- **Please give details of your answer. A direct answer without explanation is not counted.**

- **Your answers must be in English.**

- **Please carefully read problem statements.**

- **During the exam you are not allowed to borrow others' class notes.**

- **Try to work on easier questions first.**

Problem 1 (40 pts)

Consider $\Sigma = \{0, 1\}$ and the following NFA:

$$\text{start} \longrightarrow (q_0) \xrightarrow{\ 0\ } ((q_1)) \tag{1}$$

(a) (5pts) What's the formal definition?

(b) (5pts) Convert this NFA to DFA by the procedure described in our lecture. (procedure in example 1.41 of the textbook) No need to give the formal definition (also in other problems unless specified)

(c) (5pts) Consider another NFA

$$\text{start} \longrightarrow (q_2) \xrightarrow{\ 1\ } ((q_3)) \tag{2}$$

and let A, B be the languages of (1), (2) respectively. What's the NFA of $A \circ B$ by using the procedure in theorem 1.47 of the textbook?

(d) (15pts) Give a DFA with the smallest number of states to recognize $A \circ B$. You must explain why your solution has the smallest number of states. In particular, we want you to sequentially check

- 1 state can not work

- 2 states can not work

⋮

(e) (10pts) Transform NFA in (c) to DFA by the same procedure in (b). Also see if results in (e) can be simplified to results in (d).

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Answer

(a) N_a:
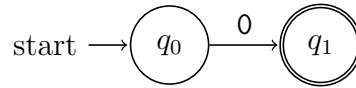
Formal definition: $M = (\{q_0, q_1\}, \{0, 1\}, \delta, q_0, \{q_1\})$
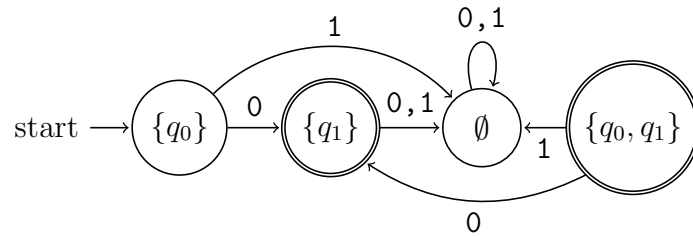
δ is described as

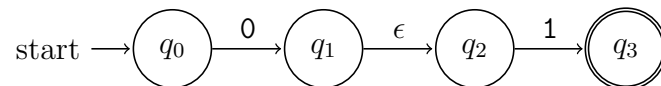|	0	1	ϵ
q_0	$\{q_1\}$	$\emptyset$	$\emptyset$
q_1	$\emptyset$	$\emptyset$	$\emptyset$

Common mistake:

1. The ϵ column is not shown.

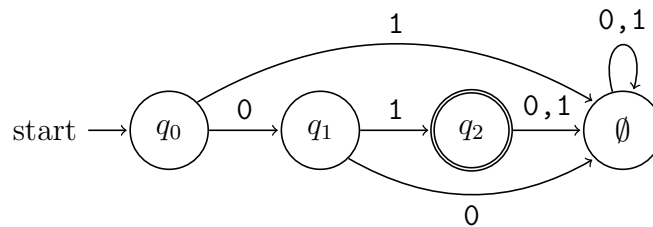2. Every output of δ must be a set, so it is $\{q_1\}$ rather than q_1.

(b) By following the procedure in the textbook, we can get a DFA with 4 states.

(c) By following the procedure in the textbook, we can get a NFA with 4 states.
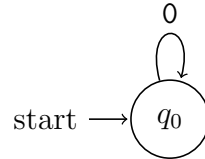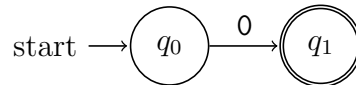

(d) N_d:

First, we show that the DFA must satisfy the following properties.

1. It has at least an accept state because 01 is accepted.

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2. The start state cannot be accept state. Otherwise, ϵ is accepted.

