- **Please give details of your answer. A direct answer without explanation is not counted.**
- **Your answers must be in English.**
- **Please carefully read problem statements.**
- **During the exam you are not allowed to borrow others' class notes.**
- **Try to work on easier questions first.**

Problem 1 (10 pts)

In our lecture, we designed a two-tape Turing Machine to generate an output ww from an input w of even length. At that time we assume in the beginning the first tape has

$$\llcorner\!\lrcorner w$$

(a) (5 pts) Redo this task without such an assumption. That is,

$$w$$

is the input in the first tape and the first element is already w_1. For simplicity, let's assume $\Sigma = \{0\}, \Gamma = \{0, \llcorner\!\lrcorner\}$.

Some notes and requirements:

1. We now have that in the definition of multi-tape TM, S is allowed. That is,

$$\delta : Q \times \Gamma \rightarrow Q \times \Gamma \times \{R, L, S\}$$

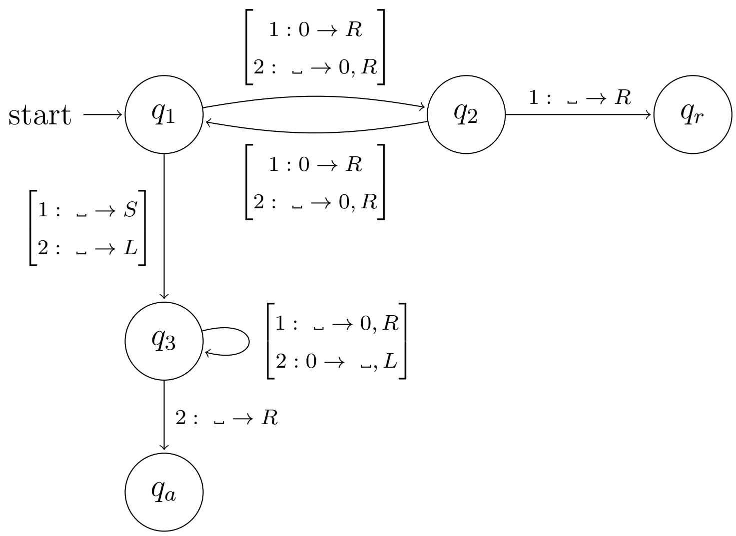
2. Whether w has even length must be checked. We assume w is already 0^* so no need to check if the input is 0^* or not.
3. Besides $\llcorner\!\lrcorner$ and 0, you cannot introduce other tape symbol.
4. The output ww must be in the 1st tape and no $\llcorner\!\lrcorner$ before it.
5. The number of states must be ≤ 5 including the accept and the reject states.

(b) (5 pts) Run a simulation on the string 00.

Answer

(a) The following description shows our idea in constructing the machine.

1. Copy the string in the 1st tape to the 2nd tape.
2. In step 1, check whether the length of w is even.
3. Concatenate the string in the 2nd tape to the 1st tape.



Links not shown go to q_r

If for a link we show only the operation on one tape, it means that for the other tape we have

$$\Sigma \to S$$

Common Mistake: You must check if w has even length.

(b) The upper one is the 1st tape and the lower one is the 2nd tape.

$$\begin{bmatrix} q_1 00 \sqcup \sqcup \sqcup \\ q_1 \sqcup \sqcup \sqcup \sqcup \sqcup \end{bmatrix} \to \begin{bmatrix} 0 q_2 0 \sqcup \sqcup \sqcup \\ 0 q_2 \sqcup \sqcup \sqcup \sqcup \end{bmatrix} \to \begin{bmatrix} 00 q_1 \sqcup \sqcup \sqcup \\ 00 q_1 \sqcup \sqcup \sqcup \end{bmatrix} \to \begin{bmatrix} 00 q_3 \sqcup \sqcup \sqcup \\ 0 q_3 0 \sqcup \sqcup \sqcup \end{bmatrix} \to$$

$$\begin{bmatrix} 000 q_3 \sqcup \sqcup \\ q_3 0 \sqcup \sqcup \sqcup \sqcup \end{bmatrix} \to \begin{bmatrix} 0000 q_3 \sqcup \\ q_3 \sqcup \sqcup \sqcup \sqcup \end{bmatrix} \to \begin{bmatrix} 0000 q_a \sqcup \\ q_a \sqcup \sqcup \sqcup \sqcup \end{bmatrix}$$

Problem 2 (20 pts)

Consider functions of

$$N \to R$$

Use the **definition** of limit to check if

(a) (10 pts) $n^2 = o(n \log n)$

(b) (10 pts) $2^n = o(3^n)$

We assume log is natural log. Do not directly calculate the limit. We want you to prove things by the definition of limit.

Answer

(a) $n^2 \neq o(n\text{log}n)$

To prove that, we need to show

$$\exists c_0 > 0 \; \forall n_0 > 0 \; \exists n \geq n_0, \text{ such that } \frac{n^2}{n \log n} \geq c_0 \tag{1}$$

We pick $c_0 = 1$. Then, for all $n_0 > 0$, we pick $n = n_0$ then we have

$$\frac{n^2}{n \log n} = \frac{n}{\log n} \geq 1$$

for $n = n_0$. Thus $n^2 \neq o(n \log n)$.

Common Mistake: some wrote a wrong statement different from (1).

