

Digital Systems Design and Laboratory

[3. Boolean Algebra (Continued)]

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Outline

- ❑ **Multiplying Out and Factoring Expressions**
- ❑ Exclusive OR and Equivalence Operations
- ❑ Consensus Theorem
- ❑ Algebraic Simplification of Switching Expressions
- ❑ Proving Validity of an Equation

Multiplying Out and Factoring

□ Distributive laws

- $X(Y + Z) = XY + XZ$
- $(X + Y)(X + Z) = X + YZ$

□ Another useful theorem

- $(X + Y)(X' + Z) = XZ + X'Y$
 - Proof? Check two cases: $X = 0$ or $X = 1$

□ Multiplying out

- Apply them from left terms to right terms

□ Factoring

- Apply them from right terms to left terms

Multiplying Out: from POS to SOP

□ From product-of-sums (POS) to sum-of-products (SOP)

□ Example

$$\frac{(A + B + C')(A + B + D)(A + B + E)(A + D' + E)(A' + C)}{2. \qquad 3.}$$

$$= \frac{(A + B + C'D)(A + B + E)[AC + A'(D' + E)]}{2.}$$

$$= (A + B + C'DE)(AC + A'D' + A'E)$$

$$= AC + ABC + A'BD' + A'BE + A'C'DE$$

$$= AC + A'BD' + A'BE + A'C'DE$$

1. $X(Y + Z) = XY + XZ$
 2. $(X + Y)(X + Z) = X + YZ$
 3. $(X + Y)(X' + Z) = XZ + X'Y$

Factoring

❑ From sum-of-products (SOP) to product-of-sums (POS)

❑ Example

$$\begin{aligned} & AC + A'BD' + A'BE + A'C'DE \\ = & \underline{AC} + \underline{A'}(BD' + BE + C'DE) \end{aligned}$$

3.

$$\begin{aligned} &= (A + BD' + BE + C'DE)(A' + C) \\ = & \underline{[A + C'DE + B(D' + E)]}(A' + C) \end{aligned}$$

2.

$$\begin{aligned} &= (A + C'DE + B)(A + C'DE + D' + E)(A' + C) \\ = & \underline{(A + B + C'DE)}(A + D' + E)(A' + C) \end{aligned}$$

2.

$$= (A + B + C')(A + B + D)(A + B + E)(A + D' + E)(A' + C)$$

1. $X(Y + Z) = XY + XZ$
2. $(X + Y)(X + Z) = X + YZ$
3. $(X + Y)(X' + Z) = XZ + X'Y$

Outline

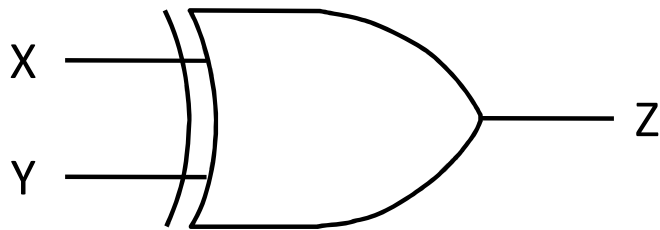
- ❑ Multiplying Out and Factoring Expressions
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Exclusive-OR (1/2)

□ Exclusive-OR (XOR) (\oplus)

➤ $0 \oplus 0 = 0, 0 \oplus 1 = 1, 1 \oplus 0 = 1, 1 \oplus 1 = 0$

➤ Symbol (XOR gate)



➤ Truth table

X	Y	$Z = X \oplus Y$
0	0	0
0	1	1
1	0	1
1	1	0

Exclusive-OR (2/2)

