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Consistency between S&P500 and VIX derivatives: Insights from model-free VIX futures pricing

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This paper studies the interdependencies between the VIX futures market and the S&P500 and VIX options markets using a model-free pricing method for VIX futures. We show that the replication strategy for the VIX futures greatly deviates from observed prices. Limited strike ranges do not suffice to explain these deviations, whereas the options' bid–ask spreads can explain most of it. After controlling for the spreads, we find a lead–lag structure between markets segmented by product, not by its underlying. If options markets imply higher volatility risks than VIX futures, options prices in both markets adjust and vice versa.

KEYWORDS

model-free pricing, S&P500 options, VIX futures, VIX options, volatility risk

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1 | INTRODUCTION

Since the financial crisis, the markets for VIX derivatives, futures and options alike, have experienced an enormous upswing in trading volume. Theoretically, VIX derivatives are directly connected to the stock market, because the VIX^2 is constructed from a portfolio of S&P500 (SPX) options. As a result, the price of volatility should be the same in both markets. In this paper, we analyse whether the theoretical connection between the S&P500 and VIX market is also empirically valid. To do so, we study two possible ways to trade volatility. Investors can either use a portfolio of S&P500 and volatility (VIX) options, which is a model-free replication of the (squared) VIX futures price, or investors can take a position in the VIX futures itself. Both strategies have the same payoff and thus have the same price in theory. All empirical deviations from this fundamental no-arbitrage relation have to be justified by market restrictions. Otherwise, this type of inefficiency could result in a superior way for investors to trade, hedge and measure volatility risks. Namely, it could be that either the options- or the futures-based alternative is the better investment vehicle.[1]

We find that the law of one price is violated and price deviations between the options and VIX futures markets can be quite large. The mispricings, however, can mainly be explained by liquidity risk, measured by the bid–ask spreads in the option markets. The latter explain most of the price dispersions before and during the financial crisis. Afterwards, the explanatory power is dampened for contracts with a short time to maturity. However, for longer maturities, the results remain unchanged. For short-term contracts, we find a lead–lag structure when price deviations cannot be explained by the bid–ask spreads. Implied volatility smiles show that if options are more costly than VIX futures, priced volatility risks in options adjust. If the prices of VIX futures are above their option implied arbitrage bounds, the VIX futures price adjusts. Therefore, the information flow

[1] Analysing the validity of the empirical relation between the SPX and VIX options market is also important from a modeling perspective. Most recent models, for example Bardgett, Gourier, and Leippold (2016) among others, jointly price S&P500 and VIX options based on the assumption that the theoretical connection between both markets also holds empirically.

between the two markets (VIX futures and SPX/VIX options) depends on which product implies a higher volatility risk and, as a result, the lead–lag structure is not just one-sided. However, we observe that VIX futures lead the option markets more often, especially after 2010. As a consequence, our results suggest that VIX futures are the better tool when it comes to the discovery of volatility risks.

Our analysis is mainly model-free and relies on the theoretical finding that squared VIX futures can be replicated as the difference between a portfolio of SPX options and one of VIX options. The first captures expected forward volatility risk, the second captures the expected variation of the VIX futures, and thus volatility-of-volatility risk. We show that the SPX portfolio makes up a large part of the short-term futures price, whereas the VIX options portfolio is important for long-term futures. The composition of the replication portfolio emerges directly from no-arbitrage pricing arguments. Limits of arbitrage could stem from limited strike ranges in the involved option portfolios. Therefore, we study the impact of this form of market incompleteness on the performance of the replication, which is similar to the analysis of Jiang and Tian (2007) for the VIX. We generate stock and volatility option prices in a jump-diffusion model for different strike grids. Our results indicate that without interpolation techniques, market incompleteness yields relative pricing errors in the range of 2–5%. If we use interpolation methods, the error becomes less than 1%. Especially for empirical strike ranges, we find the errors to be small. Our analysis shows that market incompleteness is unlikely to explain all the pricing errors we observe empirically.

The portfolio of SPX options captures parts of the term structure of the risk-neutral expected variance and thus relies on its measurement. We compare two well-known methods to build the variance term structure: the method of the CBOE (2016) and the formula of Bakshi, Kapadia, and Madan (1990). Our results show that if the method of the CBOE is used, the model-free VIX future replication has significant biases: prices are on average too low and differences between the two markets are significantly negatively skewed and highly non-normally distributed. This finding holds for all maturities. In contrast, when using the formula of Bakshi et al. (2003), pricing errors are on average zero and more symmetrically distributed. Thus, the induced price deviations are substantially less skewed and therefore more normal.

Contemporaneous regressions show that the price deviations between the futures and options markets can be well explained by liquidity. Until 2010, it explains about 48–67% of the variation in price deviations. In particular, we find that the bid–ask spreads in the option markets provide all the explanatory power, leaving various volume measures, the TED spread and market-wide volatility risks (VIX and VVIX) insignificant. After 2010, the explanatory power of the bid–ask spreads declines to 22% for short-term contracts. For longer maturities, liquidity risk is still the main driver with R^2s between 50% and 53%. In all regressions, the bid–ask spread of SPX options has the highest explanatory power.

To study market reactions to obvious mispricings, we define model-free upper and lower bounds for the short-term VIX futures price in dependence on the bid- and ask-prices of its replicating portfolio. A price deviation beyond the bounds is a signal for market movements on subsequent days. If futures are more costly than options, the implied volatilities do not react and, consequently, futures become less expensive. If options imply higher volatility risks than the VIX futures, the level of implied volatility smiles of both option markets (SPX and VIX) decrease the following day. For the SPX options we also find a significant change in the smile's slope. The different effects on the level and slopes can be explained by the weights of the SPX and VIX options in the replication portfolio.

Our research is related to the literature on model-free pricing of VIX futures, pioneered by Carr and Wu (2006). They show that futures can be replicated by the expected variation in stock prices over the next 30 days from some future point in time and the expected variance of the futures itself. In addition, the CBOE suggests pricing VIX futures using the VIX term structure and estimating the variance of the futures using historic data.[2] However, the authors remain silent about the performance of the replication. We show that using the CBOE's VIX term structure leads to systematic biases in the replicated VIX futures price. Further, the present paper is related to the literature on robust estimation of expected volatility from option prices. Similar to Jiang and Tian (2005, 2007) for the VIX, we conduct a sensitivity analysis for model-free futures pricing.

The paper which is closest to ours is Park (2015). Park researches lead–lag structures between the SPX and VIX derivatives markets. Thereby, he uses the model-free valuation method as well, but rewrites it and looks at *price dislocations* between the SPX and VIX markets. He then calibrates a model to his time-series and thereby merges the manifold information from different maturities into a single output. As a result, all his findings stem from a model-based analysis. We deviate from his procedure in several aspects. First, our analysis is model-free and grounded on nothing more than a replication strategy. Therefore our analysis relies solely on no-arbitrage conditions and does not use a parametrized model. Second,

[2] See http://cfe.cboe.com/education/vixprimer/features.aspx

we look at different maturity buckets separately and analyse the replication quality for each of them. Third, we compare the VIX futures and the options markets. Fourth, we are the first to research the sensitivity of the model-free approach to the input data.

This paper is structured as follows. The next section describes the model-free valuation method and the impact of limited strike ranges. Section 3 discusses and explains the empirical performance. There, we also show the market's reaction to large price deviations between markets. Section 4 concludes.

2 | MODEL-FREE VIX FUTURES EVALUATION

2.1 | Theory

VIX futures can be evaluated in a model-free way as the difference between an SPX and a VIX options portfolio. This evaluation method has been known since the introduction of the VIX. Its theory is described by, for example, Carr and Wu (2006). Since the model-free formula for futures relies only on the prices of S&P500 and VIX derivatives, it provides a measure of the integration of the futures and options markets when it comes to trading volatility. The formula is essentially a hedge for futures. Consequently, a violation of the pricing formula would directly lead to arbitrage opportunities between both markets by exploiting the hedge relation. According to no-arbitrage pricing theory, the replication formula should work perfectly. However, little is known about its real-world performance. If pricing errors of significant magnitude exist, the question arises if these errors can be explained by market frictions or if they indeed provide arbitrage opportunities. The latter would imply that the options and futures markets are not perfectly integrated. The following theorem describes the pricing of VIX futures using portfolios of S&P500 and VIX derivatives.

Theorem 1 (Model-free valuation of VIX futures) *The squared VIX futures price* $\left(F_t^T\right)^2$ *is the difference between the expected forward variance* $\mathbb{E}_t\left[\left(VIX_T^{30D}\right)^2\right]$ *and a convexity correction*

$$\left(F_t^T\right)^2 = \mathbb{E}_t\left[\left(VIX_T^{30D}\right)^2\right] - \left[Convexity\ Correction\right]_t, \tag{1}$$

$$\mathbb{E}_t\left[\left(VIX_T^{30D}\right)^2\right] = \frac{1}{30D}\left((T+30D)\left(VIX_t^{T+30D}\right)^2 - T\left(VIX_t^T\right)^2\right), \tag{2}$$

$$\left[Convexity\ Correction\right]_t = 2e^{rT}\left(\int_{F_t^T}^{\infty} C_t^{VIX}(T,K)dK + \int_0^{F_t^T} \mathcal{P}_t^{VIX}(T,K)dK\right), \tag{3}$$

where $\{C_t^{VIX}(T,K)\}_K$ *and* $\{\mathcal{P}_t^{VIX}(T,K)\}_K$ *are the prices of the puts and calls for VIX options with strike* K. VIX_t^T *is, depending on our later analysis, the volatility index for maturity* T *calculated using the method of the CBOE (2016) or Bakshi et al. (2003), respectively. For a proof of the theorem, see Appendix A.1.*

The model-free formula not only depends on the expectation about future values of VIX^2, but on a *convexity correction* as well. This correction is given by a portfolio of VIX options, which leads to a direct exposure to volatility-of-volatility risk. In our empirical section we will show that both parts of the replication portfolio are important.

In Equation 1 the right-hand side depends on the VIX futures price, because it is needed for the calculation of the convexity correction. Since our focus is on the consistency of futures and option prices, both sides of the equation should only depend on one asset class. As a work-around, we approximate the futures price via put/call parity.

