulse to change.

- If $\beta > \delta$ the population $N(t)$ grows exponentially away from $N = 0$, without bound. This makes sense as births outweigh deaths. We say the equilibrium $N = 0$ is *unstable* or *repelling*.

- If $\beta < \delta$ the population $N(t)$ shrinks *asymptotically* (in infinite time) towards $N = 0$. This makes sense as deaths overwhelm births. We say the equilibrium $N = 0$ is *stable* or *attracting*.

- We could re-define the crucial parameter as $\alpha = \beta - \delta$. Then there is only one parameter in the system, α = the difference between the birth and death rates. The actual values of β and δ don't matter, only their difference α (so $\beta = 2$, $\delta = 1$, behaves the same as $\beta = 10$, $\delta = 9$, as in both cases $\alpha = 1$).

A more realistic model is to say that the death rate increases if the population gets too large (typical in a contained environment), so it becomes $\delta = \gamma N$ for a constant γ, then

$$\dot{N} = (\beta - \gamma N)N \tag{2.4}$$

with $\beta, \gamma > 0$.

In this case to minimize the number of parameters we can re-scale (re-scaling is like non-dimensionalization):

- Let $N = x/\gamma$ for a scaled population x, then

  $$\dot{x} = (\beta - x)x \tag{2.5}$$

  which has only one parameter. Now the constant γ just acts like a scale for measuring the population, a system of units for N if you like.

- You can see immediately that this now has two *equilibria*, as there are two solutions to $\dot{x} = 0$. There is still an equilibrium at $x = 0$, and now a new one at $x = \beta$.

- We can still solve the nonlinear system (2.5) by separation of variables

  $$dt = \tfrac{dx}{(\beta - x)x} = \left(\tfrac{1}{x} + \tfrac{1}{\beta - x}\right)\tfrac{dx}{\beta} \quad \Rightarrow \quad t = \tfrac{1}{\beta}\ln\tfrac{(\beta - x_0)x(t)}{(\beta - x(t))x_0} \tag{2.6}$$
  $$\Rightarrow \quad x(t) = \frac{\beta x_0 e^{\beta t}}{\beta - x_0 + x_0 e^{\beta t}}$$

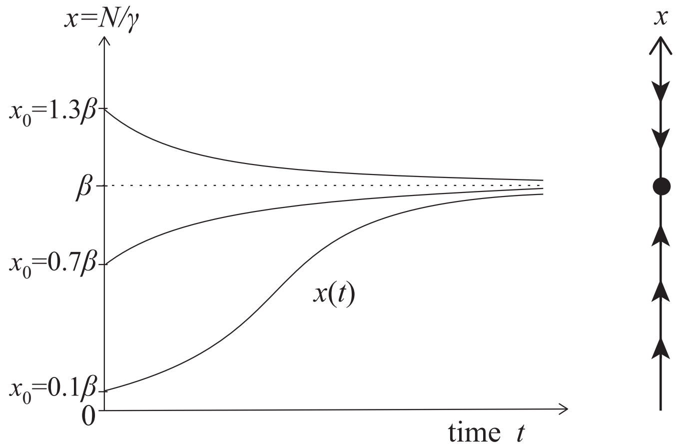
- The long time behaviour is now bounded as

  $$x(t) \to \frac{\beta x_0 e^{\beta t}}{x_0 e^{\beta t}} \to \beta \qquad \text{as} \quad t \to \infty\ . \tag{2.7}$$

- The nonlinear term stops the population exploding to infinity and instead cuts it off at $x = \beta$ (or $N = \beta/\gamma$). So
  - the equilibrium $x = 0$ is unstable,
  - the equilibrium $x = \beta$ is stable.

- Note this all assumed $\beta, \gamma > 0$. What would happen for $\beta < 0$ or $\gamma < 0$?

- Here's what these solutions look like, graphing $x(t)$ for different x_0 values...



  Note that changing the parameters β and γ would just change the scale on the vertical axis.

- On the right we've done away with the time axis, and just represented the flow of time by arrows on the x-axis. This is called the **phase portrait** of the system. It will be much more useful than the graph when we study systems with multiple variables $x, y, z, ...$

These are about the last systems we'll be able to solve exactly . . .

