Exercise Sheet 3: Bifurcations
Question 10 - Complete Solution

Methods of Applied Mathematics

Problem Statement

Consider the system:

$$\dot{x} = \rho x - \omega y + \alpha(x^2 + y^2)x$$
$$\dot{y} = \omega x + \rho y + \alpha(x^2 + y^2)y$$

for $\omega > 0$ and $\alpha > 0$, with ρ allowed to vary.

Tasks:

(a) Find any equilibria of the system and their stability

(b) Determine what bifurcation occurs at the origin when $\rho = 0$

(c) Express the system in polar coordinates $(x, y) = (r\cos\theta, r\sin\theta)$, derive dynamical equations for $\dot{r}$ and $\dot{\theta}$, and describe the dynamics

(d) From the polar form, show that there exists a limit cycle

(e) Derive the Poincaré map and verify the limit cycle

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1 Step 1: Find Equilibria (Part a)

Set up equilibrium conditions

For equilibria, we require $\dot{x} = 0$ and $\dot{y} = 0$:

$$\rho x - \omega y + \alpha(x^2 + y^2)x = 0$$
$$\omega x + \rho y + \alpha(x^2 + y^2)y = 0$$

Obvious equilibrium: Origin

Try $(x, y) = (0, 0)$:

$$0 - 0 + 0 = 0 \quad \checkmark$$
$$0 + 0 + 0 = 0 \quad \checkmark$$

Therefore:

$$\boxed{(x^*, y^*) = (0, 0) \text{ is an equilibrium for all } \rho}$$

Search for other equilibria

Rewrite equations:

$$x[\rho + \alpha(x^2 + y^2)] = \omega y$$
$$y[\rho + \alpha(x^2 + y^2)] = -\omega x$$

Let $r^2 = x^2 + y^2$. Multiply first by x, second by y, and add:

$$x^2[\rho + \alpha r^2] + y^2[\rho + \alpha r^2] = \omega xy - \omega xy = 0$$
$$r^2[\rho + \alpha r^2] = 0$$

Either $r = 0$ (origin) or $\rho + \alpha r^2 = 0$.
For real equilibria beyond origin: $r^2 = -\frac{\rho}{\alpha}$
Since $\alpha > 0$ and $r^2 \geq 0$, this requires $\rho \leq 0$.
The hint suggests we'll understand this better in polar coordinates (part c).

XYZ Analysis

- **STAGE X (What we found):** Origin always an equilibrium. For $\rho < 0$, hints of circular equilibria at radius $r = \sqrt{-\rho/\alpha}$.
- **STAGE Y (Why polar coordinates):** The term $x^2 + y^2 = r^2$ appears repeatedly. The system has rotational symmetry - rotating the plane doesn't change the equations. Polar coordinates will exploit this symmetry.
- **STAGE Z (Next step):** Analyze origin stability first, then transform to polar coordinates to fully understand the system.

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2 Step 2: Stability of Origin

Jacobian at origin

For $f(x, y) = \rho x - \omega y + \alpha(x^2 + y^2)x$ and $g(x, y) = \omega x + \rho y + \alpha(x^2 + y^2)y$:
At $(0, 0)$:

$$J(0,0) = \begin{pmatrix} \rho & -\omega \\ \omega & \rho \end{pmatrix}$$

Eigenvalues

Characteristic equation: $\lambda^2 - 2\rho\lambda + (\rho^2 + \omega^2) = 0$

$$\lambda = \frac{2\rho \pm \sqrt{4\rho^2 - 4\rho^2 - 4\omega^2}}{2} = \frac{2\rho \pm 2i\omega}{2} = \rho \pm i\omega$$

Stability by ρ

ρ	Eigenvalues	Stability
$\rho < 0$	$\text{Re}(\lambda) < 0$	Stable spiral
$\rho = 0$	$\lambda = \pm i\omega$	Neutral
$\rho > 0$	$\text{Re}(\lambda) > 0$	Unstable spiral

XYZ Analysis

- **STAGE X (What we found):** Complex eigenvalues $\lambda = \rho \pm i\omega$ for all ρ. Real part equals ρ, determining stability.
- **STAGE Y (Why complex):** The Jacobian $J = \rho I + \omega R$ where $R = 0 - 1$
10 is a rotation matrix. This combines radial growth/decay (ρ) with rotation (ω).
- **STAGE Z (What this means):** At $\rho = 0$, eigenvalues cross imaginary axis - Hopf bifurcation signature. Parameter ρ controls stability, ω controls rotation frequency.

