In [25]:
```
import scipy
import scipy.signal
import numpy as np
import pandas as pd
import matplotlib.pyplot as plt
```

In [26]:
```
np.random.seed(0)
```

Problem 2.1

Let us consider one pixel centered at (x_0, y_0).

The signal measured by the CCD for this pixel is the integral of the incoming light over the pixel area:

$$\iint_{\text{area of pixel}} s(x, y)\, dx\, dy$$

This can be modeled as:

$$r(x_0, y_0) = \iint s(x, y)\, h(x - x_0, y - y_0)\, dx\, dy$$

where $h(x - x_0, y - y_0)$ is the shifted filter to center on the pixel.

To make this expression resemble convolution, we change variables:

$$r(x, y) = \iint s(x_0, y_0)\, h(x - x_0, y - y_0)\, dx_0\, dy_0$$

Now substitute $u = x - x_0$, $v = y - y_0$:

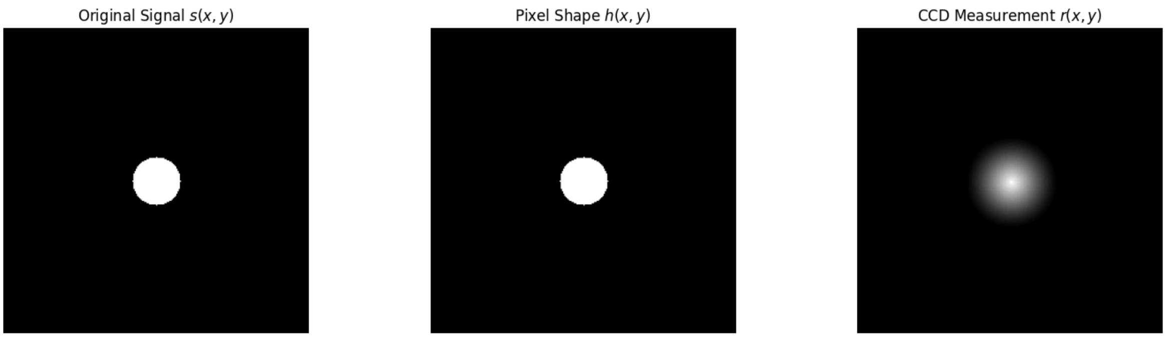
$$r(x, y) = \iint s(x_0, y_0)\, h(-(x_0 - x), -(y_0 - y))\, dx_0\, dy_0$$

Which yields the convolution form:

$$r(x, y) = h(-x, -y) * s(x, y)$$

In [27]:
```
def circ(x, y, a):
    """2D circular mask"""
    return ((x**2 + y**2) <= a**2).astype(float)

# Grid definition
N = 256
a = 20  # radius of circular support
x = np.linspace(-N//2, N//2 - 1, N)
y = np.linspace(-N//2, N//2 - 1, N)
X, Y = np.meshgrid(x, y)

# Signal and pixel shape
s = circ(X, Y, a)
h = circ(X, Y, a)
```

```python
# CCD measurement = convolution of signal and pixel shape
r = scipy.signal.convolve2d(s, h, mode='same', boundary='fill')
r /= r.max()  # normalize

# Plotting
plt.figure(figsize=(15, 4))

plt.subplot(1, 3, 1)
plt.imshow(s, cmap='gray')
plt.title("Original Signal $s(x, y)$")
plt.axis('off')

plt.subplot(1, 3, 2)
plt.imshow(h, cmap='gray')
plt.title("Pixel Shape $h(x, y)$")
plt.axis('off')

plt.subplot(1, 3, 3)
plt.imshow(r, cmap='gray')
plt.title("CCD Measurement $r(x, y)$")
plt.axis('off')

plt.tight_layout()
plt.show()
```



A CCD measures:

$$r(x, y) = (h(-x, -y) * s)(x, y)$$

This is a 2D convolution between the signal and the pixel shape.
Since both the signal and the pixel shape are circular functions:

$$s(x, y) = \text{circ}\left(\frac{x}{a}, \frac{y}{a}\right), \quad h(x, y) = \text{circ}\left(\frac{x}{a}, \frac{y}{a}\right)$$

The CCD output becomes:

$$r(x, y) = \text{circ}\left(\frac{x}{a}, \frac{y}{a}\right) * \text{circ}\left(\frac{x}{a}, \frac{y}{a}\right)$$

This results in:

- **Blurring**: smoothing of the signal edges
- **Amplitude reduction** as energy spreads across a wider area

Problem 2.2

Projection Slice Theorem

$$p_\theta(t) \longleftrightarrow P_\theta(f) = S(f\cos\theta, f\sin\theta) = S(f_x, f_y) = S_1(f_x)S_2(f_y)$$

For $\theta = 0$:

$$P_0(f) = S(f, 0) = S_1(f_x) \cdot S_2(0) = S_1(f) \cdot (\text{constant})$$

For $\theta = \frac{\pi}{2}$:

$$P_{\frac{\pi}{2}}(f) = S(0, f) = S_1(0) \cdot S_2(f_y)$$

Since $s(x, y)$ is separable:

$$s(x, y) = s_1(x) \cdot s_2(y) \Rightarrow S(f_x, f_y) = S_1(f_x) \cdot S_2(f_y)$$

From the two projections, we recover:

- $S(f_x, 0) = S_1(f_x) \cdot S_2(0)$
- $S(0, f_y) = S_1(0) \cdot S_2(f_y)$

Since the function is separable, we reconstruct the full 2D spectrum as:

$$S(f_x, f_y) = \frac{S(f_x, 0)}{S_2(0)} \cdot \frac{S(0, f_y)}{S_1(0)} = S_1(f_x)S_2(f_y)$$

This completely determines $S(f_x, f_y)$, and the inverse Fourier transform gives:

$$s(x, y) = \mathcal{F}_2^{-1}\left\{S(f_x, f_y)\right\}$$

Thus, the function $s(x, y)$ is uniquely determined by its projections at $\theta = 0$ and $\theta = \frac{\pi}{2}$.

Problem 2.3

(a) Adjoint of an Integral Operator

We are given the operator:

$$Lx(t) = \int_a^b K(t, z)x(z)\,dz$$

and the inner product:

$$\langle y, Lx\rangle_{\mathcal{Y}} = \int_a^b y(t)Lx(t)\,dt$$

Substitute (Lx(t)):

$$= \int_a^b y(t)\left(\int_a^b K(t, z)x(z)\,dz\right)dt = \int_a^b \int_a^b y(t)K(t, z)x(z)\,dz\,dt$$

Change the order of integration:

$$= \int_a^b \left(\int_a^b y(t) K(t,z)\, dt \right) x(z)\, dz = \int_a^b L^* y(z) \cdot x(z)\, dz = \langle L^* y, x \rangle_{\mathcal{X}}$$

So the adjoint operator is:

$$L^* y(z) = \int_a^b K(t,z) y(t)\, dt$$

Now change variables to express in terms of (x(t)):

$$L^* x(t) = \int_a^b K(z,t) x(z)\, dz$$

Self-adjoint condition:

$$L = L^* \Rightarrow K(t,z) = K(z,t)$$

That is, the kernel must be **symmetric**.