- From hereon we'll need something smarter — more qualitative — to study how things behave.

- We'll keep using the population model to illustrate more general and powerful ways to find the behaviour of systems, especially when we cannot solve them like we did above.

A small but extremely important thing we did above was to reduce the number of constants, which works like *non-dimensionalization*.

[Side Notes:] Rescaling / Non-dimensionalization

Given a system $\dot{X} = F(X; a, b, ...)$ in terms of a variable x and parameters $a, b, ...$, try to define new scaled quantities to reduce the number of parameters in the equation.

- You can scale any of the variables and/or parameters, say $x = AX$ and let some $\alpha, \beta, ...$ be new combinations of the old parameters $a, b, ...$, to give some $\dot{x} = f(x; \alpha, \beta, ...)$
- The object is for the number of parameters $\alpha, \beta, ...$ to be less than the number of the original parameters $a, b, ...$
- This can be an incredibly powerful tool, and is *vitally important* to do in mathematical modeling. The behaviour in the population models above only really depended on one parameter. The other just behaved like a scaling or 'set of units' for the system.
- Sometimes we start off with a physical or biological problem with many rate constants, material coefficients, and so on, which can be reduced to just one or two parameters that define the system's behaviour.
- When you've studied the system in the scaled quantities, *after* you've understood its behaviour, then you just work out what that all looks like back in the original unscaled variables and parameters.

3 ODEs and their flows

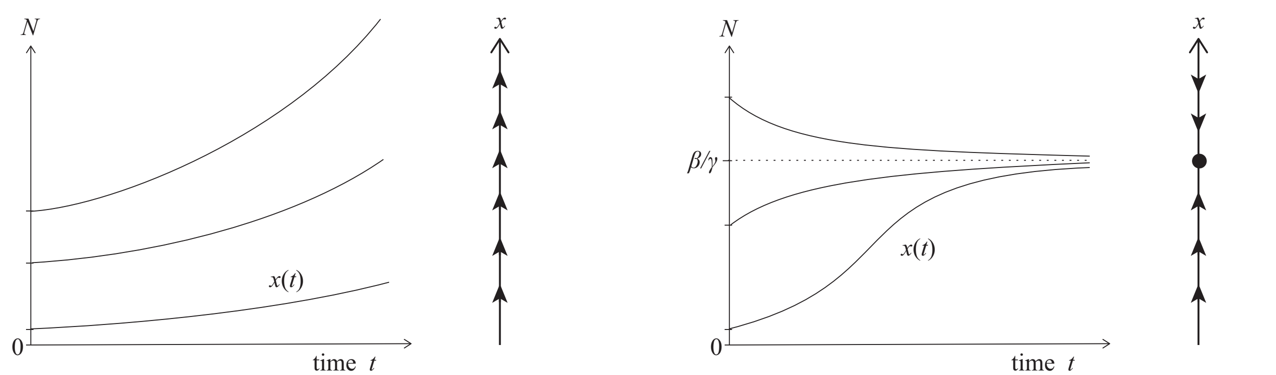
A lot of what we do will be **qualitative** dynamics, forming a conceptual sketch of a system.

That means not having an exact solution like we've plotted in the graphs above, but forming a picture of the geometry of solutions using things like equilibria, as in the phase portraits above.

This can actually be *more* powerful than having exact solutions.

Take the population models:

- Look at the arrows I've drawn on the graphs from the population models, showing the direction of travel according to the ODE, i.e. the vector $\dot{N}$ or $\dot{x}$. In higher dimensions we can't draw the graph (left of each picture below), but we can still understand things in terms of these vectors, drawn just in the space of x (right of each picture below), which are the **phase portraits** of the system.



- The phase portraits are often not only easier to sketch, and actually more useful, than the graphs of solutions. Still, as we get to more than two dimensions even these will be difficult to sketch, but fortunately we'll learn concepts to understand them that will work just as well in higher dimensions.

[Side Notes:] **Flows, orbits and phase portraits**

Consider an ODE

$$\dot{x} = f(x) \tag{3.1}$$

for $x \in D$ and $f \in R$, from the **domain** $D \subset \mathbb{R}^n$ to the **range** $R \subset \mathbb{R}^n$, that is $f : D \mapsto R$. If we are given an initial condition $x(0) = x_0$ we call this an **initial value problem**.

