8. Conservative Forces and Potential Energy

If a mass is acted on by a conservative force the total energy in the system, which is the sum of the kinetic and potential energies, will be conserved. In this Lab, the motion of such a mass will computed numerically using both the Euler-Cromer and Rungga Kutta methods.

Theory:
Consider the oscillatory motion of a mass attached to a spring in the context of energy conservation:

When the spring is initially compressed, the system has spring potential energy. When the mass is free to move, this potential energy is converted into kinetic energy, $K = \frac{1}{2}mv^2$. The spring then stretches past its equilibrium position and the potential energy increases again until it equals its initial value.

Consider now the more complicated situation in which the force on the particle is given by

$$F(x) = \kappa x - 4qx^3$$

This is a conservative force and the potential energy is given by:

$$U(x) = -\frac{1}{2}\kappa x^2 + qx^4$$

From this, we can calculate the motion using Newton's second law. Given the force, F, on an object (of mass m), its position and velocity may be found by solving the two ordinary differential equations:

$$\frac{dv}{dt} = \frac{1}{m}F(t) \quad ; \quad \frac{dx}{dt} = v(t)$$

If we replace the derivatives with their right derivative approximations, we have:

$$\frac{v(t+\Delta t) - v(t)}{\Delta t} = \frac{1}{m}F(t) \quad ; \quad \frac{x(t+\Delta t) - x(t)}{\Delta t} = v(t)$$

or

$$\frac{v_f - v_i}{\Delta t} = \frac{1}{m}F_i \quad ; \quad \frac{x_f - x_i}{\Delta t} = v_i$$

where the subscripts i and f refer to the initial (time t) and final (time $t + \Delta t$) values. Of course, the approximation is only accurate when Δt is small. Solving each equation for the final values of velocity and position, we have:

$$v_f = v_i + \frac{1}{m}F_i\Delta T \quad ; \quad x_f = x_i + v_i\Delta t$$

Using these equations repeatedly, we may iterate from any initial condition to compute the motion of the object. This scheme for computing the motion is called the *Euler method.* Unfortunately, the Euler method is not always accurate. A better way to compute the motion is to use the equations:

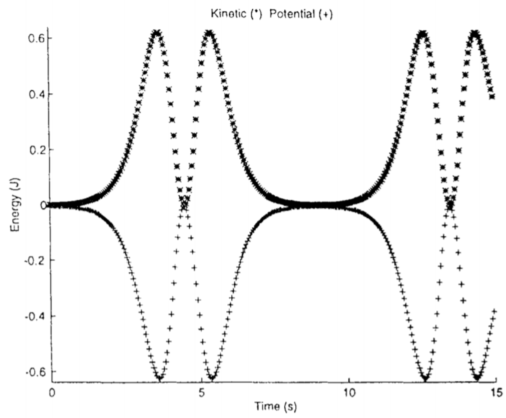
$$v_f = v_i + \frac{1}{m}F_i\Delta T \quad ; \quad x_f = x_i + v_f\Delta t$$

Notice the subtle difference: In the computation of the position the new velocity v_f is used instead of the old velocity v_i. This scheme is known as the *Euler-Cromer method* (It is a simple variant of the leap-frog method).

Hints to implementing the Euler-Cromer method:

- Initialize variables (e.g., initial position, N, K, q,m).
- Set up plot of kinetic and potential energy versus time.
- Loop for desired number of steps.
- Compute kinetic and potential energy.
- Update the plot by adding new values to the graph.
- Compute force and acceleration.
- Update position and velocity using Euler-Cromer.
- Notice that when the position and velocity are updated, since the velocity is updated first and then used to update the position, the Euler-Cromer method is implemented. If we reverse the order of these operations (i.e., compute the position first) then we would be using the Euler method.

1. With $\kappa = 1.0$ N/m, $q = 0.1$ N/m^3 and an initial displacement of $0.1m$, show that the kinetic and potential energies vary with time as:



Notice that initially the kinetic energy is zero since the initial velocity is $v = 0$. The particle speeds up as it moves into the potential "well", reaching a maximum kinetic energy when the particle reaches the minimum of the potential (near $t = 4s$). The kinetic and potential energies mirror each other since their sum, the total energy, should remain constant.

2. Plot similar graphs for the following initial displacements: (a) 10^{-3} m; (b) 0.01 m; (c) 0.1 m; (d) 1.0 m; (e) 10.0 m.
3. Repeat the previous exercise but set $q = 0$ and $\kappa = -1$ N/m, making the force that of an ideal spring.
4. Modify your code to also plot the total energy for the above displacements.
5. Modify your code to use the 2nd order Rungga Kutta method instead of the Euler-Cromer method. Comment on the difference between the plots produced by the two methods.
6. Modify your code to plot position and velocity versus time. Run it for the displacements given in exercise 2 and print the resulting graphs.

7. The period of an oscillation is the time it takes for the mass to return to its original position. Write code that computes the period for various displacements (from 10^{-3} m to 10.0 m). Plot the period versus displacement using a semi-log scale.

8. (a) Write code that accepts an arbitrary expression for the force law as a function of displacement and, using the symbolic manipulation capabilities of SymPy, obtains an expression for the potential energy. (b) Plot $U(x)$.

9. Devise and execute an extension to this lab. For example, it could be based on examining the effect of the the parameters which have been kept fixed, or investigate damping effects by introducing a damping term in the equation of motion.