Boundary Condition for Entry Interface

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1 (Geo)Detic Latitude / Altitude [1][2]

Given a probe position in fixed coordinate as (x, y, z), the (geo)centric latitude is written as

$$\phi_c = \arctan \frac{y}{x} \tag{1}$$

Using the equatorial radius R and flattening f, x_a in Figure 1 is written as

$$x_a = \frac{(1-f)R}{\sqrt{\tan^2 \phi_c + (1-f)^2}} \tag{2}$$

Using the relationship between (geo)centric latitude at the planet's surface and (geo)detic latitude, ϕ_{dg} is written as

$$\phi_{dg} = \arctan \left(\frac{\tan \phi_c}{(1-f)^2} \right) \tag{3}$$

The radius r_a from the center of the planet (O) to the surface of the planet (S) in Figure 1 is calculated by using trigonometric relationship.

$$r_a = \frac{x_a}{\cos \phi_c} \tag{4}$$

The distance from (S) to (P) in Figure 1 is defined by

$$l = r - r_a \tag{5}$$

The angular difference between (geo)centric latitude and (geo)detic latitude at (S) in Figure 1 is defined by

$$\delta\phi_g = \phi_{dg} - \phi_c \tag{6}$$

The equation for the radius of curvature in the Meridian at ϕ_{dg} (distance between (S) and (W) in Figure 1) is written as

$$\rho_a = \frac{R(1-f)^2}{\left(1-(2f-f^2)\sin^2\phi_{dg}\right)^{3/2}} \tag{7}$$

Then the (geo)detic latitude is calculated with

$$\phi_d = \phi_{dg} - \delta\phi \tag{8}$$

,where

$$\delta\phi = \arctan\left(\frac{l\sin\delta\phi_g}{\rho_a + l\cos\delta\phi_g}\right) \tag{9}$$

The (geo)detic altitude above the planetary ellipsoid is calculated with

$$h = \sqrt{x^2+y^2}\cos\phi_d + \left(z + (2f-f^2)N\sin\phi_d\right)\sin\phi_d - N \tag{10}$$

,where the radius of curvature in the vertical prime N (distance between (T) and (V) in Figure 1) is written as

$$N = \frac{R}{\sqrt{1-(2f-f^2)\sin^2\phi_d}} \tag{11}$$

2 Velocity vector

2.1 Magnitude

The magnitude of the entry inertial velocity vector and its derivatives are written as

$$\begin{aligned} v = |\vec{v}| &= \sqrt{v_x^2+v_y^2+v_z^2} \\ \frac{\partial v_r}{\partial \vec{v}} &= \frac{\vec{v_r}}{v_r} \end{aligned} \tag{12}$$

2.2 Azimuth in spherical local coordinate

The azimuth angle Az of the inertial velocity vector in spherical coordinate is written as

