Math 134: Cryptography

Lecture 7: RSA encryption

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Definition

Let $\phi : \mathbb{N} \to \mathbb{N}$ be defined on $n \in \mathbb{N}$ by

$$\phi(n) = \#\{d \in \mathbb{N} : 1 \leq d \leq n \text{ and } \gcd(d, n) = 1\}.$$

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Let $n = 30$. The integers between 1 and 30 are as follows:

$$1\ 2\ 3\ 4\ 5\ 6\ 7\ 8\ 9\ 10\ 11\ 12\ 13\ 14\ 15\ 16\ 17\ 18\ 19$$

$$20\ 21\ 22\ 23\ 24\ 25\ 26\ 27\ 28\ 29\ 30.$$

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We can write $30 = 2 \cdot 3 \cdot 5$. So a number d with $1 \leq d \leq 30$ which is coprime to 30 is not a multiple of $2, 3$ or 5. So
$\phi(30) = \#\{1, 7, 11, 13, 17, 19, 23, 29\} = 8$

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- If p^r is a prime power, for some $r \geq 1$, then
$$\phi(p^r) = \left(1 - \frac{1}{p}\right) p^r.$$
- If $a, b \in \mathbb{N}$ are coprime, then $\phi(ab) = \phi(a) \cdot \phi(b)$. In particular, for any number $n \in \mathbb{N}$ we have
$$\phi(n) = n \cdot \prod_{p \mid n,\, p \text{ prime}} \left(1 - \frac{1}{p}\right).$$

Example

For $30 = 2 \cdot 3 \cdot 5$ we have

$$\phi(30) = 30 \cdot (1 - 1/2)(1 - 1/3)(1 - 1/5)$$

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Theorem (Euler's Theorem on modular exponentiation)

For integers n and a as above, we have

$$a^{\phi(n)} \equiv 1 \pmod{n}$$

where $\phi(n)$ is Euler's totient function.

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Public vs private

- In private key encryption, Bob can be sure that the message he receives came from Alice (or someone with Alice's key).
- On the other hand, in a public key encryption scheme, Bob has no way to know if Alice really sent the message he receives.

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Note

$\phi(n) = \phi(pq) = \phi(p)\phi(q) = (p-1)(q-1)$

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Remark

Where did these numbers come from? The number $1125897758834689 = 524287 \cdot 2147483647$ is the product of two Mersenne primes. The number 65537 is a Fermat prime. In general, picking these numbers is its own problem.

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The phrase `Banana slugs actually do have known predators` converts to the following Binary using Ascii encoding:

```
01000010 01100001 01101110 01100001 01101110
01100001 00100000 01110011 01101100 01110101
01100111 01110011 00100000 01100001 01100011
01110100 01110101 01100001 01101100 01101100
01111001 00100000 01100100 01101111 00100000
01101000 01100001 01110110 01100101 00100000
01101011 01101110 01101111 01110111 01101110
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Converting each row of binary (a concatenation of 5 bytes) to an integer gives the message:

```
285102465390 417156263029 444313133411 500185525356
520234495776 448311747872 461414299502 139325498724
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And we send the encrypted ciphertext message:
`1060290675860014 68371161852617 781739137181126 1061754703811626 791604339591861 222862021768304 236496462528961 540900973748831 1023668662869907`.

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4. Bob combines these integers to obtain Alice's message $m = m_1 m_2 m_3 \ldots.$

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Example

Let's decrypt the ciphertext message:

`1123302066462145 451538274036662 247266778462632`
`514155951398710 812340948062451`

using an RSA encryption scheme with the same public key $(n, e) = (1125897758834689, 65537)$.

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Example

Let's decrypt the ciphertext message:

`1123302066462145 451538274036662 247266778462632`
`514155951398710 812340948062451`

using an RSA encryption scheme with the same public key $(n, e) = (1125897758834689, 65537)$.

Using the Euclidean algorithm, we can find $d = 600476513804837$ is an integer with

$$65537 \cdot 600476513804837 \equiv 1 \pmod{524286 \cdot 2147483646}.$$

The RSA decryption algorithm

Example

Let's decrypt the ciphertext message:

112330206462145 451538274036662 247266778462632 514155951398710 812340948062451

using an RSA encryption scheme with the same public key $(n, e) = (1125897758834689, 65537)$.

Using the Euclidean algorithm, we can find $d = 600476513804837$ is an integer with

$$65537 \cdot 600476513804837 \equiv 1 \pmod{524286 \cdot 2147483646}.$$

Here we are using the factorization

$$1125897758834689 = 524287 \cdot 2147483647$$

to compute $\phi(1125897758834689) = 524286 \cdot 2147483646.$

The RSA decryption algorithm

Example (Ciphertext)

`1123302066462145 451538274036662 247266778462632 514155951398710 812340948062451`.

