The Journal of Risk Finance

A multi-factor HJM and PCA approach to risk management of VIX futures

Philippe Bélanger, Marc-André Picard,

Received 21 July 2017
Revised 22 May 2018
Accepted 3 July 2018

A multi-factor HJM and PCA approach to risk management of VIX futures

Philippe Bélanger
Finance, Assurance et Immobilier, Laval University, Québec, Canada, and
Marc-André Picard
FINCAD, Surrey, Canada

Abstract

Purpose – Previous studies have shown the VIX futures tend to roll-down the term structure and converge towards the spot as they grow closer to maturity. The purpose of this paper is to propose an approach to improve the volatility index fear factor-level (VIX-level) prediction.

Design/methodology/approach – First, the authors use a forward-looking technique, the Heath–Jarrow–Morton (HJM) no-arbitrage framework to capture the convergence of the futures contract towards the spot. Second, the authors use principal component analysis (PCA) to reduce dimensionality and save substantial computational time. Third, the authors validate the model with selected VIX futures maturities and test on value-at-risk (VAR) computations.

Findings – The authors show that the use of multiple factors has a significant impact on the simulated VIX futures distribution, as well as the computations of their VAR (gain in accuracy and computing time). This impact becomes much more compelling when analysing a portfolio of VIX futures of multiple maturities.

Research limitations/implications – The authors' approach assumes the variance to be stationary and ignores the volatility smile. Nevertheless, they offer suggestions for future research.

Practical implications – The VIX-level prediction (the fear factor) is of paramount importance for market makers and participants, as there is no way to replicate the underlying asset of VIX futures. The authors propose a procedure that provides efficiency to both pricing and risk management.

Originality/value – This paper is the first to apply a forward-looking method by way of a HJM framework combined with PCA to VIX-level prediction in a portfolio context.

Keywords Heath-Jarrow-Morton (HJM) no-arbitrage framework, Principal component analysis (PCA), Roll-down the term structure, Value-at-Risk (VaR), VIX futures

Paper type Research paper

1. Introduction

Developed by Robert E. Whaley in 1993, the VIX volatility index has become a trademark product of the Chicago Board Option Exchange (CBOE). The VIX index represents the market's expectation of the S&P 500 index annualized volatility over a 30-day period. In simpler terms, it represents the cost of portfolio insurance with a 30-day maturity.

JEL classification – G17, G21, G28

Thanks to Prof Van Son Lai for his mentoring. The authors would like to thank the anonymous referee and Dr Bonnie Buchanan for their very constructive and insightful comments and suggestions, which have greatly enhanced the quality of the paper.

Disclaimer: The views and opinions expressed in this publication are those of the authors alone and do not necessarily represent the views of the FinancialCAD corporation or any of its affiliates.

The Journal of Risk Finance
Vol. 19 No. 5, 2018
pp. 524-547

1526-5943
DOI 10.1108/JRF-07-2017-0114

The CBOE released the VIX futures in 2004 and the VIX options in 2006. Since then, the VIX derivatives have been growing in popularity. In January 2009, smaller investors became able to benefit from those volatility hedging tools through a wide variety of exchange-traded products (ETPs) that track the VIX Futures return index. These volatility ETPs appear in a wide array of terms and formats.

VIX derivatives provide a simple and cost-efficient way to hedge against volatility risk. Nevertheless, they have a number of properties which make their pricing problematic compared to equity index derivatives (as the standard cost-of-carry model cannot be applied). Recent studies (Shu and Zhang, 2012; Fassas and Siriopoulos, 2012; Simon and Campasano, 2014) show a phenomenon in which the VIX futures returns are a correlated, but a somewhat dampened measure of the spot VIX returns. In addition, VIX futures are bound to converge towards the spot which makes their daily returns dependent on the actual term structure. These observations raise questions on how we should model the behaviour of VIX futures/ETPs for risk management purposes.

Our main goal is to focus on quantifying the risk of holding VIX futures by modelling the real-world dynamics of the VIX futures term structure while taking the prices of the VIX futures available on the market for granted. This in turn allows real-world scenario analysis and obtain more realistic measures than we would using risk-neutral probabilities.

For the considered model, the Girsanov's theorem states that the change of measure between the real-world probability measure P and the risk-neutral probability measure Q only involves a change in the drift term, while the volatility of the underlying process remains the same. Our results show that we are able to price options under the Q measure to calibrate the model for volatility. By translating the model to the P measure, we keep the same volatility specifications but simulate changes in the drift by allowing the curve to take different shapes. By doing so, the VIX futures "roll" down the curve and converge towards the spot VIX (as documented empirically).

Based on previous literature, we establish specifications for our model which leads to the adaptation of a consistent, multi-factorial, HJM-type model. By doing so, we can rely on available market information for the VIX futures volatility specifications. We also show how the principal component analysis (PCA) method may be used to model the curve's dynamics to reduce computing time. We demonstrate that the use of multiple factors has a significant impact on the drift (roll) distribution when compared to only using one factor.

By modelling the VIX futures with a HJM-type model, we take for granted the term structure of VIX futures prices. By using quoted VIX options and defining a correlation structure between the maturities, we are able to back-out the volatility specifications for the model. Therefore, we can quantify the risk of holding a VIX futures position in a portfolio, while at the same time, taking into account the roll-down associated with the shape of the VIX futures curve. This approach allows the curve to move more freely by using multiple factors to drive its dynamics. As mentioned in Lu and Zhu (2010), at least three factors are necessary to properly capture the VIX curve dynamics. To get a model that generates plausible scenarios for the VIX futures roll-down, we make use of this statement along with the intuition behind Whaley (2013), which empirically shows that there is a statistically significant relation between the VIX futures returns and the shape/slope of the term structure. The use of additional factors allows the curve to move more freely and takes into account the correlation structure between the contracts. This should ultimately lead to a better representation of the roll-down effect, which can become quite substantial when holding VIX futures/ETPs positions for a relatively long time.

This paper contributes to the literature as it (1) presents a forward-looking technique to apply HJM to volatility term structure, (2) applies PCA to model the curve dynamic and (3) illustrates an approach with selected VIX future maturities.

The rest of the paper is structured as follows. In Section 2, we describe the literature that investigated the VIX modelling. In Section 3, we specify the model. In Section 4, we present the result for an illustration portfolio. In Section 5, we discuss the results. In Section 6, we conclude the paper.

2. Literature review

2.1 VIX derivatives and ETPs

Volatility derivatives[2] were first traded at the beginning of the 1990s on the over-the-counter (OTC) market and rapidly became popular amongst traders because they could provide a cost-effective way for hedging volatility risk for portfolio insurers, options market makers and covered call writers. Whaley (1993) provides an excellent demonstration of the cost effectiveness of hedging for market volatility risks by using volatility derivatives as compared to hedging with index options.

With this increasing popularity of OTC volatility derivatives in the 1990s, many argued for the growing need of simple and standardized volatility hedging tools. The arrival of an index such as the VIX did provide a benchmark on which such contracts could be written. However, it is only after the release of the new VIX in 2003 that the Chicago Futures Exchange (CFE) launched the exchange-traded VIX futures in March 2004. The CFE then introduced exchange-traded VIX options in February 2006. These are now amongst CBOE's most liquid contracts, second only to the SPX index options.

Since January 2009, smaller investors have also been able to benefit from those volatility hedging tools through a wide variety of ETPs that track the VIX Futures return index along with their respective set of derivatives. The accessibility of these products seems to have created a new volatility trading trend with investors. More than 30 VIX ETPs are listed and seeing a fair amount of success with an aggregated market investment value of nearly US $4bn and a trading volume that averages US$800m daily.

