Methods of Applied Mathematics - Part 1
Exercise Sheet 2: Question 5
Classification of Equilibria in 3D

Complete Solution with XYZ Methodology

Problem Statement

Classify all hyperbolic equilibria of a linear vector field in three dimensions, i.e., draw phase portraits for all topologically different cases when the origin is a hyperbolic equilibrium of the vector field.

Hint: Start from the 2D cases (e.g., attracting node, attracting spiral, saddle, etc.), and bear in mind that a 3D system has 3 eigenvalues; where in the complex plane can they be?

1 Step 1: Foundation - Definition and Constraints

Define the System and Hyperbolicity

Solution 1. • **STAGE X (What we have):** A linear 3D vector field near the origin:

$$\dot{\mathbf{x}} = A\mathbf{x}, \quad \mathbf{x} \in \mathbb{R}^3 \tag{1}$$

where A is a 3×3 real matrix with equilibrium at $\mathbf{x}^* = \mathbf{0}$.

- **STAGE Y (Why hyperbolicity matters):** From Lecture Notes (Section 11, page 38), an equilibrium is **hyperbolic** if none of its eigenvalues lie on the imaginary axis, i.e., $\mathrm{Re}(\lambda_i) \neq 0$ for all i.

  Significance: Hyperbolic equilibria are structurally stable - small perturbations don't change their topological type. The Hartman-Grobman Theorem (page 38) guarantees that the nonlinear system near a hyperbolic equilibrium is topologically equivalent to its linearization.

- **STAGE Z (Our approach):** We'll systematically enumerate all possible eigenvalue configurations for a 3×3 real matrix, excluding non-hyperbolic cases (eigenvalues on imaginary axis).

Fundamental Constraints on Eigenvalues

Explanation 1 (Eigenvalue Structure for Real Matrices)**.** *For a real matrix $A \in \mathbb{R}^{3\times3}$:*

Complex Conjugate Pairs:

- *Complex eigenvalues must occur in conjugate pairs: if $\lambda = a + bi$ is an eigenvalue, then $\bar{\lambda} = a - bi$ is also an eigenvalue*
- *This is because the characteristic polynomial has real coefficients*

Parity Constraint:

- *A 3×3 matrix has exactly 3 eigenvalues (counting multiplicity)*
- *Complex eigenvalues come in pairs (even count)*
- *Therefore: Either **3 real** eigenvalues OR **1 real + 2 complex conjugate** eigenvalues*
- *Cannot have 3 complex eigenvalues (would need 4 or 6 with conjugate pairing)*

Eigenvalue Location in Complex Plane

From Lecture Notes (Section 8, pages 29-31):

- Eigenvalues with $\mathrm{Re}(\lambda) < 0$: contribute **stable** directions (attraction)
- Eigenvalues with $\mathrm{Re}(\lambda) > 0$: contribute **unstable** directions (repulsion)
- Real eigenvalues: exponential approach/departure without rotation
- Complex eigenvalues: spiral approach/departure with frequency $|\mathrm{Im}(\lambda)|$

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2 Step 2: Enumeration Strategy

Classification Tree

Solution 2.

- **STAGE X (Systematic approach):** We classify by eigenvalue structure:
  1. **Case A**: Three real eigenvalues $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$
  2. **Case B**: One real + two complex conjugate $\lambda_1 \in \mathbb{R}$, $\lambda_{2,3} = a \pm bi$ with $b \neq 0$
- **STAGE Y (Why this suffices):** These two cases exhaust all possibilities for a 3×3 real matrix. Within each case, we further classify by the signs of the real parts, which determine stability.
- **STAGE Z (Counting hyperbolic types):**
  - Case A: $2^3 = 8$ sign combinations, but exclude all-zero (non-hyperbolic) $\Rightarrow$ actually we have 4 distinct topological types
  - Case B: $2 \times 2 = 4$ sign combinations for (real eigenvalue sign) $\times$ (complex real part sign)
  - **Total**: 8 topologically distinct hyperbolic equilibria in 3D

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3 Step 3: Case A - Three Real Eigenvalues

Solution 3. For three real eigenvalues $\lambda_1, \lambda_2, \lambda_3$, hyperbolicity requires all $\lambda_i \neq 0$.