(b) $2^n = o(3^n)$

To prove that, we need to show

$$\forall c_0 > 0 \; \exists n_0 > 0, \text{ such that } \frac{2^n}{3^n} < c_0 \; \forall n \geq n_0$$

If $c_0 \geq 1$, let $n_0 = 1$ and we have

$$\frac{2^n}{3^n} < 1 \leq c_0, \forall n \geq n_0$$

If $c_0 < 1$, $\frac{2^n}{3^n} < c_0$ is equivanlent to

$$n \log \frac{2}{3} < \log c_0 < 0.$$

Thusm by choosing

$$n_0 = \max\{\lceil \frac{\log c_0}{\log \frac{2}{3}} \rceil + 1, 1\}.$$

Then we have

$$\frac{2^n}{3^n} < c_0, \forall n \geq n_0.$$

Problem 3 (20 pts)

Consider $\log(n)$ to be the natural log . Define

$$f(n) = \log O\left(g(n)\right)$$

if $\exists\ c_0,\ n_0$ such that

$$f(n) < \log\left(c_0 g(n)\right)\ \forall n > n_0$$

We consider $g(n) \geq 2$, $\forall n$.

(a) (10pts) If $f(n) = \log\ O\left(g(n)\right)$, then does $f(n) = O\left(\log\ g(n)\right)$?

(b) (10pts) If $f(n) = O\left(\log\ g(n)\right)$, then does $f(n) = \log\ O\left(g(n)\right)$?

Answer

(a) Yes. $f(n) = O(\log g(n))$

Because $f(n) = \log O(g(n))$, $\exists c_0 > 0,\ n_0 > 0$ such that

$$f(n) \leq \log\left(c_0 g(n)\right),\ \forall n \geq n_0$$

We choose $c_1 = 1 + \max\{\log_2 c_0, 1\}$. Then, we have

$$c_0 g(n) = 2^{\log_2 c_0} g(n) \leq g(n)^{\log_2 c_0} g(n) = g(n)^{1+\log_2 c_0} \leq g(n)^{c_1}.$$

Thus,

$$\log\left(c_0 g(n)\right) \leq c_1 \log g(n),\ \forall n \geq 1.$$

$\exists c_1 = 1 + \max\{\log_2 c_0, 1\}$, $n_1 = n_0$, such that

$$f(n) \leq c_1 \log g(n),\ \forall n \geq n_1$$

By the definition of big-O, $f(n) = O(\log g(n))$.

Common Mistakes:

(i) some state that

· $g(n)$ is an increasing function, or

· $g(n) >$ any given constant.

Neither is correct.

(ii) some wrongly choose c_1 to be related to $g(n)$.

(b) No. $f(n) \neq \log O(g(n))$

To prove that, we need to show

$$\forall c_0 > 0,\ n_0 > 0\ \exists n \geq n_0 \text{ such that } f(n) > \log(c_0 g(n)), \text{ where } f(n) = O(\log g(n)) \tag{2}$$

Consider $f(n) = \log n^2$ and $g(n) = \max\{n, 2\}$. First, we prove $f(n) = O(\log g(n))$.

Let $c_0 = 2$ and $n_0 = 1$. Then, we have

$$f(n) = \log n^2 = 2\log n = c_0 \log n \geq c_0 g(n)\ \forall n \geq n_0$$

We then prove (2). $\forall c_0 > 0,\ n_0 > 0$ we can pick $n = \max\{c_0,\ n_0\} + 2 > n_0$ such that

$$f(n) = \log n^2 > \log(c_0 n) = \log(c_0 g(n))$$

By the statement above, $f(n) \neq \log O(g(n))$.

Common Mistake:

Some try to compare $c_0 \log g(n)$ and $\log(c_1 g(n))$. This is incorrect because what we need to check is $f(n)$.

Note that $c_0 \log g(n)$ is only an upper bound of $f(n)$.

Problem 4 (40 pts)

In the 2nd exam, you were asked to design a Turing machine to determine whether a given binary string represents the length of another string. Now we want you to use a two-tape TM to do a similar task.

(a) (10 pts) An input includes two strings,

$$u \in \{0,1\}^*, v \in \{a\}^*,$$

for the 1st tape and the 2nd tape respectively. Design a two-tape TM that accpets the input if the **reverse** of $u_1 u_2 \cdots u_m$, i.e. $u_m \cdots u_2 u_1$, is a binary representation of the length of v and rejects it otherwise. For example, the following inputs are accpeted,

- $u = 0, v = \epsilon$
- $u = 00, v = \epsilon$
- $u = 10, v = a$
- $u = 011, v = aaaaaa$

