DURHAM UNIVERSITY

Logic Coursework 2024/25: Written Work

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Module: *Computational Thinking* – Professor: *Prof. Barnaby Martin*
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Question 1

Answer the following questions about complete sets of logical connectives, in each case justifying your answer.
(i) Show $\{\neg, \rightarrow\}$ is a complete set of connectives.
(ii) Show $\{\rightarrow, 0\}$ is a complete set of connectives (where 0 is the constant false).
(iii) Is $\{\text{NAND}, \wedge\}$ a complete set of connectives?
(iv) Is $\{\wedge, \vee\}$ a complete set of connectives?

Answer. In order to determine whether the sets are complete, I will be showing whether $\wedge, \vee$ and $\neg$ can be expressed using the connectives in the set, in which case any logical expression can be written in CNF or DNF, meaning that it's a complete set.

Proof for part (i).
Can $\vee$ be expressed using $\{\neg, \rightarrow\}$?
Yes: $\neg p \rightarrow q \equiv \neg(\neg p) \vee q \equiv p \vee q$
Can $\wedge$ be expressed using $\{\neg, to\}$?
Yes: $\neg(p \rightarrow \neg q) \equiv \neg(\neg p \vee \neg q) \equiv p \wedge q$
Since $\neg$ is already in our set of logical connectives, we can then conclude that $\{\neg, \rightarrow\}$ is a complete set of logical connectives, as any logical expression can be expressed in CNF/DNF using the connectives within the set. □

Proof for part (ii).
Can $\neg$ be expressed using $\{\rightarrow, 0\}$?
Yes: $p \rightarrow 0 \equiv \neg p \vee 0 \equiv \neg p$
Can $\vee$ be expressed using $\{\rightarrow, 0\}$?
Yes: $(p \rightarrow 0) \rightarrow q \equiv \neg(p \rightarrow 0) \vee q \equiv \neg(\neg p) \vee q \equiv p \vee q$
Can $\wedge$ be expressed using $\{\rightarrow, 0\}$?
Yes: $(p \rightarrow (q \rightarrow 0)) \rightarrow 0 \equiv (p \rightarrow (\neg q \vee 0)) \rightarrow 0 \equiv (p \rightarrow \neg q) \rightarrow 0 \equiv (\neg p \vee \neg q) \rightarrow 0 \equiv \neg(\neg p \vee \neg q) \vee 0 \equiv p \wedge q$
Therefore $\{\rightarrow, 0\}$ is a complete set of logical connectives □

Proof for part (iii).
To denote NAND, I will use the symbol: $\barwedge$
Can $\neg$ be expressed using $\{\barwedge, \wedge\}$?
Yes: $p \barwedge p \equiv \neg(p \wedge p) \equiv \neg p$
Can $\vee$ be expressed using $\{\barwedge, \wedge\}$?
Yes: $(p \barwedge p) \barwedge (q \barwedge q) \equiv \neg p \barwedge \neg q \equiv \neg(\neg p \wedge \neg q) \equiv p \vee q$

$\wedge$ is already in the set of logical connectives, therefore the set of logical connectives $\{\text{NAND}, \wedge\}$ is complete. $\square$

Proof for part (iv).
The set of logical connectives $\{\wedge, \vee\}$ is not complete as there is no way to represent one of the propositional variables being equal to zero (i.e: negation) when writing an expression in CNF/DNF. That is to say there is no way to express a tautology or contradiction using these logical connectives due to their property of idempotence. $\square$

Question 2

Convert $(((p \to q) \to r) \to (s \to t))$ to (i) Conjunctive Normal Form (CNF) (ii) Disjunctive Normal Form (DNF)

Answer.
This question is easier to approach by writing the expression $\varphi = (((p \to q) \to r) \to (s \to t))$ in DNF first:

$$\begin{aligned}
\varphi &\equiv (((\neg p \vee q) \to r) \to (s \to t)) \\
&\equiv ((\neg(\neg p \vee q) \vee r) \to (s \to t)) \\
&\equiv (((p \wedge \neg q) \vee r) \to (s \to t)) \\
&\equiv (\neg((p \wedge \neg q) \vee r) \vee (s \to t)) \\
&\equiv ((\neg(p \wedge \neg q) \wedge \neg r) \vee (s \to t)) \\
&\equiv (((\neg p \vee q) \wedge \neg r) \vee (s \to t)) \\
&\equiv (((\neg p \wedge \neg r) \vee (q \wedge \neg r)) \vee (s \to t)) \\
&\equiv ((\neg p \wedge \neg r) \vee (q \wedge \neg r) \vee (\neg s \vee t)) \\
&\equiv (\neg p \wedge \neg r) \vee (q \wedge \neg r) \vee \neg s \vee t \\
\therefore \varphi_{DNF} &= (\neg p \wedge \neg r) \vee (q \wedge \neg r) \vee \neg s \vee t
\end{aligned}$$

