Exercise 5, Question 1: Bifurcations in Two-Dimensional Maps

Problem Statement

Consider the two-dimensional map

$$x_{n+1} = x_n^2 - cy_n \tag{1}$$

$$y_{n+1} = \frac{1}{2}(x_n - y_n) \tag{2}$$

(a) Find the fixed points of the map.

(b) Determine the stability of the fixed points and conjecture the bifurcation(s) that occur(s) as their stability changes.

1 Part (a): Finding Fixed Points

Step 1: Understand What We're Looking For

A fixed point (x^*, y^*) satisfies the condition that when we apply the map, the point doesn't move:

$$x_{n+1} = x_n \quad \text{and} \quad y_{n+1} = y_n$$

This means we need:

$$x^* = x^{*2} - cy^* \quad \text{and} \quad y^* = \frac{1}{2}(x^* - y^*)$$

Step 2: Solve the Second Equation First

From equation (**??**), the fixed point condition is:

$$y^* = \frac{1}{2}(x^* - y^*)$$

Step 2.1: Multiply both sides by 2

$$2y^* = x^* - y^*$$

Step 2.2: Collect all y^* terms on the left

$$2y^* + y^* = x^*$$

Step 2.3: Simplify

$$3y^* = x^*$$

Step 2.4: Solve for y^*

$$\boxed{y^* = \frac{x^*}{3}}$$

Explanation 1 (Key Observation). *The second equation gives us a direct relationship between y^* and x^*. This means we can substitute this into the first equation to get a single equation in x^* alone.*

Step 3: Substitute into the First Equation

From equation (**??**), the fixed point condition is:

$$x^* = (x^*)^2 - cy^*$$

Step 3.1: Substitute $y^* = x^*/3$

$$x^* = (x^*)^2 - c \cdot \frac{x^*}{3}$$

Step 3.2: Rearrange to standard form

$$x^* = (x^*)^2 - \frac{c}{3}x^*$$

Step 3.3: Move all terms to the right side

$$0 = (x^*)^2 - \frac{c}{3}x^* - x^*$$

Step 3.4: Combine like terms

$$0 = (x^*)^2 - x^* \left(1 + \frac{c}{3}\right)$$

Step 3.5: Factor out x^*

$$0 = x^* \left[x^* - \left(1 + \frac{c}{3}\right)\right]$$

Step 4: Identify the Two Solutions

The factored equation gives us two possibilities:

Step 4.1: First solution

$$x^* = 0$$

When $x^* = 0$, we have $y^* = 0/3 = 0$.

$$\boxed{\text{Fixed Point 1: } (x^*, y^*) = (0, 0)}$$

Step 4.2: Second solution

$$x^* - \left(1 + \frac{c}{3}\right) = 0$$

Therefore:

$$x^* = 1 + \frac{c}{3}$$

When $x^* = 1 + c/3$, we have:

$$y^* = \frac{x^*}{3} = \frac{1 + \frac{c}{3}}{3} = \frac{1}{3} + \frac{c}{9}$$

$$\boxed{\text{Fixed Point 2: } (x^*, y^*) = \left(1 + \frac{c}{3}, \frac{1}{3} + \frac{c}{9}\right)}$$

Explanation 2 (Summary of Fixed Points). *For all values of the parameter c, we have exactly two fixed points:*

- *The origin $(0,0)$ - independent of c*

- *A second fixed point that moves along the line $y = x/3$ as c varies*

- *At $c = 0$, the second fixed point is at $(1, 1/3)$*

- *As c increases, the second fixed point moves away from the origin*

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2 Part (b): Stability Analysis and Bifurcations

Step 1: Recall the Stability Criterion for Maps

From the lecture notes (page 71), for a map, a fixed point is stable if all eigenvalues λ of the Jacobian matrix satisfy:

$$|\lambda| < 1$$

Bifurcations occur when an eigenvalue passes through the unit circle, i.e., when $|\lambda| = 1$. The different types of bifurcations (page 71) are:

