Conversions Between Cartesian and B-Plane States

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Abstract

This document describes conversions between Cartesian and B-Plane state variables. Transformations in both directions and associated Jacobians are given. Primary emphasis is given to transformations for the "incoming" velocity case. Later, the changes required to adapt the equations to the "outgoing" velocity case are given.

Contents

Nomenclature

α Right ascension of hyperbolic asymptote

δ Declination of hyperbolic asymptote

μ Gravitational parameter of flyby body

ν	True anomaly
ν_∞	True anomaly at infinity
θ	B-Plane clock angle
$\hat{\boldsymbol{R}}$	B-Plane R unit vector
$\hat{\boldsymbol{S}}$	Unit vector in direction of incoming asymptote
$\hat{\boldsymbol{T}}$	B-Plane T unit vector
$\boldsymbol{\phi}$	B-Plane reference vector
$\boldsymbol{B}$	B-Plane B vector
$\boldsymbol{e}$	Eccentricity vector
$\boldsymbol{h}$	Angular momentum vector
$\boldsymbol{I}$	Identity matrix
$\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}$	Unit vectors
$\boldsymbol{P}$	Shorthand for $\boldsymbol{h} \times \boldsymbol{e}$
$\boldsymbol{r}$	Position vector
$\boldsymbol{v}$	Velocity vector
$\boldsymbol{v}_{\infty,in}$	Incoming velocity vector at infinity
$\boldsymbol{x}$; $\hat{\boldsymbol{x}}$; x	Arbitrary vector; its unit vector; its magnitude
$\boldsymbol{x}_c$	Cartesian state vector
b	Magnitude of $\boldsymbol{B}$ vector (sometimes called Δ)
B_R	$\boldsymbol{B}^T\hat{\boldsymbol{R}}$
B_T	$\boldsymbol{B}^T\hat{\boldsymbol{T}}$
r_p	Periapsis radius
v_∞	Magnitude of velocity at infinity
x, y, z	As subscripts: represent components of 3D vector

1 State Representations

1.1 Cartesian State

The Cartesian state consists of the position and velocity vector of the spacecraft in an assumed inertial reference frame whose origin is the spacecraft's flyby body:

$$\boldsymbol{x}_c = \begin{pmatrix} \boldsymbol{r} \\ \boldsymbol{v} \end{pmatrix}_{6\times 1}. \tag{1}$$

1.2 B-Plane State

The B-Plane state is given by the vector

$$\boldsymbol{x}_b = \begin{pmatrix} v_\infty \\ \alpha \\ \delta \\ b \\ \theta \\ \nu \end{pmatrix}_{6\times 1}. \tag{2}$$

Full definition of the B-Plane state requires setting a reference vector (frequently some inertial $\boldsymbol{k}$). In this document, the reference vector is denoted $\boldsymbol{\phi}$ and left undefined further.

2 Cartesian State to B-Plane State Transformation

First, define standard convenience variables:

$$\boldsymbol{e} = \frac{1}{\mu}\left[\left(v^2 - \frac{\mu}{r}\right)\boldsymbol{r} - \left(\boldsymbol{r}^T\boldsymbol{v}\right)\boldsymbol{v}\right] \tag{3}$$

$$\boldsymbol{h} = \boldsymbol{r} \times \boldsymbol{v} \tag{4}$$

$$\boldsymbol{P} = \boldsymbol{h} \times \boldsymbol{e}. \tag{5}$$

Then, $\hat{\boldsymbol{S}}$, in the direction of the incoming asymptote (i.e., $\boldsymbol{v}_{\infty,in}$) is given by

$$\hat{\boldsymbol{S}} = \frac{1}{e}\hat{\boldsymbol{e}} + \sqrt{1 - \frac{1}{e^2}}\hat{\boldsymbol{P}}. \tag{6}$$

Additional variables are defined by

$$\hat{\boldsymbol{T}} = \frac{\hat{\boldsymbol{S}} \times \boldsymbol{\phi}}{||\hat{\boldsymbol{S}} \times \boldsymbol{\phi}||} \tag{7}$$

$$\hat{\boldsymbol{R}} = \frac{\hat{\boldsymbol{S}} \times \boldsymbol{T}}{||\hat{\boldsymbol{S}} \times \boldsymbol{T}||} \tag{8}$$

$$\hat{\boldsymbol{B}} = \frac{\hat{\boldsymbol{S}} \times \boldsymbol{h}}{||\hat{\boldsymbol{S}} \times \boldsymbol{h}||} \tag{9}$$

$$\boldsymbol{B} = b\left[\sqrt{1 - \frac{1}{e^2}}\hat{\boldsymbol{e}} - \frac{1}{e}\hat{\boldsymbol{P}}\right] \tag{10}$$

$$b = \frac{h^2}{\mu\sqrt{e^2 - 1}}. \tag{11}$$

Then, the B-Plane dot products are

$$B_T = \boldsymbol{B}^T\hat{\boldsymbol{T}} \tag{12}$$

$$B_R = \boldsymbol{B}^T\hat{\boldsymbol{R}}. \tag{13}$$

The B-Plane clock angle is given by

$$\theta = \text{atan2}\left(B_R, B_T\right). \tag{14}$$

The magnitude of the velocity at infinity is

$$v_\infty = \sqrt{v^2 - \frac{2\mu}{r}}. \tag{15}$$

The radius of periapsis is

$$r_p = \frac{\mu\left(e - 1\right)}{v_\infty^2}. \tag{16}$$

The right ascension and declination of the incoming asymptote are given by

$$\alpha = \text{atan2}\left(S_y, S_x\right) \tag{17}$$

$$\delta = \text{asin}\left(\frac{S_z}{S}\right) \tag{18}$$

$$= \text{asin}\left(S_z\right). \tag{19}$$

True anomaly is given by the angle between e and r:

$$\nu = \operatorname{atan2}\left(||\boldsymbol{e} \times \boldsymbol{r}||, \boldsymbol{e}^T \boldsymbol{r}\right), \tag{20}$$

with the quadrant check:

$$\text{if } \boldsymbol{r}^T \boldsymbol{v} < 0 : \quad \nu \leftarrow 2\pi - \nu. \tag{21}$$

