Exercise Sheet 3: Bifurcations
Question 7 - Complete Solution

Methods of Applied Mathematics

Problem Statement

Determine what bifurcation happens as μ changes in the systems:

(a) $\frac{dx}{dt} = \mu x - x^3$

(b) $\frac{dx}{dt} = \mu x + (1+\mu)x^2 - x^3$

(c) $\frac{dx}{dt} = \tanh(x) - \mu x$

(d) $\frac{d^2x}{dt^2} + \frac{dx}{dt} + \mu x + x^3 = 0$

(e) $\frac{dx}{dt} = \mu y - x, \quad \frac{dy}{dt} = \frac{1}{3}y^3 + y^2 - y + x$

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1 Part (a): $\dot{x} = \mu x - x^3$

Step 1: Find equilibria

Set $\dot{x} = 0$:

$$\mu x - x^3 = 0 \quad \Rightarrow \quad x(\mu - x^2) = 0$$

Solutions: $x = 0$ or $x^2 = \mu$
For $\mu < 0$: Only $x = 0$ (real)
For $\mu = 0$: Only $x = 0$
For $\mu > 0$: Three equilibria: $x = 0, \pm\sqrt{\mu}$

Step 2: Analyze stability

Compute $f'(x) = \mu - 3x^2$
At $x = 0$:

$$f'(0) = \mu$$

- $\mu < 0$: stable
- $\mu = 0$: neutral
- $\mu > 0$: unstable

At $x = \pm\sqrt{\mu}$ (for $\mu > 0$):

$$f'(\pm\sqrt{\mu}) = \mu - 3\mu = -2\mu < 0 \quad \Rightarrow \quad \text{stable}$$

Step 3: Identify bifurcation

Characteristics:

- One equilibrium becomes three
- Origin loses stability at $\mu = 0$
- Two stable equilibria emerge symmetrically
- System has symmetry: $f(-x) = -(\mu x - x^3) = -f(x)$

$$\boxed{\text{SUPERCRITICAL PITCHFORK BIFURCATION at } \mu = 0}$$

XYZ Analysis

- **STAGE X (What we found):** Classic pitchfork structure: $1 \to 3$ equilibria, symmetric emergence of stable branches.
- **STAGE Y (Why pitchfork):** The odd symmetry $f(-x) = -f(x)$ forces equilibria to appear in $\pm$ pairs. The new equilibria are stable (supercritical) rather than unstable (subcritical). This is the canonical example from lecture notes (pages 48-49).
- **STAGE Z (Meaning):** Spontaneous symmetry breaking: for $\mu > 0$, system must "choose" between $x > 0$ or $x < 0$ states. Common in physics (ferromagnetism), mechanics (buckling), and biology (pattern formation).

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2 Part (b): $\dot{x} = \mu x + (1+\mu)x^2 - x^3$

Step 1: Find equilibria

Set $\dot{x} = 0$:

$$\mu x + (1+\mu)x^2 - x^3 = 0 \quad \Rightarrow \quad x[\mu + (1+\mu)x - x^2] = 0$$

Either $x = 0$ or $x^2 - (1+\mu)x - \mu = 0$
For the quadratic: $x = \frac{(1+\mu)\pm\sqrt{(1+\mu)^2+4\mu}}{2}$
Simplify discriminant:

$$(1+\mu)^2 + 4\mu = 1 + 2\mu + \mu^2 + 4\mu = \mu^2 + 6\mu + 1$$

This is always positive for reasonable μ (discriminant of $\mu^2+6\mu+1 = 0$ gives $\mu = -3\pm 2\sqrt{2}$). So we always have three real equilibria:

$$x_0 = 0, \quad x_\pm = \frac{(1+\mu) \pm \sqrt{\mu^2+6\mu+1}}{2}$$

Step 2: Analyze stability

Compute $f'(x) = \mu + 2(1+\mu)x - 3x^2$
At $x = 0$:

$$f'(0) = \mu$$

- $\mu < 0$: stable

- $\mu = 0$: neutral (bifurcation point)
- $\mu > 0$: unstable

At $x = x_+$ (upper root):
For $\mu = 0$: $x_+ = \frac{1+1}{2} = 1$
Check: $f'(1) = 0 + 2(1)(1) - 3(1)^2 = 2 - 3 = -1 < 0 \to$ stable
At $x = x_-$ (lower root):
For $\mu = 0$: $x_- = \frac{1-1}{2} = 0$ (coincides with origin)
For small $\mu < 0$: $x_- \approx -\mu/1 = -\mu > 0$ (small positive)
Check sign at $\mu = -0.1$: $x_- \approx 0.1$, and we need to verify stability.

