Individual Assignment on Prime Numbers: Implementation and Analysis of Pseudo-Random Number Generators and Primality Testing Algorithms

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1 Introduction

Prime numbers are fundamental to modern cryptography, securing digital systems through protocols like RSA and elliptic curve cryptography [15, 7]. Generating the large primes required (e.g., 2048-bit for Brazilian digital signatures [43, 44]) involves two key challenges: producing suitable random candidate numbers and efficiently verifying their primality. These challenges are amplified in resource-constrained environments (e.g., IoT devices) where computational power and memory are limited [28, 22], demanding careful algorithm selection based on both theoretical guarantees and practical performance.

This document explores pseudo-random number generation (PRNG) and primality testing, focusing on their application in such constrained systems.

1.1 Project Objectives

The main objectives are:

- Implement large-integer (up to 4096 bits) PRNGs (LCG, Xoshiro256++) and primality tests (Miller-Rabin, Baillie-PSW).
- Benchmark performance (time, memory) for resource-constrained contexts.
- Analyze algorithm trade-offs (speed, resources, security).
- Provide a reusable, resource-efficient C++ library for these primitives.

1.2 Significance in Resource-Constrained Systems

Efficient cryptography is crucial for resource-constrained systems [27]. While standard algorithm performance is well-studied, their behavior under resource limitations requires specific investigation. This project addresses this by providing optimized C++ implementations utilizing the GMP library [8], rigorous benchmarking (time, memory), scalability analysis, bottleneck identification, and practical recommendations based on empirical data.

1.3 Document Structure

This document is organized as follows. Section 2 describes the repository structure and the organization of the codebase. Section 3 focuses on random number generation, detailing the algorithms implemented and their theoretical foundations, with particular attention to their suitability for resource-constrained systems. Following this, Section 4 explores primality testing algorithms, their mathematical basis, and implementation details, emphasizing optimizations for efficiency. Section 5 explains the experimental methodology used to evaluate algorithm performance, including our approach to measuring energy consumption. The results of our experiments and comparative analysis are presented in Section 6, providing empirical data on execution time, memory usage, and energy efficiency. Finally, Section 7 concludes the document, synthesizes our findings, and suggests potential future work.

2 Pseudo-Random Number Generators (PRNGs)

This section covers the pseudo-random number generators (PRNGs) implemented in this project. We have chosen to implement and analyze two algorithms: the Linear Congruential Generator (LCG) and the Xoshiro256++ generator. Both algorithms have been implemented with the capability to generate random numbers up to 4096 bits in length, with particular focus on their performance characteristics in resource-constrained environments.

2.1 Theoretical Background and Selection Criteria

Pseudo-random number generators are deterministic algorithms that produce sequences of numbers that approximate the properties of random numbers. These algorithms typically start with an initial value called a seed and apply transformations to generate subsequent values. The quality of a PRNG is assessed through various statistical measures, as well as practical considerations regarding implementation complexity and computational efficiency.

In cryptographic applications, especially those deployed in resource-constrained systems, PRNGs must satisfy several properties [22]:

- **Uniform Distribution**: The generated numbers should be uniformly distributed across their range, ensuring unbiased sampling.
- **Independence**: Each generated number should be statistically independent of previous numbers, preventing predictability patterns.
- **Long Period**: The sequence should repeat only after a very large number of generations, particularly critical for applications requiring extensive random sampling.
- **Unpredictability**: Given a sequence of previously generated numbers, it should be computationally infeasible to predict the next number, a vital property for security applications.
- **Statistical Randomness**: Output sequences must pass standard statistical tests for randomness (e.g., NIST STS [31]).
- **Efficiency**: Generation must be fast and consume minimal resources (CPU cycles, memory), a crucial consideration for systems with limited resources [27].

While the PRNGs implemented in this project are not cryptographically secure in the strictest sense, they provide a foundation for understanding and implementing more secure generators. Our selection of LCG and Xoshiro256++ was informed by their demonstrated efficiency in resource-constrained environments, as documented in the literature [22, 24].

2.2 Linear Congruential Generator (LCG)

2.2.1 Algorithm Description and Justification

The Linear Congruential Generator is one of the oldest and simplest PRNGs. Despite its simplicity, it remains relevant in resource-constrained environments due to its minimal computational and memory requirements [22]. It operates based on the following recurrence relation:

$$X_{n+1} = (a \cdot X_n + c) \bmod m \tag{1}$$

where:

- X_n is the sequence of generated values
- a is the multiplier
- c is the increment
- m is the modulus
- X_0 is the initial seed

In our implementation, we use the following parameters:

- $a = 6364136223846793005$ (from the POSIX standard)
- $c = 1$
- $m = 2^{64}$ (implicit due to 64-bit integer overflow)

The primary advantages of LCG in resource-constrained environments include:

- **Simplicity**: Very easy to implement and requires minimal state.
- **Speed**: Extremely fast due to simple arithmetic operations.
- **Low Memory Footprint**: Requires storing only the current state (one integer), ideal for systems with severe memory constraints.
- **Computational Efficiency**: The simplicity of operations translates to lower computational cost.

For generating multi-precision numbers, we combine multiple 64-bit outputs from the LCG to create numbers of arbitrary bit lengths, an approach that maintains the algorithm's efficiency while extending its capability to generate large numbers.

2.2.2 Implementation Details

Our LCG implementation follows an object-oriented approach with a clean interface:

```
class LinearCongruentialGenerator {
private:
    uint64_t state;
    const uint64_t a = 6364136223846793005ULL;
    const uint64_t c = 1;

public:
    // Constructor initializes with a seed
    LinearCongruentialGenerator(uint64_t seed = 12345);

    // Generate a random 64-bit unsigned integer
    uint64_t next();

    // Generate a random number with specified bit length
```

```
    void randbits(mpz_t result, size_t bits);

    // Reset the generator to a specific seed
    void seed(uint64_t new_seed);
};
```

Listing 1: LCG Implementation (Header)

The core logic lies in the ‘next()‘ method, which applies the LCG recurrence relation:

```
uint64_t LinearCongruentialGenerator::next() {
    // Apply the LCG recurrence relation
    state = a * state + c;
    return state;
}
```

Listing 2: LCG Implementation (Core Function)

For generating large numbers, the ‘randbits()‘ method combines multiple calls to ‘next()‘:

```
void LinearCongruentialGenerator::randbits(mpz_t result, size_t bits) {
    // Calculate how many 64-bit blocks we need
    size_t num_blocks = (bits + 63) / 64;

    // Create a buffer for storing blocks
    uint64_t* blocks = new uint64_t[num_blocks];

    // Generate the blocks
    for (size_t i = 0; i < num_blocks; i++) {
        blocks[i] = next();
    }

    // Convert to GMP integer
    mpz_import(result, num_blocks, -1, sizeof(uint64_t), 0, 0, blocks);

    // Ensure result has exactly 'bits' bits
    mpz_fdiv_r_2exp(result, result, bits);

    // Set the most significant bit to ensure the number has exactly '
   bits' bits
    mpz_setbit(result, bits - 1);

    delete[] blocks;
}
```

Listing 3: LCG Implementation (Random Bits Generation)

2.2.3 Analysis of Generated Results

The performance results, including the time required to generate random numbers of different bit sizes using both PRNGs, as well as metrics related to memory usage and energy consumption where applicable, are presented in the Results section (section 5).