Using this, we can repeatedly use the distributive property to convert this to conjunctive normal form:

$$\begin{aligned}
\varphi_{DNF} &= (\neg p \wedge \neg r) \vee (q \wedge \neg r) \vee \neg s \vee t \\
&\equiv ((\neg p \vee q) \wedge \neg r) \vee \neg s \vee t \\
&\equiv (((\neg p \vee q) \vee \neg s) \wedge (\neg r \vee \neg s)) \vee t \\
&\equiv (\neg p \vee q \vee \neg s \vee t) \wedge (\neg r \vee \neg s \vee t) \\
\therefore \varphi_{CNF} &= (\neg p \vee q \vee \neg s \vee t) \wedge (\neg r \vee \neg s \vee t)
\end{aligned}$$

(i) $\varphi_{CNF} = (\neg p \vee q \vee \neg s \vee t) \wedge (\neg r \vee \neg s \vee t)$
(ii) $\varphi_{DNF} = (\neg p \wedge \neg r) \vee (q \wedge \neg r) \vee \neg s \vee t$

Question 3

What is the purpose of Tseitin's Algorithm? Apply Tseitin's Algorithm to turn the propositional formula $(((x_1 \wedge x_2 \wedge x_3) \rightarrow (y_1 \wedge y_2 \wedge y_3)) \vee z)$ to CNF.

Answer. The purpose of Tseitin's Algorithm is to take an arbitrary propositional formula φ, and transform it to a new propositional formula φ' which is equisatisfiable with φ, and in conjunctive normal form.

Question 4

State with justification if each of the following sentences of predicate logic is logically valid.
(i) $(\forall x \exists y \forall z\ E(x,y) \wedge E(y,z)) \rightarrow (\forall x \forall z \exists y\ E(x,y) \wedge E(y,z))$
(ii) $(\forall x \exists y \exists u \forall v\ E(x,y) \wedge E(u,v)) \rightarrow (\exists u \forall v \forall x \exists y\ E(x,y) \wedge E(u,v))$
(iii) $(\forall x \exists y \forall z\ R(x,y,z)) \rightarrow (\exists x \forall y \exists z\ R(x,y,z))$
(iv) $((\forall x \forall y \exists z (E(x,y) \wedge E(y,z))) \rightarrow (\forall x \forall y \forall z (E(x,y) \vee E(y,z))))$

Question 5

Evaluate the given sentence on the respective relation E over domain $\{0,1,2\}$
(i) $\forall x \forall y \forall z \exists w (E(x,w) \wedge E(y,w) \wedge E(z,w))$
(ii) $\exists x \forall y \forall z \exists w (E(x,w) \wedge E(y,w) \wedge E(z,w))$
(iii) $\forall y \exists x \forall z \exists w (E(x,w) \wedge E(y,w) \wedge E(z,w))$
(iv) $\exists x \exists y \exists z \forall w (E(x,w) \wedge E(y,w) \wedge E(z,w))$
(v) $\forall x_1 \exists x_2 \forall y_1 \exists y_2 \forall z_1 \exists z_2 \forall z \exists y\ E(x_1,x_2) \wedge E(x_2,w) \wedge E(y_1,y_2) \wedge E(y_2,w) \wedge E(z_1,z_2) \wedge E(z_2,w) \wedge E(z,w)$
(vi) $\forall x_1 \exists x_2 \forall y_1 \exists y_2 \forall z_1 \forall z \exists z_2 \exists y\ E(x_1,x_2) \wedge E(x_2,w) \wedge E(y_1,y_2) \wedge E(y_2,w) \wedge E(z_1,z_2) \wedge E(z_2,w) \wedge E(z,w)$