And the following inputs are rejected,

- $u = \epsilon, v = \epsilon$
- $u = \epsilon, v = a$

Assume

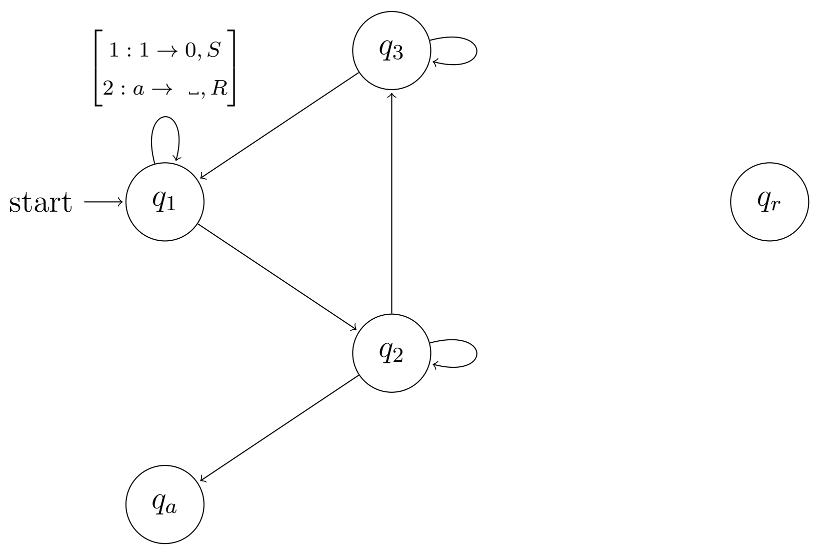
$$\Sigma_1 = \{0, 1\}, \Gamma_1 = \{0, 1, \ \sqcup\}$$

$$\Sigma_2 = \{a\}, \Gamma_2 = \{a, \ \sqcup\}$$

$$\delta : Q \times \Gamma_1 \times \Gamma_2 \to Q \times \Gamma_1 \times \Gamma_2 \times \{R, L, S\}$$

Note that the number of states of your TM, including accept and reject states, must be ≤ 5 and you can only use the symbols contained in Γ_i in the ith tape.

To make the problem easier, we tell you a diagram is like



Finish this diagram by giving details of all links and ignore links that go to q_r.

(b) (5 pts) Use the TM obtained in (a) to simulate the following inputs

(i) $u = 00$
$v = \epsilon$

(ii) $u = 11$
$v = aaa$

(iii) $u = 111$
$v = aaaaaaa$

(c) (10 pts) Assume the input is

$$u = \overbrace{111\cdots111}^{n} \text{ (i.e. string of all 1's)}$$
$$v = \overbrace{aaa\cdots aaa}^{2^n-1}$$

Please calculate the exact number of operations of the TM obtained from (a) from the start state to q_a. You must simplify your answer so any summation term like $\sum_{i=1}^{n}\cdots$ is not allowed.

(d) (5 pts) The following single-tape TM can recognize

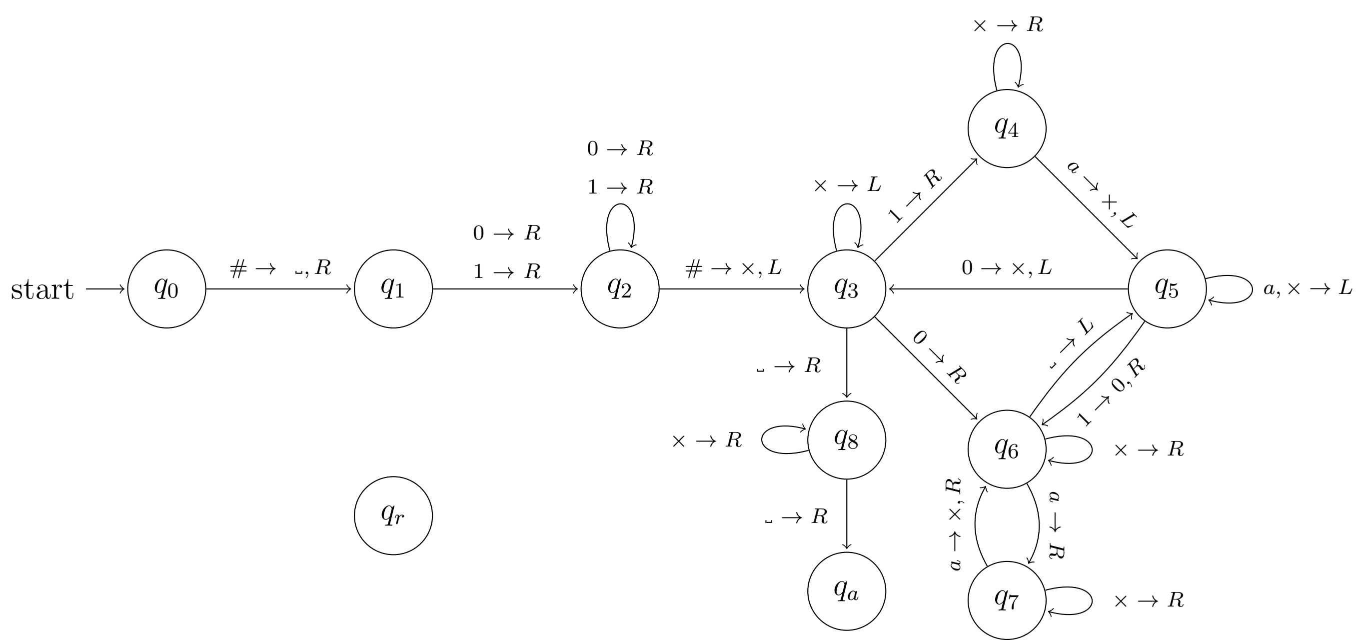
$$\{\#u\#v | v \in \{a\}^*, u \in \{0,1\}^* \text{ and is a binary representation for the length of } v\},$$

