DURHAM UNIVERSITY

Logic Coursework 2024/25: Written Work

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Module: *Computational Thinking* – Professor: *Prof. Barnaby Martin*
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Question 1

Answer the following questions about complete sets of logical connectives, in each case justifying your answer.
(i) Show $\{\neg, \to\}$ is a complete set of connectives.
(ii) Show $\{\to, 0\}$ is a complete set of connectives (where 0 is the constant false).
(iii) Is $\{\text{NAND}, \wedge\}$ a complete set of connectives?
(iv) Is $\{\wedge, \vee\}$ a complete set of connectives?

Answer. In order to determine whether the sets are complete, I will be showing whether $\wedge, \vee$ and $\neg$ can be expressed using the connectives in the set, in which case any logical expression can be written in CNF or DNF, meaning that it's a complete set.

Proof for part (i).
Can $\vee$ be expressed using $\{\neg, \to\}$?
Yes: $\neg p \to q \equiv \neg(\neg p) \vee q \equiv p \vee q$
Can $\wedge$ be expressed using $\{\neg, to\}$?
Yes: $\neg(p \to \neg q) \equiv \neg(\neg p \vee \neg q) \equiv p \wedge q$
Since $\neg$ is already in our set of logical connectives, we can then conclude that $\{\neg, \to\}$ is a complete set of logical connectives, as any logical expression can be expressed in CNF/DNF using the connectives within the set. □

Proof for part (ii).
Can $\neg$ be expressed using $\{\to, 0\}$?
Yes: $p \to 0 \equiv \neg p \vee 0 \equiv \neg p$
Can $\vee$ be expressed using $\{\to, 0\}$?
Yes: $(p \to 0) \to q \equiv \neg(p \to 0) \vee q \equiv \neg(\neg p) \vee q \equiv p \vee q$
Can $\wedge$ be expressed using $\{\to, 0\}$?
Yes: $(p \to (q \to 0)) \to 0 \equiv (p \to (\neg q \vee 0)) \to 0 \equiv (p \to \neg q) \to 0 \equiv (\neg p \vee \neg q) \to 0 \equiv \neg(\neg p \vee \neg q) \vee 0 \equiv p \wedge q$
Therefore $\{\to, 0\}$ is a complete set of logical connectives □

Proof for part (iii).
To denote NAND, I will use the symbol: $\overline{\wedge}$
Can $\neg$ be expressed using $\{\overline{\wedge}, \wedge\}$?
Yes: $p \,\overline{\wedge}\, p \equiv \neg(p \wedge p) \equiv \neg p$
Can $\vee$ be expressed using $\{\overline{\wedge}, \wedge\}$?
Yes: $(p \,\overline{\wedge}\, p) \,\overline{\wedge}\, (q \,\overline{\wedge}\, q) \equiv \neg p \,\overline{\wedge}\, \neg q \equiv \neg(\neg p \wedge \neg q) \equiv p \vee q$

$\wedge$ is already in the set of logical connectives, therefore the set of logical connectives $\{\text{NAND}, \wedge\}$ is complete. $\square$

Proof for part (iv).
The set of logical connectives $\{\wedge, \vee\}$ is not complete as there is no way to represent one of the propositional variables being equal to zero (i.e: negation) when writing an expression in CNF/DNF. That is to say there is no way to express a tautology or contradiction using these logical connectives due to their property of idempotence. $\square$

Question 2

Convert $(((p \rightarrow q) \rightarrow r) \rightarrow (s \rightarrow t))$ to (i) Conjunctive Normal Form (CNF) (ii) Disjunctive Normal Form (DNF)

Answer.
This question is easier to approach by writing the expression $\varphi = (((p \rightarrow q) \rightarrow r) \rightarrow (s \rightarrow t))$ in DNF first:

$$\begin{aligned}
\varphi &\equiv (((\neg p \vee q) \rightarrow r) \rightarrow (s \rightarrow t)) \\
&\equiv ((\neg(\neg p \vee q) \vee r) \rightarrow (s \rightarrow t)) \\
&\equiv (((p \wedge \neg q) \vee r) \rightarrow (s \rightarrow t)) \\
&\equiv (\neg((p \wedge \neg q) \vee r) \vee (s \rightarrow t)) \\
&\equiv ((\neg(p \wedge \neg q) \wedge \neg r) \vee (s \rightarrow t)) \\
&\equiv (((\neg p \vee q) \wedge \neg r) \vee (s \rightarrow t)) \\
&\equiv (((\neg p \wedge \neg r) \vee (q \wedge \neg r)) \vee (s \rightarrow t)) \\
&\equiv ((\neg p \wedge \neg r) \vee (q \wedge \neg r) \vee (\neg s \vee t)) \\
&\equiv (\neg p \wedge \neg r) \vee (q \wedge \neg r) \vee \neg s \vee t \\
\therefore \varphi_{DNF} &= (\neg p \wedge \neg r) \vee (q \wedge \neg r) \vee \neg s \vee t
\end{aligned}$$

