Digital Systems Design and Laboratory
[4. Applications of Boolean Algebra]

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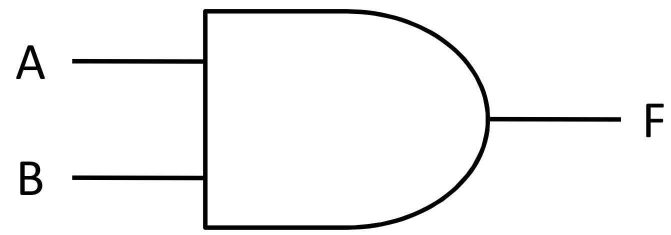
Outline

- **Conversion of English Sentences to Boolean Equations**
- Combinational Logic Design Using a Truth Table
- Minterm and Maxterm Expansions
- General Minterm and Maxterm Expansions
- Incompletely Specified Functions
- Examples of Truth Table Construction
- Design of Binary Adders and Subtracters

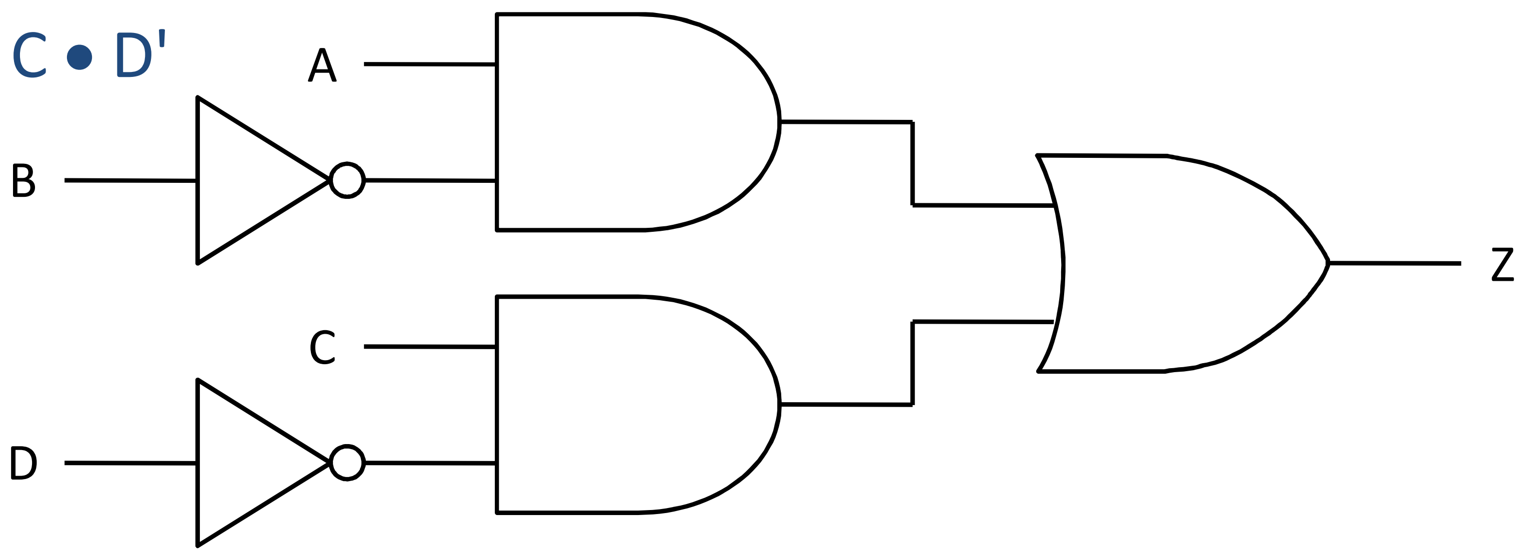
Objectives

- Design a combinational logic circuit starting with a **word description (specification)** of the desired circuit behavior
- Steps
  - Translate the word description into a switching function
    - Boolean expression or truth table
  - Simplify the function
  - Realize it using available logic gates
- Example
  - Mary watches TV **if and only if** it is Monday night **and** she has finished her homework
    - F: Mary watches TV
    - A: It is Monday night
    - B: Mary has finished her homework
  - F = A • B



Another Example

- The alarm will ring **if and only if** the alarm switch is turned on **and** the door is **not** closed, **or** it is after 6pm **and** the window is **not** closed
  - Z: The alarm will ring
  - A: the alarm switch is on
  - B: The door is closed
  - C: It is after 6pm
  - D: The window is closed
- Z = A • B' + C • D'



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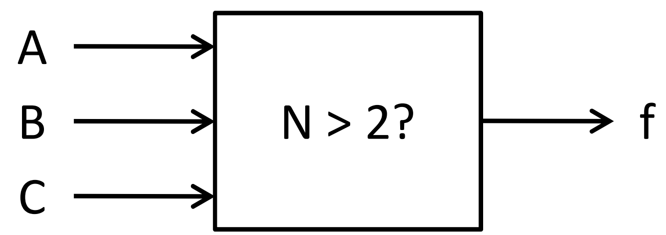
Threshold Detector (1/2)

- Design a detector that outputs 1 when input is greater than 2
  - Inputs $(A, B, C)_2$ represent a binary number N
  - If $N = (A, B, C)_2 \geq 3$, output $f = 1$; otherwise $f = 0$

A	B	C	f	f'
0	0	0	0	1
0	0	1	0	1
0	1	0	0	1
0	1	1	1	0
1	0	0	1	0
1	0	1	1	0
1	1	0	1	0
1	1	1	1	0