□ $X \oplus Y = X'Y + XY'$

□ Theorem

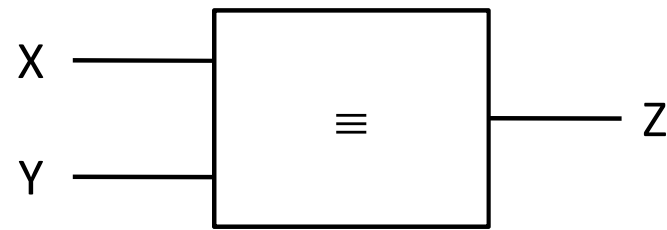
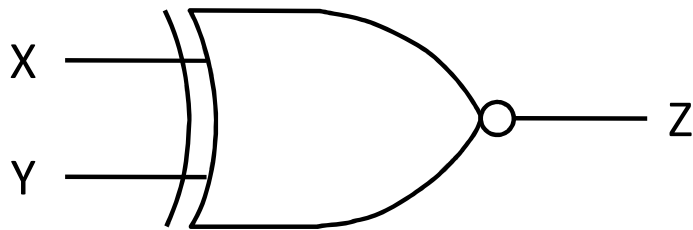
- $X \oplus 0 = X$
- $X \oplus 1 = X'$
- $X \oplus X = 0$
- $X \oplus X' = 1$
- $X \oplus Y = Y \oplus X$ (commutative)
- $(X \oplus Y) \oplus Z = X \oplus (Y \oplus Z)$ (associative)
- $X(Y \oplus Z) = XY \oplus XZ$ (distributive)
- $(X \oplus Y)' = X'Y' \oplus XY$
 - X and Y must be the same

Exclusive-NOR (1/2)

□ Exclusive-NOR (XNOR) (\equiv)

➤ $0 \equiv 0 = 1, 0 \equiv 1 = 0, 1 \equiv 0 = 0, 1 \equiv 1 = 1$

➤ Symbol (XNOR gate)



➤ Truth table

X	Y	$Z = X \oplus Y$
0	0	1
0	1	0
1	0	0
1	1	1

Simplification of XOR and XNOR

□ $X \oplus Y = X'Y + XY'$

□ $X \equiv Y = X'Y' + XY$

□ Examples

$$\begin{aligned} F &= (A'B \equiv C) + (B \oplus AC') \\ &= [A'BC + (A'B)'C'] + [B'AC' + B(AC')'] \\ &= A'BC + (A + B')C' + B'AC' + B(A' + C) \\ &= B(A'C + A' + C) + C'(A + B' + AB') \\ &= B(A' + C) + C'(A + B') \\ &= \dots \text{ can be further simplified} \\ &= B + C' \end{aligned}$$

$$\begin{aligned} F &= A' \oplus B \oplus C \\ &= (A'B' + AB) \oplus C \\ &= (A'B' + AB)C' + (A'B' + AB)'C \\ &= A'B'C' + ABC' + A'BC + AB'C \\ &= \dots \text{ can be further simplified?} \end{aligned}$$

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Consensus Theorem

□ $XY + X'Z + YZ = XY + X'Z$

➤ YZ is redundant, referred to as the consensus term

□ Example

$$\begin{aligned} & A'C'D + A'BD + BCD + ABC + ACD' \\ &= A'C'D + \underline{A}BD + BCD + \underline{A}BC + ACD' \\ &= A'C'D + A'BD + ABC + ACD' \end{aligned}$$

□ Example

$$\begin{aligned} & A'B' + AC + BC' + B'C + AB \\ &= \underline{A}B' + \underline{A}C + BC' + B'C + AB \\ &= A'B' + AC + BC' + AB \\ &= A'B' + \underline{A}\underline{C} + \underline{B}\underline{C}' + AB \\ &= A'B' + AC + BC' \end{aligned}$$

Ordering Does Matter

□ Example (one ordering)

$$\begin{aligned} & A'C'D + A'BD + BCD + ABC + ACD' \\ = & A'C'D + \underline{A'}BD + BCD + \underline{A}BC + ACD' \\ = & A'C'D + A'BD + ABC + ACD' \end{aligned}$$

□ Example (another ordering)

$$\begin{aligned} & A'\underline{C'}D + A'BD + B\underline{C}D + ABC + ACD' \\ = & A'C'D + BCD + ABC + ACD' \\ = & A'C'D + BCD + ABC + ACD' \\ = & A'C'D + BCD + ACD' \end{aligned}$$

Dual Form of Consensus Theorem

□ $XY + X'Z + YZ = XY + X'Z$

➤ YZ is redundant

□ $(X + Y)(X' + Z)(Y + Z) = (X + Y)(X' + Z)$

➤ $(Y + Z)$ is redundant

□ Example

$$\begin{aligned} & (A + B + C')(A + B + D')(B + C + D') \\ &= (A + B + \underline{C}')(A + B + D')(B + \underline{C} + D') \\ &= (A + B + C')(B + C + D') \\ &= \dots \text{ can be further simplified} \\ &= B + AC + C'D' \end{aligned}$$