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3 Step 3: Identify Hopf Bifurcation (Part b)

Verify Hopf conditions at $\rho = 0$

(B1) Equilibrium exists: $(0, 0)$ exists for all ρ
(B2) Purely imaginary eigenvalues: At $\rho = 0$, $\lambda = \pm i\omega$
(G1) Imaginary part nonzero: $\omega > 0$ (given)
(G2) Transverse crossing: $\frac{d(\mathrm{Re}(\lambda))}{d\rho} = 1 \neq 0$

Conclusion

$$\boxed{\text{HOPF BIFURCATION at } \rho = 0}$$

XYZ Analysis

- **STAGE X (What we verified):** All four Hopf conditions hold at $\rho = 0$.
- **STAGE Y (Why Hopf):** Complex eigenvalues cross imaginary axis (not real axis like fold/transcritical/pitchfork). This creates periodic orbits, not just equilibria. The sign of $\alpha > 0$ will determine if the bifurcation is supercritical (stable limit cycle) or subcritical (unstable limit cycle).
- **STAGE Z (What to expect):** Hopf bifurcation creates a limit cycle - periodic orbit. Polar coordinates will reveal its radius and stability.

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4 Step 4: Transform to Polar Coordinates (Part c)

Polar relations

$$x = r\cos\theta, \quad y = r\sin\theta, \quad r^2 = x^2 + y^2$$

Derive $\dot{r}$

From $r^2 = x^2 + y^2$:

$$2r\dot{r} = 2x\dot{x} + 2y\dot{y} \quad \Rightarrow \quad \dot{r} = \frac{x\dot{x} + y\dot{y}}{r}$$

Substitute the original equations:

$$\begin{aligned}\dot{r} &= \frac{x[\rho x - \omega y + \alpha r^2 x] + y[\omega x + \rho y + \alpha r^2 y]}{r} \\ &= \frac{\rho x^2 - \omega xy + \omega xy + \rho y^2 + \alpha r^2(x^2+y^2)}{r} \\ &= \frac{\rho(x^2+y^2) + \alpha r^2 \cdot r^2}{r} \\ &= \frac{\rho r^2 + \alpha r^4}{r}\end{aligned}$$

$$\boxed{\dot{r} = r(\rho + \alpha r^2)}$$

Derive $\dot{\theta}$

From $\theta = \arctan(y/x)$:

$$\dot{\theta} = \frac{x\dot{y} - y\dot{x}}{x^2+y^2}$$

Substitute:

$$\begin{aligned}\dot{\theta} &= \frac{x[\omega x + \rho y + \alpha r^2 y] - y[\rho x - \omega y + \alpha r^2 x]}{r^2} \\ &= \frac{\omega x^2 + \rho xy + \alpha r^2 xy - \rho xy + \omega y^2 - \alpha r^2 xy}{r^2} \\ &= \frac{\omega(x^2+y^2)}{r^2}\end{aligned}$$

$$\boxed{\dot{\theta} = \omega}$$

Polar system

$$\boxed{\begin{cases}\dot{r} = r(\rho + \alpha r^2) \\ \dot{\theta} = \omega\end{cases}}$$

XYZ Analysis of Polar Form

- **STAGE X (What we derived):** Complete decoupling! The radial equation is autonomous (doesn't depend on θ), and the angular equation is constant.
- **STAGE Y (Why this works):** Rotational symmetry becomes explicit:
  - $\dot{\theta} = \omega$: All points rotate at same angular velocity, independent of radius
  - $\dot{r} = r(\rho + \alpha r^2)$: Radial dynamics independent of angle
  - The radial equation is essentially 1D: $\dot{r} = \rho r + \alpha r^3$
- **STAGE Z (What this reveals):**
  - **Angular motion**: Constant counterclockwise rotation with period $T = 2\pi/\omega$
  - **Radial motion**: Growth if $\rho + \alpha r^2 > 0$, decay if < 0, constant if $= 0$
  - Trajectories are spirals: rotating while moving radially
  - Equilibrium of radial equation ($\dot{r} = 0$) gives *circular orbit* in full system