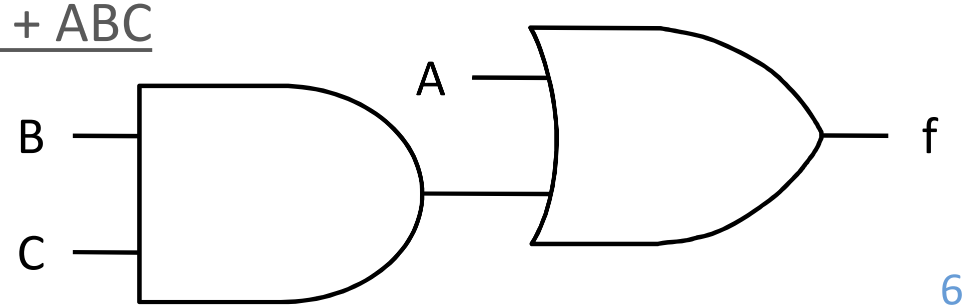
Show the condition to make output = 1



  - $f = A'BC + AB'C' + AB'C + ABC' + ABC$ (SOP)

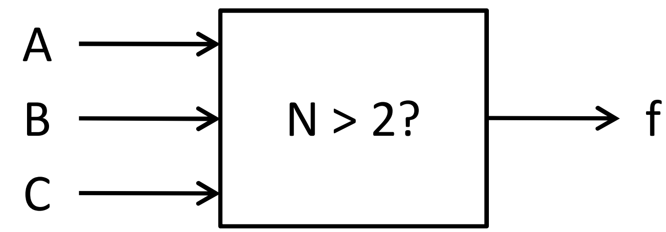
    $= \underline{A'BC + ABC} + \underline{AB'C' + AB'C + ABC' + ABC}$

    $= A + BC$



Threshold Detector (2/2)

- By counting 1's, we have SOP
- What if counting 0's



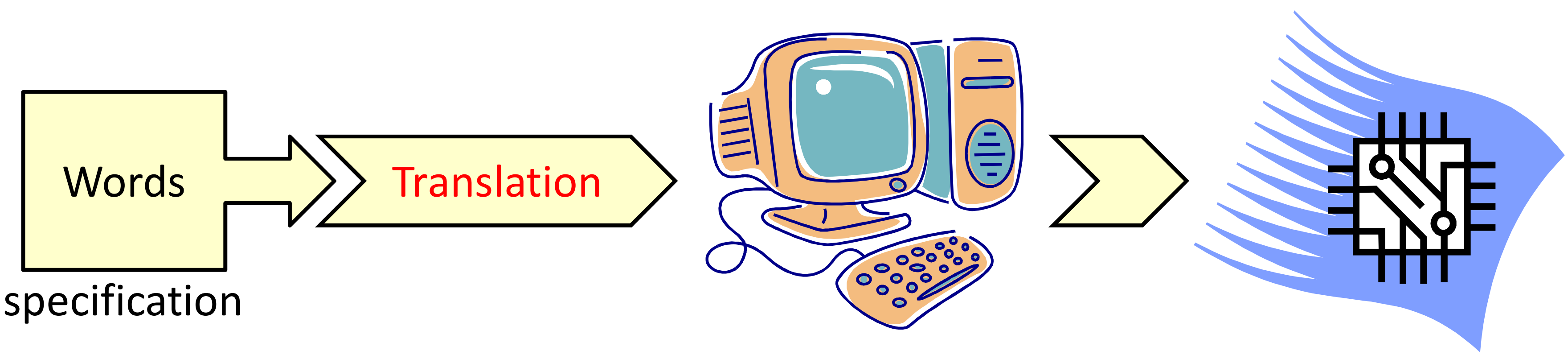
A	B	C	f	f'
0	0	0	0	1
0	0	1	0	1
0	1	0	0	1
0	1	1	1	0
1	0	0	1	0
1	0	1	1	0
1	1	0	1	0
1	1	1	1	0

Show the condition to make output = 0

- $f = (ABC \neq 000) \bullet (ABC \neq 001) \bullet (ABC \neq 010)$
  $= (A'B'C')' \bullet (A'B'C)' \bullet (A'BC')' = (A + B + C) \bullet (A + B + C') \bullet (A + B' + C)$
- $f = (A'B'C' + A'B'C + A'BC')'$
  $= (A'B'C')' \bullet (A'B'C)' \bullet (A'BC')' = (A + B + C) \bullet (A + B + C') \bullet (A + B' + C)$

Logic Design Using a Truth Table

- Steps
  - Make a truth table according to the word description
  - Generate a Boolean expression
    - Sum-of-products (SOP): check 1's
    - Product-of-sums (POS): check 0's
      - Have f' in SOP and then derive f in POS
  - Simplify the Boolean expression



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Minterm and Maxterm

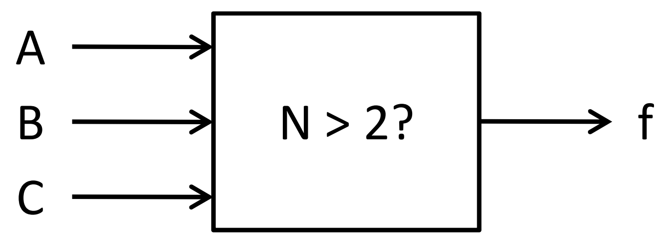
- Definition: A **minterm/maxterm** of n variables is a product/sum of n literals in which each variable appears exactly once in either true or complement form (but not both)
  - A literal is a variable or its complement (A or A')
  - Examples of 3 variables
    - Minterm: A'BC, AB'C'
    - Maxterm: A+B+C, A+B+C'