A recent study from Whaley, 2013 discusses the motives of volatility ETPs holders but also demonstrate the ineffectiveness of holding such a security in a long-term portfolio. When buying these volatility ETPs, many investors believe that they are directly holding the VIX Index in their portfolio. Holding such a security, which is negatively correlated with the market, seems promising for portfolio diversification. However, they are bound to find out that it is sadly not the case: it has been shown that the losses incurred over the holding period will wash away any diversification effect.

Unlike most exchange-traded securities, the VIX ETPs are not suitable for buy-and-hold investments and are expected to lose money. These ETPs are, in fact, created from VIX futures portfolios that demand daily rebalancing and even when leaving out the management fees and expenses to which they are subjected, they are bound to lose money from a contango trap which makes the VIX futures constantly drawn downward towards the VIX level.

Although these ETPs seem to be effective in tracking their respective indices, the problem remains in the VIX futures index being tracked. These indices are intended to mimic the behaviour of futures trading strategies (portfolio) that involve rolling a VIX futures position in a manner that maintains a constant maturity.

The main problem with this kind of strategy becomes obvious when looking at the VIX futures curve. The contango effect mentioned earlier can be seen in Figure 1 where the slope of the curve is consistently upwards. If one assumes the curve remains the same shape throughout time, then with every passing day, the dots on the curve (the futures prices) should descend down the curve as the VIX futures expirations grown shorter, ultimately converging at the current VIX value. The VIX futures curve being more often than not in

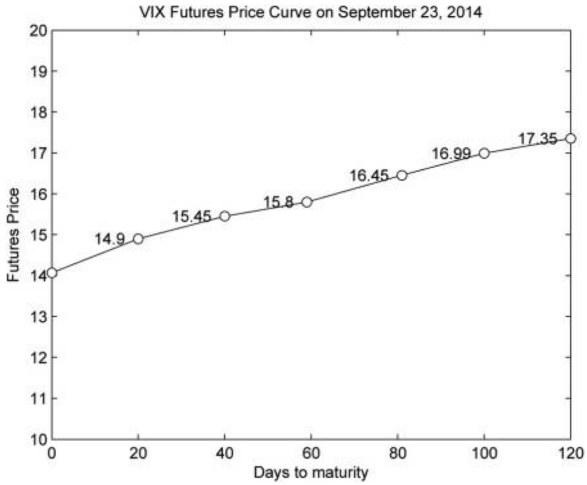

Figure 1.
VIX futures curve

contango will cause a long position in the futures to systematically lose money. For the strategies used with tracking indices, the losses incurred are predictable by taking the slope straddling the constant maturity. For example, for a 30-day constant maturity with daily rebalancing, the slope between the first and the second maturity is (15.54 – 14.90)/20 = 0.032. Holding everything else equal, this position is expected to lose US\$0.032 or 0.21 per cent every day, which yields, in this particular case, a whopping 52.9 per cent annual loss.

Utilizing a regression model, Whaley (2013) shows that the slope between the futures contract has a statistically significant role in forecasting the VIX ETPs returns. In light of these findings, when trying to measure the risk of a portfolio of VIX derivatives, the model should take into account the shape of the actual term structure and the roll-down associated with it.

This brings us to question which type of model should be used, as there are more than a few in the VIX derivatives literature.

2.2 Volatility derivatives valuation

Even before the VIX futures official launch, research started to address the problem of volatility derivatives valuation. The theoretical effectiveness of hedging with such products had been demonstrated in Whaley (1993), but the solution to pricing volatility futures remains problematic to this day. Volatility futures have a number of interesting properties that differentiate them from other futures contracts, one of which is that the standard cost-of-carry model does not apply.

The first VIX option pricing model used was the very simplistic Black (1976) futures option pricing formula. Since then, a number of models has been presented within the equilibrium model framework. The scope of this framework is mainly to explain the VIX futures curve and attempt to price the VIX futures from the instantaneous variance of the S&P 500. This usually involves an analytical formula that can be solved into a closed-form pricing formula.

The literature on equilibrium models for the VIX futures began appearing soon after the inception of VIX. Grünbichler and Longstaff (1996) proposed a model similar to the Heston (1993) model that would directly model the dynamics of the VIX term structure.

They also provided closed-form formulas to price the VIX futures and options. After the VIX methodology changed in 2003 to become the square-root of a 30-day variance swap on the S&P 500, Zhang and Zhu (2006) tried modelling the dynamics of the VIX futures within the Heston framework, arguing that modelling the instantaneous variance of the S&P 500 should allow pricing of the VIX futures. A few years later, Sepp (2008), as well as Lian and Zhu (2013), brought a more sophisticated approach of the Heston version to model the VIX futures by adding jump components in the volatility and the underlying's processes.

The results obtained with Heston-type models appear to be inconclusive as to the internal consistency of the implicit parameter when comparing the model's implied parameters with those implicit in the time series of the VIX futures prices. In conclusion of their study comparing different VIX option pricing models, Wang and Daigler (2011) suggest that the VIX generates different implied volatility surfaces than stock index options, and therefore, it should be better to model the underlying process of the VIX directly rather than modelling the volatility of the SPX (as it is done within the Heston framework). They also find that there is no direct VIX model that clearly outperforms the others in pricing out-of-sample options across all moneyness and maturities. Following those results, they also suggest that "simpler might be better" when it comes to the choice of a VIX model. Kokholm and Stisen (2015) use a Heston model with jumps in returns and the Heston model with simultaneous jumps in returns and variance. They also find that the fit is not satisfactory.

This leads us to believe that equilibrium models may be less suitable for our application. One of the main reasons being the need for the VIX futures pricing model to fit the actual term structure which is hardly attainable with this type of model while maintaining parameter stability. Even more so, the need to exactly fit the actual curve becomes paramount when considering that the shape of the curve plays a significant role in determining the VIX futures return. Keeping in mind that we are not interested in pricing the VIX futures but rather interested in real-world risk metrics for the futures prices, a HJM-type model would be the most logical choice because, within this framework, the initial curve is taken for granted.

In 1990, David Heath, Bob Jarrow and Andy Morton (HJM) published an important paper describing the no arbitrage condition that must be satisfied by a yield curve (Heath *et al.*, 1992). The HJM framework has been most commonly used in the pricing of complex interest rates derivatives. Considering the similarities between forward interest rates and forward variance, the framework may be adapted to the variance term structure, as in Dupire (1992). The approach basically consists of taking today's variance curve and adding a stochastic process where the drift should exactly follow today's given term-structure. Buehler (2006) presents the general adaptation of the framework for consistent variance curve models.

Within probability space (Ω, F, F_T, P) with a d-dimensional Brownian motion $W = (W^j)_{1,\ldots,d}$, defining the forward variance swap $V_t(T)$ with maturity T as:

$$V_t(T) = \int_0^T v_t(s)ds \tag{2.1}$$

Given that $V_t(T)$ is a martingale (see *P*2.3 from Buehler (2006)), a function $\varphi(T)$ can be found such that:

$$V_t(T) = V_0(T) + \sum_{j=1}^{d} \int_0^t \varphi_s^j(T) dW_s^j$$

$$\rho_{ij} dt = dW^i dW^j \tag{2.2}$$

To have a variance curve that is free of arbitrage, the forward variance swap rate has to remain positive. A way to ensure the respect of this constraint is to make the assumption that the forward variance swap rate is log-normally distributed.

From Itô's lemma:

$$\ln[V_t(T)] = \ln[V_0(T)] - \frac{1}{2} \sum_{j=1}^{d} \int_0^t \varphi_s^j(T)^2 ds + \sum_{j=1}^{d} \int_0^t \varphi_s^j(T) dW_s^j \tag{2.3}$$

The results shown above are also similar to those given by Dupire (1992), but in this case, we find a process, which includes multiple Brownian motions driving the curve. It is important to note that care should be taken when choosing the parameters φ_t^j to ensure that $V_t(T)$ remains a martingale.

