Methods of Applied Mathematics - Part 1
Exercise Sheet 2: Question 2
Multiple Equilibria in a 1-Dimensional System

Complete Solution with XYZ Methodology

Problem Statement

Consider the dynamical system:

$$\dot{x} = x^4 - 17x^3 + 101x^2 - 247x + 210 \tag{1}$$

You are told that this has four equilibria, at $x = 2, 3, 5, 7$.

1 Question 2(a): Factorized Form of the ODE

Step 1: Understand the Relationship Between Roots and Factorization

Solution 1. • **STAGE X (What we know):** We have a polynomial $p(x) = x^4 - 17x^3 + 101x^2 - 247x + 210$, and we're told it has roots at $x = 2, 3, 5, 7$. These are the equilibria where $\dot{x} = 0$.

- **STAGE Y (Why factorization works):** By the **Fundamental Theorem of Algebra**, a degree-4 polynomial with roots r_1, r_2, r_3, r_4 can be written as:

$$p(x) = A(x - r_1)(x - r_2)(x - r_3)(x - r_4) \tag{2}$$

  where A is the leading coefficient. Since our polynomial has leading coefficient 1 (the x^4 term), we have $A = 1$.

- **STAGE Z (What we'll show):** We'll verify that $(x-2)(x-3)(x-5)(x-7)$ equals the given polynomial, thus identifying $a = 2, b = 3, c = 5, d = 7$.

Step 2: Write the Factorized Form

Since the equilibria are at $x = 2, 3, 5, 7$, the factorized form must be:

$$\dot{x} = (x - 2)(x - 3)(x - 5)(x - 7) \tag{3}$$

Therefore:

$$\boxed{a = 2, \quad b = 3, \quad c = 5, \quad d = 7} \tag{4}$$

Step 3: Verify by Expansion (ESSENTIAL)

We must verify that $(x-2)(x-3)(x-5)(x-7)$ expands to $x^4-17x^3+101x^2-247x+210$.

Step 3A: Expand in Pairs

First pair:

$$\begin{aligned}(x-2)(x-3) &= x^2-3x-2x+6 \quad (5)\\ &= x^2-5x+6 \quad (6)\end{aligned}$$

Second pair:

$$\begin{aligned}(x-5)(x-7) &= x^2-7x-5x+35 \quad (7)\\ &= x^2-12x+35 \quad (8)\end{aligned}$$

Step 3B: Multiply the Results

$$(x^2-5x+6)(x^2-12x+35) = x^2(x^2-12x+35) - 5x(x^2-12x+35) + 6(x^2-12x+35) \quad (9)$$

$$\begin{aligned}&= x^4-12x^3+35x^2 \quad (10)\\ &\quad -5x^3+60x^2-175x \quad (11)\\ &\quad +6x^2-72x+210 \quad (12)\end{aligned}$$

Collecting like terms:

$$\begin{aligned}&= x^4+(-12-5)x^3+(35+60+6)x^2+(-175-72)x+210 \quad (13)\\ &= x^4-17x^3+101x^2-247x+210 \quad \checkmark \quad (14)\end{aligned}$$

Explanation 1 (Verification Confirms Factorization)**.** *The expansion matches the original polynomial exactly. This confirms that:*

$$x^4-17x^3+101x^2-247x+210 = (x-2)(x-3)(x-5)(x-7) \quad (15)$$

Step 4: Alternative Verification Using Vieta's Formulas

For completeness, we can verify using Vieta's formulas. For a monic polynomial $x^4+px^3+qx^2+rx+s$ with roots $\alpha,\beta,\gamma,\delta$:

$$\begin{aligned}\alpha+\beta+\gamma+\delta &= -p \quad (16)\\ \alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta &= q \quad (17)\\ \alpha\beta\gamma+\alpha\beta\delta+\alpha\gamma\delta+\beta\gamma\delta &= -r \quad (18)\\ \alpha\beta\gamma\delta &= s \quad (19)\end{aligned}$$

With roots $2,3,5,7$:

$$\begin{aligned}\text{Sum: } & 2+3+5+7 = 17 = -(-17) \quad \checkmark \quad (20)\\ \text{Pairwise products: } & 2\cdot3+2\cdot5+2\cdot7+3\cdot5+3\cdot7+5\cdot7 \quad (21)\\ & = 6+10+14+15+21+35 = 101 \quad \checkmark \quad (22)\\ \text{Triple products: } & 2\cdot3\cdot5+2\cdot3\cdot7+2\cdot5\cdot7+3\cdot5\cdot7 \quad (23)\\ & = 30+42+70+105 = 247 = -(-247) \quad \checkmark \quad (24)\\ \text{Product of all: } & 2\cdot3\cdot5\cdot7 = 210 \quad \checkmark \quad (25)\end{aligned}$$

All checks pass!

