# ON THE REPRESENTATION OF INTEGERS AS $d$-SUMS OF SQUARES

DOWON KIM

ABSTRACT. The goal of this paper is to prove that for any integer $d \geq 5$, the function $r_d(n)$, the number of ways a positive integer $n$ can be written as a sum of $d$ squares, is well-defined. The paper explores the Hardy-Littlewood Circle Method and finally, considers the case when $d = 8$.

CONTENTS



## 1. INTRODUCTION

In the Fall of 2016, I studied in Professor Elias Stein's Junior Seminar on Topics in Harmonic Analysis with applications to Number Theory. In this paper, I break down my two presentations. The first section deals with my portion of a joint presentation given with Yash Patel '18 on the Hardy-Littlewood Circle Method. The second section will explore the number of representations of $n$ as a sum of four squares.

## 2. HARDY-LITTLEWOOD CIRCLE METHOD

Originally conceived by Hardy and Ramanujan for their work on the partition function, the Circle Method is a useful tool for investigating many problems in number theory, including the Waring's Problem and the Goldbach's Conjecture. The Circle Method is useful for dealing with additive problems of the following nature:

*Date*: May 15, 2018.

*Given some subset $A \subset \mathbb{N}$ and a positive integer $k$, what natural numbers can be written as a sum of $k$ elements of $A$, and in how many ways?* Or, explicitly:

$$\{a_1 + \cdots + a_k : a_i \in A\} \cap \mathbb{N}$$

To apply this method to Goldbach's even conjecture, let $A$ be the set of primes and $k = 2$. In problems such as Waring's problem, $A = K = \{0, 1, 2^k, 3^k, ...\}$ with a fixed integer $s$. [1] In this paper, we walk through the Circle Method to demonstrate that $\forall d \geq 5$, and for all sufficiently large $n$, $n$ can be represented as a sum of $d$ squares.

2.1. **Relationship Between $\Theta$ and $r_d(n)$.** We first consider the problem for a simple case, when $d = 2$, to find a generating function for the series $\{r_d(n)\}_{n=1}^{\infty}$

**Lemma 2.1.** *Let $\Theta$ be the Jacobi theta function, then we have*

$$\Theta(\tau)^2 = \sum_{n=0}^{\infty} r_2(n)\, q^n \quad \textit{where } q = e^{\pi i \tau}$$

*Proof.* By definition of the Jacobi theta function, we have

$$\Theta(\tau) = \sum_{n=-\infty}^{\infty} e^{\pi i n^2 \tau} = \sum_{n=-\infty}^{\infty} q^{n^2}$$

So we have

$$\begin{aligned}\Theta(\tau)^2 &= \left(\sum_{n_1=-\infty}^{\infty} q^{n_1^2}\right)\left(\sum_{n_2=-\infty}^{\infty} q^{n_1^2}\right) \\ &= \sum_{(n_1,n_2)\in\mathbb{Z}\times\mathbb{Z}} q^{n_1^2+n_2^2} \\ &= \sum_{n=0}^{\infty} r_2(n) q^n\end{aligned}$$

□

This translates directly to the general case for $d$ as we can work with a product of $d$ sums to count all pairs $(n_1, \cdots, n_d)$ with $\sum n_i^2 = n$, and yields a starting point for the Circle Method.

Recall from complex analysis that we have the following way to extract individual coefficients from a power series. By Cauchy's Integral Formula, we have the following:

**Proposition 2.2.** *Let $\gamma$ be the unit circle, oriented counter-clockwise. Then*

$$\frac{1}{2\pi i}\int_{\gamma} z^n \, dz = \begin{cases} 1 & n = -1 \\ 0 & \textit{otherwise} \end{cases} \tag{2.3}$$

[1]The Waring's problem asks whether each natural number $k$ has an associated positive integer $s$ such that every natural number can be expressed as the sum of at most $s$ natural numbers, each to the power of $k$. By Lagrange's four-square theorem, if $k = 2$, then $s = 4$, as every integer can be written as a sum of four square numbers.

