Ellipsoid Interface Constraints for EMTGv9

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Abstract

This document describes the constraint relations (and associated derivatives) for entry/exit interface with an ellipsoid. Constraints considered are bodycentric latitude, longitude, heading angle, flight path angle, and velocity magnitude. Derivatives are desired with respect to a decision variable consisting of time and the state at the interface in the body-centered, body-fixed reference frame.

Contents

Nomenclature

γ Flight path angle

λ Bodycentric longitude in BCF

λ' Bodydetic longitude in BCF

$\mathcal{H}$ Heading angle

ϕ Bodycentric latitude in BCF

ϕ' Bodydetic latitude in BCF

$\hat{\boldsymbol{x}}$ Unit vector in direction of arbitrary vector $\boldsymbol{x}$

$\hat{\boldsymbol{x}}$, $\hat{\boldsymbol{y}}$, $\hat{\boldsymbol{z}}$ Unit vectors defining a coordinate frame

$\boldsymbol{r}$ Position vector

$\boldsymbol{x}_A$ Vector expressed in A frame coordinates

${}^A[\boldsymbol{v}]$ Velocity vector with respect to A frame

${}^A\left[\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t}\right]$ Time derivative of vector $\boldsymbol{x}$ with respect to the A reference frame

${}^B\boldsymbol{\omega}^A$ Angular velocity vector of frame B with respect to frame A

${}_P\boldsymbol{S}_{BCF}, {}_P\boldsymbol{E}_{BCF}, \boldsymbol{n}_{BCF}$ Unit vectors of polar frame, expressed in BCF coordinates

${}_T\boldsymbol{S}_{BCF}, {}_T\boldsymbol{E}_{BCF}, \boldsymbol{n}_{BCF}$ Unit vectors of topocentric frame, expressed in BCF coordinates

a, b, c The lengths of the semimajor, semiminor, and semiintermediate axes of the ellipsoid

P Polar south-east-up frame

T Topocentric south-east-up frame

w Rotation angle about the $\hat{\boldsymbol{z}}$ axis relating the BCI and BCF frames.

BCF Body-centered, body-fixed reference frame; rotates with central body, but is not necessarily aligned with the principal axes of the ellipsoid

BCI Body-centered inertial reference frame; does not rotate with the central body

PA Body-centered, body-fixed reference frame aligned with the principal axes of the ellipsoid; rotates with central body

1 Reference Frames

Several reference frames are used to derive these equations.

1.1 Body-centered, Body-fixed Frame

The body-centered, body-fixed (BCF) frame has its origin at the centroid of the ellipsoid and rotates with the central body. However, the BCF frame is not necessarily aligned with the principal axes of the ellipsoid.

1.2 Body-centered, Principal-axes Frame

The body-centered, principal-axes (PA) frame has its origin at the centroid of the ellipsoid and rotates with the central body. Its axes are aligned with the principal axes of the ellipse such that $\hat{\boldsymbol{x}}$ aligns with the semimajor axis, $\hat{\boldsymbol{y}}$ aligns with the semiminor axis, and $\hat{\boldsymbol{z}}$ aligns with the semiintermediate axis.

The PA frame is related to the BCF frame by a 3-1-3 Euler angle sequence such that

$$\boldsymbol{r}_{PA} = \boldsymbol{R}^{BCF\to PA}\boldsymbol{r}_{BCF} \tag{1}$$

$$\boldsymbol{R}^{BCF\to PA} = \boldsymbol{R}^{F''\to PA}\boldsymbol{R}^{F'\to F''}\boldsymbol{R}^{BCF\to F'} \tag{2}$$

$$\boldsymbol{R}^{BCF\to F'} = \begin{bmatrix} \cos\theta_1 & \sin\theta_1 & 0 \\ -\sin\theta_1 & \cos\theta_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{3}$$

$$\boldsymbol{R}^{F'\to F''} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_2 & \sin\theta_2 \\ 0 & -\sin\theta_2 & \cos\theta_2 \end{bmatrix} \tag{4}$$

$$\boldsymbol{R}^{F''\to PA} = \begin{bmatrix} \cos\theta_3 & \sin\theta_3 & 0 \\ -\sin\theta_3 & \cos\theta_3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{5}$$

$$\boldsymbol{R}^{BCF\to PA} = \begin{bmatrix} -S_{\theta_1}S_{\theta_3}C_{\theta_2} + C_{\theta_1}C_{\theta_3} & S_{\theta_3}C_{\theta_1}C_{\theta_2} + S_{\theta_1}C_{\theta_3} & S_{\theta_2}S_{\theta_3} \\ -S_{\theta_1}C_{\theta_2}C_{\theta_3} - S_{\theta_3}C_{\theta_1} & C_{\theta_1}C_{\theta_2}C_{\theta_3} - S_{\theta_1}S_{\theta_3} & S_{\theta_2}C_{\theta_3} \\ S_{\theta_1}S_{\theta_2} & -S_{\theta_2}C_{\theta_1} & C_{\theta_2} \end{bmatrix} \tag{6}$$

1.3 Body-centered Inertial Frame

The body-centered inertial (BCI) frame has its origin at the centroid of the ellipsoid and does not rotate with the central body. The BCF frame is assumed to be related to the BCI frame by a rotation about their common $\hat{\boldsymbol{z}}$ axis by an angle w.

$$\boldsymbol{r}_{BCF} = \boldsymbol{R}^{BCI\to BCF}\boldsymbol{r}_{BCI} \tag{7}$$

$$\boldsymbol{R}^{BCI\to BCF} = \begin{bmatrix} \cos w & \sin w & 0 \\ -\sin w & \cos w & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{8}$$

1.3.1 Derivatives

The angle w depends only on time, so the partial derivatives of $\boldsymbol{R}^{BCI\to BCF}$ with respect to $\boldsymbol{r}_{BCF}$ and ${}^{BCF}\boldsymbol{v}_{BCF}$ are zero. The derivative with respect to time is

$$\frac{\partial \boldsymbol{R}^{BCI\to BCF}}{\partial t} = \frac{\partial \boldsymbol{R}^{BCI\to BCF}}{\partial w}\frac{\mathrm{d}w}{\mathrm{d}t} \tag{9}$$

$$= \begin{bmatrix} -\sin w & \cos w & 0 \\ -\cos w & -\sin w & 0 \\ 0 & 0 & 0 \end{bmatrix} \frac{\mathrm{d}w}{\mathrm{d}t} \tag{10}$$

1.4 Topocentric Frame

The topocentric frame is a south-east-up frame centered as the ellipsoidal interface. The up vector is the outward normal of the ellipsoid. (See Section 2.) The east vector is defined to be tangent to the ellipsoid and point in a direction of constant z in the BCF frame:

$$_T\boldsymbol{E}_{BCF} = \hat{\boldsymbol{k}}_{BCF} \times \left[\boldsymbol{R}^{PA\to BCF}\frac{\partial \boldsymbol{n}_{PA}}{\partial \boldsymbol{r}_{PA}}\boldsymbol{R}^{BCF\to PA}\boldsymbol{r}_{BCF}\right] \tag{11}$$

$$_T\hat{\boldsymbol{E}}_{BCF} = \frac{_T\boldsymbol{E}_{BCF}}{_TE} \tag{12}$$

(See Eq. 26.) The south vector completes the right-handed system: $\hat{\boldsymbol{S}} = \hat{\boldsymbol{E}} \times \hat{\boldsymbol{n}}$. Note that, based on this definition, the $_T\hat{\boldsymbol{S}}\hat{\boldsymbol{n}}$ plane does not necessarily contain the north or south pole.

The rotation matrix is

$$\boldsymbol{R}^{BCF\to T} = \begin{bmatrix} _T\hat{\boldsymbol{S}}^T_{BCF} \\ _T\hat{\boldsymbol{E}}^T_{BCF} \\ \hat{\boldsymbol{n}}^T_{BCF} \end{bmatrix} \tag{13}$$

When defining a topocentric frame for a triaxial ellipsoid using STK, these unit vectors are the unit vectors that are returned.