Corollary 1 (Approximative valuation of VIX futures) *In the setting of Theorem 1,*

$$\left(F_t^T\right)^2 \approx MF_t^2(T) \equiv \mathbb{E}_t\left[\left(VIX_T^{30D}\right)^2\right] - \left[Approx.\ Conv.\ Corr.\right]_t, \tag{4}$$

$$\left[Approx.\ Conv.\ Corr.\right]_t = 2e^{rT}\left(\int_{\hat{F}_t^T}^{\infty} C_t^{VIX}(T,K)dK + \int_0^{\hat{F}_t^T} \mathcal{P}_t^{VIX}(T,K)dK\right), \tag{5}$$

where $\hat{F}_t^T$ is the VIX futures price implied by put/call parity of VIX options prices.[3]

Our approximation does not depend on VIX future prices as an input.[4] So we provide an independent model-free approach for pricing these products. It is worth mentioning that the proposed pricing formula in Theorem 1 has to hold even if only the right-hand side of Equation 1 is a portfolio which is directly investable. Squared futures are not traded. However, market participants can get an exposure to it, as can be seen relatively easily by Itô's lemma. The lemma implies that the dynamics of $\left(\mathrm{F}^T\right)^2$ can be replicated by taking a long position in a VIX future and by buying a portfolio of VIX options, since

$$d\left(\mathrm{F}_t^T\right)^2 = 2\mathrm{F}_t^T d\mathrm{F}_t^T + \left(d\mathrm{F}_t^T\right)^2 \approx 2\mathrm{F}_t^T \Delta\mathrm{F}_{t+1}^T + \Delta\left[\text{Convexity Correction}\right]_{t+1}, \tag{6}$$

where $\Delta X_{t+1} = X_{t+1} - X_t$. The approximation in Equation 6 is valid, because the convexity correction equals the expected variance of the futures price. A replicable squared futures price implies that a violation of the model-free pricing formula indeed leads to an imbalance between the futures and options markets.

2.2 | Measures of expected variance

To determine the squared VIX futures price solely from option prices, we need to determine $(VIX_t^T)^2$ and $(VIX_t^{T+30D})^2$ of Equation 2. We analyse two different approaches in this paper. First, we follow the calculation method of the CBOE for the VIX, which is based on the seminal work of Demeterfi, Derman, Kamal, and Zou (1999) on the fair value of variance, or equivalently, on the model-free implied variance of Britten-Jones and Neuberger (2000). Assuming a continuous price process, they show that the variance swap rate approximately equals the price of a portfolio of out-of-the-money (OTM) options $O_t(K, T)$, where each option is inversely weighted by its squared strike price (K):

$$VS_t^T = \frac{2}{T} e^{rT} \int_0^\infty \frac{O_t(K, T)}{K^2} dK + \varepsilon_t^T. \tag{7}$$

Importantly, ε_t^T is the approximation error due to discontinuous movements in the underlying price process. The CBOE's $(VIX_t^T)^2$ measure is a discretized version of (7). It is thus a biased estimate of the variance swap rate over time T:

$$VS_t^T \approx (\mathrm{VIX}_t^T)^{2,\mathrm{CBOE}} = \frac{2e^{rT}}{T} \sum_i^N \left[\frac{\Delta K_i}{K_i^2} O_i^{SPX}(K_i, T)\right] - \frac{1}{T}\left(\frac{\mathcal{F}_t(T)}{K_0} - 1\right)^2. \tag{8}$$

In the first term, $O_i^{SPX}(K_i, T)$ is the ith SPX option price, which is the average of the bid- and ask-prices. The second part of Equation 8 is a correction term which accounts for the fact that usually no option is directly traded at-the-money (ATM). The term comprises $\mathcal{F}_t(T)$, which is the S&P500 forward index level derived from SPX option prices, and K_0, which is the first strike price below $\mathcal{F}_t(T)$.[5] The CBOE's volatility index $\mathrm{VIX}^{30\mathrm{D}}$ then follows from linear interpolation, using two maturities with $T^- \leq$30D and $T^+ \geq$30D

$$\mathrm{VIX}_t^{30\mathrm{D}} = \sqrt{\left\{T^-\left(VIX_t^{T^-}\right)^2 \frac{T^+ - 30\mathrm{D}}{T^+ - T^-} + T^+\left(VIX_t^{T^+}\right)^2 \frac{30\mathrm{D} - T^-}{T^+ - T^-}\right\} \frac{365}{30}}. \tag{9}$$

Although VIX futures are written on the $\mathrm{VIX}^{30\mathrm{D}}$, we address the question whether its jump-induced error ε_t^T also affects the model-free VIX futures pricing. Therefore, we construct $(VIX_t^T)^2$ using the model-free measure of implied variance of Bakshi et al. (2003), which comprises the possibilities of jumps in the S&P500. Based on the quadratic variation, the no-arbitrage relation and the assumption that the price process grows at the risk-free rate under the risk-neutral measure, Bakshi et al. (2003)

[3] VIX options are written on the VIX future. Thus, we can rely on put/call parity to infer the futures price. In line with CBOE (2016), we use the option pair where the bid-prices are closest.

[4] We also run our later empirical analysis with the real futures price to compute the convexity adjustment. We find no significant changes in results, because the relative error of the VIX futures price, coming from put/call parity almost never exceeds ±1%.

[5] We follow Carr and Wu (2007) and interpolate the available strike range, making the correction term dispensable.

demonstrate that the implied measure of variance can be estimated by

$$(VIX_t^T)^{2,BKM} = \frac{e^{rT}}{T} \sum_i^N \left[2\left(1 - \ln\left(\frac{K_i}{S_t}\right)\right) \frac{\Delta K}{K_i^2} O_i^{SPX}(K_i, T) - \mu_t(T)^2 \right]. \tag{10}$$

Since $\mu_t(T)^2$ is normally quite small, the sensitivity to return jumps (compared to the CBOE's measure in Equation 8) manifests itself in the additional weighting term $1 - \ln(K_i/S_t)$, which depends on the options' moneyness. Since only OTM options enter the calculation, this shifts weight from OTM calls to OTM puts, whereby the latter are more sensitive to negative jumps in the S&P500.

Clearly, both measures, $(\mathrm{VIX}_t^T)^{2,\mathrm{CBOE}}$ and $(\mathrm{VIX}_t^T)^{2,\mathrm{BKM}}$, are subject to estimation errors. In the sense of Jiang and Tian (2005, 2007), we will analyse next how market incompleteness in the form of limited and discontinuous strike ranges affects the model-free valuation of VIX futures. This will give an initial intuition about *unavoidable* pricing errors. Finally, the use of mid-prices instead of actual trade-prices leaves us with a time-varying range of uncertainty about market participants' notion of expected variance. We will address this issue further in the empirical analysis.

2.3 | Theoretical impact of limited strike ranges

Markets are not complete. For options, only a discrete set of strikes within some minimal and maximal range is traded. This induces an error in the replicating portfolios from Equation 1, because it is not possible to trade all options that are theoretically needed. The model-free valuation method depends in two ways on the available strikes in both markets. First, options are not traded in a continuum and, thus, a discretization error enters the above integrals. Second, the limited strike range leads to a truncation error. Consequently, for the analysis of the performance of the model-free pricing approach, it is indispensable to know about the effects of market incompleteness with respect to the strike range and to the number of available strikes. Jiang and Tian (2005, 2007) show that for the VIX itself the error can be reduced substantially by using interpolation over strikes in the implied volatility space and extrapolation by using the implied volatility of the minimal and maximal available strikes. However, they do not consider the implications for errors in the implied VIX future pricing. From a practical point of view, it is further of special importance to look at pricing errors if no interpolation method is employed and only the available set of strikes is used.

In the following, we look at the impact of the availability of strikes on pricing errors in a model with simulated data and consider different degrees of market incompleteness. We abstract from transaction costs, bid–ask spreads, liquidity constraints, and other market frictions. This study will provide an initial intuition for the pricing errors which can be expected in our later empirical study of the model-free formula. Furthermore, it will give a range of how much of the error can be explained by market incompleteness. To generate the data we use the jump-diffusion model (SVJJ) with simultaneous jumps in the stock and its volatility introduced by Duffie, Pan, and Singleton (2000). The model dynamics of the log stock price s_t and variance V_t are

$$ds_t = \left(r - \bar{\mu}\lambda_J - \frac{1}{2}V_t\right)dt + \sqrt{V_t}dW_t^s + Z_t^s dN_t, \tag{11}$$

$$dV_t = \kappa_V\left(\bar{V} - V_t\right)dt + \sigma_V\sqrt{V_t}dW_t^V + Z_t^V dN_t, \tag{12}$$

$$dW_t^s dW_t^V = \rho,\ \ Z_t^V \sim \mathrm{Exp}\left(\mu_V\right),\ \ Z_t^s \mid Z_t^V \sim \mathcal{N}\left(\mu_J + \rho_J Z_t^V, \sigma_J^2\right), \tag{13}$$

where N_t is a Poisson process with constant intensity $\lambda_C > 0$, $\bar{\mu} = \frac{\mu_J + \frac{1}{2}\sigma_J^2}{1-\rho_J\mu_V} - 1$ and the remaining coefficients are chosen such that $r, \lambda_J, \sigma_V, \kappa_V, \bar{V}, \mu_V, \sigma_J > 0$ and $\rho, \rho_J, \mu_J < 0$. Since the model features stochastic volatility and jumps, it provides a valid market framework to study the model-free pricing formula for VIX futures.[6] For our analysis, we use a reasonable calibration which yields a mean model-implied VIX of 25.01% and where the jumps occur once every 6 months with an average jump size of 8% for the variance and −12% for stocks.[7] In this model setup, SPX options and VIX options and futures are known in semi-closed form. The SPX options prices are calculated as in Duffie et al. (2000), and the VIX derivatives are computed as in Branger, Kraftschik, and Völkert (2016). From the latter, we take the analytical VIX futures price and compare it with the model-free futures prices, which follow from Equations 4 and 10, for different strike grids.

[6] Several authors find evidence for jumps in stock returns and volatility. See, for example, Cremers, Halling, and Weinbaum (2015), Eraker (2004), Todorov and Tauchen (2011) among others.