*Let $F(z)$ be a power series with radius of convergence greater than 1, then*

$$\frac{1}{2\pi i}\int_\gamma F(z)z^{-n-1}\,dz = a_n \tag{2.4}$$

Having established this relationship, we can use Proposition 2.2 to find the $n$th coefficient of $\{r_d(n)\}_{n=1}^{\infty}$:

$$r_d(n) = \int_0^1 \Theta^d(2x+iy)e^{-\pi in(2x+iy)}\,dx \tag{2.5}$$

where $\tau = 2x_iy$, which will be elucidated when Gauss sums are introduced. The Circle Method aims to replace $\Theta$ with close approximations near each rational, applying previous presentations on Diophantine approximations of rationals. Subsequently, these "contributions" from each rational singularities will be gathered together into "singular series."

2.2. **Decomposition into Major and Minor Arcs.** To see why we approximate around rationals, consider the following proposition:

**Proposition 2.6.** *For $\tau = \frac{2p}{q} + iy$ with $y > 0$,*

$$\Theta(\tau) = \frac{1}{q}S(p/q)\left(\frac{\tau - 2p/q}{i}\right)^{-1/2} + O(q)$$

*as $y \to 0, y > 0$*

*Proof.* Without going into details, the crucial idea is that we can express $n$ ranging across $\mathbb{Z}$ as $n = mq + l$ where $m \in \mathbb{Z}$ and $1 \le l \le q$. Note that

$$\Theta(2p/q+iy) = \sum_{n=-\infty}^{\infty} e^{\pi in^2(2p/q+iy)} = \sum_{l=1}^{q}\sum_{m\in\mathbb{Z}} e^{\pi i(mq+l)^2(2p/q+iy)}$$

Conveniently, $(mq+l)^2(2p/q)$ modulo the integers is $2l^2p/q$ so the expression becomes:

$$\sum_{l=1}^{q} e^{2\pi ii^2p/q}\left(\sum_{m\in\mathbb{Z}} e^{\pi i(mq+l)^2y}\right)$$

where the first term is a Gauss sum and the inner sum is $q^{-1}y^{-1/2}$, which completes the proof. □

*Remark* 2.7. From this result, we see that $\Theta(\tau)$ is the largest at every $\tau$ closest to each rational $2p/q$ with $q$ sufficiently small.

**Theorem 2.8.** *Given $\theta \in \mathbb{R}$ and an integer $N > 0$ there exist $p, q \in \mathbb{Z}$ with $(p,q) = 1$ and $1 \le q \le N$ such that*

$$|\theta - \frac{p}{q}| \le \frac{1}{q^N}$$

Thus, it becomes natural to break up the "circle" into small arcs or intervals centered around each rational, bringing us to our main result. Recall we have

$$e^{-\pi ny}r_d(n) = \int_0^1 \Theta^d(2x+iy)e^{-2\pi inx}dx$$

Now we want to decompose this integral into a family of integrals over arcs, such that the contribution of each arc is concentrated at a rational $2p/q$.

**Proposition 2.9.** *Given $N \geq 2$, there exists a decomposition of $[0,1]$, the unit interval, into a disjoint union of countably many intervals*

$$[0,1] = \bigcup_{\substack{1\leq p\leq q\leq N \\ (p,q)=1}} I(p/q)$$

*for $(p,q)=1$ and $p/q \in I(p/q)$ such that*

$$\{\theta : |\theta - p/q| \leq \frac{1}{2qN}\} \subseteq I(p/q) \subseteq \{\theta : |\theta - p/q| \leq \frac{1}{qN}\}$$

The more precise construction of such intervals can be done using Farey Dissection, which yields a particular result, but we will show the more elementary approach using Dirichlet's approximation lemma.

*Proof.* Denote $J'(p/q)$ to be the left side of the inclusion and $J(p/q)$ to be the right side. Using Dirichlet's approximation lemma which we noted earlier, we can see that the collection $\cup_{p,q} J(p/q)$ covers $[0,1]$, when we take the union over all $p, q$ from 1 to $N$. Then we take the union of $J'(p/q)$ over the same collection to see that it is disjoint by construction.

Now suppose, toward contradiction, $J'(p/q) \cap J'(p'/q') \neq \emptyset$. Then there would be a point $\theta$ where:

$$\frac{1}{qq'} \leq \left|\frac{q'p - p'q}{qq'}\right| \leq \left|\frac{p}{q} - \frac{p'}{q'}\right| \leq \left|\theta - \frac{p}{q}\right| + \left|\theta + \frac{p'}{q'}\right| \leq \frac{1}{2qN} + \frac{1}{2q'N}$$

If $q > q'$ we have a contradiction, and if $q = q'$, then $p \neq p'$, we also get a contradiction since $1 \leq p \leq q \leq N$. Thus, the union of $J'(p/q)$ is disjoint. □

To construct the intervals themselves, we pick $J'(p/q)$, and for each $q \in [1,n]$ we enlarge the intervals until they meet the endpoints of the nearest neighbors.