$(m_i)' = M_i$

Row No.	ABC	Minterm m_i ⟷	Maxterm M_i
0	000	m_0 = A'B'C'	M_0 = A + B + C
1	001	m_1 = A'B'C	M_1 = A + B + C'
2	010	m_2 = A'BC'	M_2 = A + B' + C
3	011	m_3 = A'BC	M_3 = A + B' + C'
4	100	m_4 = AB'C'	M_4 = A' + B + C
5	101	m_5 = AB'C	M_5 = A' + B + C'
6	110	m_6 = ABC'	M_6 = A' + B' + C
7	111	m_7 = ABC	M_7 = A' + B' + C'

Minterm Expansion

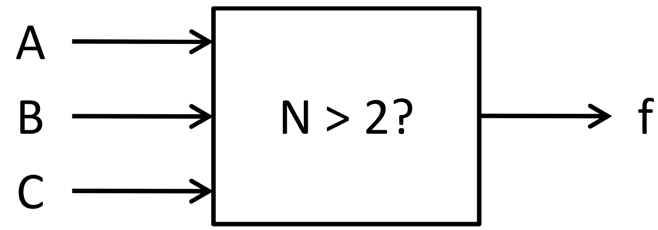
- A **minterm expansion** or a **standard sum of products** is a function is written as a sum of minterms
  - Counting 1's
- Example
  - f = A'BC + AB'C' + AB'C + ABC' + ABC
    $= m_3 + m_4 + m_5 + m_6 + m_7$ (m-notation)
    $= \sum m(3, 4, 5, 6, 7)$



Row No.	ABC	Minterm m_i	Maxterm M_i
0	000	m_0 = A'B'C'	M_0 = A + B + C
1	001	m_1 = A'B'C	M_1 = A + B + C'
2	010	m_2 = A'BC'	M_2 = A + B' + C
3	011	m_3 = A'BC	M_3 = A + B' + C'
4	100	m_4 = AB'C'	M_4 = A' + B + C
5	101	m_5 = AB'C	M_5 = A' + B + C'
6	110	m_6 = ABC'	M_6 = A' + B' + C
7	111	m_7 = ABC	M_7 = A' + B' + C'

Maxterm Expansion

- A **maxterm expansion** or a **standard product of sums** is a function written as a product of maxterms
  - Counting 0's
- Example
  - $F = (A + B + C) \bullet (A + B + C') \bullet (A + B' + C)$
    $= M_0 M_1 M_2$ (M-notation)
    $= \prod M(0, 1, 2)$



Row No.	ABC	Minterm m_i	Maxterm M_i
0	000	$m_0 = A'B'C'$	$M_0 = A + B + C$
1	001	$m_1 = A'B'C$	$M_1 = A + B + C'$
2	010	$m_2 = A'BC'$	$M_2 = A + B' + C$
3	011	$m_3 = A'BC$	$M_3 = A + B' + C'$
4	100	$m_4 = AB'C'$	$M_4 = A' + B + C$
5	101	$m_5 = AB'C$	$M_5 = A' + B + C'$
6	110	$m_6 = ABC'$	$M_6 = A' + B' + C$
7	111	$m_7 = ABC$	$M_7 = A' + B' + C'$

Complement by Minterms/Maxterms

- $(m_i)' = M_i$
- Complement of f
  - Counting 0's in f (find f' directly)
    - $f' = m_0 + m_1 + m_2 = \sum m(0, 1, 2)$
    - $f' = M_3M_4M_5M_6M_7 = \prod M(3, 4, 5, 6, 7)$
  - Counting 1's in f (find f and then complement it)
    - $f' = (m_3 + m_4 + m_5 + m_6 + m_7)'$
      $= m_3'm_4'm_5'm_6'm_7'$
      $= M_3M_4M_5M_6M_7$
      $= \prod M(3, 4, 5, 6, 7)$
    - $f' = (M_0M_1M_2)'$
      $= M_0' + M_1' + M_2'$
      $= m_0 + m_1 + m_2$
      $= \sum m(0, 1, 2)$

A	B	C	f	f'
0	0	0	0	1
0	0	1	0	1
0	1	0	0	1
0	1	1	1	0
1	0	0	1	0
1	0	1	1	0
1	1	0	1	0
1	1	1	1	0

Another Example

- Example: $f(a, b, c, d) = a'(b' + d) + acd'$

$$
\begin{aligned}
& f(a, b, c, d) \\
&= a'(b' + d) + acd' \\
&= a'b' + a'd + acd' \\
&= a'b'(c + c')(d + d') + a'd(b + b')(c + c') + acd'(b + b') \\
&= \underset{0000}{a'b'c'd'} + \underset{0001}{a'b'c'd} + \underset{0010}{a'b'cd'} + \underset{0011}{a'b'cd} + \underset{0101}{a'bc'd} + \underset{0111}{a'bcd} + \underset{1110}{abcd'} + \underset{1010}{ab'cd'} \\
&= \sum m(0, 1, 2, 3, 5, 7, 10, 14) \qquad \text{... Minterm Expansion}
\end{aligned}
$$

$$
\begin{aligned}
& f(a, b, c, d) \\
&= (a' + cd')(a + b' + d) = (a' + c)(a' + d')(a + b' + d) \\
&= (a' + bb' + c + dd')(a' + bb' + cc' + d')(a + b' + cc' + d) \\
&= \dots \\
&= \prod M(4, 6, 8, 9, 11, 12, 13, 15) \qquad \text{... Maxterm Expansion}
\end{aligned}
$$