Final Answer for Part (a)

$$\boxed{\begin{array}{r}\dot{x} = (x-a)(x-b)(x-c)(x-d) \\ \text{where } a = 2, \quad b = 3, \quad c = 5, \quad d = 7\end{array}} \tag{26}$$

---

2 Question 2(b): Stability of Each Equilibrium

Step 1: Method Selection for Stability Analysis

Solution 2. • **STAGE X (What we need):** Determine whether each equilibrium at $x^* = 2, 3, 5, 7$ is stable or unstable.

- **STAGE Y (Why linearization):** From Lecture Notes (Section 9, page 32), for a 1D system $\dot{x} = f(x)$ with equilibrium at x^*:

$$\text{Stability coefficient: } \lambda = \left.\frac{df}{dx}\right|_{x=x^*} = f'(x^*) \tag{27}$$

  - If $\lambda < 0$: equilibrium is **stable** (attractor)
  - If $\lambda > 0$: equilibrium is **unstable** (repeller)

- **STAGE Z (Our approach):** Calculate $f'(x)$ in factored form (easier than expanding), then evaluate at each equilibrium point.

Step 2: Compute the Derivative Using Product Rule

Given $f(x) = (x-2)(x-3)(x-5)(x-7)$, we need $f'(x)$.

Product Rule for Four Factors:

For $f(x) = u_1 \cdot u_2 \cdot u_3 \cdot u_4$ where $u_i = (x - a_i)$:

$$f'(x) = u_1' u_2 u_3 u_4 + u_1 u_2' u_3 u_4 + u_1 u_2 u_3' u_4 + u_1 u_2 u_3 u_4' \tag{28}$$

Since $u_i' = 1$ for all factors:

$$f'(x) = (x-3)(x-5)(x-7)+(x-2)(x-5)(x-7)+(x-2)(x-3)(x-7)+(x-2)(x-3)(x-5) \tag{29}$$

Explanation 2 (Key Observation for Evaluation)**.** *At an equilibrium $x^* = a_i$, one factor $(x - a_i)$ equals zero. When computing $f'(x^*)$, all terms containing $(x^* - a_i)$ vanish, leaving only the term where that factor was differentiated.*

For example, at $x^* = 2$:

$$f'(2) = \underbrace{(2-3)(2-5)(2-7)}_{(x-2)' \text{ term, survives}} + \underbrace{(2-2)(\ldots)}_{=0} + \underbrace{(2-2)(\ldots)}_{=0} + \underbrace{(2-2)(\ldots)}_{=0} \tag{30}$$

$$= (-1)(-3)(-5) = -15 \tag{31}$$

Step 3: Evaluate $f'(x)$ at Each Equilibrium

At $x^* = 2$:

$$\begin{aligned} f'(2) &= (2-3)(2-5)(2-7) \quad (32)\\ &= (-1)(-3)(-5) \quad (33)\\ &= -15 < 0 \quad \Rightarrow \quad \boxed{\text{STABLE}} \quad (34) \end{aligned}$$

At $x^* = 3$:

$$\begin{aligned} f'(3) &= (3-2)(3-5)(3-7) \quad (35)\\ &= (1)(-2)(-4) \quad (36)\\ &= 8 > 0 \quad \Rightarrow \quad \boxed{\text{UNSTABLE}} \quad (37) \end{aligned}$$

At $x^* = 5$:

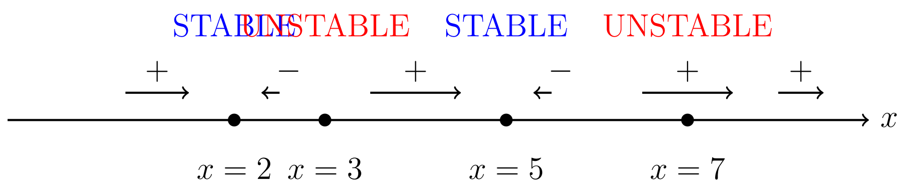
$$\begin{aligned} f'(5) &= (5-2)(5-3)(5-7) \quad (38)\\ &= (3)(2)(-2) \quad (39)\\ &= -12 < 0 \quad \Rightarrow \quad \boxed{\text{STABLE}} \quad (40) \end{aligned}$$