- The solution $x(t)$ to (3.1) traces out a **trajectory** through D as t changes.
- Often the solution we get depends crucially on the initial condition x_0, so it can help to write a solution of the initial value problem as

$$x(t) = \Phi_t(x_0) \quad s.t. \quad \frac{d}{dt}\Phi_t(x_0) = f\left(\Phi_t(x_0)\right) \quad \& \quad \Phi_0(x_0) = x_0 \ . \tag{3.2}$$

- The function $\Phi_t(x_0)$ is called the **flow operator** of the ODE.
- A complete trajectory $\{\Phi_t(x_0) \ : \ t \in [0, T]\}$ is called an **orbit** of the ODE through the point x_0.
- The collection of all orbits is called the **flow field** (or simply the **flow**).
- Its depiction in the **state space** or **phase space** of x is called the **phase portrait**.
- The system (3.1) is **autonomous** (time-independent).
  If instead time appears on the righthand side, say $\dot{x} = f(x, t)$, then the system is **non-autonomous**.
- An **equilibrium** is a point x_* where the system is stationary, i.e. where

$$f(x_*) = 0 \ . \tag{3.3}$$

All this is just the same if instead of $x \in \mathbb{R}$, we have a multivariable system with a vector $\mathbf{x} = (x, y, z, ...) \in \mathbb{R}^n$. Let's start with $\mathbf{x} = (x, y) \in \mathbb{R}^2$ . . .

To use these methods we have to work with **first order differential equations**.

- Typically we can turn a one-dimensional n^{th} order ODE, into an n-dimensional first order ODE, just by associating each derivative with a spatial coordinate, so . . .
- the ODE $\dot{x} = f(x)$ is a first order ODE.
- the second order ODE $\ddot{x} + b(x)\dot{x} + a(x) = 0$ becomes a first order ODE by letting $y = \dot{x}$, giving

$$\dot{x} = y \; , \qquad \dot{y} = -b(x)y - a(x) \; .$$

- the third order ODE $\dddot{x} + c(x)\ddot{x} + b(x)\dot{x} + a(x) = 0$ becomes a first order ODE by letting $y = \dot{x}$ and $z = \ddot{x}$, giving

$$\dot{x} = y \; , \qquad \dot{y} = z \; , \qquad \dot{z} = -c(x)z - b(x)y - a(x) \; .$$

- and so on. These are quite easy to understand, as y is then the speed, z the acceleration, etc. and in a first order system we include these to form the system's **state space**.
- Particularly with high order (large dimensional) systems, we sometimes prefer indexed variables, so for the last example $\dddot{x} + c(x)\ddot{x} + b(x)\dot{x} + a(x) = 0$ we might instead let $x_1 = x$, $x_2 = \dot{x}$, and $x_3 = \ddot{x}$, giving

$$\dot{x}_1 = x_2 \; , \qquad \dot{x}_2 = x_3 \; , \qquad \dot{x}_3 = -c(x_1)x_3 - b(x_1)x_2 - a(x_1) \; .$$

  This **index form** (called by some the **state space** form) is particularly useful for computer simulations.
- The **state space** is the space occupied by the variables $(x, y, z, ...)$ or $(x_1, x_2, x_3, ...)$ of the n-dimensional first order ODE, typically $\mathbb{R}^n$ or some subset of it (e.g. the population model's state space is $\mathbb{R}_+$ (the positive part of $\mathbb{R}$), the predator-prey model's state space is $\mathbb{R}^2_+$).

4 Two populations

An important model with two populations is the Lotka-Volterra predator-prey model. This considers a number of prey X, and number of predators Y.

$$\dot{X} = \alpha X - \beta XY \ , \qquad \dot{Y} = \delta XY - \gamma Y \ , \tag{4.1}$$

- The X equation says the prey population grows exponentially, minus the rate at which it is preyed on by Y.
- The Y equation says the predator population grows proportional to its rate of feeding on X, minus its natural death rate.

These equations are based on a number of assumptions (see box below).

