CLIFFORD ALGEBRAS

DOWON KIM

Abstract. The object of this paper is to provide an expository introduction to Clifford Algebras. In particular, it will explore the induced subalgebra of spin groups which are double covers of **SO** and complete the representations of orthogonal Lie algebras. In addition, this paper examines the periodicity of Clifford Algebras in higher dimensions.

Contents

1. Brief History

Following the extension from $\mathbb{R}$ to $\mathbb{C}$, mathematicians sought to find a further extension of the real and complex numbers. A natural place to start was a three dimensional algebra over the reals, since $\dim(\mathbb{C}) = 2$. However, the search proved elusive and a major breakthrough occurred in 1843, when William Hamilton discovered the quaternions of dimension 4. The quaternions, denoted $\mathbb{H}$ is an associative, non-commutative division algebra with a real identity and three imaginary units **i,j,k** as the basis that obey the following:

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -\mathbf{1}, \quad \mathbf{ij} = -\mathbf{ji} = \mathbf{k}$$

As a logical extension of the complex numbers, the geometric application of quaternions was realized when Cayley showed that a unit quaternion operator written as

$$\mathbf{q} = q_0\mathbf{1} + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k}$$

represents a three-dimensional rotation with the appropriate coefficients that encode the angle and axis of rotation.

Shortly after, efforts to find higher dimensional normed division algebras led to the discovery $\mathbb{O}$, which are neither commutative nor associative. In 1876, William K. Clifford in his paper, *On the Classification of Geometric Algebras* generalized $\mathbb{H}$ to higher dimensions in a different way from $\mathbb{O}$ through *Clifford Algebras*, which naturally completed the sequence $\mathbb{R}, \mathbb{C}, \mathbb{H}$.[4] Observing subalgebras of Clifford algebras

Date: May 11, 2018.

led to surprising relationships with topology and geometry via spinor groups. Furthermore, it turns out that Clifford algebras C_k can be entirely classified by virtue of their periodic nature which is analogous to Bott-Periodicity in stable homotopy groups.

2. Preliminaries

Let V be a vector space over the commutative field k and let Q be a symmetric bilinear form on V. The *Clifford algebra* $Cl(V, Q)$ is an associative algebra with unit 1, which is generated by and contains V. Clifford algebras obey the relation:

$$v \cdot v = Q(v, v) \cdot 1$$

By polarizing, equivalently [1]

$$v \cdot w + w \cdot v = 2Q(v, w) \cdot 1$$

for all $v, w \in V$. It can be generalized to be the *universal algebra* for any unital associative algebra E over field k and a linear map $j : V \mapsto E$ that obeys for all $v, w \in V$

$$j(v) \cdot j(w) + j(w) \cdot j(v) = 2Q(v, w) \cdot 1$$

Then there exists a unique homomorphism of algebras that from $Cl(V, Q) \to E$ that extends the linear map j.

To construct the Clifford algebra, consider the tensor algebra over E

$$T(E) = \sum_{i=0}^{\infty} T^i = k \oplus E \oplus (E \otimes E) \oplus (E \otimes E \otimes E) \oplus \cdots$$

and let $I(Q)$ be the two-sided ideal generated by elements $x \otimes x - Q(x, x) \cdot 1$ in $T(E)$. In particular, let $E = V, k = \mathbb{C}$ and we can set $Cl(V, Q) = T(V)/I(Q)$ where by construction $Cl(V, Q)$ satisfies the equivalence relation. Since Clifford algebras are generated by a vector field, naturally we want to consider the basis of $Cl(V, Q)$:

Proposition 2.1. *Let $e_1, ..., e_m$ be the basis for V. The products $e_I = e_{i_1} e_{i_2} e_{i_3} \cdots e_{i_k}$ where each I is a k-subset of the basis of V, $1 \leq i_1 < i_2 < \cdots < i_k \leq m$ and $0 \leq k \leq m$ form the basis of $Cl(V, Q)$.*

Since $e_i e_j + e_j e_i = 2Q(e_i, e_j)$, it follows that e_I generate $Cl(V, Q)$. Note that I spans over each subset of the m-basis of V, thus we have the dimension of a Clifford algebra over V

$$\dim(Cl(V, Q)) = \sum_{k=0}^{m} \binom{m}{k} = 2^m$$

Furthermore, note that if $Q = 0$, by definition $Cl(V, Q)$ is the exterior algebra $\wedge^k V$, i.e. the square of any element in V is 0. We can see this by constructing a filtration on $Cl(V, Q)$, since a tensor algebra $T(V)$ has a natural filtration $F^0 \subset F^1 \subset \cdots \subset F^m$, where F^k are elements that can be written as sums of at most k products of elements in V. This induces a filtration on $Cl(V, Q)$, whose associated graded algebra is isomorphic to the exterior algebra on V, namely $F^k / F^{k+1} \simeq \wedge^k V$.

