# Asymptotics 2025/2026

Problem Sheet 3, Question 4

Watson's Lemma Application

Solution

## 1 Problem Statement

**Question 4:** Use Watson's lemma to find an infinite asymptotic expansion of

$$I(X) = \int_0^{\pi} e^{-Xt} t^{-1/3} \cos t \, \mathrm{d}t. \tag{1}$$

## 2 Preliminary Analysis: Understanding the Problem Structure

### 2.1 What Do We Have?

We are given an integral of the form

$$I(X) = \int_0^{\pi} e^{-Xt} t^{-1/3} \cos t \, \mathrm{d}t, \tag{2}$$

and we seek its asymptotic expansion as $X \to \infty$.

**WHY is this the correct starting point?** Because the problem explicitly states we must find the asymptotic behavior as $X \to \infty$, and we are given a specific integral form to analyze.

### 2.2 What Form Is This Integral?

This integral has the structure

$$I(X) = \int_0^{b} f(t) e^{-Xt} \mathrm{d}t \tag{3}$$

where $b = \pi$ and $f(t) = t^{-1/3} \cos t$.

**WHY do we identify this structure?** Because Watson's lemma (from Section 4.2.2 of the lecture notes) applies specifically to integrals of the form $\int_0^b f(t) e^{-Xt} \mathrm{d}t$ where the exponential has argument $-Xt$.

### 2.3 Why Watson's Lemma?

**WHY use Watson's lemma?** The problem explicitly instructs us to use Watson's lemma. Moreover, from the lecture notes, we know Watson's lemma is the appropriate tool when:

1. We have a Laplace-type integral $\int_0^b f(t) e^{-Xt} \mathrm{d}t$
2. The function $f(t)$ may not have a Taylor expansion at $t = 0$ (due to singularities)
3. We need an asymptotic expansion as $X \to \infty$

Here, $f(t) = t^{-1/3} \cos t$ has a singularity at $t = 0$ (specifically, $t^{-1/3} \to \infty$ as $t \to 0^+$), so standard integration by parts would fail. Watson's lemma is designed precisely for this scenario.

## 3 Watson's Lemma: Statement from Lecture Notes

**Theorem 3.1** (Watson's Lemma, Section 4.2.2)**.** *Given an asymptotic sequence* $\{\phi_n(t)\}$ *where* $\phi_n(t) = t^{\alpha+n\beta}$ *with* $\alpha > -1$ *and* $\beta > 0$*, if* $f(t)$ *admits the asymptotic expansion*

$$f(t) \sim t^{\alpha} \sum_{n=0}^{\infty} a_n t^{n\beta} \quad \text{as } t \to 0^+, \tag{4}$$

*then for the integral*

$$I(X) = \int_0^b f(t)e^{-Xt}\mathrm{d}t, \quad b > 0, \tag{5}$$

*we have the asymptotic expansion*

$$I(X) \sim \sum_{n=0}^{\infty} a_n \frac{\Gamma(\alpha + n\beta + 1)}{X^{\alpha+n\beta+1}} \quad \text{as } X \to \infty. \tag{6}$$

**WHY this theorem?** This is the exact statement from our lecture notes (equation 177), which provides the formula for converting a series expansion of $f(t)$ near $t = 0$ into an asymptotic expansion of the integral as $X \to \infty$.

## 4 Strategy: Applying Watson's Lemma

*Strategy* 4.1. To apply Watson's lemma to our integral, we must:

1. **Expand** $f(t)$ in the form $t^{\alpha} \sum_{n=0}^{\infty} a_n t^{n\beta}$ near $t = 0$
2. **Identify parameters** $\alpha$, $\beta$, and coefficients $\{a_n\}$
3. **Verify conditions** $\alpha > -1$ and $\beta > 0$
4. **Apply the formula** to obtain the asymptotic expansion
5. **Simplify** the resulting expression

**WHY this strategy?** This systematic approach ensures we correctly identify all components needed for Watson's lemma and apply the theorem in the proper sequence.

## 5 Step 1: Expanding $f(t) = t^{-1/3} \cos t$

### 5.1 What Is $f(t)$?

We have

$$f(t) = t^{-1/3} \cos t. \tag{7}$$

**WHY start here?** Watson's lemma requires us to express $f(t)$ as a series in powers of $t$ near $t = 0$. We must first understand the behavior of each component.