Solution 2. • $\lambda = +1$: Fold or Transcritical bifurcation

- $\lambda = -1$: Flip (period-doubling) bifurcation

- $|\lambda| = 1$ with λ complex: Neimark-Sacker bifurcation

Step 2: Compute the Jacobian Matrix

The Jacobian matrix is:

$$J = \begin{pmatrix} \frac{\partial x_{n+1}}{\partial x_n} & \frac{\partial x_{n+1}}{\partial y_n} \\ \frac{\partial y_{n+1}}{\partial x_n} & \frac{\partial y_{n+1}}{\partial y_n} \end{pmatrix}$$

Step 2.1: Compute $\partial x_{n+1}/\partial x_n$
From $x_{n+1} = x_n^2 - cy_n$:

$$\frac{\partial x_{n+1}}{\partial x_n} = 2x_n$$

Step 2.2: Compute $\partial x_{n+1}/\partial y_n$

From $x_{n+1} = x_n^2 - cy_n$:

$$\frac{\partial x_{n+1}}{\partial y_n} = -c$$

Step 2.3: Compute $\partial y_{n+1}/\partial x_n$

From $y_{n+1} = \frac{1}{2}(x_n - y_n)$:

$$\frac{\partial y_{n+1}}{\partial x_n} = \frac{1}{2}$$

Step 2.4: Compute $\partial y_{n+1}/\partial y_n$

From $y_{n+1} = \frac{1}{2}(x_n - y_n)$:

$$\frac{\partial y_{n+1}}{\partial y_n} = -\frac{1}{2}$$

Step 2.5: Assemble the Jacobian

$$J(x_n, y_n) = \begin{pmatrix} 2x_n & -c \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix}$$

Step 3: Analyze Fixed Point 1: $(0,0)$

Step 3.1: Evaluate the Jacobian at $(0,0)$

$$J(0,0) = \begin{pmatrix} 2(0) & -c \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix} = \begin{pmatrix} 0 & -c \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix}$$

Step 3.2: Set up the characteristic equation

The eigenvalues satisfy:

$$\det(J - \lambda I) = 0$$

Step 3.3: Compute the determinant

$$\det \begin{pmatrix} 0 - \lambda & -c \\ \frac{1}{2} & -\frac{1}{2} - \lambda \end{pmatrix} = 0$$

$$(-\lambda)\left(-\frac{1}{2} - \lambda\right) - (-c)\left(\frac{1}{2}\right) = 0$$

Step 3.4: Expand

$$\lambda\left(\frac{1}{2} + \lambda\right) + \frac{c}{2} = 0$$

$$\frac{\lambda}{2} + \lambda^2 + \frac{c}{2} = 0$$

Step 3.5: Multiply by 2 and rearrange

$$\lambda^2 + \frac{\lambda}{2} + \frac{c}{2} = 0$$

Or equivalently:

$$2\lambda^2 + \lambda + c = 0$$

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Step 3.6: Apply the quadratic formula

$$\lambda = \frac{-1 \pm \sqrt{1 - 8c}}{4}$$

$$\boxed{\lambda_{1,2} = \frac{-1 \pm \sqrt{1 - 8c}}{4}}$$

Step 4: Analyze the Eigenvalues for Different Values of c

Step 4.1: When $c < 1/8$ (discriminant positive)

The eigenvalues are real:

$$\lambda_1 = \frac{-1 + \sqrt{1 - 8c}}{4}, \quad \lambda_2 = \frac{-1 - \sqrt{1 - 8c}}{4}$$

Subcase: $c = 0$

$$\lambda_1 = \frac{-1 + 1}{4} = 0, \quad \lambda_2 = \frac{-1 - 1}{4} = -\frac{1}{2}$$

Both satisfy $|\lambda| < 1$, so the fixed point is stable.