(b) Matrix Case

Given inner product:

$$\langle y, Ax \rangle = y^T A x$$

Then:

$$y^T A x = (A^T y)^T x = \langle A^T y, x \rangle \Rightarrow L^* = A^T$$

Self-adjoint condition:

$$L = L^* \Rightarrow A = A^T$$

Also, to be self-adjoint, (A) must be a **square matrix** and **symmetric**.

Problem 2.3 (c)

$$Lx(t) = \int_{-\infty}^{\infty} h(t-z)\, x(z)\, dz$$

Let (y = Lx(t))

$$\langle y, Lx \rangle = \int_{-\infty}^{\infty} y(t) \cdot Lx(t)\, dt = \int_{-\infty}^{\infty} y(t) \left(\int_{-\infty}^{\infty} h(t-z) x(z)\, dz \right) dt$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} y(t) h(t-z) x(z)\, dz\, dt = \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} y(t) h(t-z)\, dt \right) x(z)\, dz$$

$$\Rightarrow L^* y(z) = \int_{-\infty}^{\infty} y(t) h(t-z)\, dt$$

Change variables:

$$\Rightarrow L^*x(t) = \int_{-\infty}^{\infty} h(-(t-z)) \cdot x(z)\, dz$$

To impulse response be self-adjoint:

$$h(t) = h(-t)$$

That is, the impulse response (h(t)) must be **even**

Problem 2.4 (a)

Let $\vec{h} \in \mathbb{R}^P$, $\vec{x} \in \mathbb{R}^L$ be input vectors, and let their linear convolution be:

$$\vec{y} = \vec{x} * \vec{h} \in \mathbb{R}^{P+L-1}$$

Then we can express this as a matrix-vector product:

$$\vec{y} = C\vec{x}, \quad C \in \mathbb{R}^{(P+L-1)\times L}$$

Impulse-based Column Construction

Let $\delta_{x_1} = [1, 0, 0, \ldots, 0]^T$ (the unit impulse vector). Then:

$$C = [\, \vec{c}_1 \quad \vec{c}_2 \quad \cdots \quad \vec{c}_L \,] \quad \text{where} \quad \vec{c}_i = \vec{h} * \delta_{x_i}$$

Each column $\vec{c}_i$ is the convolution of $\vec{h}$ with a shifted impulse.

Examples:

- $\vec{c}_1 = \vec{h} * \delta_{x_1} = [h_0, h_1, \ldots, h_{P-1}, 0, \ldots, 0]^T$
- $\vec{c}_2 = \vec{h} * \delta_{x_2} = [0, h_0, h_1, \ldots, h_{P-1}, 0, \ldots]^T$

Full Convolution Matrix C

$$C = \begin{bmatrix} h_0 & 0 & 0 & \cdots & 0 \\ h_1 & h_0 & 0 & \cdots & 0 \\ h_2 & h_1 & h_0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ h_{P-1} & h_{P-2} & h_{P-3} & \cdots & h_0 \\ 0 & h_{P-1} & h_{P-2} & \cdots & h_1 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & h_{P-1} \end{bmatrix}_{(P+L-1)\times L}$$

This is a **full Toeplitz matrix**, where each column is a right-shifted and zero-padded version of the impulse response $\vec{h}$.

In [28]:
```
# Lengths
L = 5   # x
P = 4   # h

# Impulse response
h = np.random.randint(1,10,P)
h
```

Out[28]:
```
array([6, 1, 4, 4], dtype=int32)
```

In [29]:
```
# Initialize the first column of the convolution matrix C
# The total number of rows is L + P - 1 (length of the linear convolution output
column_C = np.zeros((L + P - 1, 1))

# Fill the first P entries of the column with the impulse response h
# Reshape h as a column vector to match dimensions
column_C[:P, :] = h.reshape(-1, 1)

# Initialize the first row of the Toeplitz matrix
row_C = np.zeros((1, L))
# First row starts with the first element of h
row_C[0][0] = h[0]

row_C
```

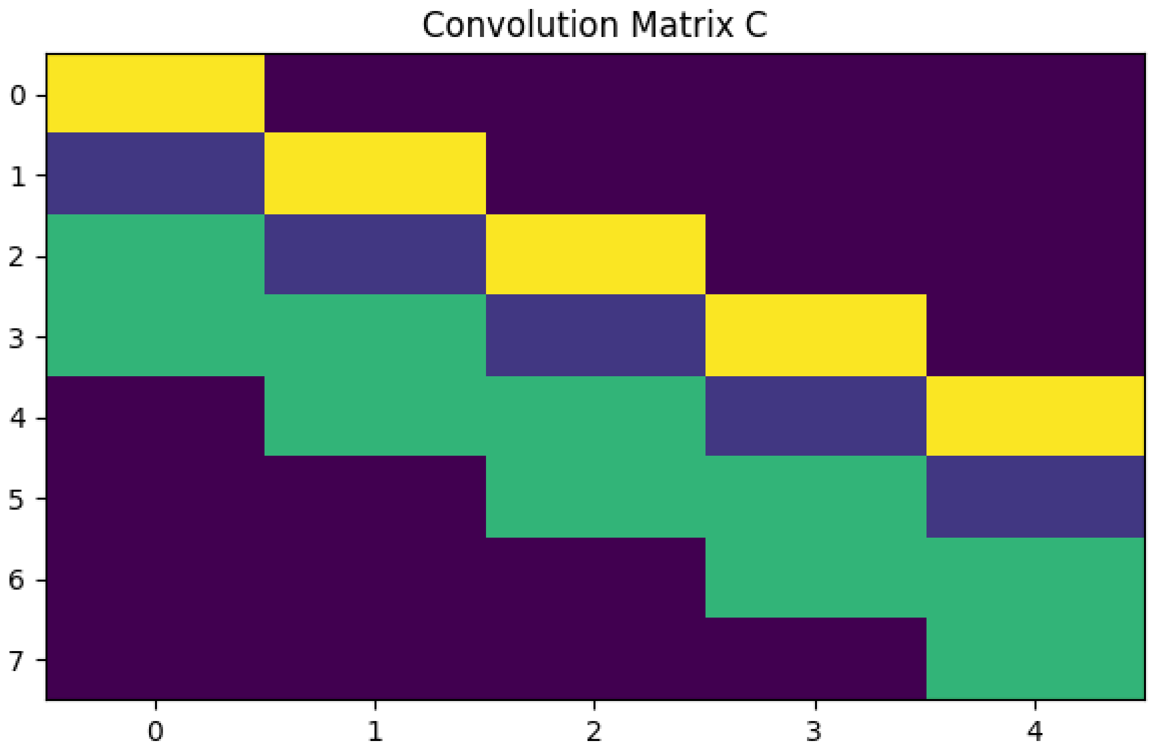
Out[29]:
```
array([[6., 0., 0., 0., 0.]])
```

In [30]:
```
# Convolution matrix
C = scipy.linalg.toeplitz(column_C, row_C)
C

def convmtx(h, L):
    column_C = np.zeros((L+len(h)-1,1))
    column_C[:len(h), :] = h.reshape(-1, 1)
    row_C = np.zeros((1, L))
    row_C[0][0] = h[0]

    return scipy.linalg.toeplitz(column_C, row_C)

convmtx(h,L), C
```

```
C:\Users\eozka\AppData\Local\Temp\ipykernel_26868\3581256877.py:2: FutureWarning:
Beginning in SciPy 1.17, multidimensional input will be treated as a batch, not `
ravel`ed. To preserve the existing behavior and silence this warning, `ravel` arg
uments before passing them to `toeplitz`.
  C = scipy.linalg.toeplitz(column_C, row_C)
C:\Users\eozka\AppData\Local\Temp\ipykernel_26868\3581256877.py:11: FutureWarnin
g: Beginning in SciPy 1.17, multidimensional input will be treated as a batch, no
t `ravel`ed. To preserve the existing behavior and silence this warning, `ravel`
arguments before passing them to `toeplitz`.
  return scipy.linalg.toeplitz(column_C, row_C)
```