For example, the following inputs are accpeted,

- $\#0\#$
- $\#00\#$
- $\#01\#a$
- $\#110\#aaaaaa$

and the following inputs are rejected,

- $\#\#$
- $\#\#a$



Links not shown go to q_r

Use the TM above to simulate the following inputs

(i) $\#1\#a$

(ii) $\#11\#aaa$

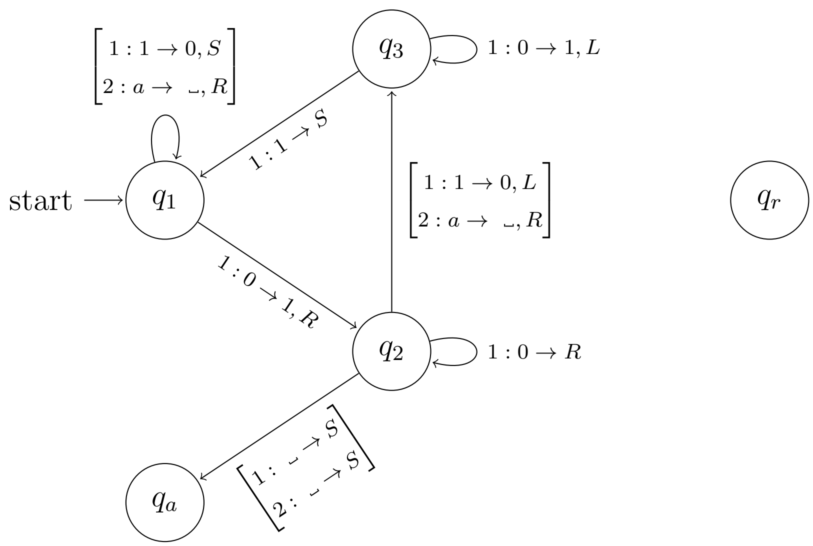
(e) (10 pts) Consider a similar input as in (c), but for single-tape TM, it is

$$\#\overbrace{111\cdots111}^{n}\#\overbrace{aaa\cdots aaa}^{2^n-1}$$

Please calculate the exact number of operations of the TM in (d) from q_0 to q_a. You must simplify your answer so any summation term like $\sum_{i=1}^{n}\cdots$ is not allowed.

Answer

(a) The concept is to minus the reverse of u by 1 and delete an a in v until $u = 0$ and $v = \epsilon$.



Links not shown go to q_r

If for a link we show only the operation on one tape, it means that for the other tape we have

$$\Sigma \to S$$

We explain the process of

$$q_1 \to q_2 \to q_3 \to q_1$$

in detail. It aims to change

$$0\cdots01\cdots$$

to

$$1\cdots 10\cdots$$

Let us consider the example

$$q_1 0001$$

From q_1 to q_2, we have

$$1q_2 001$$

This "1" lets us know the first element. We then use $q_2 \to q_2$ to find the second "1". This "1" is changed to 0 by $q_2 \to q_3$:

$$10q_3 00$$

From q_3 we move left to change every 0 to 1 ($q_3 \to q_3$) until we hit 1 ($q_3 \to q_1$). In the end we get

$$q_1 1110$$

Common Mistake: Some wronly think that at the beginning of the tape if we move left, then a $\sqcup$ will appear. This is not true under our definition.

(b) The upper one is the 1st tape and the lower one is the 2nd tape.

(i)

$$\begin{bmatrix} q_1 00 \\ q_1 \sqcup \end{bmatrix} \to \begin{bmatrix} 1q_2 0 \\ q_2 \sqcup \end{bmatrix} \to \begin{bmatrix} 10q_2 \sqcup \\ q_2 \sqcup \end{bmatrix} \to \begin{bmatrix} 10q_a \sqcup \\ q_a \sqcup \end{bmatrix}$$

(ii)

$$\begin{bmatrix} q_1 11 \\ q_1 aaa \end{bmatrix} \to \begin{bmatrix} q_1 01 \\ \sqcup q_1 aa \end{bmatrix} \to \begin{bmatrix} 1q_2 1 \\ \sqcup q_2 aa \end{bmatrix} \to$$

$$\begin{bmatrix} q_3 10 \\ \sqcup \sqcup q_3 a \end{bmatrix} \to \begin{bmatrix} q_1 10 \\ \sqcup \sqcup q_1 a \end{bmatrix} \to \begin{bmatrix} q_1 00 \\ \sqcup \sqcup \sqcup q_1 \sqcup \end{bmatrix} \to$$

$$\begin{bmatrix} 1q_2 0 \\ \sqcup \sqcup \sqcup q_2 \sqcup \end{bmatrix} \to \begin{bmatrix} 10q_2 \sqcup \\ \sqcup \sqcup \sqcup q_2 \sqcup \end{bmatrix} \to \begin{bmatrix} 10q_a \sqcup \\ \sqcup \sqcup \sqcup q_a \sqcup \end{bmatrix}$$

(iii)

$$\begin{bmatrix} q_1 111 \\ q_1 a^7 \end{bmatrix} \to \begin{bmatrix} q_1 011 \\ \sqcup q_1 a^6 \end{bmatrix} \to \begin{bmatrix} 1 q_2 11 \\ \sqcup q_2 a^6 \end{bmatrix} \to \begin{bmatrix} q_3 101 \\ \sqcup^2 q_3 a^5 \end{bmatrix} \to$$

$$\begin{bmatrix} q_1 101 \\ \sqcup^2 q_1 a^5 \end{bmatrix} \to \begin{bmatrix} q_1 001 \\ \sqcup^3 q_1 a^4 \end{bmatrix} \to \begin{bmatrix} 1 q_2 01 \\ \sqcup^3 q_2 a^4 \end{bmatrix} \to \begin{bmatrix} 10 q_2 1 \\ \sqcup^3 q_2 a^4 \end{bmatrix} \to$$

