

Pricing the CBOE VIX Futures with the Heston–Nandi GARCH Model

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We propose a closed-form pricing formula for the Chicago Board Options Exchange Volatility Index (CBOE VIX) futures based on the classic discrete-time Heston–Nandi GARCH model. The parameters are estimated using several sets of data, including the S&P 500 returns, the CBOE VIX, VIX futures prices and combinations of these data sources. Based on the resulting empirical pricing performances, we recommend the use of both VIX and VIX futures prices for a joint estimation of model parameters. Such estimation method can effectively capture the variations of the market VIX and the VIX futures prices simultaneously for both in-sample and out-of-sample analysis. © 2016 Wiley Periodicals, Inc. *Jrl Fut Mark*

1. INTRODUCTION

The idea of using derivatives of market volatility to manage financial risk can be traced back to long before the Chicago Board Options Exchange (CBOE) developed its Volatility Index (VIX). Brenner and Galai (1989) introduced a volatility index (the Sigma index) and discussed derivatives such as options and futures in relation to this index. Following this idea, Whaley (1993) introduced the old version of the VIX, which depended on the inversion of the Black–Scholes formula. However, standardized derivative contracts on the VIX were not available until the CBOE calculated the VIX on a model-free basis in 2003. Since the introduction of VIX futures in 2004 and of VIX options in 2006, volatility derivatives have become a popular set of derivatives in the market, especially after the subprime crisis.

Several models have been proposed for pricing VIX futures and other volatility derivatives. Zhang and Zhu (2006) first studied VIX futures with the Heston model.¹ Zhu and

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JEL Classification: C19, C22, C80

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Received December 2015; Accepted August 2016

¹Proposed by Heston (1993) for option pricing.

Zhang (2007) also proposed a non-arbitrage model for VIX futures, based on VIX term structures. In considering the possible jumps in log-returns, Duffie, Pan, and Singleton (2000) proposed an affine jump-diffusion process for log-returns, which soon became a new benchmark process in the asset pricing literature. On the basis of this process and its modifications, Lin (2007) proposed a stochastic volatility model with simultaneous jumps in both returns and volatility to price VIX futures, which yielded an approximation formula. Sepp (2008) added jumps into the square root mean-reverting process to price VIX options. Zhang, Shu, and Brenner (2010) included additional stochastic long-run variance into the square root mean-reverting process that linked the VIX and VIX futures prices. By adding jumps to the classical Heston model, Zhu and Lian (2012) found an analytical pricing formula for VIX futures.² Using intraday data, Frijns, Tourani-Rad, and Webb (2016) found strong evidence for bi-directional Granger causality between the VIX and the VIX futures.

Despite the development of a significant literature on the stochastic process for volatility derivatives, little attention has been paid to discrete-time GARCH family models. Studies on volatility derivatives under the GARCH framework have mainly focused on equity option pricing (e.g., Christoffersen, Jacobs, Ornathanalai, & Wang, 2008; Christoffersen, Feunou, Jacobs, & Meddahi, 2014; Duan, 1995, 1999; Heston & Nandi, 2000; Duan, Ritchken, & Sun, 2005).³ To the best of our knowledge, little (if any) literature exists on the pricing of VIX derivatives under the GARCH framework. One possible reason is that the conventional local risk-neutral valuation relationship (LRNVR) only compensates for the equity risk premium, and thus there is no room for an independent variance risk premium within a single shock in the GARCH models. To overcome this problem, recent studies have estimated parameters with information from both the underlyings and the risk-neutral measures (such as option prices and the VIX). For example, Hao and Zhang (2013) suggested that such a joint estimation can significantly improve the GARCH model's ability in fitting the market VIX. Kannianen, Lin, and Yang, (2014) showed that joint estimation with VIX data can greatly improve the GARCH model's option pricing performance. These results suggest that the GARCH models could potentially be used in volatility derivatives pricing, when an appropriate estimation method is adopted.

To fill this gap in the literature on volatility derivatives pricing, we investigate the pricing of VIX futures with discrete-time GARCH-type models. One appealing advantage of GARCH-type models is their convenience in conducting parameter estimations. Unlike stochastic volatility models with their unobservable volatility shocks, the underlying volatility process in GARCH models is recursively observable, and thus the estimation is straightforward using the maximum likelihood estimation (MLE). For a large sample of futures prices with a substantial cross-sectional dimension over a long period, it is important to use a less computationally demanding model to implement the estimation procedure. In this paper, we discuss VIX futures pricing under the classic discrete-time Heston–Nandi GARCH model of Heston and Nandi (2000), which is very popular in the option pricing literature. We derive an explicit pricing formula for VIX futures via an integration of a transformed moment-generation function of the conditional volatility. Several estimation methods are provided that differ in terms of the data used, and their pricing performances are investigated. Among these methods, the model estimated by jointly using the VIX and VIX futures prices yields a satisfying pricing performance and a good balance in fitting both the VIX and VIX futures. We also conduct a rolling window out-of-sample analysis and find similar empirical results. Such facts indicate the model is free from in-sample overfitting when proper joint estimation is used.

²Luo and Zhang (2014) provide a good discussion on the literature of VIX derivatives pricing.

³Christoffersen, Jacobs, and Ornathanalai (2012) provided an extensive review of equity option pricing with GARCH family models.

The remainder of the paper is structured as follows. Section 2 derives the pricing formula for the model-implied VIX and VIX futures. Section 3 discusses several model estimation methods. Section 4 summarizes and discusses the empirical results, and Section 5 offers conclusions and recommendations.

2. THE MODEL

2.1. Heston–Nandi GARCH Model and Risk Neutralization

We assume that the returns of the S&P 500 index follow the Heston–Nandi GARCH model under the physical (P) measure.

$$R_{t+1} = r_{t+1} + \lambda h_{t+1} - \frac{1}{2} h_{t+1} + \sqrt{h_{t+1}} \varepsilon_{t+1} \quad h_{t+1} = \omega + \beta h_t + \alpha (\varepsilon_t - \delta \sqrt{h_t})^2 \quad (2.1)$$

where ε_t follows a standard normal distribution, r_t is the risk-free interest rate, R_t is the log-return of the index, and λ is the equity premium parameter that is associated with the conditional variance. The second equation defines the volatility dynamics and nests both the leverage effect and the volatility clustering effect, which are common effects in financial asset returns.