3. We cannot have the following sub-graph. Because 01 is accepted, 0*1 must be accepted as well.
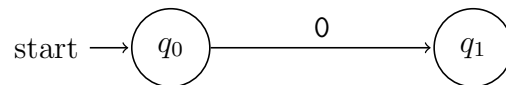

4. We cannot have the following sub-graph. Otherwise, 0 is accepted.

Now we check the following situations

- 1 state: By 1 and 2, we get a contracdiction.

- 2 states: We have a start state q_s and an accept state q_a. However, from 3 and 4, q_s can't have an out link for input 0. There is a contradiction.

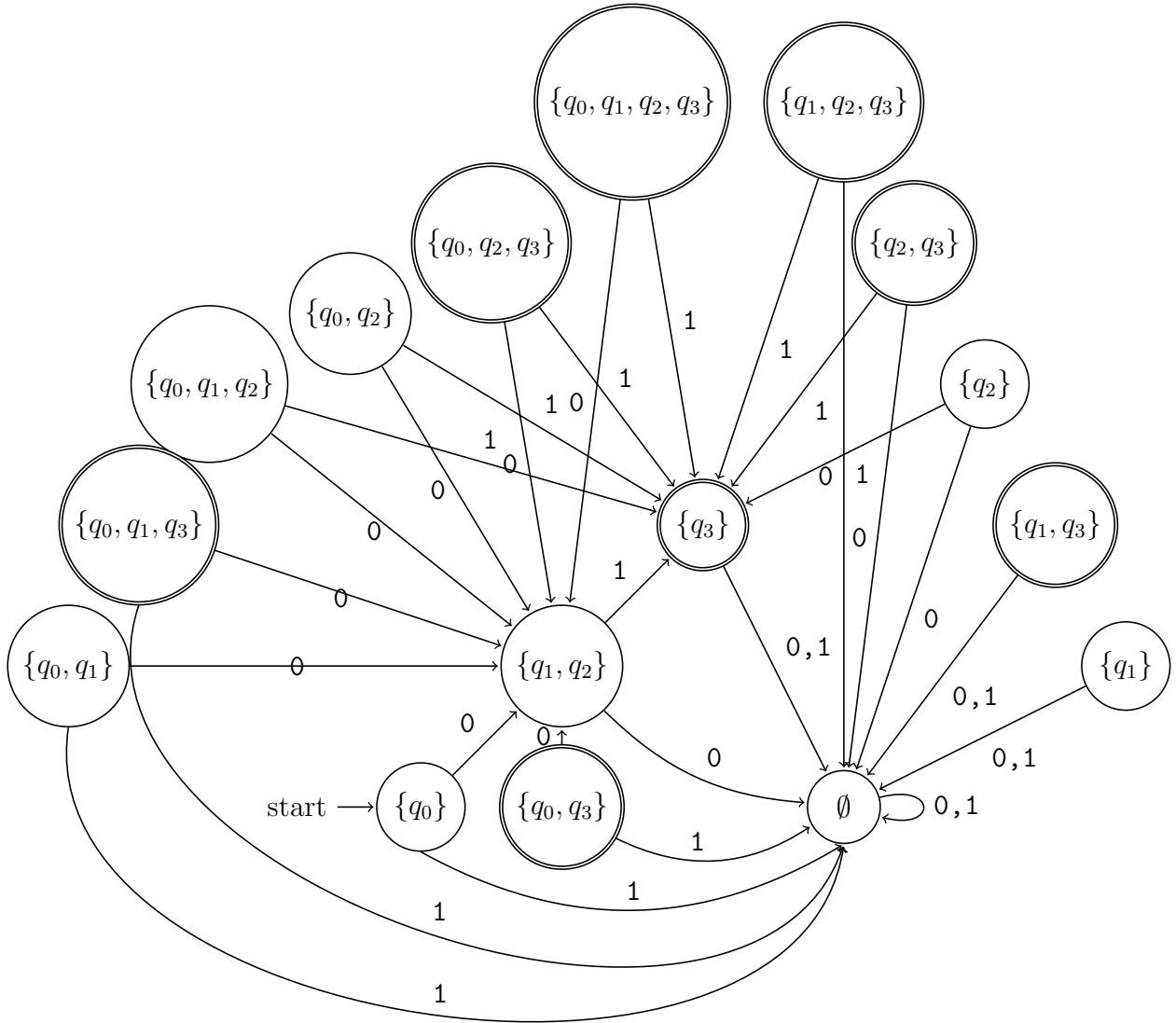- 3 states: Let q_0 be the start state. From 2, 3 and 4,