Literature developed within the same framework. Developing from Bergomi (2005, 2008, 2009), Cont and Kokholm (2013) include stochastic jumps to the model. The basics of their model for variance swap dynamics is similar to equation (2.3), with one Brownian motion to which a stochastic jump modelled with a random Poisson process is added. By doing so, the Brownian motion affects the short-term end of the curve, while the jumps impact the entire curve. They also use affine specifications that allow Fourier-based pricing of European options. The results that they obtain show that the model is able to fit VIX option prices across time to maturity and moneyness with a standard error that is less than 1 per cent on the market price.

3. Model specifications

Because the cost-of-carry model cannot be applied for VIX futures, the motivation to model the VIX futures directly to quantify the risk of holding these contracts becomes obvious.

We now establish our term structure model specifications to model the VIX term structure. As the returns of the VIX futures contracts depend on the shape of the curve, we want to allow it to take as many shapes as possible. Using a multi-factorial HJM-type model that is distributed log-normally should allow to back out a process for the VIX futures curve at any time t. We will use Monte-Carlo simulation to solve for each maturity T. The approach that is used is derived from the general framework from Buehler (2006). For scenario generation, we work under certain assumptions that will make the futures contracts converge towards the spot VIX. We use the log-normal assumption for simplicity; however, this can easily be adapted to add a jump component to match the volatility smiles.

To reduce the computing time, PCA may be used. PCA reduces the dimensionality of the problem for correlated Brownian motions. The volatility of the fixed VIX maturities can be directly backed out from VIX options implied (Black) volatility which eliminates the need for parameter calibration. We are, thus, limiting the use of empirical data only to get the correlations between each fixed maturity which is needed to define the curve dynamics by PCA.

3.1 Multi-factorial model for VIX futures

Following Buehler (2006)[3] approach for consistent variance curve within the HJM framework, we make the assumption that the variance swap rate $V_t(T)$ is log-normally distributed under Q which leads to:

$$\ln\left(V_t(T)\right) = \ln\left[V_0(T)\right] - \frac{1}{2}\sum_{j=1}^{d}\int_0^t \varphi_s^j(T)^2 ds + \sum_{j=1}^{d}\int_0^t \varphi_s^j(T) dW_s^j \tag{3.1}$$

Where φ_s^j is the factor loadings for $V_t(T)$ and dW_{s1}^j is a d-dimensional Brownian motion.

To keep a consistency between variance swap rates and VIX futures, and similarly to Cont and Kokholm (2013), we use the process for $\sqrt{V_t(T)}$ to represent VIX futures. From Itô's lemma, the VIX futures prices $VIX_t(T)$ with time to maturity T should be given by:

$$\ln\left[VIX_t(T)\right] = \ln\left[VIX_0(T)\right] - \frac{1}{8}\sum_{j=1}^{d}\int_0^t \varphi_s^j(T)^2 ds + \sum_{j=1}^{d}\int_0^t \frac{\varphi_s^j(T)}{2} dW_s^j \tag{3.2}$$

With this model, a link can still be made between the squared VIX futures and variance swaps with convexity adjustment while being able to use the VIX volatility for either case. However, the upcoming sections will solely focus on modelling the VIX futures.

The next section defines $\varphi_j^t(T)$ by using the PCA.

3.2 Curve dynamics

Many authors in interest rate literature, notably Litterman and Scheinkman (1991), Rebonato (1996), Scherer and Avellaneda (2002) and Driessen *et al.* (2003), use the PCA method to model the term structure dynamics within the HJM framework.

The main idea behind the method is to look at the curve as a whole instead of correlated individual maturities. This in turn brings a dimensionality reduction of the problem.

The similarities between forward variance swap rates and forward interest rates should enable us to use this method to extract the main factors driving the evolution of the forward variance term structure. Following Lu and Zhu (2010), it seems that, at least, the first three factors are statistically significant to correctly capture the dynamics of the curve.

Using historical VIX futures data,[4] PCA is done on the correlation matrix of the log returns of the forward variance term structure.

Let $VIX_{t_k}(T_i)$ be the VIX futures at maturity T_i where $i = 0,\ldots, n$ and let $r_{t_k}(T_i)$ be the log return of the VIX Futures with maturity (T_i) at time t_k be given by:

$$r_{t_k}(T_i) = log\left(\frac{VIX_{t_k}(T_i)}{VIX_{t_{k-1}}(T_i)}\right)$$

Let Σ be the covariance matrix of the log returns of the forward variance term structure where:

$$\Sigma_{i,j} = \mathbb{E}\left[\left(r_{t_k}(T_i) - \mathbb{E}[r_t(T_i)]\right)\left(r_{t_k}(T_j) - \mathbb{E}\left[r_t(T_j)\right]\right)\right]$$

For $i = 0,\ldots,n$ and $j = 0,\ldots,n$.

Taking the eigenvalues and eigenvectors for Σ will allow to obtain factor loadings for each maturity.

Let Φ_i be the i^{th} eigenvector corresponding to λ_i, the i^{th} largest eigenvalue of Σ, respectively, and $\varphi_i(T_j)$ be the i^{th} component for maturity T_j; the three factor loadings corresponding to each maturity are then given by:

$$\frac{\varphi_i(T_j)}{2} = \sigma_{VIX}(T_j)\sqrt{\lambda_i\left(\Phi_i^2\right)_j} \tag{3.3}$$

With this result, the only remaining parameter left to define is $\sigma_{VIX}(T)$ which is presented in the next section.

3.3 Volatility specifications

Using the HJM framework intuition, we should be able to calibrate the model to price vanilla products. Considering that there is an option market for the VIX that is fairly liquid, this offers information in regards to the VIX futures implied distribution. By taking the implied distribution from the market, we are taking the consensus from all the market's participants in regards to the estimation of the upcoming volatility.

We only have one volatility parameter, $\sigma_{VIX}(T)$ that is a deterministic function of time. Given $\sigma_L^2(u)$ a piecewise constant deterministic function, we specify:

$$\sigma_{VIX}(T) = \sqrt{\frac{1}{T}\int_0^T \sigma_L(u)^2 du} \tag{3.4}$$

We are looking to calibrate a function $\sigma_L(T)$ that satisfies equation (3.4) where $\sigma_{VIX}(T)$ is a strip of ATM Black implied volatilities on VIX options with the same maturity as the futures.

Solving for σ_{VIX}^2 and discretizing the equation, we obtain:

$$\begin{aligned}\sigma_{VIX}^2(T_n) &= \sum_{i=1}^{n}\left(\frac{T_i - T_{i-1}}{T_n}\right)^2 \sigma_L^2(T_{i-1,i}) \\ &\quad + 2\sum_{i=1}^{n}\sum_{1\le i<j\le n}\rho_{i,j}\frac{T_i - T_{i-1}}{x_n}\sigma_L(T_{i-1,i})\frac{T_j - T_{j-1}}{x_n}\sigma_L(T_{j-1,j})\end{aligned} \tag{3.5}$$

Having already defined $\rho_{i,j}$ in the previous section, we solve by iteration to find $\sigma_L^2(T_{i-1,i})$ for each maturity.

We can now present the VIX model from equation (3.2):

$$d\ln[VIX_t(T)] = -\frac{1}{2}\int_0^T \sigma_L^2(u)\sum_{j=1}^{d}\lambda_j\left(\Phi_j^2\right)_u dtdu + \int_0^T \sigma_L(u)\sum_{j=1}^{d}\sqrt{\lambda_j\left(\Phi_j^2\right)_u}dW_{t,u}^j \tag{3.6}$$

We now have a VIX curve model that allows different shapes for the term structure and also fits the initial curve perfectly.