[7] We choose the calibration to be similar to the reported parameters in Duffie et al. (2000). Those parameters are $r = 0$, $\rho = -0.8$, $\rho_J = -0.4$, $\sigma_J = 10^{-4}$, $\sigma_V = 0.14$, $\kappa_V = 3$. The initial variance is set to its long-run mean.

We start by analysing the truncation error, that is, the limited strike range. The upper panel of Table 1 (truncation method) gives the relative valuation error of the replication portfolio if we do not extrapolate in the moneyness dimension. In the lower panel (extrapolation method), we give the results if the volatility surface is extrapolated constantly as in Jiang and Tian (2007). We report errors for different moneyness ranges from ATM ± 10% to ATM ± 50% as well as for the average empirical ranges.[8] The upper panel shows that the truncation error decreases rapidly for wider ranges. For a range of ATM ± 10% the relative error for the short maturity is −15.53%, which drops with increasing moneyness range to −2.18% for ATM ± 20%. With the empirically observed ranges, the relative error for 1-month futures is −0.97%. The results are similar for the long-term VIX futures, but the decrease in pricing error is weaker. The error is at first enormous, with −51.02% (ATM ± 10%), and decreases to −4.08% (ATM ± 40%). For the empirical moneyness ranges we find an error of −3.24%. Thus, we expect the (truncation) error to be larger for long-term futures in the empirical analysis. The lower panel shows that the errors can be reduced substantially if we extrapolate the implied volatility space constantly by using the volatility of the highest and lowest strike. For the short and long maturities, the relative pricing error is below one percentage point for all moneyness ranges that are wider than ATM ± 10%. All in all, the analysis shows that a truncated strike range has little impact on the model-free VIX future valuation, especially when using empirical ranges. Our results suggest that the truncation error is about −1% to −3% for short-term to long-term VIX futures. As in Jiang and Tian (2005, 2007), the small relative pricing errors almost vanish if a simple extrapolation method is used.

Next, we concentrate on the discretization error and study the impact of a limited number of strikes. In Table 2, we give the discretization error for different combinations of SPX (rows) and VIX (columns) grids for a moneyness range of ATM ±50%. In Panel A of Table 2, we refrain from interpolation. For 1 and 12 months to maturity, the valuation errors are −5.37 and −2.23%, respectively, if SPX options and VIX options are available at their empirical moneyness steps ($1.2 \times S_0$ and $0.08 \times F_0^T$). If we widen the grid for VIX options, the columns in Panel A show that the error worsens. Hence, the number of available volatility derivatives is more important for the long-term than for the short-term futures.[9] Subtracting now the valuation errors for the empirical grids in Panel A from the errors for the truncation error in Table 1 for the moneyness range of ATM ±50% gives us an estimate for the discretization error. We find for short-term futures an error of −5.92% (= −5.37% − 0.55%) and for long-term futures −2.28% (= −2.23% − 0.05%). The decreasing error in the futures' maturity is plausible since the volatility smile becomes flatter for long-term options, which lowers the discretization error. If we interpolate between strikes, Panel B shows that the available number of SPX and VIX options hardly matters. For various grids, the resulting error is not different from the corresponding truncation error in Table 1 if we are in the medium state of the economy. To test whether the results depend significantly on the shape of the volatility term structure, we also look at regimes of high and low volatility in Panel C of Table 2. We find the relative pricing error in each volatility regime to be quite small, and slightly increasing in the level of volatility. For each volatility regime, the relative pricing error is below 1% in an absolute sense.

To sum up, we find that the discretization error can amount to roughly −6% for short-term VIX futures and roughly −2.25% for long-term futures. If we interpolate in the strike dimension, the error almost vanishes. Therefore, we will also interpolate between strikes in the next section, where we look at the empirical performance of the model-free VIX futures pricing. Since we refrain from extrapolation in the moneyness (and maturity) space, we expect relative pricing errors of at least −1% to −3%, due to the truncation error.

3 | EMPIRICAL ANALYSIS

3.1 | Data

The model-free VIX futures valuation requires a set of S&P500 options for the VIX^2 term structure and VIX options for the convexity correction term. We obtained monthly and weekly S&P500 options, as well as VIX options from Option Metrics. Since the VIX and its futures are quoted on an annual basis, these option quotes have to be treated with care, since small measurement errors can lead to substantial biases. For brevity, we relegate a more detailed discussion of our data treatment to Appendix A.2. Besides interpolating in the moneyness dimension to avoid the discretization bias, we also need to straddle the S&P500 option portfolios in order to obtain the maturities of interest. This is due to different settlement days and hours, compared to VIX derivatives. Thus, our data still refers to real VIX future quotes.[10] The latter started trading in 2004 and VIX

[8] For the SPX market, we find for our sample average empirical ranges of 0.69–1.15 and 0.56–1.30 for short- and long-term options, respectively. For VIX options we find empirical ranges of 0.60–1.80 and 0.64–1.80 for short- and long-term options, respectively.

[9] This is in line with our later findings in the empirical analysis, that the convexity correction matters more for long-term futures.

[10] VIX future quotes are available from CBOE's website.

TABLE 1 Relative errors of model-free VIX futures in the SVJJ model

Min. moneyness	0.9	0.8	0.7	0.6	0.5	Emp.
Max. moneyness	1.1	1.2	1.3	1.4	1.5	Ranges
Truncation method						
1 month to mat.	−15.53	−2.18	0.15	0.53	0.55	−0.97
12 months to mat.	−51.02	−28.47	−13.55	−4.08	0.05	−3.24
Extrapolation method						
1 month to mat.	−15.53	−0.31	0.38	0.54	0.55	0.12
12 months to mat.	0.24	−0.12	0.29	0.92	0.03	0.88

The table shows the relative pricing errors in percentage points of the model-free formula for VIX futures for different available strike ranges in the SVJJ model of Duffie et al. (2000). Available minimal and maximal strikes are quoted in moneyness terms (K/F). For the column Emp. Ranges we use average empirical strike ranges for the SPX and VIX market. For the SPX (VIX) market, the average ranges in our sample are 0.69–1.15 (0.60–1.80) and 0.56–1.30 (0.64–1.80) for short-term and long-term options, respectively. In each case we interpolate strikes in the implied volatility space using linear interpolation with 1,000 nodes. The initial stock price is $S_0 = 100$ and the initial variance V_0 is at its long-term mean.

TABLE 2 Relative errors of model-free VIX futures in the SVJJ model

Panel A	**Without interpolation**					
Δ Strike	**Emp. F_0^T**	**0.025 F_0^T**	**0.05 F_0^T**	**0.075 F_0^T**	**0.1 F_0^T**	**0.25 F_0^T**
1 month to maturity						
Emp. S_0	−5.37	−5.37	−5.39	−5.41	−5.45	−5.81
12 months to maturity						
Emp. S_0	−2.23	−2.23	−2.25	−2.27	−2.31	−2.76
Panel B	**With interpolation**					
Δ Strike	**Emp. F_0^T**	**0.025 F_0^T**	**0.05 F_0^T**	**0.075 F_0^T**	**0.1 F_0^T**	**0.25 F_0^T**
1 month to maturity						
Emp. S_0	0.55	0.55	0.55	0.55	0.55	0.55
0.050 S_0	0.55	0.55	0.55	0.55	0.55	0.55
0.075 S_0	0.53	0.53	0.53	0.53	0.53	0.54
0.100 S_0	0.46	0.46	0.46	0.46	0.46	0.46
12 months to maturity						
Emp. S_0	0.03	0.03	0.03	0.03	0.03	0.04
0.050 S_0	0.03	0.03	0.03	0.03	0.03	0.03
0.075 S_0	0.04	0.04	0.04	0.04	0.04	0.04
0.100 S_0	0.04	0.04	0.04	0.04	0.04	0.05

Panel C	**Different volatility regimes**		
	Volatility regime		
Time to maturity	**High**	**Medium**	**Low**
1 month	0.71	0.55	0.14
12 months	−0.09	0.03	0.25

The table shows the relative pricing errors in percentage points of the model-free formula for VIX futures for different available strikes in the SVJJ model of Duffie et al. (2000). Available strikes are quoted in moneyness terms. For the interpolation in Panels B and C, we interpolate in the volatility space. For both markets we choose the available strike ranges as ATM ± 50%[$Underlying$]. We use $V_0 = 0.053$ (its long-term mean) for the sensitivity analyses with respect to the strike grid. For the analysis over different volatility regimes we use empirical strike grids with step-size $0.012 \times S_0$ for stock options and $0.008 \times F_0^T$ for VIX options. In the High (Low) regime, we define the initial variance V_0 as 0.17 (0.01). In the Medium regime the initial variance is at its long-term mean.

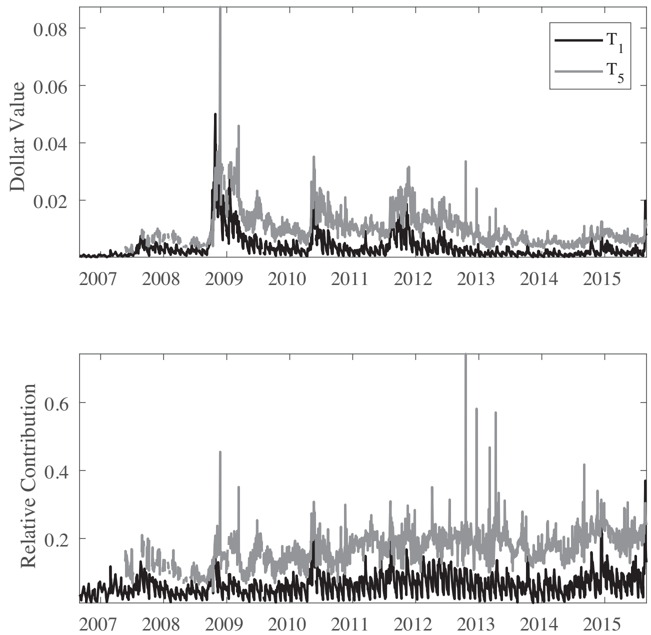

FIGURE 1 Importance of the convexity correction. The figure shows the daily volume of VIX futures, averaged over 1 month for different maturities. For the plot we use daily data from September 1, 2006 till August 31, 2015

options followed in February 2006, but their liquidity was at first very low. To obtain reliable data points, our available time-span therefore starts on September 1, 2006.[11] The end of our sample is August 28, 2015, leaving us with 2,168 days of data and 11,859 futures prices.