Out[30]:
```
(array([[6., 0., 0., 0., 0.],
        [1., 6., 0., 0., 0.],
        [4., 1., 6., 0., 0.],
        [4., 4., 1., 6., 0.],
        [0., 4., 4., 1., 6.],
        [0., 0., 4., 4., 1.],
        [0., 0., 0., 4., 4.],
        [0., 0., 0., 0., 4.]]),
 array([[6., 0., 0., 0., 0.],
        [1., 6., 0., 0., 0.],
        [4., 1., 6., 0., 0.],
        [4., 4., 1., 6., 0.],
        [0., 4., 4., 1., 6.],
        [0., 0., 4., 4., 1.],
        [0., 0., 0., 4., 4.],
        [0., 0., 0., 0., 4.]]))
```

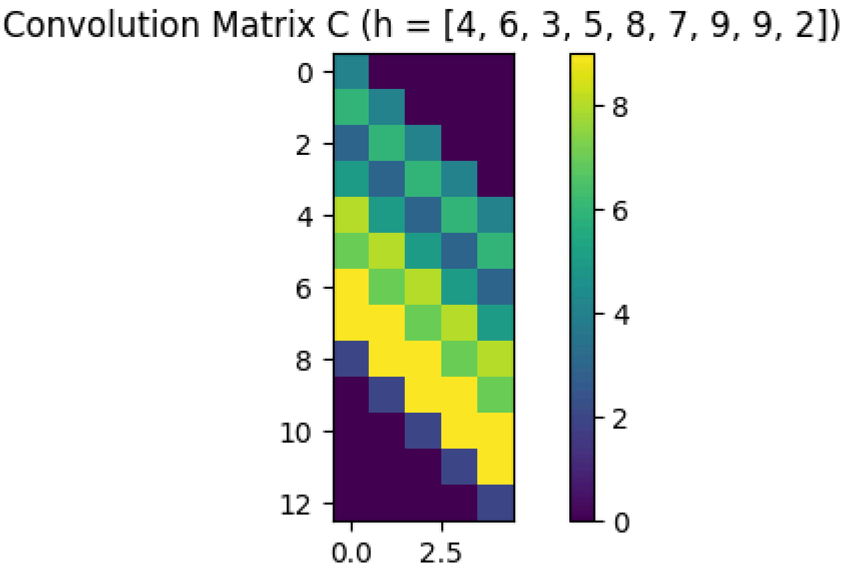
In [31]:
```
plt.figure(figsize=(6, 4))
plt.imshow(C, cmap='viridis', aspect='auto')
plt.title('Convolution Matrix C')
plt.grid(False)
plt.tight_layout()
plt.show()
```

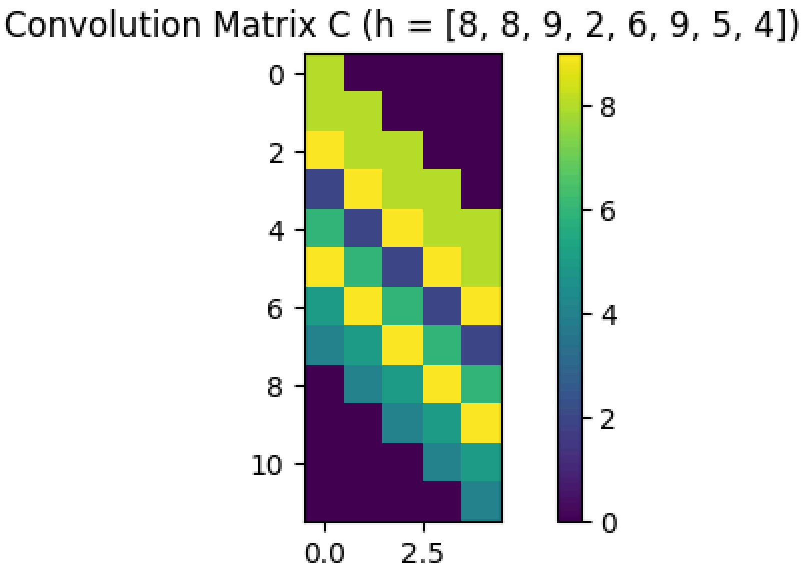


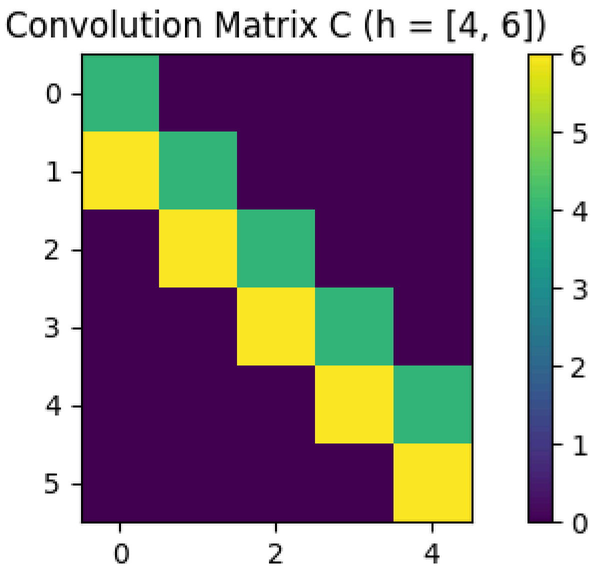
In [32]:
```
# Number of different filters
num_filters = 3

# Create subplot
fig, axs = plt.subplots(num_filters, 1, figsize=(6, 3 * num_filters))

for i in range(num_filters):
    # Random filter length between 2 and 5
    P = np.random.randint(2, 10)

    # Random integer impulse response h
    h = np.random.randint(1, 10, size=P)
```

```
    # Initialize first column (length L + P - 1)
    column_C = np.zeros((L + P - 1, 1))
    column_C[:P, :] = h.reshape(-1, 1)

    # Initialize first row (length L)
    row_C = np.zeros((1, L))
    row_C[0][0] = h[0]

    # Build Toeplitz convolution matrix
    C = scipy.linalg.toeplitz(column_C, row_C)

    # Plot the matrix
    ax = axs[i]
    im = ax.imshow(C, cmap='viridis')
    ax.set_title(f"Convolution Matrix C (h = {h.tolist()})")
    fig.colorbar(im, ax=ax)

plt.tight_layout()
plt.show()
```

```
C:\Users\eozka\AppData\Local\Temp\ipykernel_26868\388580017.py:23: FutureWarning:
Beginning in SciPy 1.17, multidimensional input will be treated as a batch, not `
ravel`ed. To preserve the existing behavior and silence this warning, `ravel` arg
uments before passing them to `toeplitz`.
  C = scipy.linalg.toeplitz(column_C, row_C)
```







Problem 2.4 (b)

When Does Circular Convolution Match Linear Convolution?