Subcase A1: All Three Negative ($\lambda_1, \lambda_2, \lambda_3 < 0$)

- **STAGE X (Eigenvalue configuration):**

$$\lambda_1 < 0, \quad \lambda_2 < 0, \quad \lambda_3 < 0 \tag{2}$$

  Example: $\lambda_1 = -3, \lambda_2 = -2, \lambda_3 = -1$

- **STAGE Y (Why this gives stability):** All three eigendirections attract. The general solution is:

$$\mathbf{x}(t) = c_1 e^{\lambda_1 t}\mathbf{v}_1 + c_2 e^{\lambda_2 t}\mathbf{v}_2 + c_3 e^{\lambda_3 t}\mathbf{v}_3 \tag{3}$$

  Since $\lambda_i < 0$ for all i, we have $e^{\lambda_i t} \to 0$ as $t \to \infty$, so $\mathbf{x}(t) \to \mathbf{0}$.

- **STAGE Z (Classification): STABLE NODE** (or attracting node)

  Stability Manifolds:

  - Stable manifold: $W^s = \mathbb{R}^3$ (entire space)
  - Unstable manifold: $W^u = \{\mathbf{0}\}$ (just the origin)
  - Dimensions: $\dim(W^s) = 3$, $\dim(W^u) = 0$

  Phase Portrait Description:

  - All trajectories approach the origin
  - Fastest approach along eigenvector with most negative λ (largest $|\lambda|$)
  - Slowest approach along eigenvector with least negative λ (smallest $|\lambda|$)
  - No rotation - purely exponential decay

  [Phase portrait: 3D stable node - all arrows point toward origin from all directions]

Subcase A2: All Three Positive ($\lambda_1, \lambda_2, \lambda_3 > 0$)

- **STAGE X (Eigenvalue configuration):**

$$\lambda_1 > 0, \quad \lambda_2 > 0, \quad \lambda_3 > 0 \tag{4}$$

  Example: $\lambda_1 = 1, \lambda_2 = 2, \lambda_3 = 3$

- **STAGE Y (Why this gives instability):** All three eigendirections repel. The solution:

$$\mathbf{x}(t) = c_1 e^{\lambda_1 t}\mathbf{v}_1 + c_2 e^{\lambda_2 t}\mathbf{v}_2 + c_3 e^{\lambda_3 t}\mathbf{v}_3 \tag{5}$$

  Since $\lambda_i > 0$ for all i, we have $e^{\lambda_i t} \to \infty$ as $t \to \infty$, so $\mathbf{x}(t) \to \infty$.

- **STAGE Z (Classification): UNSTABLE NODE** (or repelling node)

  Stability Manifolds:

  - Stable manifold: $W^s = \{\mathbf{0}\}$ (just the origin)
  - Unstable manifold: $W^u = \mathbb{R}^3$ (entire space)
  - Dimensions: $\dim(W^s) = 0$, $\dim(W^u) = 3$

  Phase Portrait Description:

  - All trajectories repel from the origin (except $\mathbf{x} = \mathbf{0}$)
  - Fastest escape along eigenvector with most positive λ
  - Slowest escape along eigenvector with least positive λ
  - Time-reversal of stable node

  [Phase portrait: 3D unstable node - all arrows point away from origin in all directions]