Step 3: Behavior near $\mu = 0$

At $\mu = 0$:

- $x_0 = 0$ and $x_- = 0$ coincide (two equilibria meet)
- $x_+ = 1$ exists separately

For μ slightly negative: $x_0 = 0$ stable, x_- unstable, x_+ stable
For μ slightly positive: $x_0 = 0$ unstable, x_- stable (moved to negative), x_+ stable

Step 4: Identify bifurcation

The equilibrium at origin is pinned (always exists). As μ varies, x_- passes through the origin at $\mu = 0$, and they exchange stability.

$$\boxed{\text{TRANSCRITICAL BIFURCATION at } \mu = 0}$$

XYZ Analysis

- **STAGE X (What we found):** The system always has three equilibria (for μ near 0), but one passes through the origin at $\mu = 0$, exchanging stability with it.
- **STAGE Y (Why transcritical):** The origin is pinned: $f(0, \mu) = 0$ for all μ. The moving equilibrium x_- passes through it, not annihilating (fold) or splitting (pitchfork). The number of equilibria stays constant at 3. This is transcritical behavior with a third "spectator" equilibrium at x_+.
- **STAGE Z (Meaning):** The additional quadratic term $(1 + \mu)x^2$ breaks the pitchfork symmetry of part (a), converting it to transcritical. The third equilibrium $x_+ \approx 1$ provides an additional stable state that persists through the bifurcation.

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3 Part (c): $\dot{x} = \tanh(x) - \mu x$

Step 1: Find equilibria

Set $\dot{x} = 0$:

$$\tanh(x) = \mu x$$

Graphically: intersections of $y = \tanh(x)$ (sigmoid, bounded by ± 1) with $y = \mu x$ (line through origin, slope μ).

Step 2: Analyze number of equilibria

For $\mu \leq 0$: Since $\tanh(x)$ has slope 1 at $x = 0$ and μx has slope $\mu \leq 0$, and $\tanh(x) > \mu x$ for $x > 0$, $\tanh(x) < \mu x$ for $x < 0$, only intersection at origin.

For small $\mu > 0$: Line $y = \mu x$ has small positive slope. Since $\tanh'(0) = 1$, for $\mu < 1$, the line intersects $\tanh(x)$ three times: at origin and two points $\pm x^*$.

For $\mu = 1$: Critical case. The line $y = x$ is tangent to $\tanh(x)$ at origin (both have slope 1).

For $\mu > 1$: Line too steep, only intersection at origin.

Step 3: Stability analysis

Compute $f'(x) = \text{sech}^2(x) - \mu = \frac{1}{\cosh^2(x)} - \mu$

At origin:

$$f'(0) = 1 - \mu$$

- $\mu < 1$: stable
- $\mu = 1$: neutral
- $\mu > 1$: unstable

Wait, this seems backwards. Let me reconsider.

Actually, for $\mu > 1$, we have $f'(0) = 1 - \mu < 0$, so origin is stable.

For $0 < \mu < 1$, we have $f'(0) = 1 - \mu > 0$, so origin is unstable.

Let me reconsider the equilibrium count:

For $\mu < 1$: The slope of $\tanh(x)$ at origin is 1, which exceeds the slope μ of the line. So near origin, $\tanh(x) > \mu x$ for small positive x, meaning $\dot{x} > 0$ just right of origin. But far from origin, $\tanh(x) \to 1$ while $\mu x \to \infty$, so eventually $\mu x > \tanh(x)$. Thus there must be an intersection at positive x. By symmetry (both functions are odd), three intersections total.

For $\mu > 1$: The line is steeper than $\tanh(x)$ at origin, so $\mu x > \tanh(x)$ for small positive x, meaning $\dot{x} < 0$ just right of origin. Thus only one intersection at origin.