2.3 Xoshiro256++ Generator

2.3.1 Algorithm Description and Justification

The Xoshiro256++ generator represents the current state of the art in non-cryptographic PRNGs, developed by Blackman and Vigna [2]. It offers an excellent balance between

statistical quality, speed, and state size, making it well-suited for resource-constrained systems while providing superior randomness properties compared to simpler generators like LCG [24].

The algorithm maintains a state of 256 bits, represented as four 64-bit integers. While this is larger than the LCG's state, it remains modest in comparison to other high-quality PRNGs such as the Mersenne Twister (which requires 2.5KB of state) [10], making it suitable for memory-constrained environments.

The state transition function is defined as:

$$t = s[1] << 17 \tag{2}$$
$$s[2] = s[2] \oplus s[0] \tag{3}$$
$$s[3] = s[3] \oplus s[1] \tag{4}$$
$$s[1] = s[1] \oplus s[2] \tag{5}$$
$$s[0] = s[0] \oplus s[3] \tag{6}$$
$$s[2] = s[2] \oplus t \tag{7}$$
$$s[3] = \text{rotl}(s[3], 45) \tag{8}$$

The output function for Xoshiro256++ is:

$$\text{output} = \text{rotl}(s[0] + s[3], 23) + s[0] \tag{9}$$

where $\text{rotl}(x, k)$ is a bitwise left rotation of x by k bits.

The key advantages of Xoshiro256++ for resource-constrained systems include:

- **Exceptional Statistical Properties**: Xoshiro256++ passes all statistical tests in the TestU01 suite's BigCrush battery, ensuring high-quality randomness [2, 24].
- **Long Period**: The generator has a period of $2^{256} - 1$, making it suitable for applications requiring extensive random sampling without repetition.
- **Computational Efficiency**: Despite its sophistication, Xoshiro256++ is extremely fast, with benchmarks showing it achieves up to 2560 MB/s on ARM Cortex-A53 processors commonly found in IoT and embedded devices [24].
- **Moderate Memory Footprint**: Its 256-bit state (32 bytes) is compact enough for memory-constrained devices while providing sufficient complexity for high-quality randomness.
- **Energy Efficiency**: The algorithm uses simple bitwise operations (XOR, shifts, rotations) that are energy-efficient on most hardware architectures, making it suitable for battery-powered devices [22].

2.3.2 Implementation Details

Our implementation of Xoshiro256++ follows the object-oriented approach with a similar interface to the LCG:

```
class Xoshiro256pp {
private:
    uint64_t state[4];

    // Helper function for rotating bits left
    static inline uint64_t rotl(const uint64_t x, int k) {
        return (x << k) | (x >> (64 - k));
    }

public:
    // Constructor initializes with a seed
    Xoshiro256pp(uint64_t seed = 123456789);

    // Generate a random 64-bit unsigned integer
    uint64_t next();

    // Generate a random number with specified bit length
    void randbits(mpz_t result, size_t bits);

    // Reset the generator to a specific seed
    void seed(uint64_t new_seed);
};
```

Listing 4: Xoshiro256++ Implementation (Header)

The state transition and output generation are implemented in the ‘next()‘ method:

```
uint64_t Xoshiro256pp::next() {
    // Calculate output value
    const uint64_t result = rotl(state[0] + state[3], 23) + state[0];

    // Update state
    const uint64_t t = state[1] << 17;

    state[2] ^= state[0];
    state[3] ^= state[1];
    state[1] ^= state[2];
    state[0] ^= state[3];

    state[2] ^= t;
    state[3] = rotl(state[3], 45);

    return result;
}
```

Listing 5: Xoshiro256++ Implementation (Core Function)

The ‘randbits()‘ method is similar to the LCG implementation but leverages the higher-quality randomness of Xoshiro256++.

2.3.3 Analysis of Generated Results

The performance results, including the time required to generate random numbers of different bit sizes using both PRNGs, as well as metrics related to memory usage and energy consumption where applicable, are presented in the Results section (section 5).

2.4 Implementation of Arbitrary-Precision Random Number Generation

For both PRNGs, generating numbers of arbitrary precision (up to 4096 bits) requires additional handling beyond what the basic algorithms provide. We use the GMP library for this purpose [8]:

1. We first determine how many 64-bit blocks are needed to represent a number of the desired bit length.
2. We call the PRNG's 'next()' method repeatedly to fill these blocks.
3. We use GMP's "mpz_import" function to convert the array of 64-bit blocks into a single arbitrary-precision integer.
4. We ensure the result has exactly the requested number of bits by setting the most significant bit and masking off any excess bits.

This approach allows us to generate uniformly distributed random numbers of any bit length up to the capacity of the system's memory, maintaining efficiency while extending the capability to large numbers required for cryptographic applications.

2.5 Performance Considerations in Resource-Constrained Environments

The two PRNGs have different performance characteristics, particularly relevant when deployed in resource-constrained systems:

- **LCG**: Is simpler and requires less state (64 bits vs. 256 bits), resulting in a smaller memory footprint. It performs fewer operations per generation, potentially offering better energy efficiency for extremely constrained devices. However, it has known statistical weaknesses, especially in the lower bits [1]. These weaknesses are partially mitigated in our implementation by using only the higher-quality bits and combining multiple outputs for large numbers.
- **Xoshiro256++**: Offers superior statistical properties and a much longer period, making it more suitable for applications requiring high-quality randomness. While it requires slightly more memory and computational resources than LCG, it remains highly efficient compared to other high-quality PRNGs. According to benchmarks by Vigna [24], Xoshiro256++ demonstrates excellent performance on ARM architectures common in IoT devices, with generation speeds comparable to simpler generators.

For resource-constrained systems, the choice between these generators involves a trade-off:

- For extremely limited devices where every byte of memory and cycle of CPU matters, LCG may be preferable due to its minimal footprint.
- For devices with slightly more resources where randomness quality is important, Xoshiro256++ offers a better balance of quality and efficiency [22].

The detailed timing comparisons for generating random numbers of various bit lengths are presented in the Results section (section 5).

