Algebraic Representation of Optimization Models

Linear Programming Problems

$$\text{minimize} \quad c_1 x_1 + c_2 x_2 + \cdots + c_N x_N$$

$$\text{subject to:} \quad a_{11} x_1 + a_{12} x_2 + \cdots + a_{1N} x_N \leq b_1$$

$$a_{21} x_1 + a_{22} x_2 + \cdots + a_{2N} x_N \leq b_2$$

$$\vdots \qquad \vdots \qquad \ddots \qquad \vdots \qquad \vdots$$

$$a_{M1} x_1 + a_{M2} x_2 + \cdots + a_{MN} x_N \leq b_M$$

$$l_1 \leq x_1 \leq u_1$$

$$\vdots$$

$$l_N \leq x_N \leq u_N$$

M = # of constraints

N = # of decision variables

$$\text{minimize} \quad \sum_{i=1}^{N} c_i x_i$$

$$\text{subject to:} \quad \sum_{i=1}^{N} a_{mi} x_i \leq b_m \quad \forall m$$

$$l_i \leq x_i \leq u_i \quad \forall i$$

$$c = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_N \end{bmatrix}_{N\times 1} \qquad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix}_{N\times 1}$$

minimize $c^T x$

$$[c_1 \; c_2 \; \cdots \; c_N]_{1\times N} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix}_{N\times 1} = c_1 x_1 + c_2 x_2 + \cdots + c_N x_N$$

$$a_1 = \begin{bmatrix} a_{11} \\ a_{12} \\ \vdots \\ a_{1N} \end{bmatrix}$$

subject to $a_{11}x_1 + a_{12}x_2 + \cdots + a_{1N}x_N \leq b_1$

$$\underbrace{[a_{11} \; a_{12} \; \cdots \; a_{1N}]}_{a_1^T}{}_{1\times N} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix}_{N\times 1} \leq b_1 \Rightarrow a_1^T x \leq b_1$$

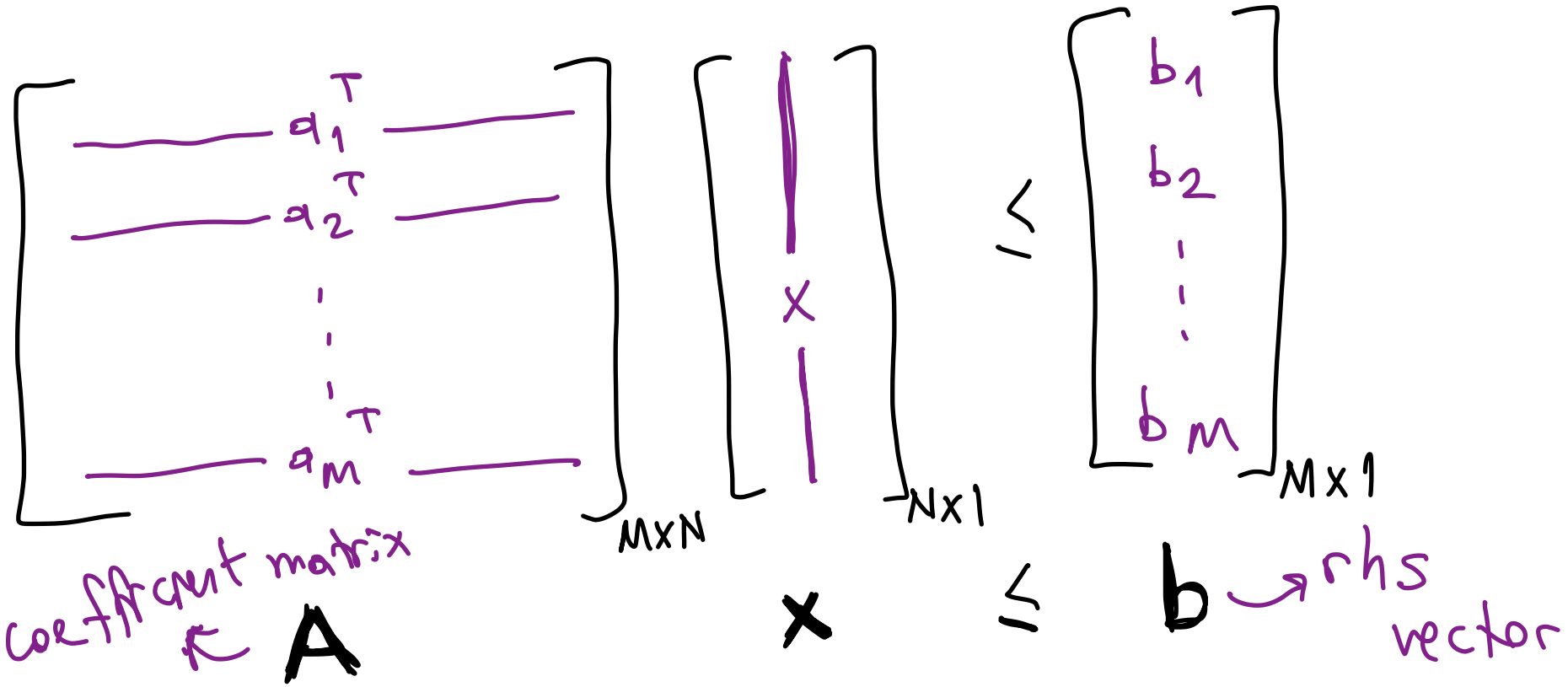
subject to:

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1N}x_N \le b_1 \Rightarrow a_1^T \cdot x \le b_1$$

$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2N}x_N \le b_2 \Rightarrow a_2^T \cdot x \le b_2$$

$$\vdots$$

$$a_{M1}x_1 + a_{M2}x_2 + \cdots + a_{MN}x_N \le b_M \Rightarrow a_M^T \cdot x \le b_M$$



minimize $c^T x$

subject to: $Ax \leq b$

$l \leq x \leq u$

$c \in \mathbb{R}^N$

$x \in \mathbb{R}^N$

$A \in \mathbb{R}^{M \times N}$

$b \in \mathbb{R}^M$

$l \in \mathbb{R}^N$

$u \in \mathbb{R}^N$

$$c = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_N \end{bmatrix}_{N \times 1} \quad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix}_{N \times 1}$$

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1N} \\ a_{21} & a_{22} & \cdots & a_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ a_{M1} & a_{M2} & \cdots & a_{MN} \end{bmatrix}_{M \times N} \quad b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_M \end{bmatrix}_{M \times 1}$$

$$l = \begin{bmatrix} l_1 \\ l_2 \\ \vdots \\ l_N \end{bmatrix}_{N \times 1} \quad u = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_N \end{bmatrix}_{N \times 1}$$

Model Parameters: c, A, b, l, u

	P_1	P_2	
M_1	1	0	4 hours
M_2	0	2	12 hours
M_3	3	2	18 hours
	\$3	\$5	

x_1 = # of P_1 that we produce

x_2 = # of P_2 that we produce.

our aim was maximizing the total profit.

maximize $3x_1 + 5x_2 \Rightarrow$ minimize $-3x_1 - 5x_2$

Subject to:

$$1x_1 + 0x_2 \leq 4$$
$$0x_1 + 2x_2 \leq 12$$
$$3x_1 + 2x_2 \leq 18$$
$$0 \leq x_1 \leq +\infty$$
$$0 \leq x_2 \leq +\infty$$

$M = 3$

$N = 2$

$3x_1 + 4x_2 = 12$

$\Downarrow$

$$\begin{cases} 3x_1 + 4x_2 \leq 12 \\ 3x_1 + 4x_2 \geq 12 \end{cases}$$

$\Downarrow$

$$\begin{cases} 3x_1 + 4x_2 \leq 12 \\ -3x_1 - 4x_2 \leq -12 \end{cases}$$

$$\mathbf{c} = \begin{bmatrix} -3 \\ -5 \end{bmatrix}_{2\times 1} \quad \mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 3 & 2 \end{bmatrix}_{3\times 2} \quad \mathbf{b} = \begin{bmatrix} 4 \\ 12 \\ 18 \end{bmatrix}_{3\times 1}$$

$$\mathbf{l} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}_{2\times 1} \quad \mathbf{u} = \begin{bmatrix} +\infty \\ +\infty \end{bmatrix}_{2\times 1}$$

minimize $\underbrace{\begin{bmatrix} -3 & -5 \end{bmatrix}}_{c^T} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{x}$ $\quad c^T \cdot x$

subject to: $\underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 3 & 2 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{x} \leq \underbrace{\begin{bmatrix} 4 \\ 12 \\ 18 \end{bmatrix}}_{b}$ $\quad Ax \leq b$

$\underbrace{\begin{bmatrix} 0 \\ 0 \end{bmatrix}}_{l} \leq \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{x} \leq \underbrace{\begin{bmatrix} +\infty \\ +\infty \end{bmatrix}}_{u}$ $\quad l \leq x \leq u$

```
def LP_solver(A, b, c, l, u):
    .
    .
    .
    return(x)
```

Problem Instance = {A, b, c, l, u}

Solution = {x}

minimize -5x
subject to: 0 ≤ x
x ≤ -2

→ might be an empty set

→ might be an unbounded problem

minimize -5x
subject to: 0 ≤ x ≤ 3

→ NO SOLUTION!

Quadratic Programming Problems

$$\text{minimize} \quad \sum_{i=1}^{N}\sum_{j=1}^{N} q_{ij}\, x_i x_j + \sum_{i=1}^{N} c_i . x_i$$

quadratic terms — linear terms

$$x_1^2 + 2x_1x_2 + x_2^2 = (x_1+x_2)^2$$

$$x_1.x_1 + x_1.x_2 + x_2.x_1 + x_2.x_2$$

$$\sum_{i=1}^{2}\sum_{j=1}^{2} 1.x_i.x_j \Rightarrow$$

$i=1$	$j=1$	$x_1.x_1$	$q_{11}=1$
$i=1$	$j=2$	$x_1.x_2$	$q_{12}=1$
$i=2$	$j=1$	$x_2.x_1$	$q_{21}=1$
$i=2$	$j=2$	$x_2 x_2$	$q_{22}=1$

$$x^T . Q . x$$

$$\begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1^2 + x_1.x_2 + x_2.x_1 + x_2^2$$

minimize $\underbrace{x^T \cdot Q \cdot x}_{A} + \underbrace{c^T \cdot x}_{B}$

(A) $$\begin{bmatrix} x_1 & x_2 & \cdots & x_N \end{bmatrix} \begin{bmatrix} q_{11} & q_{12} & \cdots & q_{1N} \\ q_{21} & q_{22} & \cdots & q_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ q_{N1} & q_{N2} & \cdots & q_{NN} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix}$$

(B) $$\begin{bmatrix} c_1 & c_2 & \cdots & c_N \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix}$$