Summary

- Convert a Boolean expression to a minterm/maxterm expansion
  - Use truth table
    - Sometimes there are too many terms
- Use Boolean algebra
  - SOP: multiply out and use $(X + X') = 1 \rightarrow$ minterm expansion
  - POS: factor and use $XX'=0 \rightarrow$ maxterm expansion

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General Truth Table

- Given n Boolean variables, how many different Boolean functions can you produce?
  - Each a_i can be assigned with either 0 or 1
  - $2^{(2^n)}$

A	B	C	f
0	0	0	a_0
0	0	1	a_1
0	1	0	a_2
0	1	1	a_3
1	0	0	a_4
1	0	1	a_5
1	1	0	a_6
1	1	1	a_7

AND of Minterm Expansions

- Given $f_1 = \sum m(0, 2, 3, 5, 9, 11)$ and $f_2 = \sum m(0, 3, 9, 11, 13, 14)$, find $f_1 f_2 = ?$
  - AND: take the numbers that appear in both expansions:
  - $f_1 f_2 = \sum m(0, 3, 9, 11)$
- AND for two maxterm expansions?
- OR for two minterm expansions?
- OR for two maxterm expansions?

Conversion of Forms (1/2)

- Convert between a minterm and a maxterm expansion

TABLE 4-3 Conversion of Forms

GIVEN FORM \ DESIRED FORM	Minterm Expansion of F	Maxterm Expansion of F	Minterm Expansion of F'	Maxterm Expansion of F'
Minterm Expansion of F	————	maxterm nos. are those nos. not on the minterm list for F	list minterms not present in F	maxterm nos. are the same as minterm nos. of F
Maxterm Expansion of F	minterm nos. are those nos. not on the maxterm list for F	————	minterm nos. are the same as maxterm nos. of F	list maxterms not present in F

Conversion of Forms (2/2)

- Convert between a minterm and a maxterm expansion

TABLE 4-4 Application of Table 4.3

GIVEN FORM	DESIRED FORM: Minterm Expansion of f	Maxterm Expansion of f	Minterm Expansion of f'	Maxterm Expansion of f'
$f = \Sigma\, m(3, 4, 5, 6, 7)$	__________	$\Pi\, M(0, 1, 2)$	$\Sigma\, m(0, 1, 2)$	$\Pi\, M(3, 4, 5, 6, 7)$
$f = \Pi\, M(0, 1, 2)$	$\Sigma\, m(3, 4, 5, 6, 7)$	__________	$\Sigma\, m(0, 1, 2)$	$\Pi\, M(3, 4, 5, 6, 7)$

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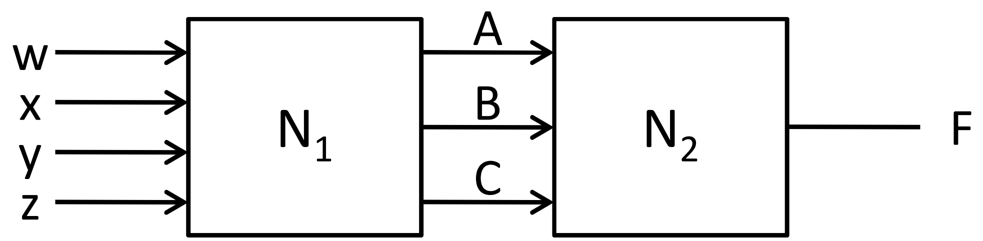
Incompletely Specified Functions (1/2)

- A large digital system is usually divided into subcircuits
- Assume N_1 never generates ABC = 001/110 for any w, x, y, z

A	B	C	F
0	0	0	1
0	0	1	X
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	0
1	1	0	X
1	1	1	1



- F: Incompletely specified function
- A'B'C, ABC': don't care terms
  - "don't care" (DC) terms can be assigned with either 0 or 1

Incompletely Specified Functions (2/2)

❑ Impact of don't care terms on Boolean simplification

- Try exhaustive combinations of DCs to find the best
  - (may be stupid but works for now)
- Assign 0 to both "X"
  - F = A'B'C' + A'BC + ABC = A'B'C' + BC
- Assign 1 to 1st "X" and 0 to 2nd "X" (seems to be the simplest)
  - F = A'B'C' + A'B'C + A'BC + ABC = A'B' + BC
- Assign 1 to both "X"
  - F = A'B'C' + A'B'C + A'BC + ABC' + ABC = A'B' + BC + AB
- …
- 2. is the simplest solution

❑ Notation

- F = ∑ m(0, 3, 7) + ∑ d(1, 6)
- F = ∏ M(2, 4, 5) • ∏ D(1, 6)

A	B	C	F
0	0	0	1
0	0	1	X
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	0
1	1	0	X
1	1	1	1

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Error Detector for 6-3-1-1 Codes (1/2)

- Design an error detector for 6-3-1-1 codes:
  - The output F = 1 iff inputs (A, B, C, D) represent an **invalid** code combination
  - Step 1: construct the truth table

