Laplace-type integrals

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1. Integration by parts yields

$$I(X) = \int_1^\infty e^{-X(t^2+1)}\,dt = \left[-\frac{1}{2tX}e^{-X(t^2+1)}\right]_1^\infty + \left[\frac{1}{4t^3X^2}e^{-X(t^2+1)}\right]_1^\infty + \int_1^\infty \frac{3}{4t^4X^2}e^{-X(t^2+1)}\,dt\,.$$

We conclude that

$$I(x) \sim \frac{1}{2X}e^{-2X} - \frac{1}{4X^2}e^{-2X} \quad \text{as} \quad X\to\infty\,.$$

2. (a) The function $\Phi(t) = t^3$ in the exponent has only one stationary point at $t = 0$ which is outside the integration range. Therefore the main contribution comes from the lower boundary of the integral. Integration by parts yields

$$I(X) = \int_X^\infty e^{-t^3}\,dt = \left[-\frac{1}{3t^2}e^{-t^3}\right]_X^\infty - \int_X^\infty \frac{2}{3t^3}e^{-t^3}\,dt \sim \frac{1}{3X^2}e^{-X^3} \quad \text{as} \quad X\to\infty\,.$$

(b) Again the stationary point of the exponent at $t = 0$ lies outside the integration range, and we obtain the leading asymptotic form by integration by parts

$$I(X) = \int_3^6 e^{-Xt^2}\sqrt{1+t^2}\,dt = \left[-\frac{\sqrt{1+t^2}}{2tX}e^{-Xt^2}\right]_3^6 - \int_3^6 e^{-Xt^2}\frac{1}{2t^2X\sqrt{1+t^2}}\,dt\,.$$

We conclude that

$$I(X) \sim \frac{\sqrt{10}}{6X}e^{-9X} \quad \text{as} \quad X\to\infty\,.$$

(c) The function in the exponent

$$\Phi(t) = -\sin t - \cos t = -\sqrt{2}\left(\sin t\,\sin\frac{\pi}{4} + \cos t\,\cos\frac{\pi}{4}\right) = -\sqrt{2}\cos\left(t - \frac{\pi}{4}\right)$$

has its minimum at $t = \pi/4$ where $\Phi(\pi/4) = -\sqrt{2}$ and $\Phi''(\pi/4) = \sqrt{2}$. The leading order form of the integral follows by expanding the exponent around the minimum up to second order, the exponential prefactor up to zeroth order, and extending the integration range to $\pm\infty$

$$I(X) = \int_0^{\pi/2} e^{X(\sin t+\cos t)}\sqrt{t}\,dt \sim \int_{-\infty}^{\infty} \exp\left(X\sqrt{2} - \frac{X}{\sqrt{2}}(t-\pi/4)^2\right)\sqrt{\frac{\pi}{4}}\,dt\,.$$

The evaluation of the Gaussian integral leads to

$$I(X) \sim \frac{\pi\, 2^{1/4}}{2\sqrt{X}}\, e^{X\sqrt{2}} \quad \text{as} \quad X \to \infty\,.$$

(d) The function in the exponent $\Phi(t) = t^2 - 2t = (t-1)^2 - 1$ is a quadratic function and has a minimum at $t = 1$. The leading order asymptotic behaviour of the integral is obtained by taking the value of the logarithm at $t = 1$ and extending the integration range to $\pm\infty$

$$I(X) = \int_0^\infty e^{-X(t^2-2t)} \log(1+t^2)\, dt \sim \int_{-\infty}^\infty e^{X - X(t-1)^2} \log 2\, dt\,.$$

The evaluation of the Gaussian integral leads to

$$I(X) \sim \sqrt{\frac{\pi}{X}}\, e^X \log 2 \quad \text{as} \quad X \to \infty\,.$$

(e) The function in the exponent $\Phi(t) = \cosh t + 1$ has its mimimum at $t = 0$ where $\Phi(0) = 2$ and $\Phi''(0) = 1$. The leading order form of the integral is obtained by expanding the exponent around the minimum up to second order, the exponential prefactor up to zeroth order, and extending the integration range to $\pm\infty$

$$I(X) = \int_{-1}^1 e^{-X(\cosh t+1)} e^t\, dt \sim \int_{-\infty}^\infty e^{-2X - Xt^2/2}\, dt \sim \sqrt{\frac{2\pi}{X}}\, e^{-2X} \quad \text{as} \quad X \to \infty\,.$$

3. In order to apply Watson's lemma we set $t^2 = s + 1$ and obtain

$$I(X) = \int_1^\infty e^{-X(t^2+1)}\, dt = \int_0^\infty \frac{e^{-X(s+2)}}{2\sqrt{1+s}}\,.$$

This is in a form where Watson's lemma can be applied. The result is

$$\frac{1}{\sqrt{1+s}} = \sum_{n=0}^\infty \frac{(-s)^n\, \Gamma(n+\frac{1}{2})}{n!\, \Gamma(\frac{1}{2})} \qquad \Longrightarrow \qquad I(X) \sim e^{-2X} \sum_{n=0}^\infty \frac{(-1)^n\, \Gamma(n+\frac{1}{2})}{2\, \Gamma(\frac{1}{2})\, X^{n+1}} \quad \text{as} \quad X \to \infty\,.$$

4. The contributions to the asymptotic expansion come from the lower boundary of the integration range. We extend the integration to infinity and expand the cosine into a Taylor series

$$I(X) = \int_0^\pi e^{-Xt}\, t^{-1/3} \cos t\, dt \sim \int_0^\infty e^{-Xt}\, t^{-1/3} \sum_{n=0}^\infty (-1)^n \frac{t^{2n}}{(2n)!}\, dt\,.$$

Now we can apply Watson's lemma and obtain

$$I(X) \sim \sum_{n=0}^\infty \frac{(-1)^n\, \Gamma\left(2n+\frac{2}{3}\right)}{(2n)!\, X^{2n+2/3}} \quad \text{as} \quad X \to \infty\,.$$

5. We can write the integral in the form $I(X) = \int_0^\infty \exp(-X\log(1+u/X) - u)\,du$. After an expansion of the logarithm one notices that the exponent is of the form $-2u + O(X^{-1})$. This integral is not of the Laplace type and the contribution does not come from a small neighbourhood of $u = 0$, but from the complete integration range. In this case we can obtain the large X asymptotics by expanding the integrand in inverse powers of X

$$I(X) = \int_0^\infty e^{-2u}\left(1 + \frac{u^2}{2X} - \frac{u^3}{3X^2} + \frac{u^4}{8X^2} + \ldots\right) du = \frac{1}{2} + \frac{1}{8X} - \frac{1}{32X^2} + \ldots .$$