# Logic Coursework 2024/25: Written Work

Student name: *Huseyin Emre Ozden*

Module: *Computational Thinking* – Professor: *Prof. Barnaby Martin*
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### Question 1

> Answer the following questions about complete sets of logical connectives, in each case justifying your answer.
> (i) Show $\{\neg, \rightarrow\}$ is a complete set of connectives.
> (ii) Show $\{\rightarrow, 0\}$ is a complete set of connectives (where 0 is the constant false).
> (iii) Is $\{\text{NAND}, \wedge\}$ a complete set of connectives?
> (iv) Is $\{\wedge, \vee\}$ a complete set of connectives?

**Answer.** In order to determine whether the sets are complete, I will be showing whether $\wedge, \vee$ and $\neg$ can be expressed using the connectives in the set, in which case any logical expression can be written in CNF or DNF, meaning that it's a complete set.

*Proof for part (i).*
Can $\vee$ be expressed using $\{\neg, \rightarrow\}$?
Yes: $\neg p \rightarrow q \equiv \neg(\neg p) \vee q \equiv p \vee q$
Can $\wedge$ be expressed using $\{\neg, to\}$?
Yes: $\neg(p \rightarrow \neg q) \equiv \neg(\neg p \vee \neg q) \equiv p \wedge q$
Since $\neg$ is already in our set of logical connectives, we can then conclude that $\{\neg, \rightarrow\}$ is a complete set of logical connectives, as any logical expression can be expressed in CNF/DNF using the connectives within the set. $\qquad\square$

*Proof for part (ii).*
Can $\neg$ be expressed using $\{\rightarrow, 0\}$?
Yes: $p \rightarrow 0 \equiv \neg p \vee 0 \equiv \neg p$
Can $\vee$ be expressed using $\{\rightarrow, 0\}$?
Yes: $(p \rightarrow 0) \rightarrow q \equiv \neg(p \rightarrow 0) \vee q \equiv \neg(\neg p) \vee q \equiv p \vee q$
Can $\wedge$ be expressed using $\{\rightarrow, 0\}$?
Yes: $(p \rightarrow (q \rightarrow 0)) \rightarrow 0 \equiv (p \rightarrow (\neg q \vee 0)) \rightarrow 0 \equiv (p \rightarrow \neg q) \rightarrow 0 \equiv (\neg p \vee \neg q) \rightarrow 0 \equiv \neg(\neg p \vee \neg q) \vee 0 \equiv p \wedge q$
Therefore $\{\rightarrow, 0\}$ is a complete set of logical connectives $\qquad\square$

*Proof for part (iii).*
To denote NAND, I will use the symbol: $\overline{\wedge}$
Can $\neg$ be expressed using $\{\overline{\wedge}, \wedge\}$?
Yes: $p \mathbin{\overline{\wedge}} p \equiv \neg(p \wedge p) \equiv \neg p$
Can $\vee$ be expressed using $\{\overline{\wedge}, \wedge\}$?
Yes: $(p \mathbin{\overline{\wedge}} p) \mathbin{\overline{\wedge}} (q \mathbin{\overline{\wedge}} q) \equiv \neg p \mathbin{\overline{\wedge}} \neg q \equiv \neg(\neg p \wedge \neg q) \equiv p \vee q$

$\wedge$ is already in the set of logical connectives, therefore the set of logical connectives $\{\text{NAND}, \wedge\}$ is complete. $\qquad\square$

*Proof for part (iv).*
The set of logical connectives $\{\wedge, \vee\}$ is not complete as there is no way to represent one of the propositional variables being equal to zero (i.e: negation) when writing an expression in CNF/DNF. That is to say there is no way to express a tautology or contradiction using these logical connectives due to their property of idempotence. $\quad\square$

## Question 2

Convert $(((p \to q) \to r) \to (s \to t))$ to
(i) Conjunctive Normal Form (CNF)
(ii) Disjunctive Normal Form (DNF)