3 Cartesian State to B-Plane State Transformation Jacobian

3.1 Derivatives of Position Vector

$$\frac{\partial \boldsymbol{r}}{\partial \boldsymbol{x}_c} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \tag{22}$$

3.2 Derivatives of Velocity Vector

$$\frac{\partial \boldsymbol{v}}{\partial \boldsymbol{x}_c} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{23}$$

3.3 Derivatives of Right Ascension of Velocity at Infinity

$$\frac{\partial \alpha}{\partial \hat{\boldsymbol{S}}} = \begin{bmatrix} -\dfrac{\hat{S}_y}{\hat{S}_x^2 + \hat{S}_y^2} & \dfrac{\hat{S}_x}{\hat{S}_x^2 + \hat{S}_y^2} & 0 \end{bmatrix} \tag{24}$$

$$\frac{\partial \alpha}{\partial \boldsymbol{x}_c} = \frac{\partial \alpha}{\partial \hat{\boldsymbol{S}}} \frac{\partial \hat{\boldsymbol{S}}}{\partial \boldsymbol{x}_c} \tag{25}$$

3.4 Derivatives of Declination of Velocity at Infinity

$$\frac{\partial \delta}{\partial \hat{\boldsymbol{S}}} = \begin{bmatrix} 0 & 0 & \dfrac{1}{\sqrt{1 - \hat{S}_z^2}} \end{bmatrix} \tag{26}$$

$$\frac{\partial \delta}{\partial \boldsymbol{x}_c} = \frac{\partial \delta}{\partial \hat{\boldsymbol{S}}} \frac{\partial \hat{\boldsymbol{S}}}{\partial \boldsymbol{x}_c} \tag{27}$$

3.5 Derivatives of Velocity at Infinity Magnitude

$$\frac{\partial v_\infty}{\partial \boldsymbol{x}_c} = \frac{1}{2}\left(v^2 - \frac{2\mu}{r}\right)^{-\frac{1}{2}}\left(2\boldsymbol{v}^T\frac{\partial \boldsymbol{v}}{\partial \boldsymbol{x}_c} + \frac{2\mu}{r^3}\boldsymbol{r}^T\frac{\partial \boldsymbol{r}}{\partial \boldsymbol{x}_c}\right) \tag{28}$$

3.6 Derivatives of Periapsis Radius

$$\frac{\partial r_p}{\partial \boldsymbol{x}_c} = \mu\left[\hat{\boldsymbol{e}}^T\frac{\partial \boldsymbol{e}}{\partial \boldsymbol{x}_c}v_\infty^{-2} - 2\left(e-1\right)v_\infty^{-3}\frac{\partial v_\infty}{\partial \boldsymbol{x}_c}\right] \tag{29}$$

3.7 Derivatives of B-Plane Clock Angle

$$\frac{\partial \theta}{\partial \boldsymbol{x}_c} = \frac{B_T}{B_R^2 + B_T^2}\frac{\partial B_R}{\partial \boldsymbol{x}_c} - \frac{B_R}{B_R^2 + B_T^2}\frac{\partial B_T}{\partial \boldsymbol{x}_c} \tag{30}$$

3.8 Derivatives of B_T

$$\frac{\partial B_T}{\partial \boldsymbol{x}_c} = \hat{\boldsymbol{T}}^T\frac{\partial \boldsymbol{B}}{\partial \boldsymbol{x}_c} + \boldsymbol{B}^T\frac{\partial \hat{\boldsymbol{T}}}{\partial \boldsymbol{x}_c} \tag{31}$$

3.9 Derivatives of B_R

$$\frac{\partial B_R}{\partial \boldsymbol{x}_c} = \hat{\boldsymbol{R}}^T\frac{\partial \boldsymbol{B}}{\partial \boldsymbol{x}_c} + \boldsymbol{B}^T\frac{\partial \hat{\boldsymbol{R}}}{\partial \boldsymbol{x}_c} \tag{32}$$

3.10 Derivatives of B Vector Magnitude

$$\boldsymbol{\xi}_1 \triangleq -\left(e^2-1\right)^{-\frac{3}{2}}\boldsymbol{e}^T\frac{\partial \boldsymbol{e}}{\partial \boldsymbol{x}_c} \tag{33}$$

$$\boldsymbol{\xi}_2 \triangleq 2\boldsymbol{h}^T\frac{\partial \boldsymbol{h}}{\partial \boldsymbol{x}_c} \tag{34}$$

$$\frac{\partial b}{\partial \boldsymbol{x}_c} = \frac{1}{\mu}\left[h^2\boldsymbol{\xi}_1 + \left(e^2-1\right)^{-\frac{1}{2}}\boldsymbol{\xi}_2\right] \tag{35}$$

3.11 Derivatives of B Vector

$$\frac{\partial \boldsymbol{B}}{\partial \boldsymbol{x}_c} = \hat{\boldsymbol{B}}\frac{\partial b}{\partial \boldsymbol{x}_c} + b\frac{\partial \hat{\boldsymbol{B}}}{\partial \boldsymbol{x}_c} \tag{36}$$

