

# Logic Coursework 2024/25: Written Work

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## Question 1

Answer the following questions about complete sets of logical connectives, in each case justifying your answer.

- (i) Show  $\{\neg, \rightarrow\}$  is a complete set of connectives.
- (ii) Show  $\{\rightarrow, 0\}$  is a complete set of connectives (where 0 is the constant false).
- (iii) Is  $\{\text{NAND}, \wedge\}$  a complete set of connectives?
- (iv) Is  $\{\wedge, \vee\}$  a complete set of connectives?

**Answer.** In order to determine whether the sets are complete, I will be showing whether  $\wedge, \vee$  and  $\neg$  can be expressed using the connectives in the set, in which case any logical expression can be written in CNF or DNF, meaning that it's a complete set.

*Proof for part (i).*

Can  $\vee$  be expressed using  $\{\neg, \rightarrow\}$ ?

Yes:  $\neg p \rightarrow q \equiv \neg(\neg p) \vee q \equiv p \vee q$

Can  $\wedge$  be expressed using  $\{\neg, \rightarrow\}$ ?

Yes:  $\neg(p \rightarrow \neg q) \equiv \neg(\neg p \vee \neg q) \equiv p \wedge q$

Since  $\neg$  is already in our set of logical connectives, we can then conclude that  $\{\neg, \rightarrow\}$  is a complete set of logical connectives, as any logical expression can be expressed in CNF/DNF using the connectives within the set.  $\square$

*Proof for part (ii).*

Can  $\neg$  be expressed using  $\{\rightarrow, 0\}$ ?

Yes:  $p \rightarrow 0 \equiv \neg p \vee 0 \equiv \neg p$

Can  $\vee$  be expressed using  $\{\rightarrow, 0\}$ ?

Yes:  $(p \rightarrow 0) \rightarrow q \equiv \neg(p \rightarrow 0) \vee q \equiv \neg(\neg p) \vee q \equiv p \vee q$

Can  $\wedge$  be expressed using  $\{\rightarrow, 0\}$ ?

Yes:  $(p \rightarrow (q \rightarrow 0)) \rightarrow 0 \equiv (p \rightarrow (\neg q \vee 0)) \rightarrow 0 \equiv (p \rightarrow \neg q) \rightarrow 0 \equiv (\neg p \vee \neg q) \rightarrow 0 \equiv \neg(\neg p \vee \neg q) \vee 0 \equiv p \wedge q$

Therefore  $\{\rightarrow, 0\}$  is a complete set of logical connectives  $\square$

*Proof for part (iii).*

To denote NAND, I will use the symbol:  $\bar{\wedge}$

Can  $\neg$  be expressed using  $\{\bar{\wedge}, \wedge\}$ ?

Yes:  $p \bar{\wedge} p \equiv \neg(p \wedge p) \equiv \neg p$

Can  $\vee$  be expressed using  $\{\bar{\wedge}, \wedge\}$ ?

Yes:  $(p \bar{\wedge} p) \bar{\wedge} (q \bar{\wedge} q) \equiv \neg p \bar{\wedge} \neg q \equiv \neg(\neg p \wedge \neg q) \equiv p \vee q$

$\wedge$  is already in the set of logical connectives, therefore the set of logical connectives  $\{\text{NAND}, \wedge\}$  is complete.  $\square$

*Proof for part (iv).*

The set of logical connectives  $\{\wedge, \vee\}$  is not complete as there is no way to represent one of the propositional variables being equal to zero (i.e: negation) when writing an expression in CNF/DNF. That is to say there is no way to express a tautology or contradiction using these logical connectives due to their property of idempotence.  $\square$

## Question 2

Convert  $((p \rightarrow q) \rightarrow r) \rightarrow (s \rightarrow t)$  to

- (i) Conjunctive Normal Form (CNF)
- (ii) Disjunctive Normal Form (DNF)

### Answer.

This question is easier to approach by writing the expression  $\varphi = ((p \rightarrow q) \rightarrow r) \rightarrow (s \rightarrow t)$  in DNF first:

$$\begin{aligned}
 \varphi &\equiv (((\neg p \vee q) \rightarrow r) \rightarrow (s \rightarrow t)) \\
 &\equiv ((\neg(\neg p \vee q) \vee r) \rightarrow (s \rightarrow t)) \\
 &\equiv (((p \wedge \neg q) \vee r) \rightarrow (s \rightarrow t)) \\
 &\equiv (\neg((p \wedge \neg q) \vee r) \vee (s \rightarrow t)) \\
 &\equiv ((\neg(p \wedge \neg q) \wedge \neg r) \vee (s \rightarrow t)) \\
 &\equiv (((\neg p \vee q) \wedge \neg r) \vee (s \rightarrow t)) \\
 &\equiv (((\neg p \wedge \neg r) \vee (q \wedge \neg r)) \vee (s \rightarrow t)) \\
 &\equiv ((\neg p \wedge \neg r) \vee (q \wedge \neg r) \vee (\neg s \vee t)) \\
 &\equiv (\neg p \wedge \neg r) \vee (q \wedge \neg r) \vee \neg s \vee t \\
 \therefore \varphi_{DNF} &= (\neg p \wedge \neg r) \vee (q \wedge \neg r) \vee \neg s \vee t
 \end{aligned}$$