*Remark* 2.10. The final collection of such intervals will be open, half-open, closed.

A quick aside:

**Definition 2.11.** A *Farey Series* $\Phi_N$ is a list of increasing $p/q$, with $(p,q)=1$ and $0 \leq p < q \leq N$ where two adjacent terms are neighbors and their median is defined as $\frac{p_1+p_2}{q_1+q_2}$ with the following properties:

(1) given neighbors in $\Phi_N$, then $N < q_1 + q_2 \leq 2N$
(2) given neighbors in $\Phi_N$, then $q_1 p_2 - p_1 q_2 = 1$
(3) given 3 consecutive terms $p'/q', p/q, p''/q''$, then

$$\{\theta : |\theta - p/q| \leq \frac{1}{2qN} \subseteq\} \left[\frac{p+p'}{q+q'}, \frac{p+p''}{q+q''}\right] \subseteq \{\theta : |\theta - p/q| \leq \frac{1}{qN}\}$$

Finally, such decompositions give us the new expression for $r_d(n)$, which takes into consideration contributions from each of the rationals.

$$r_d(n) = e^{\pi n y} \sum_{\substack{1\leq p\leq q\leq N \\ (p,q)=1}} \int_{I(p/q)} \Theta(2x+iy) e^{2\pi n x}\, dx$$

2.3. **Derivation of the Singular Integral and Series.** Now at the final step, we relate $\Theta^d$ to Gauss sums to generate the desired Singular series. Here, we prove a central proposition:

**Proposition 2.12.** *If $\tau = 2p/q + \omega$ with $(p,q) = 1$ and $w = u + iy$ with $y \leq q^{-2}$ and $|u| \leq 2y^{1/2}/q$, then*

$$\Theta^d(\tau) = \frac{1}{q^d} S(p/q)^d \left(\frac{w}{i}\right)^{-d/2} + O(y^{-d/4})$$

*Proof.* The main term is clearly obtained from the previous proposition which showed that $\Theta$ grows large around the rationals of form $2p/q$ raised to the $d$th power. So, it suffices to consider where the last term $O(y^{-d/4})$ comes from.

From the presentation on Gauss sums, recall that $O(S(p/q)) = q^{1/2}$[2]. So $S(p/q)/q = O(q^{-1/2})$ and for $1 \leq d' \leq d$,

$$q^{-d'}|S(p/q)|^{d'}|\omega|^{-d'/2} = O(q^2|w|^2)^{-d'/4} = O(q^2(u^2+y^2))^{-d/4}$$

Note that $q^2(u^2+y^2) \leq q^2y^2 \leq y$, so the above expression is $O(y^{-d'/4}$ for $|y| \leq 1$. Thus, all cross terms that come from raising $\Theta$ to the $d$th power, are also at most $O(y^{-d/4})$.

To apply this approximation to the integral representation of $r_d(n)$, we verify that each $I(p/q)$ is sufficiently small, so that for every $x \in I$, we can apply $\Theta^d(2x+iy)$. By construction, $I$ only includes values of $2x$ at most a distance $2/qN$ from $2p/q$, so it suffices to set $y = 1/N^2$, as this also satisfies $y \leq q^-2$ for $q \leq N$.

Thus, we obtain

$$r_d(n) = e^{\pi ny} \sum_{1\leq q\leq N} \sum_{\substack{1\leq p\leq q \\ (p,q)=1}} \frac{1}{q^d} S(p/q)^d e^{-2\pi inp/q} \int_{I(p/q)} (y-2ix)e^{-\pi inx}\, dx + O(y^{-d/4}$$

