Relations

Relations

Let S be a set. A **relation** on S is a subset R of $S \times S$.

R consists of ordered pairs (s, t) with $s, t \in S$. For those ordered pairs $(s, t) \in R$, we write $s \sim t$ or sRt and say s is related to t.

> Example: Let $S = \mathbb{R}$ and define $a \sim b \iff a < b$. Here:
>
> $$R = \{(s, t) \in \mathbb{R} \times \mathbb{R} \mid s < t\}$$
>
> ---
>
> Example: Let $S = \mathbb{Z}$ and let m be a positive integer. Define $a \sim b \iff a \equiv b \pmod{m}$, then:
>
> $$R = \{(s, t) \in \mathbb{Z} \times \mathbb{Z} \mid s \equiv t \pmod{m}\}$$

Equivalence relations

Let S be a set, and let $\sim$ be a relation on S. Then $\sim$ is an equivalence relation if the following three properties hold $\forall a, b, c \in S$:

- **Reflexivity**: $a \sim a$
- **Symmetry**: $a \sim b \implies b \sim a$
- **Transitivity**: $(a \sim b) \wedge (b \sim c) \implies a \sim c$

> Consider the two examples of relations in the previous section:
>
> - $R = \{(s, t) \in \mathbb{R} \times \mathbb{R} \mid s < t\}$
>   - **Reflexivity**: If $(x, x) \in R$, then for reflexivity we need $x < x$. However, this is clearly not the case, so this relation is **not reflexive**.
>   - **Symmetry**: If $(x, y) \in R$, then for symmetry we need $(x < y \implies y < x)$. Again, this is clearly not the case, so this relation is **not symmetric**.
>   - **Transitivity**: If $(x, y), (y, z) \in R$, then for transitivity we need $(x, z) \in R \iff x < z$. This is clearly the case, as $x < y < z \iff x < z$, so this relation **is transitive**.
>   - The relation is **not an equivalence relation** because not all three properties hold.
> - $R = \{(s, t) \in \mathbb{Z} \times \mathbb{Z} \mid s \equiv t \pmod{m}\}$
>   - **Reflexivity**: If $(x, x) \in R$, then for reflexivity, we need $x \equiv x \pmod{m}$, which is clearly the case, so this relation **is reflexive**.
>   - **Symmetry**: If $(x, y) \in R$, then for symmetry, we need $x - y \equiv 0 \pmod{m} \implies y - x \equiv 0 \pmod{m}$. Note that $x - y = -(y - x)$,

> so this relation **is symmetric**.
> - **Transitivity**: If $(x,y),(y,z) \in R$, then for transitivity we need $(x,z) \in R \iff x \equiv z \pmod{m}$. We have $x - y \equiv 0 \pmod{m}$ and $y - z \equiv 0 \pmod{m}$. Therefore, $(x-y)+(y-z) \equiv 0 \pmod{m}$, giving us $x \equiv z \pmod{m}$, so this relation **is transitive**.
> - The relation **is an equivalence relation** because all three properties hold.

Equivalence classes

Let S be a set and $\sim$ an equivalence relation on S. For $a \in S$, define

$$cl(a) = \{s \mid s \in S, s \sim a\}$$

Thus, $cl(a)$ is the set of things that are related to a. The subset $cl(a)$ of S is called an equivalence class of $\sim$. The equivalence classes of $\sim$ are the subsets $cl(a)$ as a ranges over the elements of S.

> Example: Consider the equivalence relation
>
> $$R = \{(s,t) \in \mathbb{Z} \times \mathbb{Z} \mid s \equiv t \pmod{m}\}$$
>
> Some various equivalence classes are:
>
> - $cl(0) = \{s \in \mathbb{Z} \mid s \equiv 0 \pmod{m}\}$
> - $cl(1) = \{s \in \mathbb{Z} \mid s \equiv 1 \pmod{m}\}$
>
> $$\dots$$
>
> - $cl(m-1) = \{s \in \mathbb{Z} \mid s \equiv m-1 \pmod{m}\}$
>
> We claim that these are all the equivalence classes. For if n is any integer, then $\exists q, r \in \mathbb{Z} : n = qm + r$ with $0 \leq r < m$. Then $n \equiv r \pmod{m}$, so $n \in cl(r)$, which is one of the classes listed above.