3.4 Scenario generation

To generate scenarios where we capture the roll-down of the VIX futures along the term structure, we make the assumption that the S&P 500 implied volatility surface for relative maturities and moneyness at time t is expected to be the same as the S&P 500 surface at $t = 0$. This, in turn, would indicate that the VIX term structure with constant maturities is also expected to remain unchanged while the VIX futures contract $VIX_t(T)$ have now $T - t$ left to maturity, thus converging towards the spot. On a longer horizon, one could also adjust the drift to make the expected term structure converge to some pre-determined values.

We make adjustments taking into account the aforementioned roll-down effect and convergence towards the spot. This adjustment will yield (in most cases) a negative mean return when the VIX curve is in its normal contango shape. This yields the following modification to the model:

$$ln\left[VIX_t(T)\right] = ln\left[VIX_0(T-t)\right] - \int_0^{T-t}\int_0^t \frac{1}{2}\sigma_L^2(T)\sum_{j=1}^{d}\lambda_j\left(\Phi_j^2\right)_u dsdu$$
$$+\int_0^{T-t}\int_0^t\sum_{j=1}^{d}\sigma_L(x)\sqrt{\lambda_j\left(\Phi_j^2\right)_u}dW_{s,u}^j \tag{3.7}$$

for $t \leq T$.

To properly capture the roll-down effect, we also need to model the spot VIX (along with the futures) to capture the slope between the spot and the first maturity. For the sake of this exercise, the spot VIX volatility is extrapolated by fitting the volatility of volatility curve with the same volatility specification used by Cont and Kokholm (2013) which is given by:

$$\sigma_{VIX}(x) = \omega e^{kx} \tag{3.8}$$

Where we solve numerically for ω and k to minimise the squared error and then proceed to extrapolate for spot volatility.

4. Results

4.1 Model validation

To validate the model, we must ensure that the simulated VIX futures $VIX_t(T)$ remain martingales for all specified maturities. The mean and volatility of the simulated data for each maturity on the curve should match what is given in input at any time t. The model should also be able to price European VIX options correctly in the Q measure. To be consistent, the price of the ATM options should, therefore, be consistent between the Black model and proposed model for the chosen options.

We have simulated the VIX curve for $t = 0$ to 90 days with time steps dt of one day. The test is done with 1,000 simulations of 25,000 paths for a total of 250,000 simulated paths. Figure 2 show a sample of the simulated VIX Futures Curve. The initial term structure and volatilities used as inputs are shown in Table I.

The error on the mean for a simulation run is calculated as follows:

$$MeanError(\$) = \mathbb{E}\left[VIX_t(T)\right] - VIX_0(T)$$

$$MeanError(\%) = 100 \times \frac{MeanError(\$)}{VIX_0(T)}$$

Where $VIX_t(T)$ is the simulated sample for maturity T at time step t and $VIX_0(T)$ is the initial VIX term structure.

The error on the sample volatility is given by:

$$VolError(\$) = \sqrt{\mathbb{E}\left[\log\left(\frac{VIX_{t_i}(T)}{VIX_{t_{i-1}}(T)}\right)^2\right]} - VIXVOL_0(T)$$

$$VolError(\%) = 100 \times \frac{VolError(\$)}{VIXVOL_0(T)}$$

The distribution of the mean and standard deviation error on each maturity of the simulated samples are shown, respectively, in Figures A1 and A2 in Appendix 2.

The last step is to validate the model under the Q measure. To do so, we ensure that the calibrated European option valuation model matches the calibration instruments. This exercise will enable us to verify that the model follows the process specifications.

In Section 3.3, we use the implied Black/Whaley volatility on ATM calls with the same maturity as the forward to back out the volatility specifications for the forward variance. The model's options prices should then yield the same options prices as the Whaley model

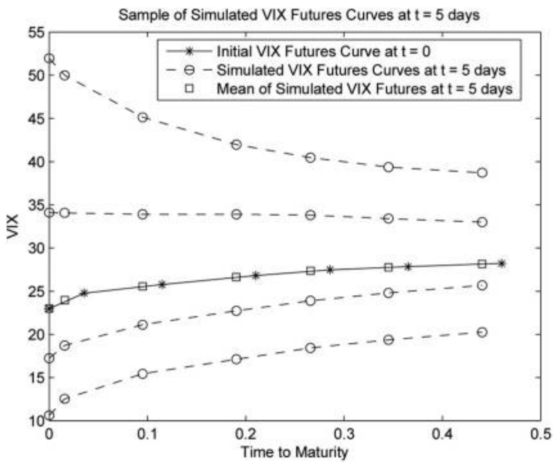

Figure 2.
Sample of the simulated VIX Futures Curve at $t = 5$ days

Maturity	Spot	First	Second	Third	Fourth	Fifth	Sixth
Days to maturity		9	29	53	72	92	116
Price	22.97	24.77	25.77	26.80	27.48	27.82	28.20
Volatility	1.2910	1.1049	0.9175	0.7951	0.7120	0.6552	0.6040

Table I.
VIX Term structure

across time to maturity and moneyness, as we are making the same assumptions of log-normality of the underlying[5]. The model implementations which allow us to capture the skew of the implied distribution will be further discussed in Section 5.

Because we are making the same assumptions (under Q) on the VIX futures distribution as the Whaley model, we should be able to price European VIX options and obtain the same results for the same VIX volatility inputs. Table A1 shows the results. A set of options of 9, 29 and 53 days to maturity was priced according to different moneyness. Being able to comparatively price vanilla options to the Whaley model should confirm that our model follows the process specifications that we have initially set.

Table II shows the volatility inputs for both models, starting with the square-root of the VIX variance swap as volatility specification that is used as input for the Whaley model. Having defined the forward variance correlations, we are then able to find the volatility specifications of the forward variance from equation (3.5) of Section 3.3.

As expected, the results indicate that the two models give similar results for the same VIX futures volatility inputs. The discrepancy between the two models across days to maturity and moneyness is (in most cases) less than 1 per cent. In any case, the absolute error is significantly smaller than the bid-ask spread of the traded VIX options. This might simply be caused by numerical error of the Monte-Carlo simulation. Adding more simulations or using variance reduction methods should yield more precise results. But this shows, however, that by taking the black volatility parameters directly from the VIX options implied volatility, we should then be able to obtain the correct market prices across moneyness and time to maturity for the VIX options with a relatively small error.

4.2 Distributions with one vs multiple factors

This section puts the emphasis on the importance of having multiple factors to simulate the curve and compares results for the distributions and VaR given by two variations of the model from equation (3.7), one of which uses multiple factors against one that uses only one factor (perfectly correlated maturities). First, we will take a look at the distributions and VaR of a long and a short position of one contract[6] in a VIX futures. We then compare the distributions and VaR on a VIX futures portfolio. The evaluation date was 3 January 2012. The initial VIX term structure details are given in Table I. The implied volatility is calculated on the day of valuation with the square-root of the variance swap on the VIX options as in the VIX methodology in equation (A.2).

	First	Second	Third
Maturity			
Days to maturity	9	29	53
Volatility	1.1049	0.9175	0.7951
Forward variance volatility	2.2098	1.4065	1.2745
Correlations between maturities			*Third*
First	1	0.8493	0.6982
Second	0.8493	1	0.8221
Third	0.6982	0.8221	1

Notes: The table shows the volatility inputs, starting with the square-root of the VIX variance swap as volatility specification that is used as input for the Whaley model. Having defined the forward variance correlations, we are then able to find the volatility specifications of the forward variance from equation (3.5) of Section 3.3

Table II.
Model inputs

Table III shows the computing time along with the explanatory power of the PCA method. The explanatory power indicates the accuracy of the approximation we are making on the curve movements, considering the correlations between maturities and how many factors we use from the PCA. The maximum number of factors is seven because the correlation matrix is in this case 7×7 (spot VIX plus six other maturities). Using seven factors is the equivalent to simulating the seven correlated maturities by using a more classical approach of the Cholesky factorisation for correlated random samples, hence why it explains 100% of the curve movements. Using fewer factors means dimensionality reduction, as well as reducing the computing time. However, we should expect a certain decrease to be introduced in the explanatory power. This decrease is dependent on the correlations between forward variance maturities. If the correlations are strong, i.e. close to 1, then fewer factors are required for a good approximation. The less maturities are correlated, the more factors are then necessary for a good approximation.