must occur and neither q_0 nor q_1 is an accept state. To accept 01 we must have
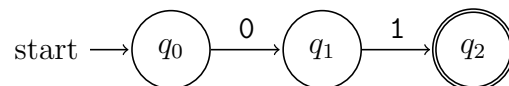
However, q_0's any out link for "1" cause a string other than 01 to be accepted, so there is a contradiction.

Therefore, we can't have a DFA with less than 4 states.

(e) By following the procedure in the textbook, we can get a DFA with 16 states.

Note that q_1 and q_2 must appear together. By removing nodes without in-links we have
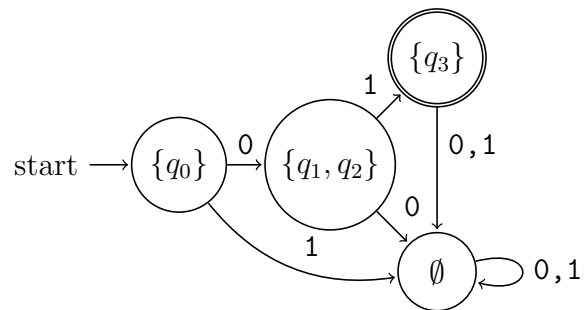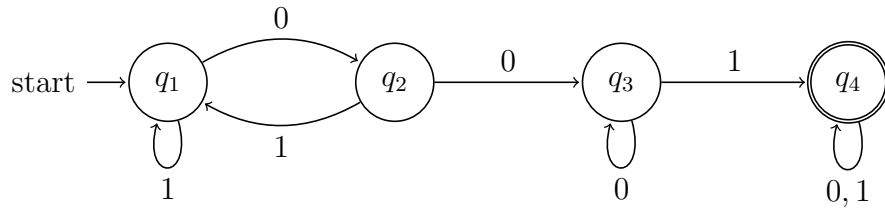
It is the same as (d)

Common mistake: you cannot say it is the same without details.
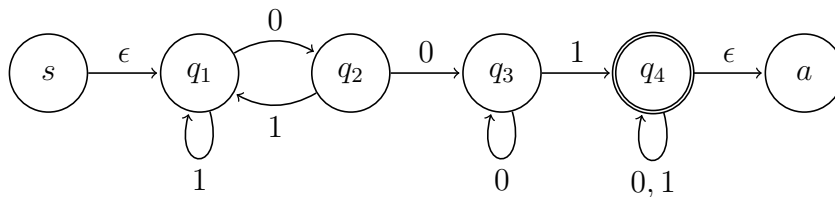
Problem 2   (10 pts)
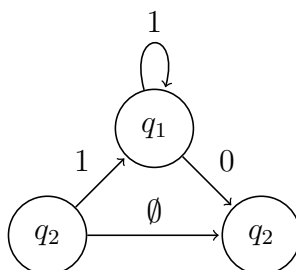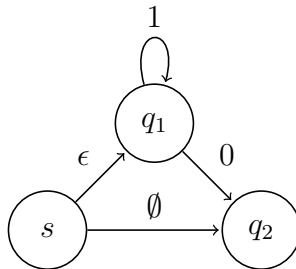
Consider the following DFA