We cannot solve these equations exactly. Instead, the way we study systems like this then typically starts the same way.

- First thing: try to reduce the number of parameters.

  Let $x = \delta X$ and $y = \beta Y$, giving

  $$\dot{x} = (\alpha - y)x \ , \qquad \dot{y} = (x - \gamma)y \ , \tag{4.2}$$

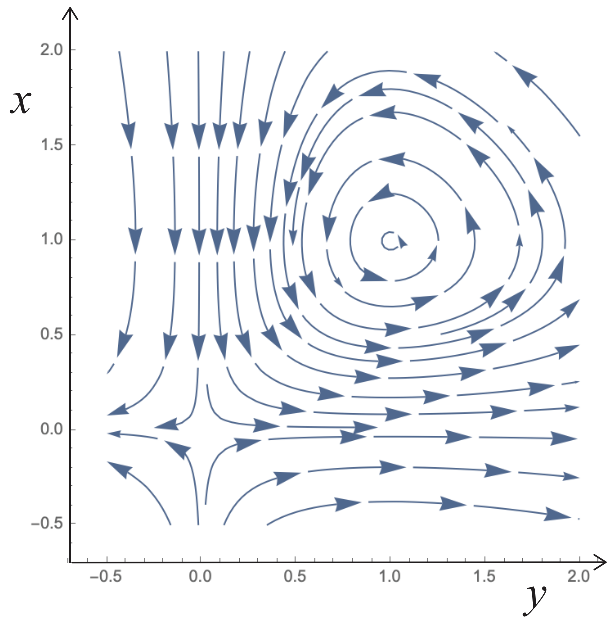
  so now we just have the two parameters α, γ (with δ and β just being the 'units' of x and y).

- Second thing: find any equilibria. There are two places that $\dot{x} = 0$ and $\dot{y} = 0$, and these are the 'trivial' state at $(x, y) = (0, 0)$, and a second state at $(x, y) = (\gamma, \alpha)$.
  At these points (and *only* at these points) the system won't change, elsewhere it will usually be traveling towards or away from these equilibria.

- Third thing: sketch. . .

Plot the **vector field** $(\dot{x}, \dot{y})$ defined by the ODE, by drawing an arrow at each point (x, y) representing the length and direction of $(\dot{x}, \dot{y})$:

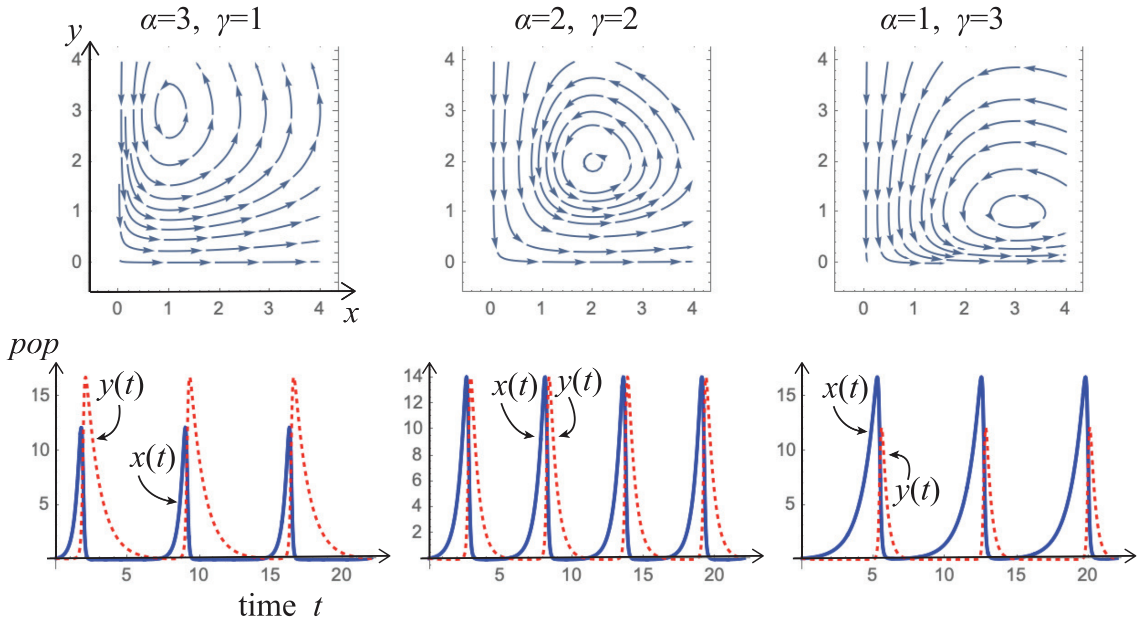
This shows the **phase portrait** of the system in (x, y) space.