3.12 Derivatives of B Unit Vector

$$\frac{\partial \hat{\boldsymbol{P}}}{\partial \boldsymbol{x}_c} = \frac{1}{P}\left(\boldsymbol{I} - \frac{1}{P^2}\boldsymbol{P}\boldsymbol{P}^T\right)\frac{\partial \boldsymbol{P}}{\partial \boldsymbol{x}_c} \tag{37}$$

$$\frac{\partial \hat{\boldsymbol{e}}}{\partial \boldsymbol{x}_c} = \frac{1}{e}\left(\boldsymbol{I} - \frac{1}{e^2}\boldsymbol{e}\boldsymbol{e}^T\right)\frac{\partial \boldsymbol{e}}{\partial \boldsymbol{x}_c} \tag{38}$$

$$\zeta_1 \triangleq \sqrt{1 - \frac{1}{e^2}} \tag{39}$$

$$\frac{\partial \zeta_1}{\partial \boldsymbol{x}_c} = \frac{1}{e^3}\left(1 - \frac{1}{e^2}\right)^{-\frac{1}{2}} \hat{\boldsymbol{e}}^T \frac{\partial \boldsymbol{e}}{\partial \boldsymbol{x}_c} \tag{40}$$

$$\boldsymbol{\zeta}_2 \triangleq -\frac{1}{e^2}\hat{\boldsymbol{e}}^T \frac{\partial \boldsymbol{e}}{\partial \boldsymbol{x}_c} \tag{41}$$

$$\boldsymbol{\xi}_1 \triangleq \zeta_1 \frac{\partial \hat{\boldsymbol{e}}}{\partial \boldsymbol{x}_c} \tag{42}$$

$$\boldsymbol{\xi}_2 \triangleq \hat{\boldsymbol{e}} \frac{\partial \zeta_1}{\partial \boldsymbol{x}_c} \tag{43}$$

$$\boldsymbol{\xi}_3 \triangleq -\frac{1}{e}\frac{\partial \hat{\boldsymbol{P}}}{\partial \boldsymbol{x}_c} \tag{44}$$

$$\boldsymbol{\xi}_4 \triangleq -\hat{\boldsymbol{P}}\boldsymbol{\zeta}_2 \tag{45}$$

$$\frac{\partial \hat{\boldsymbol{B}}}{\partial \boldsymbol{x}_c} = \boldsymbol{\xi}_1 + \boldsymbol{\xi}_2 + \boldsymbol{\xi}_3 + \boldsymbol{\xi}_4 \tag{46}$$

3.13 Derivatives of R Unit Vector

$$\boldsymbol{R} \triangleq \hat{\boldsymbol{S}} \times \boldsymbol{T} \tag{47}$$

$$\frac{\partial \boldsymbol{R}}{\partial \boldsymbol{x}_c} = -\{\boldsymbol{T}\}^\times \frac{\partial \hat{\boldsymbol{S}}}{\partial \boldsymbol{x}_c} + \left\{\hat{\boldsymbol{S}}\right\}^\times \frac{\partial \boldsymbol{T}}{\partial \boldsymbol{x}_c} \tag{48}$$

$$\frac{\partial \hat{\boldsymbol{R}}}{\partial \boldsymbol{x}_c} = \frac{1}{R}\left(\boldsymbol{I} - \frac{1}{R^2}\boldsymbol{R}\boldsymbol{R}^T\right)\frac{\partial \boldsymbol{R}}{\partial \boldsymbol{x}_c} \tag{49}$$

3.14 Derivatives of T Unit Vector

$$\boldsymbol{T} \triangleq \hat{\boldsymbol{S}} \times \boldsymbol{\phi} \tag{50}$$

$$\frac{\partial \boldsymbol{T}}{\partial \boldsymbol{x}_c} = -\{\boldsymbol{\phi}\}^\times \frac{\partial \hat{\boldsymbol{S}}}{\partial \boldsymbol{x}_c} + \left\{\hat{\boldsymbol{S}}\right\}^\times \frac{\partial \boldsymbol{\phi}}{\partial \boldsymbol{x}_c} \tag{51}$$

$$\frac{\partial \hat{\boldsymbol{T}}}{\partial \boldsymbol{x}_c} = \frac{1}{T}\left(\boldsymbol{I} - \frac{1}{T^2}\boldsymbol{T}\boldsymbol{T}^T\right)\frac{\partial \boldsymbol{T}}{\partial \boldsymbol{x}_c} \tag{52}$$

3.15 Derivatives of S Unit Vector

$$\frac{\partial \hat{\boldsymbol{P}}}{\partial \boldsymbol{x}_c} = \frac{1}{P}\left(\boldsymbol{I} - \frac{1}{P^2}\boldsymbol{P}\boldsymbol{P}^T\right)\frac{\partial \boldsymbol{P}}{\partial \boldsymbol{x}_c} \tag{53}$$

$$\zeta_1 \triangleq \sqrt{1 - \frac{1}{e^2}} \tag{54}$$

$$\frac{\partial \zeta_1}{\partial \boldsymbol{x}_c} = \frac{1}{e^3}\left(1 - \frac{1}{e^2}\right)^{-\frac{1}{2}} \hat{\boldsymbol{e}}^T \frac{\partial \boldsymbol{e}}{\partial \boldsymbol{x}_c} \tag{55}$$

$$\boldsymbol{\xi}_1 \triangleq -\frac{1}{e^2}\hat{\boldsymbol{e}}\hat{\boldsymbol{e}}^T \frac{\partial \boldsymbol{e}}{\partial \boldsymbol{x}_c} \tag{56}$$

$$\boldsymbol{\xi}_2 \triangleq \frac{1}{e}\frac{\partial \hat{\boldsymbol{e}}}{\partial \boldsymbol{x}_c} \tag{57}$$

$$\boldsymbol{\xi}_3 \triangleq \hat{\boldsymbol{P}}\frac{\partial \zeta_1}{\partial \boldsymbol{x}_c} \tag{58}$$

$$\boldsymbol{\xi}_4 \triangleq \zeta_1 \frac{\partial \hat{\boldsymbol{P}}}{\partial \boldsymbol{x}_c} \tag{59}$$

$$\frac{\partial \hat{\boldsymbol{S}}}{\partial \boldsymbol{x}_c} = \boldsymbol{\xi}_1 + \boldsymbol{\xi}_2 + \boldsymbol{\xi}_3 + \boldsymbol{\xi}_4 \tag{60}$$

3.16 Derivatives of Angular Momentum Vector

$$\frac{\partial \boldsymbol{h}}{\partial \boldsymbol{x}_c} = \begin{bmatrix} -\left\{\boldsymbol{v}\right\}^{\times} & \left\{\boldsymbol{r}\right\}^{\times} \end{bmatrix} \tag{61}$$