1.5 Polar Frame

The polar frame is a south-east-up frame centered as the ellipsoidal interface. The up vector is the outward normal of the ellipsoid. (See Section 2.) The east vector is tangent to the ellipsoid but does not necessarily point in a direction of constant z in the BCF frame. The $_P\hat{\boldsymbol{S}}\hat{\boldsymbol{n}}$ plane does contain the north and south pole.

$$\tilde{\boldsymbol{E}}_{BCF} = \hat{\boldsymbol{k}}_{BCF} \times \boldsymbol{r}_{BCF} \tag{14}$$

$$\hat{\tilde{\boldsymbol{E}}}_{BCF} = \frac{\tilde{\boldsymbol{E}}_{BCF}}{\tilde{E}} \tag{15}$$

$$_P\boldsymbol{S}_{BCF} = \tilde{\boldsymbol{E}}_{BCF} \times \boldsymbol{n}_{BCF} \tag{16}$$

$$_P\hat{\boldsymbol{S}}_{BCF} = \frac{_P\boldsymbol{S}_{BCF}}{_PS} \tag{17}$$

$$_P\boldsymbol{E}_{BCF} = \boldsymbol{n}_{BCF} \times_P \boldsymbol{S}_{BCF} \tag{18}$$

$$_P\hat{\boldsymbol{E}}_{BCF} = \frac{_P\boldsymbol{E}_{BCF}}{_PE} \tag{19}$$

The rotation matrix is

$$\boldsymbol{R}^{BCF\to T} = \begin{bmatrix} _T\hat{\boldsymbol{S}}^T_{BCF} \\ _T\hat{\boldsymbol{E}}^T_{BCF} \\ \hat{\boldsymbol{n}}^T_{BCF} \end{bmatrix} \tag{20}$$

2 Ellipsoid

An ellipsoid is defined by the equation

$$\frac{r^2_{x,PA}}{a^2} + \frac{r^2_{y,PA}}{b^2} + \frac{r^2_{z,PA}}{c^2} = 1. \tag{21}$$

The vector normal to the surface of the ellipsoid is found by taking the (transpose of the) gradient of the equation of the ellipsoid:

$$\boldsymbol{n}_{PA} = \begin{bmatrix} 2\frac{r_{x,PA}}{a^2} & 2\frac{r_{y,PA}}{b^2} & 2\frac{r_{z,PA}}{c^2} \end{bmatrix}^T. \tag{22}$$

Define the auxiliary matrix

$$\boldsymbol{\epsilon} = \begin{bmatrix} 1/a^2 & 0 & 0 \\ 0 & 1/b^2 & 0 \\ 0 & 0 & 1/c^2 \end{bmatrix}. \tag{23}$$

Then $\boldsymbol{n}_{PA}$ can also be written as

$$\boldsymbol{n}_{PA} = 2\boldsymbol{\epsilon}\boldsymbol{r}_{PA} \tag{24}$$

The normal vector can be expressed in terms of the BCF frame by expressing the PA position vector as a function of the BCF position vector.

$$\boldsymbol{n}_{PA} = 2\boldsymbol{\epsilon}\boldsymbol{R}^{BCF\to PA}\boldsymbol{r}_{BCF}. \tag{25}$$

2.1 Derivatives

$$\frac{\partial \boldsymbol{n}_{PA}}{\partial \boldsymbol{r}_{PA}} = \begin{bmatrix} \frac{2}{a^2} & 0 & 0 \\ 0 & \frac{2}{b^2} & 0 \\ 0 & 0 & \frac{2}{c^2} \end{bmatrix} \tag{26}$$

$$= 2\boldsymbol{\epsilon} \tag{27}$$

We wish to have the derivative with respect to the BCF frame. For position,

$$\frac{\partial \boldsymbol{n}_{BCF}}{\partial \boldsymbol{r}_{BCF}} = \frac{\partial \boldsymbol{n}_{BCF}}{\partial \boldsymbol{r}_{PA}} \frac{\partial \boldsymbol{r}_{PA}}{\partial \boldsymbol{r}_{BCF}} \tag{28}$$

where

$$\frac{\partial \boldsymbol{r}_{PA}}{\partial \boldsymbol{r}_{BCF}} = \boldsymbol{R}^{BCF \to PA} \tag{29}$$

and

$$\frac{\partial \boldsymbol{n}_{BCF}}{\partial \boldsymbol{r}_{PA}} = \frac{\partial}{\partial \boldsymbol{r}_{PA}} \left[\boldsymbol{R}^{PA \to BCF} \boldsymbol{n}_{PA} \right] \tag{30}$$

$$= \boldsymbol{R}^{PA \to BCF} \frac{\partial \boldsymbol{n}_{PA}}{\partial \boldsymbol{r}_{PA}} \tag{31}$$

where $\frac{\partial \boldsymbol{n}_{PA}}{\partial \boldsymbol{r}_{PA}}$ is given by Eq. (26). $\frac{\partial}{\partial \boldsymbol{r}_{PA}} \left[\boldsymbol{R}^{PA \to BCF} \right]$ is zero because the orientation of the PA and BCF frames is independent of the position of entry interface.

The vector normal to the ellipsoid is independent of the velocity of the spacecraft at the interface, so

$$\frac{\partial \boldsymbol{n}_{BCF}}{\partial \left[{}^{BCF}\boldsymbol{v}_{BCF} \right]} = \boldsymbol{0} \tag{32}$$

The PA and BCF frames are related through the Euler angles $\theta_1, \theta_2, \theta_3$ as described in Section 1. If these angles are changing in time, then $\boldsymbol{R}^{PA \to BCF}$ has nonzero time derivatives:

$$\frac{\partial \boldsymbol{n}_{PA}}{\partial t} = 2\boldsymbol{\epsilon} \frac{\partial \boldsymbol{R}^{PA \to BCF}}{\partial t} \boldsymbol{r}_{BCF} \tag{33}$$

$$\frac{\partial \boldsymbol{n}_{BCF}}{\partial t} = \frac{\partial}{\partial t} \left[\boldsymbol{R}^{PA \to BCF} \right] \boldsymbol{n}_{PA} \tag{34}$$

$$= \frac{\partial}{\partial \theta_1} \left[\boldsymbol{R}^{PA \to BCF} \right] \frac{\mathrm{d}\theta_1}{\mathrm{d}t} + \frac{\partial}{\partial \theta_2} \left[\boldsymbol{R}^{PA \to BCF} \right] \frac{\mathrm{d}\theta_2}{\mathrm{d}t} + \frac{\partial}{\partial \theta_3} \left[\boldsymbol{R}^{PA \to BCF} \right] \frac{\mathrm{d}\theta_3}{\mathrm{d}t} \tag{35}$$

where the derivatives of the rotation matrices with respect to the angles can be obtained from Eq. (6).