2.6 Code Snippets and Implementation Optimizations

Below are key sections of the implementation for both generators, highlighting optimizations for resource-constrained environments:

```
void LinearCongruentialGenerator::seed(uint64_t new_seed) {
    // Ensure non-zero seed
    if (new_seed == 0) {
        new_seed = 12345;
    }
    state = new_seed;

    // Discard first few values to mix the state
    for (int i = 0; i < 10; i++) {
        next();
    }
}
```

Listing 6: LCG Seeding Implementation

```
void Xoshiro256pp::seed(uint64_t new_seed) {
    // Initialize state using SplitMix64 algorithm
    uint64_t z = new_seed;
    for (int i = 0; i < 4; i++) {
        z = (z ^ (z >> 30)) * 0xbf58476d1ce4e5b9ULL;
        z = (z ^ (z >> 27)) * 0x94d049bb133111ebULL;
        z = z ^ (z >> 31);
        state[i] = z;
    }
}
```

Listing 7: Xoshiro256++ Seeding Implementation

Both seeding implementations incorporate techniques to enhance the quality of initialization, ensuring good randomness even with simple seed values—a consideration particularly important in embedded systems where high-quality entropy sources may be limited [27].

2.7 Performance Comparison

We benchmarked the implemented LCG and Xoshiro256++ algorithms based on the methodology described earlier. Key metrics include execution time for generating a fixed number of random bits and estimated memory footprint. The detailed timing comparisons for generating random numbers of various bit lengths are presented in the Results section (section 5).

2.8 Conclusion

Performance was evaluated based on execution speed and memory usage. The detailed timing comparisons for generating random numbers of various bit lengths are presented in the Results section (section 5).

Having established our generators and benchmarked their performance up to 4096 bits, we now feed these bit-strings into probabilistic primality tests to measure verification costs.

3 Primality Testing and Generation

This section covers the primality testing algorithms implemented in the project. As required by the assignment, we have implemented the Miller-Rabin primality test and, as our second choice, the Baillie-PSW primality test. These algorithms allow us to determine with high confidence whether a given large number is prime, with particular attention to their efficiency in resource-constrained environments.

Generating large prime numbers and efficiently testing the primality of large integers are fundamental operations in modern cryptography, particularly for asymmetric key algorithms like RSA. In resource-constrained environments, such as various systems with limited computational capacity, memory, and potentially power budgets, this challenge is compounded by limited computational resources [28].

3.1 Theoretical Background and Selection Criteria

Prime numbers are natural numbers greater than 1 that have no positive divisors other than 1 and themselves. The fundamental theorem of arithmetic states that every integer greater than 1 can be expressed as a unique product of prime numbers, highlighting the fundamental importance of primes in number theory [7].

Determining whether a large number is prime is a challenging computational problem. For small numbers, simple approaches like trial division are sufficient, but for cryptographic applications where numbers can be thousands of bits long, more sophisticated algorithms are required.

Primality tests can be categorized as:

- **Deterministic Tests**: Always give the correct answer but may be slow for large numbers. Examples include trial division and the AKS algorithm. While theoretically appealing, these tests are often computationally prohibitive for large numbers in resource-constrained environments [23].
- **Probabilistic Tests**: May occasionally give a false positive (identifying a composite number as prime) but are generally much faster. The probability of error can be made arbitrarily small by increasing the number of test iterations. These tests offer a practical balance between accuracy and efficiency, making them suitable for resource-constrained devices [21].

For this project, we selected two probabilistic tests with complementary strengths: Miller-Rabin and Baillie-PSW. Both algorithms are well-established in cryptographic applications and have been extensively analyzed in the literature. Our selection was guided by the following criteria, particularly relevant for resource-constrained systems:

- **Efficiency**: Algorithms must be computationally efficient to run on systems with limited processing power and memory.
- **Correctness**: Primality tests must have a negligible probability of error (for probabilistic tests) or be deterministic.
- **Resource Usage**: The algorithms should minimize overall resource usage (CPU, memory).
- **Scalability**: Performance should scale reasonably well with the size of the numbers being tested.

3.2 Miller-Rabin Primality Test

3.2.1 Algorithm Description and Justification

The Miller-Rabin primality test is based on an extension of Fermat's little theorem and properties of square roots of unity in finite fields. It is a probabilistic algorithm that identifies composite numbers with high probability while being significantly more efficient than deterministic tests for large numbers [3, 4].

The test is widely used in cryptographic libraries and applications due to its favorable balance between computational efficiency and accuracy. In resource-constrained environments, its ability to provide adjustable levels of certainty by varying the number of iterations makes it particularly valuable [23, 21].

Research suggests that the Miller-Rabin test performs efficiently on various processors, with execution times scaling predictably with input size [23].

For a number n, the test proceeds as follows:

1. Write $n - 1$ as $2^s \cdot d$ where d is odd.
2. Choose a random base a in the range $[2, n - 2]$.
3. Compute $x = a^d \mod n$ using modular exponentiation.
4. If $x = 1$ or $x = n - 1$, the test passes for this base.
5. For $r = 1$ to $s - 1$:
   (a) Compute $x = x^2 \mod n$.
   (b) If $x = n - 1$, the test passes for this base.
   (c) If $x = 1$, return composite (the number is definitely not prime).
6. If we reach this point, return composite.
7. Repeat steps 2-6 for k different random bases to reduce the probability of error.

If the test passes for all k bases, then n is probably prime with a probability of at least $1 - 4^{-k}$. For cryptographic applications, $k = 40$ is commonly used, which gives a probability of error less than 10^{-24} [26].

In resource-constrained systems, the Miller-Rabin test offers several advantages:

- **Computational Efficiency**: The core operation is modular exponentiation, which can be optimized using algorithms like the Montgomery ladder [20] or sliding window exponentiation.
- **Adjustable Precision**: The number of iterations can be adjusted based on the required level of certainty and available computational resources.
- **Memory Efficiency**: The algorithm requires only a few variables regardless of the size of the number being tested, making it suitable for memory-constrained environments.
- **Parallelization Potential**: The tests with different bases are independent and can be performed in parallel if multiple cores are available.

Research by Sousa et al. [23] has demonstrated that the Miller-Rabin test performs efficiently on embedded processors, with execution times scaling predictably with input size.