- Let the input signal be:

$$x \in \mathbb{R}^L$$

- Let the impulse response be:

$$h \in \mathbb{R}^P$$

Then their **linear convolution** has length:

$$\text{len}(x * h) = L + P - 1$$

To make **circular convolution** behave like **linear convolution**, we must compute the circular convolution over a length:

$$\boxed{N \geq L + P - 1}$$

How Are `x̃` and `h̃` Defined?

To perform length-(N) circular convolution that matches linear convolution, we define the zero-padded versions of the sequences:

$$\tilde{x} = [x_0, x_1, \ldots, x_{L-1}, \underbrace{0, \ldots, 0}_{N-L}]$$

$$\tilde{h} = [h_0, h_1, \ldots, h_{P-1}, \underbrace{0, \ldots, 0}_{N-P}]$$

These padded sequences are of length (N). Then:

$$\tilde{y} = \mathcal{F}^{-1}\left(\mathcal{F}(\tilde{x}) \cdot \mathcal{F}(\tilde{h})\right) = \tilde{x} \circledast \tilde{h} = x * h$$

In [33]:
```
def cconv(x,h,N):

    lin_len = len(x) + len(h) - 1
    x_padded = np.pad(x, (0, lin_len - len(x)))
    h_padded = np.pad(h, (0, lin_len - len(h)))
    y_circular = np.zeros(N)
    X = np.fft.fft(x_padded)
    H = np.fft.fft(h_padded)
    y = np.real_if_close(np.fft.ifft(X * H))

    for i in range(len(y)):
        y_circular[i % N] += y[i]
    return np.round(y_circular,decimals=10)
```

In [34]:
```
x = [1,3,2,1]

h = [1,1,-1,1]

y = cconv(x,h,3)
y
```

Out[34]:
```
array([3., 6., 5.])
```

(c) Matrix Form of Circular Convolution

Let

$$\tilde{h} \in \mathbb{R}^N$$

be the zero-padded impulse response (assumption N > P).

$$\tilde{h} = [h_0, h_1, \ldots, h_{P-1}, \underbrace{0, \ldots, 0}_{N-P}]$$

Let the unit impulse vector be:

$$\delta_{x_1} = [1, 0, 0, \ldots, 0]^T$$

Then the circular convolution matrix is:

$$\tilde{C} = [\, \vec{c}_1 \quad \vec{c}_2 \quad \cdots \quad \vec{c}_N \,] \quad \text{where} \quad \vec{c}_i = \tilde{h} \circledast \delta_{x_i}$$

Each column $\vec{c}_i$ is the circular convolution of $\tilde{h}$ with a circularly shifted impulse.

Let the zero-padded filter be:

$$\tilde{h} = [h_0, h_1, h_2, h_3, h_4]^T$$

Then:

$$\delta_{x_1} = [1, 0, 0, 0, 0]^T$$

So,

$$\vec{c}_1 = \tilde{h} \circledast \delta_{x_1} = [h_0, h_1, h_2, h_3, 0]^T$$

Impulse vector:

$$\delta_{x_2} = [0, 1, 0, 0, 0]^T$$

Circular convolution (right circular shift of $\vec{c}_1$):

$$\vec{c}_2 = \tilde{h} \circledast \delta_{x_2} = [0, h_0, h_1, h_2, h_3]^T$$

This means that the **first column of $\tilde{C}$** is simply the original filter $\tilde{h}$. Then the circular convolution can be written as a matrix-vector product:

$$\tilde{y} = \tilde{C} \cdot \tilde{x}$$

where

$$\tilde{C} \in \mathbb{R}^{N \times N}$$

is the circulant matrix:

$$\tilde{C} = \begin{bmatrix} \tilde{h}_0 & \tilde{h}_{N-1} & \cdots & \tilde{h}_1 \\ \tilde{h}_1 & \tilde{h}_0 & \cdots & \tilde{h}_2 \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{h}_{N-1} & \tilde{h}_{N-2} & \cdots & \tilde{h}_0 \end{bmatrix}$$

- $\tilde{C}$ is a circulant matrix.
- Every row is a right circular shift of the row above.
- $\tilde{C}$ is structured wrapping around rows of C.

For example,

$$C = \begin{bmatrix} h_0 & 0 & 0 \\ h_1 & h_0 & 0 \\ h_2 & h_1 & h_0 \\ 0 & h_2 & h_1 \\ 0 & 0 & h_2 \end{bmatrix}$$

the corresponding **circular convolution matrix** is:

$$\tilde{C} = \begin{bmatrix} h_0 & h_2 & h_1 \\ h_1 & h_0 & h_2 \\ h_2 & h_1 & h_0 \end{bmatrix}$$

In [35]:
```
def cconvmtx(h,N):
    C_conv = np.zeros((N,N))
    if N > len(h):
        h_padded = np.pad(h, (0, N - len(h)))
    else:
        h_padded = h
    for i in range(N):
        unit_vector = np.zeros(N)
        unit_vector[i] = 1
        C_conv[:, i] = cconv(h_padded, unit_vector, N)

    return C_conv

C_conv = cconvmtx(h,5)
C_conv
```

Out[35]:
```
array([[ 1.,  0.,  1., -1.,  1.],
       [ 1.,  1., -0.,  1., -1.],
       [-1.,  1.,  1.,  0.,  1.],
       [ 1., -1.,  1.,  1., -0.],
       [ 0.,  1., -1.,  1.,  1.]])
```

(d) DFT Matrix Construction (with positive exponents)

Let the Discrete Fourier Transform (DFT) of a length-(N) sequence be defined as:

$$X(k) = \sum_{n=0}^{N-1} x(n) \cdot W_N^{kn} \quad \text{where} \quad W_N = e^{-j\frac{2\pi}{N}}$$

Vector Form

For (k = 0):

$$X(0) = \sum_{n=0}^{N-1} x(n) = (\vec{W}_N^0)^T \cdot \vec{x}$$

where:

$$\vec{W}_N^0 = [1\ 1\ \dots\ 1]^T \in \mathbb{R}^N$$

For (k = 1):

The DFT coefficient is:

$$X(1) = \sum_{n=0}^{N-1} x(n) \cdot W_N^n = x(0)W_N^0 + x(1)W_N^1 + \cdots + x(N-1)W_N^{N-1}$$

We can also express (X(1)) as the inner product of a row vector and a column vector:

$$X(1) = \begin{bmatrix} W_N^0 & W_N^1 & \cdots & W_N^{N-1} \end{bmatrix} \cdot \begin{bmatrix} x(0) \\ x(1) \\ \vdots \\ x(N-1) \end{bmatrix}$$

That is:

$$X(1) = \vec{w}_1 \cdot \vec{x}$$

where:

- $\vec{w}_1 \in \mathbb{C}^{1\times N}$: the 2nd row of the DFT matrix (for $k = 1$)
- $\vec{x} \in \mathbb{C}^{N\times 1}$: the input signal as a column vector

General DFT Matrix (F)

The ((k,n))-th entry of the DFT matrix is:

$$F[k,n] = W_N^{kn} = e^{-j\frac{2\pi}{N}kn}$$

So the full DFT matrix is:

$$F = \begin{bmatrix} W_N^0 & W_N^0 & \cdots & W_N^0 \\ W_N^0 & W_N^1 & \cdots & W_N^{(N-1)} \\ W_N^0 & W_N^2 & \cdots & W_N^{2(N-1)} \\ \vdots & \vdots & \ddots & \vdots \\ W_N^0 & W_N^{(N-1)} & \cdots & W_N^{(N-1)^2} \end{bmatrix}$$