Step 4: Identify bifurcation

Summary:

- $\mu < 1$: Three equilibria (origin unstable, $\pm x^*$ stable)
- $\mu = 1$: One equilibrium (origin, marginally stable)
- $\mu > 1$: One equilibrium (origin stable)

This is a supercritical pitchfork in reverse (as μ increases, three equilibria merge into one).

$$\boxed{\text{SUPERCRITICAL PITCHFORK BIFURCATION at } \mu = 1}$$

(Direction reversed: stable equilibrium splits into three as μ decreases through 1)

XYZ Analysis

- **STAGE X (What we found):** For $\mu < 1$: three equilibria. For $\mu > 1$: one equilibrium. Change at $\mu = 1$.

- **STAGE Y (Why pitchfork):** The system has odd symmetry: $f(-x) = \tanh(-x) - \mu(-x) = -\tanh(x) + \mu x = -f(x)$. This forces equilibria to appear in $\pm$ pairs. At $\mu = 1$, the tangency condition $\tanh'(0) = \mu$ is satisfied, marking the bifurcation point. For $\mu < 1$, the origin loses stability and two stable equilibria emerge. This is the "backwards" version of standard pitchfork - or equivalently, a standard supercritical pitchfork as μ decreases.
- **STAGE Z (Meaning):** The function $\tanh(x)$ represents a saturating nonlinearity (common in neural networks, control systems). The parameter μ represents linear damping/feedback. For weak feedback ($\mu < 1$), the nonlinearity dominates and system exhibits bistability (two stable states $\pm x^*$). For strong feedback ($\mu > 1$), linear damping dominates and only origin is stable.

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4 Part (d): $\ddot{x} + \dot{x} + \mu x + x^3 = 0$

Step 1: Convert to first-order system

Let $y = \dot{x}$. Then:

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -\dot{x} - \mu x - x^3 = -y - \mu x - x^3 \end{aligned}$$

Step 2: Find equilibria

Set $\dot{x} = 0$ and $\dot{y} = 0$:

$$\begin{aligned} y &= 0 \\ -y - \mu x - x^3 &= 0 \end{aligned}$$

From second equation with $y = 0$:

$$\mu x + x^3 = 0 \quad \Rightarrow \quad x(\mu + x^2) = 0$$

Solutions: $x = 0$ or $x^2 = -\mu$
For $\mu > 0$: Only $(0, 0)$
For $\mu = 0$: Only $(0, 0)$
For $\mu < 0$: Three equilibria: $(0, 0)$ and $(\pm\sqrt{-\mu}, 0)$

Step 3: Jacobian analysis

$$J = \begin{pmatrix} 0 & 1 \\ -\mu - 3x^2 & -1 \end{pmatrix}$$

At origin:

$$J(0, 0) = \begin{pmatrix} 0 & 1 \\ -\mu & -1 \end{pmatrix}$$

Trace: $\tau = -1$, Determinant: $\Delta = \mu$
Eigenvalues: $\lambda^2 + \lambda + \mu = 0$, so $\lambda = \frac{-1 \pm \sqrt{1-4\mu}}{2}$
For $\mu > 1/4$: Complex eigenvalues $\lambda = -\frac{1}{2} \pm i\frac{\sqrt{4\mu-1}}{2}$ with negative real part $\rightarrow$ stable spiral
For $\mu = 0$: $\lambda = \frac{-1 \pm 1}{2}$, so $\lambda = 0, -1 \rightarrow$ neutral
For $\mu < 0$: Real eigenvalues. $\Delta = \mu < 0$ means opposite signs $\rightarrow$ saddle

At $(\pm\sqrt{-\mu}, 0)$ for $\mu < 0$:

$$J = \begin{pmatrix} 0 & 1 \\ -\mu - 3(-\mu) & -1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 2\mu & -1 \end{pmatrix}$$

Since $\mu < 0$: $\Delta = -2\mu > 0$ and $\tau = -1 < 0 \to$ stable (node or spiral depending on discriminant)

Step 4: Identify bifurcation

Characteristics:

- One equilibrium becomes three as μ decreases through 0
- Origin changes from stable to saddle
- Two stable equilibria emerge symmetrically
- System has Hamiltonian structure (conservative for $\dot{x} = 0$)

$$\boxed{\text{SUPERCRITICAL PITCHFORK BIFURCATION at } \mu = 0}$$

(In reverse direction: as μ decreases through 0)