At $x^* = 7$:

$$\begin{aligned} f'(7) &= (7-2)(7-3)(7-5) \quad (41)\\ &= (5)(4)(2) \quad (42)\\ &= 40 > 0 \quad \Rightarrow \quad \boxed{\text{UNSTABLE}} \quad (43) \end{aligned}$$

Step 4: Physical Interpretation via Phase Line

To understand the dynamics, construct the phase line showing $\dot{x}$ in each region:



Explanation 3 (Understanding the Phase Line). ***Sign Analysis:***
In each region, determine the sign of $\dot{x} = (x-2)(x-3)(x-5)(x-7)$:

- $x < 2$: *All four factors negative* $\Rightarrow (-)(-)(-)(-) = (+) \Rightarrow \dot{x} > 0$ *(moving right)*
- $2 < x < 3$: *Three factors negative, one positive* $\Rightarrow (+)(-)(-)(-) = (-) \Rightarrow \dot{x} < 0$ *(moving left toward 2)*
- $3 < x < 5$: *Two factors negative, two positive* $\Rightarrow (+)(+)(-)(-) = (+) \Rightarrow \dot{x} > 0$ *(moving right)*
- $5 < x < 7$: *One factor negative, three positive* $\Rightarrow (+)(+)(+)(-) = (-) \Rightarrow \dot{x} < 0$ *(moving left toward 5)*

- *$x > 7$: All factors positive* $\Rightarrow (+)(+)(+)(+) = (+) \Rightarrow \dot{x} > 0$ *(moving right to* ∞*)*

Stability Pattern:

- *At $x = 2$: Arrows point toward it from both sides* $\Rightarrow$ *STABLE*
- *At $x = 3$: Arrows point away from it on both sides* $\Rightarrow$ *UNSTABLE*
- *At $x = 5$: Arrows point toward it from both sides* $\Rightarrow$ *STABLE*
- *At $x = 7$: Arrows point away from it on both sides* $\Rightarrow$ *UNSTABLE*

This confirms our linearization analysis!

Step 5: Pattern Recognition

Explanation 4 (General Pattern for Multiple Equilibria)**.** *Notice the alternating stability pattern: STABLE - UNSTABLE - STABLE - UNSTABLE.*

Why this happens:

- *Between two adjacent equilibria, $\dot{x}$ must have constant sign (continuous function, no zeros in between)*
- *At each equilibrium, the function crosses zero (changes sign)*
- *Starting from $x \to -\infty$: For large negative x, the leading term is $x^4 > 0$, so eventually $\dot{x} > 0$*
- *Each zero crossing alternates the sign of $\dot{x}$*
- *Stability alternates: if arrows approach from the left and leave to the right (stable), the next equilibrium must have arrows approach from left and right (unstable), and so on*

General Rule (Lecture Notes, Section 6): *In 1D systems, adjacent equilibria must have opposite stability.*

Final Answer for Part (b)

Equilibrium x^*	$f'(x^*)$	Sign	Stability
2	-15	$-$	STABLE (attractor)
3	$+8$	$+$	UNSTABLE (repeller)
5	-12	$-$	STABLE (attractor)
7	$+40$	$+$	UNSTABLE (repeller)

(44)

3 Question 2(c): Long-Term Behavior from $x_0 = 6$

Step 1: Identify Initial Position on Phase Line

Solution 3. • **STAGE X (What we know):** The initial condition is $x_0 = 6$, which lies between the equilibria at $x = 5$ and $x = 7$.

- **STAGE Y (Why location matters):** From the phase line analysis in part (b), we determined the sign of $\dot{x}$ in the interval $(5, 7)$. This tells us which direction the trajectory moves.

- **STAGE Z (What we'll determine):** We'll find which equilibrium the trajectory approaches as $t \to \infty$.

Step 2: Determine Sign of $\dot{x}$ at $x_0 = 6$

From part (b), in the region $5 < x < 7$:

$$\begin{aligned} \dot{x} &= (x-2)(x-3)(x-5)(x-7) && (45)\\ &= (+)(+)(+)(-) && (46)\\ &= (-) < 0 && (47) \end{aligned}$$

At $x = 6$ specifically:

$$\begin{aligned} \dot{x}\big|_{x=6} &= (6-2)(6-3)(6-5)(6-7) && (48)\\ &= (4)(3)(1)(-1) && (49)\\ &= -12 < 0 \quad \checkmark && (50) \end{aligned}$$

Step 3: Determine Direction of Motion

Since $\dot{x} < 0$ at $x_0 = 6$:

$$\frac{dx}{dt} < 0 \quad \Rightarrow \quad x \text{ is } \textbf{decreasing} \tag{51}$$

The trajectory moves to the **left** (toward smaller values of x).