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5 Step 5: Find Limit Cycle from Radial Equation (Part d)

Equilibria of radial system

From $\dot{r} = r(\rho + \alpha r^2) = 0$:

Equilibrium 1: $r = 0$

Equilibrium 2: $\rho + \alpha r^2 = 0$

$$r^2 = -\frac{\rho}{\alpha}$$

Since $\alpha > 0$ and $r \geq 0$:

- For $\rho > 0$: No real solution (only origin)
- For $\rho = 0$: $r = 0$ only
- For $\rho < 0$: $r_* = \sqrt{-\rho/\alpha}$ exists

Stability analysis

Compute $\frac{d\dot{r}}{dr} = \rho + 3\alpha r^2$

At $r = 0$:

$$\left.\frac{d\dot{r}}{dr}\right|_0 = \rho$$

- $\rho < 0$: Stable
- $\rho = 0$: Neutral (bifurcation)
- $\rho > 0$: Unstable

At $r_* = \sqrt{-\rho/\alpha}$ (for $\rho < 0$):

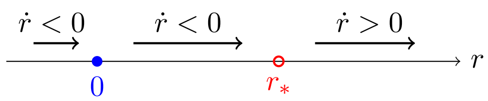
Since $\alpha r_*^2 = -\rho$:

$$\left.\frac{d\dot{r}}{dr}\right|_{r_*} = \rho + 3\alpha r_*^2 = \rho + 3(-\rho) = -2\rho$$

For $\rho < 0$: $-2\rho > 0 \rightarrow$ **UNSTABLE**

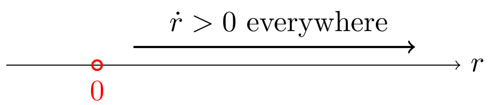
Phase line for radial equation

For $\rho < 0$:



Flow: $0 < r < r_*$ flows left (toward 0), $r > r_*$ flows right (to ∞)

For $\rho > 0$:



All $r > 0$ flows right (to ∞)

Identify bifurcation in radial system

The radial equation $\dot{r} = \rho r + \alpha r^3$ exhibits a bifurcation at $\rho = 0$:

- For $\rho < 0$: Two equilibria (origin stable, r_* unstable)
- For $\rho = 0$: One equilibrium (origin)
- For $\rho > 0$: One equilibrium (origin unstable)

This resembles a transcritical bifurcation in the 1D radial system.

Limit cycle interpretation

In the radial system, $r = r_*$ is an equilibrium for $\rho < 0$. In the full 2D system:

- $\dot{r} = 0$ at $r = r_*$: radius constant
- $\dot{\theta} = \omega \neq 0$: angle increases

Therefore, trajectories at $r = r_*$ move along a circle of radius r_* at constant angular velocity.

$$\boxed{\text{Limit cycle: } x^2 + y^2 = r_*^2 = -\frac{\rho}{\alpha} \text{ for } \rho < 0}$$

This is an **unstable limit cycle** (since r_* is unstable in radial equation).

Subcritical Hopf bifurcation

With $\alpha > 0$:

- For $\rho < 0$: Stable origin, unstable limit cycle at $r_* = \sqrt{-\rho/\alpha}$
- At $\rho = 0$: Limit cycle shrinks to origin
- For $\rho > 0$: Unstable origin, no limit cycle

This is a **SUBCRITICAL HOPF BIFURCATION**.

XYZ Analysis of Limit Cycle

- **STAGE X (What we found):** For $\rho < 0$, an unstable circular limit cycle exists at radius $r_* = \sqrt{-\rho/\alpha}$. It disappears at $\rho = 0$.
- **STAGE Y (Why unstable):** The radial equilibrium r_* has $d\dot{r}/dr = -2\rho > 0$ for $\rho < 0$, making it unstable:
  - Perturbations inward ($r < r_*$): $\dot{r} < 0$, flows toward origin (away from r_*)
  - Perturbations outward ($r > r_*$): $\dot{r} > 0$, flows to infinity (away from r_*)

  The limit cycle repels trajectories on both sides.