3.17 Derivatives of P Vector

$$\frac{\partial \boldsymbol{P}}{\partial \boldsymbol{x}_c} = -\left\{\boldsymbol{e}\right\}^{\times}\frac{\partial \boldsymbol{h}}{\partial \boldsymbol{x}_c} + \left\{\boldsymbol{h}\right\}^{\times}\frac{\partial \boldsymbol{e}}{\partial \boldsymbol{x}_c} \tag{62}$$

3.18 Derivatives of Eccentricity Vector

$$\boldsymbol{\zeta}_1 \triangleq \boldsymbol{r}^T \frac{\partial \boldsymbol{v}}{\partial \boldsymbol{x}_c} + \boldsymbol{v}^T \frac{\partial \boldsymbol{r}}{\partial \boldsymbol{x}_c} \tag{63}$$

$$\frac{\partial r}{\partial \boldsymbol{r}} = \frac{\boldsymbol{r}^T}{r} \tag{64}$$

$$\frac{\partial v}{\partial \boldsymbol{v}} = \frac{\boldsymbol{v}^T}{v} \tag{65}$$

$$\boldsymbol{\xi}_1 \triangleq \boldsymbol{r} \left(2v \frac{\partial v}{\partial \boldsymbol{v}} \frac{\partial \boldsymbol{v}}{\partial \boldsymbol{x}_c} + \frac{\mu}{r^2} \frac{\partial r}{\partial \boldsymbol{r}} \frac{\partial \boldsymbol{r}}{\partial \boldsymbol{x}_c} \right) + \left(v^2 - \frac{\mu}{r} \right) \frac{\partial \boldsymbol{r}}{\partial \boldsymbol{x}_c} \tag{66}$$

$$\boldsymbol{\xi}_2 \triangleq \boldsymbol{v} \boldsymbol{\zeta}_1 + \left(\boldsymbol{r}^T \boldsymbol{v} \right) \frac{\partial \boldsymbol{v}}{\partial \boldsymbol{x}_c} \tag{67}$$

$$\frac{\partial \boldsymbol{e}}{\partial \boldsymbol{x}_c} = \frac{1}{\mu} \left(\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2 \right) \tag{68}$$

3.19 Derivatives of True Anomaly

The derivatives of true anomaly with respect to the Cartesian state are obtained by differentiating Eq. (20).

$$\frac{\partial \left(\boldsymbol{e} \times \boldsymbol{r} \right)}{\partial \boldsymbol{e}} = - \left\{ \boldsymbol{r} \right\}^{\times} \tag{69}$$

$$\frac{\partial \left(\boldsymbol{e} \times \boldsymbol{r} \right)}{\partial \boldsymbol{r}} = \left\{ \boldsymbol{e} \right\}^{\times} \tag{70}$$

$$\frac{\partial \left(\boldsymbol{e} \times \boldsymbol{r} \right)}{\partial \boldsymbol{x}_c} = \frac{\partial \left(\boldsymbol{e} \times \boldsymbol{r} \right)}{\partial \boldsymbol{e}} \frac{\partial \boldsymbol{e}}{\partial \boldsymbol{x}_c} + \frac{\partial \left(\boldsymbol{e} \times \boldsymbol{r} \right)}{\partial \boldsymbol{r}} \frac{\partial \boldsymbol{r}}{\partial \boldsymbol{x}_c} \tag{71}$$

$$\xi_1 \triangleq || \boldsymbol{e} \times \boldsymbol{r} || \tag{72}$$

$$\xi_2 \triangleq \boldsymbol{e}^T \boldsymbol{r} \tag{73}$$

$$\frac{\partial \xi_1}{\partial \boldsymbol{x}_c} = \frac{1}{\xi_1} \left(\boldsymbol{e} \times \boldsymbol{r} \right)^T \frac{\partial \left(\boldsymbol{e} \times \boldsymbol{r} \right)}{\partial \boldsymbol{x}_c} \tag{74}$$

$$\frac{\partial \xi_2}{\partial \boldsymbol{x}_c} = \boldsymbol{e}^T \frac{\partial \boldsymbol{r}}{\partial \boldsymbol{x}_c} + \boldsymbol{r}^T \frac{\partial \boldsymbol{e}}{\partial \boldsymbol{x}_c} \tag{75}$$

$$\frac{\partial \nu}{\partial \xi_1} = \frac{\xi_2}{\xi_1^2 + \xi_2^2} \tag{76}$$

$$\frac{\partial \nu}{\partial \xi_2} = - \frac{\xi_1}{\xi_1^2 + \xi_2^2} \tag{77}$$

$$\frac{\partial \nu}{\partial \boldsymbol{x}_c} = \frac{\partial \nu}{\partial \xi_1} \frac{\partial \xi_1}{\partial \boldsymbol{x}_c} + \frac{\partial \nu}{\partial \xi_2} \frac{\partial \xi_2}{\partial \boldsymbol{x}_c} \tag{78}$$

Like with the calculation of true anomaly itself, a quadrant check is required at the end of the derivatives calculations:

$$\text{if } \boldsymbol{r}^T \boldsymbol{v} < 0 : \quad \frac{\partial \nu}{\partial \boldsymbol{x}_c} \leftarrow -\frac{\partial \nu}{\partial \boldsymbol{x}_c} \tag{79}$$

4 B-Plane State to Cartesian State Transformation

The transformation from B-Plane state to Cartesian state is accomplished by expressing the Cartesian state as a function of $\boldsymbol{e}$, $\boldsymbol{h}$, and ν:

$$\boldsymbol{r} = \frac{h^2}{\mu\,(1 + e\cos\nu)} \left[\hat{\boldsymbol{e}}\cos\nu + \hat{\boldsymbol{P}}\sin\nu\right] \tag{80}$$

$$\boldsymbol{v} = -\frac{\mu}{h} \left[\hat{\boldsymbol{e}}\sin\nu - (e + \cos\nu)\,\hat{\boldsymbol{P}}\right]. \tag{81}$$