XYZ Analysis

- **STAGE X (What we found):** The second-order ODE exhibits pitchfork bifurcation when converted to phase-plane system. For $\mu > 0$: stable equilibrium at origin. For $\mu < 0$: saddle at origin, two stable equilibria at $\pm\sqrt{-\mu}$.
- **STAGE Y (Why pitchfork):** This is Duffing's equation with damping. The term $\mu x + x^3$ represents a potential $V(x) = \frac{\mu x^2}{2} + \frac{x^4}{4}$:
  - For $\mu > 0$: Single-well potential (minimum at $x = 0$)
  - For $\mu < 0$: Double-well potential (minima at $x = \pm\sqrt{-\mu}$, maximum at $x = 0$)

  The damping term $\dot{x}$ dissipates energy, causing trajectories to settle at potential minima. The pitchfork occurs when the single well bifurcates into double well at $\mu = 0$.
- **STAGE Z (Meaning):** This models mechanical systems like buckled beams, magnetic pendulums, or nonlinear springs. For $\mu > 0$ (stiff restoring force), unique stable position at origin. For $\mu < 0$ (negative stiffness + cubic hardening), two stable positions emerge - the system buckles. Common in structural mechanics and nonlinear oscillators.

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5 Part (e): $\dot{x} = \mu y - x, \quad \dot{y} = \frac{1}{3}y^3 + y^2 - y + x$

Step 1: Find equilibria

Set $\dot{x} = 0$ and $\dot{y} = 0$:

$$\mu y - x = 0 \quad \Rightarrow \quad x = \mu y \quad \cdots (1)$$
$$\frac{1}{3}y^3 + y^2 - y + x = 0 \quad \cdots (2)$$

Substitute (1) into (2):

$$\frac{1}{3}y^3 + y^2 - y + \mu y = 0$$

$$\frac{1}{3}y^3 + y^2 + (\mu - 1)y = 0$$

$$y\left[\frac{1}{3}y^2 + y + (\mu - 1)\right] = 0$$

Either $y = 0$ or $\frac{1}{3}y^2 + y + (\mu - 1) = 0$
Equilibrium 1: $y = 0$, then $x = 0$

$$(x, y) = (0, 0)$$

Equilibria 2, 3: From quadratic:

$$y = \frac{-1 \pm \sqrt{1 - 4 \cdot \frac{1}{3}(\mu - 1)}}{2 \cdot \frac{1}{3}} = \frac{-3 \pm \sqrt{9 - 4(\mu - 1)}}{2}$$

$$y = \frac{-3 \pm \sqrt{13 - 4\mu}}{2}$$

For real solutions: $13 - 4\mu \geq 0$, i.e., $\mu \leq 13/4$

Step 2: Analyze equilibrium count

For $\mu < 13/4$**:** Three equilibria
For $\mu = 13/4$**:** Two equilibria coincide at $y = -3/2$
For $\mu > 13/4$**:** One equilibrium at origin

Step 3: Jacobian at origin

$$J = \begin{pmatrix} -1 & \mu \\ 1 & y^2 + 2y - 1 \end{pmatrix}$$

At origin:

$$J(0, 0) = \begin{pmatrix} -1 & \mu \\ 1 & -1 \end{pmatrix}$$

Trace: $\tau = -2$
Determinant: $\Delta = (-1)(-1) - \mu \cdot 1 = 1 - \mu$
Eigenvalues: $\lambda^2 + 2\lambda + (1 - \mu) = 0$

$$\lambda = \frac{-2 \pm \sqrt{4 - 4(1 - \mu)}}{2} = -1 \pm \sqrt{\mu}$$

For $\mu < 0$**:** Complex eigenvalues $\lambda = -1 \pm i\sqrt{|\mu|} \to$ stable spiral
For $\mu = 0$**:** $\lambda = -1$ (repeated) $\to$ stable node
For $0 < \mu < 1$**:** Complex eigenvalues $\lambda = -1 \pm i\sqrt{\mu} \to$ stable spiral
For $\mu = 1$**:** $\lambda = 0, -2 \to$ neutral (one zero eigenvalue)
For $\mu > 1$**:** Real eigenvalues, one positive ($\lambda = -1 + \sqrt{\mu} > 0$) $\to$ saddle

Step 4: Identify bifurcation

At $\mu = 1$, the origin has one zero eigenvalue, suggesting a codimension-1 bifurcation.