Step 4: Identify the Attractor

Starting at $x_0 = 6$ and moving left:

- The trajectory is in the region $(5, 7)$ where $\dot{x} < 0$ throughout
- Moving left, the trajectory approaches $x = 5$
- From part (b), $x = 5$ is a STABLE equilibrium (attractor)
- The trajectory cannot cross the equilibrium (by uniqueness of solutions)

Explanation 5 (Why $x = 5$ is the Long-Term Destination)**.** ***Basin of Attraction:*** *The basin of attraction of $x = 5$ consists of all initial conditions that eventually approach $x = 5$.*

From the phase line:

- *For $x \in (3,5)$: $\dot{x} > 0$, trajectories move right toward $x = 5$*
- *For $x \in (5,7)$: $\dot{x} < 0$, trajectories move left toward $x = 5$*

Therefore, the basin of attraction is the entire interval $(3,7)$. Since $x_0 = 6 \in (3,7)$, the trajectory must approach $x = 5$.

Step 5: Characterize the Approach

Near the stable equilibrium $x = 5$, the linearization gives:

$$\dot{x} \approx f'(5) \cdot (x-5) = -12(x-5) \tag{52}$$

This has solution:

$$x(t) - 5 \approx (x_0 - 5)e^{-12t} = (6-5)e^{-12t} = e^{-12t} \tag{53}$$

Therefore:

$$x(t) \approx 5 + e^{-12t} \tag{54}$$

The approach is **exponential decay** with rate constant $\lambda = 12$, giving a time scale $\tau = 1/12 \approx 0.083$.

Final Answer for Part (c)

$$\boxed{\begin{array}{l} \text{Starting from } x_0 = 6: \\ \text{Long-term behavior: } x(t) \to 5 \text{ as } t \to \infty \\ \text{Approach: Exponential decay with rate } \lambda = 12 \\ \text{Time scale: } \tau = 1/12 \approx 0.083 \end{array}} \tag{55}$$

Physical Description: The trajectory decreases monotonically from $x_0 = 6$ toward the stable equilibrium at $x = 5$, approaching it exponentially with approximately 95% of convergence achieved by $t \approx 0.25$.

---

4 Question 2(d): Long-Term Behavior from $x_0 = 8$

Step 1: Identify Initial Position on Phase Line

Solution 4. • **STAGE X (What we know):** The initial condition is $x_0 = 8$, which lies beyond all equilibria: $8 > 7 > 5 > 3 > 2$.

- **STAGE Y (Why this is different):** Unlike part (c), we're not between two equilibria. We're in the unbounded region $x > 7$. This means the trajectory might escape to infinity.
- **STAGE Z (What we'll determine):** Whether the trajectory approaches an equilibrium or diverges to $+\infty$.

Step 2: Determine Sign of $\dot{x}$ for $x > 7$

From part (b), in the region $x > 7$:

$$\begin{aligned} \dot{x} &= (x-2)(x-3)(x-5)(x-7) && (56)\\ &= (+)(+)(+)(+) && (57)\\ &= (+) > 0 && (58) \end{aligned}$$

At $x = 8$ specifically:

$$\begin{aligned} \dot{x}\Big|_{x=8} &= (8-2)(8-3)(8-5)(8-7) && (59)\\ &= (6)(5)(3)(1) && (60)\\ &= 90 > 0 \quad \checkmark && (61) \end{aligned}$$

Step 3: Determine Direction of Motion

Since $\dot{x} > 0$ at $x_0 = 8$:

$$\frac{dx}{dt} > 0 \quad \Rightarrow \quad x \text{ is } \textbf{increasing} \tag{62}$$

The trajectory moves to the **right** (toward larger values of x).