### 5.2 Expanding $\cos t$

The cosine function has the Taylor series (valid for all $t \in \mathbb{R}$):

$$\cos t = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{(2n)!} = 1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \frac{t^6}{6!} + \cdots \tag{8}$$

**WHY use the Taylor series?** Because:

1. The Taylor series of $\cos t$ is an exact representation (infinite radius of convergence)
2. This series expresses $\cos t$ in powers of $t$, which is the form required by Watson's lemma
3. Near $t = 0$, this series converges rapidly

**WHY only even powers?** The cosine function is even, so $\cos(-t) = \cos(t)$, which means only even powers of $t$ appear in its Taylor expansion.

### 5.3 Multiplying by $t^{-1/3}$

Now we multiply the Taylor series by $t^{-1/3}$:

$$f(t) = t^{-1/3} \cos t = t^{-1/3} \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{(2n)!}. \tag{9}$$

**WHY multiply?** Because $f(t) = t^{-1/3} \cos t$ is the product of these two factors, and we need the expansion of the entire function $f(t)$.

### 5.4 Combining the Powers

Distributing $t^{-1/3}$ into the sum:

$$f(t) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} t^{2n-1/3}. \tag{10}$$

**WHY combine powers?** Because when we multiply $t^{-1/3}$ by $t^{2n}$, we use the law of exponents: $t^{-1/3} \cdot t^{2n} = t^{-1/3+2n} = t^{2n-1/3}$.

### 5.5 Explicit First Terms

Let us write out the first few terms explicitly to verify our expansion:

$$\begin{aligned} f(t) &= \frac{(-1)^0}{0!} t^{-1/3} + \frac{(-1)^1}{2!} t^{2-1/3} + \frac{(-1)^2}{4!} t^{4-1/3} + \frac{(-1)^3}{6!} t^{6-1/3} + \cdots & (11) \\ &= t^{-1/3} - \frac{1}{2} t^{5/3} + \frac{1}{24} t^{11/3} - \frac{1}{720} t^{17/3} + \cdots & (12) \end{aligned}$$

**WHY write explicit terms?** To verify:

1. The pattern is correct
2. The algebraic manipulations are accurate
3. The series has the required form for Watson's lemma

## 6 Step 2: Identifying Watson's Lemma Parameters

### 6.1 Matching to Standard Form

Watson's lemma requires the form

$$f(t) \sim t^{\alpha} \sum_{n=0}^{\infty} a_n t^{n\beta}. \tag{13}$$

Our expansion is

$$f(t) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} t^{2n-1/3}. \tag{14}$$

We can rewrite this as

$$f(t) = t^{-1/3} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} t^{2n}. \tag{15}$$

**WHY rewrite in this form?** To clearly separate the leading power $t^\alpha$ from the sum, making the comparison with Watson's lemma formula explicit.

### 6.2 Parameter Identification

By comparing

$$t^{-1/3} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} t^{2n} \quad \text{with} \quad t^\alpha \sum_{n=0}^{\infty} a_n t^{n\beta}, \tag{16}$$

we identify:

$$\alpha = -\frac{1}{3}, \tag{17}$$

$$\beta = 2, \tag{18}$$

$$a_n = \frac{(-1)^n}{(2n)!}. \tag{19}$$

**WHY these values?**

- $\alpha = -1/3$: This is the leading power of $t$ when we factor out the smallest power
- $\beta = 2$: Each successive term in the sum increases the power by 2 (we have $t^{2n}$, so the increment is 2)
- $a_n = (-1)^n/(2n)!$: These are the coefficients in front of $t^{2n}$ in our expansion

## 7 Step 3: Verifying Watson's Lemma Conditions

### 7.1 Condition 1: $\alpha > -1$

We have $\alpha = -1/3$.
Is $-1/3 > -1$? Yes, since $-1/3 \approx -0.333 > -1$.
**WHY must $\alpha > -1$?** From the lecture notes, this condition ensures that $\int_0^b t^\alpha e^{-Xt} \mathrm{d}t$ converges at $t = 0$. Specifically, near $t = 0$, we have $t^\alpha \sim t^{-1/3}$, and

$$\int_0^\epsilon t^{-1/3} \mathrm{d}t = \left[\frac{3}{2} t^{2/3}\right]_0^\epsilon = \frac{3}{2}\epsilon^{2/3} < \infty. \tag{20}$$

If $\alpha \leq -1$, the integral would diverge at the lower limit.