Decimal Digit	6-3-1-1 Code
0	0000
1	0001
2	0011
3	0100
4	0101
5	0111
6	1000
7	1001
8	1011
9	1100

⇨

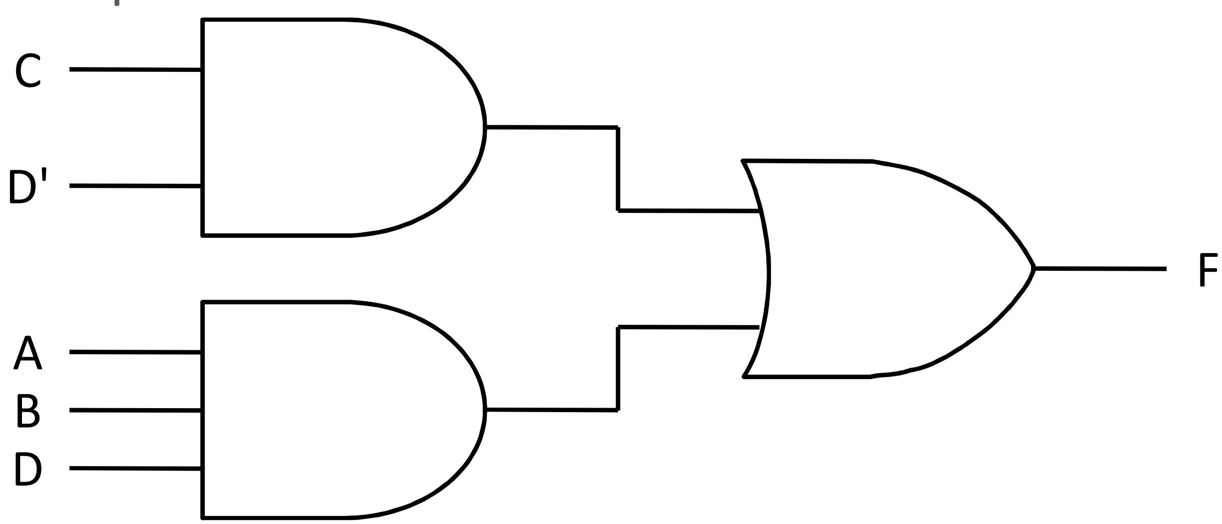
ABCD	F
0000	0
0001	0
0010	1
0011	0
0100	0
0101	0
0110	1
0111	0
1000	0
1001	0
1010	1
1011	0
1100	0
1101	1
1110	1
1111	1

Error Detector for 6-3-1-1 Codes (2/2)

- Design an error detector for 6-3-1-1 codes:
  - The output F = 1 iff inputs (A, B, C, D) represent an **invalid** code combination
  - Step 2: simplify the function

$$
\begin{aligned}
& F(A, B, C, D) \\
&= \sum m(2, 6, 10, 13, 14, 15) \\
&= \underline{A'B'CD' + A'BCD'} + \underline{AB'CD' + ABCD'} + \underline{ABC'D + ABCD} \\
&= \underline{A'CD' + ACD'} + ABD \\
&= CD' + ABD
\end{aligned}
$$

  - Step 3: realize it



ABCD	F
0000	0
0001	0
0010	1
0011	0
0100	0
0101	0
0110	1
0111	0
1000	0
1001	0
1010	1
1011	0
1100	0
1101	1
1110	1
1111	1

Another Example

- The output Z = 1 iff the 8-4-2-1 BCD number (A, B, C, D) is divisible by 3
  - Step 1: construct the truth table
  - Step 2: simplify the function

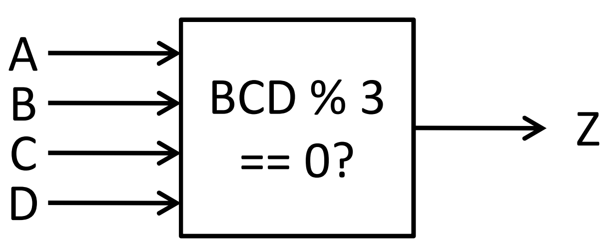
    Z(A, B, C, D)

    $= \sum m(0, 3, 6, 9) +$

    $\sum d(10, 11, 12, 13, 14, 15)$
  - Step 3: realize it
    - ?
    - Unit 5!

Decimal Digit	6-3-1-1 Code
0	0000
1	0001
2	0010
3	0011
4	0100
5	0101
6	0110
7	0111
8	1000
9	1001

ABCD	Z
0000	1
0001	0
0010	0
0011	1
0100	0
0101	0
0110	1
0111	0
1000	0
1001	1
1010	X
1011	X
1100	X
1101	X
1110	X
1111	X



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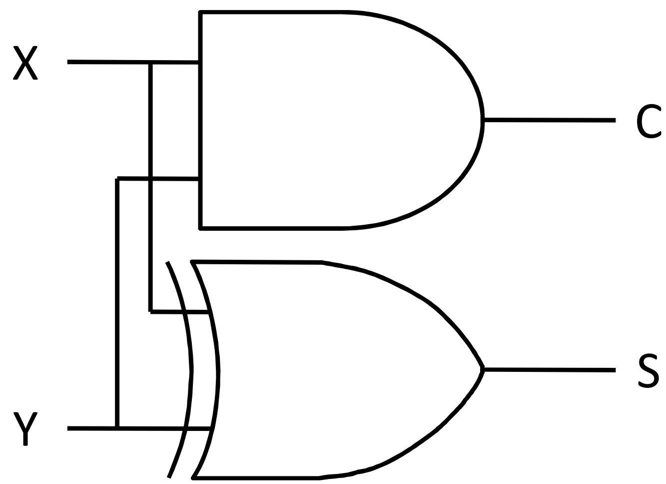
1-Bit Half Adder (HA)