The RSA decryption algorithm

Example (Ciphertext)

`1123302066462145 451538274036662 247266778462632 514155951398710 812340948062451`.

We compute:

The RSA decryption algorithm

Example (Ciphertext)

1123302066462145 451538274036662 247266778462632 514155951398710 812340948062451.

We compute:

$$112330206646214 5^{600476513804837} \equiv 289397830432 \pmod{1125897758834689}$$

Example (Ciphertext)

`112330206646214545153827403666224726677846263251415595139871081234094806245 1`

$$1123302066462145^{600476513804837} \equiv 289397830432 \pmod{1125897758834689}$$
$$451538274036662^{600476513804837} \equiv 418531057766 \pmod{1125897758834689}$$

The RSA decryption algorithm

Example (Ciphertext)

1123302066462145 451538274036662 247266778462632 514155951398710 812340948062451.

We compute:

$$
\begin{aligned}
112330206646 2145^{600476513804837} &\equiv 289397830432 \pmod{1125897758834689} \\
451538274036662^{600476513804837} &\equiv 418531057766 \pmod{1125897758834689} \\
247266778462632^{600476513804837} &\equiv 504363907360 \pmod{1125897758834689}
\end{aligned}
$$

The RSA decryption algorithm

Example (Ciphertext)

`1123302066462145 451538274036662 247266778462632 514155951398710 812340948062451`.

We compute:

$$\begin{aligned}
112330206646214 5^{600476513804837} &\equiv 289397830432 \pmod{1125897758834689} \\
451538274036662^{600476513804837} &\equiv 418531057766 \pmod{1125897758834689} \\
247266778462632^{600476513804837} &\equiv 504363907360 \pmod{1125897758834689} \\
&\vdots
\end{aligned}$$

The RSA decryption algorithm

Example (Ciphertext)

`112330206646214`5 `451538274036662` `247266778462632` `514155951398710` `812340948062451`.

We compute:

$$
\begin{aligned}
112330206646214 5^{600476513804837} &\equiv 289397830432 \pmod{1125897758834689} \\
451538274036662^{600476513804837} &\equiv 418531057766 \pmod{1125897758834689} \\
247266778462632^{600476513804837} &\equiv 504363907360 \pmod{1125897758834689} \\
&\vdots
\end{aligned}
$$

Example (Cont'd)

We find the plaintext message: `289397830432` `418531057766` `504363907360` `418464230753` `27763`.

Example (Cont'd)

We find the plaintext message: 289397830432 418531057766 504363907360 418464230753 27763.

Example (Cont'd)

We find the plaintext message: 289397830432 418531057766 504363907360 418464230753 27763.

Converting these decimal numbers into binary gives:

```
100001101100001011101000111001100100000
110000101110010011001010010000001100110
111010101101110011011100111100100100000
110000101101110011010010110110101100001
110110001110011
```

The RSA decryption algorithm

Example (Cont'd)

We find the plaintext message: 289397830432 418531057766 504363907360 418464230753 27763.

Converting these decimal numbers into binary gives:

```
1000011011000010111010001110011001000000
1100001011100100110010100100000001100110
1110101011011100110111001111001001000000
1100001011011100110100101101101011000001
110110001110011
```

Processing this binary sequence to a sequence of bytes gives:

```
01000011 01100001 01110100 01110011 00100000
01100001 01110010 01100101 00100000 01100110
01110101 01101110 01101110 01111001 00100000
01100001 01101110 01101001 01101101 01100001
01101100 01110011
```

The RSA decryption algorithm

Example (Cont'd)

We find the plaintext message: 289397830432 418531057766 504363907360 418464230753 27763.

Processing this binary sequence to a sequence of bytes gives:

```
01000011 01100001 01110100 01110011 00100000
01100001 01110010 01100101 00100000 01100110
01110101 01101110 01101110 01111001 00100000
01100001 01101110 01101001 01101101 01100001
01101100 01110011
```

The RSA decryption algorithm

Example (Cont'd)

We find the plaintext message: `289397830432 418531057766 504363907360 418464230753 27763`.

Processing this binary sequence to a sequence of bytes gives:

```
01000011 01100001 01110100 01110011 00100000
01100001 01110010 01100101 00100000 01100110
01110101 01101110 01101110 01111001 00100000
01100001 01101110 01101001 01101101 01100001
01101100 01110011
```

Example (Cont'd)

Converting this sequence of bytes to a string gives: `Cats are funny animals`.