**Answer.**
This question is easier to approach by writing the expression $\varphi = (((p \to q) \to r) \to (s \to t))$ in DNF first:

$$
\begin{aligned}
\varphi &\equiv (((\neg p \vee q) \to r) \to (s \to t)) \\
&\equiv ((\neg(\neg p \vee q) \vee r) \to (s \to t)) \\
&\equiv (((p \wedge \neg q) \vee r) \to (s \to t)) \\
&\equiv (\neg((p \wedge \neg q) \vee r) \vee (s \to t)) \\
&\equiv ((\neg(p \wedge \neg q) \wedge \neg r) \vee (s \to t)) \\
&\equiv (((\neg p \vee q) \wedge \neg r) \vee (s \to t)) \\
&\equiv (((\neg p \wedge \neg r) \vee (q \wedge \neg r)) \vee (s \to t)) \\
&\equiv ((\neg p \wedge \neg r) \vee (q \wedge \neg r) \vee (\neg s \vee t)) \\
&\equiv (\neg p \wedge \neg r) \vee (q \wedge \neg r) \vee \neg s \vee t \\
\therefore \varphi_{DNF} &= (\neg p \wedge \neg r) \vee (q \wedge \neg r) \vee \neg s \vee t
\end{aligned}
$$

    Using this, we can repeatedly use the distributive property to convert this to conjunctive normal form:

$$
\begin{aligned}
\varphi_{DNF} &= (\neg p \wedge \neg r) \vee (q \wedge \neg r) \vee \neg s \vee t \\
&\equiv ((\neg p \vee q) \wedge \neg r) \vee \neg s \vee t \\
&\equiv (((\neg p \vee q) \vee \neg s) \wedge (\neg r \vee \neg s)) \vee t \\
&\equiv (\neg p \vee q \vee \neg s \vee t) \wedge (\neg r \vee \neg s \vee t) \\
\therefore \varphi_{CNF} &= (\neg p \vee q \vee \neg s \vee t) \wedge (\neg r \vee \neg s \vee t)
\end{aligned}
$$

(i) $\varphi_{CNF} = (\neg p \vee q \vee \neg s \vee t) \wedge (\neg r \vee \neg s \vee t)$
(ii) $\varphi_{DNF} = (\neg p \wedge \neg r) \vee (q \wedge \neg r) \vee \neg s \vee t$

**Question 3**

What is the purpose of Tseitin's Algorithm? Apply Tseitin's Algorithm to turn the propositional formula $(((x_1 \wedge x_2 \wedge x_3) \rightarrow (y_1 \wedge y_2 \wedge y_3)) \vee z)$ to CNF.

**Answer.**

The purpose of Tseitin's Algorithm is to take an arbitrary propositional formula $\varphi$, and transform it to a new propositional formula $\varphi'$ which is equisatisfiable with $\varphi$, and in conjunctive normal form.

$$\text{Let } \varphi = (((x_1 \wedge x_2 \wedge x_3) \rightarrow (y_1 \wedge y_2 \wedge y_3)) \vee z)$$

Introduce new variables for each subformula:

$$\alpha_1 \leftrightarrow x_1 \wedge x_2 \wedge x_3$$
$$\alpha_2 \leftrightarrow y_1 \wedge y_2 \wedge y_3$$
$$\alpha_3 \leftrightarrow \alpha_1 \rightarrow \alpha_2$$
$$\alpha_4 \leftrightarrow \alpha_3 \vee z$$

Write each expression as conjunctions

From $\alpha_1$ :

$$\alpha_1 \leftrightarrow (x_1 \wedge x_2 \wedge x_3) \equiv (\alpha_1 \rightarrow (x_1 \wedge x_2 \wedge x_3)) \wedge (\alpha_1 \leftarrow (x_1 \wedge x_2 \wedge x_3))$$
$$\equiv (\neg\alpha_1 \vee (x_1 \wedge x_2 \wedge x_3)) \wedge (\alpha_1 \vee \neg(x_1 \wedge x_2 \wedge x_3))$$
$$\equiv (\neg\alpha_1 \vee x_1) \wedge (\neg\alpha_1 \vee x_2) \wedge (\neg\alpha_1 \vee x_3) \wedge (\alpha_1 \vee \neg x_1 \vee \neg x_2 \vee \neg x_3)$$