True anomaly ν is known because it is a member of $\boldsymbol{x}_b$. The rest of the elements needed to calculate the Cartesian state are given by:

$$e = \sqrt{1 + \frac{v_\infty^4 b^2}{\mu^2}} \tag{82}$$

$$h = v_\infty b \tag{83}$$

$$\boldsymbol{v}_\infty = v_\infty \begin{bmatrix} \cos\delta\cos\alpha \\ \cos\delta\sin\alpha \\ \sin\delta \end{bmatrix} \tag{84}$$

$$B_R = \boldsymbol{B}^T \hat{\boldsymbol{R}} = b\sin\theta \tag{85}$$

$$B_T = \boldsymbol{B}^T \hat{\boldsymbol{T}} = b\cos\theta \tag{86}$$

$$\hat{\boldsymbol{S}} = \hat{\boldsymbol{v}}_\infty \tag{87}$$

$$\hat{\boldsymbol{T}} = \frac{\hat{\boldsymbol{S}} \times \boldsymbol{\phi}}{||\hat{\boldsymbol{S}} \times \boldsymbol{\phi}||} \tag{88}$$

$$\hat{\boldsymbol{R}} = \frac{\hat{\boldsymbol{S}} \times \hat{\boldsymbol{T}}}{||\hat{\boldsymbol{S}} \times \hat{\boldsymbol{T}}||} \tag{89}$$

$$\boldsymbol{B} = B_R \hat{\boldsymbol{R}} + B_T \hat{\boldsymbol{T}} \tag{90}$$

$$\hat{\boldsymbol{h}} = \frac{\boldsymbol{B} \times \hat{\boldsymbol{S}}}{||\boldsymbol{B} \times \hat{\boldsymbol{S}}||} \tag{91}$$

$$\boldsymbol{h} = h\hat{\boldsymbol{h}} \tag{92}$$

$$\nu_{\infty,in} = -\text{acos}\left(-\frac{1}{e}\right) \tag{93}$$

$$\hat{\boldsymbol{e}} = \frac{\hat{\boldsymbol{S}}\cos(\pi - \nu_{\infty,in}) - \hat{\boldsymbol{B}}\sin(\pi - \nu_{\infty,in})}{||\hat{\boldsymbol{S}}\cos(\pi - \nu_{\infty,in}) - \hat{\boldsymbol{B}}\sin(\pi - \nu_{\infty,in})||} \tag{94}$$

$$\boldsymbol{e} = e\hat{\boldsymbol{e}} \tag{95}$$

5 B-Plane State to Cartesian State Transformation Jacobian

5.1 Derivatives of Magnitude of B Vector

$$\frac{\partial b}{\partial \boldsymbol{x}_b} = [0 \quad 0 \quad 0 \quad 1 \quad 0 \quad 0] \tag{96}$$

5.2 Derivatives of B-Plane Clock Angle

$$\frac{\partial \theta}{\partial \boldsymbol{x}_b} = [0 \quad 0 \quad 0 \quad 0 \quad 1 \quad 0] \tag{97}$$

5.3 Derivatives of True Anomaly

With $\boldsymbol{x}_b$ defined as in Eq. (2), the derivatives of true anomaly are

$$\frac{\partial \nu}{\partial \boldsymbol{x}_b} = [0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 1] \tag{98}$$

5.4 Derivatives of Eccentricity Vector

$$\frac{\partial \boldsymbol{e}}{\partial \boldsymbol{x}_b} = \hat{\boldsymbol{e}} \frac{\partial e}{\partial \boldsymbol{x}_b} + e \frac{\partial \hat{\boldsymbol{e}}}{\partial \boldsymbol{x}_b} \tag{99}$$

5.5 Derivatives of Eccentricity Magnitude

$$\frac{\partial e}{\partial \boldsymbol{x}_b} = \frac{1}{2}\left(1 + \frac{v_\infty^4 b^2}{\mu^2}\right)^{-\frac{1}{2}} \left[\frac{4 v_\infty^3 b^2}{\mu^2} \quad 0 \quad 0 \quad \frac{2 v_\infty^4 b}{\mu^2} \quad 0 \quad 0\right] \tag{100}$$

5.6 Derivatives of Angular Momentum Vector

$$\frac{\partial \boldsymbol{h}}{\partial \boldsymbol{x}_b} = \hat{\boldsymbol{h}} \frac{\partial h}{\partial \boldsymbol{x}_b} + h \frac{\partial \hat{\boldsymbol{h}}}{\partial \boldsymbol{x}_b} \tag{101}$$

5.7 Derivatives of Angular Momentum Magnitude

$$\frac{\partial h}{\partial \boldsymbol{x}_b} = [b \quad 0 \quad 0 \quad v_\infty \quad 0 \quad 0] \tag{102}$$

5.8 Derivatives of Angular Momentum Unit Vector

$$\boldsymbol{\gamma} \triangleq \boldsymbol{B} \times \hat{\boldsymbol{S}} \tag{103}$$

$$\frac{\partial \hat{\boldsymbol{h}}}{\partial \boldsymbol{x}_b} = \left(-\frac{1}{\gamma^3}\boldsymbol{\gamma}\boldsymbol{\gamma}^T + \frac{1}{\gamma}\boldsymbol{I}\right)\left(-\left\{\hat{\boldsymbol{S}}\right\}^\times \frac{\partial \boldsymbol{B}}{\partial \boldsymbol{x}_b} + \{\boldsymbol{B}\}^\times \frac{\partial \hat{\boldsymbol{S}}}{\partial \boldsymbol{x}_b}\right) \tag{104}$$

5.9 Derivatives of S Unit Vector

$$\frac{\partial \hat{\boldsymbol{S}}}{\partial \boldsymbol{x}_b} = \begin{bmatrix} 0 & -\cos\delta\sin\alpha & -\sin\delta\cos\alpha & 0 & 0 & 0 \\ 0 & \cos\delta\cos\alpha & -\sin\delta\sin\alpha & 0 & 0 & 0 \\ 0 & 0 & \cos\delta & 0 & 0 & 0 \end{bmatrix} \tag{105}$$