This matrix performs the forward DFT:

$$\vec{X} = F \cdot \vec{x}$$

In Full Vector Form:

$$\vec{X} = \begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ \vdots \\ X(N-1) \end{bmatrix} = \begin{bmatrix} W_N^{0\cdot 0} & W_N^{0\cdot 1} & \cdots & W_N^{0\cdot(N-1)} \\ W_N^{1\cdot 0} & W_N^{1\cdot 1} & \cdots & W_N^{1\cdot(N-1)} \\ W_N^{2\cdot 0} & W_N^{2\cdot 1} & \cdots & W_N^{2\cdot(N-1)} \\ \vdots & \vdots & \ddots & \vdots \\ W_N^{(N-1)\cdot 0} & W_N^{(N-1)\cdot 1} & \cdots & W_N^{(N-1)(N-1)} \end{bmatrix} \cdot \begin{bmatrix} x(0) \\ x(1) \\ \vdots \\ x(N-1) \end{bmatrix}$$

Inverse of the DFT Matrix (F^{-1})

From the inverse DFT definition:

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) \cdot W_N^{-kn}$$

where $W_N = e^{-j\frac{2\pi}{N}}$

So the inverse DFT matrix uses **negative exponents**:

- The (n,k)-th entry of F^{-1} is:

$$W_N^{-nk} = e^{+j\frac{2\pi}{N}nk}$$

Thus, the inverse DFT matrix is:

$$\boxed{F^{-1} = \frac{1}{N} F^H}$$

where F^H is the **Hermitian (conjugate transpose)** of the DFT matrix F.

$$\vec{x} = \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ \vdots \\ x(N-1) \end{bmatrix} = \frac{1}{N} \begin{bmatrix} W_N^{0\cdot 0} & W_N^{0\cdot 1} & \cdots & W_N^{0\cdot(N-1)} \\ W_N^{-1\cdot 0} & W_N^{-1\cdot 1} & \cdots & W_N^{-1\cdot(N-1)} \\ W_N^{-2\cdot 0} & W_N^{-2\cdot 1} & \cdots & W_N^{-2\cdot(N-1)} \\ \vdots & \vdots & \ddots & \vdots \\ W_N^{-(N-1)\cdot 0} & W_N^{-(N-1)\cdot 1} & \cdots & W_N^{-(N-1)\cdot(N-1)} \end{bmatrix} \cdot \begin{bmatrix} X(0) \\ X(1) \\ \vdots \\ X(N-1) \end{bmatrix}$$

In [36]:
```
# Test vectors
import scipy.linalg


x = np.array([1, 2, 3])
h = np.array([4, 5, 6])
N = 3

F = scipy.linalg.dft(N)
y = cconv(x,h,N)
```

```
y_ = np.real(np.linalg.inv(F) @ ((F @ x) * (F @ h)))
y, y_
```

Out[36]: (array([31., 31., 28.]), array([31., 31., 28.]))

In [37]:

```
# Test with varying N values
def run_multiple_cconv_tests_varied_N(N_values=[3, 4, 5, 6], num_tests_per_N=3,
    results = []

    for N in N_values:
        F = scipy.linalg.dft(N)
        for _ in range(num_tests_per_N):
            x = np.random.randint(0, 10, size=N)
            h = np.random.randint(0, 10, size=N)

            y_time = cconv(x, h, N)
            y_freq = np.round(np.real(np.linalg.inv(F) @ ((F @ x) * (F @ h))))

            match = np.allclose(y_time, y_freq, atol=tol)
            results.append((N, x, h, y_time, y_freq, match))

    return results

# Run the test
test_results = run_multiple_cconv_tests_varied_N()

# Display the results
df_varied = pd.DataFrame(
    [(i+1, N, x.tolist(), h.tolist(),y_time.tolist(), y_freq.tolist(), " PASS" i
     for i, (N, x, h,y_time, y_freq, match) in enumerate(test_results)],
    columns=["Test#", "N", "x", "h","y_time", "y_freq", "Result"]
)
df_varied
```

Out[37]:

	Test#	N	x	h	y_time	y_freq	Result
0	1	3	[0, 2, 3]	[8, 1, 3]	[9.0, 25.0, 26.0]	[9.0, 25.0, 26.0]	PASS
1	2	3	[3, 3, 7]	[0, 1, 9]	[34.0, 66.0, 30.0]	[34.0, 66.0, 30.0]	PASS
2	3	3	[9, 0, 4]	[7, 3, 2]	[75.0, 35.0, 46.0]	[75.0, 35.0, 46.0]	PASS
3	4	4	[7, 2, 0, 0]	[4, 5, 5, 6]	[40.0, 43.0, 45.0, 52.0]	[40.0, 43.0, 45.0, 52.0]	PASS
4	5	4	[8, 4, 1, 4]	[9, 8, 1, 1]	[109.0, 105.0, 53.0, 56.0]	[109.0, 105.0, 53.0, 56.0]	PASS
5	6	4	[7, 9, 9, 3]	[6, 7, 2, 0]	[81.0, 109.0, 131.0, 99.0]	[81.0, 109.0, 131.0, 99.0]	PASS
6	7	5	[3, 5, 9, 4, 4]	[6, 4, 4, 3, 4]	[97.0, 106.0, 114.0, 105.0, 103.0]	[97.0, 106.0, 114.0, 105.0, 103.0]	PASS
7	8	5	[4, 8, 4, 3, 7]	[5, 5, 0, 1, 5]	[99.0, 83.0, 82.0, 74.0, 78.0]	[99.0, 83.0, 82.0, 74.0, 78.0]	PASS
8	9	5	[9, 3, 0, 5, 0]	[1, 2, 4, 2, 0]	[29.0, 31.0, 42.0, 35.0, 16.0]	[29.0, 31.0, 42.0, 35.0, 16.0]	PASS
9	10	6	[3, 2, 0, 7, 5, 9]	[0, 2, 7, 2, 9, 2]	[71.0, 142.0, 102.0, 111.0, 63.0, 83.0]	[71.0, 142.0, 102.0, 111.0, 63.0, 83.0]	PASS
10	11	6	[3, 3, 2, 3, 4, 1]	[2, 9, 1, 4, 6, 8]	[67.0, 84.0, 86.0, 77.0, 75.0, 91.0]	[67.0, 84.0, 86.0, 77.0, 75.0, 91.0]	PASS
11	12	6	[2, 3, 0, 0, 6, 0]	[6, 3, 3, 8, 8, 8]	[54.0, 72.0, 63.0, 73.0, 76.0, 58.0]	[54.0, 72.0, 63.0, 73.0, 76.0, 58.0]	PASS

(e) Show that the matrix

$$\Lambda = \frac{F\tilde{C}F^H}{N}$$

is a diagonal matrix

Say $P' = \frac{F}{N}$ and $P = F^H$

$$\Lambda = P\tilde{C}P' \Rightarrow P\Lambda = \tilde{C}P$$

Column vectors of P:

$$P = \begin{bmatrix} \vec{f}_0 & \vec{f}_1 & \dots & \vec{f}_{N-1} \end{bmatrix}$$

Then:

$$P\Lambda = \begin{bmatrix} \vec{f}_0 & \vec{f}_1 & \dots & \vec{f}_{N-1} \end{bmatrix} \begin{bmatrix} \lambda_0 & & & \\ & \lambda_1 & & \\ & & \ddots & \\ & & & \lambda_{N-1} \end{bmatrix} = \tilde{C} \begin{bmatrix} \vec{f}_0 & \vec{f}_1 & \dots & \vec{f}_{N-1} \end{bmatrix}$$

So:

$$\tilde{C}\vec{f}_k = \lambda_k \vec{f}_k$$

That is, columns of P are eigenvectors of $\tilde{C}$.