3 Bodycentric Latitude

The bodycentric latitude is calculated as angle between the BCF xy plane and the interface position vector (in BCF):[1]

$$
\begin{aligned}
\phi &= \operatorname{atan2}\left[r_{z,BCF}, r_{xy,BCF}\right] && (36)\\
&= \operatorname{atan2}\left[r_{z,BCF}, \left(r_{x,BCF}\cos\lambda + r_{y,BCF}\sin\lambda\right)\right] && (37)
\end{aligned}
$$

3.1 Derivatives

Let

$$
\begin{aligned}
\phi_x &= r_{x,BCF}\cos\lambda + r_{y,BCF}\sin\lambda && (38)\\
\phi_y &= r_{z,BCF}. && (39)
\end{aligned}
$$

Then

$$
\frac{\partial \phi}{\partial \boldsymbol{r}_{BCF}} = \frac{\partial \phi}{\partial \phi_x}\frac{\partial \phi_x}{\partial \boldsymbol{r}_{BCF}} + \frac{\partial \phi}{\partial \phi_y}\frac{\partial \phi_y}{\partial \boldsymbol{r}_{BCF}}. \quad (40)
$$

where

$$
\begin{aligned}
\frac{\partial \phi_x}{\partial r_{x,BCF}} &= \cos\lambda + r_{x,BCF}\frac{\partial \cos\lambda}{\partial r_{x,BCF}} + r_{y,BCF}\frac{\partial \sin\lambda}{\partial r_{x,BCF}} && (41)\\
\frac{\partial \phi_x}{\partial r_{y,BCF}} &= r_{x,BCF}\frac{\partial \cos\lambda}{\partial r_{y,BCF}} + \sin\lambda + r_{y,BCF}\frac{\partial \sin\lambda}{\partial r_{y,BCF}} && (42)\\
\frac{\partial \phi_y}{\partial \boldsymbol{r}_{BCF}} &= \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} && (43)
\end{aligned}
$$

where

$$
\begin{aligned}
\frac{\partial \cos\lambda}{\partial \boldsymbol{r}_{BCF}} &= \frac{\partial \cos\lambda}{\partial \lambda}\frac{\partial \lambda}{\partial \boldsymbol{r}_{BCF}} && (44)\\
&= -\sin\lambda\frac{\partial \lambda}{\partial \boldsymbol{r}_{BCF}} && (45)
\end{aligned}
$$

where $\frac{\partial \lambda}{\partial r_{x,BCF}}$ is given in Eq. (58). Additionally,

$$
\begin{aligned}
\frac{\partial \sin\lambda}{\partial \boldsymbol{r}_{BCF}} &= \frac{\partial \sin\lambda}{\partial \lambda}\frac{\partial \lambda}{\partial \boldsymbol{r}_{BCF}} && (46)\\
&= \cos\lambda\frac{\partial \lambda}{\partial \boldsymbol{r}_{BCF}} && (47)
\end{aligned}
$$

[1] Note that body*centric* is emphasized to differentiate between it and body*detic*.

The latitude is independent of the velocity, so

$$\boxed{\frac{\partial \phi}{\partial \boldsymbol{v}_{BCF}} = \mathbf{0}^T}. \tag{48}$$

The latitude is independent of time, so

$$\boxed{\frac{\partial \phi}{\partial t} = 0}. \tag{49}$$

4 Bodydetic Latitude

The bodydetic latitude is calculated as the angle between the ellipsoid normal vector and the equatorial plane. This angle may be calculated as the complement of the angle betweeen the normal vector and the $\boldsymbol{k}_{BCF}$ vector. Using Eq. (183),

$$\phi' = \frac{\pi}{2} - \operatorname{atan2}\left[||\boldsymbol{k}_{BCF} \times \boldsymbol{n}_{BCF}||, \boldsymbol{k}_{BCF}^T \boldsymbol{n}_{BCF}\right] \tag{50}$$

4.1 Derivatives

ϕ' has no velocity dependence, so

$$\frac{\partial \phi'}{\partial^{BCF} \boldsymbol{v}_{BCF}} = \mathbf{0}^T \tag{51}$$

For position and time, we have, from the derivative of atan2:

$$\frac{\partial \phi'}{\partial \boldsymbol{x}} = -\left[\frac{\partial \phi'}{\partial ||\hat{\boldsymbol{k}} \times \boldsymbol{n}_{BCF}||} \frac{\partial ||\hat{\boldsymbol{k}} \times \boldsymbol{n}_{BCF}||}{\partial \boldsymbol{x}} + \frac{\partial \phi'}{\partial \hat{\boldsymbol{k}}^T \boldsymbol{n}_{BCF}} \frac{\partial \hat{\boldsymbol{k}}^T \boldsymbol{n}_{BCF}}{\partial \boldsymbol{x}}\right] \tag{52}$$

$$\frac{\partial \phi'}{\partial ||\hat{\boldsymbol{k}} \times \boldsymbol{n}_{BCF}||} = \frac{\hat{\boldsymbol{k}}^T \boldsymbol{n}_{BCF}}{||\hat{\boldsymbol{k}} \times \boldsymbol{n}_{BCF}||^2 + \left(\hat{\boldsymbol{k}}^T \boldsymbol{n}_{BCF}\right)^2} \tag{53}$$

$$\frac{\partial \phi'}{\partial \hat{\boldsymbol{k}}^T \boldsymbol{n}_{BCF}} = -\frac{||\hat{\boldsymbol{k}} \times \boldsymbol{n}_{BCF}||}{||\hat{\boldsymbol{k}} \times \boldsymbol{n}_{BCF}||^2 + \left(\hat{\boldsymbol{k}}^T \boldsymbol{n}_{BCF}\right)^2} \tag{54}$$

$$\frac{\partial ||\hat{\boldsymbol{k}} \times \boldsymbol{n}_{BCF}||}{\partial \boldsymbol{x}} = \frac{\left(\hat{\boldsymbol{k}}_{BCF} \times \boldsymbol{n}_{BCF}\right)^T}{||\hat{\boldsymbol{k}} \times \boldsymbol{n}_{BCF}||} \left\{\hat{\boldsymbol{k}}_{BCF}\right\}^{\times} \frac{\partial \boldsymbol{n}_{BCF}}{\partial \boldsymbol{x}} \tag{55}$$

$$\frac{\partial \hat{\boldsymbol{k}}^T \boldsymbol{n}_{BCF}}{\partial \boldsymbol{x}} = \hat{\boldsymbol{k}}_{BCF}^T \frac{\partial \boldsymbol{n}_{BCF}}{\partial \boldsymbol{x}} \tag{56}$$

The derivatives of $\boldsymbol{n}_{BCF}$ are given in Section 2.1.

5 Bodycentric Longitude

The bodycentric longitude is calculated as the angle in the BCF xy plane from the BCF x axis to the interface position vector:

$$\lambda = \operatorname{atan2}\left(r_{y,BCF}, r_{x,BCF}\right). \tag{57}$$

5.1 Derivatives

The derivative of longitude with respect to position is

$$\boxed{\frac{\partial \lambda}{\partial \boldsymbol{r}_{BCF}} = \begin{bmatrix} -\dfrac{r_{y,BCF}}{r_{x,BCF}^2 + r_{y,BCF}^2} & \dfrac{r_{x,BCF}}{r_{x,BCF}^2 + r_{y,BCF}^2} & 0 \end{bmatrix}}. \tag{58}$$

The longitude is independent of the velocity, so

$$\boxed{\frac{\partial \lambda}{\partial \boldsymbol{v}_{BCF}} = \boldsymbol{0}^T}. \tag{59}$$

The longitude is independent of time, so

$$\boxed{\frac{\partial \lambda}{\partial t} = 0}. \tag{60}$$

6 Bodydetic Longitude

Bodydetic longitude is calculated as the angle between the BCF x axis (i.e., $\boldsymbol{i}_{BCF}$) and the vector normal to the surface of the ellipsoid at the projection of $\boldsymbol{r}$ into the BCF xy plane. Let

$$\boldsymbol{r}_{proj_{xy},BCF} = \sqrt{r_{x,BCF}^2 + r_{y,BCF}^2} \begin{bmatrix} \cos\lambda \\ \sin\lambda \\ 0 \end{bmatrix}. \tag{61}$$

Then, transforming $\boldsymbol{r}_{proj_{xy}}$ into the PA frame gives

$$\boldsymbol{r}_{proj_{xy},PA} = \boldsymbol{R}^{BCF \to PA} \boldsymbol{r}_{proj_{xy},BCF}. \tag{62}$$

The normal vector itself is calculated in the PA frame using the equation of an ellipsoid:

$$\boldsymbol{n}_{proj_{xy},PA} = 2\boldsymbol{\epsilon}\boldsymbol{r}_{proj_{xy},PA}. \tag{63}$$

Transforming back to the BCF frame:

$$\boldsymbol{n}_{proj_{xy},BCF} = \boldsymbol{R}^{PA\rightarrow BCF}\boldsymbol{n}_{proj_{xy},PA} \tag{64}$$