Let $\sum = \{0, 1\}$

Convert DFA to GNFA and sequentially remove states in the order q_1, q_2, q_3, q_4 to generate a regular expression.
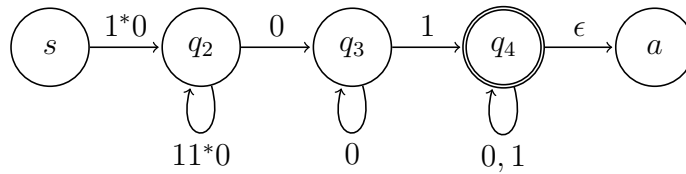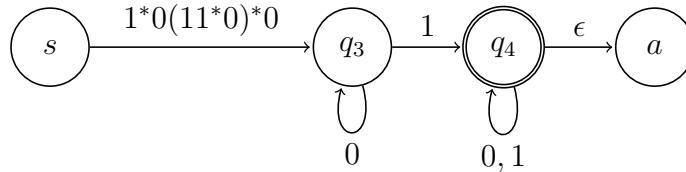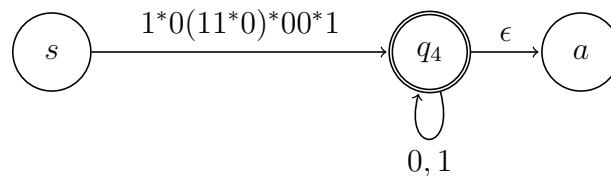
Answer

From

First remove q_1 by

we have

Similarly, remove q_2

remove q_3

remove q_4
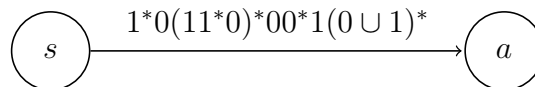
Problem 3   (20 pts)

Consider the language $\emptyset 1^*$ with $\Sigma = \{1\}$

(a) (5 pts) Generate a 4-state NFA for this language by using the procedure in theorem 1.54 of the textbook.

(b) (10 pts) Convert this 4-state NFA to a 16-state DFA by the procedure in example 1.41 of the textbook.

(c) (5 pts) Simplify the result in (b) to a DFA with the smallest number of states. Show details and explain why yours has the smallest number of states.

Answer

(a) For ∅
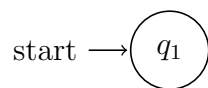
For 1*
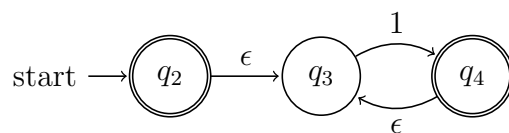
Concatenate the two diagrams above, we have the diagram for ∅1*