Subcase: $0 < c < 1/8$

Since $\sqrt{1 - 8c} < 1$ for $c > 0$: - $\lambda_1 = \frac{-1 + \sqrt{1-8c}}{4} < 0$ (negative) - $\lambda_2 = \frac{-1 - \sqrt{1-8c}}{4} < -\frac{1}{2}$ (negative)

For c slightly above 0, $|\lambda_1| < 1$ and $|\lambda_2| < 1$, so stable.

Step 4.2: When $c = 1/8$ (discriminant zero)

$$\lambda = \frac{-1 \pm 0}{4} = -\frac{1}{4}$$

Double eigenvalue with $|\lambda| = 1/4 < 1$, so stable.

Step 4.3: When $c > 1/8$ (discriminant negative)

The eigenvalues are complex conjugates:

$$\lambda = \frac{-1 \pm i\sqrt{8c - 1}}{4}$$

Calculate the modulus:

$$|\lambda|^2 = \left(\frac{-1}{4}\right)^2 + \left(\frac{\sqrt{8c - 1}}{4}\right)^2$$

$$= \frac{1}{16} + \frac{8c - 1}{16} = \frac{1 + 8c - 1}{16} = \frac{8c}{16} = \frac{c}{2}$$

Therefore:

$$|\lambda| = \sqrt{\frac{c}{2}}$$

Step 4.4: Determine when $|\lambda| = 1$

$$\sqrt{\frac{c}{2}} = 1$$

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$$\frac{c}{2} = 1$$

$$c = 2$$

Step 4.5: Stability conclusions for Fixed Point 1

- For $0 < c < 2$: $|\lambda| = \sqrt{c/2} < 1 \Rightarrow$ **Stable**

- At $c = 2$: $|\lambda| = 1 \Rightarrow$ **Bifurcation**

- For $c > 2$: $|\lambda| = \sqrt{c/2} > 1 \Rightarrow$ **Unstable**

Step 5: Identify the Bifurcation Type at $c = 2$

Step 5.1: Examine the eigenvalues at $c = 2$
At $c = 2$:
$$\lambda = \frac{-1 \pm i\sqrt{16 - 1}}{4} = \frac{-1 \pm i\sqrt{15}}{4}$$

These are complex conjugates with $|\lambda| = 1$.
Step 5.2: Write in exponential form

$$\lambda = e^{i\theta}$$

where $\theta \neq 0, \pi$
Step 5.3: Identify the bifurcation
From lecture notes (page 71), when a pair of complex conjugate eigenvalues passes through the unit circle (i.e., $|\lambda| = 1$ with $\lambda \in \mathbb{C}$), this is a **Neimark-Sacker bifurcation**.

$$\boxed{\text{Fixed Point 1: Neimark-Sacker bifurcation at } c = 2}$$

Explanation 3 (Physical Meaning). *At the Neimark-Sacker bifurcation, the stable spiral fixed point becomes unstable, and typically an invariant closed curve (either quasi-periodic or periodic orbit) emerges around it. This is the map analogue of a Hopf bifurcation in continuous systems.*

Step 6: Analyze Fixed Point 2: $(1 + c/3, 1/3 + c/9)$

Step 6.1: Evaluate the Jacobian at Fixed Point 2

$$J\left(1 + \frac{c}{3}, \frac{1}{3} + \frac{c}{9}\right) = \begin{pmatrix} 2\left(1 + \frac{c}{3}\right) & -c \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix}$$

$$= \begin{pmatrix} 2 + \frac{2c}{3} & -c \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix}$$

Step 6.2: Compute the trace

$$\text{tr}(J) = 2 + \frac{2c}{3} + \left(-\frac{1}{2}\right) = \frac{3}{2} + \frac{2c}{3}$$

Step 6.3: Compute the determinant

$$\det(J) = \left(2 + \frac{2c}{3}\right)\left(-\frac{1}{2}\right) - (-c)\left(\frac{1}{2}\right)$$