The geodetic longitude angle calculation is then

$$\lambda' = \text{atan2}\left[||\boldsymbol{n}_{proj_{xy},BCF} \times \boldsymbol{i}_{BCF}||, \boldsymbol{n}^T_{proj_{xy},BCF}\boldsymbol{i}_{BCF}\right] \tag{65}$$

Because $\lambda' \in [0, 2\pi)$, a quadrant check is required:

$$\lambda' = 2\pi - \lambda' \quad \text{if} \quad \boldsymbol{n}^T_{proj_{xy},BCF}\boldsymbol{j}_{BCF} < 0 \tag{66}$$

6.1 Derivatives

λ' has no velocity dependence, so

$$\frac{\partial \lambda'}{\partial^{BCF}\boldsymbol{v}_{BCF}} = \boldsymbol{0}^T \tag{67}$$

For position and time, we have, from the derivative of atan2:

$$\frac{\partial \lambda'}{\partial \boldsymbol{x}} = \left[\frac{\partial \lambda'}{\partial||\boldsymbol{n}_{proj_{xy},BCF} \times \hat{\boldsymbol{i}}_{BCF}||}\frac{\partial||\boldsymbol{n}_{proj_{xy},BCF} \times \hat{\boldsymbol{i}}_{BCF}||}{\partial \boldsymbol{x}} + \frac{\partial \lambda'}{\partial \hat{\boldsymbol{i}}^T\boldsymbol{n}_{proj_{xy},BCF}}\frac{\partial \hat{\boldsymbol{i}}^T\boldsymbol{n}_{proj_{xy},B}}{\partial \boldsymbol{x}} \tag{68}$$

$$\frac{\partial \lambda'}{\partial||\boldsymbol{n}_{proj_{xy},BCF} \times \hat{\boldsymbol{i}}_{BCF}||} = \frac{\hat{\boldsymbol{i}}^T\boldsymbol{n}_{proj_{xy},BCF}}{||\boldsymbol{n}_{proj_{xy},BCF} \times \hat{\boldsymbol{i}}_{BCF}||^2 + \left(\hat{\boldsymbol{k}}^T\boldsymbol{n}_{BCF}\right)^2} \tag{69}$$

$$\frac{\partial \lambda'}{\partial \hat{\boldsymbol{i}}^T\boldsymbol{n}_{proj_{xy},BCF}} = -\frac{||\boldsymbol{n}_{proj_{xy},BCF} \times \hat{\boldsymbol{i}}_{BCF}||}{||\boldsymbol{n}_{proj_{xy},BCF} \times \hat{\boldsymbol{i}}_{BCF}||^2 + \left(\hat{\boldsymbol{i}}^T\boldsymbol{n}_{proj_{xy},BCF}\right)^2} \tag{70}$$

$$\frac{\partial||\boldsymbol{n}_{proj_{xy},BCF} \times \hat{\boldsymbol{i}}_{BCF}||}{\partial \boldsymbol{x}} = -\frac{\left(\boldsymbol{n}_{proj_{xy},BCF} \times \hat{\boldsymbol{i}}_{BCF}\right)^T}{||\boldsymbol{n}_{proj_{xy},BCF} \times \hat{\boldsymbol{i}}_{BCF}||}\left\{\hat{\boldsymbol{i}}_{BCF}\right\}^{\times}\frac{\partial \boldsymbol{n}_{proj_{xy},BCF}}{\partial \boldsymbol{x}} \tag{71}$$

$$\frac{\partial \hat{\boldsymbol{i}}^T\boldsymbol{n}_{proj_{xy},BCF}}{\partial \boldsymbol{x}} = \hat{\boldsymbol{i}}^T_{BCF}\frac{\partial \boldsymbol{n}_{proj_{xy},BCF}}{\partial \boldsymbol{x}} \tag{72}$$

Then, the derivatives of $\boldsymbol{n}_{proj_{xy},BCF}$ are:

$$\frac{\partial \boldsymbol{n}_{proj_{xy},BCF}}{\partial \boldsymbol{x}} = \frac{\partial}{\partial \boldsymbol{x}} \left[\boldsymbol{R}^{PA\to BCF} \boldsymbol{n}_{proj_{xy},PA}\right] \tag{73}$$

$$= 2\frac{\partial}{\partial \boldsymbol{x}} \left[\boldsymbol{R}^{PA\to BCF} \epsilon \boldsymbol{r}_{proj_{xy},PA}\right] \tag{74}$$

$$= 2\frac{\partial}{\partial \boldsymbol{x}} \left[\boldsymbol{R}^{PA\to BCF} \epsilon \boldsymbol{R}^{BCF\to PA} \boldsymbol{r}_{proj_{xy},BCF}\right] \tag{75}$$

$$= 2\frac{\partial}{\partial \boldsymbol{x}} \left[\boldsymbol{R}^{PA\to BCF} \epsilon \boldsymbol{R}^{BCF\to PA} \left(r_{x,BCF}^2 + r_{y,BCF}^2\right)^{1/2} \begin{bmatrix} \cos\lambda \\ \sin\lambda \\ 0 \end{bmatrix}\right] \tag{76}$$

$$\begin{aligned} &= 2\left[\frac{\partial \boldsymbol{R}^{PA\to BCF}}{\partial \boldsymbol{x}} \epsilon \boldsymbol{R}^{BCF\to PA} \left(r_{x,BCF}^2 + r_{y,BCF}^2\right)^{1/2} \begin{bmatrix} \cos\lambda \\ \sin\lambda \\ 0 \end{bmatrix}\right] + \\ &+ 2\left[\boldsymbol{R}^{PA\to BCF} \epsilon \frac{\partial \boldsymbol{R}^{BCF\to PA}}{\partial \boldsymbol{x}} \left(r_{x,BCF}^2 + r_{y,BCF}^2\right)^{1/2} \begin{bmatrix} \cos\lambda \\ \sin\lambda \\ 0 \end{bmatrix}\right] + \\ &+ 2\left[\boldsymbol{R}^{PA\to BCF} \epsilon \boldsymbol{R}^{BCF\to PA} \frac{\partial \left[\left(r_{x,BCF}^2 + r_{y,BCF}^2\right)^{1/2}\right]}{\partial \boldsymbol{x}} \begin{bmatrix} \cos\lambda \\ \sin\lambda \\ 0 \end{bmatrix}\right] + \\ &+ 2\left[\boldsymbol{R}^{PA\to BCF} \epsilon \boldsymbol{R}^{BCF\to PA} \left(r_{x,BCF}^2 + r_{y,BCF}^2\right)^{1/2} \frac{\partial}{\partial \boldsymbol{x}} \begin{bmatrix} \cos\lambda \\ \sin\lambda \\ 0 \end{bmatrix}\right] \end{aligned} \tag{77}$$

The derivatives of $\boldsymbol{R}^{BCF\to PA}$ and $\boldsymbol{R}^{PA\to BCF}$ with respect to the θ_i are given in Section 2.1. The θ_i are functions of time only with constant derivatives, so

$$\frac{\partial \boldsymbol{R}^{PA\to BCF}}{\partial t} = \sum_{i=1}^{3} \frac{\partial \boldsymbol{R}^{PA\to BCF}}{\partial \theta_i} \frac{\mathrm{d}\theta_i}{\mathrm{d}t} \tag{78}$$

The other intermediate derivatives are:

$$\frac{\partial \left(r_{x,BCF}^2 + r_{y,BCF}^2\right)^{1/2}}{\partial \boldsymbol{r}_{BCF}} = \frac{1}{\left(r_{x,BCF}^2 + r_{y,BCF}^2\right)^{1/2}} \begin{bmatrix} r_{x,BCF} & r_{y,BCF} & 0 \end{bmatrix} \tag{79}$$

$$\frac{\partial \left(r_{x,BCF}^2 + r_{y,BCF}^2\right)^{1/2}}{\partial^{BCF} \boldsymbol{v}_{BCF}} = \boldsymbol{0}^T \tag{80}$$

$$\frac{\partial \left(r_{x,BCF}^2 + r_{y,BCF}^2\right)^{1/2}}{\partial t} = 0 \tag{81}$$

$$\frac{\partial}{\partial \boldsymbol{x}} \begin{bmatrix} \sin\lambda \\ \cos\lambda \\ 0 \end{bmatrix} = \begin{bmatrix} -\sin\lambda \\ \cos\lambda \\ 0 \end{bmatrix} \frac{\partial \lambda}{\partial \boldsymbol{x}} \tag{82}$$