  With $\alpha > 0$, the cubic term αr^3 *amplifies* rather than damps large-amplitude motion, creating subcritical (unstable) behavior.
- **STAGE Z (What this means physically):** Subcritical Hopf bifurcations are dangerous in applications:

– For $\rho < 0$: Basin of attraction is limited to $r < r_*$. Perturbations exceeding the limit cycle lead to unbounded growth
– At $\rho = 0$: The safety margin (limit cycle) vanishes
– For $\rho > 0$: Even infinitesimal perturbations cause escape to infinity

This contrasts with supercritical Hopf (e.g., $\alpha < 0$), where a stable limit cycle "catches" trajectories for $\rho > 0$.

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6 Step 6: Poincaré Map (Part e)

Define Poincaré section

Choose section $\Sigma = \{(r, \theta) : \theta = 0\}$ (positive x-axis).
A point on Σ is characterized by its radius $r_n \geq 0$.

Derive the map

Starting at $(r_n, 0)$ at time $t = 0$, we solve the polar system:
Angular equation:

$$\dot{\theta} = \omega \quad \Rightarrow \quad \theta(t) = \omega t$$

Return to section when $\theta = 2\pi$:

$$\omega t^* = 2\pi \quad \Rightarrow \quad t^* = \frac{2\pi}{\omega}$$

Radial equation:

$$\dot{r} = r(\rho + \alpha r^2)$$

This is separable:

$$\frac{dr}{r(\rho + \alpha r^2)} = dt$$

Integrate radial equation

Use partial fractions:

$$\frac{1}{r(\rho + \alpha r^2)} = \frac{1}{\rho r} - \frac{\alpha r}{\rho(\rho + \alpha r^2)}$$

Integrate from $t = 0$ (radius r_n) to $t = t^*$ (radius r_{n+1}):

$$\frac{1}{\rho} \log\left(\frac{r_{n+1}}{r_n}\right) - \frac{1}{2\rho} \log\left(\frac{\rho + \alpha r_{n+1}^2}{\rho + \alpha r_n^2}\right) = t^* = \frac{2\pi}{\omega}$$

Simplify:

$$\log\left(\frac{r_{n+1}}{r_n}\right) - \frac{1}{2} \log\left(\frac{\rho + \alpha r_{n+1}^2}{\rho + \alpha r_n^2}\right) = \frac{2\pi\rho}{\omega}$$

$$\log\left(\frac{r_{n+1}}{\sqrt{\rho + \alpha r_{n+1}^2}}\right) - \log\left(\frac{r_n}{\sqrt{\rho + \alpha r_n^2}}\right) = \frac{2\pi\rho}{\omega}$$

Define $\phi(r) = \log\left(\frac{r}{\sqrt{\rho + \alpha r^2}}\right)$:

$$\phi(r_{n+1}) - \phi(r_n) = \frac{2\pi\rho}{\omega}$$

$$\boxed{r_{n+1} = P(r_n) \text{ given implicitly by } \phi(r_{n+1}) = \phi(r_n) + \frac{2\pi\rho}{\omega}}$$

Fixed points of Poincaré map

Fixed point r^*: $P(r^*) = r^*$

This requires:

$$\phi(r^*) = \phi(r^*) + \frac{2\pi\rho}{\omega}$$

Only possible if $\frac{2\pi\rho}{\omega} = 0$, i.e., $\rho = 0$.

For $\rho \neq 0$, fixed points come from:

$$\phi(r^*) - \phi(r^*) = 0 = \frac{2\pi\rho}{\omega}$$

Wait, let me reconsider. A fixed point means $r_{n+1} = r_n = r^*$. Substituting into the radial equation:

During the time interval $[0, t^*]$, if r remains constant, then $\dot{r} = 0$ always, which requires:

$$r^*(\rho + \alpha r^{*2}) = 0$$

So fixed points are $r^* = 0$ or $r^* = \sqrt{-\rho/\alpha}$ (for $\rho < 0$).