- Step 1: construct the truth table

X	Y	Carry	Sum
0	0	0	0
0	1	0	1
1	0	0	1
1	1	1	0

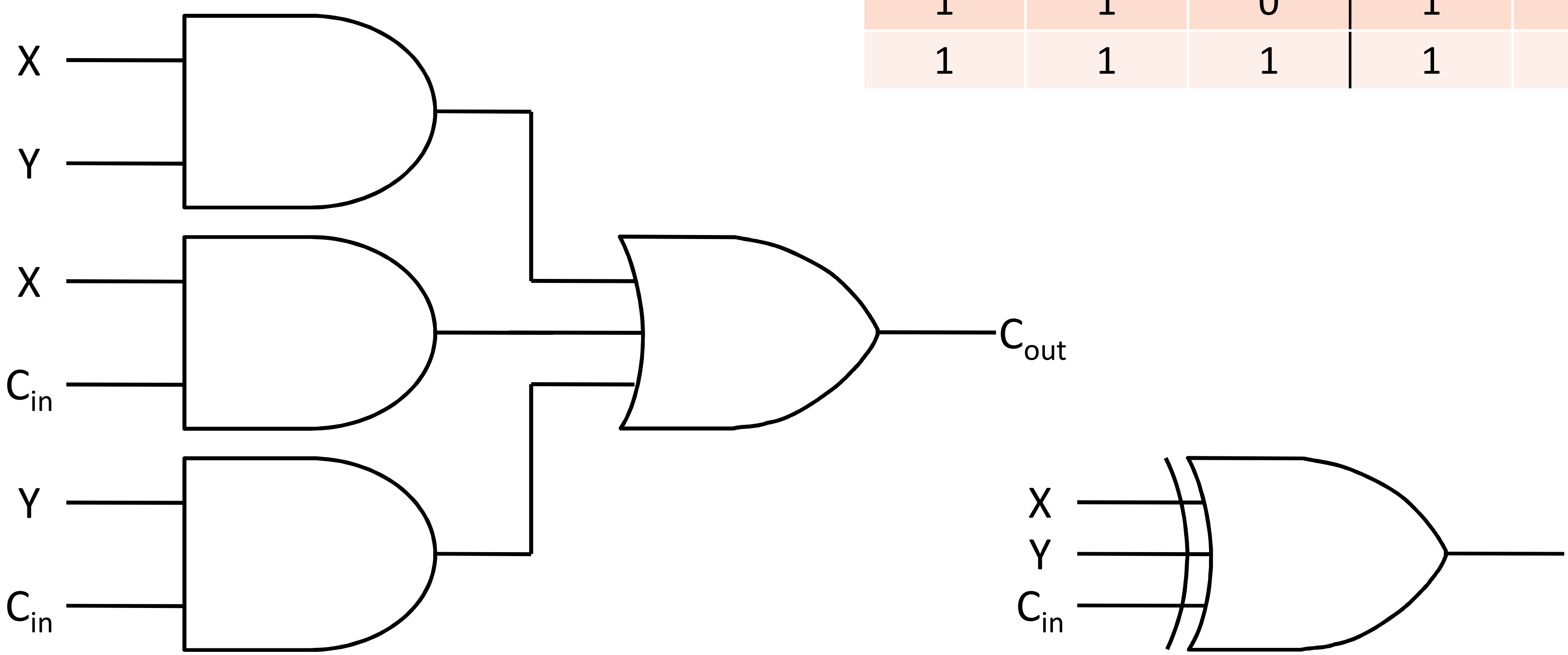
- Step 2: simplify the function
  - $C = XY$
  - $S = X'Y + XY' = X \oplus Y$
- Step 3: realize it



1-Bit Full Adder (FA)

- Step 1: construct the truth table
- Step 2: simplify the function
  - $C_{out} = X'YC_{in} + XY'C_{in} + XYC_{in}' + XYC_{in}$
    $= XY + XC_{in} + YC_{in}$
  - $S = X'Y'C_{in} + X'YC_{in}' + XY'C_{in}' + XYC_{in}$
- Step 3: realize it

X	Y	C_{in}	C_{out}	S
0	0	0	0	0
0	0	1	0	1
0	1	0	0	1
0	1	1	1	0
1	0	0	0	1
1	0	1	1	0
1	1	0	1	0
1	1	1	1	1