5.10 Derivatives of B Vector

$$\frac{\partial \sin\nu}{\partial \boldsymbol{x}_b} = [0 \quad 0 \quad 0 \quad 0 \quad \cos\theta \quad 0] \tag{106}$$

$$\frac{\partial \cos\nu}{\partial \boldsymbol{x}_b} = [0 \quad 0 \quad 0 \quad 0 \quad 0 - \sin\theta \quad 0] \tag{107}$$

$$\frac{\partial \boldsymbol{B}}{\partial \boldsymbol{x}_b} = \sin\theta \hat{\boldsymbol{R}} \frac{\partial b}{\partial \boldsymbol{x}_b} + b\hat{\boldsymbol{R}} \frac{\partial \sin\theta}{\partial \boldsymbol{x}_b} + b\sin\theta \frac{\partial \hat{\boldsymbol{R}}}{\partial \boldsymbol{x}_b} + \cos\theta \hat{\boldsymbol{T}} \frac{\partial b}{\partial \boldsymbol{x}_b} + b\hat{\boldsymbol{T}} \frac{\partial \cos\theta}{\partial \boldsymbol{x}_b} + b\cos\theta \frac{\partial \hat{\boldsymbol{T}}}{\partial \boldsymbol{x}_b} \tag{108}$$

5.11 Derivatives of T Unit Vector

The derivatives of $\hat{\boldsymbol{T}}$ cannot be fully defined until the reference vector $\boldsymbol{\phi}$ is chosen. In this section, the derivatives are left in terms of the derivatives of $\boldsymbol{\phi}$.

$$\boldsymbol{T} \triangleq \hat{\boldsymbol{S}} \times \boldsymbol{\phi} \tag{109}$$

$$\frac{\partial \boldsymbol{T}}{\partial \boldsymbol{x}_b} = -\left\{\boldsymbol{\phi}\right\}^{\times} \frac{\partial \hat{\boldsymbol{S}}}{\partial \boldsymbol{x}_b} + \left\{\hat{\boldsymbol{S}}\right\}^{\times} \frac{\partial \boldsymbol{\phi}}{\partial \boldsymbol{x}_b} \tag{110}$$

$$\boldsymbol{\xi}_2 \triangleq -\frac{1}{T^3} \left(\boldsymbol{T}^T \frac{\partial \boldsymbol{T}}{\partial \boldsymbol{x}_b} \right)^T \tag{111}$$

$$\frac{\partial \hat{\boldsymbol{T}}}{\partial \boldsymbol{x}_b} = \frac{1}{T} \frac{\partial \boldsymbol{T}}{\partial \boldsymbol{x}_b} + \boldsymbol{T}\boldsymbol{\xi}_2^T \tag{112}$$

5.12 Derivatives of R Unit Vector

$$\boldsymbol{R} \triangleq \hat{\boldsymbol{S}} \times \hat{\boldsymbol{T}} \tag{113}$$

$$\frac{\partial \boldsymbol{R}}{\partial \boldsymbol{x}_b} = -\left\{\hat{\boldsymbol{T}}\right\}^{\times} \frac{\partial \hat{\boldsymbol{S}}}{\partial \boldsymbol{x}_b} + \left\{\hat{\boldsymbol{S}}\right\}^{\times} \frac{\partial \hat{\boldsymbol{T}}}{\partial \boldsymbol{x}_b} \tag{114}$$

$$\boldsymbol{\xi}_2 \triangleq -\frac{1}{R^3} \left(\boldsymbol{R}^T \frac{\partial \boldsymbol{R}}{\partial \boldsymbol{x}_b} \right)^T \tag{115}$$

$$\frac{\partial \hat{\boldsymbol{R}}}{\partial \boldsymbol{x}_b} = \frac{1}{R} \frac{\partial \boldsymbol{R}}{\partial \boldsymbol{x}_b} + \boldsymbol{R}\boldsymbol{\xi}_2^T \tag{116}$$

5.13 Derivatives of Eccentricity Unit Vector

$$\beta \triangleq \pi - \nu_{\infty,in} \tag{117}$$

$$c_\beta \triangleq \cos\beta \tag{118}$$

$$s_\beta \triangleq \sin\beta \tag{119}$$

$$\frac{\partial c_\beta}{\partial \boldsymbol{x}_b} = s_\beta \frac{\partial \nu_{\infty,in}}{\partial \boldsymbol{x}_b} \tag{120}$$

$$\frac{\partial s_\beta}{\partial \boldsymbol{x}_b} = -c_\beta \frac{\partial \nu_{\infty,in}}{\partial \boldsymbol{x}_b} \tag{121}$$

$$\boldsymbol{\xi}_1 \triangleq c_\beta \hat{\boldsymbol{S}} - s_\beta \hat{\boldsymbol{B}} \tag{122}$$

$$\frac{\partial \hat{\boldsymbol{B}}}{\partial \boldsymbol{x}_b} = \frac{\partial \hat{\boldsymbol{B}}}{\partial \boldsymbol{B}} \frac{\partial \boldsymbol{B}}{\partial \boldsymbol{x}_b} \tag{123}$$

$$\frac{\partial \hat{\boldsymbol{B}}}{\partial \boldsymbol{B}} = \frac{1}{B}\left(\boldsymbol{I} - \frac{1}{B^2}\boldsymbol{B}\boldsymbol{B}^T\right) \tag{124}$$

$$\frac{\partial \boldsymbol{\xi}_1}{\partial \boldsymbol{x}_b} = \frac{\partial \hat{\boldsymbol{S}}}{\partial \boldsymbol{x}_b} c_\beta + \hat{\boldsymbol{S}} \frac{\partial c_\beta}{\partial \boldsymbol{x}_b} - \frac{\partial \hat{\boldsymbol{B}}}{\partial \boldsymbol{x}_b} s_\beta - \hat{\boldsymbol{B}} \frac{\partial s_\beta}{\partial \boldsymbol{x}_b} \tag{125}$$

$$\boldsymbol{\xi}_2 \triangleq -\frac{1}{\xi_1^3} \boldsymbol{x}_1^T \frac{\partial \boldsymbol{\xi}_1}{\partial \boldsymbol{x}_b} \tag{126}$$

$$\frac{\partial \hat{e}}{\partial \boldsymbol{x}_b} = \frac{1}{\xi_1} \frac{\partial \boldsymbol{\xi}_1}{\partial \boldsymbol{x}_b} + \boldsymbol{\xi}_1 \boldsymbol{\xi}_2 \tag{127}$$