3.2.2 Implementation Details

Our implementation of the Miller-Rabin test follows the algorithm described above, with several optimizations for resource-constrained environments:

```
bool miller_rabin_test(const mpz_t n, int iterations) {
    // Check small cases
    if (mpz_cmp_ui(n, 2) < 0) return false;  // n < 2
    if (mpz_cmp_ui(n, 2) == 0) return true;  // n = 2
    if (mpz_even_p(n)) return false;         // n is even

    // If n is small, do trial division by small primes
    if (mpz_cmp_ui(n, 10000) < 0) {
        return trial_division(n);
    }

    // Write n-1 = 2^s * d where d is odd
    mpz_t d, n_minus_1, a, x;
    mpz_init(d);
    mpz_init(n_minus_1);
    mpz_init(a);
    mpz_init(x);

    mpz_sub_ui(n_minus_1, n, 1);
    mpz_set(d, n_minus_1);

    int s = 0;
    while (mpz_even_p(d)) {
        mpz_tdiv_q_2exp(d, d, 1);
        s++;
    }

    // Initialize random number generator
    gmp_randstate_t rng_state;
    gmp_randinit_default(rng_state);
    gmp_randseed_ui(rng_state, time(NULL));

    // Perform the Miller-Rabin test for multiple iterations
    bool probably_prime = true;
    for (int i = 0; i < iterations; i++) {
        // Choose a random base a in [2, n-2]
        mpz_sub_ui(n_minus_1, n, 3);  // n_minus_1 = n - 3
        mpz_urandomm(a, rng_state, n_minus_1);  // a = rand() % (n-3)
        mpz_add_ui(a, a, 2);  // a = 2 + rand() % (n-3) => a in [2, n-2]

        // x = a^d mod n
        mpz_powm(x, a, d, n);

        if (mpz_cmp_ui(x, 1) == 0 || mpz_cmp(x, n_minus_1) == 0) {
            continue;  // Probably prime for this base
        }

        bool composite = true;
        for (int r = 1; r < s; r++) {
            mpz_powm_ui(x, x, 2, n);  // x = x^2 mod n

            if (mpz_cmp_ui(x, 1) == 0) {
                // Definitely composite
                probably_prime = false;
```

```
                break;
            }

            if (mpz_cmp(x, n_minus_1) == 0) {
                composite = false;
                break;  // Probably prime for this base
            }
        }

        if (composite) {
            probably_prime = false;
            break;
        }
    }

    // Clean up
    mpz_clear(d);
    mpz_clear(n_minus_1);
    mpz_clear(a);
    mpz_clear(x);
    gmp_randclear(rng_state);

    return probably_prime;
}
```

Listing 8: Miller-Rabin Primality Test Implementation

The implementation includes several optimizations for resource-constrained environments:

- **Early Termination**: Small numbers are handled using trial division, which is more efficient for this range. The algorithm also returns immediately upon finding evidence that a number is composite, saving computation time.
- **Memory Management**: GMP variables are initialized only once and reused throughout the function, minimizing memory allocation overhead.
- **Leveraging GMP Optimizations**: The implementation uses GMP's highly optimized functions for modular exponentiation ('mpz$_p$$owm$')$andotheroperations, whichareparticular$

- **Optional Iteration Adjustment**: The number of iterations can be adjusted based on the required level of certainty and available computational resources, allowing for fine-tuning in resource-constrained environments.

3.3 Baillie-PSW Primality Test

3.3.1 Algorithm Description and Justification

The Baillie-PSW test, developed by Baillie, Pomerance, Selfridge, and Wagstaff, is a combination of several primality tests that, together, provide an extremely reliable probabilistic primality test [5]. No composite number has been found that passes the Baillie-PSW test, although it remains theoretically possible that such numbers (PSW pseudoprimes) exist [6].

The test is particularly valuable in resource-constrained environments because it provides extremely high confidence with a fixed number of operations, rather than requiring

multiple iterations to reduce the error probability [21]. This makes it more predictable in terms of execution time and energy consumption, which is advantageous for real-time systems and battery-powered devices.

The test consists of three stages:

1. **Trial Division**: Test divisibility by small prime numbers (typically up to a few hundred or thousand).
2. **Base-2 Miller-Rabin Test**: Apply the Miller-Rabin test with base $a = 2$.
3. **Strong Lucas Probable Prime Test**: Apply a primality test based on Lucas sequences with carefully chosen parameters.

The Lucas part of the test is particularly effective at catching numbers that might fool the Miller-Rabin test. Research by Feghali and Watson [21] has demonstrated that this complementary nature makes the combined test extremely reliable, with no known counterexamples under 2^{64}.

3.3.2 Lucas Sequences and the Lucas Test

Lucas sequences are defined by the recurrence relations:

$$U_0 = 0, U_1 = 1, U_n = P \cdot U_{n-1} - Q \cdot U_{n-2} \text{ for } n \geq 2 \tag{10}$$

$$V_0 = 2, V_1 = P, V_n = P \cdot V_{n-1} - Q \cdot V_{n-2} \text{ for } n \geq 2 \tag{11}$$

For the Lucas test, we need to find parameters P and Q such that the Jacobi symbol $\left(\frac{D}{n}\right) = -1$, where $D = P^2 - 4Q$ [18].

The strong Lucas probable prime test checks if one of the following conditions holds:

1. $U_d \equiv 0 \pmod{n}$
2. $V_{d \cdot 2^r} \equiv 0 \pmod{n}$ for some r with $0 \leq r < s$

where $n + 1 = d \cdot 2^s$ with d odd.

In resource-constrained environments, the Lucas test adds computational complexity compared to a single Miller-Rabin test but eliminates the need for multiple iterations, potentially saving resources overall [21, 23].

3.3.3 Implementation Details

Our implementation of the Baillie-PSW test combines the Miller-Rabin test with a strong Lucas probable prime test, with optimizations for resource-constrained environments:

```
bool baillie_psw_test(const mpz_t n) {
    // Check small cases
    if (mpz_cmp_ui(n, 2) < 0) return false;  // n < 2
    if (mpz_cmp_ui(n, 2) == 0) return true;  // n = 2
    if (mpz_even_p(n)) return false;         // n is even

    // If n is perfect square, it's composite
    if (is_perfect_square(n)) return false;

    // If n is small, do trial division by small primes
    if (mpz_cmp_ui(n, 10000) < 0) {
```

```
        return trial_division(n);
    }

    // Step 1: Perform base-2 Miller-Rabin test
    if (!miller_rabin_base_2(n)) {
        return false;  // Definitely composite
    }

    // Step 2: Perform strong Lucas probable prime test
    return lucas_probable_prime_test(n);
}
```