### 7.2 Condition 2: $\beta > 0$

We have $\beta = 2 > 0$.
**WHY must $\beta > 0$?** The condition $\beta > 0$ ensures that the sequence $\{t^{\alpha+n\beta}\}$ forms an asymptotic sequence, meaning each term is asymptotically smaller than the previous as $t \to 0$. With $\beta = 2 > 0$, we have

$$t^{\alpha+(n+1)\beta} = t^{\alpha+n\beta+2} = t^{\alpha+n\beta} \cdot t^2 = o(t^{\alpha+n\beta}) \quad \text{as } t \to 0^+. \tag{21}$$

### 7.3 Condition 3: Convergence of the Integral

Watson's lemma also implicitly requires that $I(X)$ converges. For large $t$, we need $f(t) = o(e^{ct})$ for some $c > 0$.
We have $|f(t)| = |t^{-1/3}\cos t| \leq t^{-1/3}$ for $t > 0$.
**WHY check this?** To ensure the integral converges at the upper limit $t = \pi$. Since $t^{-1/3}$ is bounded on $[0, \pi]$ (except near 0, which we've already handled), and the exponential $e^{-Xt}$ decays rapidly for large $X$, the integral converges.

# 8 Step 4: Applying Watson's Lemma Formula

## 8.1 The Formula

Watson's lemma states:

$$I(X) \sim \sum_{n=0}^{\infty} a_n \frac{\Gamma(\alpha + n\beta + 1)}{X^{\alpha+n\beta+1}} \quad \text{as } X \to \infty. \tag{22}$$

**WHY this formula?** This is equation (177) from Section 4.2.2 of the lecture notes, derived by:

1. Substituting $f(t) \sim t^\alpha \sum a_n t^{n\beta}$ into the integral
2. Interchanging sum and integral (justified for asymptotic series)
3. Recognizing $\int_0^\infty t^{\alpha+n\beta} e^{-Xt} \mathrm{d}t = \Gamma(\alpha + n\beta + 1)/X^{\alpha+n\beta+1}$

## 8.2 Substituting Our Parameters

With $\alpha = -1/3$, $\beta = 2$, and $a_n = (-1)^n/(2n)!$, we substitute:

$$I(X) \sim \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \frac{\Gamma(-1/3 + 2n + 1)}{X^{-1/3+2n+1}}. \tag{23}$$

**WHY substitute?** To apply the general formula to our specific problem, replacing abstract parameters with concrete values.

## 8.3 Simplifying the Argument

Simplifying the argument:

$$\begin{aligned} \alpha + n\beta + 1 &= -\frac{1}{3} + 2n + 1 && (24) \\ &= 2n + 1 - \frac{1}{3} && (25) \\ &= 2n + \frac{3-1}{3} && (26) \\ &= 2n + \frac{2}{3}. && (27) \end{aligned}$$

**WHY simplify?** To express the formula in its cleanest form, making the pattern clear and the result easier to interpret.

### 8.4 Final Asymptotic Expansion

Therefore:

$$\boxed{I(X) \sim \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \frac{\Gamma(2n+2/3)}{X^{2n+2/3}} \quad \text{as } X \to \infty.} \tag{28}$$

**WHY is this the answer?** This is the direct result of applying Watson's lemma with our identified parameters. Each term represents the contribution from the $n$-th term in the Taylor expansion of $\cos t$ multiplied by $t^{-1/3}$.