Similarly for $\alpha_2$ :

$$\alpha_2 \leftrightarrow (y_1 \wedge y_2 \wedge y_3) \equiv (\neg\alpha_2 \vee y_1) \wedge (\neg\alpha_2 \vee y_2) \wedge (\neg\alpha_2 \vee y_3) \wedge (\alpha_2 \vee \neg y_1 \vee \neg y_2 \vee \neg y_3)$$

For $\alpha_3$ :

$$\alpha_3 \leftrightarrow (\alpha_1 \rightarrow \alpha_2) \equiv (\alpha_3 \rightarrow (\alpha_1 \rightarrow \alpha_2)) \wedge (\alpha_3 \leftarrow (\alpha_1 \rightarrow \alpha_2))$$
$$\equiv (\neg\alpha_3 \vee (\neg\alpha_1 \vee \alpha_2)) \wedge (\neg(\neg\alpha_1 \vee \alpha_2) \vee \alpha_3) \equiv (\neg\alpha_3 \vee \neg\alpha_1 \vee \alpha_2) \wedge ((\alpha_1 \wedge \neg\alpha_2) \vee \alpha_3)$$
$$\equiv (\neg\alpha_1 \vee \alpha_2 \vee \neg\alpha_3) \wedge (\alpha_1 \vee \alpha_3) \wedge (\neg\alpha_2 \vee \alpha_3)$$

For $\alpha_4$ :

$$\alpha_4 \leftrightarrow (\alpha_3 \vee z) \equiv (\alpha_4 \rightarrow (\alpha_3 \vee z)) \wedge (\alpha_4 \leftarrow (\alpha_3 \vee z))$$
$$\equiv (\neg\alpha_4 \vee \alpha_3 \vee z) \wedge (\neg(\alpha_3 \vee z) \vee \alpha_4) \equiv (\neg\alpha_4 \vee \alpha_3 \vee z) \wedge ((\neg\alpha_3 \wedge \neg z) \vee \alpha_4)$$
$$\equiv (\neg\alpha_4 \vee \alpha_3 \vee z) \wedge (\neg\alpha_3 \vee \alpha_4) \wedge (\neg z \vee \alpha_4)$$

The conjunction of all these variables and the clause $\alpha_4$

gives us the Tseitin Transformation of $\varphi'$

To save space, I will write this as a clause set

$$\therefore \varphi' = \{\{\alpha_4\}, \{-\alpha_1, x_1\}, \{-\alpha_1, x_2\}, \{-\alpha_1, x_3\}, \{\alpha_1, -x_1, -x_2, -x_3\},$$
$$\{-\alpha_2, y_1\}, \{-\alpha_2, y_2\}, \{-\alpha_2, y_3\}, \{\alpha_2, -y_1, -y_2, -y_3\},$$
$$\{-\alpha_1, \alpha_2, -\alpha_3\}, \{\alpha_1, \alpha_3\}, \{-\alpha_2, \alpha_3\}, \{-\alpha_4, \alpha_3, z\}, \{-\alpha_3, \alpha_4\}, \{-z, \alpha_4\}\}$$

## Question 4

State with justification if each of the following sentences of predicate logic is logically valid.
(i) $(\forall x \exists y \forall z (E(x,y) \wedge E(y,z))) \rightarrow (\forall x \forall z \exists y (E(x,y) \wedge E(y,z)))$
(ii) $(\forall x \exists y \exists u \forall v (E(x,y) \wedge E(u,v))) \rightarrow (\exists u \forall v \forall x \exists y (E(x,y) \wedge E(u,v)))$
(iii) $(\forall x \exists y \forall z\, R(x,y,z)) \rightarrow (\exists x \forall y \exists z\, R(x,y,z))$
(iv) $(\forall x \forall y \exists z (E(x,y) \wedge E(y,z))) \rightarrow (\forall x \forall y \forall z (E(x,y) \vee E(y,z)))$

**Answer.**
(i) Logically Valid.

*Proof.*                                               □

(ii) Logically Valid.