If we can show that $\vec{f}_k$ is an eigenvector of $\tilde{C}$, then Λ is diagonal.

Now consider the matrix-vector product:

$$\tilde{C}\begin{bmatrix} W_N^{-0k} \\ W_N^{-1k} \\ \vdots \\ W_N^{-(N-1)k} \end{bmatrix} = \begin{bmatrix} h_0 & h_{N-1} & \cdots & h_1 \\ h_1 & h_0 & \cdots & h_2 \\ \vdots & \vdots & \ddots & \vdots \\ h_{N-1} & h_{N-2} & \cdots & h_0 \end{bmatrix}\begin{bmatrix} W_N^{-0k} \\ W_N^{-1k} \\ \vdots \\ W_N^{-(N-1)k} \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_{N-1} \end{bmatrix}$$

The n-th element of y is:

$$y_n = \sum_{m=0}^{N-1} h_{(n-m) \bmod N} \cdot W_N^{-mk}$$

Now factor W_N^{nk} from the sum:

$$y_n = W_N^{-nk} \sum_{m=0}^{N-1} h_{(n-m) \bmod N} \cdot W_N^{(n-m)k} = W_N^{-nk} \sum_{m=0}^{N-1} h_{(n-m) \bmod N} \cdot W_N^{nk-mk}$$

Since both h_m and W_N^m are periodic in m, the sum is independent of n:

$$y_n = W_N^{-nk} \sum_{m=0}^{N-1} h_m \cdot W_N^{mk} = \lambda_k \cdot W_N^{-nk}$$

$y_0 = W_N^{-0k} \cdot \lambda_k$
$y_1 = W_N^{-1k} \cdot \lambda_k$
$\vdots$
$y_n = W_N^{-nk} \cdot \lambda_k$

$$\Rightarrow \vec{y} = \vec{f}_k \cdot \lambda_k \Rightarrow \tilde{C}\vec{f}_k = \vec{f}_k \cdot \lambda_k \Rightarrow \vec{f}_k \text{ is eigenvector}$$

$$\tilde{C}P = P \cdot \text{diag}(\tilde{H}) \Rightarrow \text{diag}(\tilde{H}) = P^{-1}\tilde{C}P = \frac{F\tilde{C}F^H}{N}$$

$y = Cx = \tilde{C}\tilde{x}$

$$Y = Fy = FCx = F\tilde{C}\tilde{x}$$

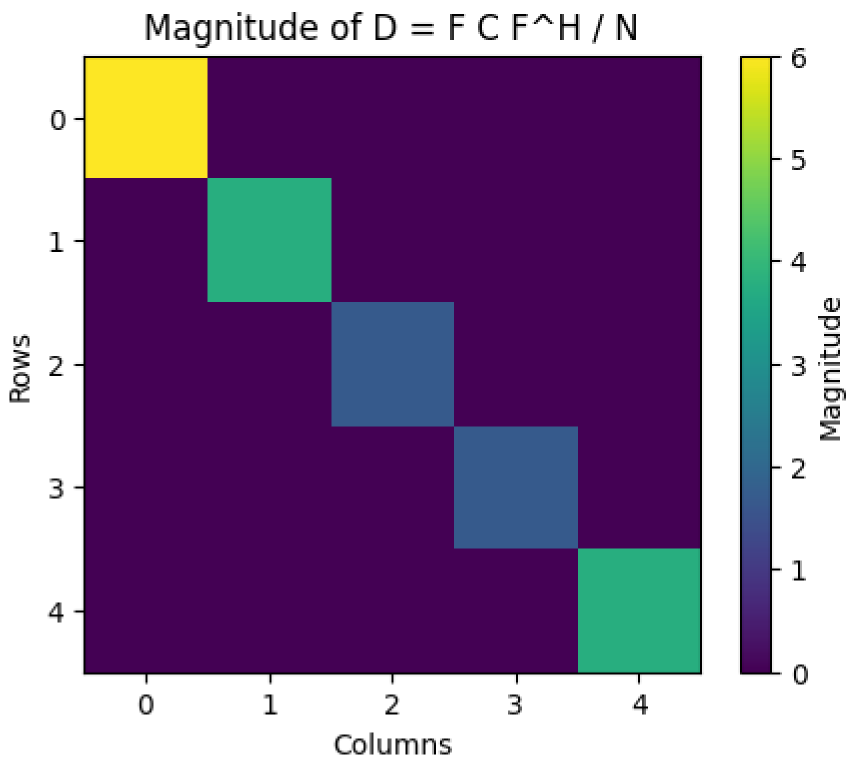
$$Y = \tilde{H}\tilde{X} = \tilde{H} \cdot F\tilde{x}$$

$$F\tilde{C}\tilde{x} = \tilde{H} \cdot \tilde{X} \Rightarrow \frac{F\tilde{C}F^H}{N} \cdot \tilde{X} = \tilde{H} \cdot \tilde{X}$$

$$\Rightarrow \frac{F\tilde{C}F^H}{N} = \text{diag}(\tilde{H}) = \text{diag}(Fh)$$

$\tilde{H} \in \mathbb{R}^{N\times 1}$
$\text{diag}(\tilde{H}) \in \mathbb{R}^{N\times N}$

In [38]:
```
N = 5
h = np.array([1,2,3])
C_conv = cconvmtx(h,N)
F = scipy.linalg.dft(N)

FH = F.conj().T
D = (F @ C_conv @ FH) / N

# Visualize the magnitude of matrix D
plt.figure(figsize=(5, 4))
plt.imshow(np.abs(D), cmap='viridis')
plt.colorbar(label='Magnitude')
plt.title("Magnitude of D = F C F^H / N")
plt.xlabel("Columns")
plt.ylabel("Rows")
plt.xticks(np.arange(N))
plt.yticks(np.arange(N))
plt.grid(False)
plt.show()
```



In [39]:
```
# Test function for verifying: y = Cx = h ⊛ x = IDFT(H .* X), and FCF^H/N is di
def verify_circular_convolution_dft(N_values=[3, 4, 5, 6], num_tests_per_N=3, to
    results = []

    for N in N_values:
        F = scipy.linalg.dft(N)
        FH = F.conj().T
        for _ in range(num_tests_per_N):
            x = np.random.randint(0, 10, size=N)
```

```
            h = np.random.randint(0, 10, size=N)

            # DFTs
            X = F @ x
            H = F @ h
            Y_freq = H * X

            # Compute y_time via circular convolution
            y_cconv = cconv(x, h, N)

            # Compute y from frequency domain
            y_freq = np.real(np.linalg.inv(F) @ Y_freq)

            # Check if matrix FCF^H/N is diagonal
            C = cconvmtx(h,N)
            D = (F @ C @ FH) / N
            is_diag = np.allclose(D, np.diag(np.diag(D)), atol=tol)

            # Match check
            match = np.allclose(y_cconv, y_freq, atol=tol)

            results.append((N, x.tolist(), h.tolist(), y_cconv.round(3).tolist()

    return results

# Run the test and format as DataFrame
test_data = verify_circular_convolution_dft()
df_test = pd.DataFrame(test_data, columns=["N", "x", "h", "y_cconv", "y_freq", "
df_test
```