For 25,000 simulated paths over 90 time steps, the computing goes from 3.084 s using seven factors to 1.843 s when using only one factor. However, there is a substantial loss in explanatory power going from 100 per cent to 71.8 per cent. In the remainder of the section, we will document how this loss of explanatory power may affect the P&L distribution of the simulated scenarios of using only one factor and multiple factors.

Let us begin with a position in a single futures contract. We first take a look at a long position in the second maturity contract which has 29 business days to maturity. The analysis is made at $t = 5$ days. Figures A3 and A4 in Appendix 3 present the distributions and VaR obtained with the one factor model and the multiple factors model, respectively. Table IV presents each distribution's parameters for the two model variations. As we are able to see in Table IV, the use of multiple factors has

Factors	CPU time(s)	Explained (%)
1	1.843	71.80
2	2.026	87.79
3	2.232	93.26
4	2.548	95.98;
5	2.700	97.81
6	2.952	99.11
7	3.084	100

Notes: This table shows the average computing time for 25,000 simulations for 90 time steps and the explanatory power using a different number of factors from the PCA method to approximate the correlation between each maturity

Table III. Computing time vs number of factors used and explanatory power of PCA

Model	Mean	SD	Skewness	Kurtosis
One-factor	−251.45	3,488.52	0.5108	3.4544
Multiple factors	−250.74	3,443.43	0.4873	3.4373

Table IV. Distribution parameters on the second maturity contract P&L

little impact on the distribution parameters where all four presented moments show slight differences. The VaR measures at 95 and 99 per cent are substantially the same for both model variations, with no significant differences. The next case in Figures A5 and A6 presents the distribution of a short position in the same contract. The distribution parameters remain the same as for the long position but the VaR changes. Once again, the VaR between the two model variations shows almost no significant differences.

Although the distributions for a position in one contract are undeniably similar from one variation of the model to another, let us take a look at the distribution of the roll-down effect (drift) for both. Figures A7 and A8 show the distribution obtained with the one factor model and multiple factors model, respectively. Despite the fact that both means are very similar, the shape of the distributions and higher moments differ significantly. From the descriptive statistics in Table V and the shapes of the distributions, we can make the statement that using only one factor is in fact limiting when it comes to the number of possible scenarios.

Following on the fact that the number of factors has an impact on the roll-down distribution, this explains the slight differences in the single contract distributions and should consequently have a more pronounced effect on a portfolio holding multiple contracts of different maturities.

Let us now consider a portfolio that is short-one futures contract of the first maturity and long-one contract of the fourth maturity with the market conditions in Table I.

Following earlier discussions and empirical evidence from Whaley (2013), this strategy should have a positive mean return caused by the roll-down effect because the VIX curve is currently in contango and the slope is steeper between the first month and the spot than between the third and fourth months. Distributions for both model variations methods are presented in Figures A9 and A10 along with the parameters of each distributions in Table VI.

Once more, even with similar means, Table VI shows that the higher moments differ between both distributions.

When comparing both distributions, we can see that the multiple factors model enables more possible outcomes on both the up and down side. The VaR for the one

Table V. Distribution parameters on roll-down effect in the second maturity contract

Model	Mean	SD	Skewness	Kurtosis
One-factor	−251.45	136.04	1.0916	4.9722
Multiple factors	−250.74	166.16	0.6361	3.9215

Table VI. Distribution parameters on calendar spread for each method

Model	Mean	SD	Skewness	Kurtosis
One-factor	784.35	1,239.13	−0.9840	4.6133
Multiple factors	778.98	1,431.21	−0.6452	3.8756

factor model is undervalued at 95 and 99 per cent when compared to the three factors model VaR.

5. Discussion

As the shape of the term structure has a direct impact on the VIX futures returns, it is, therefore, important to take it into account. By using the no arbitrage framework, we are able to build the VIX curve dynamics that relies on actual given market data instead of parameter calibration, but also fit the initial term structure perfectly in any given situation. The use of multiple Brownian motions to drive the curve dynamics also plays an important role in allowing the simulated term structure to take different shapes.

Taking into consideration the previous research, the presented model seems to be a more realistic approach for risk management when dealing with a portfolio of VIX futures. As we are dealing with a forward-looking measure, the usefulness of historical data is arguable when estimating distribution parameters. We can, however, rely on market information to build the distributions of VIX futures or price other VIX derivatives. As previously mentioned in Section 5.4, the relation between the realised VIX and its implied volatility should be studied to determine whether the VIX implied volatility is, in fact, an efficient predictor of its realised volatility, as it is the case with stock indices.

Aside from obtaining a more realistic VIX futures VaR model, the model could also be used to price VIX exotic options or VIX ETPs with their respective options. Like the VIX, the VXX (or any other VIX ETP) has a constant time to maturity. This fixed time to maturity can change from one rolling period to the other, depending on the number of business days between the first and second month contract. The usual time to maturity ranges between 18 and 25 business days.

More sophisticated modifications could also be investigated to obtain implied distributions of the VIX futures closer to the market expectations. For example, we have overlooked the skewness aspect of the distribution by using a deterministic volatility of volatility that is only a function of time to maturity. By using a volatility surface in the model (instead of a variance swap term structure for the volatility of the forward volatility), we should be able to capture the positive correlation between the VIX and its volatility, and therefore the skewness of the distribution. Staying consistent with the market model approach, this could be achieved by using local volatility as in Dupire (1997). We would, however, have to undergo certain modifications because we would have to back out the local volatility of the forward variance from the implied volatility surface of the VIX options. Another solution to capture the positive skewness of the VIX distribution would be to include a stochastic jump in the model as in Cont and Kokholm (2013), but further model calibrations would be needed to have the proper parameters for the Poisson process used for the jump to fit the implied volatility of the VIX futures.

6. Conclusion

Knowing from Whaley (2013) that the slope of the VIX futures term structure plays an important role in predicting their returns, we have argued that the no arbitrage framework should be better suited to capture this measure in risk analysis. By taking the term structure as an input, the model fits the initial term structure perfectly and the use of the PCA for the curve dynamics allows it to take different shapes.

Our results show that, when comparing our multi-factorial model to a one factor model, there is a significant difference in the distribution of the roll-down of the VIX futures contract. The differences in the roll-down distribution when using a multi-factorial model has a slight impact on the skewness and kurtosis of the distribution when holding a position in a single VIX futures contract, but becomes far more apparent when holding a VIX futures portfolio of different maturities. This, in turn, significantly impacts the value-at-risk (VaR) of a portfolio holding multiple maturities.

The results also show that the use of PCA to model the curve's dynamics reduces the number of factor needed to drive the curve while still capturing most of the possible curve scenarios (93 per cent vs 71 per cent when using only one factor). When compared to more classical methods, such as correlated random samples generation using Cholesky factorisation – which can capture 100 per cent of the possible curve's movements but only uses one factor per simulated maturity – the PCA significantly reduces the computing time (approximately 38 per cent) when simulating the entire VIX curve that usually includes the spot VIX along with six additional forward maturities.

Using an approach similar to Bergomi (2005), the model can be used jointly with a process for the S&P 500 that gives a complete market free of arbitrage and can also be used to price VIX exotic options, as well as VIX ETPs options.

Modifications are still to come regarding the volatility of volatility, which, in this case, was a deterministic function of time to maturity calculated by the square-root of a variance swap from the VIX European options. Further developments are expected to include a skewness component that should be added by taking into account the volatility across moneyness on the VIX options. This, in turn, would allow us to capture the correlation between the volatility and the volatility of volatility.