Under the LRNVR proposed by Duan (1995), we obtain the following risk-neutral (Q) dynamics:

$$R_{t+1} = r_{t+1} - \frac{1}{2} h_{t+1} + \sqrt{h_{t+1}} \varepsilon_{t+1}^* \quad h_{t+1} = \omega + \beta h_t + \alpha (\varepsilon_t^* - \delta^* \sqrt{h_t})^2 \quad (2.2)$$

where $\varepsilon_t^* = \varepsilon_t + \lambda \sqrt{h_t}$, $\delta^* = \delta + \lambda$. The unconditional expectation of h_t under the Q measure (also called the long-run variance) is

$$\bar{h} = E^Q(h_t) = \frac{\tilde{\omega}}{1 - \tilde{\beta}} \quad (2.3)$$

where $\tilde{\omega} = \omega + \alpha$, and $\tilde{\beta} = \beta + \alpha(\delta + \lambda)^2$. As the persistency of the conditional variance is related to $\tilde{\beta}$, we call $\tilde{\beta}$ as the persistent parameter of the model under the Q measure. The corresponding persistence parameter under the P measure is $\beta + \alpha\delta^2$.

2.2. The Model-Implied VIX and Volatility Term Structure

According to the CBOE (2015) and related papers, such as the study by Hao and Zhang (2013), the VIX can be calculated as the annualized arithmetic average of the expected daily variance over the following month under the risk-neutral measure, which is

$$\left(\frac{VIX_t}{100} \right)^2 = \frac{1}{n} \sum_{k=1}^n E_t^Q[h_{t+k}] \times AF \quad (2.4)$$

where h_t is the instantaneous daily variance of the return of the S&P 500. AF is the annualizing factor that converts daily variance into annualized variance by holding the daily variance constant over a year. For simplicity, the implied volatility term structure at time t with maturity of n , or $V_t(n)$, is defined as the average expected volatility over the next n trading days:

$$V_t(n) = \frac{1}{n} \sum_{k=1}^n E_t^Q[h_{t+k}] \quad (2.5)$$

Assume that there are 22 trading days in 1 month and 252 trading days in 1 year. The VIX_t is then linked to $V_t(22)$ through $VIX_t = 100\sqrt{252V_t(22)}$.

The affine structure of the Heston–Nandi GARCH model provides the following linear relationship between the model-implied volatility term structure and the conditional variance.

Proposition 1: If the S&P 500 return follows the Heston–Nandi GARCH model presented in (2.2), then the implied volatility term structure at time t is a weighted average of the current conditional variance of the next period and the long-run variance of the model under the risk-neutral measure,

$$V_t(n) = (1 - \Gamma(n))\bar{h} + \Gamma(n)h_{t+1} \quad (2.6)$$

where $\bar{h} = \frac{\tilde{\omega}}{1 - \tilde{\beta}}$ and $\Gamma(n) = \frac{1 - \tilde{\beta}^n}{n(1 - \tilde{\beta})}$. Specifically, the model-implied VIX_t is then given by

$$VIX_t = 100 \times \sqrt{252V_t(22)} = 100 \times \sqrt{252((1 - \Gamma(22))\bar{h} + \Gamma(22)h_{t+1})} \quad (2.7)$$

Proof: See the Appendix.

Proposition 1 develops the model, which suggests a no-arbitrage pricing relationship between the implied volatility term structure and the return information, as the current conditional variance of the next period (h_{t+1}) is a function of the current and past daily returns. This relationship could also be seen as the link between the information from the physical measure (i.e., returns) and the information from the risk-neutral measure (i.e., the VIX).

In Equation (2.6), the maturity-dependent weighting coefficients of the current conditional variance, $\Gamma(n)$, determine the shape of the implied volatility forward curve. This parameter converges to zero when the time to maturity goes to infinity. As a result, the volatility term structure will eventually become flat at the level of long-run variance of the return. The $\Gamma(n)$ is determined by both the time to maturity and the persistent parameter, $\tilde{\beta}$, which controls the speed of convergence of the implied volatility to the long-run variance. The higher the $\tilde{\beta}$, the slower the convergence speed. The level of the current conditional variance determines the slope of the volatility term structure curve. The term structure curve will be upward sloping when the current conditional volatility is lower than the long-run variance ($h_{t+1} < \bar{h}$), and it will be downward sloping when the current conditional volatility level is relatively higher ($h_{t+1} > \bar{h}$). Luo and Zhang (2012) has derived a similar formula for implied volatility term structures as in *Proposition 1* under continuous-time stochastic volatility models.

2.3. The Pricing Formula of VIX Futures

The advantage of using the Heston–Nandi GARCH model is that it ensures a closed-form pricing formula for the VIX futures. This formula substantially simplifies the estimation procedure when the objective function of parameter optimization includes the VIX futures pricing error. For the sake of convenience, we rewrite (2.7) as

$$VIX_t = 100 \times \sqrt{a + bh_{t+1}}$$

where a and b are defined as $a = 252 \times (1 - \Gamma(22))\bar{h}$ and $b = 252 \times \Gamma(22)$.

The price of the VIX futures can be interpreted as its conditional expectation at maturity time T , evaluated at the current time t under the risk-neutral measure, which allows the following expression given by *Proposition 1* of Zhu and Lian (2012):

$$F(t, T) = E_t^Q[VIX_T] = E_t^Q[100\sqrt{a + bh_{T+1}}] = \frac{100}{2\sqrt{\pi}} \int_0^\infty \frac{1 - e^{-sa} E_t^Q[e^{-sbh_{T+1}}]}{s^{3/2}} ds \quad (2.8)$$

where t is the current date, and $T - t$ is the time to maturity.

The last term, $E_t^Q[e^{-sbh_{T+1}}]$, can be expressed as the moment-generating function of the conditional variance at time $T + 1$. Under the Heston–Nandi GARCH model, this function allows a closed-form solution, as stated in the following proposition.

Proposition 2: The moment-generating function of the conditional variance h_{t+m} (m days from now) at time t is exponentially affine in h_{t+1} , and it allows the following expression:

$$f(\phi, m, h_{t+1}) = E_t^Q[e^{\phi h_{t+m+1}}] = e^{C(\phi, m) + H(\phi, m)h_{t+1}} \quad (2.9)$$

where the functions $C(\phi, m)$ and $H(\phi, m)$ are given by an iterative relationship:

$$C(\phi, n + 1) = C(\phi, n) - \frac{1}{2} \ln(1 - 2\alpha H(\phi, n)) + \omega H(\phi, n) \quad (2.10)$$

$$H(\phi, n + 1) = \frac{\alpha \delta^* H(\phi, n)}{1 - 2\alpha H(\phi, n)} + \beta H(\phi, n) \quad (2.11)$$

with the initial conditions of

$$C(\phi, 0) = 0, H(\phi, 0) = \phi \quad (2.12)$$

where the parameters are defined in the Heston–Nandi GARCH model. Proof: See the Appendix.

