- **Please give details of your calculation. A direct answer without explanation is not counted.**
- **Your answers must be in English.**
- **Please carefully read problem statements.**
- **During the exam you are not allowed to borrow others' class notes.**
- **Try to work on easier questions first.**

Problem 1 (20 pts)

We say w_1 is w_2's subsequence if we can obtain w_1 by removing some symbols in w_2. For example, ad is a subsequence of $abcdc$ by removing the second, the third, and the fifth symbols. As a result, ϵ is a subsequence of any string. Is

$$A = \{\langle C, w\rangle \mid C \text{ is a CFG which can generate at least one subsequence of } w^k \text{ for some } k\}$$

Turing decidable?

If you use any lemma which is not taught in the class, even if it is in the textbook, you need to prove it. This problem can be solved without using any lemma which is not taught.

Answer

Yes, this language is decidable. Assuming $w = w_1 w_2 \cdots w_n$, where $w_i \in \Sigma, \forall i = 1 \ldots n$, we denote $S \equiv \{\text{subsequences of } w^k \text{ for some } k\}$ and $T \equiv \{w_1, w_2, \cdots, w_n\}^*$. Now we show $S = T$. Trivially, $S \subset T$. Besides, for any string $w' \in T$, w' is a subsequence of $w^{|w'|}$ because we can obtain w' by removing all symbols excepts $(w')_i$ in the i-th w of $w^{|w'|}$. Therefore, $T \subset S$. The following Turning Machine decides language A by check if C can accept any string in T.

1. Mark all terminals in $\{w_1, w_2, \ldots, w_n\}$.
2. Repeat until no new variables get marked:
   - Mark any variable V where C has a rule $V \rightarrow U_1 U_2 \cdots U_k$ and each symbol $U_1, \ldots, U_k$ has already been marked.
3. If the start variable is not marked, reject; otherwise, accept.

Problem 2 (20 pts)

Assume

$$f_1(n) = O(2^n) \text{ and } f_2(n) = O(3^n).$$

Is

$$f_1(n) + f_2(n) = O(3^n)?$$

You need to formally prove your answer. You get 0 point if only answering yes/no.

Answer

Yes. Since $f_1(n) = O(2^n)$ and $f_2(n) = O(3^n)$, there exists positive numbers c_1, c_2, n_1, n_2, such that

$$f_1(n) \leq c_1 2^n, \forall n > n_1, \text{ and} \tag{1}$$

$$f_2(n) \leq c_2 3^n, \forall n > n_2. \tag{2}$$

Take $n_0 = \max(n_1, n_2)$ and $c = c_1 + c_2$, then for all $n > n_0$,

$$f_1(n) + f_2(n) < c_1 2^n + c_2 3^n \leq (c_1 + c_2) 3^n.$$

So we have $f_1(n) + f_2(n) = O(3^n)$.

Common mistakes

1. You cannot give c_1, n_1, c_2, n_2 in (1) and (2).
2. You cannot say what $f_1(n)$ and $f_2(n)$ are. For example, you cannot say that $f_1(n) = 2^n$.

Problem 3 (30 pts)

Assume

$$f_1(n) = O(2^n) \text{ and } f_2(n) = o(3^n).$$

Is

$$f_1(n) + f_2(n) = o(3^n)?$$

You need to formally prove your answer. You get 0 point if only answering yes/no.

Answer

Yes. Since $f_1(n) = O(2^n)$, there exists c_1, n_1 such that

$$f_1(n) \leq c_1 2^n, \forall n > n_1.$$

Then

$$\begin{aligned}
&\lim_{n\to\infty} \frac{f_1(n) + f_2(n)}{3^n} \\
\leq &\lim_{n\to\infty} \frac{c_1 2^n + f_2(n)}{3^n} \\
= &\ 0 + 0 = 0.
\end{aligned}$$

Therefore, $f_1(n) + f_2(n) = o(3^n)$.

Problem 4 (20 pts)

Prove whether the following two statements are true or false

(a) If A and B are countable, then $A \cup B$ is also countable

(b) All irrational numbers are not countable. (hint: use the first part of this question)

Answer

(a) Assuming $A = \{a_1, a_2, a_3, \dots\}$ and $B = \{b_1, b_2, b_3, \dots\}$, then $A \cup B = \{a_1, b_1, a_2, b_2, \dots\}$ is also countable. And because $A \cap B$ may not be empty set, we have to skip the elements that have been counted before.

(b) We prove by contradiction. If irrational numbers are countable. Then the union of irrational numbers and rational numbers is also a countable set because rational numbers are countable. However, the union is real number, so it is a contradiction.

Common Mistakes

If you world like to use diagonalization method, you must be able to argue that the obtained number is irrational. Note that a rational number may have infinitely many digits.

Problem 5 (5 pts or -10 pts)

In Chapter Seven (or in the lecture last week) we mentioned one of the greatest unsolved computer science problem. What is it?

If you correctly answer this question, you get 5 points. Otherwise, you get -10.

Answer

NP = P or NP $\neq$ P.

Problem 6 (5 or 25 pts)

(a) (5 pts) Calculate $48,023 \times 89,363$

(b) (Bonus 20 pts) Find $512,973,211 = p \times q$, where $p, q > 1$ and $p, q \in \mathbf{N}$.

Answer

(a) $48,023 \times 89,363 = 4,291,479,349$.

(b) $512,973,211 = 9,133 \times 56,167$.