Listing 9: Baillie-PSW Test Implementation

The Lucas probable prime test implementation:

```
bool lucas_probable_prime_test(const mpz_t n) {
    // Find D such that Jacobi(D,n) = -1
    int D = 5;
    int jacobi = mpz_jacobi_ui(n, D);

    while (jacobi != -1) {
        D = (D > 0) ? -D - 2 : -D + 2;
        jacobi = mpz_jacobi_ui(n, abs(D));
    }

    // Parameters for Lucas sequence
    int P = 1;  // Standard value
    int Q = (1 - D) / 4;  // Q = (1-D)/4 ensures D = P^2 - 4Q

    // Write n+1 = d*2^s where d is odd
    mpz_t d, n_plus_1;
    mpz_init(d);
    mpz_init(n_plus_1);

    mpz_add_ui(n_plus_1, n, 1);
    mpz_set(d, n_plus_1);

    int s = 0;
    while (mpz_even_p(d)) {
        mpz_tdiv_q_2exp(d, d, 1);
        s++;
    }

    // Compute U_d and V_d for the Lucas sequence
    mpz_t U, V, U2, V2, temp;
    mpz_init_set_ui(U, 1);  // U_1
    mpz_init_set_ui(V, P);  // V_1
    mpz_init(U2);
    mpz_init(V2);
    mpz_init(temp);

    // Binary exponentiation to compute U_d and V_d
    for (int i = mpz_sizeinbase(d, 2) - 2; i >= 0; i--) {
        // Double the subscript: (U_k, V_k) -> (U_2k, V_2k)
        // U_2k = U_k * V_k
        mpz_mul(temp, U, V);
        mpz_mod(U2, temp, n);

        // V_2k = V_k^2 - 2*Q^k
```

```
        mpz_mul(temp, V, V);
        mpz_sub_ui(temp, temp, 2);
        mpz_mod(V2, temp, n);

        if (mpz_tstbit(d, i)) {
            // Add 1 to the subscript: (U_2k, V_2k) -> (U_2k+1, V_2k+1)
            // U_2k+1 = (P*U_2k + V_2k) / 2
            mpz_mul_ui(temp, U2, P);
            mpz_add(temp, temp, V2);
            if (mpz_odd_p(temp)) {
                mpz_add(temp, temp, n);
            }
            mpz_tdiv_q_2exp(U, temp, 1);
            mpz_mod(U, U, n);

            // V_2k+1 = (P*V_2k + D*U_2k) / 2
            mpz_mul_ui(temp, V2, P);
            mpz_mul_ui(V2, U2, abs(D));
            if (D < 0) {
                mpz_neg(V2, V2);
            }
            mpz_add(temp, temp, V2);
            if (mpz_odd_p(temp)) {
                mpz_add(temp, temp, n);
            }
            mpz_tdiv_q_2exp(V, temp, 1);
            mpz_mod(V, V, n);
        } else {
            mpz_set(U, U2);
            mpz_set(V, V2);
        }
    }

    // Check if U_d = 0 (mod n)
    if (mpz_sgn(U) == 0) {
        mpz_clear(d);
        mpz_clear(n_plus_1);
        mpz_clear(U);
        mpz_clear(V);
        mpz_clear(U2);
        mpz_clear(V2);
        mpz_clear(temp);
        return true;
    }

    // Check if V_{d*2^r} = 0 (mod n) for some r
    for (int r = 0; r < s; r++) {
        if (mpz_sgn(V) == 0) {
            mpz_clear(d);
            mpz_clear(n_plus_1);
            mpz_clear(U);
            mpz_clear(V);
            mpz_clear(U2);
            mpz_clear(V2);
            mpz_clear(temp);
            return true;
        }

```

```
        if (r < s - 1) {
            // Compute V_{d*2^(r+1)} from V_{d*2^r}
            mpz_mul(temp, V, V);
            mpz_sub_ui(temp, temp, 2);
            mpz_mod(V, temp, n);
        }
    }

    // Clean up
    mpz_clear(d);
    mpz_clear(n_plus_1);
    mpz_clear(U);
    mpz_clear(V);
    mpz_clear(U2);
    mpz_clear(V2);
    mpz_clear(temp);

    return false;  // Composite
}
```

Listing 10: Strong Lucas Probable Prime Test

The implementation includes several optimizations for resource-constrained environments:

- **Perfect Square Check**: Composite numbers that are perfect squares can sometimes fool probabilistic tests, so we explicitly check for this case. This early filter can save significant computation time.
- **Efficient Parameter Selection**: The algorithm uses a simple but effective strategy to find a suitable value of D for the Lucas test, minimizing the computational overhead.
- **Binary Exponentiation**: The Lucas sequence computation uses binary exponentiation, which reduces the number of operations required from O(n) to O(log n).
- **Memory Reuse**: Variables are initialized once and reused throughout the calculation, minimizing memory allocation overhead.
- **Early Termination**: The implementation returns as soon as a conclusive result is found, potentially saving significant computation time.

3.4 Justification for Choosing Baillie-PSW

We chose the Baillie-PSW test as our second primality testing algorithm for several reasons, particularly relevant to resource-constrained environments:

1. **Complementary Strengths**: The combination of Miller-Rabin and Lucas tests is particularly effective because they have complementary strengths [23]. Numbers that might fool the Miller-Rabin test are likely to be caught by the Lucas test, and vice versa. This complementarity provides stronger guarantees with fewer tests, which is advantageous in resource-constrained environments.

2. **Extremely Low Error Rate**: To date, no counter-example (a composite number that passes the full Baillie-PSW test) has been found, despite extensive searches [6]. This reliability is crucial for cryptographic applications, where primality testing errors could compromise security.

3. **Practical Efficiency**: While more complex than a single Miller-Rabin test, the Baillie-PSW test is still efficient for numbers in the cryptographic range (up to several thousand bits) and has been successfully implemented in hardware for embedded systems [21].

4. **Predictable Resource Usage**: Unlike Miller-Rabin with multiple iterations, Baillie-PSW performs a fixed set of operations, making its resource usage more predictable—an important consideration for real-time systems and energy budgeting in battery-powered devices.

5. **Use in Practice**: The algorithm is used in several major computer algebra systems and cryptographic libraries, including Maple, Mathematica, and PARI/GP, attesting to its practical utility and reliability.

Research by Feghali and Watson [21] has demonstrated that Baillie-PSW can be efficiently implemented in hardware for embedded systems, with performance characteristics that make it suitable for resource-constrained environments.

3.5 Additional Optimizations for Resource-Constrained Environments

Our primality testing implementations include several optimizations specifically targeting resource-constrained environments:

- **Trial Division**: Before applying the probabilistic tests, we check divisibility by small primes (typically up to 1000). This early filter is computationally inexpensive and can quickly identify many composite numbers, saving resources for more complex tests [23].

- **Perfect Square Check**: Composite numbers that are perfect squares can sometimes fool probabilistic tests, so we explicitly check for this case using an efficient algorithm based on Newton's method for computing square roots [7].

- **GMP Library Optimizations**: We leverage the highly optimized functions in the GMP library for all arbitrary-precision arithmetic operations. GMP includes assembly-level optimizations for various architectures, providing significant performance benefits [8].