$$Az = \arctan\frac{v_{east}}{v_{north}} \tag{13}$$

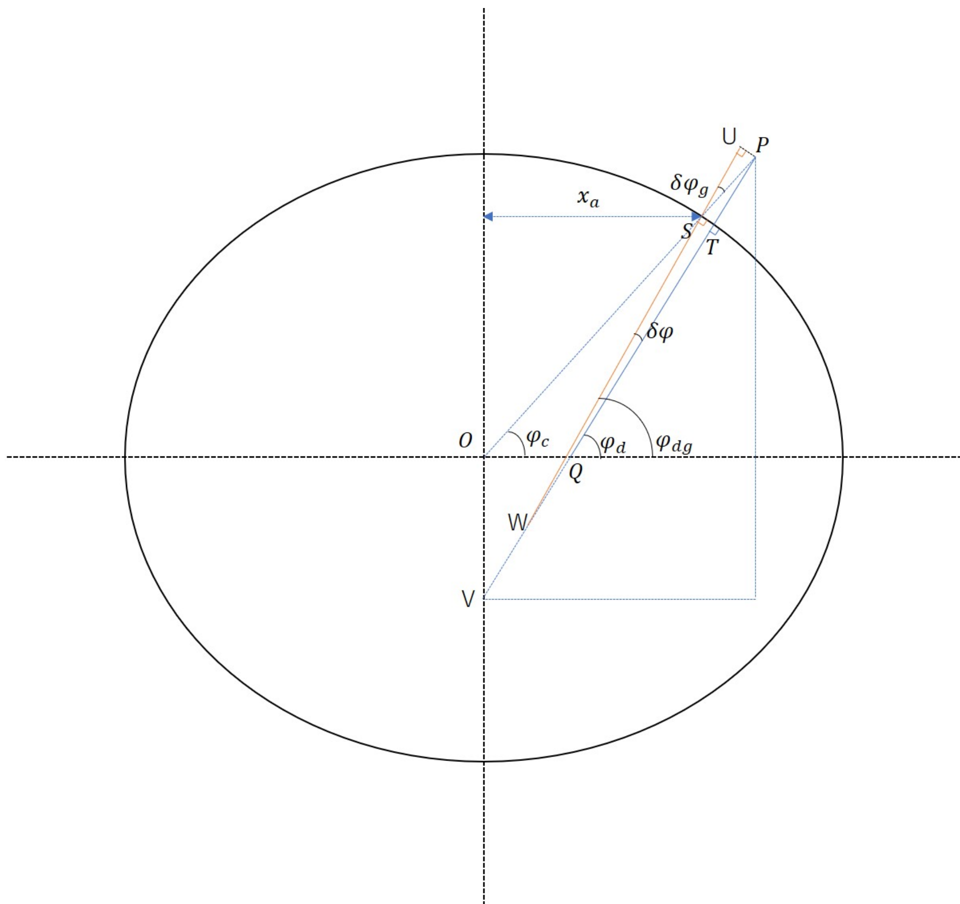

Figure 1: Geometry of (geo)detic latitude and altitude

,where v_{east} and v_{north} are inertial velocity elements at the spherical local coordinate as

$$\begin{bmatrix} v_{up} \\ v_{east} \\ v_{north} \end{bmatrix} = \begin{bmatrix} \cos\phi & 0 & \sin\phi \\ 0 & 1 & 0 \\ -\sin\phi & 0 & \cos\phi \end{bmatrix} \begin{bmatrix} \cos\lambda & \sin\lambda & 0 \\ -\sin\lambda & \cos\lambda & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} v_r \\ v_r \\ v_r \end{bmatrix} \tag{14}$$

Here, ϕ is a (geo)centric latitude and λ is a longitude.
The derivative of azimuth angle is written as

$$\frac{\partial Az}{\partial *} = \frac{v_{north}^2}{v_{north}^2 + v_{east}^2} \left(\frac{1}{v_{north}} \frac{\partial v_{east}}{\partial *} - \frac{v_{east}}{v_{north}^2} \frac{\partial v_{north}}{\partial *} \right) \tag{15}$$

,where derivatives of these local velocity elements are written as

$$\begin{aligned} \frac{\partial}{\partial *} \begin{bmatrix} v_{up} \\ v_{east} \\ v_{north} \end{bmatrix} &= \begin{bmatrix} -\sin\phi \frac{\partial\phi}{\partial *} & 0 & \cos\phi \frac{\partial\phi}{\partial *} \\ 0 & 0 & 0 \\ -\cos\phi \frac{\partial\phi}{\partial *} & 0 & -\sin\phi \frac{\partial\phi}{\partial *} \end{bmatrix} \begin{bmatrix} \cos\lambda & \sin\lambda & 0 \\ -\sin\lambda & \cos\lambda & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix} \\ &+ \begin{bmatrix} \cos\phi & 0 & \sin\phi \\ 0 & 1 & 0 \\ -\sin\phi & 0 & \cos\phi \end{bmatrix} \begin{bmatrix} -\sin\lambda \frac{\partial\lambda}{\partial *} & \cos\lambda \frac{\partial\phi}{\partial *} & 0 \\ -\cos\lambda \frac{\partial\phi}{\partial *} & -\sin\lambda \frac{\partial\phi}{\partial *} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix} \end{aligned} \tag{16}$$

$$\frac{\partial}{\partial \vec{v}} \begin{bmatrix} v_{up} \\ v_{east} \\ v_{north} \end{bmatrix} = \begin{bmatrix} \cos\phi & 0 & \sin\phi \\ 0 & 1 & 0 \\ -\sin\phi & 0 & \cos\phi \end{bmatrix} \begin{bmatrix} \cos\lambda & \sin\lambda & 0 \\ -\sin\lambda & \cos\lambda & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{17}$$