$\frac{\partial \lambda}{\partial \boldsymbol{x}}$ is given in Section 5.1.
Aside: Note that derivatives may be simplified somewhat because:

$$\frac{\partial \boldsymbol{r}_{proj_{xy},BCF}}{\partial t} = \boldsymbol{0} \tag{83}$$

$$\frac{\partial \boldsymbol{r}_{proj_{xy},BCF}}{\partial \boldsymbol{r}_{BCF}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{84}$$

The derivatives can switch sign because of the quadrant check:

$$\frac{\partial \lambda'}{\partial \boldsymbol{x}} = -\frac{\partial \lambda'}{\partial \boldsymbol{x}} \quad \text{if} \quad \boldsymbol{n}_{proj_{xy},BCF}^T \boldsymbol{j}_{BCF} < 0 \tag{85}$$

7 Velocity Magnitude

The velocity magnitude is calculated as the 2 norm of the velocity vector: $v = \left(\boldsymbol{v}^T \boldsymbol{v}\right)^{1/2}$. There are two velocities of interest: the velocity with respect to the BCF frame and the velocity with respect to the BCI frame. The two velocities are related via

$$^{BCI}\left[\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}\right] = ^{BCF}\left[\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}\right] + ^{BCF}\boldsymbol{\omega}^{BCI} \times \boldsymbol{r} \tag{86}$$

The frame in which the quantities in Eq. (86) are written does not matter as long as all quantities are written in the same frame.

7.1 Derivatives

7.1.1 Velocity in BCF Frame

Given a velocity with respect to the BCF frame, the derivatives are simple because the decision variables are also in the BCF frame.

$$\boxed{\frac{\partial \left[{}^{BCF}v_{BCF}\right]}{\partial \boldsymbol{r}_{BCF}} = \boldsymbol{0}^T} \tag{87}$$

$$\boxed{\frac{\partial \left[{}^{BCF}v_{BCF}\right]}{\partial \left[{}^{BCF}\boldsymbol{v}_{BCF}\right]} = \frac{{}^{BCF}\boldsymbol{v}_{BCF}^T}{v_{BCF}}} \tag{88}$$

$$\boxed{\frac{\partial \left[{}^{BCF}v_{BCF}\right]}{\partial t} = 0} \tag{89}$$

7.1.2 Velocity in BCI Frame

Given a velocity with respect to the BCI frame, the velocity must be expressed instead with respect to the BCF frame using Eq. (86), then differentiated with respect to the BCF state.

$$\frac{\partial \left[{}^{BCI}v_{BCF}\right]}{\partial \boldsymbol{r}_{BCF}} = \frac{\partial \left[{}^{BCI}v_{BCF}\right]}{\partial \left[{}^{BCI}\boldsymbol{v}_{BCF}\right]} \frac{\partial \left[{}^{BCI}\boldsymbol{v}_{BCF}\right]}{\partial \boldsymbol{r}_{BCF}} \tag{90}$$

$$= \frac{{}^{BCI}\boldsymbol{v}_{BCF}^T}{{}^{BCI}v_{BCF}} \frac{\partial \left[{}^{BCI}\boldsymbol{v}_{BCF}\right]}{\partial \boldsymbol{r}_{BCF}} \tag{91}$$

Similarly for the velocity and time derivatives:

$$\frac{\partial \left[{}^{BCI}v_{BCF}\right]}{\partial \left[\boldsymbol{v}_{BCF}\right]} = \frac{{}^{BCI}\boldsymbol{v}_{BCF}^T}{{}^{BCI}v_{BCF}} \frac{\partial \left[{}^{BCI}\boldsymbol{v}_{BCF}\right]}{\partial \boldsymbol{v}_{BCF}} \tag{92}$$

$$\frac{\partial \left[{}^{BCI}v_{BCF}\right]}{\partial t} = \frac{{}^{BCI}\boldsymbol{v}_{BCF}^T}{{}^{BCI}v_{BCF}} \frac{\partial \left[{}^{BCI}\boldsymbol{v}_{BCF}\right]}{\partial t} \tag{93}$$

$$\tag{94}$$

The intermediate derivatives are:

$$\frac{\partial \left[{}^{BCI}\boldsymbol{v}_{BCF}\right]}{\partial \boldsymbol{r}_{BCF}} = \left\{{}^{BCF}\boldsymbol{\omega}^{BCI}\right\}^{\times} \tag{95}$$

$$\frac{\partial \left[{}^{BCI}\boldsymbol{v}_{BCF}\right]}{\partial \boldsymbol{v}_{BCF}} = \boldsymbol{I} \tag{96}$$

$$\frac{\partial \left[{}^{BCI}\boldsymbol{v}_{BCF}\right]}{\partial t} = \frac{\partial \left\{{}^{BCF}\boldsymbol{\omega}^{BCI}\right\}^{\times}}{\partial t} \boldsymbol{r}_{BCF} \tag{97}$$

where the skew-symmetric cross matrix $\left\{ {}^{BCF}\boldsymbol{\omega}^{BCI} \right\}^{\times}$ is given by Eq. (182).

The derivative with respect to time is not currently written completely because it depends on the time derivative of ${}^{BCF}\boldsymbol{\omega}^{BCI}$, whose form I don't know yet.

If $\left[{}^{BCI}v_{BCI} \right]$ is known instead of $\left[{}^{BCI}v_{BCF} \right]$, then a coordinate transformation is also required:

$$\frac{\partial \left[{}^{BCI}v_{BCI} \right]}{\partial \boldsymbol{r}_{BCF}} = \frac{\partial \left[{}^{BCI}v_{BCI} \right]}{\partial \left[{}^{BCI}\boldsymbol{v}_{BCI} \right]} \frac{\partial \left[{}^{BCI}\boldsymbol{v}_{BCI} \right]}{\partial \left[{}^{BCI}\boldsymbol{v}_{BCF} \right]} \frac{\partial \left[{}^{BCI}\boldsymbol{v}_{BCF} \right]}{\partial \boldsymbol{r}_{BCF}} \tag{98}$$

$$= \frac{{}^{BCI}\boldsymbol{v}_{BCI}^{T}}{{}^{BCI}v_{BCI}} \boldsymbol{R}^{BCF \to BCI} \frac{\partial \left[{}^{BCI}\boldsymbol{v}_{BCF} \right]}{\partial \boldsymbol{r}_{BCF}} \tag{99}$$

$$= \frac{{}^{BCI}\boldsymbol{v}_{BCI}^{T}}{{}^{BCI}v_{BCI}} \boldsymbol{R}^{BCF \to BCI} \left\{ {}^{BCF}\boldsymbol{\omega}^{BCI} \right\}^{\times} \tag{100}$$