Hence, the VIX futures price under the Heston–Nandi GARCH model can be calculated by

$$F(t, T) = \frac{100}{2\sqrt{\pi}} \int_0^\infty \frac{1 - e^{-sa} f(-sb, T - t, \frac{(VIX_t/100)^2 - a}{b})}{s^{3/2}} ds \quad (2.13)$$

where the function $f(\cdot)$ is defined in *Proposition 2*. It is worth to mention that equation (2.13) does include β , α , and δ^* through the moment generating function $f(\cdot)$ even when the market VIX is used for calculating futures prices. Thus, these parameters are identifiable.

Equation (2.13) develops the pricing relationship between the contemporaneous spot price of the VIX and its futures prices. This equation also defines the whole forward curve, whose shape is determined by the current VIX, the long-run variance, and $\Gamma(22)$ as a function of the persistence parameter $\tilde{\beta}$.

3. DATA AND ESTIMATION

Basically, we have three daily frequency data sources: the log-return series of the S&P 500 index, the quote series of the CBOE VIX and the panel of VIX futures prices with various maturities. The first two series are collected from Yahoo Finance, and the last one is collected

from the CBOE's website. All of them range from March 2004 to December 2013 with 2451 returns/VIX observations and 14501 VIX futures prices. A summary of statistics for the dataset is shown in Tables I and II.

Given the three sources of information, namely the S&P 500 index return, the CBOE VIX and the VIX futures, we have a number of options for estimating model parameters. We propose five methods, which differ in terms of the information used. The first three methods each use only one of the information sources, and the last two methods use combinations of two information sources.

The first approach, which is the most straightforward, is to run a maximum likelihood estimation (MLE) with the S&P 500 returns only (hereafter we denote this as the "Return only" approach). The likelihood function is obtained from the \mathbf{P} dynamics:

$$\ln L_R = -\frac{M}{2} \ln(2\pi) - \frac{1}{2} \left\{ \sum_{t=1}^M \ln(h_t) + [R_t - r_t - \lambda h_t + \frac{1}{2} h_t]^2 / h_t \right\} \quad (3.1)$$

where h_t is updated by the conditional variance process, and M is the number of observations of returns. Such estimation provides us with parameters under physical dynamics. Due to the simplicity of both the Heston–Nandi GARCH model and the LRNV, this kind of estimation can obtain all of the parameters needed to recover its risk-neutral dynamics and, therefore, this approach is sufficient to calculate the model-implied VIX and VIX futures prices.

The second approach is to calibrate⁴ the parameters with the VIX data only (we denote this hereafter as the "VIX only" approach). We assume the following distribution for pricing errors as in Hao and Zhang (2013):

$$u_t = (VIX_t^{Mkt} - VIX_t^{Mod}) / (\sqrt{252} \times 100) \quad u_t \sim i.i.dN(0, s_V^2)$$

s_V^2 is estimated by using the sample variance of pricing errors.⁵ Then, we have the log-likelihood function:

$$\ln L_V = -\frac{M}{2} \ln(2\pi s_V^2) - \frac{1}{2 s_V^2} \sum_{t=1}^M u_t^2 \quad (3.2)$$

This method focuses on fitting the VIX only, instead of the S&P500 return. As the VIX is the underlying spot asset of the VIX futures, we postulate that this index contains more relevant information than the S&P 500 returns.

The third approach is to calibrate the parameters with the VIX futures prices only (which we denote hereafter as the "Futures only" approach). This approach is direct, and common in practice, regardless of rationality. In terms of dealing with pricing error, this approach is supposed to have the best performance, because its objective function only concerns the pricing error of VIX futures. However, this approach may lead to substantial distortions when we use the calibrated parameters for implied VIX calculation. The corresponding likelihood function is

⁴Although we are using a maximum likelihood estimation (MLE) notation, this method is essentially a calibration method that matches the model-implied variables to the corresponding market variables. The name of each model suggests the objection function during parameter estimation. For example, "Futures only" maximizes the likelihood function of VIX futures data, whereas "VIX + Futures" maximizes the joint likelihood function of both VIX and VIX futures data.

⁵The model implied VIX is calculated with Equation (2.7) and h_{t+1} is filtered from return series by Equation (2.2).

TABLE I
Summary of Statistics for S&P 500 Returns and VIX

	<i>N</i>	<i>Mean</i>	<i>Median</i>	<i>Std</i>	<i>Max</i>	<i>Min</i>	<i>Skew</i>	<i>Kurt</i>
Return	2451	2.0E-04	7.8E-04	0.01	0.11	−0.09	−0.33	10.9
VIX	2451	20.27	17.25	10.00	80.86	9.89	2.29	6.72

Note. Kurt is the excess kurtosis.

TABLE II
Summary of Statistics for VIX Futures

	<i>N</i>	<i>Mean</i>	<i>Std</i>	<i>Max</i>	<i>Min</i>	<i>Skewn</i>	<i>Kurt</i>
VIX							
<15	3809	15.54	2.51	27.60	10.37	1.44	3.42
15–20	4496	21.46	3.37	31.00	13.50	0.05	−0.54
20–25	2848	25.52	2.98	33.35	17.38	0.30	−0.58
25–30	1339	27.96	2.92	34.55	21.10	−0.13	−0.83
>30	2009	36.02	6.98	66.23	22.15	0.92	1.13
Basis							
<−6	692	36.09	8.40	66.23	22.15	0.59	0.10
−6 to −3	520	31.29	9.23	63.88	14.42	0.53	0.33
−3 to +3	7311	21.43	7.56	63.71	10.37	1.16	1.49
3–6	3899	21.99	4.89	51.13	13.08	0.46	0.44
>6	2079	26.20	3.26	34.20	17.35	0.13	−0.38
Maturity							
<50	3641	21.42	8.87	66.23	10.37	1.70	3.53
50–100	3365	23.21	7.77	59.77	12.21	1.15	1.74
100–150	3082	24.06	6.97	50.58	13.16	0.70	0.40
150–200	2785	24.77	6.36	45.26	13.52	0.41	0.03
>200	1628	23.90	5.50	38.80	14.32	0.17	−0.56

Note. Basis = VIX index—VIX futures price. Kurt is the excess kurtosis.

$$\ln L_F = -\frac{N}{2} \ln(2\pi \hat{s}_F^2) - \frac{1}{2\hat{s}_F^2} \sum_{i=1}^N (Fut_i^{Mkt} - Fut_i^{Mod})^2 \quad (3.3)$$

where N is the total number of observations of the VIX future prices panels, and \hat{s}_F^2 is the sample variance of the futures pricing errors.⁶

In addition to the three above-described straightforward estimation approaches, we consider two additional estimation methods that use combined information. The fourth approach is to run a joint estimation with both the S&P 500 returns and the VIX (which we denote hereafter as the “Return + VIX” approach). This kind of joint method is popular in current pricing studies that aim to better reconcile \mathbf{P} and \mathbf{Q} measure information.⁷ The parameters can be obtained by maximizing the joint log-likelihood function as follows:

$$\ln L_{VR} = \ln L_V + \ln L_R \quad (3.4)$$

⁶The market VIX is used when calculating the VIX futures prices with Equation (2.13).