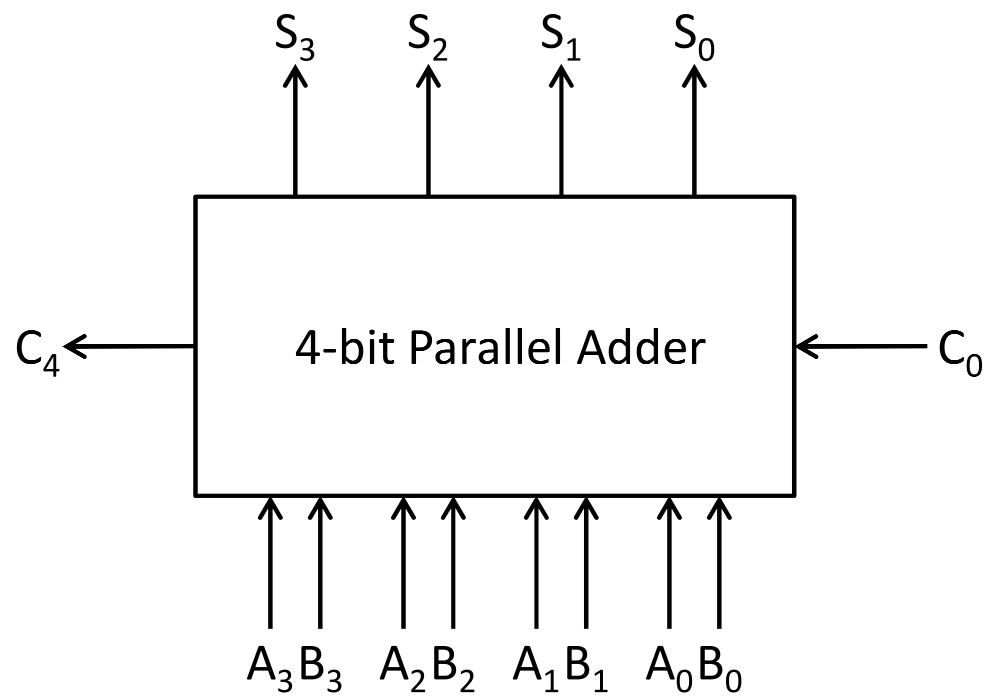


4-Bit Parallel Adder (1/3)

- A = ($A_3A_2A_1A_0$), B = ($B_3B_2B_1B_0$)

$$
\begin{array}{ccccccccc}
 & & A_3 & A_2 & A_1 & A_0 & & \\
 & + & B_3 & B_2 & B_1 & B_0 & & \\
\hline
C_4 & \leftarrow & S_3 & S_2 & S_1 & S_0 & \leftarrow & C_0
\end{array}
$$

carry-out carry-in



- Example

$$
\begin{array}{cccccccccc}
 & & 1 & 0 & 1 & 1 & 0 & \leftarrow & \text{carries} \\
 & & & 1 & 0 & 1 & 1 & & \\
 & + & & 1 & 0 & 1 & 1 & & \\
\hline
1 & \leftarrow & 1 & 0 & 1 & 1 & 0 & \leftarrow & 0
\end{array}
$$

- How?
  - Step 1: construct the truth table
  - ...

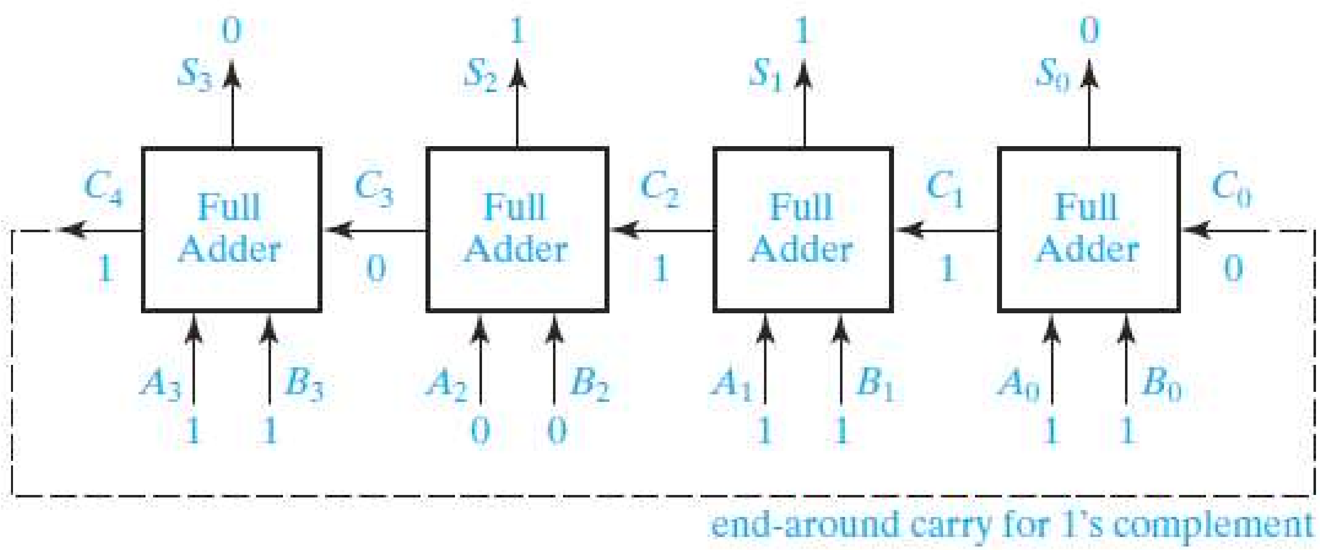
4-Bit Parallel Adder (2/3)

- Decompose the 4 bit adder into four modules
  - Each module adds two bits and a carry → use full adder
- Extend to negative numbers
  - Consider 1's complement
    - Add just as if all numbers are positive
    - Add the carry out back to the rightmost bit
  - How to detect overflow?
    - Check the sign
      - (+) + (+) becomes (-)
      - (-) + (-) becomes (+)