5.14 Derivatives of Incoming True Anomaly at Infinity

$$\frac{\partial \nu_{\infty,in}}{\partial \boldsymbol{x}_b} = \frac{1}{e\sqrt{e^2-1}} \frac{\partial e}{\partial \boldsymbol{x}_b} \tag{128}$$

5.15 Derivatives of Position Vector

The final derivatives of the position vector utilize the derivatives of $\boldsymbol{h}$, $\boldsymbol{e}$, and ν:

$$\frac{\partial \boldsymbol{r}}{\partial \boldsymbol{x}_b} = \frac{\partial \boldsymbol{r}}{\partial \boldsymbol{h}} \frac{\partial \boldsymbol{h}}{\partial \boldsymbol{x}_b} + \frac{\partial \boldsymbol{r}}{\partial \boldsymbol{e}} \frac{\partial \boldsymbol{e}}{\partial \boldsymbol{x}_b} + \frac{\partial \boldsymbol{r}}{\partial \nu} \frac{\partial \nu}{\partial \boldsymbol{x}_b} \tag{129}$$

5.15.1 Derivatives of Position Vector with Respect to Angular Momentum Vector

$$\xi_1 \triangleq 2\cos\nu \hat{\boldsymbol{e}}\boldsymbol{h}^T \tag{130}$$

$$\xi_2 \triangleq \frac{\sin\nu}{P}\left[2\boldsymbol{P}\boldsymbol{h}^T + h^2\left(-\boldsymbol{I} + \frac{1}{P^2}\boldsymbol{P}\boldsymbol{P}^T\right)\{\boldsymbol{e}\}^\times\right] \tag{131}$$

$$\frac{\partial \boldsymbol{r}}{\partial \boldsymbol{h}} = \frac{1}{\mu\left(1 + e\cos\nu\right)}\left(\xi_1 + \xi_2\right) \tag{132}$$

5.15.2 Derivatives of Position Vector with Respect to Eccentricity Vector

$$\xi_1 \triangleq \left[\hat{\boldsymbol{e}}\cos\nu + \hat{\boldsymbol{P}}\sin\nu\right]\left[\hat{\boldsymbol{e}}^T\frac{-\cos\nu}{\left(1 + e\cos\nu\right)^2}\right] \tag{133}$$

$$\xi_2 \triangleq \frac{1}{1 + e\cos\nu}\left[\frac{\cos\nu}{e}\left(\boldsymbol{I} - \frac{1}{e^2}\boldsymbol{e}\boldsymbol{e}^T\right) + \frac{\sin\nu}{P}\left(\boldsymbol{I} - \frac{1}{P^2}\boldsymbol{P}\boldsymbol{P}^T\right)\{\boldsymbol{h}\}^\times\right] \tag{134}$$

$$\frac{\partial \boldsymbol{r}}{\partial \boldsymbol{e}} = \frac{h^2}{\mu}\left(\xi_1 + \xi_2\right) \tag{135}$$

5.15.3 Derivatives of Position Vector with Respect to True Anomaly

$$\xi_1 \triangleq \frac{e\sin\nu}{\left(1 + e\cos\nu\right)^2}\left(\hat{\boldsymbol{e}}\cos\nu + \hat{\boldsymbol{P}}\sin\nu\right) \tag{136}$$

$$\xi_2 \triangleq \frac{1}{1 + e\cos\nu}\left(-\hat{\boldsymbol{e}}\sin\nu + \hat{\boldsymbol{P}}\cos\nu\right) \tag{137}$$

$$\frac{\partial \boldsymbol{r}}{\partial \nu} = \frac{h^2}{\mu}\left(\xi_1 + \xi_2\right) \tag{138}$$

5.16 Derivatives of Velocity Vector

The final derivatives of the velocity vector utilize the derivatives of $\boldsymbol{h}$, $\boldsymbol{e}$, and ν:

$$\frac{\partial \boldsymbol{v}}{\partial \boldsymbol{x}_b} = \frac{\partial \boldsymbol{v}}{\partial \boldsymbol{h}}\frac{\partial \boldsymbol{h}}{\partial \boldsymbol{x}_b} + \frac{\partial \boldsymbol{v}}{\partial \boldsymbol{e}}\frac{\partial \boldsymbol{e}}{\partial \boldsymbol{x}_b} + \frac{\partial \boldsymbol{v}}{\partial \nu}\frac{\partial \nu}{\partial \boldsymbol{x}_b} \tag{139}$$

5.16.1 Derivatives of Velocity Vector with Respect to Angular Momentum Vector

$$\xi_1 \triangleq -\frac{1}{h^3}\left[\hat{\boldsymbol{e}}\sin\nu - (e+\cos\nu)\,\hat{\boldsymbol{P}}\right]\boldsymbol{h}^T \tag{140}$$

$$\xi_2 \triangleq -\frac{e+\cos\nu}{hP}\left[-\{\boldsymbol{e}\}^{\times} + \frac{1}{P^2}\boldsymbol{P}\boldsymbol{P}^T\{\boldsymbol{e}\}^{\times}\right] \tag{141}$$

$$\frac{\partial \boldsymbol{v}}{\partial \boldsymbol{h}} = -\mu\left(\xi_1 + \xi_2\right) \tag{142}$$