3.2 | Performance of the model-free formula

In this section, we test the performance of the model-free VIX futures pricing formula (4) with real world SPX and VIX options data. First, we discuss the importance of the convexity adjustment (CC_t). Figure 1 illustrates the convexity adjustment for the shortest and the longest maturity bucket. The top panel gives the value of CC_t, that is, the VIX options portfolio, in dollar terms. It shows that the convexity adjustment is rather stable across all maturities if markets are calm.[12] But in times of market stress, the convexity adjustments become quite large. The lower panel plots the convexity correction in relative terms compared to the true VIX futures price. Interestingly, the importance of the convexity correction is increasing in maturity. While for short-term futures it is rather unimportant with an average relative share of 5.96%, its relative contribution increases to roughly 20% for the longest maturity (T_5). In the course of the financial crisis of 2008 and at the peaks of the European sovereign debt crisis in 2010 and 2011, CC_t made up 17% of the short-term VIX futures price and 35% of the long-term VIX futures price. Altogether, this is model-free evidence that stochastic volatility-of-volatility risk makes up a significant part of VIX future prices and that it is especially important for longer maturities.

We now analyse the pricing errors of the model-free futures replication. We compare the method of the CBOE with that of Bakshi et al. (2003) to construct the VIX term structure. Table 3 presents the relative and dollar errors, as well as summary statistics for six different maturity buckets.[13] Throughout this paper, we focus on short-term futures with maturities of no more than 30 days, on the mid-term bucket with maturities between 60 and 90 days, and on long-term futures with times to maturity between 120 and 150 days.

Panel A of Table 3 documents the pricing errors if the VIX term structure is calculated using the approach of the CBOE. For short-term futures the average (median) relative error is −2.79% (−2.82%), for mid-term futures it is −7.20% (−4.39%), and for long-term futures it is −7.22% (−5.08%). Further, we find the quantiles for these errors to be unbalanced. The lower 5%

[11] This is due to our filtering method, where we require in each options market (S&P500 and VIX) at least 5 traded OTM contracts.

[12] Equation 5 shows that the convexity correction is not annualized. Thus, the seasonal pattern of CC_t stems from the decreasing maturity of the replicated VIX future.

[13] The relative error is the model-free futures price divided by the real price minus one.

TABLE 3 Errors of model-free VIX futures

Maturity	Mean	Median	$\widetilde{\text{Mean}}$	$\widetilde{\text{Median}}$	Std	$q^{0.95}$	$q^{0.05}$
Panel A	**CBOE method for VIX TS**						
	Relative error [%]						
7D < T_1 ≤ 30D	−2.79	−2.82	3.73	3.19	3.74	3.29	−8.79
30D < T_2 ≤ 60D	−4.10	−3.83	5.14	4.17	5.89	3.76	−11.94
60D < T_3 ≤ 90D	−7.20	−4.39	9.30	5.41	14.62	6.57	−29.97
90D < T_4 ≤ 120D	−0.85	−0.58	7.78	4.60	13.02	15.94	−19.14
120D < T_5 ≤ 150D	−4.30	−2.69	7.17	4.59	11.81	8.08	−20.70
150D < T_6	−7.22	−5.08	9.76	6.31	14.41	7.82	−29.24
	Actual error [$]						
7D < T_1 ≤ 30D	−0.0064	−0.0056	0.0084	0.0064	0.0096	0.0065	−0.0227
30D < T_2 ≤ 60D	−0.0101	−0.0080	0.0123	0.0089	0.0163	0.0074	−0.0333
60D < T_3 ≤ 90D	−0.0192	−0.0094	0.0243	0.0118	0.0461	0.0150	−0.0841
90D < T_4 ≤ 120D	−0.0025	−0.0012	0.0206	0.0101	0.0393	0.0425	−0.0525
120D < T_5 ≤ 150D	−0.0114	−0.0059	0.0188	0.0105	0.0363	0.0214	−0.0514
150D < T_6	−0.0187	−0.0115	0.0251	0.0144	0.0428	0.0193	−0.0772
Panel B	**BKM method for VIX TS**						
	Relative error [%]						
7D < T_1 ≤ 30D	0.08	−0.09	3.18	2.38	4.32	7.36	−6.66
30D < T_2 ≤ 60D	−0.30	0.00	4.51	2.91	7.16	9.98	−10.25
60D < T_3 ≤ 90D	−2.89	0.24	10.22	5.55	17.83	16.03	−31.66
90D < T_4 ≤ 120D	6.44	6.78	12.09	8.52	16.90	27.68	−15.44
120D < T_5 ≤ 150D	3.15	4.60	10.06	7.07	15.08	20.97	−18.50
150D < T_6	0.11	2.40	11.55	7.33	18.76	21.97	−30.75
	Actual error [$]						
7D < T_1 ≤ 30D	0.0013	−0.0002	0.0074	0.0046	0.0123	0.0198	−0.0135
30D < T_2 ≤ 60D	−0.0003	0.0000	0.0109	0.0061	0.0202	0.0255	−0.0238
60D < T_3 ≤ 90D	−0.0083	0.0005	0.0275	0.0118	0.0568	0.0438	−0.0884
90D < T_4 ≤ 120D	0.0165	0.0139	0.0320	0.0188	0.0532	0.0793	−0.0413
120D < T_5 ≤ 150D	0.0079	0.0100	0.0268	0.0157	0.0463	0.063	−0.0397
150D < T_6	0.0003	0.0052	0.0305	0.0168	0.0556	0.0643	−0.0731

The table shows moments for the actual and relative pricing errors (in percentage points) of the model-free formula for VIX futures for different maturity buckets. We report results for the cases when the VIX is calculated following CBOE's White Paper and Bakshi et al. (2003), respectively. The actual error is defined as $\epsilon_i^{\text{abs}} \equiv MF_t(T_i) - F_t^{T_i}$ and the relative error as $\epsilon_i^{\text{rel}} = MF_t(T_i)/F_t^{T_i} - 1$. The columns $\widetilde{\text{Mean}}$ and $\widetilde{\text{Median}}$ report the mean and median of $|\epsilon_i^{\text{abs}}|$ and $|\epsilon_i^{\text{rel}}|$, respectively. To calculate the moments we use daily data from September 1, 2006 till August 31, 2015.

quantiles are much larger than the upper 5% quantiles. Thus, for all maturities, the distributions of the relative error are highly skewed, which implies that model-free futures prices, computed by the CBOE method, are systematically too low. The pattern is the same for the errors in dollar terms.

Panel B of Table 3 gives the pricing errors of the method of Bakshi et al. (2003) for calculating the VIX term structure. In this case, we find average and median pricing errors for all maturity buckets closer to zero than with the CBOE method. The average (median) relative errors are 0.80% (−0.90%) for the short-term bucket, −2.89% (0.24%) for the mid-term bucket, and 3.15% (4.60%) for the long-term bucket. Further, we find the imbalance between the upper and lower 5% quantiles to be less pronounced than for the CBOE method. Thus, in contrast to the CBOE method, the pricing errors are overall more symmetrically distributed.

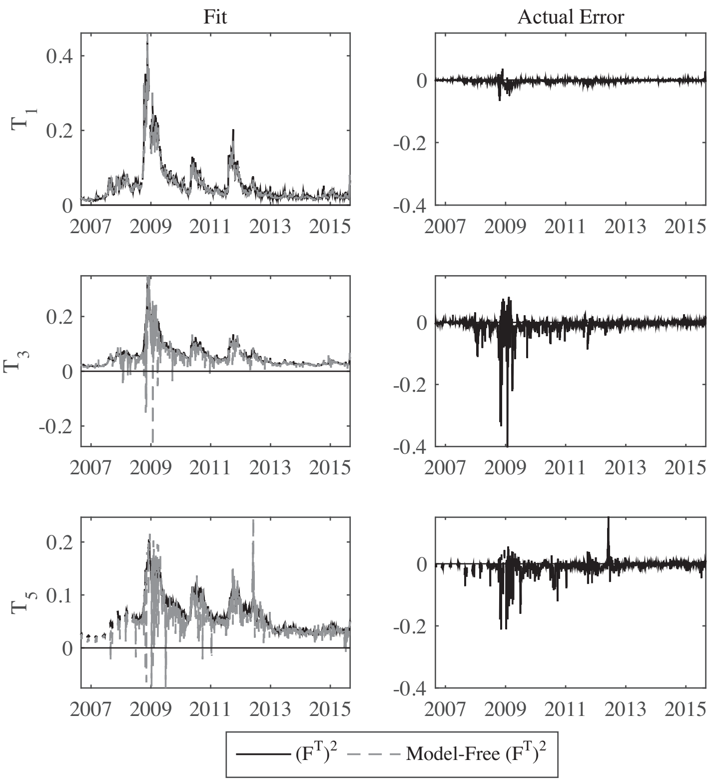

FIGURE 2 Model-free VIX futures pricing errors. The figure shows the fit for the model-free futures calculation on the left and the difference error on the right when the VIX is calculated using the approach from Bakshi et al. (2003). For the plots we use daily data from September 1, 2006 till August 31, 2015

Unreported tests show that the differences between the two approaches are highly significant with *t*-statistics exceeding 9 for all maturities. Still, the standard deviations of the errors are quite large (for either method). Figure 2, which plots the replication portfolio and the dollar errors over time, shows that the high standard deviation stems from the highly volatile periods and that the errors are heavily pronounced for mid to long maturities.[14] Especially in the financial crisis, the pricing errors became quite large. At that time, the model-free futures prices even became negative for maturities of more than 30 days. We find the reason for these negative prices to be the *de-annualized* VIX^2 term structure, which is at these dates decreasing in maturity. This implication from the data is theoretically impossible and shows an inconsistent pricing of the underlying SPX options.[15]

Our overall results indicate that the method for the calculation of the VIX term structure is crucial for model-free futures pricing and that the method of Bakshi et al. (2003) is superior in this application to the CBOE's method.[16] However, from the standard deviation and quantiles it is clear that the relative errors are still too large to be fully explained by market incompleteness in terms of option availability. The ranges of errors well exceed the values for the truncation error suggested by our theoretical analysis in section 2.3.