Similarly,

$$\frac{\partial \left[{}^{BCI}v_{BCI} \right]}{\partial \boldsymbol{v}_{BCF}} = \frac{\partial \left[{}^{BCI}v_{BCI} \right]}{\partial \left[{}^{BCI}\boldsymbol{v}_{BCI} \right]} \frac{\partial \left[{}^{BCI}\boldsymbol{v}_{BCI} \right]}{\partial \left[{}^{BCI}\boldsymbol{v}_{BCF} \right]} \frac{\partial \left[{}^{BCI}\boldsymbol{v}_{BCF} \right]}{\partial \boldsymbol{v}_{BCF}} \tag{101}$$

$$= \frac{{}^{BCI}\boldsymbol{v}_{BCI}^{T}}{{}^{BCI}v_{BCI}} \boldsymbol{R}^{BCF \to BCI} \frac{\partial \left[{}^{BCI}\boldsymbol{v}_{BCF} \right]}{\partial \left[{}^{BCF}\boldsymbol{v}_{BCF} \right]} \tag{102}$$

$$= \frac{{}^{BCI}\boldsymbol{v}_{BCI}^{T}}{{}^{BCI}v_{BCI}} \boldsymbol{R}^{BCF \to BCI} \boldsymbol{I} \tag{103}$$

$$= \frac{{}^{BCI}\boldsymbol{v}_{BCI}^{T}}{{}^{BCI}v_{BCI}} \boldsymbol{R}^{BCF \to BCI} \tag{104}$$

and

$$\frac{\partial \left[{}^{BCI}v_{BCI} \right]}{\partial t} = \frac{\partial \left[{}^{BCI}v_{BCI} \right]}{\partial \left[{}^{BCI}\boldsymbol{v}_{BCI} \right]} \frac{\partial \left[{}^{BCI}\boldsymbol{v}_{BCI} \right]}{\partial \left[{}^{BCI}\boldsymbol{v}_{BCF} \right]} \frac{\partial \left[{}^{BCI}\boldsymbol{v}_{BCF} \right]}{\partial t} \tag{105}$$

$$= \frac{{}^{BCI}\boldsymbol{v}_{BCI}^{T}}{{}^{BCI}v_{BCI}} \boldsymbol{R}^{BCF \to BCI} \frac{\partial \left[{}^{BCI}\boldsymbol{v}_{BCF} \right]}{\partial t} \tag{106}$$

$$= \frac{{}^{BCI}\boldsymbol{v}_{BCI}^{T}}{{}^{BCI}v_{BCI}} \boldsymbol{R}^{BCF \to BCI} \frac{\partial \left\{ {}^{BCF}\boldsymbol{\omega}^{BCI} \right\}^{\times}}{\partial t} \boldsymbol{r}_{BCF} \tag{107}$$

$$= \frac{{}^{BCI}\boldsymbol{v}_{BCI}^{T}}{{}^{BCI}v_{BCI}} \boldsymbol{R}^{BCF \to BCI} \frac{\partial \left\{ {}^{BCF}\boldsymbol{\omega}^{BCI} \right\}^{\times}}{\partial t} \boldsymbol{r}_{BCF} \tag{108}$$

8 Heading Angle

The heading angle may take on one of several values depending on reference frame choices. In all cases, the heading angle is measured with $\mathcal{H} = 0$ corresponding to a definition of south and measured positively toward a definition of east.

8.1 Topocentric Heading Angle Using Velocity with Respect to BCF Frame

For this definition,

$$\mathcal{H} = \text{atan2}\left({}^{BCF}v_{y,T}, {}^{BCF}v_{x,T}\right), \tag{109}$$

where

$${}^{BCF}\boldsymbol{v}_T = \boldsymbol{R}^{BCF\to T}\left[{}^{BCF}\boldsymbol{v}_{BCF}\right] \tag{110}$$

8.1.1 Derivatives

Chain rule:

$$\frac{\partial \mathcal{H}}{\partial \boldsymbol{x}} = \frac{\partial \mathcal{H}}{\partial {}^{BCF}v_{y,T}}\frac{\partial {}^{BCF}v_{y,T}}{\partial \boldsymbol{x}} + \frac{\partial \mathcal{H}}{\partial {}^{BCF}v_{x,T}}\frac{\partial {}^{BCF}v_{x,T}}{\partial \boldsymbol{x}} \tag{111}$$

Derivatives of atan2:

$$\frac{\partial \mathcal{H}}{\partial {}^{BCF}v_{y,T}} = \frac{{}^{BCF}v_{x,T}}{{}^{BCF}v_{x,T}^2 + {}^{BCF}v_{y,T}^2} \tag{112}$$

$$\frac{\partial \mathcal{H}}{\partial {}^{BCF}v_{x,T}} = -\frac{{}^{BCF}v_{y,T}}{{}^{BCF}v_{x,T}^2 + {}^{BCF}v_{y,T}^2} \tag{113}$$

Next, get derivatives of the velocity vector in the T frame because we need two components of it.

$$\frac{\partial {}^{BCF}\boldsymbol{v}_T}{\partial \boldsymbol{x}} = \frac{\partial \boldsymbol{R}^{BCF\to T}}{\partial \boldsymbol{x}}\left[{}^{BCF}\boldsymbol{v}_{BCF}\right] + \boldsymbol{R}^{BCF\to T}\frac{\partial {}^{BCF}\boldsymbol{v}_{BCF}}{\partial \boldsymbol{x}} \tag{114}$$

$$\frac{\partial {}^{BCF}\boldsymbol{v}_{BCF}}{\partial \boldsymbol{x}} = \begin{bmatrix} \boldsymbol{0}_{3\times3} & \boldsymbol{I}_{3\times3} & \boldsymbol{0}_{3\times1} \end{bmatrix} \tag{115}$$

$$\frac{\partial \boldsymbol{R}^{BCF\to T}}{\partial \boldsymbol{x}} = \frac{\partial}{\partial \boldsymbol{x}}\begin{bmatrix} {}_T\hat{\boldsymbol{S}}_{BCF}^T \\ {}_T\hat{\boldsymbol{E}}_{BCF}^T \\ \hat{\boldsymbol{n}}_{BCF}^T \end{bmatrix} \tag{116}$$

(Recall that $\hat{\boldsymbol{n}}_{BCF} = {}_T\hat{\boldsymbol{U}}_{BCF}^T$.) $\partial\hat{\boldsymbol{n}}_{BCF}/\partial\boldsymbol{x}$ is given in Section 2.1. East and south need to be derived, though.

$$\frac{\partial_T \boldsymbol{E}_{BCF}}{\partial \boldsymbol{x}} = \frac{\partial}{\partial \boldsymbol{x}}\left[2\left\{\hat{\boldsymbol{k}}_{BCF}\right\}^{\times}\boldsymbol{R}^{PA\to BCF}\boldsymbol{\epsilon}\boldsymbol{R}^{BCF\to PA}\boldsymbol{r}_{BCF}\right] \tag{117}$$

The elements $\left\{\hat{\boldsymbol{k}}_{BCF}\right\}^{\times}$ and $\boldsymbol{\epsilon}$ are constant with respect to the decision vector and their derivatives with respect to the decision vector are zero. Thus,

$$\frac{\partial_T \boldsymbol{E}_{BCF}}{\partial \boldsymbol{x}} = 2\left\{\hat{\boldsymbol{k}}_{BCF}\right\}^{\times} \tag{118}$$

$$\left\{\left[\frac{\partial}{\partial \boldsymbol{x}}\boldsymbol{R}^{PA\rightarrow BCF}\boldsymbol{\epsilon}\boldsymbol{R}^{BCF\rightarrow PA} + \boldsymbol{R}^{PA\rightarrow BCF}\epsilon\frac{\partial}{\partial \boldsymbol{x}}\boldsymbol{R}^{BCF\rightarrow PA}\right]\boldsymbol{r}_{BCF} + \boldsymbol{R}^{PA\rightarrow BCF}\boldsymbol{\epsilon}\boldsymbol{R}^{BCF\rightarrow PA}\frac{\partial \boldsymbol{r}_{BCF}}{\partial \boldsymbol{x}}\right\} \tag{119}$$

The individual derivatives are:

$$\frac{\partial \boldsymbol{r}_{BCF}}{\partial \boldsymbol{x}} = \begin{bmatrix} \boldsymbol{I}_{3\times 3} & \boldsymbol{0}_{3\times 3} & \boldsymbol{0}_{3\times 1} \end{bmatrix} \tag{120}$$

$$\frac{\partial}{\partial \boldsymbol{r}_{BCF}}\boldsymbol{R}^{PA\rightarrow BCF} = \boldsymbol{0} \tag{121}$$

$$\frac{\partial}{\partial^{BCF}\boldsymbol{v}_{BCF}}\boldsymbol{R}^{PA\rightarrow BCF} = \boldsymbol{0} \tag{122}$$

$$\frac{\partial}{\partial t}\boldsymbol{R}^{PA\rightarrow BCF} = \frac{\partial}{\partial \theta_1}\left[\boldsymbol{R}^{PA\rightarrow BCF}\right]\frac{\mathrm{d}\theta_1}{\mathrm{d}t} + \frac{\partial}{\partial \theta_2}\left[\boldsymbol{R}^{PA\rightarrow BCF}\right]\frac{\mathrm{d}\theta_2}{\mathrm{d}t} + \frac{\partial}{\partial \theta_3}\left[\boldsymbol{R}^{PA\rightarrow BCF}\right]\frac{\mathrm{d}\theta_3}{\mathrm{d}t} \tag{123}$$

Note that $\boldsymbol{R}^{BCF\rightarrow PA} = \left[\boldsymbol{R}^{PA\rightarrow BCF}\right]^T$, so the derivatives are also transposes of one another.