[1] If the characteristic on the ground field k is not 2. When the $char(k) = 2$, the quadratic form does not uniquely determine a symmetric bilinear form. In this paper, we will ignore this particular case and assume in general the characteristic is not 2

Another important property of Clifford algebras is that it inherits a $\mathbb{Z}/2\mathbb{Z}$ *gradation*, naturally as the underlying construction involves direct sums of tensor algebra.

Proposition 2.2. *Let $Cl^0(Q)$ be the image of $\sum_{i=0}^{\infty} T^{2i}(V)$ in $Cl(Q)$ and set $Cl^1(Q)$ equal to the image of $\sum_{i=0}^{\infty} T^{2i+1}(V)$ in $Cl(Q)$. Then this decomposition defines $Cl(Q)$ as a $\mathbb{Z}/2\mathbb{Z}$-graded algebra.*[2]
Equivalently:
(1) $Cl(Q) = Cl^0(Q) \oplus Cl^1(Q)$
(2) *If $x_i \in Cl^i(Q), y_j \in Cl^j(Q)$ then*

$$x_i y_j \in Cl^k(Q), \quad k = i + j \mod 2$$

An alternative interpretation of this decomposition follows from an *involution* map of Clifford algebras. Note that the linear map on V defined by $v \mapsto -v$ (i.e. flipping across the origin) preserves the quadratic form Q. Thus, using the universal property, this extends to an algebraic automorphism $\alpha : Cl(Q) \to Cl(Q)$. Since α is an involution, we can decompose $Cl(Q)$ into positive and negative eigenspaces of α which yields (1)

$$Cl(Q) = Cl^0(Q) \oplus Cl^1(Q)$$

since $Cl^i(Q)$ is equivalently $\{x \in Cl(Q) | \alpha(x)^i = (-1)^i x\}$ under this reflection. As α is an automorphism, we arrive at (2)

$$Cl^{i*}(Q)Cl^{j*}(Q) = Cl^{(i+j)*}(Q)$$

where $*$ denotes mod 2. The importance of $\mathbb{Z}_2$-graded structure of $Cl(Q)$ can be seen from the following [1].

Proposition 2.3. *Let $V = V_1 \oplus V_2$ be an orthogonal decomposition of the vector space V relative to Q, i.e. $Q(v_1 + v_2) = Q(v_1) + Q(v_2) \forall v_1 \in V, v_2 \in V_2$. Then there is a natural isomorphism of Clifford algebras*

$$Cl(V, Q) \simeq Cl(V_1, Q_1) \hat{\otimes} Cl(V_2, Q_2)$$

of the graded tensor-product of $Cl(V_1, Q_1)$ and $Cl(V_2, Q_2)$ with $Cl(V, Q)$, where Q_i denotes the restriction of Q to V_i.

Proof. The graded tensor product of two graded algebras $A = \sum_{\alpha=0,1} A^\alpha, B = \sum_{\alpha=0,1} B^\alpha$ is by definition the algebra over the vector space $\sum_{\alpha,\beta=0,1} A^\alpha \otimes B^\beta$ where tensor multiplication is defined as:

$$(u \otimes x_i) \cdot (y_j \otimes v) = (-1)^{ij} u y_j \otimes x_i v, \quad x_i \in Cl^i(Q), y_j \in Cl^j(Q)$$

The graded tensor product is again a graded algebra and denoted

$$(A \hat{\otimes} B)^k = \sum A^i \times B^j \quad (i + j) \equiv k \mod 2$$

Now, we define the map $\phi : V \to Cl(Q_1) \hat{\otimes} Cl(Q_2)$ by explicitly as

$$\phi(v) = v_2 \otimes 1 + 1 \otimes v_2$$

where v_1, v_2 are the orthogonal projections of v on V_1, V_2. Then

$$\phi(v)^2 = (v_1 \otimes 1 + 1 \otimes v_2)^2 = \{Q_1(v_1) + Q_2(v_2)\}(1 \otimes 1) = Q(v)(1 \otimes 1)$$