Using this, we can repeatedly use the distributive property to convert this to conjunctive normal form:

$$\begin{aligned}
 \varphi_{DNF} &= (\neg p \wedge \neg r) \vee (q \wedge \neg r) \vee \neg s \vee t \\
 &\equiv ((\neg p \vee q) \wedge \neg r) \vee \neg s \vee t \\
 &\equiv (((\neg p \vee q) \vee \neg s) \wedge (\neg r \vee \neg s)) \vee t \\
 &\equiv (\neg p \vee q \vee \neg s \vee t) \wedge (\neg r \vee \neg s \vee t) \\
 \therefore \varphi_{CNF} &= (\neg p \vee q \vee \neg s \vee t) \wedge (\neg r \vee \neg s \vee t)
 \end{aligned}$$

$$(i) \varphi_{CNF} = (\neg p \vee q \vee \neg s \vee t) \wedge (\neg r \vee \neg s \vee t)$$

$$(ii) \varphi_{DNF} = (\neg p \wedge \neg r) \vee (q \wedge \neg r) \vee \neg s \vee t$$

### Question 3

What is the purpose of Tseitin's Algorithm? Apply Tseitin's Algorithm to turn the propositional formula  $((x_1 \wedge x_2 \wedge x_3) \rightarrow (y_1 \wedge y_2 \wedge y_3)) \vee z$  to CNF.

#### Answer.

The purpose of Tseitin's Algorithm is to take an arbitrary propositional formula  $\varphi$ , and transform it to a new propositional formula  $\varphi'$  which is equisatisfiable with  $\varphi$ , and in conjunctive normal form.

Let  $\varphi = (((x_1 \wedge x_2 \wedge x_3) \rightarrow (y_1 \wedge y_2 \wedge y_3)) \vee z)$

Introduce new variables for each subformula:

$$\alpha_1 \leftrightarrow x_1 \wedge x_2 \wedge x_3$$

$$\alpha_2 \leftrightarrow y_1 \wedge y_2 \wedge y_3$$

$$\alpha_3 \leftrightarrow \alpha_1 \rightarrow \alpha_2$$

$$\alpha_4 \leftrightarrow \alpha_3 \vee z$$

Write each expression as conjunctions

From  $\alpha_1$  :

$$\begin{aligned} \alpha_1 \leftrightarrow (x_1 \wedge x_2 \wedge x_3) &\equiv (\alpha_1 \rightarrow (x_1 \wedge x_2 \wedge x_3)) \wedge (\alpha_1 \leftarrow (x_1 \wedge x_2 \wedge x_3)) \\ &\equiv (\neg \alpha_1 \vee (x_1 \wedge x_2 \wedge x_3)) \wedge (\alpha_1 \vee \neg(x_1 \wedge x_2 \wedge x_3)) \\ &\equiv (\neg \alpha_1 \vee x_1) \wedge (\neg \alpha_1 \vee x_2) \wedge (\neg \alpha_1 \vee x_3) \wedge (\alpha_1 \vee \neg x_1 \vee \neg x_2 \vee \neg x_3) \end{aligned}$$

Similarly for  $\alpha_2$  :

$$\alpha_2 \leftrightarrow (y_1 \wedge y_2 \wedge y_3) \equiv (\neg \alpha_2 \vee y_1) \wedge (\neg \alpha_2 \vee y_2) \wedge (\neg \alpha_2 \vee y_3) \wedge (\alpha_2 \vee \neg y_1 \vee \neg y_2 \vee \neg y_3)$$

For  $\alpha_3$  :

$$\begin{aligned} \alpha_3 \leftrightarrow (\alpha_1 \rightarrow \alpha_2) &\equiv (\alpha_3 \rightarrow (\alpha_1 \rightarrow \alpha_2)) \wedge (\alpha_3 \leftarrow (\alpha_1 \rightarrow \alpha_2)) \\ &\equiv (\neg \alpha_3 \vee (\neg \alpha_1 \vee \alpha_2)) \wedge (\neg(\neg \alpha_1 \vee \alpha_2) \vee \alpha_3) \equiv (\neg \alpha_3 \vee \neg \alpha_1 \vee \alpha_2) \wedge ((\alpha_1 \wedge \neg \alpha_2) \vee \alpha_3) \\ &\equiv (\neg \alpha_1 \vee \alpha_2 \vee \neg \alpha_3) \wedge (\alpha_1 \vee \alpha_3) \wedge (\neg \alpha_2 \vee \alpha_3) \end{aligned}$$