⁷See Kannianen et al. (2014).

The fifth estimation approach is to run a joint estimation with the VIX and the VIX futures prices (hereafter denoted as the “VIX + Futures” approach). That is, we estimate parameters by maximizing the joint log-likelihood function as

$$\ln L_{VF} = \ln L_V + \ln L_F \quad (3.5)$$

Compared with the other estimation methods provided above, this last method takes into account the fitting errors of both the VIX and the VIX futures prices, and it finds a good balance between fittings of the underlying spot assets (VIX) and the derivatives (VIX futures).

To insure the stationarity of the volatility process, the following constraint on the parameters is imposed, in addition to the positivity constraint of all parameters.

$$\beta + \alpha \delta^* < 1$$

Note that $\delta^* = \delta + \lambda$. The positivity of λ and δ implies that this \mathbf{Q} measure stationary condition also ensures \mathbf{P} measure stationarity.

For estimation approaches in which returns data are not used, only parameters under the \mathbf{Q} measure could be identified. More specifically, the equity risk premium parameter λ cannot be identified separately from δ^* for “VIX”, “Futures”, and “VIX + Futures”. For those three methods, $\ln L_R$ could not be calculated either.

4. EMPIRICAL RESULTS

4.1. Estimated Parameters

In Table III, we present our parameter estimation results for the Heston–Nandi GARCH model using the five proposed estimation approaches. The first row denotes the estimation approach used. For example, the “Ret” column denotes the estimation by “Return only.” The last five rows present the values of the log-likelihood functions from each estimation

TABLE III
Parameters of the Heston–Nandi GARCH Model Under Different Methods

	<i>Ret</i>	<i>Ret + VIX</i>	<i>VIX</i>	<i>Fut</i>	<i>VIX + Fut</i>
β	0.7638 (0.003)	0.6763 (0.002)	0.6819 (0.001)	0.9954 (0.000)	0.7743 (0.002)
α (10E-6)	3.4109 (0.009)	1.1314 (0.007)	2.3235 (0.005)	1.2264 (0.001)	1.4468 (0.000)
δ^*	249.3476 (1.490)	529.3705 (3.787)	365.2518 (1.125)	5.4066 (0.005)	390.7377 (4.748)
\bar{h} (10E-4)	1.2007 (0.091)	1.708 (0.092)	2.8511 (0.082)	2.6863 (0.028)	2.9990 (0.038)
λ	2.5189 (1.098)	3.0367 (0.370)			
$\bar{\beta}$	0.9759	0.9934	0.9919	0.9954	0.9952
logL					
L_R	7895	7736	—	—	—
L_{RV}	17989	18419	—	—	—
L_V	9953	10476	10956	9852	10775
L_F	−49836	−42521	−39517	−37964	−38155
L_{VF}	−39883	−32045	−28561	−28113	−27380

Note. Robust standard errors are in parenthesis. The bold number in Log-likelihood indicates the largest within each row. Ret, Return only; Ret + VIX, Return + VIX; VIX, VIX only; Fut, Futures only; VIX + Fut, VIX + Futures. For methods without return data, the log-likelihood of return is not available as λ is not identified.

approach, in which the bolded value is the conventionally reported likelihood value.⁸ Robust standard errors are provided in parentheses. We report the values of λ and $\ln L_R$ only when they are identified.

The most notable finding in Table III is the obvious parameter distortion in the estimation by the “Futures only” method. The β increases significantly, and gets very close to one, which is its upper limit. In contrast, the δ^* drops greatly to 5.41.⁹ Such parameter distortion indicates a great smoothness in the model-implied VIX. With such large β values, the conditional volatility at time t almost totally determines the conditional volatility at time $t + 1$, whereas the “shock” at time t has little effect (due to a much weaker leverage effect δ^*). Other notable findings among the “single information source” estimations are (i) a significant rise of δ^* from the return-based estimate to the VIX-based estimate; (ii) a significant rise of the persistent parameter $\tilde{\beta}$ from \mathbf{P} information-based parameters (only with returns) to the \mathbf{Q} information-based parameters (with VIX and/or VIX futures). These two findings are consistent with the existing option pricing literature (e.g., Christoffersen et al., 2014; Kanniainen et al., 2014).

The parameters determined by the “Return + VIX” method provide larger equity premium parameter (λ) and leverage parameter (δ^*) compared with “Return only” method. This suggests that the estimated equity risk premium has to be inflated to fit the VIX data, which is consistent with Hao and Zhang (2013). The joint estimates parameters in the “VIX + Futures” method also significantly differ from those calibrated only from the VIX futures or the VIX itself.

From *Proposition 1*, we have pointed out that the key parameter controlling the mean reversion speed of the implied volatility term structure is $\Gamma(n)$, which is related to the persistent parameter β . Based on information given in Table III, Figure 1 plots $\Gamma(n)$ for the different estimation methods.

Three groups are formed, of which the “Return only” method delivers the lowest $\Gamma(n)$, the VIX futures-related methods (“Futures only” and “VIX + Futures”) deliver much higher $\Gamma(n)$, and the VIX-related methods (“VIX only” and “Return + VIX”) deliver the modest $\Gamma(n)$.

The following subsections are provided to investigate the model’s ability to match the market VIX and price VIX futures, in which a desirable parameter set should perform well in both aspects.¹⁰

4.2. Model-Implied VIX

Table IV summarizes the pricing performance of the five estimation methods for the model-implied VIX. The first four columns are related to pricing errors, which are defined as the market VIX minus the corresponding model-implied VIX. Conventional measures are provided, namely the mean error (ME), the root mean square error (RMSE), the mean absolute error (MAE), and the standard deviation of error (StdErr). The last column is the correlation coefficient (CorrCoef) between the model-implied VIX and the market VIX. Plots on both series are also provided¹¹ in Figure 2.

⁸The bolded diagonal elements denote the elements to be maximized, whereas the off-diagonal elements denote the likelihood values of the corresponding parameters in the column evaluated along with the likelihood function in the row.

⁹We tried several different initial values for this case, including the estimated parameters from the VIX/VIX futures. A significantly lower δ^* is robust with respect to initial values.

¹⁰A more desirable case is that a model can, with one set of parameters, match all three data series: the S&P 500 returns, the VIX, and the VIX futures. Unfortunately, the existing literature indicates that it is hard for GARCH-type models to reconcile both \mathbf{P} and \mathbf{Q} dynamics. Therefore, this paper only focuses on the fitting of VIX and the VIX futures, and we leave the additional fit of returns for further research.