3.3 | Liquidity and pricing errors

In this section, we aim to explain the emerging pricing errors documented in the previous section. We choose to analyse the pricing errors using the method of Bakshi et al. (2003) to calculate the VIX term structure, since it leads to more symmetric price deviations, as expected from a valid replication strategy. In general, we observe that the price deviations are less pronounced after the financial crisis. This coincides with a strong increase in the VIX futures' trading volume, depicted in Figure 3. The volume remained rather low for all maturities until the middle of 2009. Since then, market participants' interest in short-term

[14] Figure 2 shows the errors for the method of Bakshi et al. (2003). The plot looks similar if the method of the CBOE is used.

[15] By theory, it holds that $\int_0^{t+s} \left(d \ln S_u\right)^2 \geq \int_0^t \left(d \ln S_u\right)^2$ for all $s \geq 0$.

[16] This result is unchanged if we consider the absolute errors, as can be seen from $\widetilde{\text{Mean}}$ and $\widetilde{\text{Median}}$ in Table 3.

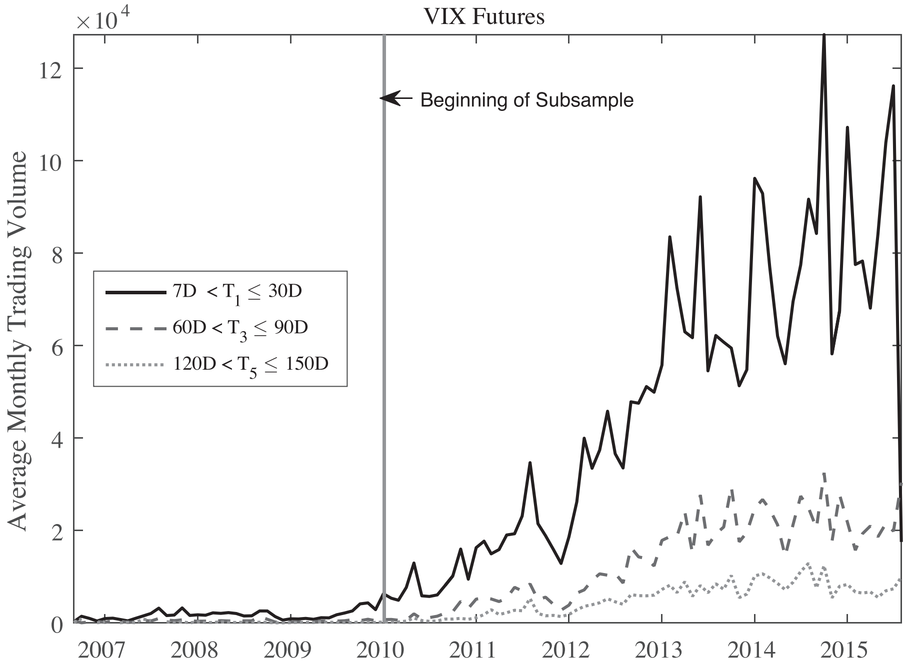

FIGURE 3 VIX futures trading volume. The figure shows the daily volume of VIX futures, averaged over one month for different maturities. For the plot we use daily data from September 1, 2006 till August 31, 2015

futures has risen sharply and from January 2010 onwards, the trading of VIX futures increased for all maturities.[17] We account for this structural change in the VIX futures market by dividing our sample at January 4, 2010. We refer to the two subsamples as the *pre- and in-crisis* and the *post crisis* samples.

To analyse the driving forces behind the pricing errors, we conduct contemporaneous regressions of the changes in the dollar pricing errors for our two subsamples. The regressions take the form

$$\Delta\epsilon_t^{MF^2,i} = \alpha + \beta\,\Delta X_t + \eta_t\ ,$$

where $\Delta\epsilon_t^{MF^2,i}$ is the change in the dollar pricing error of the squared VIX futures price, given by $MF_t^2(T_i) - \left(F_t^{T_i}\right)^2$. We choose to regress on the changes of the pricing errors in order to avoid problems of multicollinearity and auto-correlation of the regressors X_t. Further, we normalize ΔX_t by its sample's standard deviations to ensure the comparability of the betas.

For the regressors, we choose liquidity measures and control for different states of the economy by volatility risk measures. A widely used measure of liquidity is the bid–ask spread. However, we have multiple bid–ask spreads, since the model-free futures price consists of SPX and VIX options portfolios. We thus calculate the spreads of the SPX and VIX options portfolios as a weighted sum of the involved bid and ask prices. We define the weight of each option by its contribution to the VIX term structure and the convexity correction, respectively. The two spreads are then given by

$$\begin{aligned}\text{Spread}_{t,T}^{\text{SPX}} &\equiv \mathbb{E}_t\left[\left(\text{VIX}_T^{30\text{D}}\right)^2\right]^{\text{ask}} - \mathbb{E}_t\left[\left(\text{VIX}_T^{30\text{D}}\right)^2\right]^{\text{bid}}\\ \text{Spread}_{t,T}^{\text{VIX}} &\equiv \text{CC}_{t,T}^{\text{ask}} - \text{CC}_{t,T}^{\text{bid}}\ ,\end{aligned}$$

which can also be interpreted as the spread of the expected forward VIX^2 and the convexity correction. Further, we include the logarithm of the mean volume of the SPX and VIX options, as well as the log-volume of the VIX ETN (VXX). The latter was introduced at the end of January 2009 and is thus only relevant for our post-crises sample. We include all these variables to proxy for hedging costs and liquidity constraints. The VXX is the most liquid exchange traded product with which to take a position in VIX futures. As this ETN can either hedge the VIX future or the replicating portfolio, it also provides a

[17] Note that the drop in short-term futures volume at the end of our sample period reflects the CBOE's introduction of weekly VIX futures on July 23, 2015. Since weekly VIX options followed later on October 8, 2015, we can only study the prices of monthly VIX futures.

natural candidate for investors to exploit price discrepancies.[18] In addition, we include the TED-Spread as a proxy for funding liquidity.[19] To control for the state of the economy, we include the CBOE indices VIX and VVIX. The former captures the overall volatility risk and the latter volatility-of-volatility risk, which is especially relevant for VIX option prices.[20] All our variables are only mildly correlated, with correlations almost never exceeding 0.38.[21] Similarly the autocorrelations of all our variables are always below 0.45 in an absolute sense. Still, we include the innovations in the pricing error ($\Delta\epsilon_{t-1}^{MF^2,i}$) into our multi-variate regressions for robustness reasons.[22]

We begin with univariate regressions on the pricing errors. This is to determine all relevant variables first, which we will then test in multivariate regressions. Table 4 reports the intercepts, betas, and R^2s of the univariate regressions for short-, mid-, and long-term VIX futures. It shows that Spread$^{\text{SPX}}$ is highly significant for all maturities, with R^2 values ranging in the pre- and in-crisis (post-crisis) sample in Panel A (B) between 47.8% and 67.1% (21.2% and 50.1%). Results for the significance of Spread$^{\text{VIX}}$ are mixed and its R^2s are much lower, with values ranging between 0.3% and 7.4%. Market volatility, measured by the VIX, is only significant for short-term futures in the pre- and in-crisis period, with R^2 of 9.2%. The VVIX index, all volume measures (Vol$^{\text{SPX}}$, Vol$^{\text{VIX}}$, Vol$^{\text{ETN}}$), and the TED Spread have no or only little explanatory power, with R^2s well below 1%. So we drop them from our analysis and concentrate on the remaining variables.[23]

Table 5 shows multivariate regressions for both spread measures and for the VIX. In the pre- and in-crisis period, reported in Panel A, Spread$^{\text{SPX}}$ has a positive sign and remains highly significant for all maturities. In contrast, Spread$^{\text{VIX}}$ loses its significance for short-term contracts and is only significant for mid-term futures, with a negative beta. Similarly, the VIX is only significant for long-term futures and has a negative coefficient as well.[24]

The different signs for the spreads are quite reasonable if we reconsider Equation 4 for the replication portfolio. The less liquid the SPX options, the more expensive the VIX forward, which leads to higher pricing errors and is reflected in a positive coefficient for Spread$^{\text{SPX}}$. In contrast, the negative beta for the VIX options spread can be explained by its short position in the replicating portfolio. A lower liquidity of the VIX options leads to a higher price of the convexity correction, which in turn lowers the value of the model-free futures price.[25] The negative and significant sign for the VIX in Panel A stems from the highly volatile period within the financial crisis. As discussed in section 3.2, the de-annualized, long-term VIX term structure was inconsistent at this time, leading to negative prices of the replication portfolio.[26] If we discard theses days from our sample, the VIX becomes insignificant and at the same time Spread$^{\text{VIX}}$ becomes highly significant for long-term futures, with a beta of -0.0055 and a t-statistic of over 3. So we conclude that Spread$^{\text{VIX}}$ is especially important for longer maturities.

Panel B of Table 5 presents the results for the post-crisis period. For this period, the results for the signs and significances of the betas are similar to those in Panel A. So the overall pattern remains unchanged. However, we identify three major differences. First, the scaled beta estimates are lower in absolute terms. The reason is that the pricing errors are smaller after the financial crisis. Second, the spread in VIX options is only significant for long-term contracts, which supports our conclusion for Spread$^{\text{VIX}}$ from Panel A. Third, and most importantly, we find that the maximal explanatory power of our variables decreases for short-term contracts, whereas it remains rather high for the others. The R^2 for contracts with times to maturity between 7 and 30 days decreases from 60% to 27%, whereas the R^2s for mid- and long-term contracts in the post-crisis period remain high: between 55% and 59%.

All in all, the spreads in the option markets explain large parts of the pricing errors for all maturities and both samples. Comparing the R^2 values of the univariate and multivariate regressions, we find that the spread of the S&P500 index options market is the strongest driver. Meanwhile, the level of market volatility (measured by the VIX) has no explanatory power. The only exception to our results are short-term contracts, for which a big part of the price deviations cannot be explained after the financial crisis. In the next section we explore how market participants react to most inconsistent prices for these contracts.