2.3 Horizontal Flight Path Angle in spherical local coordinate

The horizontal flight path angle $HFPA$ of the inertial velocity vector in spherical coordinate is written as

$$\begin{aligned} HFPA &= \arctan \frac{v_{up}}{\sqrt{v_{north}^2 + v_{east}^2}} \\ &= \arccos \frac{\vec{r} \cdot \vec{v}}{|\vec{r}||\vec{v}|} \end{aligned} \tag{18}$$

The derivative of horizontal flight path angle is written as

$$\frac{\partial HFPA}{\partial *} = \frac{\sqrt{v_{north}^2 + v_{east}^2}}{v_{north}^2 + v_{east}^2 + v_{up}^2} \left(\frac{\partial v_{up}}{\partial *} - \frac{v_{up}}{v_{north}^2 + v_{east}^2} \left(v_{north} \frac{\partial v_{north}}{\partial *} + v_{east} \frac{\partial v_{east}}{\partial *} \right) \right) \tag{19}$$

The derivatives of the local velocity elements are same with those in 2.2

2.4 Vertical Flight Path Angle in spherical local coordinate

The vertical flight path angle $VFPA$ of the inertial velocity vector in spherical coordinate is written as

$$VFPA = \arccos \frac{\vec{r} \cdot \vec{v}}{|\vec{r}||\vec{v}|} \tag{20}$$

The derivative of $\cos(VFPA)$ is written as

$$\begin{aligned} \frac{\partial \cos(VFPA)}{\partial \vec{r}} &= \frac{\vec{v}}{|\vec{r}||\vec{v}|} - \frac{\vec{r} \cdot \vec{v}}{|\vec{r}|^2|\vec{v}|} \frac{\partial |\vec{r}|}{\partial \vec{r}} \\ \frac{\partial \cos(VFPA)}{\partial \vec{v}} &= \frac{\vec{r}}{|\vec{r}||\vec{v}|} - \frac{\vec{r} \cdot \vec{v}}{|\vec{r}||\vec{v}|^2} \frac{\partial |\vec{v}|}{\partial \vec{v}} \end{aligned} \tag{21}$$

3 Relative velocity vector

The entry velocity vector of a vehicle relative to the Earth ground (or atmosphere) is written as

$$\vec{v_r} = \vec{v} - \vec{\omega} \times \vec{r} = \vec{v} - \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} \vec{r} \tag{22}$$

3.1 Magnitude

The magnitude of the relative velocity vector and its derivatives are written as

$$\begin{aligned} v_r &= |\vec{v_r}| = \sqrt{v_{rx}^2 + v_{ry}^2 + v_{rz}^2} \\ \frac{\partial v_r}{\partial \vec{r}} &= \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} \frac{\partial v_r}{\partial \vec{v_r}} \\ \frac{\partial v_r}{\partial \vec{v}} &= \frac{\vec{v_r}}{v_r} \end{aligned} \tag{23}$$

3.2 Azimuth in (geo)detic local coordinate

The azimuth angle of the relative velocity vector Az is written as

$$Az = \arctan \frac{v_{r_{east}}}{v_{r_{north}}} \tag{24}$$

,where $v_{r_{east}}$ and $v_{r_{north}}$ are relative velocity elements at the horizontal local coordinate as

$$\begin{bmatrix} v_{r_{up}} \\ v_{r_{east}} \\ v_{r_{north}} \end{bmatrix} = \begin{bmatrix} \cos\phi & 0 & \sin\phi \\ 0 & 1 & 0 \\ -\sin\phi & 0 & \cos\phi \end{bmatrix} \begin{bmatrix} \cos\lambda & \sin\lambda & 0 \\ -\sin\lambda & \cos\lambda & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} v_{rx} \\ v_{ry} \\ v_{rz} \end{bmatrix} \tag{25}$$