Out[39]:

	N	x	h	y_cconv	y_freq	FCF^H/N Diagonal?	Match
0	3	[2, 3, 2]	[0, 8, 8]	[40.0, 32.0, 40.0]	[40.0, 32.0, 40.0]	True	True
1	3	[3, 8, 2]	[8, 4, 3]	[56.0, 82.0, 57.0]	[56.0, 82.0, 57.0]	True	True
2	3	[0, 4, 3]	[6, 9, 8]	[59.0, 48.0, 54.0]	[59.0, 48.0, 54.0]	True	True
3	4	[0, 8, 5, 9]	[0, 9, 6, 5]	[151.0, 79.0, 117.0, 93.0]	[151.0, 79.0, 117.0, 93.0]	True	True
4	4	[3, 1, 8, 0]	[4, 9, 6, 5]	[65.0, 71.0, 59.0, 93.0]	[65.0, 71.0, 59.0, 93.0]	True	True
5	4	[7, 8, 8, 9]	[2, 8, 6, 6]	[182.0, 174.0, 176.0, 172.0]	[182.0, 174.0, 176.0, 172.0]	True	True
6	5	[9, 1, 6, 8, 8]	[3, 2, 3, 6, 3]	[106.0, 111.0, 119.0, 117.0, 91.0]	[106.0, 111.0, 119.0, 117.0, 91.0]	True	True
7	5	[6, 5, 7, 0, 8]	[4, 6, 5, 8, 2]	[138.0, 110.0, 152.0, 131.0, 119.0]	[138.0, 110.0, 152.0, 131.0, 119.0]	True	True
8	5	[3, 9, 7, 5, 3]	[4, 5, 3, 3, 7]	[126.0, 124.0, 126.0, 112.0, 106.0]	[126.0, 124.0, 126.0, 112.0, 106.0]	True	True
9	6	[9, 9, 9, 7, 3, 2]	[3, 9, 7, 7, 5, 1]	[169.0, 187.0, 207.0, 241.0, 245.0, 199.0]	[169.0, 187.0, 207.0, 241.0, 245.0, 199.0]	True	True
10	6	[2, 2, 8, 1, 5, 8]	[4, 0, 2, 5, 5, 0]	[63.0, 54.0, 101.0, 58.0, 56.0, 84.0]	[63.0, 54.0, 101.0, 58.0, 56.0, 84.0]	True	True
11	6	[8, 1, 1, 0, 3, 8]	[8, 4, 4, 0, 9, 3]	[120.0, 75.0, 71.0, 89.0, 124.0, 109.0]	[120.0, 75.0, 71.0, 89.0, 124.0, 109.0]	True	True

(f) Relationship between the elements on the diagonal of Λ and the DTFT of the original impulse response

Elements on the diagonal of Λ are the DFT of $\tilde{h}$.
Therefore, they are samples of the DTFT of h:

$$\tilde{H}_k = H(e^{j\omega})\Big|_{\omega=\frac{2\pi}{N}k}$$

DFT of the first column of $\tilde{C}$ is the DFT of $\tilde{h}$.
Therefore, it gives eigenvalues of $\tilde{C}$.

In part (e), we showed that:

$$\Lambda = \frac{F\tilde{C}F^H}{N} \quad \text{is diagonal}$$

Then:

$$\tilde{C} = F^H \Lambda F = P \Lambda P^{-1}$$

where:

$$P = F^H, \quad P' = \frac{F}{N}$$

So:

$$\Lambda = \text{diag}(\lambda_k) \quad \text{are the eigenvalues of } \tilde{C}$$

and:

$\text{eigenvectors of } \tilde{C} \text{ are the columns of } P$

Specifically, the eigenvectors are:

$$\vec{f}_k = \left\{ e^{j\frac{2\pi}{N}kn} \right\}_{n=0}^{N-1}$$

In [40]:

```
# Sample impulse response
h = np.array([1, 2, 3, 4])
N = len(h)

# Circulant matrix from h
C = cconvmtx(h,N)
diag_C = np.diag(C)

# DFT of h (≈ sampled DTFT at k = 2pi/N * k)
H_dft = np.fft.fft(h)

# DFT matrix
F = scipy.linalg.dft(N)
FH = F.conj().T

# Diagonalized version: F C F^H / N
D = (F @ C @ FH) / N
diag_D = np.diag(D)

# Eigenvector test: check if each DFT column is eigenvector of C
eigenvalues = []
for k in range(N):
    v_k = F[:, k]
    lambda_k = np.round(np.dot(C @ v_k, np.conj(v_k)) / np.dot(v_k, np.conj(v_k)
    eigenvalues.append(lambda_k)

# Print relationships
print("Impulse response h:", h)
print("\nDiagonal of circulant matrix C:", diag_C)
print("\nDFT of h (H_dft):", H_dft)
print("\nDiagonal of F C F^H / N:", diag_D)
print("\nEigenvalues from eigenvector test (C v_k = lambda_k v_k):", eigenvalues
```

```
Impulse response h: [1 2 3 4]

Diagonal of circulant matrix C: [1. 1. 1. 1.]

DFT of h (H_dft): [10.+0.j -2.+2.j -2.+0.j -2.-2.j]

Diagonal of F C F^H / N: [10.+0.j -2.+2.j -2.+0.j -2.-2.j]

Eigenvalues from eigenvector test (C v_k = lambda_k v_k): [np.complex128(10+0j),
np.complex128(-2-2j), np.complex128(-2+0j), np.complex128(-2+2j)]
```

(g) What changes if the system is not shift-invariant?

If the system is no longer shift-invariant, then the system matrix C is no longer circulant.
This is because a **circulant structure directly reflects shift-invariance**: shifting the input simply results in an equally shifted output.

In the circulant case, each column of C is a circular shift of the impulse response $\tilde{h}$.
That allows us to build C by convolving $\tilde{h}$ with shifted unit impulses δ_{x_i}:

$$C = [\vec{c}_1 \; \vec{c}_2 \; \cdots \; \vec{c}_N], \quad \text{where} \quad \vec{c}_i = \tilde{h} \circledast \delta_{x_i}$$

However, if the system is **not shift-invariant**, then the impulse response varies with position.
So the columns of C are no longer circular shifts of each other.

As a result:

- C is no longer circulant — it may become a **Toeplitz** matrix (shift-variant but structured) or even fully unstructured.
- We **cannot construct** the matrix C by simply shifting the input and observing the impulse response.
- Circular convolution modeling breaks down.

In the shift-invariant (circulant) case, the DFT matrix F diagonalizes C:

$$FCF^{-1} = \Lambda$$

Here, Λ is a diagonal matrix, and the columns of F (the DFT basis vectors) are the eigenvectors of C.

But when C is no longer circulant, its rows are not circular shifts anymore.
This means the DFT basis vectors are **no longer eigenvectors** of C, so:

$$FCF^{-1} \neq \text{diagonal}$$

In other words, the DFT cannot simplify or diagonalize the system anymore.
This is because a non shift-invariant system treats each input location differently,
so the DFT — which assumes uniform, periodic structure — is no longer valid for spectral analysis.