Then by the universal algebra property, ϕ extends to an algebra homomorphism that maps $Cl(V, Q) \to Cl(Q_1) \hat{\otimes} Cl(Q_2)$. Bijectivity follows by checking the action of ϕ with basis elements. □

[2] Let $Cl(V, Q)$ and $Cl(Q)$ be equivalent for simplicity

Remark 2.4. When considering Clifford algebras over a real vector space of $\dim V = m$, non-degenerate quadratic forms can be expressed into a diagonal form with $p + q = m$ where p denotes the number of positive terms and q the negative terms. The quadratic form corresponding to (p, q) is denoted $Q_{p,q}$ and we call (p, q) the *signature* of $Q_{p,q}$. For instance, physicists may be interested in the case $p = 3, q = 1$, which corresponds to the *Minkowski space*, four-dimensional Euclidean spacetime. We will primarily focus on Clifford algebras equipped the positive definite quadratic forms where $p = m$. But later we show that there exists isomorphisms that relate Clifford algebras equipped with $Q_{p,q}$.
Over complex vector spaces, there is only one non-degenerate quadratic form Q up to isomorphism (i.e. all diagonal elements may be chosen to be $+1$).

Examples 2.5. Let us consider a few examples of Clifford algebras over $V = \mathbb{R}^k$, which we denote Cl_k.

For $k = 0$, Cl_0 is the algebra of dimension $2^0 = 1$, over $\mathbb{R}$ generated by $e_\emptyset$ which is a unit by definition. Clearly, $Cl_0 = \mathbb{R}$.

For $k = 1$, Cl_1 is the algebra of dimension 2 over $\mathbb{R}$ generated by $\{1, e_1\}$ obeying $e_1^2 = -1$. Note that this familiar algebra is equivalent to the complex numbers $\mathbb{C}$! Thus, $Cl_1 = \mathbb{C}$.

For $k = 2$, Cl_2 is the algebra of dimension $2^2 = 4$ generated by the basis elements $\{1, e_1, e_2, e_1e_2\}$ with the relations

$$e_1^2 = -1, e_2^2 = -1, (e_1e_2)^2 = -1 \Rightarrow e_1e_2 = -e_2e_1$$

This is precisely the generalization of the quaternions $\mathbb{H}$ where

$$i = e_1, j = e_2 = k = e_1e_2$$

thus, we see that $Cl_2 = \mathbb{H}$.

We will see later that Clifford algebras in higher dimensions exhibit a certain 8-periodicity

$$C_{k+8} \simeq C_k \otimes \mathbb{R}(16)$$

which remains true over all nondegenerate quadratic forms with signature (p, q)

$$C_{0,q+8} \simeq C_{0,q} \otimes \mathbb{R}(16)$$

Remark 2.6. Of the 4 normed division algebras: $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$. The octonions cannot be described through Clifford algebras because they are *nonassociative*. However, Clifford algebras can be used to prove that there are only 4 such algebraic systems, in dimensions $1, 2, 4, 8$, which was proven by Chevalley in 1954.

3. Spin and Pin Groups

In representations of orthogonal Lie algebras, $\mathbf{SL}(n)$ and $\mathbf{Sp}(2n)$ can be found inside tensor powers of the standard irreducible representation, but this is not the case for $\mathbf{SO}(m)$ [3]. One difference is that the topology of $\mathbf{SL}(n)$ and $\mathbf{Sp}(2n)$ are simply connected, while $\mathbf{SO}(m)$ is not. Furthermore, $\mathbf{SO}(m)$ has a cyclic fundamental group $\mathbb{Z}/2$. Therefore, $\mathbf{SO}(m)$ has a double covering, called the *spin group*, *Spin*(m), which also provide the missing representations of the special orthogonal group.

To motivate the definition of spin groups, we can observe the lower dimensional cases of the special orthogonal group. It turns out that rotations in $\mathbf{SO}(3)$ can be represented by the group of unit quaternions [3] $\mathbf{SU}(2)$ on $\mathbb{R}^3$, and rotations in $\mathbf{SO}(4)$ can be represented by the action of $\mathbf{SU}(2) \times \mathbf{SU}(2)$ on $\mathbb{R}^4$. Thus, a generalization of the quaternions would be a natural place to induce $Spin(m)$ that cover $\mathbf{SO}(m)$, and we will see that *Clifford algebras* provide the way to construct these spin groups as groups of invertible elements or units of Clifford groups.