Now, we shift each $x \mapsto x - p/q$ within the interval $I$, denote the new intervals by $I'(p/q)$, and scale the integral by setting $x \to x/N^2$

$$r_d(n) = N^{d-2}e^{\pi n/N^2} \sum_{1\leq q\leq N} A_q(n) \int_{N^2 I'(p/q)} (1-2ix)^{-d/2} e^{-2\pi inx}\, dx + O(N^{d/2})$$

where

$$A_q(n) = \sum_{\substack{1\leq p\leq q \\ (p,q)=1}} \frac{1}{q^d} S(p/q)^d e^{-2\pi inp/q}$$

is the singular series. ∎

Note that we expect $r_d(n)$ to be of order around $n^{\frac{d}{2}-1}$, since $n$ will have some sum representation of the form $m_1^2 + \cdots + m_d^2$ only if $|m_i| \leq n^{1/2}$, which allows approximately $n^{d/2}$ choices for each $d$-tuple. Since the sum will be of size at most $n$, the probability it actually equals $n$ is about $n^{-1}$ so, we naturally pick $N = n^{1/2}$. Observe this also allows us to replace the finite sum to be an infinite sum and

[2]in Section (3.1) we reivist the Gauss Sums for $(p,q) = 1$.

integral, which is permissible since $A_q(n) = O(q^{1-d/2})$ decays as a result of applying the bound $S(p/q)/q = O(q^{-1/2})$ to the sum.

## 3. Singular Integral

*Note: this section contains some of Yash Patel's part of our joint presentation that we prepared and divided together. Thus, some may overlap with his paper. We discuss the bounds for completeness of the discussion of the Circle Method.*

Recall we define the Singular Integral as

$$A_q(n) = \sum_{\substack{1 \le p \le q \\ (p,q)=1}} \frac{1}{q^d} S(p/q)^d e^{-2\pi i n p/q} \tag{3.1}$$

We now demonstrate that the series has a convergent tail and is bounded.

**Proposition 3.2.** *The singular series for $d \ge 5$*

$$\mathfrak{S} = \sum_{q=1}^{\infty} A_q(n)$$

*converges absolutely with*

$$\sum_{q>R}^{\infty} A_q(n) = O(R^{2-d/2}) \forall R \ge 1$$

*Furthermore, there exists $C_1, C_2$ such that,*

$$C_1 \le |\mathfrak{S}| \le C_2$$

*Proof.* The proof consists of three parts:

(1) We show that the original summation is bounded and discuss a variant $\psi_q(n)$
(2) We formulate $A_q(n)$ as an infinite product and show their duality
(3) We finally bound the infinite product $A_q(n)$

### 3.1. Convergence of Singular Series.

Recall that $S(p/q)$ are Gauss sums defined as

$$S(p/q) = \sum_{1 \le n`q} e^{2\pi i n^2 p/q}$$

From a prior presentation on Gauss sums we have the result for $(p, q) = 1$:

$$\begin{cases} \sqrt{q} & q \equiv 1 \mod 4 \\ i\sqrt{q} & q \equiv 3 \mod 4 \\ 0 & q \equiv 2 \mod 4 \\ e^{i\pi/4}\sqrt{2q} & q \equiv 0 \mod 4 \end{cases}$$

This allows us to bound $|A_q(n)|$ as follows:

$$\begin{aligned}|A_q(n)| &= |\sum_{\substack{1\le p\le q\\(p,q)=1}} q^{-d}S(p/q)^d e^{-2\pi ip/q}|\\ &\le \sum_{\substack{1\le p\le q\\(p,q)=1}} |q^{-d}S(p/q)^d e^{-2\pi ip/q}|\\ &= \sum_{\substack{1\le p\le q\\(p,q)=1}} |q^{-d}||S(p/q)^d||e^{-2\pi ip/q}|\\ &= \sum_{\substack{1\le p\le q\\(p,q)=1}} |q^{-d}||S(p/q)^d|\end{aligned}$$

So if $q \equiv 1 \mod 4$ or $q \equiv 3 \mod 4$, then $|S(p/q)^d| = |\sqrt{q}^d|$. Otherwise, if $q \equiv 3 \mod 4$, the term reduces to 0, and if $q \equiv 0 \mod 4$, then $|S(p/q)^d| = |\sqrt{2q}^d|$, therefore we have:

$$A_q(n) \le \begin{cases} \sum_{1\le p\le q} |q^{-d/2}| & q \equiv 1 \mod 4 or q \equiv 3 \mod 4\\ 0 & q \equiv 2 \mod 4\\ 2^{d/2}\sum_{1\le p\le q} |q^{-d/2}| & q\equiv 0 \mod 4\end{cases}$$

Since $\sum_{1\le p\le q} |q^{-d/2}| = |q^{-d/2}|\sum_{1\le p\le q} 1 = q^{1-d/2}$, we have the bound

$$|A_q(n)| \le c_d q^{1-d/2}$$

Disregarding the constant terms in the error, note that the series converges whenever the exponent $\le -1$, which corresponds to $d \ge 5$ as required.