5.16.2 Derivatives of Velocity Vector with Respect to Eccentricity Vector

$$\xi_1 \triangleq \frac{\sin\nu}{e}\left(\boldsymbol{I} - \frac{1}{e^2}\boldsymbol{e}\boldsymbol{e}^T\right) \tag{143}$$

$$\xi_{21} \triangleq \hat{\boldsymbol{P}}\hat{\boldsymbol{e}}^T \tag{144}$$

$$\xi_{22} \triangleq (e+\cos\nu)\left(\frac{1}{P}\right)\left[\{\boldsymbol{h}\}^{\times} - \frac{1}{P^2}\boldsymbol{P}\boldsymbol{P}^T\{\boldsymbol{h}\}^{\times}\right] \tag{145}$$

$$\xi_2 \triangleq -\left(\xi_{21} + \xi_{22}\right) \tag{146}$$

$$\frac{\partial \boldsymbol{v}}{\partial \boldsymbol{e}} = -\frac{\mu}{h}\left(\xi_1 + \xi_2\right) \tag{147}$$

5.16.3 Derivatives of Velocity Vector with Respect to True Anomaly

$$\xi_1 \triangleq \cos\nu\,\hat{\boldsymbol{e}} \tag{148}$$

$$\xi_2 \triangleq \sin\nu\,\hat{\boldsymbol{P}} \tag{149}$$

$$\frac{\partial \boldsymbol{v}}{\partial \nu} = -\frac{\mu}{h}\left(\xi_1 + \xi_2\right) \tag{150}$$

6 Outgoing Transformations

6.1 Cartesian State to B-Plane State Transformation

For the Cartesian state to B-Plane transformation, the substantive change is that the vector $\hat{\boldsymbol{S}}$ – which is aligned with the hyperbolic asymptote – changes from being parallel to the incoming asymptote to being parallel to the outgoing asymptote. Consequently, Eq. (6) becomes

$$\hat{\boldsymbol{S}} = -\frac{1}{e}\hat{\boldsymbol{e}} + \sqrt{1 - \frac{1}{e^2}}\hat{\boldsymbol{P}} \tag{151}$$

All subsequent calculations proceed as described in Section 2 using the expression for $\hat{\boldsymbol{S}}$ given in Eq. (151) with the exception of Eq. (9), which becomes

$$\hat{\boldsymbol{B}} = \frac{1}{e}\hat{\boldsymbol{P}} + \sqrt{1 - \frac{1}{e^2}}\hat{\boldsymbol{e}} \tag{152}$$

6.2 Cartesian State to B-Plane State Transformation Jacobian

Changes to the Cartesian state to B-Plane state transformation Jacobian relative to the expressions presented in Section 3 arise due to the changes presented in Section 6.1. Specifically, Eq. (60) becomes

$$\frac{\partial \hat{\boldsymbol{S}}}{\partial \boldsymbol{x}_c} = -\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2 + \boldsymbol{\xi}_3 + \boldsymbol{\xi}_4 \tag{153}$$

using the variable definitions of Section 3.15, overridden where applicable by Section 6.1.

Additionally, Eq. (46) becomes

$$\frac{\partial \hat{\boldsymbol{B}}}{\partial \boldsymbol{x}_c} = \boldsymbol{\xi}_1 + \boldsymbol{\xi}_2 - \boldsymbol{\xi}_3 - \boldsymbol{\xi}_4 \tag{154}$$

using the variable definitions of Section 3.12, overridden where applicable by Section 6.1.

6.3 B-Plane State to Cartesian State Transformation

For the B-Plane state to Cartesian state transformation, the true anomaly at infinity is calculated for the outgoing asymptote rather than for the incoming asymptote:

$$\nu_{\infty,out} = \text{acos}\left(-\frac{1}{e}\right) \tag{155}$$

$\nu_{\infty,out}$ then replaces $\nu_{\infty,in}$ in Eq. (94), which becomes

$$\hat{\boldsymbol{e}} = \hat{\boldsymbol{B}}\sin\left(\nu_{\infty,out}\right) + \hat{\boldsymbol{S}}\cos\left(\nu_{\infty,out}\right) \tag{156}$$

All other equations of Section 4 still hold with the important note that α and δ must be interpreted as the right ascension and declination, respectively, of the outgoing asymptote. (For the incoming B-Plane transformation, α and δ are the right ascension and declination, respectively, of the incoming asymptote.)

6.4 B-Plane State to Cartesian State Transformation Jacobian

Changes to the Cartesian state to B-Plane state transformation Jacobian relative to the expressions presented in Section 5 arise due to the changes presented in Section 6.3. Specifically, Eq. (128) becomes

$$\frac{\partial \nu_{\infty,out}}{\partial \boldsymbol{x}_b} = -\frac{1}{e\sqrt{e^2-1}}\frac{\partial e}{\partial \boldsymbol{x}_b} \tag{157}$$

Additionally, Eq. (127) becomes

$$\frac{\partial \hat{e}}{\partial \boldsymbol{x}_b} = \frac{\partial \hat{\boldsymbol{S}}}{\partial \boldsymbol{x}_b}\cos\left(\nu_{\infty,out}\right) + \hat{\boldsymbol{S}}\frac{\partial \cos\left(\nu_{\infty,out}\right)}{\partial \boldsymbol{x}_b} + \frac{\partial \hat{\boldsymbol{B}}}{\partial \boldsymbol{x}_b}\sin\left(\nu_{\infty,out}\right) + \hat{\boldsymbol{B}}\frac{\partial \sin\left(\nu_{\infty,out}\right)}{\partial \boldsymbol{x}_b} \tag{158}$$

with

$$\frac{\partial \cos\left(\nu_{\infty,out}\right)}{\partial \boldsymbol{x}_b} = -\sin\left(\nu_{\infty,out}\right)\frac{\partial \nu_{\infty,out}}{\partial \boldsymbol{x}_b} \tag{159}$$

$$\frac{\partial \sin\left(\nu_{\infty,out}\right)}{\partial \boldsymbol{x}_b} = \cos\left(\nu_{\infty,out}\right)\frac{\partial \nu_{\infty,out}}{\partial \boldsymbol{x}_b} \tag{160}$$

The expressions for $\frac{\partial \hat{\boldsymbol{S}}}{\partial \boldsymbol{x}_b}$ and $\frac{\partial \hat{\boldsymbol{B}}}{\partial \boldsymbol{x}_b}$ do not change from those presented in Section 5.