(b) The 16-state DFA

(c) Remove all states without in-arrow

and we don't need state $\{q_1\}$, then we have

It has only one state and we must have at least one state as the starting state. Therefore it has the smallest number of states.
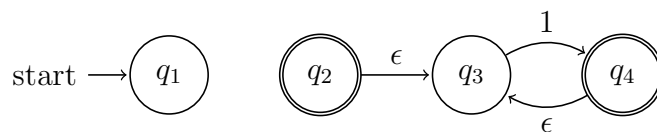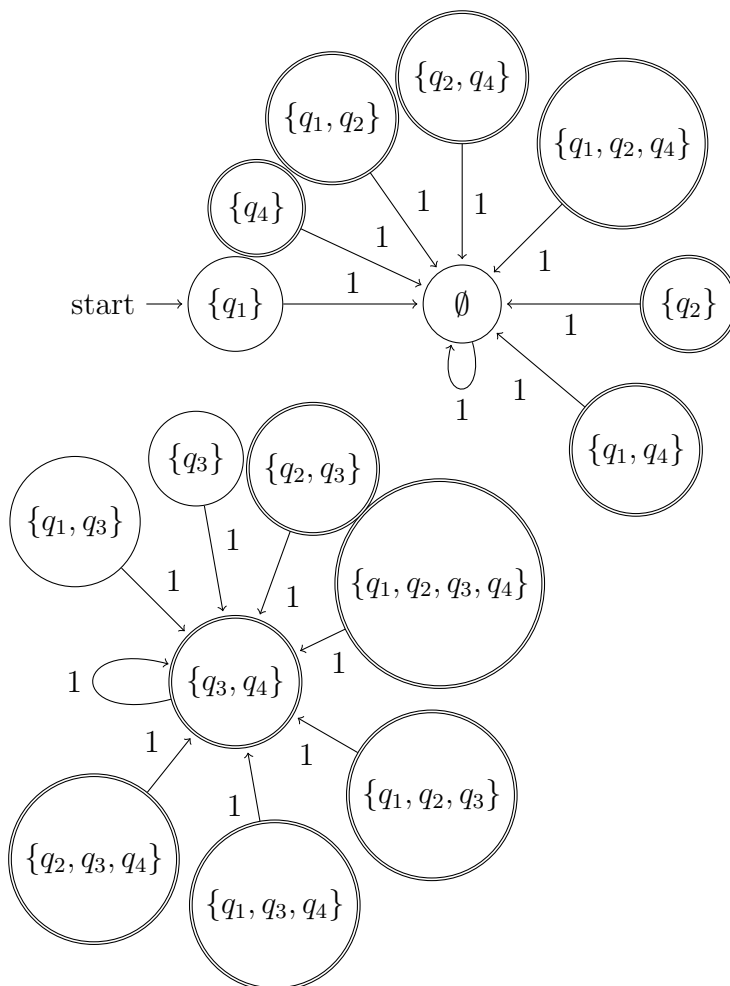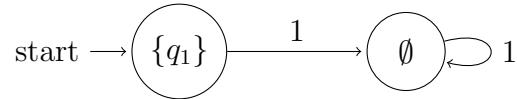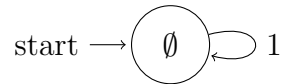
Problem 4   (10 pts)

We would like to prove that the following language is not regular.

$$\{1^{(n-1)^2} | n \geq 1\}$$

For any pumping length p, we consider

$$s = 1^{(p-1)^2}$$

Could you use this s to finish the proof? If yes, do it. If not explain why and find another way to finish the proof.

Answer

s cannot be used for pumping lemma as when $p = 1$, $|s| = 0 < 1 = p$. Thus we find another way.

Suppose $\{1^{(n-1)^2} | n \geq 1\} \equiv \mathbb{A}$ is regular. Let $s = 1^{p^2} = 1^{((p+1)-1)^2} \in \mathbb{A}$ and $|s| = p^2 \geq p$. Then by pumping lemma, for any x, y, z s.t.

$$s = xyz, \ |xy| = q \leq p, \ |y| > 0$$

we have

$$xy^2z \in \mathbb{A}$$

However

$$p^2 < p^2 + |y| = |xy^2z| \leq |x^2y^2z| = p^2 + q \leq p^2 + p < (p+1)^2$$

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implies

$$xy^2z = 1^r \text{ where } p^2 < r < (p+1)^2.$$

Hence $xy^2z \notin \mathbb{A}$ and leads to a contradiction.

Common mistake:

Some ignore the question whether $s = 1^{(p-1)^2}$ is suitable. However, it is the main question asked by this problem.

Problem 5   (10 pts)

Is the following language regular or not?

$$\{\sigma_1\sigma_2\sigma_3\sigma_4\cdots\sigma_{2n-1}\sigma_{2n} \mid (\sigma_1\sigma_3\cdots\sigma_{2n-1}) \in L \text{ and } \sigma_i \in \Sigma\},$$

where $\Sigma = \{0,1\}$, $n \geq 1$ and $L = \{w \mid \text{every odd position of } w \text{ is a } 1\}$ is a language over Σ .
You cannot just answer yes or no. You need to give details of your proof.