Case 1	Case 2	Case 3	Case 4	Case 5	Case 6
+3 0011 +4 0100 +7 0111	+5 0101 +6 0110 1011	+5 0101 −6 1001 −1 1110	−5 1010 +6 0110 +1 (1) 0000 1 0001	−3 1100 −4 1011 −7 (1) 0111 1 1000	−5 1010 −6 1001 (1) 0011 1 0100

4-Bit Parallel Adder (3/3)

FIGURE 4-3
Parallel Adder Composed of Four Full Adders

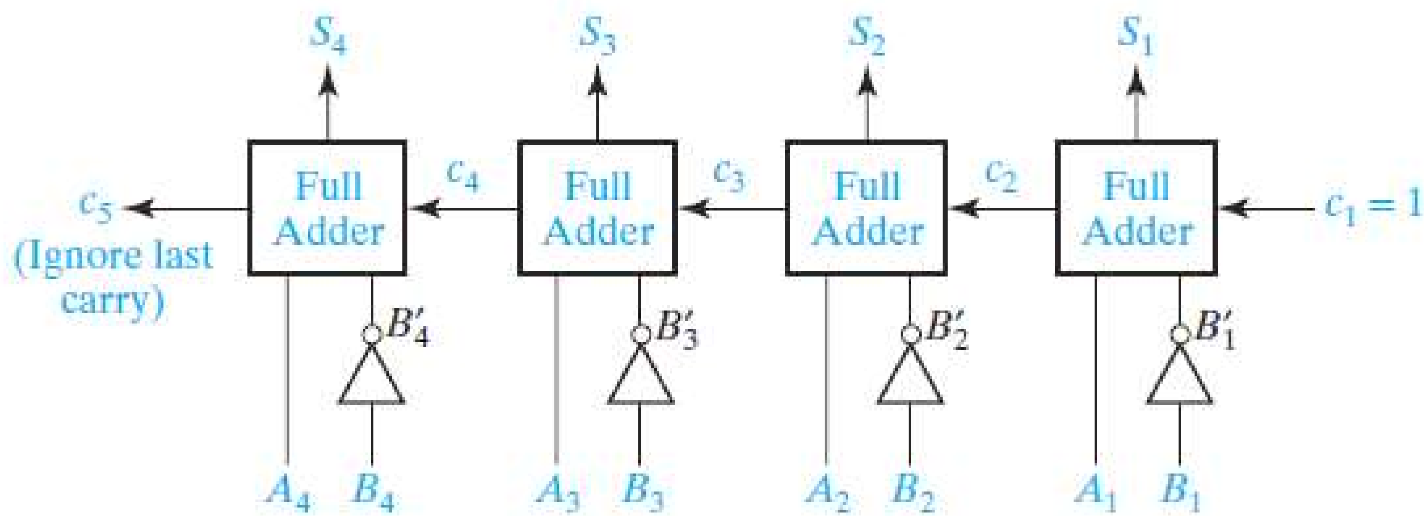


- Overflow detection?
  - $V = A_3'B_3'S_3 + A_3B_3S_3'$
  - Why?

Binary Subtracter (1/2)

- Consider A – B = A + (-B) in 2's complement
  - A – B = A + (-B) = A + B* = A + $\overline{\text{B}}$ + 1
  - Convert B to 2's complement: inverse and then add 1
- Discard the carry from the sign bit

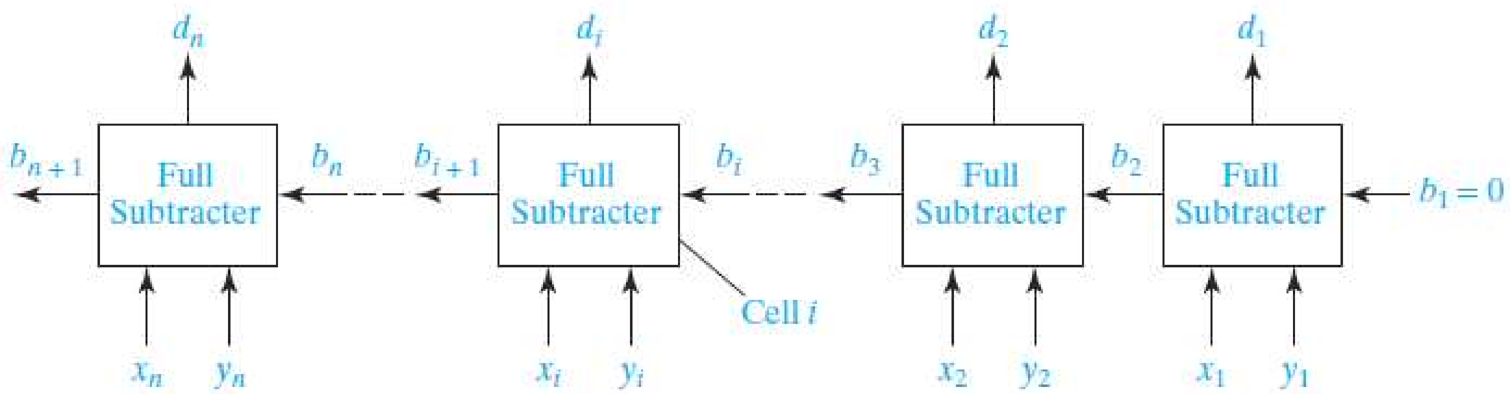
FIGURE 4-6
Binary Subtracter Using Full Adders



Binary Subtracter (2/2)

- Or design a full subtracter
  - D = X – Y: difference
  - B: borrow

FIGURE 4-7
Parallel Subtracter



Q&A