Subcase A3: Two Negative, One Positive

- **STAGE X (Eigenvalue configuration):**

$$\lambda_1 < 0, \quad \lambda_2 < 0, \quad \lambda_3 > 0 \tag{6}$$

  Example: $\lambda_1 = -2, \lambda_2 = -1, \lambda_3 = 3$

- **STAGE Y (Why this gives saddle):** Two directions attract, one repels. The solution:

$$\mathbf{x}(t) = c_1 e^{\lambda_1 t}\mathbf{v}_1 + c_2 e^{\lambda_2 t}\mathbf{v}_2 + c_3 e^{\lambda_3 t}\mathbf{v}_3 \tag{7}$$

  – If $c_3 = 0$: trajectory stays in $\text{span}\{\mathbf{v}_1, \mathbf{v}_2\}$ and decays to origin (2D stable)

  – If $c_3 \neq 0$: $e^{\lambda_3 t}$ term dominates for large t, trajectory escapes along $\mathbf{v}_3$

- **STAGE Z (Classification): SADDLE** with 2D stable manifold, 1D unstable manifold

  Stability Manifolds:

  - Stable manifold: $W^s = \text{span}\{\mathbf{v}_1, \mathbf{v}_2\}$ (2D plane)
  - Unstable manifold: $W^u = \text{span}\{\mathbf{v}_3\}$ (1D line)
  - Dimensions: $\dim(W^s) = 2$, $\dim(W^u) = 1$

  Phase Portrait Description:

  - Trajectories starting in W^s approach origin
  - Trajectories starting on W^u (except origin) escape along the line

- Generic trajectories: approach the 2D stable manifold, then follow it toward origin, but get deflected and escape along unstable direction

- Creates characteristic "saddle surface"

  [Phase portrait: 3D saddle - 2D stable plane, 1D unstable line perpendicular]

Notation: Sometimes denoted as $(2, 1)$-saddle (2 stable dimensions, 1 unstable dimension).

Subcase A4: One Negative, Two Positive

- **STAGE X (Eigenvalue configuration):**

$$\lambda_1 < 0, \quad \lambda_2 > 0, \quad \lambda_3 > 0 \tag{8}$$

  Example: $\lambda_1 = -3, \lambda_2 = 1, \lambda_3 = 2$

- **STAGE Y (Why this gives saddle):** One direction attracts, two repel. The solution:

$$\mathbf{x}(t) = c_1 e^{\lambda_1 t}\mathbf{v}_1 + c_2 e^{\lambda_2 t}\mathbf{v}_2 + c_3 e^{\lambda_3 t}\mathbf{v}_3 \tag{9}$$

  - If $c_2 = c_3 = 0$: trajectory stays on $\mathbf{v}_1$ line and approaches origin
  - Otherwise: $e^{\lambda_2 t}$ and $e^{\lambda_3 t}$ terms dominate, trajectory escapes in 2D plane

- **STAGE Z (Classification): SADDLE** with 1D stable manifold, 2D unstable manifold

  Stability Manifolds:

  - Stable manifold: $W^s = \text{span}\{\mathbf{v}_1\}$ (1D line)
  - Unstable manifold: $W^u = \text{span}\{\mathbf{v}_2, \mathbf{v}_3\}$ (2D plane)
  - Dimensions: $\dim(W^s) = 1$, $\dim(W^u) = 2$

  Phase Portrait Description:

  - Trajectories starting on W^s approach origin along the line
  - Trajectories in W^u escape (except origin)
  - Generic trajectories: initially move toward the stable line, but get deflected and escape in the 2D unstable plane
  - Time-reversal of $(2, 1)$-saddle

  [Phase portrait: 3D saddle - 1D stable line, 2D unstable plane perpendicular]

Notation: Sometimes denoted as $(1, 2)$-saddle (1 stable dimension, 2 unstable dimensions).

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4 Step 4: Case B - One Real + Complex Conjugate Pair

Solution 4. For eigenvalues $\lambda_1 \in \mathbb{R}$ and $\lambda_{2,3} = a \pm bi$ with $b \neq 0$:

Key Concepts for Complex Eigenvalues

Explanation 2 (Complex Eigenvalues and Spiraling)**.** *From Lecture Notes (Section 7, pages 26-27):*

When eigenvalues are complex, $\lambda = a \pm bi$:

- *The real part $a = Re(\lambda)$ controls stability: $a < 0$ attracts, $a > 0$ repels*
- *The imaginary part $b = Im(\lambda)$ controls rotation frequency: $\omega = |b|$*
- *Solutions in the complex eigenspace spiral: $e^{(a+bi)t} = e^{at}(\cos(bt) + i\sin(bt))$*
- *This corresponds to spiraling in a 2D real plane (the real and imaginary parts of the eigenvector)*

The general solution in the 2D invariant plane is:

$$\mathbf{x}_{plane}(t) = e^{at}\begin{pmatrix}\cos(bt) & -\sin(bt)\\ \sin(bt) & \cos(bt)\end{pmatrix}\mathbf{x}_0 \tag{10}$$

This describes spiraling motion with exponential growth/decay.

Subcase B1: Real Negative, Complex with Negative Real Part

- **STAGE X (Eigenvalue configuration):**

$$\lambda_1 < 0 \in \mathbb{R}, \quad \lambda_{2,3} = a \pm bi \text{ with } a < 0,\, b \neq 0 \tag{11}$$

  Example: $\lambda_1 = -2$, $\lambda_{2,3} = -1 \pm 3i$

- **STAGE Y (Why this gives stable spiral node):**
  - Real eigenvalue $\lambda_1 < 0$: exponential decay along 1D line (direction $\mathbf{v}_1$)
  - Complex pair with $a < 0$: spiral decay in 2D plane (spanned by $\mathrm{Re}(\mathbf{v}_2)$, $\mathrm{Im}(\mathbf{v}_2)$)
  - Both components attract to origin

  General solution:

$$\mathbf{x}(t) = c_1 e^{\lambda_1 t}\mathbf{v}_1 + e^{at}[c_2\cos(bt) + c_3\sin(bt)]\mathbf{u} + e^{at}[c_2\sin(bt) - c_3\cos(bt)]\mathbf{w} \tag{12}$$

  where $\mathbf{u}, \mathbf{w}$ span the 2D complex eigenspace.

- **STAGE Z (Classification): STABLE SPIRAL NODE** (or stable focus-node)

  Stability Manifolds:

  - Stable manifold: $W^s = \mathbb{R}^3$ (entire space)

- Unstable manifold: $W^u = \{\mathbf{0}\}$ (just the origin)
- Dimensions: $\dim(W^s) = 3$, $\dim(W^u) = 0$

Phase Portrait Description:

- All trajectories spiral into the origin
- In 2D plane: inward spiral (focus behavior)
- Along 3rd direction: exponential decay (node behavior)
- Combined: 3D spiral converging to origin
- Frequency of rotation: $\omega = |b|$

[Phase portrait: 3D stable spiral - trajectories spiral inward like a corkscrew toward origin]

Subcase B2: Real Positive, Complex with Positive Real Part

- **STAGE X (Eigenvalue configuration):**

$$\lambda_1 > 0 \in \mathbb{R}, \quad \lambda_{2,3} = a \pm bi \text{ with } a > 0,\ b \neq 0 \tag{13}$$

  Example: $\lambda_1 = 2$, $\lambda_{2,3} = 1 \pm 3i$

- **STAGE Y (Why this gives unstable spiral node):**
  - Real eigenvalue $\lambda_1 > 0$: exponential growth along 1D line
  - Complex pair with $a > 0$: spiral growth in 2D plane
  - Both components repel from origin
- **STAGE Z (Classification): UNSTABLE SPIRAL NODE** (or unstable focus-node)

Stability Manifolds:

- Stable manifold: $W^s = \{\mathbf{0}\}$ (just the origin)
- Unstable manifold: $W^u = \mathbb{R}^3$ (entire space)
- Dimensions: $\dim(W^s) = 0$, $\dim(W^u) = 3$

Phase Portrait Description:

- All trajectories spiral away from the origin
- In 2D plane: outward spiral
- Along 3rd direction: exponential growth
- Combined: 3D spiral diverging from origin
- Time-reversal of stable spiral node

[Phase portrait: 3D unstable spiral - trajectories spiral outward like expanding corkscrew]

Subcase B3: Real Negative, Complex with Positive Real Part

- **STAGE X (Eigenvalue configuration):**

$$\lambda_1 < 0 \in \mathbb{R}, \quad \lambda_{2,3} = a \pm bi \text{ with } a > 0,\, b \neq 0 \tag{14}$$

  Example: $\lambda_1 = -2$, $\lambda_{2,3} = 1 \pm 3i$

- **STAGE Y (Why this gives saddle-focus):**
  - Real eigenvalue $\lambda_1 < 0$: attracts along 1D line
  - Complex pair with $a > 0$: spirals outward in 2D plane
  - Mixed behavior: attraction in one direction, spiraling repulsion in plane

- **STAGE Z (Classification):** **SADDLE-FOCUS** with 1D stable, 2D unstable spiral

  Stability Manifolds:

  - Stable manifold: $W^s = \text{span}\{\mathbf{v}_1\}$ (1D line)
  - Unstable manifold: $W^u = \text{span}\{\text{Re}(\mathbf{v}_2), \text{Im}(\mathbf{v}_2)\}$ (2D plane, spiral structure)
  - Dimensions: $\dim(W^s) = 1$, $\dim(W^u) = 2$

  Phase Portrait Description:

  - Trajectories on W^s approach origin along the line
  - Trajectories in W^u spiral away from origin
  - Generic trajectories: initially attracted toward stable line, but deflected by spiraling unstable plane, eventually escape while spiraling
  - Creates characteristic "spiral saddle" or "saddle-focus"

  [Phase portrait: Saddle-focus - 1D stable line with 2D unstable spiral plane perpendicular]

  Notation: $(1, 2)$-saddle-focus or saddle-focus with 1D stable manifold.

Explanation 3 (Homoclinic Connections and Chaos)**.** *From Lecture Notes (Section 10, page 36): The Shilnikov bifurcation involves a saddle-focus equilibrium where a 1D unstable manifold connects back to the 2D stable manifold (homoclinic connection). This configuration is associated with chaotic dynamics in certain parameter regimes.*

Subcase B4: Real Positive, Complex with Negative Real Part

- **STAGE X (Eigenvalue configuration):**

$$\lambda_1 > 0 \in \mathbb{R}, \quad \lambda_{2,3} = a \pm bi \text{ with } a < 0,\, b \neq 0 \tag{15}$$

  Example: $\lambda_1 = 2$, $\lambda_{2,3} = -1 \pm 3i$

- **STAGE Y (Why this gives saddle-focus):**
  - Real eigenvalue $\lambda_1 > 0$: repels along 1D line
  - Complex pair with $a < 0$: spirals inward in 2D plane
  - Mixed behavior: repulsion in one direction, spiraling attraction in plane

- **STAGE Z (Classification): SADDLE-FOCUS** with 2D stable spiral, 1D unstable

Stability Manifolds:

- Stable manifold: $W^s = \text{span}\{\text{Re}(\mathbf{v}_2), \text{Im}(\mathbf{v}_2)\}$ (2D plane, spiral structure)
- Unstable manifold: $W^u = \text{span}\{\mathbf{v}_1\}$ (1D line)
- Dimensions: $\dim(W^s) = 2$, $\dim(W^u) = 1$

Phase Portrait Description:

- Trajectories on W^u escape from origin along the line
- Trajectories in W^s spiral into origin
- Generic trajectories: initially repelled along unstable line, but attracted by spiraling stable plane, eventually spiral into origin
- Time-reversal of $(1, 2)$-saddle-focus

[Phase portrait: Saddle-focus - 2D stable spiral plane with 1D unstable line perpendicular]

Notation: $(2, 1)$-saddle-focus or saddle-focus with 2D stable manifold.

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5 Step 5: Complete Classification Summary

All Eight Hyperbolic Equilibrium Types in 3D

Solution 5.

Type	**Eigenvalue Configuration**	$\dim(W^s)$	$\dim(W^u)$	**Name**
Case A: Three Real Eigenvalues				
A1	$\lambda_1, \lambda_2, \lambda_3 < 0$	3	0	Stable Node
A2	$\lambda_1, \lambda_2, \lambda_3 > 0$	0	3	Unstable Node
A3	$\lambda_1, \lambda_2 < 0,\ \lambda_3 > 0$	2	1	$(2,1)$-Saddle
A4	$\lambda_1 < 0,\ \lambda_2, \lambda_3 > 0$	1	2	$(1,2)$-Saddle
Case B: One Real + Complex Conjugate Pair				
B1	$\lambda_1 < 0,\ \lambda_{2,3} = a \pm bi,\ a < 0$	3	0	Stable Spiral Node
B2	$\lambda_1 > 0,\ \lambda_{2,3} = a \pm bi,\ a > 0$	0	3	Unstable Spiral Node
B3	$\lambda_1 < 0,\ \lambda_{2,3} = a \pm bi,\ a > 0$	1	2	Saddle-Focus $(1,2)$
B4	$\lambda_1 > 0,\ \lambda_{2,3} = a \pm bi,\ a < 0$	2	1	Saddle-Focus $(2,1)$

Dimensional Analysis Verification

Explanation 4 (Verification via Stable/Unstable Manifold Dimensions)**.** *For each equilibrium, verify that dimensions sum correctly:*

$$\dim(W^s) + \dim(W^u) = \textit{dimension of phase space} = 3 \tag{16}$$

Checking each case:

- *A1:* $3 + 0 = 3$ ✓
- *A2:* $0 + 3 = 3$ ✓
- *A3:* $2 + 1 = 3$ ✓
- *A4:* $1 + 2 = 3$ ✓
- *B1:* $3 + 0 = 3$ ✓
- *B2:* $0 + 3 = 3$ ✓
- *B3:* $1 + 2 = 3$ ✓
- *B4:* $2 + 1 = 3$ ✓

All cases satisfy the dimension requirement.

Topological Equivalence Classes

Explanation 5 (When Are Two Equilibria Topologically Equivalent?)**.** *From Lecture Notes (Section 11, page 38), two linear systems* $\dot{\mathbf{x}} = A\mathbf{x}$ *and* $\dot{\mathbf{y}} = B\mathbf{y}$ *are topologically equivalent if and only if:*

$$n_+(A) = n_+(B) \quad \textit{and} \quad n_-(A) = n_-(B) \tag{17}$$

where n_+ *= number of eigenvalues with positive real part,* n_- *= number with negative real part.*

This means: *Only the count of positive/negative eigenvalues matters for topological equivalence, not:*

- *The specific values of eigenvalues*
- *Whether eigenvalues are real or complex*

However, we distinguish real vs. complex for qualitative behavior (spiraling vs. not).

Relationship to 2D Classification

From the hint in the problem and Lecture Notes (Section 8):

2D Equilibrium Types:

- Stable/Unstable Node: 2 real eigenvalues, same sign
- Saddle: 2 real eigenvalues, opposite signs
- Stable/Unstable Focus: 2 complex conjugate eigenvalues
- Center: 2 purely imaginary eigenvalues (not hyperbolic)

3D as Extension of 2D:

- Types A1, A2: "Node" extended to 3D (all eigenvalues same sign)
- Types A3, A4: "Saddle" extended to 3D (mixed eigenvalue signs)
- Types B1, B2: "Focus/Spiral" extended to 3D (complex pair + real with same sign)
- Types B3, B4: "Saddle-Focus" - unique to 3D+ (complex pair + real with opposite sign)

The saddle-focus types (B3, B4) **cannot occur in 2D** - they require at least 3 dimensions.

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6 Step 6: Geometric Visualization Guide

How to Sketch Phase Portraits

For each equilibrium type, follow this procedure: **Step 1: Identify Eigenspaces**
Solution 6.
- Real eigenvalues: draw eigenvector lines/planes
- Complex eigenvalues: identify 2D invariant plane

Step 2: Draw Stable/Unstable Manifolds

- W^s: manifold where trajectories approach origin as $t \to +\infty$
- W^u: manifold where trajectories approach origin as $t \to -\infty$ (equivalently, leave origin as $t \to +\infty$)

Step 3: Add Trajectory Arrows

- On W^s: arrows point toward origin
- On W^u: arrows point away from origin
- Off manifolds: show typical trajectory behavior

Step 4: Indicate Spiraling (if complex eigenvalues present)

- Draw spiral curves in the 2D invariant plane
- Indicate frequency with spiral tightness (higher $|b|$ means more rotations)

Key Features to Highlight

- **Stable Node (A1)**: All arrows inward, no preferred direction except rate differences
- **Unstable Node (A2)**: All arrows outward, time-reversal of A1
- $(2,1)$**-Saddle (A3)**: 2D attracting plane, 1D repelling line - classic saddle
- $(1,2)$**-Saddle (A4)**: 1D attracting line, 2D repelling plane - inverted saddle
- **Stable Spiral Node (B1)**: Inward spiraling corkscrew, all trajectories converge with rotation
- **Unstable Spiral Node (B2)**: Outward spiraling corkscrew, all trajectories diverge with rotation
- **Saddle-Focus** $(1,2)$ **(B3)**: 1D stable line, 2D unstable spiral - trajectories escape while spiraling
- **Saddle-Focus** $(2,1)$ **(B4)**: 2D stable spiral, 1D unstable line - trajectories approach while spiraling around unstable line

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Final Summary and Key Insights

Complete Answer to Question 5

1. **Total Count**: There are **exactly 8 topologically distinct types** of hyperbolic equilibria for 3D linear systems

2. **Eigenvalue Constraints**:

   - 3D real matrix $\Rightarrow$ either 3 real eigenvalues OR 1 real + 2 complex conjugate
   - Hyperbolic $\Rightarrow$ all eigenvalues off imaginary axis ($\text{Re}(\lambda) \neq 0$)

3. **Classification Principle**: Determined by:

   - Number of eigenvalues with $\text{Re}(\lambda) > 0$ vs. $\text{Re}(\lambda) < 0$
   - Whether eigenvalues are real or complex

4. **Stability Manifold Dimensions**: Always $\dim(W^s) + \dim(W^u) = 3$

5. **New Phenomena in 3D**: The saddle-focus types (B3, B4) are unique to dimensions ≥ 3 and cannot occur in 2D systems

Connection to Lecture Material

- **Section 7-8 (pages 24-31)**: Eigenvalue analysis, stable/unstable manifolds, node/saddle/focus classification
- **Section 10 (page 34)**: Stable/unstable manifolds in higher dimensions
- **Section 11 (pages 37-38)**: Topological equivalence and hyperbolicity
- **Hartman-Grobman Theorem (page 38)**: Guarantees local topological equivalence to linearization for hyperbolic equilibria

Practical Importance

Understanding 3D equilibrium classification is essential for:

- Analyzing dynamical systems in mechanics, electronics, population dynamics
- Predicting long-term behavior from eigenvalue calculations
- Identifying bifurcations (transitions between equilibrium types)
- Understanding chaos (saddle-focus homoclinic connections)