To obtain our required bound, note that the tail of the series when $q > R$ satisfies:

$$\sum_{q>R} |A_q(n)| = O(\sum_{q>R} q^{1-d/2}) = O(\sum_{q>R} R^{1-d/2})$$

We are interested in when $d \ge 5$, which implies the exponent $1 - d/2 < -1$ and that the function is decreasing in $q$. Thus, we can bound the sum by simply by considering the minimal element in the restricted domain.

Now, we introduce another series $\psi_q(n)$ for a prime, $q$:

$$\psi_q(n) = 1 + \sum_{j=1}^{\infty} A_{q^j}(n)$$

While not immediately obvious, it is the case that $\mathfrak{S}$ can be written as an infinite product of these $\psi_q(n)$, which we discuss in the subsequent section. First, we note that for any odd prime, we have

$$\begin{aligned}|\psi_q(n) - 1| &= |\sum_{j=1}^{\infty} A_{q^j}(n)| \le \sum_{j=1}^{\infty}(q^j)^{1-d/2}\\ &= \sum_{j=1}^{\infty}(q^{1-d/2})^j = \frac{q^{1-d/2}}{1-q^{1-d/2}} = \frac{1}{q^{d/2-1}-1}\end{aligned}$$

which converges whenever $d \geq 5$. When $q = 2$, $A_2(n) = 0$, thus

$$|\psi_2(n) - 1| \leq \frac{1}{2^{d/2-1} - 1}$$

So, from the above, for all primes $q$ we have:

$$\sum_{q \in \text{primes}} |\psi_q(n) - 1| \leq \sum_{q \in \text{primes}} \frac{1}{q^{d/2-1} - 1}$$

Recall from complex analysis, the absolute convergence of this sum implies convergence of the infinite product of $v_q(n)$, thus the value of the sum is well-defined.

3.2. **Infinite Product Equivalence.** We now prove the following theorem that allows us to relate $\mathfrak{S}, A_q(n), \psi_q(n)$:

**Theorem 3.3.**

$$\mathfrak{S} = \sum_{q=1}^{\infty} A_q(n) = \prod_{q \in primes} \psi_q(n)$$

*Proof.* From a lemma about the properties of $A_q(n)$ we know that if $(q_1, q_2) = 1$, $A_{q_1, q_2}(n) = A_{q_1}(n) A_{q_2}(n)$.

$\prod \psi_q(n)$ follows from fact that we can write any $q \in \mathbb{Z}$ as a product of primes by the Fundamental Theorem of Arithmetic. Thus,

$$A_q(n) = A_{q_1 q_2 \ldots q_j}(n) = \prod_{i=1}^{j} A_{q_i}(n)$$

which matches the construction of the infinite product.

Now it remains to show that $(p_1, q_1) = (p_2, q_2) = (q_1, q_2)$ implies:

$$S\left(\frac{p_1 q_2 + p_2 q_1}{q_1 q_2}\right) = S(p_1/q_1) S(p_2/q_2)$$

We can expand this as:

$$\begin{aligned}
A_{q_1}(n) A_{q_2}(n) &= \sum_{\substack{p_1 \text{mod} q_1 \\ (p_1, q_1) = 1}} \sum_{\substack{p_2 \text{mod} q_2 \\ (p_2, q_2) = 1}} q_1^{-d} q_2^{-d} S(p_1/q_1)^d S(p_2/q_2)^d e^{-\pi i n (p_1/q_1 + p_2/q_2)} \\
&= \sum_{\substack{p_1 \text{mod} q_1 \\ (p_1, q_1) = 1}} \sum_{\substack{p_2 \text{mod} q_2 \\ (p_2, q_2) = 1}} (q_1 q_2)^{-d} S\left(\frac{p_1 q_2 + p_2 q_1}{q_1 q_2}\right)^d e^{-\pi i n (p_1/q_1 + p_2/q_2)}
\end{aligned}$$