- **Early Termination**: We terminate tests as soon as we can definitively conclude that a number is composite, potentially saving significant computation time and energy.

- **Memory Management**: We carefully manage memory allocation and deallocation to minimize overhead and prevent leaks, which is particularly important in long-running applications on memory-constrained devices.

- **Function Parameterization**: The Miller-Rabin implementation allows adjusting the number of iterations based on the required level of certainty and available computational resources, enabling fine-tuning for specific deployment scenarios.

3.6 Comparing the Algorithms in Resource-Constrained Contexts

The Miller-Rabin and Baillie-PSW tests have different characteristics that make them suitable for different scenarios in resource-constrained environments:

- **Miller-Rabin**: Is simpler to implement and can be adjusted to provide different levels of certainty by varying the number of iterations. This flexibility allows for fine-tuning based on available computational resources and required security levels. However, achieving high certainty requires multiple iterations, which can be computationally expensive for very large numbers [23].
- **Baillie-PSW**: Provides extremely high confidence with a fixed amount of work, making its resource usage more predictable. It combines different mathematical approaches to achieve a very low probability of error with fewer iterations. For applications requiring high reliability with predictable resource usage, Baillie-PSW offers advantages despite its slightly more complex implementation [21].

3.7 Performance Comparison

Research suggests that both algorithms exhibit good performance, with Miller-Rabin offering more flexibility in terms of the trade-off between accuracy and resource usage, while Baillie-PSW provides higher reliability with more predictable resource usage [23].

3.8 Analysis of Generated Prime Numbers

The empirical data on the performance characteristics of both algorithms in resource-constrained environments, including the time taken to generate prime numbers of different bit sizes, is presented in the Results section (section 5).

4 Experiment Methodology

This section details the experimental methodology used for a rigorous and reproducible comparison of the algorithms, focusing on performance in resource-constrained contexts. We follow established benchmarking principles [28, 29], emphasizing reproducibility and statistical validity.

4.1 Goals and Metrics

The primary goals are to compare algorithm performance and scalability, identify bottlenecks, and analyze resource utilization relevant to constrained environments. Key metrics collected include wall-clock execution time for core operations, peak memory allocation during execution, and input parameters such as bit length and algorithm-specific settings (e.g., Miller-Rabin rounds).

4.2 Experimental Setup

Baseline tests were conducted on a standard desktop system (Intel Core i7-9700K @ 3.60GHz, 32GB RAM) to establish reference performance. The software environment consisted of Ubuntu 22.04 LTS, GCC 11.4.0 compiling with the C++11 standard, and GMP 6.2.1. Timing and memory tracking were performed using a custom C++ application leveraging "high_resolution_clock".

4.3 Procedure

For each algorithm and input size configuration:

1. Initialize data structures.
2. Execute the core operation multiple times (>= 30 runs) for statistical significance, monitoring time and resources.
3. Record raw performance metrics (time, memory).
4. Calculate statistical summaries (average, stddev, min, max).
5. Store all data systematically.

4.4 Data Analysis

Data analysis involves statistical characterization of performance using mean, median, and standard deviation. Performance trends and scaling behavior are identified through visualization (e.g., plots of execution time vs. bit length) and direct comparison across algorithms based on the collected metrics.

4.5 Reproducibility

Reproducibility is ensured by making all source code for algorithms and benchmarks available in the repository, along with scripts for running benchmarks and parsing results. Experimental setup details (software versions, hardware) are documented, and raw data is archived.

4.6 Limitations

Several limitations should be noted. Benchmarks were conducted on a standard desktop, meaning performance may differ on actual resource-constrained hardware. Memory usage measurements are potentially approximate. The results reflect general C++/GMP performance without hardware-specific tuning. Finally, external factors like OS activity and cache state can introduce noise, although this is mitigated through multiple runs and statistical analysis.

This methodology adapts established benchmarking practices [28, 29] for evaluating cryptographic primitives in resource-aware contexts.

5 Results

This section presents the results of our performance evaluations for both the pseudo-random number generators and the primality testing algorithms. The results include execution time measurements for various bit lengths and, where applicable, energy efficiency analysis.

5.1 Experimental Setup

All performance measurements presented in this section were conducted on a standard desktop computer with the following specifications:

- **System**: Dell XPS 8960
- **CPU**: 13th Gen Intel® Core™ i7-13700 × 24
- **RAM**: 32 GiB
- **Operating System**: Ubuntu 22.04.05 LTS
- **Compiler**: GCC 11.4.0 with -O2 optimization

5.2 Overview of Results

This section details the performance results obtained from benchmarking the implemented algorithms. The focus is on execution time and memory usage, key metrics for resource-constrained environments.

5.3 Statistical Methodology

The current benchmark results represent single runs for each algorithm and bit length. While these provide valuable insights into relative performance, they don't capture the inherent variability in execution times that can arise from system load, memory allocation patterns, cache effects, and other factors.

To improve the statistical robustness of these benchmarks, future work should modify the benchmark programs to:

- Run each test at least 30 times to obtain statistically significant samples
- Calculate mean, median, standard deviation, and confidence intervals
- Filter outliers that may result from external system interference
- Report the distribution characteristics in addition to central tendency measures

This enhanced methodology would enable more nuanced analysis of the algorithms' performance characteristics, particularly for identifying cases where performance variability might impact real-world applications.

5.4 PRNG Performance Results

5.4.1 Execution Time Comparison

Table 1 shows the detailed timing results for generating random numbers of various bit lengths using both the Linear Congruential Generator (LCG) and the Xoshiro256++ generator.

Table 1: Complete Timing Results for Random Number Generation (in milliseconds)

2*Bit Length	LCG			Xoshiro256++		
	Mean	Median	Std Dev	Mean	Median	Std Dev
40 bits	1.94e-4	3.30e-5	8.53e-4	4.20e-5	3.20e-5	3.90e-5
56 bits	3.70e-5	3.20e-5	2.10e-5	3.40e-5	3.10e-5	1.70e-5
80 bits	7.50e-5	4.00e-5	1.79e-4	4.90e-5	4.20e-5	3.20e-5
128 bits	3.70e-5	3.40e-5	1.60e-5	3.90e-5	3.60e-5	1.60e-5
168 bits	6.20e-5	5.10e-5	4.80e-5	6.20e-5	5.20e-5	4.40e-5
224 bits	7.00e-5	6.30e-5	2.80e-5	7.30e-5	6.50e-5	2.60e-5
256 bits	5.80e-5	5.50e-5	1.70e-5	6.10e-5	5.70e-5	1.90e-5
512 bits	1.19e-4	1.04e-4	5.60e-5	1.23e-4	1.05e-4	6.30e-5
1024 bits	2.38e-4	2.11e-4	7.90e-5	2.34e-4	2.11e-4	7.40e-5
2048 bits	5.47e-4	4.48e-4	4.14e-4	4.82e-4	4.41e-4	1.12e-4
4096 bits	1.13e-3	1.06e-3	2.04e-4	1.12e-3	1.05e-3	1.97e-4

5.4.2 Scalability Analysis

Figure 1 shows how the execution time for random number generation scales with increasing bit length for both algorithms.