$$\begin{bmatrix} 1 q_3 00 \\ \sqcup^4 q_3 a^3 \end{bmatrix} \to \begin{bmatrix} q_3 110 \\ \sqcup^4 q_3 a^3 \end{bmatrix} \to \begin{bmatrix} q_1 110 \\ \sqcup^4 q_1 a^3 \end{bmatrix} \to \begin{bmatrix} q_1 010 \\ \sqcup^5 q_1 a^2 \end{bmatrix} \to$$

$$\begin{bmatrix} 1 q_2 10 \\ \sqcup^5 q_2 a^2 \end{bmatrix} \to \begin{bmatrix} q_3 100 \\ \sqcup^6 q^3 a \end{bmatrix} \to \begin{bmatrix} q_1 100 \\ \sqcup^6 q_1 a \end{bmatrix} \to \begin{bmatrix} q_1 000 \\ \sqcup^7 q_1 \ \sqcup \end{bmatrix} \to$$

$$\begin{bmatrix} 1 q_2 00 \\ \sqcup^7 q_2 \ \sqcup \end{bmatrix} \to \begin{bmatrix} 10 q_2 0 \\ \sqcup^7 q_2 \ \sqcup \end{bmatrix} \to \begin{bmatrix} 100 q_2 \ \sqcup \\ \sqcup^7 q_2 \ \sqcup \end{bmatrix} \to \begin{bmatrix} 100 q_a \ \sqcup \\ \sqcup^7 q_a \ \sqcup \end{bmatrix}$$

Note that a^i means

$$\overbrace{a \cdots a}^{i}$$

(c) $6 \cdot 2^{n-1} - n - 2$

Let r_n be the number of operations from

$$1 : q_1 \overbrace{0 \cdots 0}^{n} 1 \cdots$$

to

$$1 : q_1 \overbrace{1 \cdots 1}^{n} 0 \cdots$$

s_n be the number of operations from

$$1 : q_1 \overbrace{1 \cdots 1}^{n} \cdots$$

to

$$1 : q_1 \overbrace{0 \cdots 0}^{n} \cdots$$

and t_n be the number of operations from

$$1 : q_1 \overbrace{0 \cdots 0}^{n}$$

to

$$1 : \overbrace{0 \cdots 0}^{n} q_a \llcorner\!\lrcorner$$

So immediately we know

$$
\begin{aligned}
t_n &= n + 1 \\
r_n &= 2n + 1 \\
s_n &= \begin{cases} 1 & \text{, if } n = 1 \\ 2s_{n-1} + r_{n-1} & \text{, otherwise} \end{cases} \qquad (3)
\end{aligned}
$$

where (3) is because,

$$
\begin{aligned}
s_n =& q_1 \overbrace{1 \cdots 1}^{n} \to q_1 \overbrace{0 \cdots 0}^{n} \\
=& q_1 \overbrace{1 \cdots 1}^{n-1} 1 \to q_1 \overbrace{0 \cdots 0}^{n-1} 1 \to q_1 \overbrace{1 \cdots 1}^{n-1} 0 \to q_1 \overbrace{0 \cdots 0}^{n-1} 0 \\
=& s_{n-1} + r_{n-1} + s_{n-1}, \text{ if } n > 1
\end{aligned}
$$

Now we want to solve the recursive equation (3).

$$
\begin{aligned}
s_n &= 2s_{n-1} + r_{n-1} \\
\big(&= 2(2s_{n-2} + r_{n-2}) + r_{n-1}\big) \qquad (4) \\
\big(&= 2^2 s_{n-2} + 2r_{n-2} + r_{n-1}\big) \qquad (5) \\
&= 2^{n-1} s_1 + \sum_{i=1}^{n-1} 2^{i-1} r_{n-i} \\
&= 2^{n-1} + \sum_{i=1}^{n-1} 2^{i-1} (2(n-i) + 1) \\
&= 2^{n-1} + n \sum_{i=1}^{n-1} 2^i + \sum_{i=1}^{n-1} 2^{i-1} - \sum_{i=1}^{n-1} i 2^i \\
&= 2^{n-1} + nA_{n-1} + \frac{A_{n-1}}{2} - B_{n-1} \qquad (6)
\end{aligned}
$$

where (15) and (5) are not generally correct as they are just quick notes for a better understanding, and

$$
\begin{aligned}
A_n &\equiv \sum_{i=1}^{n} 2^i \\
B_n &\equiv \sum_{i=1}^{n} i 2^i
\end{aligned}
$$

We know

$$A_n = 2^{n+1} - 2,$$

and now we want to calculate B_n.

$$B_n = 1 \cdot 2 + 2 \cdot 4 + 3 \cdot 8 + \cdots + n \cdot 2^n \tag{7}$$

$$2 \cdot B_n = \qquad 1 \cdot 4 + 2 \cdot 8 + \cdots + (n-1) \cdot 2^n + n \cdot 2^{n+1} \tag{8}$$

Minus (7) with (8), we have

$$B_n = n \cdot 2^{n+1} - A_n$$

Thus,

$$\begin{aligned}(6) &= 2^{n-1} + n(2^n - 2) - (n-1) \cdot 2^n + \frac{3}{2} A_{n-1} \\ &= 2^{n-1} - 2n + 2^n + 3(2^{n-1} - 1) \\ &= 6 \cdot 2^{n-1} - 2n - 3\end{aligned}$$