There are multiple approaches we can take:

(1) Define $Spin(m)$ in terms of invertible elements $x \in Cl^0$ that fix the space $V = \mathbb{R}^n$ to be invariant under conjugation $xVx^{-1} \subset V$

(2) Define the Lie algebra of $Spin(m)$ in terms of quadratic elements of Clifford algebra using the Lie algebra of $\mathbf{SO}(m)$.

(3) Define a map $v \mapsto uvu^{-1}$ for unit vectors $u, v \in V$, which is a reflection map in the hyperplane perdendicular to u. Then $Spin(V, Q)$ will be the restriction of $Pin(V, Q)$ (double covering of the orthogonal group) to the even elements or Cl^0. This is a result of Cartan-Dieudonne theorem which shows that any rotation can be expressed as an even number of reflections. We sketch this method here.

We define a group of units or invertible elements of [4] $Cl_{p,q}(V, Q)$ as

$$Cl_{p,q}^{\times} = \{\zeta \in Cl_{p,q} : \exists \zeta^{-1}, \zeta^{-1}\zeta = \zeta\zeta^{-1} = 1\}$$

The associated Lie algebra is denoted $\mathfrak{cl}_{p,q}^{\times} = Cl_{p,q}$ with the Lie bracket given by $[x, y] = xy - yx$.
The group of units act naturally as automorphisms of the algebra, so there is a homomorphism

$$Ad : Cl_{p,q}^{\times} \mapsto Aut(Cl_{p,q})$$

called the *adjoint representation* given by

$$Ad_{\zeta}(x) = \zeta x \zeta^{-1}$$

whose derivative is the adjoint map $ad : \mathfrak{cl}_{p,q}^{\times} \to Der(Cl_{p,q})$

$$ad_y(x) = [y, x]$$

To see the relationship between the adjoint map and the reflection map about a hyperplane, we have the following result.

[3]For a nonzero vector $u \in \mathbb{R}^3$ the reflection s_u about the hyperplane perpendicular to the vector u can be expressed as $v \mapsto -uvu^{-1}$ where u, v can be thought of as quaternions by $u = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$ and likewise for v

[4]Here we permit all non-degenerate quadratic forms $Q = Q_{p,q}$

Proposition 3.1. *Let $v \in V \subset Cl_{p,q}$ be an element with $Q(v) \neq 0$, then $Ad_v(V) = V$ and for all $u \in V$*

$$-Ad_v(u) = u - 2\frac{Q(v,u)}{Q(v,v)} \cdot v \tag{3.2}$$

Proof. Recall that $uv + vu = -Q(v,u)$. Since $v^{-1} = v/Q(v)$

$$\begin{aligned} -Q(v)Ad_v(u) &= -Q(v)vuv^{-1} = vuv \\ &= -v^2u - 2Q(v,u)v = Q(v)u - 2Q(u,v)v \end{aligned}$$

□

This suggests that we ought to consider the subgroup of elements $\zeta \in Cl^{\times}_{p,q}$ such that $Ad_\zeta(V) = V$. By the above proposition, this group contains all $v \in V$ with $Q(v) \neq 0$ and the adjoint map preserves the quadtratic form. So we define the group $P(V,Q)$ as the subgroup of $Cl^{\times}_{p,q}$ generated by elements $v \in V, Q(v) \neq 0$. From $P(V,Q)$ we define our desired subgroups by quotienting out by constants. Let $\dim V = m$.