Redundancy Insertion

□ Example

- $ABCD + B'CDE + A'B' + BCE'$
- Consensus terms
 - $\underline{A}BCD + \underline{B}'CDE \rightarrow ACDE$
 - $\underline{A}BCD + \underline{A}'B' \rightarrow 0$; $\underline{A}BCD + A'\underline{B}' \rightarrow 0$
 - $\underline{B}'CDE + \underline{B}CE' \rightarrow 0$; $B'CDE\underline{E} + BCE'\underline{E} \rightarrow 0$
 - $A'\underline{B}' + \underline{B}CE \rightarrow A'CE$
- No redundancy
- Redundancy insertion

$$\begin{aligned} & ABCD + B'CDE + A'B' + BCE' + ACDE \\ = & \text{ABCD} + B'CDE + A'B' + \underline{BCE'} + \underline{ACDE} \\ = & B'CDE + A'B' + BCE' + ACDE \\ = & B'CDE + \underline{A}'B' + BCE' + \underline{ACDE} \\ = & A'B' + BCE' + ACDE \end{aligned}$$

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Three Basic Ways

□ Goal

- Simplifying an expression reduces the cost of realizing the expression using gates

□ Combining terms: $XY + XY' = X(Y + Y') = X$

- $ABC'D' + ABCD' = ABD'$
- $AB'C + ABC + A'BC = AB'C + ABC + ABC + A'BC = AC + BC$
- $(A + BC)(D + E') + A'(B' + C')(D + E') = D + E'$
 - DeMorgan's Law

□ Eliminating terms: $X + XY = X$ and $XY + X'Z + YZ = XY + X'Z$

- $A'B + A'BC = A'B$
- $A'BC' + BCD + A'BD = A'BC' + BCD$

□ Eliminating literals: $X + X'Y = X + Y$

- $A'B + A'B'C'D' + ABCD' = A'B + A'C'D' + ABCD' = A'B + A'C'D' + BCD'$

Adding Redundant Terms

□ $Y = Y + XX'$

□ $Y = Y(X + X')$

□ $XY + X'Z = XY + X'Z + YZ$

□ $X = X + XY$

□ Example

$$\begin{aligned} & W\underline{X} + XY + \underline{X}'Z' + WY'Z' \\ &= WX + XY + X'Z' + WY'Z' + WZ' \\ &= W\underline{X} + XY + \underline{X}'Z' + WZ' \\ &= WX + XY + X'Z' \text{ ...can you derive this directly?} \end{aligned}$$

Quick Note

- ❑ No easy way to determine if a Boolean expression has a minimum number of terms or literals
 - Systematic (graphical) method will be discussed

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Equation Validity (1/2)

□ Several methods

- Construct a truth table
 - Proof by cases (equivalent for all combinations of values of variables)
- Manipulate one side until it is identical to the other side
- Reduce both sides independently to the same expression
- Perform the same operation on both sides if the operation is reversible
 - Complement is reversible
 - AND is not reversible: $XY = XZ$ does not imply $Y = Z$
 - OR is not reversible: $X + Y = X + Z$ does not imply $Y = Z$
 - $X \oplus Y = X \oplus Z$ implies $Y = Z$?

□ Prove invalidity

- Try to find one combination of values of variables such that two sides have different values

Equation Validity (2/2)

□ Example: prove $A'BD' + BCD + ABC' + AB'D = BC'D' + AD + A'BC$

$$\begin{aligned}
 & A'BD' + BCD + ABC' + AB'D \\
 &= A'BD' + BCD + ABC' + AB'D + A'BC + BC'D' + ABD \\
 &= AD + A'BD' + BCD + ABC' + A'BC + BC'D' \\
 &= AD + A'BC + BC'D'
 \end{aligned}$$

□ Example: prove the equivalence between

- $A'BC'D + (A' + BC)(A + C'D') + BC'D + A'BC'$
- $ABCD + A'C'D' + ABD + ABCD' + BC'D$

Q&A

Announcement

☐ Gradescope for homework submission and grading

- Use the entry code 9KKP26 to sign up

☐ Homework 1

- Due at noon on March 11

☐ Homework 2

- "Part of" it is online (NTU COOL)
 - 5 questions there, more coming
- Due at noon on April 1

☐ Slides version control

- Notify at lectures (white board)
- Notify after lectures (NTU COOL announcement)

☐ Office hour

- Please feel free to contact me if you need it!