3.4 | Market reactions to inconsistent pricing

In the previous section, we studied the drivers of the price deviations between the VIX futures and options markets. Our results indicate that the deviations vary a lot and can take negative as well as positive signs. Thus, there are times when the VIX futures

[18] We thank an anonymous referee for pointing this out.

[19] See, for example, Campbell and Taksler (2003) and Gupta and Subrahmanyam (2000).

[20] See, for example, Park (2016).

[21] The only exception is the correlation of changes in the VIX and VVIX, which is 0.66.

[22] If we do not include the lagged pricing error, the results remain qualitatively the same.

[23] In unreported multivariate regressions, these variables turn out to be insignificant.

[24] The variables' variance inflation factor (VIF score) never exceeds 1.55. So according to the VIF score, multicollinearity is not an issue for our results.

[25] To think of both spreads as positive transaction costs, one might also rearrange Equation 4 as $\mathbb{E}_t\left[\left(\text{VIX}_T^{30\text{D}}\right)^2\right] \approx \left(\text{F}_t^T\right)^2 + \left[\text{Approx. Conv. Corr.}\right]_t$, which consists of long positions only. In this case, the beta of the convexity correction would be positive.

[26] See the lower left plot of Figure 2.

TABLE 4 Univariate regressions on changes in pricing errors

Panel A	**Pre- and in-crisis – 09/01/2006 to 12/31/2009**								
	7D < T_1 ≤ 30D			**60D < T_3 ≤ 90D**			**120D < T_5 ≤ 150D**		
Mat. bucket	α	β	R^2	α	β	R^2	α	β	R^2
$\text{Spread}^{SPX}_{T_i}$	0.0000 (0.0002)	0.0079*** (0.0009)	0.4790	0.0001 (0.0005)	0.0549*** (0.0058)	0.6712	0.0002 (0.0006)	0.0424*** (0.0039)	0.6265
$\text{Spread}^{VIX}_{T_i}$	0.0001 (0.0002)	0.0031*** (0.0009)	0.0738	−0.0002 (0.0010)	−0.0082*** (0.0035)	0.0152	0.0001 (0.0010)	−0.0029 (0.0036)	0.0029
VIX	0.0000 (0.0003)	0.0036*** (0.0009)	0.0922	−0.0002 (0.0010)	0.0046 (0.0052)	0.0043	0.0001 (0.0010)	0.0014 (0.0035)	0.0008
VVIX	0.0000 (0.0003)	0.0010** (0.0005)	0.0070	−0.0002 (0.0010)	0.0035 (0.0029)	0.0025	0.0002 (0.0010)	0.0040 (0.0029)	0.0039
VOL^{SPX}	−0.0001 (0.0003)	−0.0004 (0.0009)	0.0011	−0.0002 (0.0010)	0.0027 (0.0032)	0.0016	0.0001 (0.0010)	0.0019 (0.0033)	0.0012
VOL^{VIX}	0.0000 (0.0003)	−0.0002 (0.0003)	0.0002	−0.0001 (0.0009)	0.0038*** (0.0018)	0.0033	0.0001 (0.0010)	−0.0014 (0.0012)	0.0009
TED-Sprd	0.0001 (0.0003)	0.0009*** (0.0004)	0.0045	−0.0002 (0.0010)	0.0009 (0.0015)	0.0001	0.0005 (0.0010)	0.0004 (0.0018)	0.0000
Panel B	**Post-crisis – 01/04/2010 to 08/31/2015**								
$\text{Spread}^{SPX}_{T_i}$	0.0000 (0.0000)	0.0015*** (0.0005)	0.2126	0.0001 (0.0001)	0.0093*** (0.0008)	0.5009	0.0000 (0.0001)	0.0124*** (0.0013)	0.4925
$\text{Spread}^{VIX}_{T_i}$	0.0000 (0.0001)	0.0003 (0.0007)	0.0098	0.0002 (0.0001)	−0.0008 (0.0006)	0.0033	−0.0001 (0.0002)	−0.0024*** (0.0011)	0.0184
VIX	0.0000 (0.0000)	0.0004 (0.0005)	0.0138	0.0002 (0.0001)	0.0002 (0.0006)	0.0004	0.0000 (0.0002)	0.0001 (0.0007)	0.0000
VVIX	0.0000 (0.0000)	0.0003 (0.0005)	0.0072	0.0002 (0.0001)	−0.0002 (0.0004)	0.0003	0.0000 (0.0002)	−0.0001 (0.0005)	0.0001
VOL^{SPX}	0.0000 (0.0001)	0.0001 (0.0001)	0.0013	0.0002 (0.0001)	−0.0003 (0.0005)	0.0005	0.0000 (0.0002)	0.0007 (0.0007)	0.0014
VOL^{VIX}	0.0000 (0.0001)	0.0001 (0.0001)	0.0009	0.0002 (0.0001)	0.0000 (0.0004)	0.0000	0.0000 (0.0002)	−0.0004 (0.0005)	0.0006
VOL^{ETN}	0.0000 (0.0000)	0.0003*** (0.0001)	0.0069	0.0002 (0.0001)	0.0005 (0.0004)	0.0013	0.0000 (0.0002)	−0.0002 (0.0005)	0.0001
TED-Sprd	0.0000 (0.0001)	−0.0002 (0.0002)	0.0031	0.0002 (0.0001)	−0.0002 (0.0004)	0.0002	0.0000 (0.0002)	−0.0001 (0.0004)	0.0000

The table shows intercepts, betas and R^2s of univariate regressions of innovations in our explanatory variables on changes in price deviations $\Delta\epsilon_t^{MF^2,i} = \Delta\left[MF_t^2(T_i) - \left(F_t^{T_i}\right)^2\right]$. All explanatory variables are normalized by their standard deviation. *, **, and *** indicate statistical significance at the 90%, 95%, and 99% confidence levels. Newey–West robust standard errors are stated in parentheses.

are more expansive than options and vice versa. This section aims to uncover the market reaction to most pronounced price dispersions. With this aim, we define model-free upper and lower bounds for VIX futures. Since the price dispersions of short-maturity contracts can be least explained by liquidity risks in the period after the financial crisis, we study in more detail the options and futures for this period with maturities between 7 and 30 days.

The main idea is to define the bounds for futures prices in dependence on the options' bid- and ask-prices, because we do not know the exact trading price for the options on a particular day. Thus, we define bounds for the VIX futures price in dependence on the options' bid- and ask-prices to get upper and lower ranges of the replication portfolio.[27] By looking at violations of these bounds, we concentrate on times when it may be difficult to explain price deviations with the spreads in the options markets, as was done in our previous regression analysis. If the replication strategy is valid, the VIX futures price has to lie within these

[27] Unfortunately, our data only covers mid-prices of the VIX futures. This is, however, no limitation on our results, because the relative bid–ask spread of VIX futures almost never exceeds 1% (see Park, 2015).

TABLE 5 Multivariate regressions on changes in pricing errors

Panel A	**Pre- and in-crisis – 09/01/2006 to 12/31/2009**					
Mat. bucket	**7D < T_1 ≤ 30D**		**60D < T_3 ≤ 90D**		**120D < T_5 ≤ 150D**	
Intercept	0.0001 (0.0003)	0.0001 (0.0003)	−0.0002 (0.0006)	−0.0002 (0.0007)	−0.0002 (0.0006)	−0.0001 (0.0006)
$\text{Spread}_{T_i}^{SPX}$	0.0066*** (0.0010)	0.0065*** (0.0009)	0.0517*** (0.0067)	0.0516*** (0.0069)	0.0388*** (0.0033)	0.0395*** (0.0031)
$\text{Spread}_{T_i}^{VIX}$	0.0021 (0.0015)	0.0020 (0.0016)	−0.0045*** (0.0020)	−0.0045*** (0.0020)	−0.0025 (0.0025)	−0.0017 (0.0026)
VIX		0.0003 (0.0008)		0.0009 (0.0040)		−0.0039* (0.0020)
$\Delta\epsilon_{t-1}^{MF^2,i}$	−0.0040*** (0.0013)	−0.0040*** (0.0013)	−0.0066 (0.0062)	−0.0067 (0.0064)	−0.0110*** (0.0022)	−0.0112*** (0.0022)
adj. R^2	0.5996	0.5994	0.6838	0.6835	0.6710	0.6764
Panel B	**Post-Crisis – 01/04/2010 to 08/31/2015**					
Intercept	−0.0001 (0.0001)	−0.0001 (0.0001)	0.0001 (0.0001)	0.0001 (0.0001)	−0.0001 (0.0001)	−0.0001 (0.0001)
$\text{Spread}_{T_i}^{SPX}$	0.0014*** (0.0006)	0.0014*** (0.0006)	0.0082*** (0.0007)	0.0082*** (0.0007)	0.0109*** (0.0012)	0.0109*** (0.0012)
$\text{Spread}_{T_i}^{VIX}$	−0.0002 (0.0005)	−0.0003 (0.0005)	−0.0006 (0.0004)	−0.0005 (0.0004)	−0.0029*** (0.0005)	−0.0031*** (0.0005)
VIX		0.0001 (0.0004)		−0.0002 (0.0004)		0.0009* (0.0005)
$\Delta\epsilon_{t-1}^{MF^2,i}$	−0.0007 (0.0005)	−0.0007 (0.0005)	−0.0033*** (0.0005)	−0.0033*** (0.0005)	−0.0048*** (0.0007)	−0.0048*** (0.0007)
adj. R^2	0.2675	0.2677	0.5579	0.5578	0.5892	0.5911

The table shows betas and intercepts of multivariate regressions of pricing errors $\Delta\epsilon_t^{MF^2,i} = \Delta\left[MF_t^2(T_i) - \left(F_t^{T_i}\right)^2\right]$ for the model-free VIX futures from Equation 4 on innovations of liquidity measures. We use the approach of Bakshi et al. (2003) to calculate the VIX term structure. All explanatory variables are normalized by their standard deviation. For the regressions we use daily data. *, **, and *** indicate statistical significance at the 90%, 95%, and 99% confidence levels. Newey–West robust standard errors are stated in parentheses.

bounds. Otherwise it is too expensive or too cheap relative to options and profits can be made by taking a position in futures and the hedge portfolios.