¹¹The graph for “VIX only” is similar to the graph for “Return + VIX”. The result is available upon request.

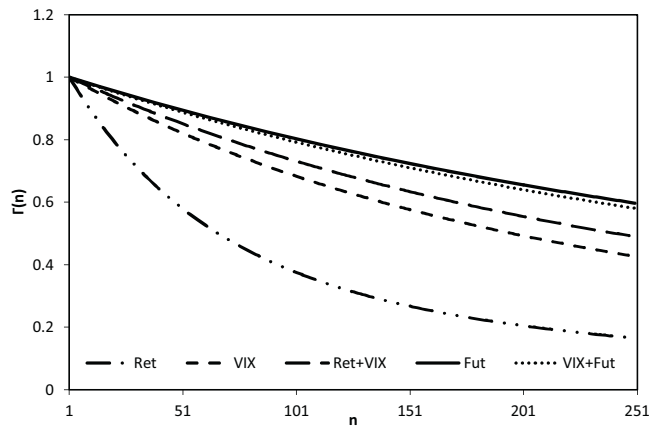


FIGURE 1

$\Gamma(n)$ for Different Parameter Sets

Note: Ret, return only; VIX, VIX only; Ret + VIX, Return + VIX; Fut, futures only; VIX + Fut, VIX + Future. All weights are declining whereas n goes larger. The speed of decline is determined by the persistent parameter. Three different groups can be found: the fastest only use return data, the slowest use VIX futures data whereas the ones with VIX index data have the modest speed.

TABLE IV

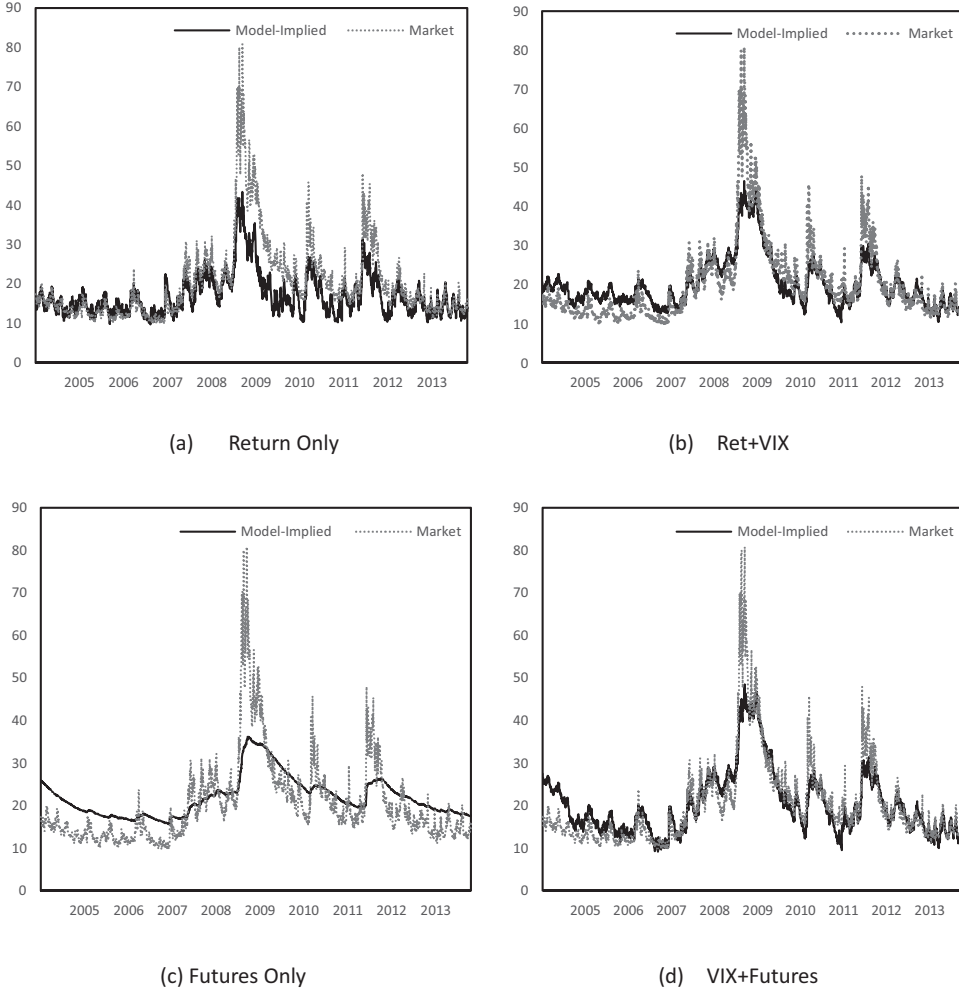
Summary of VIX Pricing

	<i>ME</i>	<i>RMSE</i>	<i>MAE</i>	<i>Std</i>	<i>CorrCoef</i>
R	3.2709	6.4742	4.1650	5.5883	0.9091
R + VIX	0.1148	4.9741	3.4786	4.9738	0.9063
VIX	−0.1198	4.3970	3.2325	4.3963	0.9060
FUT	−1.7258	6.9171	5.2036	6.6997	0.8230
VIX + FUT	−0.2613	4.7334	3.2056	4.7272	0.8923

Note. Error = market VIX index—model implied VIX. Ret, Return only; Ret + VIX, Return + VIX; VIX, VIX only; Fut, Futures only; VIX + Fut, VIX + Futures.

Not surprisingly, the model-implied VIX generated by parameters from the “Return only” method significantly underestimates the market VIX. This result reflects the fact that the single-shock GARCH-type models under LRNVR can only capture the equity risk premium, and they leave the volatility risk premium unattended. When parameters are calibrated by the VIX or VIX futures under the risk-neutral dynamic, the underestimation is no longer profound. This finding has been reported by Hao and Zhang (2013) for a number of GARCH family models other than Hestion-Nandi GARCH.

In terms of the RMSE, the model-implied VIX based on “Futures only” has an even higher RMSE than the model based on “Return only”. This pattern appears more obviously in Figure 2(c), in which the model-implied VIX is just a long-run smoothed average of the actual data. This finding confirms the result that we have discussed in our model estimation, namely that parameter distortion leads to a great smoothness in the model-implied VIX. For the “VIX + Futures” method, although the parameters are obviously different from those estimated by the “VIX only” method, the fitting performance for the implied VIX is similar, with an RMSE of 4.73 for “VIX + Futures,” compared to 4.40 for “VIX only.” Moreover, compared with the parameters from the “Futures only” method, the joint parameters show a

**FIGURE 2**

Model-Implied VIX from Different Estimation Methods

Note: The black solid line is the model-implied VIX. The gray dotted line is the CBOE market VIX level.

significant improvement in the implied VIX fitting, with the RMSE decreasing from 6.92 to 4.73. This result indicates that the parameters derived by the “VIX + Futures” method are more desirable.