The derivative of the unit vector ${}_T\hat{\boldsymbol{E}}_{BCF}$ is then calculated using Eq. (185) and

$$\frac{\partial_T \hat{\boldsymbol{E}}_{BCF}}{\partial \boldsymbol{x}} = \frac{\partial_T \hat{\boldsymbol{E}}_{BCF}}{\partial_T \boldsymbol{E}_{BCF}}\frac{\partial_T \boldsymbol{E}_{BCF}}{\partial \boldsymbol{x}}. \tag{124}$$

For the south unit vector,

$$\frac{\partial_T \hat{\boldsymbol{S}}_{BCF}}{\partial \boldsymbol{x}} = \frac{\partial_T \hat{\boldsymbol{S}}_{BCF}}{\partial_T \boldsymbol{S}_{BCF}}\frac{\partial_T \boldsymbol{S}_{BCF}}{\partial \boldsymbol{x}}. \tag{125}$$

The unit vector derivative is calculated using Eq. (185). The derivative of the non-unitized south vector is

$$\frac{\partial_T \boldsymbol{S}_{BCF}}{\partial \boldsymbol{x}} = \frac{\partial}{\partial \boldsymbol{x}}\left[\left\{{}_T\boldsymbol{E}_{BCF}\right\}^{\times}\boldsymbol{n}_{BCF}\right] \tag{126}$$

$$= \frac{\partial}{\partial \boldsymbol{x}}\left[\left\{{}_T\boldsymbol{E}_{BCF}\right\}^{\times}\right]\boldsymbol{n}_{BCF} + \left\{{}_T\boldsymbol{E}_{BCF}\right\}^{\times}\frac{\partial \boldsymbol{n}_{BCF}}{\partial \boldsymbol{x}} \tag{127}$$

$\frac{\partial \boldsymbol{n}_{BCF}}{\partial \boldsymbol{x}}$ is given in Section 2.1. $\frac{\partial}{\partial \boldsymbol{x}}\left[\left\{{}_T\boldsymbol{E}_{BCF}\right\}^{\times}\right]$ is calculated using components of $\frac{\partial_T \boldsymbol{E}_{BCF}}{\partial \boldsymbol{x}}$, which is given by Eq. (118):

$$\frac{\partial}{\partial \boldsymbol{x}}\left[\{{}_T\boldsymbol{E}_{BCF}\}^{\times}\right] = \begin{bmatrix} 0 & -\frac{\partial}{\partial \boldsymbol{x}}\left[{}_T\boldsymbol{E}_{z,BCF}\right] & \frac{\partial}{\partial \boldsymbol{x}}\left[{}_T\boldsymbol{E}_{y,BCF}\right] \\ \frac{\partial}{\partial \boldsymbol{x}}\left[{}_T\boldsymbol{E}_{z,BCF}\right] & 0 & -\frac{\partial}{\partial \boldsymbol{x}}\left[{}_T\boldsymbol{E}_{x,BCF}\right] \\ -\frac{\partial}{\partial \boldsymbol{x}}\left[{}_T\boldsymbol{E}_{y,BCF}\right] & \frac{\partial}{\partial \boldsymbol{x}}\left[{}_T\boldsymbol{E}_{x,BCF}\right] & 0 \end{bmatrix} \tag{128}$$

8.2 Topocentric Heading Angle Using Velocity with Respect to BCI Frame

For this definition,

$$\mathcal{H} = \operatorname{atan2}\left({}^{BCI}v_{y,T}, {}^{BCI}v_{x,T}\right). \tag{129}$$

where

$$\begin{aligned} {}^{BCI}\boldsymbol{v}_T &= \boldsymbol{R}^{BCF\to T}\left[{}^{BCI}\boldsymbol{v}_{BCF}\right] && (130) \\ &= \boldsymbol{R}^{BCF\to T}\boldsymbol{R}^{BCI\to BCF}\left[{}^{BCI}\boldsymbol{v}_{BCI}\right] && (131) \end{aligned}$$

8.2.1 Derivatives

Chain rule:

$$\frac{\partial \mathcal{H}}{\partial \boldsymbol{x}} = \frac{\partial \mathcal{H}}{\partial {}^{BCI}v_{y,T}}\frac{\partial {}^{BCI}v_{y,T}}{\partial \boldsymbol{x}} + \frac{\partial \mathcal{H}}{\partial {}^{BCI}v_{x,T}}\frac{\partial {}^{BCI}v_{x,T}}{\partial \boldsymbol{x}} \tag{132}$$

Derivatives of atan2:

$$\begin{aligned} \frac{\partial \mathcal{H}}{\partial {}^{BCI}v_{y,T}} &= \frac{{}^{BCI}v_{x,T}}{{}^{BCI}v_{x,T}^2 + {}^{BCI}v_{y,T}^2} && (133) \\ \frac{\partial \mathcal{H}}{\partial {}^{BCI}v_{x,T}} &= -\frac{{}^{BCI}v_{y,T}}{{}^{BCI}v_{x,T}^2 + {}^{BCI}v_{y,T}^2} && (134) \end{aligned}$$

Next, get derivatives of the velocity vector in the T frame because we need two components of it.

$$\begin{aligned} \frac{\partial {}^{BCI}\boldsymbol{v}_T}{\partial \boldsymbol{x}} &= \frac{\partial \boldsymbol{R}^{BCF\to T}}{\partial \boldsymbol{x}}\left[{}^{BCI}\boldsymbol{v}_{BCF}\right] + \boldsymbol{R}^{BCF\to T}\frac{\partial {}^{BCI}\boldsymbol{v}_{BCF}}{\partial \boldsymbol{x}} && (135) \\ &= \frac{\partial \boldsymbol{R}^{BCF\to T}}{\partial \boldsymbol{x}}\boldsymbol{R}^{BCI\to BCF}\left[{}^{BCI}\boldsymbol{v}_{BCI}\right] + \boldsymbol{R}^{BCF\to T}\frac{\partial \boldsymbol{R}^{BCI\to BCF}}{\partial \boldsymbol{x}}\left[{}^{BCI}\boldsymbol{v}_{BCI}\right] + \boldsymbol{R}^{BCF\to T}\boldsymbol{R}^{BCI\to BCF}\frac{\partial}{} && (136) \end{aligned}$$

$\frac{\partial \boldsymbol{R}^{BCF\to T}}{\partial \boldsymbol{x}}$ is given in Section 8.1.1. $\frac{\partial^{BCI}\boldsymbol{v}_{BCF}}{\partial \boldsymbol{x}}$ and $\frac{\partial^{BCI}\boldsymbol{v}_{BCI}}{\partial \boldsymbol{x}}$ are given in Section 7.1.2. $\frac{\partial \boldsymbol{R}^{BCI\to BCF}}{\partial \boldsymbol{x}}$ is given in Section 1.3.1. Thus, all the intermediate derivatives are already known.