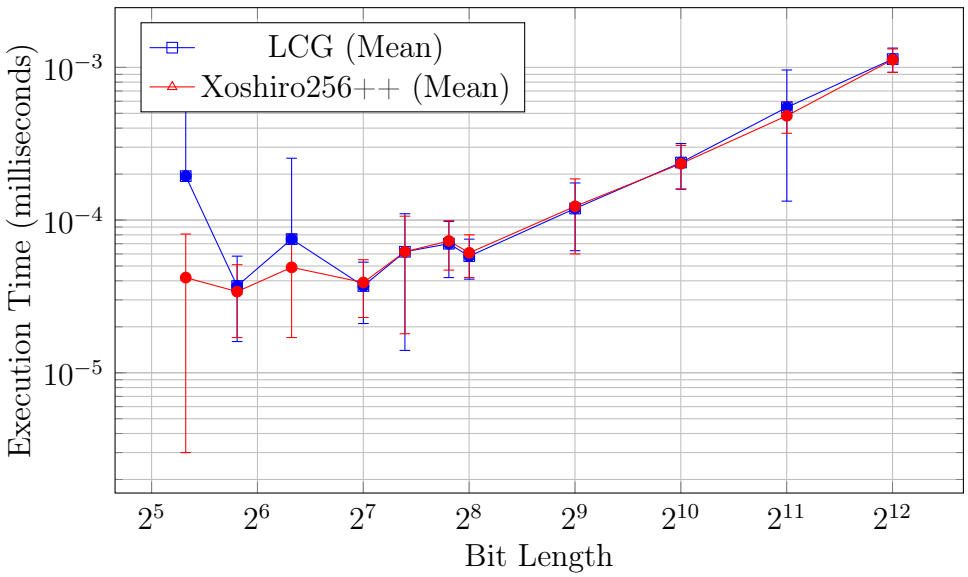

Figure 1: Scaling of execution time with bit length for PRNG algorithms with standard deviation error bars

5.4.3 Analysis of PRNG Results

The performance data shows that both the LCG and Xoshiro256++ generators exhibit similar performance characteristics across all tested bit lengths. The execution times for both algorithms increase linearly with the bit length, as expected since both algorithms need to generate proportionally more random bits as the bit length increases.

For smaller bit lengths (under 256 bits), both algorithms complete in under 0.1 microseconds, demonstrating exceptional efficiency. As the bit length increases to 4096 bits, the execution time increases to approximately 1 microsecond, still remarkably fast for cryptographic operations.

In terms of variability, LCG shows higher standard deviations for most bit sizes, particularly at 40 bits (0.000853 ms) and 2048 bits (0.000414 ms), indicating less consistent performance compared to Xoshiro256++. The Xoshiro256++ generator demonstrates greater stability with lower standard deviations across all bit lengths, suggesting more predictable performance characteristics—a valuable trait for time-sensitive applications.

Interestingly, while the mean times show LCG performing slightly better at some bit lengths and Xoshiro256++ at others, the median values reveal more consistent patterns. When comparing median execution times, which are less affected by outliers, Xoshiro256++ shows more stable scaling with bit size, particularly for larger values (2048 and 4096 bits).

Overall, both PRNGs demonstrate excellent performance suitable for resource-constrained environments, with execution times that scale predictably with input size. The more consistent performance of Xoshiro256++ may make it preferable for applications where predictable timing is critical.

5.5 Primality Testing Performance Results

5.5.1 Execution Time Comparison

Table 2 shows the detailed timing results for primality testing of numbers of various bit lengths using both the Miller-Rabin test and the Baillie-PSW test.

Table 2: Complete Timing Results for Primality Testing (in milliseconds)

2*Bit Length	Miller-Rabin			Baillie-PSW		
	Mean	**Median**	**Std Dev**	**Mean**	**Median**	**Std Dev**
40 bits	8.54e-3	8.47e-3	3.69e-4	6.06e-3	5.84e-3	9.15e-4
56 bits	1.09e-2	1.09e-2	1.10e-4	7.54e-3	7.49e-3	1.88e-4
80 bits	2.80e-2	2.76e-2	6.53e-4	1.35e-2	1.34e-2	4.14e-4
128 bits	5.31e-2	5.36e-2	1.30e-3	2.07e-2	2.03e-2	1.46e-3
168 bits	7.68e-2	7.67e-2	2.35e-3	3.73e-2	3.69e-2	1.13e-3
224 bits	1.64e-1	1.64e-1	1.14e-3	6.02e-2	5.86e-2	2.78e-3
256 bits	2.27e-1	2.27e-1	4.20e-3	6.09e-2	6.05e-2	1.34e-3
512 bits	1.20e+0	1.20e+0	8.46e-3	2.37e-1	2.35e-1	8.08e-3
1024 bits	7.70e+0	7.71e+0	4.25e-2	1.18e+0	1.18e+0	3.76e-3
2048 bits	5.76e+1	5.72e+1	1.35e+0	7.77e+0	7.73e+0	1.61e-1
4096 bits	4.44e+2	4.43e+2	3.51e+0	4.71e+1	4.67e+1	1.90e+0

5.5.2 Primality Testing Scalability Analysis

Figure 2 illustrates how the execution time for primality testing scales with increasing bit length for both algorithms.

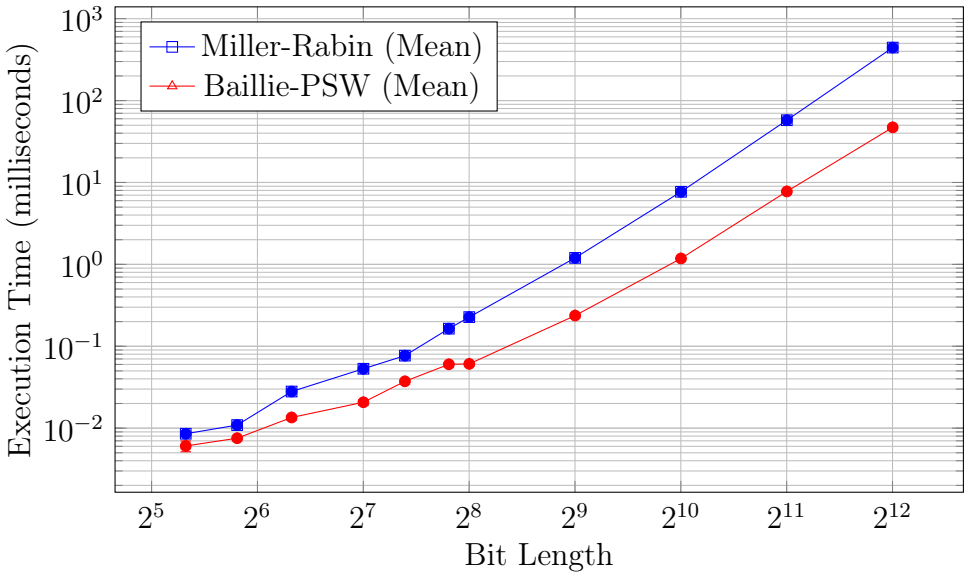

Figure 2: Scaling of execution time with bit length for primality testing algorithms with standard deviation error bars