Answer

The language is the same as

$$\{w \mid \text{position } 4p+1 \text{ of } w \text{ is } 1, \text{ where } p \geq 0, \text{ and } |w| = 2n, \, n \geq 1 \}$$

We want to construct a DFA to accept it to prove it is regular.

First, we show that the DFA must satisfy the following properties.

1. The start state cannot be accept state. Otherwise, ϵ is accepted.

2. We need a four state loop to check $4p + 1$ is 1 or 0.

3. We need a state to deal the string with position $4p + 1$ is not 1.

Then, the DFA accepting the language is as follows.
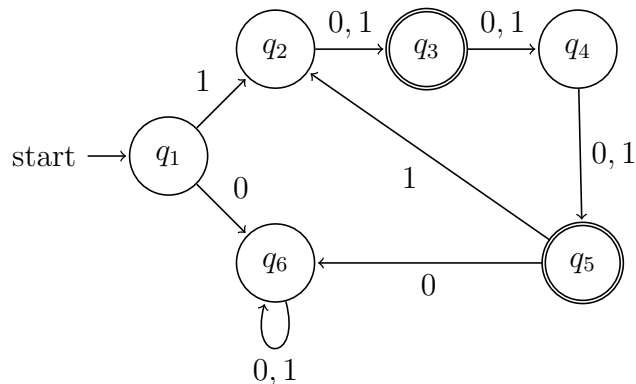
Another solution is the following NFA:
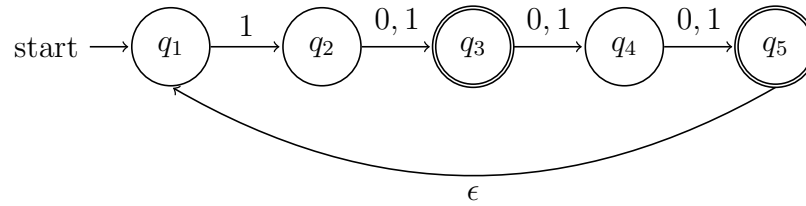
Using Regular Expression

Let $A = \{w \mid \text{position } 4p+1 \text{ of } w \text{ is } 1, \text{ where } p \geq 0, \text{ and } |w| = 2n, n \geq 1 \}$, so A can be written as $(1\Sigma\Sigma\Sigma)^*(1\Sigma \cup 1\Sigma\Sigma\Sigma)$. Thus A is regular.

Problem 6 (10 pts)

Is the following language regular or not, where $\Sigma = \{1\}$?

$$L = \{1^n \mid \text{where } n \text{ is a prime number}\}$$

Answer

No. We prove it by pumping lemma.

If it is regular language, then let p be its pumping length. Then, we pick $s = 1^{l_s}$ where l_s is the biggist prime number which is not bigger than $(p+1)! + 1$. Let $s = xyz$ where $|xy| \leq p$ and $l \equiv |y| > 0$. Let $s' = xy^2z$, then $|s'| = l_s + l$. If $l_s + l \leq (p+1)! + 1$, then s' is not a prime. Therefore $l_s + l > (p+1)! + 1$. However, $l_s + l \leq (p+1)! + 1 + p$ and there is no prime number between $(p+1)! + 2$ and $(p+1)! + 1 + p$. Thus s' is not in L and there is a contradiction. Then the language is not regular.

Another solution

let $n \geq p$ be a prime number. By pumping lemma, there exist xyz such that $1^n = xyz$, where $|xy| \leq p$, $|y| > 0$, $xy^iz \in L$ for all $i \in N$. However, when $i = n + 1$, $xy^{n+1}z = 1^{n+n|y|} = 1^{n(1+|y|)}$. Because $|y| > 0$, $n(1 + |y|)$ is not a prime. Therefore, $xy^{n+1}z \notin L$, and L is not regular.

Common mistake:

Some state that $s = 1^p$, where p is a prime number. This is wrong because the pumping length may not be a prime number.