We define the model-free bounds by relying on Equation 4. The lower bound is defined as the *bid* price of the replication portfolio and the upper bound is defined as the *ask* price of the portfolio, without subtracting the (positive) value of the convexity correction. Thus, the bounds measure the lowest (highest) possible replication prices. Formally, the lower bound is defined as

$$\mathcal{L}_t^T \equiv \mathbb{E}_t\left[\left(\text{VIX}_T^{30D}\right)\right]^{\text{bid}} - \text{CC}_{t,T}^{\text{ask}} \tag{14}$$

and the upper bound is defined as

$$\mathcal{U}_t^T \equiv \mathbb{E}_t\left[\left(\text{VIX}_T^{30D}\right)\right]^{\text{ask}} . \tag{15}$$

To calculate the bid (ask) price of the expected forward VIX $\left(\mathbb{E}_t\left[\left(\text{VIX}_T^{30D}\right)\right]\right)$, we use the necessary bid (ask) prices of SPX options with time to maturity $T+30\text{D}$ and ask (bid) prices for SPX options with time to maturity T. For the ask price of the convexity correction we simply use the ask prices of the involved options.

Figure 4 displays the relative pricing error for short-term futures with respect to the bounds, given that the futures price is either too low or too high. We find that the lower bound in the left panel is violated more often and by a larger amount than the upper bound in the right panel. Thus, in our sample, options were too expensive more often relative to VIX futures, rather than the other way around. For the whole sample we find that for 10.19% of the days, the future is below its lower bound, with an average relative error of 9.75%, whereas the upper bound is violated on 3.21% of the days, with an average relative error of

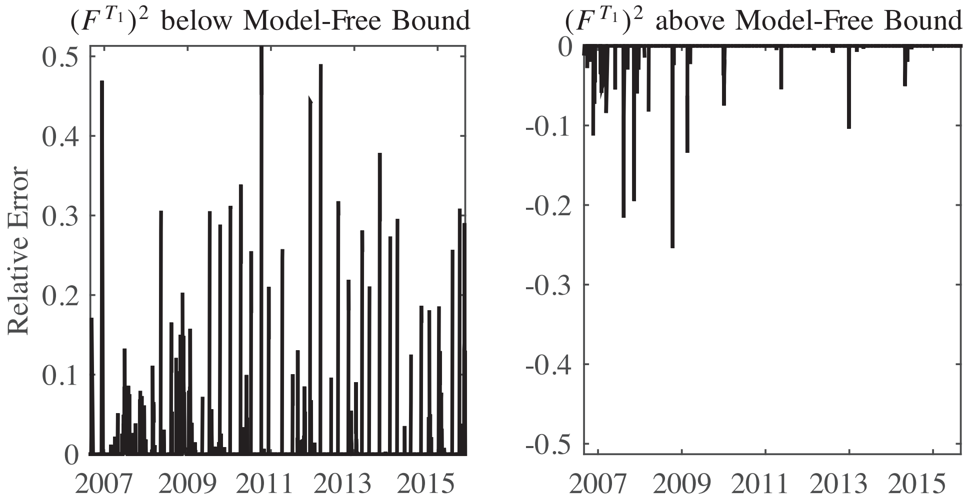

FIGURE 4 Relative errors if futures bounds are violated. The figure shows the relative pricing error when the short-term VIX future is below its lower bound (left panel) and above its upper bound (right panel)

-4.57%. Further, we find that the times when the lower bound $\mathcal{U}_t^T$ is violated are evenly spread across the sample, whereas futures exceeded their upper bound mostly in the financial crisis.

To study market reactions to these obvious price dispersions, we study the involved implied volatility smiles. Figure 5 shows the smiles for SPX and VIX short-term options from a kernel regression on moneyness $m = \log\left(K/F_0\right) / \left(\sqrt{T}\sigma_{\text{ATM}}\right)$.[28] We report the smiles of the days of the violation of the upper and lower bounds, as well as one day before and up to two days after a violation. The plots include the average smile, for comparison.

If the futures price is below its lower bound (dashed lines), we see no movement in either volatility smile (SPX and VIX) one day before and the day of the violation. However, we find a strong movement the day after. Thus, the information implied by the price deviation triggers market movements after its occurrence. Table 6 supports this finding in a statistical sense. The table documents the relative changes of implied volatilities for different moneyness buckets as well as changes of the slope of the smile around the discussed days.[29] Before the mispricing, all changes are insignificant. After the occurrence of the price deviation, implied volatilities decrease significantly for all moneyness buckets and in both markets. The levels of both smiles, measured by ATM options with $-0.5 \leq m \leq 0.5$, decrease by roughly 5–6% in both markets. Hereby, the effect in the SPX market prevails longer and we observe a further significant decrease of roughly 4% after the first day of the mispricing.[30] For the slope, the table documents a relative decrease in the SPX market by 8.5% and 4.7% in the two subsequent days after the mispricing.[31] In comparison, the slope in the VIX market remains unchanged.

To explain the observed pattern, recall that SPX options account on average for more than 94% of the value of the short-term model-free replication and they positively influence its price. Therefore, the volatility risks embedded in SPX option prices decrease, mitigating the relative overpricing. This means there is a decrease in the level of aggregate volatility risk implied by SPX options. For VIX options, it could have been expected that they would be cheaper in the case of an overpricing, since they have a negative impact on the replication portfolio. However, their implied volatilities also decrease the day after the mispricing. This means that not only does the level of volatility risk decrease, but also its uncertainty (volatility-of-volatility risk). This leads to overall lower volatility risks implied by the option markets. The observed pattern in the slopes in both markets can be explained by the weights of the options in the replication portfolios. SPX options are weighted by $1/Strike^2$. Therefore, OTM puts are more important than OTM calls for the forward VIX and their prices are more strongly ef-

[28] The definition to quote moneyness in terms of standard deviations is used by other authors as well, for example Andersen, Fusari, and Todorov (2017). Note that in section 2, we pointed out that it is necessary to use two SPX option portfolios. For illustrative purposes, we only show the smiles for the shorter end of the SPX options' maturities.

[29] We report relative changes to account for the fact that implied volatilities of VIX options are much higher compared to S&P500 options.

[30] In results not presented here, we find no significant movements after the second day.

[31] The slope on day t_i is defined as $\text{IV}_{t_i}^{\text{DeepOTM}} - \text{IV}_{t_i}^{\text{ITM}}$, where DeepOTM refers to a moneyness of $-4 \leq m < -2$ $(2 < m \leq 4)$ and ITM to a moneyness of $0.5 < m \leq 1.5$ $(-1.5 \leq m < -0.5)$ for SPX (VIX) options.

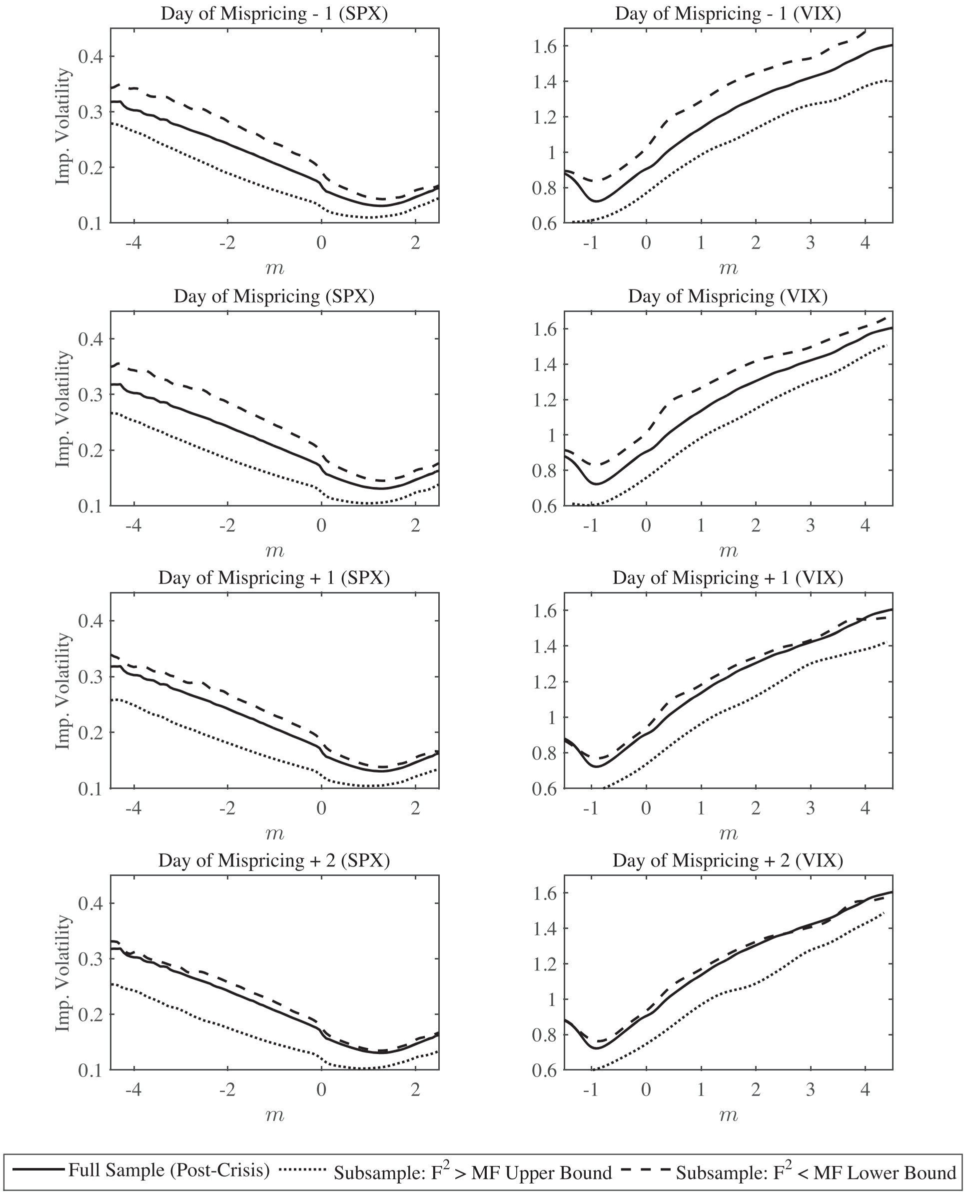

FIGURE 5 Market reactions to futures bounds violations The figure shows the market reaction to price dispersion between the VIX futures and options market. The left panels show implied volatility smiles relative to moneyness for SPX options. The right panel gives corresponding smiles for VIX options. The smiles are obtained from kernel regressions

fected.[32] VIX options, on the other hand, are weighted equally in the replication portfolio. Thus, their prices adjust evenly across strikes.