*Proof.* We can begin by considering an interpretation $I$:

$$I \models \forall x \exists y \exists u \forall v (E(x,y) \wedge E(u,v)) \iff I \models \forall x \exists y E(x,y) \wedge \exists u \forall v E(u,v)$$
$$\iff I \models \exists u \forall v \forall x \exists y (E(x,y) \wedge E(u,v))$$

Hence it follows that $\forall x \exists y \exists u \forall v (E(x,y) \wedge E(u,v)) \equiv \exists u \forall v \forall x \exists y (E(x,y) \wedge E(u,v))$. Since the antecedent and consequent are logically equivalent, it follows that this sentence is a tautology, therefore logically valid. □

(iii) Logically Invalid.

*Proof.* In order to demonstrate that this is a logically invalid sentence, I will provide a counter-model: Consider $R$ a terenary relation over the domain $\{0, 1\}$, where

$$R := \{(0,0,0), (0,0,1), (1,0,0), (1,0,1)\}$$

From this we can see, that for all values of x, there in fact is a y, for all z. However there doesn't exist an $x$ for all values of $y$, where there exists a $z$. This is because $y$ only appears as a 0 in the relation, hence $y$ cannot take each value in the domain in this relation. □

(iv) Logically Valid.

*Proof.*                                                □

## Question 5

Evaluate the given sentence on the respective relation $E$ over domain $\{0,1,2\}$ with relation $E := \{(0,1),(1,0),(1,2),(2,1),(2,0),(0,2)\}$

(i) $\forall x \forall y \forall z \exists w (E(x,w) \land E(y,w) \land E(z,w))$

(ii) $\exists x \forall y \forall z \exists w (E(x,w) \land E(y,w) \land E(z,w))$

(iii) $\forall y \exists x \forall z \exists w (E(x,w) \land E(y,w) \land E(z,w))$

(iv) $\exists x \exists y \exists z \forall w (E(x,w) \land E(y,w) \land E(z,w))$

(v) $\forall x_1 \exists x_2 \forall y_1 \exists y_2 \forall z_1 \exists z_2 \forall z \exists w \, E(x_1,x_2) \land E(x_2,w) \land E(y_1,y_2) \land E(y_2,w) \land E(z_1,z_2) \land E(z_2,w) \land E(z,w)$

(vi) $\forall x_1 \exists x_2 \forall y_1 \exists y_2 \forall z_1 \forall z \exists z_2 \exists w \, E(x_1,x_2) \land E(x_2,w) \land E(y_1,y_2) \land E(y_2,w) \land E(z_1,z_2) \land E(z_2,w) \land E(z,w)$

**Answer.**
For the purposes of this question, it will be easier to express the relation $E$ as such:

$$E := \{(u,v) \in \{0,1,2\} : u \neq v\}$$

(i) False. This sentence can be refuted by assigning the following variables:

$$x = 0, y = 1, z = 2$$

That is to say, the expression inside the brackets requires that $x \neq w, y \neq w, z \neq w$. However if we assign $x, y, z$ uniquely, by the pigeonhole principle, it follows that there will exist no such $w$, that satisfies the sentence.

(ii) False. This sentence can be refuted by considering the fact that if $x, y, z$ are unique, then there will exist no $w$, such that the conjunction of the atomic formulae $E(x,w), E(y,w), E(z,w)$ are satisfied. From the domain of the interpretation, we can deduce that there is no such value of $x$.

(iii) True.

(iv) True. In this interpretation, for the sentence to be true, there must exist $x, y, z$ for all $w$, such that none of $x, y, z$ is equal to $w$. It then becomes clear that for each $w$, we can choose $x, y, z$ such that $x = y$ or $x = z$ or $y = z$ and $x, y, z \neq w$.

(v) False.

(vi) False.