For  $\alpha_4$  :

$$\begin{aligned} \alpha_4 \leftrightarrow (\alpha_3 \vee z) &\equiv (\alpha_4 \rightarrow (\alpha_3 \vee z)) \wedge (\alpha_4 \leftarrow (\alpha_3 \vee z)) \\ &\equiv (\neg \alpha_4 \vee \alpha_3 \vee z) \wedge (\neg(\alpha_3 \vee z) \vee \alpha_4) \equiv (\neg \alpha_4 \vee \alpha_3 \vee z) \wedge ((\neg \alpha_3 \wedge \neg z) \vee \alpha_4) \\ &\equiv (\neg \alpha_4 \vee \alpha_3 \vee z) \wedge (\neg \alpha_3 \vee \alpha_4) \wedge (\neg z \vee \alpha_4) \end{aligned}$$

The conjunction of all these variables and the clause  $\alpha_4$

gives us the Tseitin Transformation of  $\varphi'$

To save space, I will write this as a clause set

$$\begin{aligned} \therefore \varphi' = \{ &\{\alpha_4\}, \{\neg \alpha_1, x_1\}, \{\neg \alpha_1, x_2\}, \{\neg \alpha_1, x_3\}, \{\alpha_1, \neg x_1, \neg x_2, \neg x_3\}, \\ &\{\neg \alpha_2, y_1\}, \{\neg \alpha_2, y_2\}, \{\neg \alpha_2, y_3\}, \{\alpha_2, \neg y_1, \neg y_2, \neg y_3\}, \\ &\{\neg \alpha_1, \alpha_2, \neg \alpha_3\}, \{\alpha_1, \alpha_3\}, \{\neg \alpha_2, \alpha_3\}, \{\neg \alpha_4, \alpha_3, z\}, \{\neg \alpha_3, \alpha_4\}, \{\neg z, \alpha_4\} \} \end{aligned}$$

### Question 4

State with justification if each of the following sentences of predicate logic is logically valid.

- (i)  $(\forall x \exists y \forall z (E(x, y) \wedge E(y, z)) \rightarrow (\forall x \forall z \exists y (E(x, y) \wedge E(y, z)))$
- (ii)  $(\forall x \exists y \exists u \forall v (E(x, y) \wedge E(u, v)) \rightarrow (\exists u \forall v \forall x \exists y (E(x, y) \wedge E(u, v)))$
- (iii)  $(\forall x \exists y \forall z (R(x, y, z)) \rightarrow (\exists x \forall y \exists z (R(x, y, z)))$
- (iv)  $((\forall x \forall y \exists z (E(x, y) \wedge E(y, z))) \rightarrow (\forall x \forall y \forall z (E(x, y) \vee E(y, z))))$

#### Answer.

(i) This sentence can be translated as such: if for every  $x$ , there exists a  $y$ , for every  $z$  the relation  $E(x, y)$  holds and  $E(y, z)$  holds, then for every  $x$  and  $z$ , there exists  $y$  such that the relation  $E(x, y)$  holds and  $E(y, z)$  holds.

### Question 5

Evaluate the given sentence on the respective relation  $E$  over domain  $\{0, 1, 2\}$  with relation  $E := \{(0, 1), (1, 0), (1, 2), (2, 1), (2, 0), (0, 2)\}$

- (i)  $\forall x \forall y \forall z \exists w (E(x, w) \wedge E(y, w) \wedge E(z, w))$
- (ii)  $\exists x \forall y \forall z \exists w (E(x, w) \wedge E(y, w) \wedge E(z, w))$
- (iii)  $\forall y \exists x \forall z \exists w (E(x, w) \wedge E(y, w) \wedge E(z, w))$
- (iv)  $\exists x \exists y \exists z \forall w (E(x, w) \wedge E(y, w) \wedge E(z, w))$
- (v)  $\forall x_1 \exists x_2 \forall y_1 \exists y_2 \forall z_1 \exists z_2 \forall z \exists y (E(x_1, x_2) \wedge E(x_2, w) \wedge E(y_1, y_2) \wedge E(y_2, w) \wedge E(z_1, z_2) \wedge E(z_2, w) \wedge E(z, w))$
- (vi)  $\forall x_1 \exists x_2 \forall y_1 \exists y_2 \forall z_1 \forall z_2 \exists z \exists y (E(x_1, x_2) \wedge E(x_2, w) \wedge E(y_1, y_2) \wedge E(y_2, w) \wedge E(z_1, z_2) \wedge E(z_2, w) \wedge E(z, w))$

#### Answer.

For the purposes of this question, it will be easier to express the relation  $E$  as such:

$$E := \{(u, v) \in \{0, 1, 2\} : u \neq v\}$$

(i) This sentence can be refuted by assigning the following variables:

$$x = 0, y = 1, z = 2$$

That is to say, the expression inside the brackets requires that  $x \neq w, y \neq w, z \neq w$ . However if we assign  $x, y, z$  uniquely, by the pigeonhole principle, it follows that there will exist no such  $w$ , that satisfies the sentence.

(ii)