Definition 3.3. The *Pin group* of (V,Q) is the subgroup $Pin(m)$ of $P(V,Q)$ generated by $v \in V, Q(v) = \pm 1$. The associated *Spin group* of (V,Q) is defined as the restriction of the Pin group to the elements with even decompositions

$$Spin(m) = Pin(m) \bigcap Cl^0_{p,q}(V,Q)$$

Recall that equation (3.2) is equivalent to a reflection across the hyperplane perpendicular to v, let us denote this map as $\rho_v : V \mapsto V$. So we can think of ρ_v as a map which fixes the hyperplane $v^{\perp} = \{w \in V : Q(v,w) = 0\}$ and maps v to $-v$. But the minus sign in (3.2) means that if $\dim V$ is odd, then Ad_v preserves the orientation. To remedy this we introduce a *twisted adjoint representation.*

$$\widetilde{Ad} : Cl^{\times}_{p,q}(V,Q) \to GL(Cl(V,Q))$$

defined by setting

$$\widetilde{Ad}_\zeta(y) = \alpha(\zeta)y\zeta^{-1} \tag{3.4}$$

where α is the involution map which flips the parity. It is easy to check that $\widetilde{Ad}$ is a homomorphism and importantly, for even elements ζ (i.e. $\zeta \in Cl^0(V,Q)$), $Ad_\zeta = \widetilde{Ad}_\zeta$. Now equation (3.2) becomes,

$$Ad_v(u) = u - 2\frac{Q(v,u)}{Q(v,v)} \cdot v \tag{3.5}$$

and note that this induces a $\widetilde{P}(V,Q)$ analogous to $P(V,Q)$ except $\widetilde{Ad}_\zeta(V) = V$.

As $\widetilde{Ad}_\zeta : V \to V$ for $\zeta \in \widetilde{P}(V,Q)$ preserve the quadratic form through the reflection, the image of $\widetilde{Ad}$ is contained in the group of isometries of V, which is described by the orthogonal group $\mathbf{O}(V,Q) = \mathbf{O}(m)$.

Theorem 3.6. *The $\widetilde{Ad}$ restricted to the Pinor group $Pin(V,Q)$ is a surjection that maps $Pin(V,Q)$ onto $\mathbf{O}(m)$ with kernel $\mathbb{Z}_2 \simeq \{-1,+1\}$.*

To see that $\widetilde{Ad}$ is onto, consider an orthogonal base $e_1 \in V$,

$$\widetilde{Ad}_{e_1}(e_i) = \alpha(e_1)e_ie_1^{-1} = \begin{cases} -e_i & i = 1 \\ e_i & i \neq 1 \end{cases}$$

Thus, $e_1 \in Pin(m)$ and $\widetilde{Ad}_{e_1}$ is the reflection in the hyperplane perpendicular to e_1. We can do this for all of the orthonormal base $\{e_i\} \in V$ to see that the unit sphere lies in $Pin(m)$ by construction, and all orthogonal reflections in the hyperplanes of V are in $\widetilde{Ad}(Pin(m))$. The kernel follows from the fact that $Pin(m)$ is a subgroup of $P(V, Q)$ which is generated by $v \in V, Q(V) = +1, -1$.

By our above definition, the spin groups are comprised of elements with even numbers of generators. As $Spin \subset Pin$ we have

Corollary 3.7. *$\widetilde{Ad}$ also maps $Spin(m) \to \mathbf{SO}(m)$ surjectively.*

Finally, the only thing remains is to show that $Spin(m)$ yields a *double covering* of $SO(m)$. It suffices to show that the elements of the kernel $1, -1$ can be joined by a path in $Spin(m)$. Such a path is given explicitly by

$$\lambda : t \to \cos t + sint \cdot e_1e_2$$

Remark 3.8. All the preceeding discussion is analogous for the spin group over the complex numbers, the formal treatment which we omit. The complex spin group is related to the real spin group by

$$Spin^c(m) \simeq Spin(m) \times \mathbf{U}(1)$$

This is significant as the homomorphism $j : \mathbf{U}(m) \to \mathbf{SO}(2m)$ does not lift to $Spin(2m)$; however the homomorphism $l : \mathbf{U}(m) \to \mathbf{SO}(2m) \times \mathbf{U}(1)$ does lift to $Spin^c(2m)$.