The difference in performance of the model-implied VIX for the VIX-based methods (the “VIX only” and “VIX + Futures” methods) and that of the “Futures only” method can be explained as follows. The persistence parameter $\tilde{\beta} = \beta + \alpha\delta^{*2}$ is a combination of β and δ^* . A better pricing result requires that $\tilde{\beta}$ be very close to one. However, the leverage parameter δ^* for the short-term shock is important for fitting the VIX dynamics. The importance of this parameter arises from the fact that “VIX only” depends on a 22-day expectation of the conditional volatility, while “VIX + Futures” depends on a much longer horizon. As shown in *Proposition 1*, the weight on the current conditional volatility becomes smaller when n is larger. This pattern indicates that fitting the VIX level forces the model parameters to align with the short-term dynamics of h_{t+1} (i.e., to focus on β and δ^* separately), but fitting VIX

futures forces the model parameters to focus on the persistence parameter (i.e., the combination of β and δ^* as a whole).

4.3. VIX Futures Pricing

Table V summarizes the performance of the five methods for VIX futures pricing. Not surprisingly, the parameters from the “Futures only” method provide the best pricing performance. Among the other approaches, the parameters estimated by the “Return only” method show the worst RMSE performance and a severe underestimation for VIX futures prices. Parameters from the “VIX only” and the “Return + VIX” methods demonstrate better pricing performance for VIX futures compared with that from the “Return only” method. Again, the “VIX + Futures” method shows a significant improvement over the VIX only methods, and the prices given by this method are very close to those estimated by the “Futures only” approach.

We present a more detailed comparison of the various models’ pricing errors in Table VI, in which the RMSE of pricing errors are summarized by different levels of VIX, basis¹² and time to maturity. First, all of the models tend to have larger pricing errors on average when the time to maturity is longer. Second, when the absolute magnitudes of basis get smaller, the models tend to have smaller RMSEs, with the best goodness-of-fit for basis between -3 and 3 . Moreover, it is interesting to find a U-shaped relation between the RMSE and the VIX level for the three models which do not fit returns data, that is, the RMSEs tend to decrease as the level of VIX gets smaller, until the VIX is less than 15 . The smallest RMSE is achieved when the VIX is between 15 and 20 .

To better illustrate our models’ fit for VIX futures along the forward curves, we follow Zhu and Lian (2012) in plotting the averaged forward curves¹³ of the model-implied VIX futures along with their market counterparts in Figure 3. Instead of evaluating the forward curve on the mean of VIX in the full sample, we evaluate the structure with a partition of futures data with respect to their underlying VIX levels. The main concern in this modification is to reconcile the fact that our sample spans a period during which the VIX changed drastically. In particular, we divide the futures into two groups. The first group,

TABLE V
Summary of VIX Futures Pricing

	<i>ME</i>	<i>RMSE</i>	<i>MAE</i>	<i>Std</i>	<i>CorrCoef</i>
R	5.9879	8.3967	6.4674	5.8867	0.6419
R + VIX	3.1164	4.9892	3.8946	3.8964	0.8568
VIX	0.0866	4.0287	3.1488	4.0279	0.8518
FUT	0.0838	3.6073	2.6793	3.6064	0.8806
VIX + FUT	0.1621	3.6566	2.7561	3.6531	0.8762

Note. Error = market VIX futures price—model VIX futures price. Ret, Return only; VIX, VIX only; Ret + VIX, Return + VIX; Fut, Futures only; VIX + Fut, VIX + Futures.

¹²Basis = VIX level—VIX futures price.
¹³We divide the futures price data into groups according to their maturities, with 10 days for each group. The average forward curve is defined as the average of market/model prices within each group, plotted against maturities. The curve for the “Futures only” method is close to that for the “VIX + Futures” method. Result is available upon request.

TABLE VI
RMSE for VIX Futures Pricing by VIX, Basis, and Maturity

VIX	<15	15–20	20–25	25–30	>30
R	2.1449	6.0282	8.7777	10.1875	15.1219
R + VIX	1.8994	4.7227	5.9626	5.9365	6.7775
VIX	4.7512	2.5988	3.3996	3.8639	5.7468
FUT	3.3965	2.6825	3.2882	3.2751	5.7273
VIX + FUT	3.5753	2.6844	3.3808	3.4190	5.6199
Basis	<–6	–6 to –3	–3 to +3	3–6	>6
R	10.1265	6.9656	6.8880	13.0984	15.8514
R + VIX	7.9272	4.7872	3.5013	4.9270	7.0460
VIX	4.1083	3.9164	3.7007	4.5240	6.4570
FUT	4.3498	3.1379	2.6182	3.0137	8.8321
VIX + FUT	4.3773	3.3150	2.7641	3.0653	8.3741
Maturity	<50	50–99	100–149	150–200	>200
R	4.3978	8.2205	9.6632	10.2600	9.1614
R + VIX	2.6689	4.5219	5.4098	6.2395	6.2910
VIX	2.4816	3.9195	4.4429	4.7371	4.7273
FUT	2.3849	3.6428	3.9892	4.0161	4.1751
VIX + FUT	2.3988	3.6774	4.0326	4.0994	4.2538

Note. Basis = VIX index – VIX futures price. Ret, Return only; Ret + VIX, Return + VIX; VIX, VIX only; Fut, Futures only; VIX + Fut, VIX + Futures.

containing futures with an underlying VIX level of less than 25, is denoted as the “normal VIX level” group. The second group, containing futures with an underlying VIX level greater than 25, is denoted as the “high VIX level” group.¹⁴

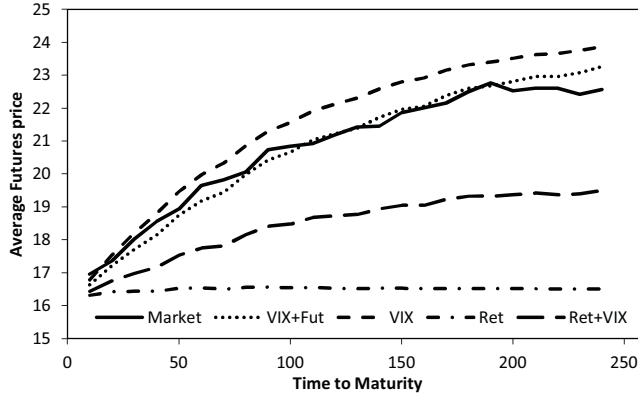
Before focusing on the comparison, it is important to note that the shape of the market forward curve is totally different when the underlying VIX is high, as it shows a downward slope instead of the commonly reported full-sample upward-sloped curve. This downward slope is a natural result when volatility follows a mean reverting process: The high VIX level is much greater than the mean VIX level, and therefore we can expect the future VIX level to fall.