References

Bergomi, L. (2005), "Smile dynamics ii", *Risk*, Vol. 18 No. 10, pp. 67-73, Copyright - Copyright Risk Waters Group Oct 2005; Caractéristique du document - graphs; references; equations; Last update - 2012-02-09.

Bergomi, L. (2008), "Smile dynamics iii", *Risk*, Vol. 21 No. 10, p. 90, Copyright - Copyright Risk Waters Group Oct 2008; Caractéristique du document – Equations; Graphs; Tables; Last update - 2012-02-09.

Bergomi, L. (2009), "Smile dynamics iv", *Risk*, Vol. 22 No. 12, pp. 94-100, Copyright - Copyright Incisive Media Plc Dec 2009; Caractéristique du document - Graphs, Equations, Last update - 2012-02-09.

Black, F. (1976), "The pricing of commodity contracts", *Journal of Financial Economics*, Vol. 3 Nos 1/2, pp. 167-179.

Buehler, H. (2006), "Consistent variance curve models", *Finance and Stochastics*, Vol. 10 No. 2, pp. 178-203.

CBOE (2003), "CBOE VIX white paper", available at: www.cboe.com

Cont, R. and Kokholm, T. (2013), "A consistent pricing model for index options and volatility derivatives", *Mathematical Finance*, Vol. 23 No. 2, pp. 248-274.

Driessen, J., Klaassen, P. and Melenberg, B. (2003), "The performance of multi-factor term structure models for pricing and hedging caps and swaptions", *Journal of Financial and Quantitative Analysis*, Vol. 38 No. 3, pp. 635-672.

Dupire, B. (1992), "Arbitrage pricing with stochastic volatility", *Société Générale*.

Dupire, B. (1997), *Pricing and Hedging with Smiles, Mathematics of Derivative Securities*, Dempster, M. A.H. and Pliska, S.R. (Eds), Cambridge University Press, Cambridge.

Fassas, A.P. and Siriopoulos, C. (2012), "The efficiency of the VIX futures market: a panel data approach", *The Journal of Alternative Investments*, Vol. 14 No. 3, pp. 55-65, 5. Copyright – Copyright Euromoney Institutional Investor PLC Winter 2012, Document characteristics – Tables, Equations; Graphs; Last update – 2015-05-25; SubjectsTermNotLitGenreText - United States-US, available at: https://acces.bibl.ulaval.ca/login?url=https://search-proquest-com.acces.bibl.ulaval.ca/docview/921633777?accountid=12008

Grünbichler, A. and Longstaff, F.A. (1996), "Valuing futures and options on volatility", *Journal of Banking and Finance*, Vol. 20 No. 6, pp. 985-1001.

Heath, D., Jarrow, R. and Morton, A. (1992), "Bond pricing and the term structure of interest rates: a new methodology for contingent claims valuation", *Econometrica: Journal of the Econometric Society*, Vol. 60 No. 1, pp. 77-105.

Heston, S.L. (1993), "A closed-form solution for options with stochastic volatility with applications to bond and currency options", *Review of Financial Studies*, Vol. 6 No. 2, pp. 327-343.

Kokholm, T. and Stisen, M. (2015), "Joint pricing of VIX and SPX options with stochastic volatility and jump models", *The Journal of Risk Finance*, Vol. 16 No. 1, pp. 27-48, available at: https://doi.org/10.1108/JRF-06-2014-0090

Lian, G.-H. and Zhu, S.-P. (2013), "Pricing VIX options with stochastic volatility and random jumps", *Decisions in Economics and Finance*, Vol. 36 No. 1, pp. 71-88, available at: https://doi.org/10.1007/s10203-011-0124-0

Litterman, R.B. and Scheinkman, J. (1991), "Common factors affecting bond returns", *The Journal of Fixed Income*, Vol. 1 No. 1, pp. 54-61.

Lu, Z. and Zhu, Y. (2010), "Volatility components: the term structure dynamics of VIX futures", *Journal of Futures Markets*, Vol. 30 No. 3, pp. 230-256.

Rebonato, R. (1996), "Interest-Rate Option Models".

Scherer, K.P., Avellaneda, M. (2002), "All for one . . . one for all? A principal component analysis of Latin American Brady bond debt from 1994 to 2000", *International Journal of Theoretical and Applied Finance*, Vol. 5 No. 1, pp. 79-106.

Sepp, A. (2008), "Pricing options on realized variance in the Heston model with jumps in returns and volatility", *The Journal of Computational Finance*, Vol. 11 No. 4, p. 33.

Shu, J. and Zhang, J.E. (2012), "Causality in the VIX futures market", *Journal of Futures Markets*, Vol. 32 No. 1, pp. 24-46, available at: http://dx.doi.org/10.1002/fut.20506

Simon, D.P. and Campasano, J. (2014), "The VIX futures basis: evidence and trading strategies", *Journal of Derivatives*, Vol. 21 No. 3, pp. 54-69, 6 Copyright - Copyright Euromoney Institutional Investor PLC Spring 2014; Document characteristics - Tables; Graphs; Equations;; Last update - 2015-05-25; SubjectsTermNotLitGenreText-United States-US, available at: https://acces.bibl.ulaval.ca/login?url=https://search-proquest-com.acces.bibl.ulaval.ca/docview/1508755938?accountid=12008

Wang, Z. and Daigler, R.T. (2011), "The performance of VIX option pricing models: empirical evidence beyond simulation", *Journal of Futures Markets*, Vol. 31 No. 3, pp. 251-281.

Whaley, R.E. (1993), "Derivatives on market volatility: hedging tools long overdue", *The Journal of Derivatives*, Vol. 1 No. 1, pp. 71-84.

Whaley, R.E. (2013), "Trading volatility: at what cost?", *The Journal of Portfolio Management*, Vol. 40 No. 1, pp. 95-108.

Zhang, J.E. and Zhu, Y. (2006), "VIX futures", *Journal of Futures Markets*, Vol. 26 No. 6, pp. 521-531.

Appendix 1

VIX methodology

A new methodology for the VIX calculation was introduced on 22 September 2003. The new methodology now uses SPX options rather than OEX options and does not rely on the Black–Scholes model. It now relies on a more robust methodology for pricing a continuously monitored variance swaps with a wider spectrum of out-of-the-money index calls and puts. These changes were made to better reflect the trading motives of participants in market index options. Including additional option series would also help make the VIX less sensitive to single-option price, and therefore less susceptible to manipulation.

As explained in CBOE(2003), the VIX follows a precise methodology where it is calculated as a continuously monitored variance swap. It is the square root of the 30-day average variance swap rate using the near- and next term SPX options. To minimise pricing anomalies, near-term options must have at least one week to expiration. When the near-term options have less than a week to expiration, the second and third months are used.

As with interest rates swaps, both counterparties will agree upon a fair value for the fixed rate at inception. The variance swap price can be calculated with the following equation:

$$\sigma_0^2(T) = -2\int_0^1 \frac{1}{K^2} Put(K,T)\,\mathrm{d}K - 2\int_1^\infty \frac{1}{K^2} Call(K,T)\,\mathrm{d}K \tag{A.1}$$

However, for equation (A.1) to hold, a continuum of strikes must be available. To solve the problem, the CBOE uses a replication method to approximate the equation with a discreet set of available strikes. Note: *Call*(K, T) and *Put*(K, T) are the forward values of the options:

$$\sigma_T^2 = \frac{2}{T}\sum_i \frac{\Delta K_i}{K_i^2} e^{RT} Q(K_i) - \frac{1}{T}\left[\frac{F}{K_0} - 1\right]^2 \tag{A.2}$$

Where:

σ_T^2 = Price of the variance swap for maturity T;
T = Time to expiration in years;
F = Forward index level derived from index option prices;
K_0 = The first strike below the forward index level F;
K_i = Strike price of the i^{th} OTM option; a call if $K_i > K_0$ and a put if $K_i < K_0$; both put and call if $K_i = K_0$;
ΔK_i = Interval between strike prices; half the difference between the strikes on either sides of K_i; $\Delta K_i = \frac{K_{i+1} - K_{i-1}}{2}$; and
R = Risk-free interest rate to expiration $Q(K_i)$: The mid-point of the bid-ask spread for each option with strike K_i.