Step 4: Analyze Long-Term Behavior

Key Observation:

- For all $x > 7$: $\dot{x} > 0$ (from phase line analysis)
- As x increases, $\dot{x}$ also increases (positive feedback)
- There are no equilibria to the right of $x = 7$ to stop the growth

Explanation 6 (Why Trajectory Escapes to Infinity)**. *Asymptotic Analysis:***
For large x, the polynomial behaves like its leading term:

$$\dot{x} = x^4 - 17x^3 + 101x^2 - 247x + 210 \sim x^4 \text{ as } x \to \infty \tag{63}$$

This gives:

$$\frac{dx}{dt} \sim x^4 \quad \text{for large } x \tag{64}$$

Separating variables:

$$\begin{aligned} \frac{dx}{x^4} &\sim dt && (65)\\ \int_{x_0}^{x(t)} \frac{ds}{s^4} &\sim \int_0^t d\tau && (66)\\ -\frac{1}{3x^3}\Bigg|_{x_0}^{x(t)} &\sim t && (67)\\ -\frac{1}{3x(t)^3} + \frac{1}{3x_0^3} &\sim t && (68) \end{aligned}$$

Solving for $x(t)$:

$$\frac{1}{x(t)^3} \sim \frac{1}{x_0^3} - 3t \tag{69}$$

This blows up (becomes singular) when:

$$t_{blow\text{-}up} \sim \frac{1}{3x_0^3} = \frac{1}{3 \cdot 8^3} = \frac{1}{1536} \approx 0.00065 \tag{70}$$

The trajectory reaches infinity in ***finite time****!*

Step 5: Verify with Qualitative Reasoning

Explanation 7 (Finite-Time Blow-Up Mechanism)**.** *From Lecture Notes (Section 6): For $\dot{x} = f(x)$, if $f(x) \to \infty$ faster than linearly as $x \to \infty$, trajectories can escape to infinity in finite time.*

Intuition:

- *At $x = 8$: $\dot{x} = 90$ (already quite large)*
- *As x grows, $\dot{x} \sim x^4$ grows even faster*
- *The rate of growth accelerates without bound*
- *The accumulated distance $\int \dot{x}\, dt$ diverges in finite time*

Contrast with Exponential Growth:

- *If $\dot{x} = x$: solution is $x(t) = x_0 e^t$, which reaches infinity only as $t \to \infty$*
- *Here $\dot{x} = x^4$: superlinear growth causes finite-time blow-up*

Step 6: Exact Blow-Up Time Estimate

For $x_0 = 8$, the asymptotic estimate gives:

$$t_{\text{blow-up}} \approx \frac{1}{3 \cdot 8^3} = \frac{1}{1536} \approx 6.5 \times 10^{-4} \tag{71}$$

This is an approximation valid for large x. The actual blow-up time might differ slightly, but the order of magnitude is correct.

Final Answer for Part (d)

$$\boxed{\begin{array}{l} \text{Starting from } x_0 = 8: \\ \text{Long-term behavior: } x(t) \to +\infty \text{ as } t \to t_{\text{blow-up}} \\ \text{Type: } \textbf{Finite-time blow-up} \\ \text{Blow-up time: } t_{\text{blow-up}} \approx \dfrac{1}{3x_0^3} = \dfrac{1}{1536} \approx 6.5 \times 10^{-4} \\ \text{Mechanism: Superlinear growth } (\dot{x} \sim x^4 \text{ for large } x) \end{array}} \tag{72}$$

Physical Description: The trajectory increases monotonically from $x_0 = 8$, accelerating rapidly as $\dot{x} \sim x^4$. The solution becomes unbounded (reaches infinity) in finite time, approximately $t \approx 0.00065$. This is a characteristic behavior of superlinearly growing systems.

Summary: Global Phase Portrait of the System

Complete Classification of Long-Term Behaviors

For the system $\dot{x} = (x-2)(x-3)(x-5)(x-7)$:

1. **Basin of** $x = 2$: Initial conditions $x_0 \in (-\infty, 3) \Rightarrow x(t) \to 2$ as $t \to \infty$
2. **Basin of** $x = 5$: Initial conditions $x_0 \in (3, 7) \Rightarrow x(t) \to 5$ as $t \to \infty$
3. **Escape to** $+\infty$: Initial conditions $x_0 \in (7, \infty) \Rightarrow x(t) \to +\infty$ in finite time
4. **Unstable equilibria**: $x = 3$ and $x = 7$ are repellers (no basin of attraction)

Key Insights from Lecture Notes

- **Alternating stability**: Adjacent equilibria have opposite stability (Section 6)
- **Basins separated by unstable equilibria**: The unstable points at $x = 3$ and $x = 7$ form boundaries between different basins of attraction
- **Finite-time blow-up**: Possible when $\dot{x}$ grows superlinearly (faster than linear) as $|x| \to \infty$
- **Phase line is the complete picture**: In 1D, the phase line fully determines all long-term behaviors (no hidden complexity like limit cycles, which are impossible in 1D)