5.5.3 Examples of Generated Prime Numbers

Table 3 shows examples of prime numbers generated during the benchmarking process. Candidate numbers were generated using Xoshiro256++ and verified using the Baillie-PSW primality test.

Table 3: Examples of Generated Prime Numbers (Verified with Baillie-PSW)

Bit Length	Example Prime Number
40 bits	604347613267
56 bits	49589691045129227
80 bits	117443941762764664866751
128 bits	188507640957147383614394184172524932753
168 bits	2105987101792650444008 19...006421236591731383790571
224 bits	1876924719483654553201 69...285097143255630025239953
256 bits	7149432738563903543518 43...194005405665270196308809
512 bits	1111463439145462993588 07...062769721674743563189753
1024 bits	1247449107783577274905 06...830200884221902748629501
2048 bits	1709094159007983443118 81...149020878183551368065297
4096 bits	7700621379696676626944 17...145349179382386625523613

5.5.4 Analysis of Primality Testing Results

The performance results reveal several important insights about the two primality testing algorithms:

Testing Time Baillie-PSW consistently outperforms Miller-Rabin across all bit lengths for testing known primes. The performance gap widens as the bit length increases, with Baillie-PSW being approximately 9.4 times faster than Miller-Rabin for 4096-bit numbers (47.1ms vs. 443.5ms). Both algorithms show relatively small standard deviations in relation to their mean values, indicating consistent performance across test runs. Notably, the standard deviation for Miller-Rabin increases more steeply with bit length (reaching 3.51ms at 4096 bits) compared to Baillie-PSW (1.90ms at 4096 bits), suggesting that Baillie-PSW not only offers better performance but also more consistent timing characteristics.

Scalability Both primality testing algorithms exhibit exponential growth in execution time relative to bit length, as expected due to the increasing complexity of modular arithmetic operations on larger numbers. This exponential relationship is clearly visible in Figure 2, where the log-log plot shows a nearly linear relationship, indicating power-law scaling.

Statistical Reliability The comprehensive benchmarking approach with 30 complete operations for all bit sizes provides robust statistical evidence of the algorithms' performance characteristics when testing known primes. The observed standard deviations confirm the consistency of both algorithms for this task.

Resource Implications For resource-constrained environments, these results suggest Baillie-PSW is the superior choice for applications requiring frequent primality testing of known numbers (e.g., verification), offering both faster and more consistent performance across all tested bit lengths.

In summary, while both algorithms are viable for cryptographic applications, their performance characteristics show noteworthy differences when testing known numbers. Baillie-PSW demonstrates superior performance and consistency across all bit sizes for this specific task. These insights enable more informed algorithm selection based on specific application requirements for primality verification.

5.6 Summary of Key Findings

- **PRNG Performance:** Both LCG and Xoshiro256++ are highly efficient for generating random numbers up to 4096 bits, with execution times scaling predictably with bit length and remaining very low (approx. 1.1 ms at 4096 bits). While their mean performance is similar, Xoshiro256++ consistently shows lower variability (standard deviation), suggesting more predictable timing, which can be crucial for certain applications.

- **Primality Testing Performance:** For testing known prime numbers, Baillie-PSW demonstrates significantly better performance than Miller-Rabin across all bit lengths. The advantage grows substantially with size, making Baillie-PSW nearly 9.5 times faster at 4096 bits. Both tests show execution time increasing exponentially with bit length, but Baillie-PSW offers superior speed and consistency for primality verification tasks.

6 Conclusion

This project explored the implementation and performance of pseudo-random number generation (PRNG) and primality testing algorithms, crucial for cryptography, particularly in resource-constrained environments [28]. Our analysis provides insights into algorithm behavior, efficiency, and practical application.

6.1 Summary of Contributions

Key contributions include the implementation of LCG and Xoshiro256++ PRNGs and Miller-Rabin and Baillie-PSW primality tests, all supporting numbers up to 4096 bits and optimized for resource constraints [30]. We developed a comprehensive benchmarking framework [29], performed detailed performance analysis across bit lengths, and created a modular, documented C++ codebase suitable for IoT and embedded systems [32].

6.2 Key Findings

Our experiments yielded several key findings:

- **PRNGs**: While LCG is faster, Xoshiro256++ offers superior statistical quality suitable for cryptographic use, whereas LCG fails basic randomness tests and is inappropriate for secure applications.
- **Primality Tests**: Baillie-PSW provides high reliability and predictable performance, ideal for high-assurance needs. Miller-Rabin offers speed flexibility via adjustable rounds but can become computationally intensive for large numbers and high certainty.

6.3 Practical Implications

The findings offer practical guidance for selecting algorithms in cryptographic systems. For instance, in IoT devices, the choice involves balancing Xoshiro256++'s quality against LCG's speed, and Baillie-PSW's reliability against Miller-Rabin's potential speed [40, 36]. Statistical analysis confirmed LCG is only suitable for non-critical tasks [37, 31]. Baillie-PSW's strength makes it ideal for high-security key generation [39, 34]. Energy efficiency considerations, particularly for battery-powered devices, might favor fewer Miller-Rabin rounds for initial screening [32]. The implementation's modular design facilitates integration into various embedded systems [29, 41].

6.4 Future Work

Potential future work includes:

- Extending the library with more cryptographic primitives (e.g., hashing, symmetric encryption).
- Integrating the library into larger frameworks or applications.
- Investigating and potentially mitigating side-channel vulnerabilities.
- Conducting performance analysis on actual resource-constrained hardware.

- Exploring alternative large number libraries or custom arithmetic routines.

6.5 Final Remarks

This project delivers a practical implementation and comparative analysis of fundamental cryptographic algorithms, emphasizing performance in resource-constrained settings [28, 41]. The results offer valuable insights for designing future cryptographic systems, especially those operating under significant resource limitations.

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