- This is plotted for $\alpha = \gamma > 0$. Try out other values, and even try with α and/or γ negative.

- Actually I've cheated a bit in this picture, and not just plotted vectors at each point, but curves with vectors on them. These show bits of solutions of $(x(t), y(t))$, or their *flow*. Some packages like Mathematica can do this (using StreamPlot), in Maple or Matlab you might have to make do with a vector field plot (VectorField in Maple or Quiver in Matlab).

- Below are a few different positive values, along with the graphs of the solutions $x(t)$ and $y(t)$ (found numerically).



[Further Reading Only:] Assumptions of the Lotka-Volterra model

It is always important to understand the simplifying assumptions that we make about a system in order to write down equations modeling it. That's was allows us to understand the limitations of a model and improve it, to compare different models, or compare our model to reality.
For a predator-prey system the equations above assume:

1. The prey population finds ample food at all times.
2. The food supply of the predator population depends entirely on the size of the prey population.
3. The rate of change of population is proportional to its size.
4. During the process, the environment does not change in favour of one species, and genetic adaptation is inconsequential.
5. Predators have limitless appetite.

Other examples, see e.g. SIR model and others on Ex.Sht.

We can form the sketch for the predator-prey model without using a computer. Taking the equations

$$\dot{x} = (\alpha - y)x \ , \qquad \dot{y} = (x - \gamma)y \ ,$$

you have the same two main bits of information:

- nullclines: where $\dot{x} = 0$ or $\dot{y} = 0$, so the vector field $(\dot{x}, \dot{y})$ is vertical on $\dot{x} = 0$ and horizontal on $\dot{y} = 0$, and either side of those curves is up/down/left/right as given by their signs.

- equilibria: where $\dot{x} = \dot{y} = 0$, hence where the nullclines cross, the vector field is zero there and is attracted to and/or repelled from them.

- That's all you need to plot the phase portraits:

  Put points at the equilibria: solutions here cannot move, and other solutions will usually move around, towards, or away from these.

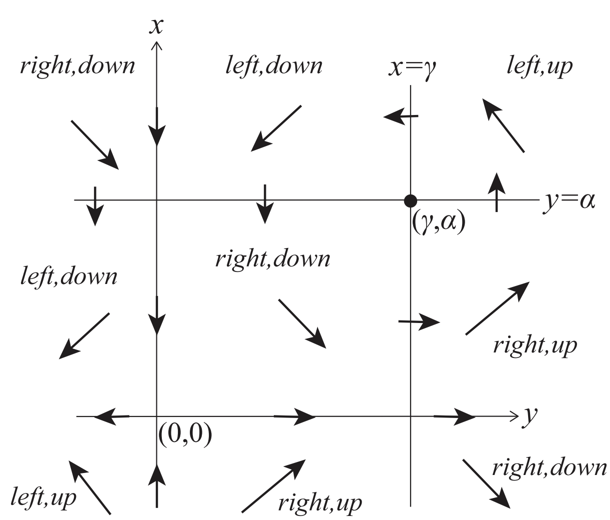
  Use the nullclines to infer whether the vector field $(\dot{x}, \dot{y})$ is pointing up/down and left/right in different regions, don't worry about the actual size of $\dot{x}$ and $\dot{y}$. Here we have 9 different regions:

	$x < 0$	$0 < x < \gamma$	$\gamma < x$
$\alpha < y$	$\dot{y} < 0 < \dot{x}$	$\dot{x}, \dot{y} < 0$	$\dot{x} < 0 < \dot{y}$
$0 < y < \alpha$	$\dot{x}, \dot{y} < 0$	$\dot{y} < 0 < \dot{x}$	$0 < \dot{x}, \dot{y}$
$y < 0$	$\dot{x} < 0 < \dot{y}$	$0 < \dot{x}, \dot{y}$	$\dot{y} < 0 < \dot{x}$

i.e.