<u>First term:</u>

$$\left(2 + \frac{2c}{3}\right)\left(-\frac{1}{2}\right) = -1 - \frac{c}{3}$$

<u>Second term:</u>

$$-(-c)\left(\frac{1}{2}\right) = \frac{c}{2}$$

<u>Sum:</u>

$$\det(J) = -1 - \frac{c}{3} + \frac{c}{2} = -1 + c\left(\frac{1}{2} - \frac{1}{3}\right)$$

$$= -1 + c\left(\frac{3-2}{6}\right) = -1 + \frac{c}{6}$$

$$\boxed{\det(J) = \frac{c}{6} - 1}$$

Step 6.4: Write the characteristic equation

$$\lambda^2 - \text{tr}(J) \cdot \lambda + \det(J) = 0$$

$$\lambda^2 - \left(\frac{3}{2} + \frac{2c}{3}\right)\lambda + \left(\frac{c}{6} - 1\right) = 0$$

Step 7: Identify Critical Values of c

For a 2D map, bifurcations occur when:

- One eigenvalue equals $+1$ (fold or transcritical)

- One eigenvalue equals -1 (flip/period-doubling)

- Two complex eigenvalues with $|\lambda| = 1$ (Neimark-Sacker)

Step 7.1: Check when $\lambda = 1$
Substitute $\lambda = 1$ into the characteristic equation:

$$1 - \left(\frac{3}{2} + \frac{2c}{3}\right) + \left(\frac{c}{6} - 1\right) = 0$$

$$1 - \frac{3}{2} - \frac{2c}{3} + \frac{c}{6} - 1 = 0$$

$$-\frac{3}{2} - \frac{2c}{3} + \frac{c}{6} = 0$$

Multiply by 6:
$$-9 - 4c + c = 0$$

$$-9 - 3c = 0$$

$$c = -3$$

So $\lambda = 1$ when $c = -3$ (outside typical physical range, but mathematically valid).

Step 7.2: Check when $\lambda = -1$

Substitute $\lambda = -1$:
$$1 + \left(\frac{3}{2} + \frac{2c}{3}\right) + \left(\frac{c}{6} - 1\right) = 0$$

$$1 + \frac{3}{2} + \frac{2c}{3} + \frac{c}{6} - 1 = 0$$

$$\frac{3}{2} + \frac{2c}{3} + \frac{c}{6} = 0$$

Multiply by 6:
$$9 + 4c + c = 0$$

$$9 + 5c = 0$$

$$c = -\frac{9}{5}$$

So $\lambda = -1$ when $c = -9/5$ (again outside typical range).

Step 7.3: Alternative approach using $\det(J)$

For eigenvalues λ_1, λ_2:
$$\lambda_1 \lambda_2 = \det(J) = \frac{c}{6} - 1$$

When does $\det(J) = 1$?
$$\frac{c}{6} - 1 = 1$$

$$\frac{c}{6} = 2$$

$$c = 12$$

At $c = 12$, if the eigenvalues are real and have product 1, then one equals $1/\lambda_1$ where the other is λ_1. If they're equal, both equal ± 1.

Check trace at $c = 12$:
$$\mathrm{tr}(J) = \frac{3}{2} + \frac{2(12)}{3} = \frac{3}{2} + 8 = \frac{19}{2}$$

Eigenvalues sum to $19/2$ and multiply to 1.

From quadratic formula:
$$\lambda = \frac{\frac{19}{2} \pm \sqrt{\left(\frac{19}{2}\right)^2 - 4}}{2}$$

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Since discriminant > 0, eigenvalues are real and distinct. One will equal $+1$ when we solve more carefully.

When does $\det(J) = -1$?

$$\frac{c}{6} - 1 = -1$$

$$\frac{c}{6} = 0$$

$$c = 0$$

At $c = 0$, one eigenvalue equals -1 (flip bifurcation).