From the figure, the forward curve implied by the “VIX + Futures” method fits the whole market forward curve reasonably well. The parameters from the “VIX only” method have generated substantial overestimation when the volatility is normal, and consistent underestimation along the forward curve when the volatility is high. The “Return only” parameters provide a flat forward curve when volatility is low, and a steep curve when volatility is high. As shown in Table III, both the persistent parameter and the long-run variance for the “Return only” method are significantly lower than those for the other methods.

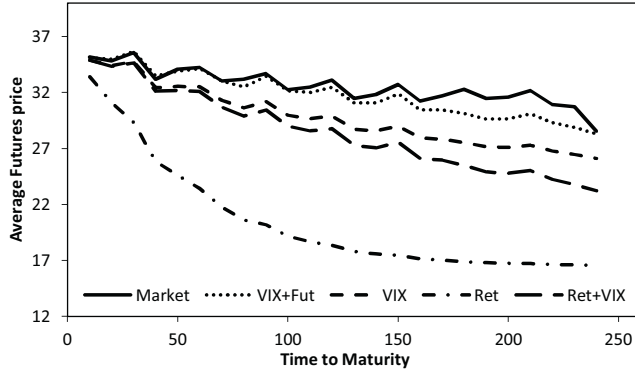
4.4. Out-of-Sample Results

Concerning the in-sample pricing performance as analyzed in the previous section, it is not very surprising that the “Futures only” method is the best in fitting VIX futures price, and the

¹⁴The first group contains some 77% of all futures contracts. VIX = 25 is also a cut-off point where RMSE is evaluated along VIX levels.



(a) Normal VIX level: VIX < 25



(b) High VIX level: VIX > 25

FIGURE 3**Model-Implied VIX Futures Term Structures for Different VIX Levels**

Note: VIX futures terms structure is constructed by averaging VIX futures price within 10 days' time to maturity groups (the first group is for $0 < T \leq 10$ and so on). Solid line is calculated with market price of VIX futures. Cutting point 25 is selected for several reasons: (i) It is a cutting point while summarizing VIX data. (ii) It is close to the third quartile (actually the 77 percentile) of VIX data. (iii) It is close to the “unconditional” VIX level (around 26) which is calculated by setting the unconditional volatility as h_{t+1} . The curve for the “Futures only” method is also close to that for the “VIX + Futures” method. The Result is available upon request.

“VIX only” method delivers the smallest pricing error for the VIX level. One natural concern is whether these methods are subject to in-sample overfitting. To check this possibility, we provide an out-of-sample performance evaluation based on a rolling window of 500 trading days, with the parameters updated on the daily basis. We evaluate the out-of-sample pricing errors of 1951 trading days with 1951 VIX prices and 12826 VIX futures prices. The RMSE for the model-implied VIX and VIX futures prices are reported in Tables VII and VIII, respectively.

The main results are as follows. (i) For the model-implied VIX, the “VIX only” method provides the smallest RMSE, and the “Futures only” method provides the largest. This result reinforces the findings in the in-sample analysis. (ii) For VIX futures pricing, the “Futures only” approach has the smallest RMSE, and the RMSE for the “VIX + Futures” method differs only slightly from the best case. When the sharp difference between these two

TABLE VII
Summary of Out-of-Sample Model-Implied VIX

	<i>ME</i>	<i>RMSE</i>	<i>MAE</i>	<i>Std</i>	<i>CorrCoef</i>
R	3.4508	6.4842	4.2849	5.4911	0.8775
R + VIX	1.3429	4.7043	2.9907	4.5097	0.9117
VIX	0.9923	4.3522	2.8785	4.2386	0.9215
FUT	0.0469	7.6728	5.4316	7.6746	0.6983
VIX + FUT	1.2704	4.6367	2.7982	4.4604	0.9151

Note. Error = market VIX index—model implied VIX. Ret, Return only; Ret + VIX, Return + VIX; VIX, VIX only; Fut, Futures only; VIX + Fut, VIX + Futures.

TABLE VIII
Summary of Out-of-Sample VIX Futures Pricing

	<i>ME</i>	<i>RMSE</i>	<i>MAE</i>	<i>Std</i>	<i>Corr</i>
R	4.4018	6.9304	5.0452	5.3533	0.6995
R + VIX	1.9579	5.6212	4.2460	5.2694	0.7226
VIX	0.6919	4.9233	3.6796	4.8747	0.7625
FUT	0.1954	3.1278	2.2062	3.1219	0.9073
VIX + FUT	0.2613	3.1293	2.2539	3.1185	0.9075

Note. Error = market VIX futures price—model VIX futures price. Ret, Return only; Ret + VIX, Return + VIX; VIX, VIX only; Fut, Futures only; VIX + Fut, VIX + Futures.

methods (as reported in Table VII) is compared with the results from Table VIII, the value of joint estimation with both VIX and VIX futures becomes apparent. As the methods that involve fitting VIX or VIX futures prices directly still deliver good out-of-sample performance, we find no evidence of in-sample overfitting. A detailed decomposition of the out-of-sample RMSE is also reported in Table IX, which confirms the findings at the aggregated level.

5. CONCLUSION

In this paper, we discuss the pricing of CBOE VIX futures under the classic discrete-time Heston–Nandi GARCH model. Concerning the LRNVR, an explicit pricing formula can be found via an integration of the transformed moment generation function of conditional volatility. We apply several estimation approaches using differing sets of information, and investigate their performances in dealing with the market data. In line with the existing literature, the S&P 500 returns provide little information on VIX futures, whereas the VIX data provide more. Although the calibration with respect to the VIX futures data provides minimum pricing errors for futures prices, it also leads to great distortion in the model-implied VIX. However, the joint information from combining the VIX and VIX futures can provide similar pricing performance concerning VIX futures without distortion on the dynamic of the model-implied VIX. We also check the pricing performance of different methods in a rolling window out-of-sample analysis. The empirical results confirm our findings, and provide no evidence of in-sample overfitting.