From equilibrium structure: for μ slightly less than 1, origin is stable and there exist two other equilibria. For μ slightly greater than 1, origin becomes saddle.

The equilibria meet at $y = (-3 \pm \sqrt{13-4})/2 = (-3 \pm 3)/2$, giving $y = 0$ or $y = -3$.

At $\mu = 1$: one equilibrium at origin, another at $y = -3$ (so $x = -3$).

This doesn't look like a simple collision at origin. Let me recalculate where equilibria collide.

Actually, at $\mu = 13/4$, the discriminant vanishes, and the two non-origin equilibria collide at:

$$y = -3/2, \quad x = \mu(-3/2) = (13/4)(-3/2) = -39/8$$

This is a fold bifurcation of the two non-origin equilibria at $\mu = 13/4$.

But at $\mu = 1$, the origin changes stability (eigenvalue crosses zero). This could be a transcritical bifurcation where one of the non-origin equilibria passes through the origin.

Let me check: at $\mu = 1$, do any of the non-origin equilibria equal $(0, 0)$?

$$y = \frac{-3 \pm \sqrt{13-4}}{2} = \frac{-3 \pm 3}{2}$$

So $y = 0$ or $y = -3$.

Yes! At $\mu = 1$, one equilibrium is at $y = 0, x = 0$ (the origin), confirming a collision.

$$\boxed{\text{TRANSCRITICAL BIFURCATION at } \mu = 1}$$

and

$$\boxed{\text{FOLD BIFURCATION at } \mu = 13/4}$$

XYZ Analysis

- **STAGE X (What we found):** Two bifurcations occur in this system as μ varies. At $\mu = 1$, a transcritical bifurcation where a moving equilibrium passes through the origin. At $\mu = 13/4$, a fold bifurcation where two equilibria collide and annihilate.
- **STAGE Y (Why this complexity):** The y-equation is cubic, giving up to three values of y for equilibrium, hence up to three equilibria total. The parameter μ enters only through the coupling term μy in the x-equation. As μ increases:
  - $\mu < 1$: Three equilibria exist; origin stable
  - $\mu = 1$: Transcritical bifurcation; moving equilibrium passes through origin
  - $1 < \mu < 13/4$: Three equilibria; origin now saddle
  - $\mu = 13/4$: Fold bifurcation; two non-origin equilibria collide
  - $\mu > 13/4$: Only origin remains (saddle)
- **STAGE Z (Meaning):** This system exhibits a sequence of bifurcations, showing that complex dynamics can have multiple qualitative transitions. The cubic nonlinearity in y combined with linear coupling creates a rich bifurcation structure. Such sequences appear in chemical reactors, neural networks, and ecological models where multiple mechanisms interact.

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6 Summary Table

System	Bifurcation Type	Critical Value
(a) $\dot{x} = \mu x - x^3$	Supercritical Pitchfork	$\mu = 0$
(b) $\dot{x} = \mu x + (1+\mu)x^2 - x^3$	Transcritical	$\mu = 0$
(c) $\dot{x} = \tanh(x) - \mu x$	Supercritical Pitchfork	$\mu = 1$
(d) $\ddot{x} + \dot{x} + \mu x + x^3 = 0$	Supercritical Pitchfork	$\mu = 0$
2*(e) $\dot{x} = \mu y - x$, $\dot{y} = \frac{y^3}{3} + y^2 - y + x$	Transcritical	$\mu = 1$
	Fold	$\mu = 13/4$

Key Insights

Identifying bifurcation types:

- **Pitchfork**: Odd symmetry, 1 3 equilibria, symmetric branches
- **Transcritical**: Pinned equilibrium, passing equilibrium, stability exchange
- **Fold**: Equilibria created/destroyed, tangent collision, 2 0 equilibria

Analysis strategy:

1. Find equilibria as functions of parameter
2. Count equilibria for different parameter ranges
3. Compute stability (1D: $f'(x)$; 2D: eigenvalues of Jacobian)
4. Look for:
   - Changes in equilibrium count $\to$ fold or pitchfork
   - Equilibria passing through each other $\to$ transcritical
   - Eigenvalues becoming complex/real $\to$ potential Hopf (if imaginary axis crossing)
5. Check for symmetries ($f(-x) = -f(x)$ suggests pitchfork)
6. Verify critical point: equilibrium exists, eigenvalue = 0