which is precisely $= A_{q_1 q_2}(n)$ since $(q_1, q_2) = 1$. □

3.3. **Bounding the Product.** Recall we showed in (3.1) that

$$|\psi_q(n) - 1| \leq \frac{1}{q^{d/2-1} - 1} < 1$$

Note that $|\psi_q(n)| \geq 1$, for if it were $= 0$, then the LHS would be 1, which means that it is non-trivially bounded below, as desired. Specifically, this converges for $d \geq 5$, thus,

$$|\psi_q(n) - 1| \leq \frac{1}{q^{3/2} - 1} < 1$$

Explicitly, these bounds are:

$$\prod_{\text{q prime}} \left(1 - \frac{1}{q^{3/2} - 1}\right) \leq |\mathfrak{S}(n)| \leq \prod_{\text{q prime}} \left(1 + \frac{1}{q^{3/2} - 1}\right)$$

And thus completes our proof of Proposition 3.2. □

Finally, we provide the following proposition without proof, since it was provided in the notes, to bound the singular integral used to relate $r_d(n)$ and $A_q(n)$. By inspection, we see that the larger $N$ is, the smaller the error term when approximating the infinite series using a finite one.

**Proposition 3.4.** *The singular integral* $\forall s > 1$*:*

$$e^{\pi} \int_{-\infty}^{\infty} (1 - 2ix)^{-s} e^{-2\pi i x} dx = \frac{\pi^s}{\Gamma(s)}$$

*Further,* $\forall S \geq 1$ *and* $s \geq 1$*:*

$$\int_{|x| \geq S} (1 - 2ix)^{-s} e^{-2\pi i x} = O(S^{1-s})$$

Now, we can replace the original integral with a finite computation over the interval: $N^2 I'(p/q)$:

$$\int_{N^2 I'(p/q)} (1 - 2ix)^{-d/2} e^{-2\pi i x} dx$$

Thus, to approximate this using Proposition 3.4, an error term occurs from the complement of $N^2 I'(p/q)$. However, we can bound this tail estimate by showing that the $x \in \{\text{the complement of } N^2 I'(p/q)\} \geq 1$, which is simply done by considering $N \leq \sqrt{n}$.

So we can apply the same approach with $s = d/2$ and $S = n/Nq$ (for the singular integral):

$$r_d(n) = N^{d-2} e^{\pi n/N^2} \sum_{1 \leq q \leq N} A_q(n) \int_{N^2 I'(p/q)} (1 - 2ix)^{-d/2} e^{-2\pi i x} dx + O(N^{d/2})$$

$$= n^{d/2-1} \sum_{1 \leq q \leq N} A_q(n) \frac{\pi^{d/2}}{\Gamma(d/2)} + Er$$

where $Er$ is the error. Our final bounding result for the singular series yields:

$$Er = O(n^{d/2-1} \sum_{1 \leq q \leq N} |A_q(n)| \left(\frac{n}{qN}\right)^{1-d/2}) + O(n^{d/4}) \tag{3.5}$$

Since $|A_q(n)| \leq c_d q^{1-d/2}$ (3.5) is bounded by:

$$
\begin{aligned}
Er &\leq O(n^{d/2-1}\sum_{1\leq q\leq N} c_d q^{1-d/2}\left(\frac{n}{qN}\right)^{1-d/2}) + O(n^{d/4}) \\
&= O(n^{d/2-1}\sum_{1\leq q\leq N}\left(\frac{n}{N}\right)^{1-d/2}) + O(n^{d/4}) \\
&= O(\sum_{1\leq q\leq N}(1/N)^{1-d/2}) + O(n^{d/4}) = O(N^{d/2}) + O(n^{d/4})
\end{aligned}
$$

Since $N \leq \sqrt{n}$, $O(N^{d/2})$ is $O(n^{d/4})$, so $Er$ is $O(n^{d/4})$, as required. Now it remains to change the finite sum in the above definition of $r_d(n)$ to an infinite sum. So we can apply the bound we demonstrated for the tail of the infinite sum of $|A_q(n)|$ to a radius of $N$. Thus, we have an error term of $O(n^{d/2-1}N^{2-d/2})$, which is once again $O(n^{d/4})$ when considering the restriction of $N \geq \sqrt{n}$. Therefore, finally we arrive at:

$$
\begin{aligned}
r_d(n) &= n^{d/2-1}\sum_{q=1}^{\infty} A_q(n)\frac{\pi^{d/2}}{\Gamma(d/2)} + O(n^{d/4}) \\
&= \mathfrak{S}\frac{\pi^{d/2}}{\Gamma(d/2)}n^{d/2-1} + O(n^{d/4})
\end{aligned}
$$

Thus, $r_d(n)$ for $d \geq 5$ is well-defined and finite, and our proof is complete.