Here, ϕ is a (geo)detic latitude and λ is a longitude.
The derivative of azimuth angle is written as

$$\frac{\partial Az}{\partial *} = \frac{v_{r_{north}}^2}{v_{r_{north}}^2 + v_{r_{east}}^2} \left(\frac{1}{v_{r_{north}}} \frac{\partial v_{r_{east}}}{\partial *} - \frac{v_{r_{east}}}{v_{r_{north}}^2} \frac{\partial v_{r_{north}}}{\partial *} \right) \tag{26}$$

,where derivatives of these local velocity elements are written as

$$\begin{aligned} \frac{\partial}{\partial *} \begin{bmatrix} v_{r_{up}} \\ v_{r_{east}} \\ v_{r_{north}} \end{bmatrix} &= \begin{bmatrix} -\sin\phi\frac{\partial\phi}{\partial *} & 0 & \cos\phi\frac{\partial\phi}{\partial *} \\ 0 & 0 & 0 \\ -\cos\phi\frac{\partial\phi}{\partial *} & 0 & -\sin\phi\frac{\partial\phi}{\partial *} \end{bmatrix} \begin{bmatrix} \cos\lambda & \sin\lambda & 0 \\ -\sin\lambda & \cos\lambda & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} v_{rx} \\ v_{ry} \\ v_{rz} \end{bmatrix} \\ &+ \begin{bmatrix} \cos\phi & 0 & \sin\phi \\ 0 & 1 & 0 \\ -\sin\phi & 0 & \cos\phi \end{bmatrix} \begin{bmatrix} -\sin\lambda\frac{\partial\lambda}{\partial *} & \cos\lambda\frac{\partial\phi}{\partial *} & 0 \\ -\cos\lambda\frac{\partial\phi}{\partial *} & -\sin\lambda\frac{\partial\phi}{\partial *} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_{rx} \\ v_{ry} \\ v_{rz} \end{bmatrix} \\ &+ \begin{bmatrix} \cos\phi & 0 & \sin\phi \\ 0 & 1 & 0 \\ -\sin\phi & 0 & \cos\phi \end{bmatrix} \begin{bmatrix} \cos\lambda & \sin\lambda & 0 \\ -\sin\lambda & \cos\lambda & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{\partial v_{rx}}{\partial *} \\ \frac{\partial v_{ry}}{\partial *} \\ \frac{\partial v_{rz}}{\partial *} \end{bmatrix} \end{aligned} \tag{27}$$

$$\frac{\partial}{\partial \vec{v}} \begin{bmatrix} v_{r_{up}} \\ v_{r_{east}} \\ v_{r_{north}} \end{bmatrix} = \begin{bmatrix} \cos\phi & 0 & \sin\phi \\ 0 & 1 & 0 \\ -\sin\phi & 0 & \cos\phi \end{bmatrix} \begin{bmatrix} \cos\lambda & \sin\lambda & 0 \\ -\sin\lambda & \cos\lambda & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{28}$$

3.3 Horizontal Flight Path Angle in (geo)detic local coordinate

The horizontal flight path angle of the relative velocity vector $HFPA$ is written as

$$HFPA = \arctan \frac{v_{r_{up}}}{\sqrt{v_{r_{north}}^2 + v_{r_{east}}^2}} \tag{29}$$

The derivative of horizontal flight path angle is written as

$$\frac{\partial HFPA}{\partial *} = \frac{\sqrt{v_{r_{north}}^2 + v_{r_{east}}^2}}{v_{r_{north}}^2 + v_{r_{east}}^2 + v_{r_{up}}^2} \left(\frac{\partial v_{r_{up}}}{\partial *} - \frac{v_{r_{up}}}{v_{r_{north}}^2 + v_{r_{east}}^2} \left(v_{r_{north}} \frac{\partial v_{r_{north}}}{\partial *} + v_{r_{east}} \frac{\partial v_{r_{east}}}{\partial *} \right) \right) \tag{30}$$

The derivatives of the local velocity elements are same with those in 3.2

References

[1] https://jp.mathworks.com/help/aeroblks/ecefpositiontolla.html

[2] https://jp.mathworks.com/help/aeroblks/geocentrictogeodeticlatitude.html