Therefore,

$$\begin{aligned}& q_1 \overbrace{1 \cdots 1}^{n} \to \overbrace{0 \cdots 0}^{n} q_a \sqcup \\ =& q_1 \overbrace{1 \cdots 1}^{n} \to q_1 \overbrace{0 \cdots 0}^{n} \to \overbrace{0 \cdots 0}^{n} q_a \sqcup \\ =& s_n + t_n \\ =& 6 \cdot 2^{n-1} - n - 2\end{aligned}$$

(d) The simulations are as follows

(i)

$$\begin{array}{ccccccccc} q_0\#1\#a & \to & \sqcup q_1 1\#a & \to & \sqcup 1 q_2 \#a & \to & \sqcup q_3 1 \times a & \to \\ \sqcup 1 q_4 \times a & \to & \sqcup 1 \times q_4 a & \to & \sqcup 1 q_5 \times \times & \to & \sqcup q_5 1 \times \times & \to \\ \sqcup 0 q_6 \times \times & \to & \sqcup 0 \times q_6 \times & \to & \sqcup 0 \times \times q_6 \sqcup & \to & \sqcup 0 \times q_5 \times & \to \\ \sqcup 0 q_5 \times \times & \to & \sqcup q_5 0 \times \times & \to & q_3 \sqcup \times \times\times & \to & \sqcup q_8 \times \times\times & \to \\ \sqcup \times q_8 \times \times & \to & \sqcup \times \times q_8 \times & \to & \sqcup \times \times \times q_8 \sqcup & \to & \sqcup \times \times \times \ \sqcup q_a \sqcup & \end{array}$$

(ii)

$$
\begin{array}{ccccccc}
q_0\#11\#aaa & \rightarrow & \sqcup q_1 11\#aaa & \rightarrow & \sqcup 1q_2 1\#aaa & \rightarrow \\
\sqcup 11q_2\#aaa & \rightarrow & \sqcup 1q_3 1 \times aaa & \rightarrow & \sqcup 11q_4 \times aaa & \rightarrow \\
\sqcup 11 \times q_4 aaa & \rightarrow & \sqcup 11q_5 \times \times aa & \rightarrow & \sqcup 1q_5 1 \times \times aa & \rightarrow \\
\sqcup 10q_6 \times \times aa & \rightarrow & \sqcup 10 \times q_6 \times aa & \rightarrow & \sqcup 10 \times \times q_6 aa & \rightarrow \\
\sqcup 10 \times \times a q_7 a & \rightarrow & \sqcup 10 \times \times a \times q_6 \sqcup & \rightarrow & \sqcup 10 \times \times a q_5 \times & \rightarrow \\
\sqcup 10 \times \times q_5 a \times & \rightarrow & \sqcup 10 \times q_5 \times a \times & \rightarrow & \sqcup 10 q_5 \times \times a \times & \rightarrow \\
\sqcup 1q_5 0 \times \times a \times & \rightarrow & \sqcup q_3 1 \times \times \times a \times & \rightarrow & \sqcup 1q_4 \times \times \times a \times & \rightarrow \\
\sqcup 1 \times q_4 \times \times a \times & \rightarrow & \sqcup 1 \times \times q_4 \times a \times & \rightarrow & \sqcup 1 \times \times \times q_4 a \times & \rightarrow \\
\sqcup 1 \times \times q_5 \times \times \times & \rightarrow & \sqcup 1 \times q_5 \times \times \times \times & \rightarrow & \sqcup 1q_5 \times \times \times \times \times & \rightarrow \\
\sqcup q_5 1 \times \times \times \times \times & \rightarrow & \sqcup 0q_6 \times \times \times \times & \rightarrow & \sqcup 0 \times q_6 \times \times \times \times & \rightarrow \\
\sqcup 0 \times \times q_6 \times \times \times & \rightarrow & \sqcup 0 \times \times \times q_6 \times \times & \rightarrow & \sqcup 0 \times \times \times \times q_6 \times & \rightarrow \\
\sqcup 0 \times \times \times \times \times q_6 \sqcup & \rightarrow & \sqcup 0 \times \times \times \times q_5 \times & \rightarrow & \sqcup 0 \times \times \times q_5 \times \times & \rightarrow \\
\sqcup 0 \times \times q_5 \times \times \times & \rightarrow & \sqcup 0 \times q_5 \times \times \times \times & \rightarrow & \sqcup 0q_5 \times \times \times \times \times & \rightarrow \\
\sqcup q_5 0 \times \times \times \times \times & \rightarrow & q_3 \sqcup \times \times \times \times \times \times & \rightarrow & \sqcup q_8 \times \times \times \times \times \times & \rightarrow \\
\sqcup \times q_8 \times \times \times \times \times & \rightarrow & \sqcup \times \times q_8 \times \times \times \times & \rightarrow & \sqcup \times \times \times q_8 \times \times \times & \rightarrow \\
\sqcup \times \times \times \times q_8 \times \times & \rightarrow & \sqcup \times \times \times \times \times q_8 \times & \rightarrow & \sqcup \times \times \times \times \times \times q_8 \sqcup & \rightarrow \\
\sqcup \times \times \times \times \times \times \sqcup q_a \sqcup & & & & &
\end{array}
$$