Note: For the lowest strike, ΔK is simply the difference between the lowest strike and next higher strike. Likewise, ΔK for the highest strike is the difference between the highest strike and next lower strike.

After finding the variance swap price for the near- and next-term SPX options, the VIX can be calculated and using equation (A.3):

$$VIX = 100 \times \sqrt{\{T_1\sigma_1^2\left[\frac{N_{T_2} - N_{30}}{N_{T_2} - N_{T_1}}\right] + T_2\sigma_2^2\left[\frac{N_{30} - N_{T_1}}{N_{T_2} - N_{T_1}}\right]\} \times \frac{N_{365}}{N_{30}}} \tag{A.3}$$

Where:

N_{T_1} = number of minutes to settlement of the near-term option;
N_{T_2} = number of minutes to settlement of the next-term option;
N_{30} = number of minutes in 30 days = 43,200; and
N_{365} = number of minutes in a year = 525,600.

Appendix 2. Model validation

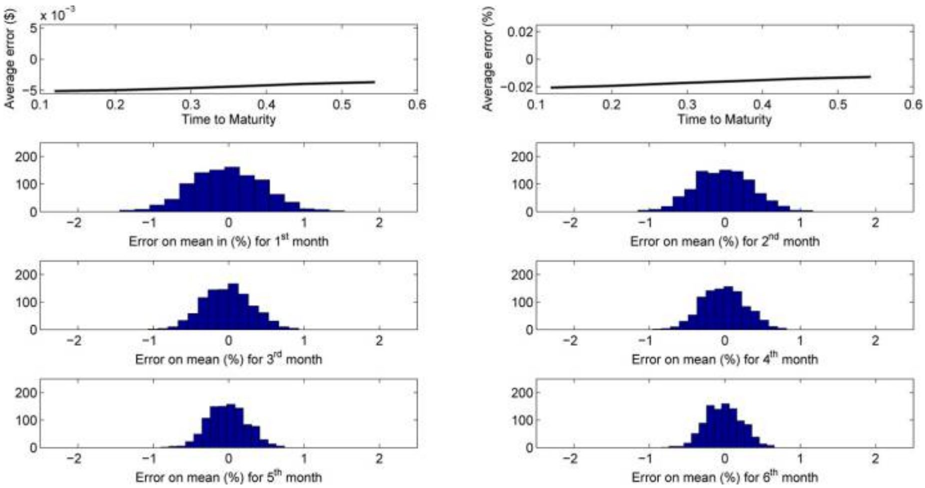

Figure A1.
Average error on the initial VIX curve Monte-Carlo approximation at t = 90 days with 1,000 simulations of 25,000 paths with distribution of error per maturity

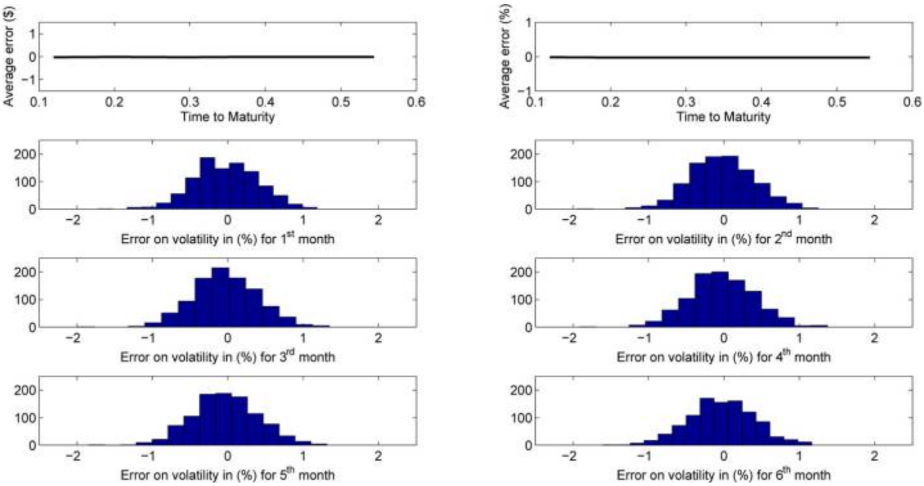

Figure A2.
Average error on the initial VIX volatility using Monte-Carlo approximation at t = 90 days with 1,000 simulations of 25,000 paths with distribution of error per maturity

Model	Days to maturity	Volatility	Option price	Absolute error	Relative error
	9	1.1049	2.9944		
Whaley	29	0.9175	4.5954		
(Black)	53	0.7951	5.8102		
HJM	9	1.1049	2.9755	−0.0189	−0.630%
Simulation	29	0.91755	4.5904	−0.0050	−0.109%
	53	0.7951	5.8209	0.0107	0.184%

Note: The table shows the comparison on European option pricing between the Whaley (closed-form) and the simulated forward variance model (100,000 simulations) for different maturities. Using the same inputs for both models, the results show an error that is relatively small (>1% on average), and in any case, significantly smaller than the usual bid-ask spread on these options

Table AI.
European VIX option pricing

Appendix 3. Distribution and VaR estimate

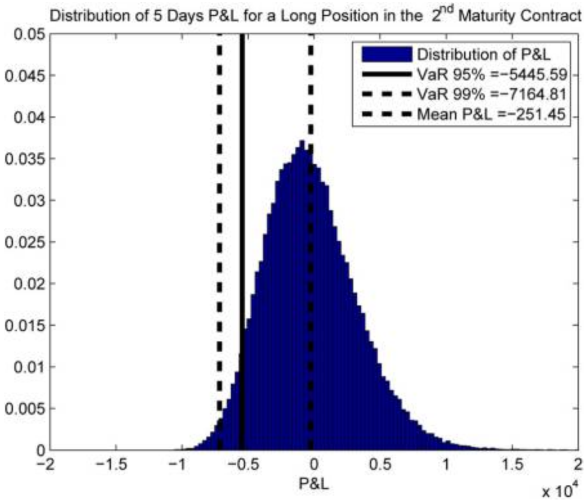

Figure A3.
One-factor model P&L distribution of on a long position consisting of one contract of the second maturity (29 business days to expiration) VIX futures at $t = 5$ days along with 95% and 99% VaR estimated with the model from equation (3)0.7 with 100,000 simulations

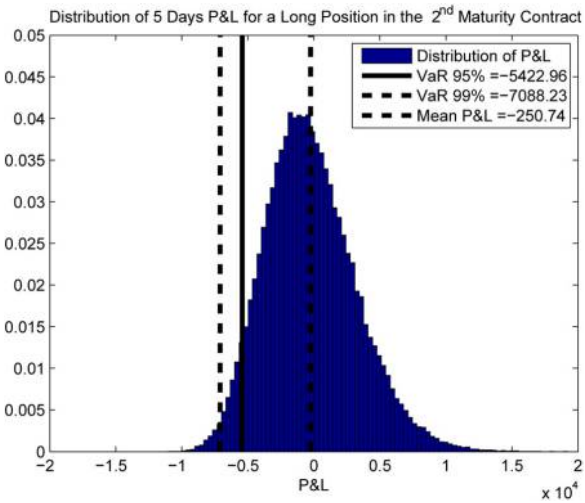

Figure A4.
Multiple factors model P&L distribution of on a long position consisting of one contract of the second maturity (29 business days to expiration) VIX futures at $t = 5$ days along with 95% and 99% VaR estimated with the model from equation (3)0.7 with 100,000 simulations