	$x < 0$	$0 < x < \gamma$	$\gamma < x$
$\alpha < y$	right,down	left,down	left,up
$0 < y < \alpha$	left,down	right,down	right,up
$y < 0$	left,up	right,up	right,down

and on the nullclines themselves the arrows are either horizontal or vertical.



- Compare this to the flow we plotted earlier. With a bit of practice you can produce accurate phase portraits from a sketch like this. You will need to know how to tell if solutions will go towards or away from an equilibrium, or just circle it as they do in the Lotka-Volterra model. We'll learn how to do this in '*Linear stability*' below.

Want an easy million dollars?

Before you worry about what solutions look like, you need to know whether there are any at all. If they do exist, you then need to know whether there is just one solution, or a whole bunch of possible solutions.

This basic detail is one of the most important parts of dynamical systems theory.

Even for a simple viscous fluid for instance, described by the Navier-Stokes equations, we don't know generally whether solutions exist, and there is a \$1*million* pound prize (from the Clay Institute) if you can find a general solution, or even just show there is more than one.

5 Existence and uniqueness

In an ODE

$$\dot{x} = f(x, t) \tag{5.1}$$

if f is differentiable on some region, then there exist unique solutions to the ODE there. For example:

- The linear population model:

  $$\dot{N} = f(N) = (\beta - \delta)N$$

  Here $f(N)$ is continuous and differentiable with respect to N. The solution through any initial point $N(0) = N_0$ clearly exists, since it is

  $$N(t) = N_0 e^{(\beta-\delta)t}$$

  and this is clearly unique for any N_0.

- The non-linear population model:

  $$\dot{N} = f(N) = (\beta - \gamma N)N$$

  Here $f(N)$ is continuous and differentiable with respect to N. The solution through any initial point $N(0) = N_0$ clearly exists, since it is (using the result we found before for $x = \gamma N$)

  $$N(t) = \frac{\beta N_0 e^{\beta t}}{\beta - \gamma N_0 + \gamma N_0 e^{\beta t}}$$

  This is also unique for any N_0, but does not exist for all N because it cannot pass the equilibrium at $N_* = \beta/\gamma$. So:

  - if $N_0 > \beta/\gamma$ then the population is restricted to $N(t) > \beta/\gamma$ for all t,
  - if $N_0 < \beta/\gamma$ then the population is restricted to $N(t) < \beta/\gamma$ for all t.

For more than two dimensions (or more than first order ODEs) things can be more complicated, but roughly speaking the same rule applies: existence and uniqueness can be expected to hold locally where the equations are continuous and differentiable. For example:

- The predator-prey model:

$$\dot{x} = f(x, y) = (\alpha - y)x \; , \qquad \dot{y} = g(x, y) = (x - \gamma)y \; ,$$

  We cannot write down explicit solutions to these, but they are continuous and differentiable in x and y, so given any initial point with (scaled) prey number x_0 and predator number y_0, we can expect solutions to exist and be unique, at least for a limited time and a limited region around (x_0, y_0).

[Side Notes:] Formal statements and sketch proofs

- **Theorem 1 [Existence]** Suppose that $f(x,t)$ is a function that is uniformly Lipschitz continuous in a region

$$R = \{(x,t) \ : \ |x - x_0| < \delta \ , \ |t - t_0| < \varepsilon\} \ . \tag{5.2}$$

  Then there exists an interval $|t - t_0| < E \leq \varepsilon$ where the solution $x(t)$ to (5.1) is defined.

- **Theorem 2 [Uniqueness]** Suppose that $f(x,t)$ and $\frac{\partial}{\partial x} f(x,t)$ are continuous on R with respect to x and t, then there exists an interval $|t - t_0| < \hat{E} \leq E$ where the solution $x(t)$ to (5.1) is unique.