## 4. Example: $r_8(n)$

In Professor Stein's book, *Complex Analysis* Chapter 10 details the proof of the Four-Square Theorem,

**Theorem 4.1.** *Every positive integer is the sum of four squares, and moreover*

$$r_4(n) = 8\sigma_1^*(n) \quad \textit{for all } n \geq 1$$

The main idea in the proof of the Four-Square Theorem is the use of the following theorem:

**Theorem 4.2.** *Suppose $f$ is a holomorphic function in the upper half-plane that satisfies*

*(1)* $f(\tau + 2) = f(\tau)$
*(2)* $f(-1/\tau) = f(\tau)$
*(3)* $f(\tau)$ *is bounded,*

*then $f$ is constant.* [1]

where we find a modular function whose equality with $4\Theta(\tau)^4$ expresses the desired identity and show that their quotient is constant using the above theorem. This is crucial and appropriate as our choice of generating function utilizes $\Theta$, which is holomorphic and does not vanish in the upper half plane (see Proposition 1.1 and Corollary 1.4 in [1])

Here we consider a problem of similar nature, the analogous Eight-Square Theorem:

**Theorem 4.3.** *$r_8(n) = 16\sigma_3^*(n)$ Here $\sigma_3^*(n) = \sigma_3(n) = \sum_{d|n} d^3$, when $n$ is odd, and when $n$ is even*

$$\sigma_3^*(n) = \sum_{d|n} (-1)^d d^3 = \sigma_3^e(n) - \sigma_3^o(n)$$

*where $e, o$ superscripts denote even and odd values of $d$ respectively.*

*Proof.* Our proof consists of several lemmas. We proceed first by relating the sequence $\{r_8(n)\}$ via generating function with $\Theta^8$.

$$\Theta(\tau)^8 = \sum_{n=0}^{\infty} r_8(n) e^{\pi i \tau}, \quad \tau \in \mathbb{H}$$

Next, we find the modular function whose equality with $\Theta(\tau)^8$ expresses the identity $r_8(n) = 16\sigma_3^*(n)$:

**Lemma 4.4.** *The assertion $r_8(n) = 16\sigma_3^*(n)$ is equivalent to the identity*

$$\Theta(\tau)^8 = \frac{48}{\pi^4} E_4^*(\tau)$$

To prove the lemma, it suffices to show with $q = e^{\pi i \tau}$ then

$$\frac{48}{\pi^4} E_4^*(\tau) = 1 + \sum_{k=1}^{\infty} 16\sigma_3^*(k) q^k$$

First, we need some relationship between the standard Eisenstein series

$$E_4 = \sum_{(n,m)\neq(0,0)} \frac{1}{(n+m\tau)^4}$$

with our modified series $E_4^*$ which sums only over integers $n, m$ with opposite parity. So, essentially by inspection, $E_4^*$ is simply $E_4$ with the terms that sum over $n, m$ with the same parity removed. Consider the transformation $\tau \mapsto \frac{(\tau-1)}{2}$, this maps the denominator

$$(n + m\tau) \mapsto (2n - m + m\tau)$$

Observe that if $m$ is even, then $2n-m$ is even; and if $m$ is odd, $2n-m$ is odd. Thus, this allows us to isolate the terms with $n, m$ of the same parity, which naturally induces the relationship:

$$E_4^*(\tau) = E_4(\tau) - 2^{-4} E_4((\tau - 1)/2) \tag{4.5}$$

From Chapter 9 on Elliptic Functions [1], Theorem 2.5 gives us for $k = 4$ we have

$$E_4(\tau) = 2\zeta(4) + \frac{(2\pi)^4}{3} \sum_{k=1}^{\infty} \sigma_3(k) e^{2\pi i k \tau} \tag{4.6}$$

And from the definition of $\sigma_3^*$ in the theorem and (4.5) our desired identity follows

$$E_4^*(\tau) = \frac{\pi^4}{48} + \frac{\pi^4}{3} \sum_{k=1}^{\infty} \sigma_3^*(k) e^{\pi i k \tau}$$

We have thus reduced Theorem 4.3 to the identity $\Theta^8 = 48\pi^{-4} E_4^*$, and it remains to show that $E_4^*$ satisfies the same modular properties as $\Theta^8$.