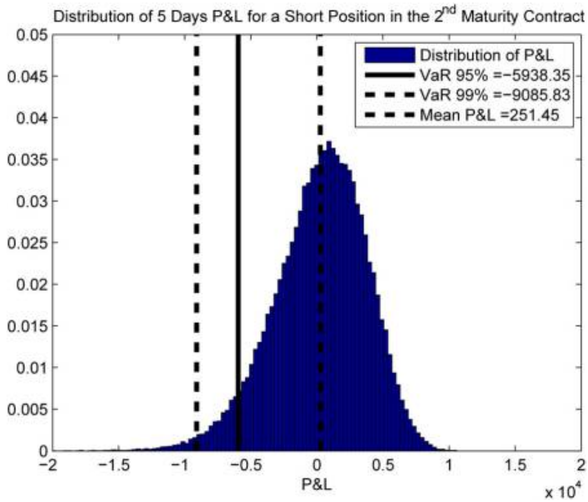

Figure A5.
One-factor model P&L distribution of on a short position consisting of one contract of the second maturity (29 business days to expiration) VIX futures at $t = 5$ days along with 95% and 99% VaR estimated with the model from equation (3)0.7 with 100,000 simulations

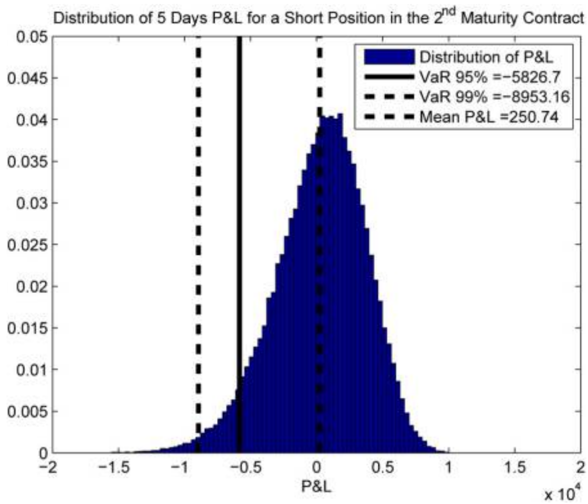

Figure A6.
Multiple factors model P&L distribution of on a short position consisting of one contract of the second maturity (29 business days to expiration) VIX futures at $t = 5$ days along with 95% and 99% VaR estimated with the model from equation (3)0.7 with 100,000 simulations

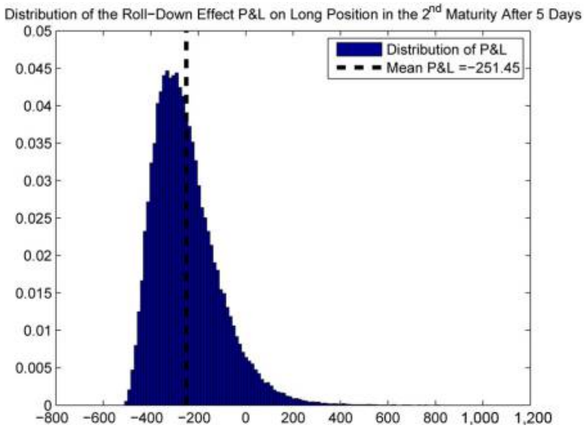

Figure A7.
One-factor roll-down effect P&L distribution on the second maturity (29 business days to expiration) VIX futures at $t = 5$ days with 100,000 simulations

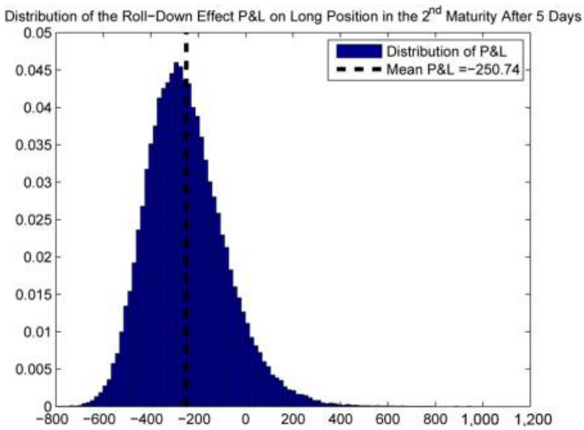

Figure A8.
Multiple factors roll-down effect P&L distribution on the second maturity (29 business days to expiration) VIX futures at $t = 5$ days with 100,000 simulations

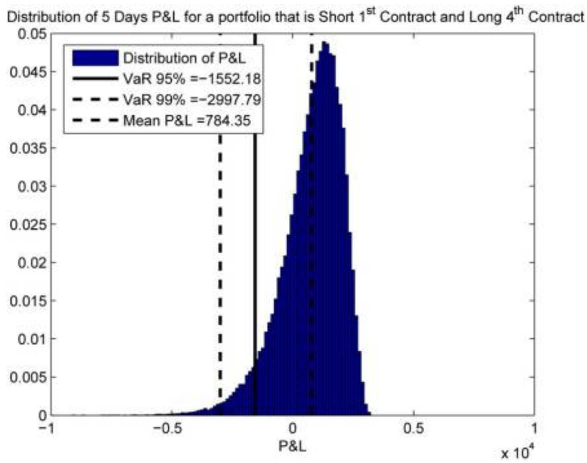

Figure A9.
One-factor P&L distribution calendar spread strategy between the first and fourth maturity VIX futures at $t = 5$ days along with 95% and 99% VaR estimated with the model from equation (3)0.7 with 100,000 simulations

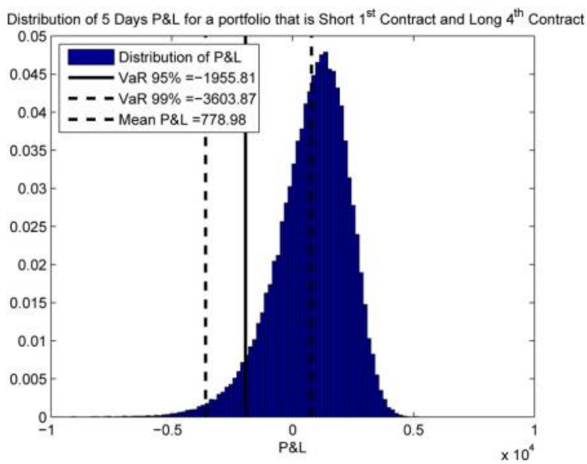

Figure A10.
Multiple factor P&L distribution for the calendar spread strategy between the first and fourth maturity VIX futures at $t = 5$ days along with 95% and 99% VaR estimated with the model from equation (3)0.7 with 100,000 simulations

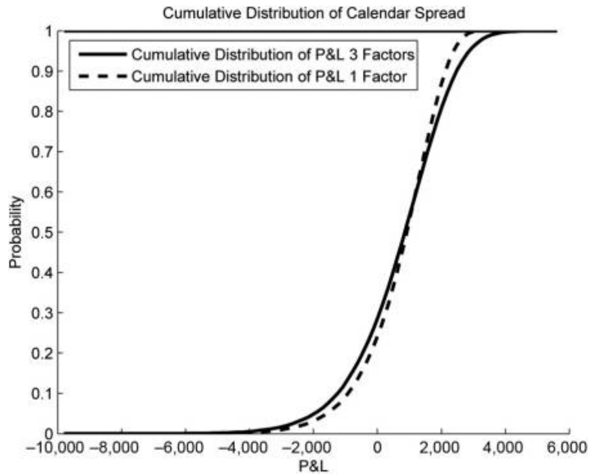

Figure A11.
Multiple and one-factor P&L cumulative distribution for the calendar spread strategy between the first and fourth maturity VIX futures at $t = 5$ days with 100,000 simulations

Corresponding author
Philippe Bélanger can be contacted at: philippe.belanger.1@ulaval.ca