(e) $(2n+3)2^n + 2n^2 + 5n + 2$

From the simulation we notice an iterative procedure is conducted so that at the ith iteration, the following configuration

$$1 \cdots 1 q_3 \overbrace{1 \times \cdots \times}^{i} \# \overbrace{\times \cdots \times}^{2^{i-1}-1} \overbrace{a \cdots}^{2^n - 2^{i-1}}$$

is changed to

$$1 \cdots 1 q_3 \overbrace{1 \times \cdots \times}^{i+1} \# \overbrace{\times \cdots \times}^{2^{i}-1} \overbrace{a \cdots}^{2^n - 2^{i}}$$

Note that in practice # has been changed to $\times$, but we still use # for easy discussion. We check how many steps are taken in the above process. From

$$1 \cdots 1 q_3 \overbrace{1 \times \cdots \times}^{i} \# \overbrace{\times \cdots \times}^{2^{i-1}-1} \overbrace{a \cdots}^{2^n - 2^{i-1}}$$

it moves right to find the first a, so

$$i + 1 + 2^{i-1} - 1 \tag{9}$$

steps are taken to have

$$1\cdots 1\overbrace{1\times\cdots\times}^{i}\#\overbrace{\times\cdots\times}^{2^{i-1}-1}q_4\ \overbrace{a\cdots}^{2^n-2^{i-1}}$$

Then a is changed to $\times$ and we move to find the last 1.

$$1\cdots 1q_5\overbrace{1\times\cdots\times}^{i}\#\overbrace{\times\cdots\times}^{2^{i-1}-1}\ \overbrace{\times\cdots}^{2^n-2^{i-1}}$$

This process takes

$$2^{i-1}-1+1+i \tag{10}$$

Then 1 is changed to 0 ($q_5\to q_6$) and we move to pass $\times$ ($q_6\to q_6$) and start cancelling out pairs of $a\cdots a\cdots$ ($q_6\to q_7\to q_6$). In the end we have

$$1\cdots 1\overbrace{0\times\cdots\times}^{i}\#\overbrace{\cdots}^{2^n-1}q_6\ \llcorner\!\lrcorner$$

This takes

$$i+1+2^n-1 \tag{11}$$

steps. Then we move left to find the 0 in the first part ($q_6\to q_5$, and then $q_5\to q_5$). In the end we have

$$1\cdots 1q_3\overbrace{1\times\cdots\times}^{i+1}\#\overbrace{\times\cdots\times}^{2^i-1}\overbrace{a\cdots}^{2^n-2^i}$$

This process takes

$$2^n-1+1+i+1 \tag{12}$$

Next we consider the initial steps. From

$$q_0\#\overbrace{1\cdots 1}^{n}\#a\cdots a$$

to

$$\llcorner\!\lrcorner\overbrace{1\cdots 1}^{n}q_2\#a\cdots a$$

and then

$$\llcorner\!\lrcorner 1\cdots 1q_3 1\times a\cdots a$$

it takes

$$1+n+1 \tag{13}$$

steps. Next we discuss the final steps of from q_3 to q_a. We have

$$q_3\ \overbrace{\llcorner\!\lrcorner\times\cdots\times}^{1+n+1+2^n-1}\ \llcorner\!\lrcorner$$

after the last iteration. We pass all elements (by $q_8 \to q_8$) and the $\sqcup$ in the end. Thus the number of steps is one more than the length of the input string:

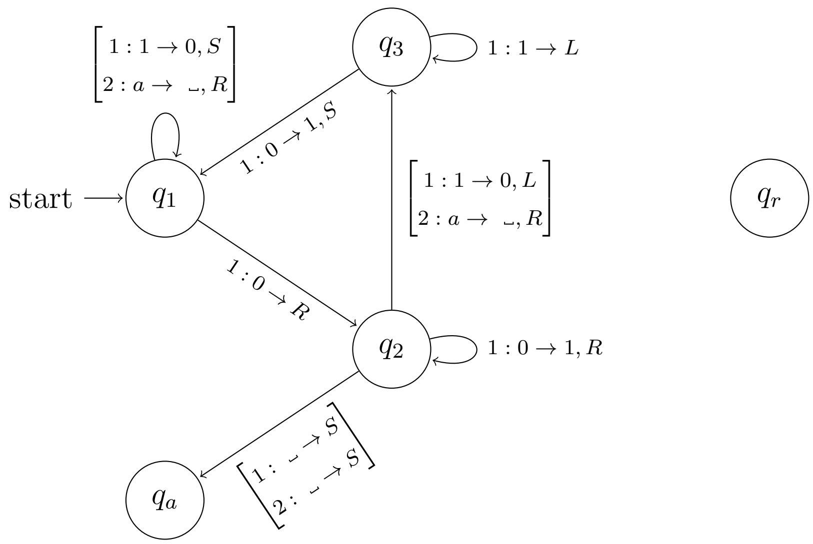
$$1 + n + 1 + 2^n - 1 + 1 \tag{14}$$

From (9) to (14), the total number of steps is

$$\begin{aligned}
&(1 + n + 1) + \\
&\sum_{i=1}^{n} \Big(2(i + 1 + 2^{i-1} - 1) + \\
&\quad (i + 1 + 2^n - 1) + \\
&\quad (2^n - 1 + 1 + i + 1) \Big) + \\
&(1 + n + 1 + 2^n - 1 + 1) \\
=&(n + 2) + \\
&\sum_{i=1}^{n} (4i + 2^{n+1} + 2^i + 1) + \\
&(2^n + n + 2) \\
=&(n + 2) + (2n(n + 1) + n2^{n+1} + 2^{n+1} - 2 + n) + (2^n + n + 2) \\
=&(2n + 3)2^n + 2n^2 + 5n + 2 \\
=&n2^{n+1} + 3 \cdot 2^n + 2n^2 + 5n + 2
\end{aligned}$$

Alternative solution for $(a)(b)$

(i) (a) This is similar to the original solution. We use 0 rather than 1 to denote the first element. Also, we change all 0's to 1's from the 2nd element to the left most 1 before the left most 1 is found ($q_2 \to q_2$) rather than after the left most 1 is found ($q_3 \to q_3$).