## 9 Step 5: Explicit First Terms

### 9.1 Computing Initial Terms

Let us compute the first few terms explicitly:

#### 9.1.1 Term $n = 0$

$$\frac{(-1)^0}{(0)!} \frac{\Gamma(2/3)}{X^{2/3}} = \frac{\Gamma(2/3)}{X^{2/3}}. \tag{29}$$

**WHY start with $n = 0$?** This is the leading-order term, which dominates the asymptotic behavior as $X \to \infty$.

#### 9.1.2 Term $n = 1$

$$\frac{(-1)^1}{(2)!} \frac{\Gamma(2+2/3)}{X^{2+2/3}} = -\frac{1}{2} \frac{\Gamma(8/3)}{X^{8/3}}. \tag{30}$$

#### 9.1.3 Term $n = 2$

$$\frac{(-1)^2}{(4)!} \frac{\Gamma(4+2/3)}{X^{4+2/3}} = \frac{1}{24} \frac{\Gamma(14/3)}{X^{14/3}}. \tag{31}$$

#### 9.1.4 Term $n = 3$

$$\frac{(-1)^3}{(6)!} \frac{\Gamma(6+2/3)}{X^{6+2/3}} = -\frac{1}{720} \frac{\Gamma(20/3)}{X^{20/3}}. \tag{32}$$

### 9.2 Expanded Form

The asymptotic expansion is:

$$I(X) \sim \frac{\Gamma(2/3)}{X^{2/3}} - \frac{\Gamma(8/3)}{2X^{8/3}} + \frac{\Gamma(14/3)}{24X^{14/3}} - \frac{\Gamma(20/3)}{720X^{20/3}} + \cdots \tag{33}$$

**WHY write explicit terms?** To:

1. Show the pattern clearly
2. Verify the formula is producing sensible results
3. Demonstrate the alternating sign structure
4. Show how rapidly the powers of $X$ increase in the denominator

# 10 Verification and Interpretation

## 10.1 Structure of the Expansion

**Observation 1:** The powers of $X$ in the denominator are $2/3, 8/3, 14/3, 20/3, \ldots$, which increase by 2 each time.
**WHY this pattern?** Because $\beta = 2$, so consecutive terms differ by $\beta = 2$ in the exponent.
**Observation 2:** The signs alternate due to $(-1)^n$.
**WHY alternating signs?** This comes from the Taylor series of $\cos t = \sum(-1)^n t^{2n}/(2n)!$, which has alternating signs.
**Observation 3:** As $X \to \infty$, each term is much smaller than the previous.
**WHY asymptotic?** For large $X$:

$$\frac{\text{Term}_{n+1}}{\text{Term}_n} \sim \frac{X^{2n+2/3}}{X^{2n+8/3}} = \frac{1}{X^2} \to 0. \tag{34}$$

## 10.2 Gamma Function Values

The Gamma function values can be computed using the recurrence $\Gamma(z+1) = z\Gamma(z)$:

$$\Gamma(2/3) \approx 1.35412, \tag{35}$$
$$\Gamma(8/3) = \frac{5}{3} \cdot \frac{2}{3} \cdot \Gamma(2/3) \approx 1.50407, \tag{36}$$
$$\Gamma(14/3) = \frac{11}{3} \cdot \frac{8}{3} \cdot \frac{5}{3} \cdot \frac{2}{3} \cdot \Gamma(2/3) \approx 5.50533. \tag{37}$$

**WHY include numerical values?** To demonstrate that the coefficients are finite and well-defined, confirming our asymptotic expansion is meaningful.

# 11 Final Answer

The infinite asymptotic expansion of

$$I(X) = \int_0^{\pi} e^{-Xt} t^{-1/3} \cos t \, \mathrm{d}t \tag{38}$$

as $X \to \infty$ is given by Watson's lemma as:

$$\boxed{I(X) \sim \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \frac{\Gamma\left(2n + \frac{2}{3}\right)}{X^{2n+2/3}} \quad \text{as } X \to \infty,} \tag{39}$$

which can also be written explicitly as:

$$\boxed{I(X) \sim \frac{\Gamma(2/3)}{X^{2/3}} - \frac{\Gamma(8/3)}{2X^{8/3}} + \frac{\Gamma(14/3)}{24X^{14/3}} - \frac{\Gamma(20/3)}{720X^{20/3}} + \cdots} \tag{40}$$

**WHY is this the complete answer?** This satisfies all requirements:

1. We used Watson's lemma as instructed
2. We obtained an *infinite* asymptotic expansion (not just leading order)
3. The expansion is valid as $X \to \infty$
4. Every step followed rigorously from the course material