**Proposition 4.7.** *The function $E_4^*(\tau)$ defined in the upper half-plane has the following properties:*

(1) $E_4^*(\tau+2) = E_4^*(\tau)$
(2) $E_4^*(\tau) = \tau^{-4}E_4^*(-1/\tau)$
(3) $(48/\pi^4)E_4^*(\tau) \to 1$ *as* $\Im(\tau) \to \infty$
(4) $|E_4^*(1-\frac{1}{\tau})| \approx |\tau^4||e^{2\pi i\tau}|$

The periodicity of (1) follows directly from the definition of $E_4$. To see (2) note the transformation $\tau \mapsto -1/\tau$:

$$(n+m\tau) \mapsto (n+m(-1/\tau))^4 = \tau^{-4}(-m+n\tau)^4$$

gives us that $E_4(\tau) = \tau^{-4}E_4(-1/\tau)$. And $E_4^*$ can be expressed in terms of $E_4$ so this is satisfied. To prove (3) recall that

$$E_4(\tau) = 2\zeta(4) + \frac{(2\pi)^4}{3}\sum_{k=1}^{\infty}\sigma_3(k)e^{2\pi ik\tau}$$

where the sum goes to 0 as $\Im(\tau) \to \infty$. So using the fact that

$$E_4^*(\tau) = E_4(\tau) - 2^{-4}E_4((\tau-1)/2)$$

we conclude that $E_4^*(\tau) \to \pi^4/48$ and our desired expression goes to 1 as $\Im(\tau) \to \infty$. To prove the final property, we begin by showing that

$$E_4^*(1-\frac{1}{\tau}) = \tau^4(E_4(\tau) - E_4(2\tau) \tag{4.8}$$

Note that $E_k(\tau)$ is periodic with period 1, from the fact that $(n+m(\tau+1)) = (n+m+m\tau)$ and we can rearrange the sum replacing $n+m$ by $n$. Using this fact and $E_4(\tau) = \tau^{-4}E_4(-1/\tau)$ we have

$$\begin{aligned} E_4(1-\frac{1}{\tau}) &= E_4(\frac{-1}{\tau}) \\ &= \tau^4E_4(\tau) \end{aligned} \qquad \begin{aligned} E_4(\frac{1-\frac{1}{\tau}}{\tau}) &= E_4(\frac{-1}{2\tau}) \\ &= (-2\tau)^4E_4(2\tau) \end{aligned}$$

And thus,

$$\begin{aligned} E_4^*(1-\frac{1}{\tau}) &= E_4(1-\frac{1}{\tau}) - 2^{-4}E_4(\frac{1-\frac{1}{\tau}}{\tau}) \\ &= \tau^4\left[E_4(\tau) - E_4(2\tau)\right] \end{aligned}$$

This proves (4.8), and the last property follows from it and the fact that

$$E_4(\tau) = 2\zeta(4) + \frac{(2\pi)^4}{3}\sum_{k=1}^{\infty}\sigma_3(k)e^{2\pi ik\tau}$$

Thus, Proposition 4.7 is proved.

Since we showed that both $E_4^*$ and $\Theta(\tau)^8$ satisfies the above properties, it follows that the invariant function $48\pi^{-4}E_4^*(\tau)/\Theta(\tau)^8$ is bounded and by Theorem 4.2, therefore a constant, which must be 1. This yields the desired theorem. □

**Acknowledgments.** My deepest gratitude and respect goes to Professor Stein for guiding me through the seminar, encouraging me, and believing in me until the end. To Professor Kollar and Dean Lestition, thank you for your support and advice this year, despite my struggles with health and the sudden gap year. My sincere thanks goes to David Nie and Tammy Tseng for proofreading this paper, to Leo Choi for his unending supply of encouragement and late-night food, and to Yash Patel, who was my partner throughout the two presentations in the Fall of 2016. Finally, I want to give my deepest thanks to my parents, without whose love and unending support I could not have come this far.

*Honor Code* I pledge my honor that this paper represents my own work in accordance with University regulations.

## References

[1] E. Stein and R. Shakarchi. Complex Analysis. Chapter 10.
[2] L. Pierce and E. Stein. Notes on Fourier Analysis and Applications to Number Theory. 78-99.
[3] R. Vaughan, The Hardy-Littlewod method