4. Periodicity of the Clifford Algebras $Cl_{p,q}$

We saw that $Cl_{i,0}$ for $i = 0, 1, 2$ are isomorphic to the algebras $\mathbb{R}, \mathbb{C}, \mathbb{H}$. In general, the real algebras $Cl_{p,q}$ can be built up as tensor products of $\mathbb{R}, \mathbb{C}, \mathbb{H}$ and there are periodic relationships that allow us to represent higher dimensional Clifford algebras.

n	$Cl_{n,0}$	$Cl_{0,n}$
0	$\mathbb{R}$	$\mathbb{R}$
1	$\mathbb{C}$	$\mathbb{R} \oplus \mathbb{R}$
2	$\mathbb{H}$	$\mathbb{R}(2)$
3	$\mathbb{H} \oplus \mathbb{H}$	$\mathbb{C}(2)$
4	$\mathbb{H}(2)$	$\mathbb{H}(2)$
5	$\mathbb{C}(4)$	$\mathbb{H}(2) \oplus \mathbb{H}(2)$
6	$\mathbb{R}(8)$	$\mathbb{H}(4)$
7	$\mathbb{R}(8) \oplus \mathbb{R}(8)$	$\mathbb{C}(8)$
8	$\mathbb{R}(16)$	$\mathbb{R}(16)$

The key to show the above classifiation of Clifford algebras follows from this lemma:

Lemma 4.1. *For all $n, p, q \geq 0$, there are the following isomorphisms:*

$$Cl_{n+2,0} \simeq Cl_{0,n} \otimes Cl_{2,0}$$

$$Cl_{0,n+2} \simeq Cl_{n,0} \otimes Cl_{0,2}$$
$$Cl_{p+1,q+1} \simeq Cl_{p,q} \otimes Cl_{1,1}$$

Before we can use this lemma to show periodicity, we note some isomorphisms between tensor products of matrix algebras.

Proposition 4.2. *The following isomorphisms hold:*

$$\begin{aligned} \mathbb{R}(m) \otimes \mathbb{R}(n) &\simeq \mathbb{R}(mn) \quad \forall m, n \geq 0 \\ \mathbb{R} \otimes_{\mathbb{R}} K &\simeq K(n) \quad K = \mathbb{C}, \mathbb{H}, \forall n \geq 0 \\ \mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} &\simeq \mathbb{C} \oplus \mathbb{C} \\ \mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} &\simeq \mathbb{C}(2) \\ \mathbb{H} \otimes_{\mathbb{R}} \mathbb{H} &\simeq \mathbb{R}(4) \end{aligned}$$

$K(n)$ denotes the algebra of $n \times n$ matrices over K. The first two isomorphisms are obvious. Detailed proofs can be found in Lawson and Michelsohn [2].
Now, we arrive at the main periodicity result:

Theorem 4.3 (Bott)**.** *For all $n \geq 0$, the following isomorphisms hold:*

$$Cl_{0,n+8} \simeq Cl_{0,n} \otimes Cl_{0,8}$$
$$Cl_{n+8,0} \simeq Cl_{n,0} \otimes Cl_{8,0}$$

and $Cl_{0,8} = Cl_{8,0} = \mathbb{R}(16)$.

Proof. The proof follows easily from the above lemma.

$$\begin{aligned} Cl_{0,n+8} &\simeq Cl_{n+6,0} \otimes Cl_{0,2} \simeq Cl_{0,n+4} \otimes Cl_{2,0} \otimes Cl_{0,2} \simeq ... \\ &\simeq Cl_{0,n} \otimes Cl_{2,0} \otimes Cl_{0,2} \otimes Cl_{2,0} \otimes Cl_{0,2} \end{aligned}$$

We know that $Cl_{0,2} = \mathbb{R}(2)$ and $Cl_{2,0} = \mathbb{H}$. Using the above proposition we have

$$Cl_{2,0} \otimes Cl_{0,2} \otimes Cl_{2,0} \otimes Cl_{0,2} \simeq \mathbb{H} \otimes \mathbb{H} \otimes \mathbb{R}(2) \otimes \mathbb{R}(2) \simeq \mathbb{R}(4) \otimes \mathbb{R}(4) \simeq \mathbb{R}(16)$$

The isomorphism of $Cl_{n+8,0}$ can be shown analogously to above. $\square$

Acknowledgments. I want to express my sincere gratitude to Professor Lin for guiding me through the Junior Seminar on Differential Topology and Professor Kollar for his support and advice during this semester. My sincere thanks goes to my friends, Hyeong-Joon Suh and Tammy Tseng for their generous help and careful proofreading of this paper, and to Jiwon Kim, who guested me into Korea University's 24-hour study rooms that I may be able to finish this paper while abroad. Finally, I want to give my deepest thanks to my parents, without whose love and unending support I could not have come this far.

Honor Code I pledge my honor that this paper represents my own work in accordance with University regulations.

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