That's why C is no longer diagonalizable by the DFT.

Problem 2.5

In [41]:
```
!pip install scikit-image
```

```
Requirement already satisfied: scikit-image in c:\users\eozka\desktop\remote-imag
e-formation\venv\lib\site-packages (0.25.2)
Requirement already satisfied: numpy>=1.24 in c:\users\eozka\desktop\remote-image
-formation\venv\lib\site-packages (from scikit-image) (2.2.5)
Requirement already satisfied: scipy>=1.11.4 in c:\users\eozka\desktop\remote-ima
ge-formation\venv\lib\site-packages (from scikit-image) (1.15.2)
Requirement already satisfied: networkx>=3.0 in c:\users\eozka\desktop\remote-ima
ge-formation\venv\lib\site-packages (from scikit-image) (3.4.2)
Requirement already satisfied: pillow>=10.1 in c:\users\eozka\desktop\remote-imag
e-formation\venv\lib\site-packages (from scikit-image) (11.2.1)
Requirement already satisfied: imageio!=2.35.0,>=2.33 in c:\users\eozka\desktop\r
emote-image-formation\venv\lib\site-packages (from scikit-image) (2.37.0)
Requirement already satisfied: tifffile>=2022.8.12 in c:\users\eozka\desktop\remo
te-image-formation\venv\lib\site-packages (from scikit-image) (2025.3.30)
Requirement already satisfied: packaging>=21 in c:\users\eozka\desktop\remote-ima
ge-formation\venv\lib\site-packages (from scikit-image) (25.0)
Requirement already satisfied: lazy-loader>=0.4 in c:\users\eozka\desktop\remote-
image-formation\venv\lib\site-packages (from scikit-image) (0.4)
```

```
[notice] A new release of pip is available: 24.0 -> 25.1.1
[notice] To update, run: python.exe -m pip install --upgrade pip
```

In [42]:
```python
from skimage import data
X = data.camera()

# Display the image
plt.imshow(X, cmap='gray')
plt.axis('off')
plt.show()
```

In [43]:
```
import scipy.signal


def sepconv2(X, h1, h2):
    h1 = h1.reshape(1, -1)  # 1D → row vector
    h2 = h2.reshape(-1, 1)  # 1D → column vector
    conv1 = scipy.signal.convolve(X,h1)
    conv2 = scipy.signal.convolve(conv1,h2)

    return conv2
```

In [44]:
```
P1 = 5
P2 = 5
h1 = np.ones(P1) / P1
h2 = np.ones(P2) / P2
X_blur = sepconv2(X,h1,h2)

# Display the image
plt.imshow(X_blur, cmap='gray')
plt.axis('off')
plt.show()
```

In [45]:
```
# Try different (P1, P2) sizes for box filter
P_pairs = [(3, 4), (5, 9), (9, 12)]
results = []

for P1, P2 in P_pairs:

    h1 = np.ones(P1) / P1
    h2 = np.ones(P2) / P2

    Y_sep = sepconv2(X, h1, h2)

    h2d = np.ones((P2,P1)) / (P1*P2)
    Y_builtin = scipy.signal.convolve2d(X, h2d)
```

```
    error = np.abs(Y_sep - Y_builtin).mean()
    results.append((P1, P2, error))

df = pd.DataFrame(results, columns=["P1", "P2", "Mean Absolute Error"])
df
```

Out[45]:

	P1	P2	Mean Absolute Error
0	3	4	1.134689e-14
1	5	9	4.239239e-14
2	9	12	7.274436e-14

The test results above show that the output of your sepconv2 function is nearly identical to that of the built-in convolve2d function. The mean absolute errors are on the order of ~10^{-14}, which are due to numerical rounding and can be considered negligible.

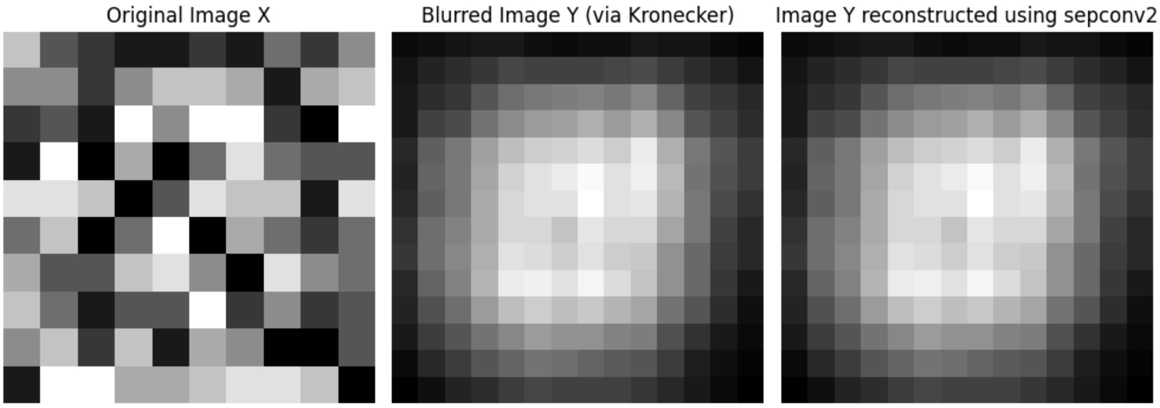
In [46]:
```
def sepconvmtx2(h1, L1, h2, L2):
    H1 = convmtx(h1,L1)
    H2 = convmtx(h2,L2)

    # Kronecker product of H1 and H2
    K = np.kron(H2,H1)

    return K
```

In [47]:
```
P1 = 5
P2 = 5
h1 = np.ones(P1) / P1
h2 = np.ones(P2) / P2
X = np.random.randint(0, 10, (10, 10))  # small 5×5 image
# Flatten it in lexicographic order
X_lex = X.flatten(order='F')
K = sepconvmtx2(h1, X.shape[0], h2, X.shape[1])
Y_lex = K @ X_lex
```

```
C:\Users\eozka\AppData\Local\Temp\ipykernel_26868\3581256877.py:11: FutureWarnin
g: Beginning in SciPy 1.17, multidimensional input will be treated as a batch, no
t `ravel`ed. To preserve the existing behavior and silence this warning, `ravel`
arguments before passing them to `toeplitz`.
  return scipy.linalg.toeplitz(column_C, row_C)
```

In [48]:
```
side = int(np.sqrt(Y_lex.shape[0]))
Y = Y_lex.reshape((side, side), order='F')
Y_sep = sepconv2(X,h1,h2)

# Plot input and output images
fig, axes = plt.subplots(1, 3, figsize=(10, 4))
axes[0].imshow(X, cmap='gray')
axes[0].set_title('Original Image X')
axes[0].axis('off')

axes[1].imshow(Y, cmap='gray')
axes[1].set_title('Blurred Image Y (via Kronecker)')
axes[1].axis('off')
```

```
axes[2].imshow(Y_sep, cmap='gray')
axes[2].set_title('Image Y reconstructed using sepconv2')
axes[2].axis('off')

plt.tight_layout()
plt.show()

# Error between the two methods
error = np.abs(Y_sep - Y).mean()
print(f"Mean Absolute Error: {error:.6f}")
```




```
Mean Absolute Error: 0.000000
```

In [ ]: