Methods of Applied Mathematics - Part 1 [SEMT30006]

Exercise Sheet 1 – ODEs, flows, and phase portraits

1. **First order ODEs**
   Rewrite the following systems as first order ODEs, making it clear what state variables you have chosen and what their state-space is.

   (a) $\frac{d^3u}{dt^3} - \frac{du}{dt} + \sin(u) = 0$

   (b) $\frac{d^2u}{dt^2} + \frac{du}{dt} + u - 2v = 0$, $\frac{d^2v}{dt^2} + \frac{dv}{dt} + v - 2u = 0$

   (c) $\frac{d^2u}{dt^2} + \frac{du}{dt} - u + u^3 - v = 0$, $\frac{dv}{dt} = u - v$

2. **Finite time blow up**
   Solve the initial value problem
   $$\dot{x} = ax^2$$
   with $x(0) = x_0$.

   (a) Does this have solutions, are they unique, and where do they exist?

   (b) Solve the equations for $x(t)$ in terms of a and x_0.

   (c) Despite your answer to (a), show that something 'goes wrong' at time $t = \frac{1}{ax_0}$, and describe what happens there. This is known as **finite time blow up**.

   (d) Sketch a solution for $a = 1$ and $x_0 = 0.2$.

3. **Autonomy (or time-independence)**
   We've seen systems that either depend on time or don't. A system that *does not* depend explicitly on its independent variable is called **autonomous**. Which of the following is autonomous? What is the independent variable?

   (a) The ODE $\ddot{u} = u + \sin(t)$

   (b) The ODE $y'' - y - \sin(x) = 0$

   (c) The ODE $\ddot{\theta} + a\dot{\theta} + b = 0$

   (d) The map $x_{n+1} = ax_n + x_n^2$

   (e) The map $x_{n+1} = nx_n + b$

4. **1D phase portraits**
   Sketch phase portraits for the following differential equations, and describe the long-time behaviour for the given starting conditions.

   (a) $\frac{du}{dt} = \frac{2}{1+u^2} - 1$, with $u(0) = 2$, $u(0) = 0$ and $u(0) = -2$.

   (b) $\frac{du}{dt} = -u^3 + 5u^2 - 6u$, with $u(0) = 1$ and $u(0) = 4$.

5. **2D phase portraits**
   Sketch phase portraits for the following differential equations and classify the equilibria.

   (a) $\frac{du}{dt} = v^2 - u$, $\frac{dv}{dt} = u^2 - v$

   (b) $\frac{d^2u}{dt^2} + \frac{du}{dt} + \sin(u) = 0$

6. **Existence and uniqueness**
   Solve the following initial value problems to find a solution $x(t)$ in terms of x_0:

   (a) $\dot{x} = x^2$ with $x(0) = x_0$.

   (b) $\dot{x} = |x|$ with $x(0) = x_0$.

   (c) $\dot{x} = |x|^{1/2}$ with $x(0) = x_0$.

   People do struggle with solving (c), so to save a bit of time you can check you're answer against the solution:

$$x(t) = \begin{cases} +\left(|x_0|^{1/2} + \frac{1}{2}t\right)^2 & \text{if } \; x_0 \geq 0 \\ 0 & \text{if } \; x_0 = 0 \\ -\left(|x_0|^{1/2} - \frac{1}{2}t\right)^2 & \text{if } \; x_0 \leq 0 \end{cases}$$

   This is a little tricky to get all the $+/-$ signs right. The best way to do it (rather than working with $|x|$ or $|^{1/2}$ which is a bit of a difficult term to use reliably, is to completely treat the cases $x, x_0 < 0$ and $x, x_0 > 0$ separately. If you're careful, you should get the answer above. If not, don't waste too much time right now, but in the long run this *is* a good exercise in reliably doing algebra with $|x|$.

   Then answer the following questions for each system (these are vital to a fundamental understanding of determinacy and uniqueness in ODEs):

   i. Consider three initial conditions $x_0 = 0$, $x_0 = -1$ and $x_0 = +1$. From each, where does the solution go and how long does it take?
   ii. Identify the different orbits of each system.
   iii. Are the orbits uniquely determined by the ODE (for a given x_0 is there only one unique solution)?
   iv. This is a little more advanced but do-able. In lectures we said that if an ODE is Lipschitz continuous then its solutions exist and are unique. Show that each system here is/isn't Lipschitz continuous, and say how this agrees with your answer to (c).