Asymptotics Problem 2(b)

Solution using Method of Steepest Descent

Problem Statement

Use the method of steepest descent to find the leading asymptotic behaviour as $X \to \infty$ of:

$$I(X) = \int_{-\infty}^{\infty} \frac{te^{iX(t^5/5+t)}}{1+t^2}\,dt$$

Solution

Step 1: Identify the integral structure

We have a complex integral of the form (Eq. 239 from notes):

$$I(X) = \int_C f(z)\,e^{X\phi(z)}\,dz$$

where:

- $f(t) = \dfrac{t}{1+t^2}$
- $\phi(t) = i\left(\dfrac{t^5}{5} + t\right)$
- Original contour C: the real axis from $-\infty$ to ∞

Step 2: Find saddle points

Saddle points occur where $\phi'(t) = 0$.

Computing the derivative:

$$\phi'(t) = i(t^4 + 1)$$

Setting $\phi'(t) = 0$:

$$i(t^4 + 1) = 0 \implies t^4 = -1 = e^{i\pi}$$

The four saddle points are:

$$t_k = e^{i\pi(2k+1)/4}, \quad k = 0, 1, 2, 3$$

Explicitly:

$$\begin{aligned}
t_0 &= e^{i\pi/4} = \frac{1+i}{\sqrt{2}} \\
t_1 &= e^{i3\pi/4} = \frac{-1+i}{\sqrt{2}} \\
t_2 &= e^{i5\pi/4} = \frac{-1-i}{\sqrt{2}} \\
t_3 &= e^{i7\pi/4} = \frac{1-i}{\sqrt{2}}
\end{aligned}$$

Step 3: Evaluate $\phi(t)$ at saddle points

For any saddle point where $t^4 = -1$:

$$\phi(t) = i\left(\frac{t^5}{5} + t\right) = i\left(\frac{t \cdot t^4}{5} + t\right) = i\left(\frac{-t}{5} + t\right) = i \cdot \frac{4t}{5}$$

Therefore:

$$\begin{aligned}
\phi(t_0) &= i \cdot \frac{4e^{i\pi/4}}{5} = \frac{4i}{\sqrt{2}}e^{i\pi/4} = \frac{2\sqrt{2}i}{5}(1+i) = \frac{2\sqrt{2}}{5}(i-1) \\
\phi(t_1) &= i \cdot \frac{4e^{i3\pi/4}}{5} = \frac{2\sqrt{2}}{5}(i+1) \\
\phi(t_2) &= i \cdot \frac{4e^{i5\pi/4}}{5} = \frac{2\sqrt{2}}{5}(-i+1) \\
\phi(t_3) &= i \cdot \frac{4e^{i7\pi/4}}{5} = \frac{2\sqrt{2}}{5}(-i-1)
\end{aligned}$$

Computing the real parts:

$$\begin{aligned}
\mathrm{Re}[\phi(t_0)] &= -\frac{2\sqrt{2}}{5} \\
\mathrm{Re}[\phi(t_1)] &= +\frac{2\sqrt{2}}{5} \\
\mathrm{Re}[\phi(t_2)] &= +\frac{2\sqrt{2}}{5} \\
\mathrm{Re}[\phi(t_3)] &= -\frac{2\sqrt{2}}{5}
\end{aligned}$$

Step 4: Determine relevant saddle point

For steepest descent, we seek saddle points with **maximum** real part of $\phi(t)$ (since the integrand contains $e^{X\phi(t)}$).

The saddle points t_1 and t_2 both have $\mathrm{Re}[\phi] = +\frac{2\sqrt{2}}{5}$, which is the maximum.

Step 5: Check contour deformation

The original contour is the real axis. We need to determine which saddle point(s) the real axis can be deformed through using Cauchy's theorem.

By examining the geometry and applying Cauchy's theorem (Section 4.4), the real axis can be deformed to pass through saddle points in the lower half-plane as $t \to -\infty$ and upper half-plane as $t \to +\infty$.

Given the symmetry of the problem and the location of saddle points:

- $t_1 = \frac{-1+i}{\sqrt{2}}$ (upper left quadrant)
- $t_2 = \frac{-1-i}{\sqrt{2}}$ (lower left quadrant)

However, we must check which gives a steepest **descent** path compatible with the original contour.