Turning to the case when futures are overpriced relative to options, that is, the upper bound is violated, we find almost no reaction in the option markets. The dotted volatility smiles in Figure 5 document that starting from the day just before the price dispersion to two days after it, the levels of the SPX and VIX option smiles remain almost unchanged.[33] The lack of an increasing pattern after the mispricing implies that prices of VIX futures need to adjust when they are relatively overpriced compared to options.

Overall, we find that the volatility product that is less expensive sets the expected volatility risks. The volatility embedded in the cheaper products adjusts. Either the prices of futures adjust or the implied volatility of options does. Our results thus indicate

[32] Similarly, Griffin and Shams (2018) find that the price movements in SPX options are related by the weighting $1/Strike^2$ to the VIX when it comes to the settlement of VIX futures.

[33] This is true for relative changes as well, which, for the sake of brevity, we do not report.

TABLE 6 Relative changes in implied volatility if futures violate the lower bound

	Relative change in implied volatility				
	SPX smile				
ΔTime	**$-4 \leq m < -2$**	**$-2 \leq m < -0.5$**	**$-0.5 \leq m \leq 0.5$**	**$0.5 < m \leq 1.5$**	**rel.ΔSlope**
$t_{-1} \rightarrow t_0$	0.0106 (0.0108)	0.0105 (0.0098)	0.0074 (0.0104)	0.0147 (0.0183)	0.0074 (0.0209)
$t_0 \rightarrow t_1$	−0.0673*** (0.0112)	−0.0609*** (0.0099)	−0.0535*** (0.0106)	−0.0470*** (0.0166)	−0.0850*** (0.0216)
$t_1 \rightarrow t_2$	−0.0372*** (0.0115)	−0.0363*** (0.0088)	−0.0403*** (0.0103)	−0.0251 (0.0165)	−0.0473*** (0.0190)
	VIX smile				
	$-1.5 \leq m < -0.5$	**$-0.5 \leq m < 0.5$**	**$0.5 < m \leq 2$**	**$2 < m \leq 4$**	
$t_{-1} \rightarrow t_0$	0.0077 (0.0115)	−0.0080 (0.0084)	−0.0223 (0.0145)	-0.0278 (0.0244)	−0.0738 (0.0587)
$t_0 \rightarrow t_1$	−0.0613*** (0.0098)	−0.0754*** (0.0060)	−0.0601*** (0.0118)	−0.0408*** (0.0197)	−0.0121 (0.0315)
$t_1 \rightarrow t_2$	0.0068 (0.0111)	−0.0120*** (0.0057)	−0.0111 (0.0134)	−0.0132 (0.0213)	−0.0398 (0.0325)

The table shows relative changes of implied volatilities for different adjusted moneyness buckets and of the smile's slope around day t_0 where $F_{t_0}^{T_1}$ violates its lower bound. Relative changes from time t_i to time t_j in volatility are defined as $\Delta \mathrm{IV}_{t_i,t_j} = \frac{\mathrm{IV}_{t_j} - \mathrm{IV}_{t_i}}{\mathrm{IV}_{t_i}}$ and relative changes in slope are measured correspondingly. The slope on day t_i is defined as $\mathrm{IV}_{t_i}^{\mathrm{DeepOTM}} - \mathrm{IV}_{t_i}^{\mathrm{ITM}}$, where DeepOTM refers to a moneyness of $-4 \leq m < -2$ ($2 < m \leq 4$) and ITM to a moneyness of $0.5 < m \leq 1.5$ ($-1.5 \leq m < -0.5$) for SPX (VIX) options. *, **, and *** indicate statistical significance at the 90%, 95%, and 99% confidence levels. Newey–West robust standard errors are stated in parentheses.

a lead–lag structure between the market for VIX futures and the markets for SPX and VIX options. Consequently, we observe a lead–lag structure between markets segmented by their product, not by their underlying.

4 | CONCLUSION

This paper studies the interdependencies between the VIX futures market and the SPX and VIX options markets. The main interest of our analysis is the model-free VIX futures replication by a long position in a portfolio of SPX options and a short position in a portfolio of VIX options. In a similar fashion to Jiang and Tian (2007), we conduct an extensive sensitivity analysis and find that limited option availability only leads to relatively small pricing errors. An application to real world data shows that the pricing errors are too large to be explained by market incompleteness (in terms of limited strikes) alone. Until 2010, we can mainly explain the price deviations of all maturities by the bid–ask spreads of the option portfolios. Afterwards, we cannot explain large parts of the price deviations for short-term VIX futures. For contracts with maturities larger than one month, we still identify the bid–ask spreads as the main driver. Thus, we further analyse market reactions to price deviations beyond the bid–ask in short-term contracts and find that either the VIX future price or the option prices in both markets (S&P500 and VIX) adjust. If VIX futures imply lower volatility risks, the level of the SPX and VIX option volatility smiles adjust the next day. For SPX options, also the slope decreases significantly at this day. On the other hand, if options imply lower volatility risks, VIX futures adjust. We thus uncover a lead–lag structure between the VIX futures and the options market in times when price dispersions are largest. Since, much more often, it is the VIX futures that lead the option markets, especially after 2010, our results suggest that VIX futures are the better investment vehicle to discover and trade aggregate market volatility risks.

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APPENDIX A

A.1. Proof of theorem 1

Carr and Madan (2001) show that for a twice differential function $f : \mathbb{R} \mapsto \mathbb{R}$,

$$f(X) = f(\bar{X}) + f'(\bar{X})(X - \bar{X}) + \int_{\bar{X}}^{\infty} f''(K)(X - K)^{+} dK + \int_{0}^{\bar{X}} f''(K)(K - X)^{+} dK, \tag{A.1}$$

for $X, \bar{X} \in \mathbb{R}$. If we set $f(X) = X^2$, $X = \mathrm{VIX}_T$ and $\bar{X} = \mathbb{E}_t\left[\mathrm{VIX}_T\right] = \mathrm{F}_t^T$, it follows that

$$\begin{aligned}\left(\mathrm{VIX}_T^{30\mathrm{D}}\right)^2 =& \left(\mathbb{E}_t\left[\mathrm{VIX}_T^{30\mathrm{D}}\right]\right)^2 + 2\mathbb{E}_t\left[\mathrm{VIX}_T^{30\mathrm{D}}\right]\left(\mathrm{VIX}_T^{30\mathrm{D}} - \mathbb{E}_t\left[\mathrm{VIX}_T^{30\mathrm{D}}\right]\right) \\ &+ 2\int_{\mathbb{E}_t\left[\mathrm{VIX}_T^{30\mathrm{D}}\right]}^{\infty} (\mathrm{VIX}_T^{30\mathrm{D}} - K)^+ dK + \int_0^{\mathbb{E}_t\left[\mathrm{VIX}_T^{30\mathrm{D}}\right]} (K - \mathrm{VIX}_T^{30\mathrm{D}})^+ dK.\end{aligned}$$

Now taking expectations on both sides and rearranging yields

$$\left(\mathrm{F}_t^T\right)^2 = \mathbb{E}_t\left[\left(\mathrm{VIX}_T^{30\mathrm{D}}\right)^2\right] - 2e^{rT}\left(\int_{\mathrm{F}_t^T}^{\infty} C_t^{\mathrm{VIX}}(T, K)dK + \int_0^{\mathrm{F}_t^T} \mathcal{P}_t^{\mathrm{VIX}}(T, K)dK\right). \tag{A.2}$$

A.2. Data

Throughout this paper, we only analyse VIX futures with non-zero volume and maturities with less than 6 months. For the corresponding model-free VIX futures price, it is important to note that SPX and VIX derivatives settle on different dates. While SPX options settle on Fridays, VIX derivatives settle on Wednesdays. We correct for this two-day difference by interpolating the VIX^2 term structure linearly, as in Equation 9, and refrain from any interpolation or extrapolation of the VIX options and futures in the maturity dimension.[34] We discard the days when no interpolation is possible. Monthly and weekly options have different settlement times. Monthlies (SPX) settle at 8.30 a.m and Weeklies (SPXW) options settle at 3.00 p.m.[35] Thus, the times to maturity used for calculating VIX^2 are

$$\begin{aligned} T_{\text{monthly}} &= (\mathrm{Date}_T - \mathrm{Date}_t - 1)/365 + \frac{(8.75 + 8.5)}{24 \times 365}\,, \\ T_{\text{weekly}} &= (\mathrm{Date}_T - \mathrm{Date}_t)/365\,. \end{aligned}$$

To obtain reliable option quotes for the calculation of VIX^2 and the convexity adjustment, we follow the CBOE (2016) white paper for filtering rules. We delete zero bids and delete all data points after two subsequent zero bids. Furthermore, we require at least five quotes for both SPX and VIX options on each day. Also, we delete contracts with maturities of less than 7 days in order to avoid microstructure effects. As does the CBOE, we obtain the forward price F_t in Equation 8 (which determines OTM options) in a model-free manner by using put/call parity and the two closest call and put prices. Lastly, we use the interpolation techniques from section 2 to calculate the forward VIX in (2) and the CC in (5), using a fine grid of strikes with $\Delta K = K_{i+1} - K_i = 1$ for SPX options and $\Delta K = 0.1$ for VIX options.

[34] In results not presented here, we find that a minimum of interpolation in the maturity dimension is inevitable to obtain reasonable results. For example, interpolating both VIX and SPX volatility surfaces in order to generate constant maturity contracts deteriorates the pricing performances sharply.

[35] We follow the CBOE in using only *standard* SPX options and include SPXW options, starting in 2014. Before, weekly options were not liquid enough. Thus we delete all options with the root JXA, JXB, JXC, JXD, JXE. Up to 2010, they referred to weekly options. Also, we delete options with the root QSE, QSZ, QZQ, SAQ, SKQ, SLQ, SQG, SQP, SZQ, and SZU. These are non-standard LEAPS options, which settle on the last trading day of the quarter. For further details, see Andersen et al. (2011).