TM_1

Links not shown go to q_r

If for a link we show only the operation on one tape, it means that for the other tape we have

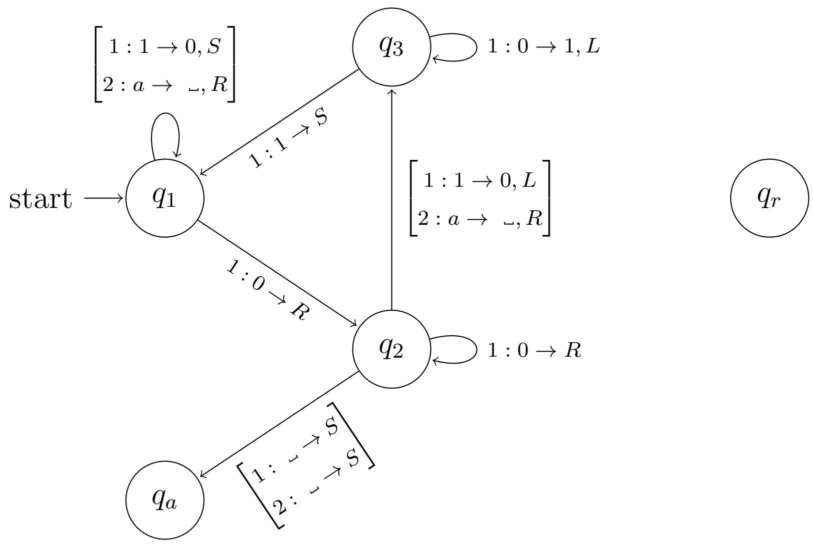
$$\Sigma \to S$$

(b) $6 \cdot 2^{n-1} - n - 2$

Discussionn is similar to the original solution.

(ii) (a) This is similar to the original solution. We do not mark the first element, but use the fact that, when the head is at the beginning of the tape and move left, it is still at the beginning of the tape. So when the head is at the beginning and the state is q_3,

$$q_3 01 \cdots 1 \to q_3 11 \cdots 1 \to q_1 11 \cdots 1$$



TM_2
Links not shown go to q_r

If for a link we show only the operation on one tape, it means that for the other tape we have

$$\Sigma \to S$$

(b) $7 \cdot 2^{n-1} - n - 3$

Comparing to the original solution, this solution takes one more step to go to q_1 after finding the left most 1. That is for the original solution,

$$1\overbrace{0\cdots 0}^{m}q_2 1\cdots \to q_1 1\overbrace{1\cdots 1}^{m}0\cdots$$

takes $m+2$ steps, and for this solution,

$$0\overbrace{0\cdots 0}^{m}q_2 1\cdots \to q_1 1\overbrace{1\cdots 1}^{m}0\cdots$$

takes $m+3$ steps. We continue using the notation from the discussion for the original solution,

$$r_n = 1 : q_1\overbrace{0\cdots 0}^{n}1\cdots \to 1 : q_1\overbrace{1\cdots 1}^{n}0\cdots$$
$$s_n = 1 : q_1\overbrace{1\cdots 1}^{n}\cdots \to 1 : q_1\overbrace{0\cdots 0}^{n}\cdots$$
$$t_n = 1 : q_1\overbrace{0\cdots 0}^{n} \to 1 : \overbrace{0\cdots 0}^{n}q_a\ \llcorner\!\lrcorner$$

Here we slightly abuse the "=" to denote the left hand side is equal to the number of the operations of the right hand side. So we know

$$
\begin{aligned}
t_n &= n+1 \\
r_n &= 2n+2 \\
s_n &= \begin{cases} 1 & \text{, if } n=1 \\ 2s_{n-1}+r_{n-1} & \text{, otherwise} \end{cases}
\end{aligned} \tag{15}
$$

Note that the only difference is, this r_n is 1 more larger than the original r_n. Now we solve (15).

$$
\begin{aligned}
s_n &= 2s_{n-1} + r_{n-1} \\
&= 2^{n-1}s_1 + \sum_{i=1}^{n-1} 2^{i-1} r_{n-i} \\
&= 2^{n-1} + \sum_{i=1}^{n-1} 2^{i-1}(2(n-i)+2) \\
&= 2^{n-1} + n\sum_{i=1}^{n-1} 2^i + \sum_{i=1}^{n-1} 2^i - \sum_{i=1}^{n-1} i2^i \\
&= 2^{n-1} + (n+2)(2^n-2) - n2^n + 2^n \\
&= 7\cdot 2^{n-1} - 2n - 4
\end{aligned}
$$

Therefore the total number of operations is

$$s_n + t_n = 7\cdot 2^{n-1} - n - 3$$