Step 6: Analyze steepest descent paths

For a saddle point where $\phi''(t_c) = ae^{i\alpha}$, steepest descent directions are (from notes, Section 4.4.2):

$$\theta_{\text{descent}} = -\frac{\alpha}{2} + \frac{(2p+1)\pi}{2}, \quad p = 0, 1$$

Computing $\phi''(t) = 4it^3$:
At $t_1 = e^{i3\pi/4}$:

$$\phi''(t_1) = 4i \cdot e^{i9\pi/4} = 4i \cdot e^{i\pi/4} = 4e^{i3\pi/4}$$

So $\alpha = 3\pi/4$, giving descent directions:

$$p = 0: \quad \theta = -\frac{3\pi/4}{2} + \frac{\pi}{2} = -\frac{3\pi}{8} + \frac{\pi}{2} = \frac{\pi}{8}$$
$$p = 1: \quad \theta = -\frac{3\pi}{8} + \frac{3\pi}{2} = \frac{9\pi}{8}$$

The contour can be deformed through t_1 along steepest descent paths.

Step 7: Evaluate contribution from t_1

Near the saddle point t_1, we use the expansion (Laplace's method, Eq. 205):

$$\phi(t) \approx \phi(t_1) + \frac{1}{2}\phi''(t_1)(t - t_1)^2$$

The leading asymptotic contribution is (from Eq. 207 for $n = 2$):

$$I(X) \sim \sqrt{\frac{2\pi i}{X\phi''(t_1)}}\, f(t_1)\, e^{X\phi(t_1)}$$

Step 8: Compute the leading term

First, evaluate $f(t_1)$:

$$f(t_1) = \frac{t_1}{1 + t_1^2} = \frac{e^{i3\pi/4}}{1 + e^{i3\pi/2}} = \frac{e^{i3\pi/4}}{1 - i}$$

Simplifying:

$$f(t_1) = \frac{e^{i3\pi/4}}{1 - i} \cdot \frac{1 + i}{1 + i} = \frac{e^{i3\pi/4}(1 + i)}{2} = \frac{e^{i3\pi/4} \cdot \sqrt{2}e^{i\pi/4}}{2} = \frac{\sqrt{2}e^{i\pi}}{2} = -\frac{\sqrt{2}}{2}$$

Now evaluate the coefficient:

$$\phi''(t_1) = 4e^{i3\pi/4}$$

Therefore:

$$\sqrt{\frac{2\pi i}{X\phi''(t_1)}} = \sqrt{\frac{2\pi i}{4Xe^{i3\pi/4}}} = \sqrt{\frac{\pi i}{2X}} \cdot e^{-i3\pi/8}$$

Note: $\sqrt{i} = e^{i\pi/4}$, so:

$$\sqrt{\frac{\pi i}{2X}} = \sqrt{\frac{\pi}{2X}} \cdot e^{i\pi/4}$$

Thus:

$$\sqrt{\frac{2\pi i}{X\phi''(t_1)}} = \sqrt{\frac{\pi}{2X}} \cdot e^{i\pi/4} \cdot e^{-i3\pi/8} = \sqrt{\frac{\pi}{2X}} \cdot e^{-i\pi/8}$$

Step 9: Final answer

Combining all terms:

$$I(X) \sim -\frac{\sqrt{2}}{2} \cdot \sqrt{\frac{\pi}{2X}} \cdot e^{-i\pi/8} \cdot \exp\left(\frac{2\sqrt{2}X}{5}(i + 1)\right)$$

Simplifying:

$$\boxed{I(X) \sim -\frac{1}{2}\sqrt{\frac{\pi}{X}} \cdot e^{-i\pi/8} \cdot e^{2\sqrt{2}X/5} \cdot e^{i2\sqrt{2}X/5} \quad \text{as } X \to \infty}$$

Or more compactly:

$$\boxed{I(X) \sim -\frac{1}{2}\sqrt{\frac{\pi}{X}} \cdot \exp\left[\frac{2\sqrt{2}X}{5}(1 + i) - \frac{i\pi}{8}\right] \quad \text{as } X \to \infty}$$

The leading order behaviour is dominated by $e^{2\sqrt{2}X/5} \cdot X^{-1/2}$.