Check: At $c = 0$:

$$\lambda^2 - \frac{3}{2}\lambda - 1 = 0$$

$$\lambda = \frac{\frac{3}{2} \pm \sqrt{\frac{9}{4} + 4}}{2} = \frac{\frac{3}{2} \pm \sqrt{\frac{25}{4}}}{2}$$

$$= \frac{\frac{3}{2} \pm \frac{5}{2}}{2}$$

$$\lambda_1 = \frac{4}{2} = 2, \quad \lambda_2 = \frac{-1}{2} = -\frac{1}{2}$$

Wait, this gives $\lambda_1 = 2$, not -1. Let me recalculate.

Actually, $\det(J) = \lambda_1 \lambda_2 = 2 \cdot (-1/2) = -1$.

So one eigenvalue is 2 (outside unit circle, unstable) and other is $-1/2$ (inside unit circle).

At $c = 0$, the determinant equals -1, but we need one eigenvalue exactly at -1 for flip bifurcation.

Let me solve for when $\lambda = -1$ directly:

If $\lambda = -1$ is an eigenvalue:

$$(-1)^2 - \left(\frac{3}{2} + \frac{2c}{3}\right)(-1) + \left(\frac{c}{6} - 1\right) = 0$$

$$1 + \frac{3}{2} + \frac{2c}{3} + \frac{c}{6} - 1 = 0$$

$$\frac{3}{2} + \frac{4c + c}{6} = 0$$

$$\frac{3}{2} + \frac{5c}{6} = 0$$

$$9 + 5c = 0$$

$$c = -\frac{9}{5}$$

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Step 8: Summary of Bifurcations for Fixed Point 2

Based on the determinant analysis:

- At $c = 0$: $\det(J) = -1$, indicating one eigenvalue crosses the negative real axis

- At $c = 12$: $\det(J) = 1$, indicating eigenvalue behavior changes

More precisely:
At $c = -9/5$: One eigenvalue equals -1

$$\boxed{\text{Flip (period-doubling) bifurcation at } c = -\frac{9}{5}}$$

At $c = -3$: One eigenvalue equals $+1$

$$\boxed{\text{Transcritical or fold bifurcation at } c = -3}$$

Explanation 4 (Interpretation for Positive c). *For physically relevant positive values of c:*

- *The second fixed point starts (at $c = 0$) with mixed stability*

- *As c increases, the determinant increases from -1 toward $+1$*

- *At $c = 0$: determinant crosses -1*

- *At $c = 12$: determinant crosses $+1$*

- *The detailed stability depends on both trace and determinant; typically one eigenvalue will be outside the unit circle for large c*

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3 Final Summary

Fixed Points

$$\boxed{(x_1^*, y_1^*) = (0, 0)}$$

$$\boxed{(x_2^*, y_2^*) = \left(1 + \frac{c}{3}, \frac{1}{3} + \frac{c}{9}\right)}$$

Bifurcations

Fixed Point 1 at origin:

- Eigenvalues: $\lambda = \frac{-1 \pm \sqrt{1 - 8c}}{4}$ for $c < 1/8$ (real)

- Eigenvalues: $\lambda = \frac{-1 \pm i\sqrt{8c - 1}}{4}$ for $c > 1/8$ (complex)

- Modulus: $|\lambda| = \sqrt{c/2}$ when complex

- **Stable for** $c < 2$

- **Neimark-Sacker bifurcation at** $c = 2$

- **Unstable for** $c > 2$

Fixed Point 2:

- Determinant: $\det(J) = c/6 - 1$

- Trace: $\operatorname{tr}(J) = 3/2 + 2c/3$

- **Flip bifurcation at** $c = -9/5$ (eigenvalue $= -1$)

- **Transcritical bifurcation at** $c = -3$ (eigenvalue $= +1$)

- Stability for positive c depends on full eigenvalue analysis