Caveats:

- These are easily generalised to higher-order ODEs

$$\dot{x} = f(x, y, z, ..., t) \ , \quad \dot{y} = g(x, y, z, ..., t) \ , ...$$

- These only apply on the intervals described.

- For an initial value problem (an ODE plus initial data) these only provide a local statement: near t_0 and x_0. They existence and uniqueness for all space and time, but do not guarantee it — the region of existence may be smaller, e.g. due to finite-time blow up (see Exercise Sheets).

- The criteria are sufficient but not necessary (we can still have existence of unique solutions even if the conditions on f are violated).

- We can estimate $\hat{E} \leq \min\{\varepsilon, \delta/D\}$ where $|f(x, ..., t)| \leq D$ on R.

- The concept of *Lipschitz continuity* is weaker than being differentiable but stronger than just being continuous, essentially it says the change in value of f is constrained between any two points $x = a$ and $x = b$, meaning that

$$|f(b) - f(a)| \leq L|b - a|$$

  where L is called the *Lipschitz constant*, and 'uniformly' means L does not depend on t . . . the most full version of this Theorem also allows f to depend on and be continuous in t, i.e. a non-autonomous system.

Sketch proof of Existence Theorem:

- Let $\phi(t)$ be a function with continuous derivative on R, then ϕ satisfies the initial value problem

$$\phi'(t) = f(\phi(t), t) \qquad \& \qquad \phi(t_0) = x_0 \tag{5.3}$$

  if and only if it satisfies the integral equation

$$\phi(t) = x_0 + \int_{t_0}^{t} f(\phi(s), s)ds \ . \tag{5.4}$$

- To prove the "if": given (5.3), integrate the lefthand side to give $\int_{t_0}^{t} \phi'(s)ds = \phi(t) - \phi(t_0)$, the initial condition gives $\int_{t_0}^{t} \phi'(s)ds = \phi(t) - x_0$, equate this to the integral of the righthand side and we get (5.4).

- To prove the "only if", assuming (5.4), the initial condition follows by substitution, and by the fundamental theorem of calculus we can differentiate, giving (5.3).

So solutions exist, but are they unique?

Sketch proof of Uniqueness Theorem:

- Assume that two functions $\phi(t)$ and $\psi(t)$ satisfy the ODE. Can they be *different* functions?

- Say $|f(\phi(t),t) - f(\psi(t),t)| \leq K\,|\phi(t) - \psi(t)|$,

$$\text{then} \qquad \begin{aligned} |\phi(t) - \psi(t)| &\leq \left| \int_{t_0}^{t} \left(f(\phi(s),s) - f(\psi(s),s)\right) ds \right| \\ &\leq \int_{t_0}^{t} |f(\phi(s),s) - f(\psi(s),s)|\, ds \\ &\leq K \int_{t_0}^{t} |\phi(s) - \psi(s)|\, ds \ := \ KU(t) \end{aligned} \tag{5.5}$$

  defining $U(t)$ where clearly $U(t) \geq 0$.

- Eq.(5.5) reads $U' - KU \leq 0$. We can multiply by $e^{-K(t-t_0)} \geq 0$ to get $(U' - KU)e^{-K(t-t_0)} \leq 0$ or $\left(Ue^{-K(t-t_0)}\right)' \leq 0$, whose integral is $U(t)e^{-K(t-t_0)} - U(t_0)e^{-K(t-t_0)} \leq 0$, but $U(t_0) = 0$ (since $\phi(t_0) = \psi(t_0) = x_0$), implying $U(t)e^{-K(t-t_0)} \leq 0$ implying $U(t) \leq 0$.

- Hence $U(t) \geq 0$ & $U(t) \leq 0 \quad \Rightarrow \quad U(t) = 0 \quad \Rightarrow \quad \phi(t) = \psi(t)$.

These important statements are sometimes referred to as the Picard-Lindelöf theorem, and date back to Émile Picard, Ernst Lindelöf, Rudolf Lipschitz, and Augustin-Louis Cauchy. The precise statement can be weakened to not require $\partial f/\partial x$ to be continuous, but only f to be 'uniformly Lipschitz continuous in x' and continuous in t, which just requires that the change in f is smaller than some Lipschitz constant L as x varies (so $\partial f/\partial x$ might not exist).