Our empirical results illustrate the capacity of the Heston–Nandi GARCH model, with appropriate information sources under the Q measure, for fitting the dynamics of both the

TABLE IX
RMSE for Out-of-Sample VIX Futures Pricing by VIX, Basis, and Maturity

VIX	<15	15–20	20–25	25–30	>30
R	2.8560	4.2205	5.7975	6.9965	13.4099
R + VIX	2.8879	3.7453	5.3028	6.7813	9.5988
VIX	2.5422	3.6263	4.1091	5.4527	8.7223
FUT	2.1717	2.0732	2.1806	2.5898	6.0444
VIX + FUT	2.2761	2.1299	2.2231	2.6564	5.9179
Basis	<−6	−6 to 3	−3 to 3	3–6	>6
R	15.3323	11.9253	6.6319	4.0449	5.0829
R + VIX	10.2977	8.2813	5.1620	5.0725	4.6754
VIX	9.5242	7.6950	4.5882	4.2098	3.6601
FUT	8.1555	3.9865	2.3952	2.7417	2.2656
VIX + FUT	7.6565	4.0721	2.4971	2.8393	2.2919
Maturity	<50	50–99	100–149	150–200	>200
R	4.6028	7.2565	7.6924	7.7067	7.1452
R + VIX	3.0142	5.2748	6.3663	7.0541	5.9933
VIX	2.7475	4.7186	5.6579	6.1020	4.9195
FUT	2.2411	3.1319	3.3452	3.6250	3.2851
VIX + FUT	2.2352	3.1206	3.3546	3.6449	3.2706

Note. Basis = VIX index—VIX futures price. Ret, Return only; Ret + VIX, Return + VIX; VIX, VIX only; Fut, Futures only; VIX + Fut, VIX + Futures.

CBOE VIX and VIX futures. With the increasing availability of high frequency financial data, realized variance (RV), as calculated from intraday asset returns, has been widely used in volatility modeling during the past decade. For example, Hansen, Huang, and Shek (2012) incorporate the RV into the GARCH framework, and show the extended model's superior performance in forecasting volatility. The intuition is that the RV contains much more accurate information about underlying volatility than the squared daily return. In this sense, RV has the most information content about volatility under the physical dynamics. Thus, it would be very interesting to investigate whether RV can be used to develop a new model that has the ability to simultaneously capture the data dynamics under both the physical and risk-neutral measures, and to provide better pricing performance for VIX derivatives. We leave this topic for further research.

APPENDIX

PROOF OF PROPOSITION 1

Proof: Following Hao and Zhang (2013), let the long-run variance $\bar{h} = \frac{\tilde{\omega}}{1-\tilde{\beta}}$. Then we have

$$E_t^Q(h_{t+2}) - \bar{h} = \tilde{\omega} + \tilde{\beta}h_{t+1} - \frac{\tilde{\omega}}{1-\tilde{\beta}} = \tilde{\beta}(h_{t+1} - \bar{h})$$

Therefore, the k-step ahead expectation of conditional variance is

$$E_t^Q(h_{t+k}) = \bar{h} + \tilde{\beta}^{k-1}(h_{t+1} - \bar{h})$$

By the definition of $V_t(n)$, we have

$$\begin{aligned}
 V_t(n) &= \frac{1}{n} \sum_{k=1}^n E_t^Q(h_{t+k}) \\
 &= \bar{h} + \frac{1}{n} \sum_{k=1}^n \tilde{\beta}^{k-1} (h_{t+1} - \bar{h}) \\
 &= \bar{h} + \frac{\frac{1}{n} (1 - \tilde{\beta}^n)}{1 - \tilde{\beta}} (h_{t+1} - \bar{h}) \\
 &= \bar{h} + \Gamma(n) (h_{t+1} - \bar{h}) \\
 &= \bar{h} (1 - \Gamma(n)) + \Gamma(n) h_{t+1}
 \end{aligned}$$

where $\Gamma(n) = \frac{1 - \tilde{\beta}^n}{n(1 - \tilde{\beta})}$. Therefore, the volatility term structure is the weighting average of the long-run variance \bar{h} and the conditional variance h_{t+1} .

PROOF OF PROPOSITION 2

Proof: Suppose that the moment-generating function has the following form:

$$E_t^Q[e^{\phi h_{t+m+1}}] = e^{C(\phi, m) + H(\phi, m) h_{t+1}}$$

Then, by the initial condition, we have

$$C(\phi, 0) = 0, \quad H(\phi, 0) = \phi$$

We also suppose that the following recursion relationship:

$$\begin{aligned}
 C(\phi, m+1) &= C(\phi, m) + C_J(H(\phi, m)) \\
 H(\phi, m+1) &= H_J(H(\phi, m))
 \end{aligned}$$

We will see under the Heston–Nandi GARCH model that we can have a closed-form solution for $C_J(\cdot)$ and $H_J(\cdot)$. We have

$$E_t^Q[e^{\phi h_{t+m+2}}] = e^{C(\phi, m+1) + H(\phi, m+1) h_{t+1}}$$

and

$$E_t^Q[e^{\phi h_{t+m+2}}] = E_t^Q[E_{t+1}^Q[e^{\phi h_{t+m+2}}]] = E_t^Q[e^{C(\phi, m) + H(\phi, m) h_{t+2}}]$$

Let

$$s = H(\phi, m)$$

Then we obtain

$$E_t^Q[e^{s h_{t+2}}] = e^{s\omega + s\beta h_{t+1} + s\delta^2 \alpha h_{t+1}} E_t^Q[e^{s\alpha \varepsilon_{t+1}^{*2} - 2s\alpha \delta^* \sqrt{h_{t+1}} \varepsilon_{t+1}^*}]$$

$$\begin{aligned}
&= e^{s\omega + s\beta h_{t+1} + s\delta^{*2} \alpha h_{t+1}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}} e^{s\alpha x^2 - 2s\alpha\delta^* \sqrt{h_{t+1}} x} dx \\
&= e^{s\omega + s\beta h_{t+1} + s\delta^{*2} \alpha h_{t+1}} \left(\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi} \frac{1}{\sqrt{1-2s\alpha}}} e^{-\frac{(x + \frac{2s\alpha\delta^* \sqrt{h_{t+1}}}{1-2s\alpha})^2}{2(\frac{1}{1-2s\alpha})}} dx \right) \frac{1}{\sqrt{1-2s\alpha}} e^{\frac{2s^2\alpha^2\delta^{*2} h_{t+1}}{1-2s\alpha}}
\end{aligned}$$

The integral part is 1, so we have

$$E_t^Q[e^{sh_{t+2}}] = e^{s\omega - \frac{1}{2}\ln(1-2s\alpha)} e^{(s\beta + \frac{s\alpha\delta^{*2}}{1-2s\alpha})h_{t+1}}$$

By comparison, we have

$$\begin{aligned}
C_J(s) &= s\omega - \frac{1}{2}\ln(1-2s\alpha) \\
H_J(s) &= s\beta + \frac{s\alpha\delta^{*2}}{1-2s\alpha}
\end{aligned}$$

Thus, we have obtained the closed-form recursion relationship for the moment-generating function.

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