Using this, we can repeatedly use the distributive property to convert this to conjunctive normal form:

$$\begin{aligned}
\varphi_{DNF} &= (\neg p \wedge \neg r) \vee (q \wedge \neg r) \vee \neg s \vee t \\
&\equiv ((\neg p \vee q) \wedge \neg r) \vee \neg s \vee t \\
&\equiv (((\neg p \vee q) \vee \neg s) \wedge (\neg r \vee \neg s)) \vee t \\
&\equiv (\neg p \vee q \vee \neg s \vee t) \wedge (\neg r \vee \neg s \vee t) \\
\therefore \varphi_{CNF} &= (\neg p \vee q \vee \neg s \vee t) \wedge (\neg r \vee \neg s \vee t)
\end{aligned}$$

(i) $\varphi_{CNF} = (\neg p \vee q \vee \neg s \vee t) \wedge (\neg r \vee \neg s \vee t)$
(ii) $\varphi_{DNF} = (\neg p \wedge \neg r) \vee (q \wedge \neg r) \vee \neg s \vee t$

Question 3

What is the purpose of Tseitin's Algorithm? Apply Tseitin's Algorithm to turn the propositional formula $(((x_1 \wedge x_2 \wedge x_3) \to (y_1 \wedge y_2 \wedge y_3)) \vee z)$ to CNF.

Answer. The purpose of Tseitin's Algorithm is to take an arbitrary propositional formula φ, and transform it to a new propositional formula φ' which is equisatisfiable with φ, and in conjunctive normal form.

$$\text{Let } \varphi = (((x_1 \wedge x_2 \wedge x_3) \to (y_1 \wedge y_2 \wedge y_3)) \vee z)$$

Introduce new variables for each subformula:

$$\alpha_1 \leftrightarrow x_1 \wedge x_2 \wedge x_3$$
$$\alpha_2 \leftrightarrow y_1 \wedge y_2 \wedge y_3$$
$$\alpha_3 \leftrightarrow z$$
$$\alpha_4 \leftrightarrow \alpha_1 \to \alpha_2$$
$$\alpha_5 \leftrightarrow \alpha_4 \vee \alpha_3$$

Write each expression as conjunctions

From α_1 :

$$\alpha_1 \leftrightarrow (x_1 \wedge x_2 \wedge x_3) \equiv (\alpha_1 \to (x_1 \wedge x_2 \wedge x_3)) \wedge (\alpha_1 \leftarrow (x_1 \wedge x_2 \wedge x_3))$$
$$\equiv (\neg\alpha_1 \vee (x_1 \wedge x_2 \wedge x_3)) \wedge (\alpha_1 \vee \neg(x_1 \wedge x_2 \wedge x_3))$$
$$\equiv (\neg\alpha_1 \vee x_1) \wedge (\neg\alpha_1 \vee x_2) \wedge (\neg\alpha_1 \vee x_3) \wedge (\alpha_1 \vee \neg x_1 \vee \neg x_2 \vee \neg x_3)$$

Similarly for α_2 :

$$\alpha_2 \leftrightarrow (y_1 \wedge y_2 \wedge y_3) \equiv (\neg\alpha_2 \vee y_1) \wedge (\neg\alpha_2 \vee y_2) \wedge (\neg\alpha_2 \vee y_3) \wedge (\alpha_2 \vee \neg y_1 \vee \neg y_2 \vee \neg y_3)$$

For α_3 :

$$\alpha_3 \leftrightarrow z \equiv (\alpha_3 \to z) \wedge (\alpha_3 \leftarrow z)$$
$$\equiv (\neg\alpha_3 \vee z) \wedge (\alpha_3 \vee \neg z)$$

For α_4 :

$$\alpha_4 \leftrightarrow \alpha_1 \to \alpha_2 \equiv (\alpha_4 \to (\alpha_1 \to \alpha_2)) \wedge (\alpha_4 \leftarrow (\alpha_1 \to \alpha_2))$$
$$\equiv (\neg\alpha_4 \vee \neg\alpha_1 \vee \alpha_2) \wedge (\neg(\alpha_1 \to \alpha_2) \vee \alpha_4)$$
$$\equiv (\neg\alpha_4 \vee \neg\alpha_1 \vee \alpha_2) \wedge (\neg(\neg\alpha_1 \vee \alpha_2) \vee \alpha_4)$$
$$\equiv (\neg\alpha_4 \vee \neg\alpha_1 \vee \alpha_2) \wedge ((\alpha_1 \wedge \neg\alpha_2) \vee \alpha_4)$$
$$\equiv (\neg\alpha_4 \vee \neg\alpha_1 \vee \alpha_2) \wedge (\alpha_1 \vee \alpha_4) \wedge (\neg\alpha_2 \vee \alpha_4)$$