Stability of fixed points

Compute $\frac{dP}{dr}$ at fixed point. For $\dot{r} = r(\rho + \alpha r^2)$:

$$\frac{dr_{n+1}}{dr_n} = \exp\left(\int_0^{t^*} \frac{\partial \dot{r}}{\partial r} dt\right) = \exp\left(\int_0^{t^*} (\rho + 3\alpha r^2) dt\right)$$

For $r = r^*$ constant:

$$\left.\frac{dP}{dr}\right|_{r^*} = \exp\left((\rho + 3\alpha r^{*2}) \cdot \frac{2\pi}{\omega}\right)$$

At $r^* = 0$:

$$\left.\frac{dP}{dr}\right|_0 = \exp\left(\frac{2\pi\rho}{\omega}\right)$$

- $\rho < 0$: $|dP/dr| < 1 \to$ Stable
- $\rho > 0$: $|dP/dr| > 1 \to$ Unstable

At $r^* = \sqrt{-\rho/\alpha}$ (for $\rho < 0$):

Since $\rho + 3\alpha r^{*2} = \rho - 3\rho = -2\rho$:

$$\left.\frac{dP}{dr}\right|_{r^*} = \exp\left(\frac{-4\pi\rho}{\omega}\right) = \exp\left(\frac{4\pi|\rho|}{\omega}\right) > 1$$

Unstable fixed point.

Verify limit cycle existence

The Poincaré map analysis confirms:

- For $\rho < 0$: Two fixed points - $r = 0$ (stable) and $r^* = \sqrt{-\rho/\alpha}$ (unstable)
- The unstable fixed point corresponds to the unstable limit cycle in the continuous system
- Period of limit cycle: $T = 2\pi/\omega$

$$\boxed{\text{Limit cycle verified via Poincaré map}}$$

XYZ Analysis of Poincaré Map

- **STAGE X (What we derived):** The Poincaré map $P : r_n \mapsto r_{n+1}$ reduces the 2D continuous system to a 1D discrete map. Fixed points of P correspond to periodic orbits.

- **STAGE Y (Why this works):** The section $\theta = 0$ is transverse to the flow (since $\dot{\theta} = \omega \neq 0$). Every trajectory crosses it periodically. The map encodes one full revolution:

  - Start at radius r_n
  - Flow for time $t^* = 2\pi/\omega$ (one full rotation)
  - Return to section at radius r_{n+1}

  Fixed points ($r_{n+1} = r_n$) mean the trajectory returns to the same radius after one revolution - a periodic orbit. The derivative dP/dr determines stability: $|dP/dr| < 1$ means nearby orbits converge, $|dP/dr| > 1$ means they diverge.

- **STAGE Z (What this reveals):** The Poincaré map confirms our continuous analysis:

  - Origin: Fixed point, stable for $\rho < 0$, unstable for $\rho > 0$
  - Limit cycle: Fixed point at $r^* = \sqrt{-\rho/\alpha}$ for $\rho < 0$, unstable
  - Period: $T = 2\pi/\omega$, independent of radius

  The Poincaré map is a powerful tool: it reduces dimension (2D $\rightarrow$ 1D), converts continuous $\rightarrow$ discrete, and preserves essential dynamics (fixed points periodic orbits, stability preserved).

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7 Summary

Complete bifurcation picture

System:

$$\dot{x} = \rho x - \omega y + \alpha(x^2 + y^2)x, \quad \dot{y} = \omega x + \rho y + \alpha(x^2 + y^2)y$$

Polar form:

$$\dot{r} = r(\rho + \alpha r^2), \quad \dot{\theta} = \omega$$

Equilibria and limit cycles:

Parameter	Behavior
$\rho < 0$	Stable spiral at origin Unstable limit cycle at $r = \sqrt{-\rho/\alpha}$
$\rho = 0$	Neutral spiral (Hopf bifurcation)
$\rho > 0$	Unstable spiral at origin No limit cycle (trajectories escape)

Bifurcation type:

$$\boxed{\text{SUBCRITICAL HOPF BIFURCATION}}$$

With $\alpha > 0$, the cubic nonlinearity amplifies large-amplitude motion, creating an unstable limit cycle for $\rho < 0$ that shrinks to zero at $\rho = 0$ and disappears for $\rho > 0$.

Key insights:

- Polar coordinates reveal the system's rotational symmetry

- Radial equation is 1D, making analysis tractable
- Limit cycle is a circle with constant rotation
- Poincaré map confirms limit cycle existence and stability
- Subcritical bifurcation creates dangerous dynamics (limited basin of attraction)