For α_5 :

$$\alpha_5 \leftrightarrow \alpha_4 \vee \alpha_3 \equiv (\alpha_5 \to (\alpha_4 \vee \alpha_3)) \wedge (\alpha_5 \leftarrow (\alpha_4 \vee \alpha_3))$$
$$\equiv (\neg\alpha_5 \vee \alpha_4 \vee \alpha_3) \wedge (\neg(\alpha_4 \vee \alpha_3) \vee \alpha_5)$$
$$\equiv (\neg\alpha_5 \vee \alpha_4 \vee \alpha_3) \wedge ((\neg\alpha_4 \wedge \neg\alpha_3) \vee \alpha_5)$$
$$\equiv (\neg\alpha_5 \vee \alpha_4 \vee \alpha_3) \wedge (\neg\alpha_4 \vee \alpha_5) \wedge (\neg\alpha_3 \vee \alpha_5)$$

The conjunction of all these variables and the clause α_5

gives us the Tseitin Transformation of φ'

To save space, I will write this as a clause set

$$\therefore \varphi' = \{\{\alpha_5\}, \{\neg\alpha_1, x_1\}, \{\neg\alpha_1, x_2\}, \{\neg\alpha_1, x_3\}, \{\alpha_1, \neg x_1, \neg x_2, \neg x_3\}, \{\neg\alpha_2, y_1\}, \{\neg\alpha_2, y_2, \}$$
$$\{\neg\alpha_2, y_3\}, \{\alpha_2, \neg y_1, \neg y_2, \neg y_3\}, \{\neg\alpha_3, z\}, \{\alpha_3, \neg z\}, \{\neg\alpha_4, \neg\alpha_1, \alpha_2\}, \{\alpha_1, \alpha_4\}, \{\neg\alpha_2, \alpha_4\}, \{\neg\alpha_5, \alpha_4, \alpha_3\}$$
$$\{\neg\alpha_4, \alpha_5\}, \{\neg\alpha_3, \alpha_5\}\}$$

Question 4

State with justification if each of the following sentences of predicate logic is logically valid.
(i) $(\forall x \exists y \forall z\ E(x,y) \wedge E(y,z)) \rightarrow (\forall x \forall z \exists y\ E(x,y) \wedge E(y,z))$
(ii) $(\forall x \exists y \exists u \forall v\ E(x,y) \wedge E(u,v)) \rightarrow (\exists u \forall v \forall x \exists y\ E(x,y) \wedge E(u,v))$
(iii) $(\forall x \exists y \forall z\ R(x,y,z)) \rightarrow (\exists x \forall y \exists z\ R(x,y,z))$
(iv) $((\forall x \forall y \exists z (E(x,y) \wedge E(y,z))) \rightarrow (\forall x \forall y \forall z (E(x,y) \vee E(y,z))))$

Question 5

Evaluate the given sentence on the respective relation E over domain $\{0,1,2\}$ with relation $E := \{(0,1),(1,0),(1,2),(2,1),(2,0),(0,2)\}$

(i) $\forall x \forall y \forall z \exists w (E(x,w) \wedge E(y,w) \wedge E(z,w))$
(ii) $\exists x \forall y \forall z \exists w (E(x,w) \wedge E(y,w) \wedge E(z,w))$
(iii) $\forall y \exists x \forall z \exists w (E(x,w) \wedge E(y,w) \wedge E(z,w))$
(iv) $\exists x \exists y \exists z \forall w (E(x,w) \wedge E(y,w) \wedge E(z,w))$
(v) $\forall x_1 \exists x_2 \forall y_1 \exists y_2 \forall z_1 \exists z_2 \forall z \exists y\ E(x_1,x_2) \wedge E(x_2,w) \wedge E(y_1,y_2) \wedge E(y_2,w) \wedge E(z_1,z_2) \wedge E(z_2,w) \wedge E(z,w)$
(vi) $\forall x_1 \exists x_2 \forall y_1 \exists y_2 \forall z_1 \forall z \exists z_2 \exists y\ E(x_1,x_2) \wedge E(x_2,w) \wedge E(y_1,y_2) \wedge E(y_2,w) \wedge E(z_1,z_2) \wedge E(z_2,w) \wedge E(z,w)$

Answer.
For the purposes of this question, it will be easier to express the relation E as such:

$$E := \{(u,v) \in \{0,1,2\} : u \neq v\}$$

(i) This sentence can be refuted by assigning the following variables:

$$x = 0, y = 1, z = 2$$

That is to say, the expression inside the brackets requires that $x \neq w, y \neq w, z \neq w$. However if we assign x, y, z uniquely, by the pigeonhole principle, it follows that there will exist no such w, that satisfies the sentence.

(ii)