8.3 Polar Heading Angle Using Velocity with Respect to BCF Frame

For this definition,

$$\mathcal{H} = \text{atan2}\left({}^{BCF}v_{y,P}, {}^{BCF}v_{x,P}\right). \tag{137}$$

where

$${}^{BCF}\boldsymbol{v}_P = \boldsymbol{R}^{BCF\to P}\left[{}^{BCF}\boldsymbol{v}_{BCF}\right] \tag{138}$$

8.3.1 Derivatives

Chain rule:

$$\frac{\partial \mathcal{H}}{\partial \boldsymbol{x}} = \frac{\partial \mathcal{H}}{\partial {}^{BCF}v_{y,P}}\frac{\partial {}^{BCF}v_{y,P}}{\partial \boldsymbol{x}} + \frac{\partial \mathcal{H}}{\partial {}^{BCF}v_{x,P}}\frac{\partial {}^{BCF}v_{x,P}}{\partial \boldsymbol{x}} \tag{139}$$

Derivatives of atan2:

$$\frac{\partial \mathcal{H}}{\partial {}^{BCF}v_{y,P}} = \frac{{}^{BCF}v_{x,P}}{{}^{BCF}v_{x,P}^2 + {}^{BCF}v_{y,P}^2} \tag{140}$$

$$\frac{\partial \mathcal{H}}{\partial {}^{BCF}v_{x,P}} = -\frac{{}^{BCF}v_{y,P}}{{}^{BCF}v_{x,P}^2 + {}^{BCF}v_{y,P}^2} \tag{141}$$

Next, get derivatives of the velocity vector in the P frame because we need two components of it.

$$\frac{\partial {}^{BCF}\boldsymbol{v}_P}{\partial \boldsymbol{x}} = \frac{\partial \boldsymbol{R}^{BCF\to P}}{\partial \boldsymbol{x}}\left[{}^{BCF}\boldsymbol{v}_{BCF}\right] + \boldsymbol{R}^{BCF\to P}\frac{\partial {}^{BCF}\boldsymbol{v}_{BCF}}{\partial \boldsymbol{x}} \tag{142}$$

$$\frac{\partial {}^{BCF}\boldsymbol{v}_{BCF}}{\partial \boldsymbol{x}} = \begin{bmatrix} \boldsymbol{0}_{3\times3} & \boldsymbol{I}_{3\times3} & \boldsymbol{0}_{3\times1} \end{bmatrix} \tag{143}$$

$$\frac{\partial \boldsymbol{R}^{BCF\to P}}{\partial \boldsymbol{x}} = \frac{\partial}{\partial \boldsymbol{x}}\begin{bmatrix} {}_P\hat{\boldsymbol{S}}_{BCF}^T \\ {}_P\hat{\boldsymbol{E}}_{BCF}^T \\ \hat{\boldsymbol{n}}_{BCF}^T \end{bmatrix} \tag{144}$$

(Recall that $\hat{\boldsymbol{n}}_{BCF} = {}_P\hat{\boldsymbol{U}}_{BCF}^T$.) $\partial\hat{\boldsymbol{n}}_{BCF}/\partial\boldsymbol{x}$ is given in Section 2.1. East and south need to be derived, though. Start with "pseudo-east."

$$\frac{\partial \tilde{\boldsymbol{E}}_{BCF}}{\partial \boldsymbol{x}} = \frac{\partial}{\partial \boldsymbol{x}}\left[\left\{\hat{\boldsymbol{k}}_{BCF}\right\}^{\times}\boldsymbol{r}_{BCF}\right] \tag{145}$$

$$= \frac{\partial\left\{\hat{\boldsymbol{k}}_{BCF}\right\}^{\times}}{\partial \boldsymbol{x}}\boldsymbol{r}_{BCF} + \left\{\hat{\boldsymbol{k}}_{BCF}\right\}^{\times}\frac{\partial \boldsymbol{r}_{BCF}}{\partial \boldsymbol{x}} \tag{146}$$

The derivatives of $\left\{\hat{\boldsymbol{k}}_{BCF}\right\}^{\times}$ are zero. $\frac{\partial \boldsymbol{r}_{BCF}}{\partial \boldsymbol{x}}$ is

$$\frac{\partial \boldsymbol{r}_{BCF}}{\partial \boldsymbol{x}} = \left[\begin{array}{ccc} \boldsymbol{I}_{3\times 3} & \boldsymbol{0}_{3\times 3} & \boldsymbol{0}_{3\times 1} \end{array}\right] \tag{147}$$

For south:

$$\frac{\partial_P \boldsymbol{S}_{BCF}}{\partial \boldsymbol{x}} = \frac{\partial}{\partial \boldsymbol{x}}\left[\left\{\tilde{\boldsymbol{E}}_{BCF}\right\}^{\times} \boldsymbol{n}_{BCF}\right] \tag{148}$$

$$= \frac{\partial \left\{\tilde{\boldsymbol{E}}_{BCF}\right\}^{\times}}{\partial \boldsymbol{x}} \boldsymbol{n}_{BCF} + \left\{\tilde{\boldsymbol{E}}_{BCF}\right\}^{\times} \frac{\partial \boldsymbol{n}_{BCF}}{\partial \boldsymbol{x}} \tag{149}$$

$\frac{\partial \boldsymbol{n}_{BCF}}{\partial \boldsymbol{x}}$ is given in Section 2.1. $\frac{\partial \left\{\tilde{\boldsymbol{E}}_{BCF}\right\}^{\times}}{\partial \boldsymbol{x}}$ is made up of components of $\frac{\partial \tilde{\boldsymbol{E}}_{BCF}}{\partial \boldsymbol{x}}$, given in Eq. (146). It is noted that the derivative of a matrix with respect to a vector gives a three-dimensional tensor. (See Section 10.6.)

For east:

$$\frac{\partial_P \boldsymbol{E}_{BCF}}{\partial \boldsymbol{x}} = \frac{\partial}{\partial \boldsymbol{x}}\left[\left\{\boldsymbol{n}_{BCF}\right\}^{\times}_{P} \boldsymbol{S}_{BCF}\right] \tag{150}$$

$$= \frac{\partial \left\{\boldsymbol{n}_{BCF}\right\}^{\times}}{\partial \boldsymbol{x}} {}_P\boldsymbol{S}_{BCF} + \left\{\boldsymbol{n}_{BCF}\right\}^{\times} \frac{\partial_P \boldsymbol{S}_{BCF}}{\partial \boldsymbol{x}} \tag{151}$$

$\frac{\partial_P \boldsymbol{S}_{BCF}}{\partial \boldsymbol{x}}$ is given in Eq. (149). $\frac{\partial \{\boldsymbol{n}_{BCF}\}^{\times}}{\partial \boldsymbol{x}}$ is a 3D tensor derived from elements of $\frac{\partial \boldsymbol{n}_{BCF}}{\partial \boldsymbol{x}}$. (See Sections 2.1 and 10.6.)

8.4 Polar Heading Angle Using Velocity with Respect to BCI Frame

For this definition,

$$\mathcal{H} = \operatorname{atan2}\left({}^{BCI}v_{y,P}, {}^{BCI}v_{x,P}\right). \tag{152}$$

where

$${}^{BCI}\boldsymbol{v}_P = \boldsymbol{R}^{BCF\to P}\left[{}^{BCI}\boldsymbol{v}_{BCF}\right] \tag{153}$$

$$= \boldsymbol{R}^{BCF\to P}\boldsymbol{R}^{BCI\to BCF}\left[{}^{BCI}\boldsymbol{v}_{BCI}\right] \tag{154}$$

8.4.1 Derivatives

Chain rule:

$$\frac{\partial \mathcal{H}}{\partial \boldsymbol{x}} = \frac{\partial \mathcal{H}}{\partial^{BCI} v_{y,P}} \frac{\partial^{BCI} v_{y,P}}{\partial \boldsymbol{x}} + \frac{\partial \mathcal{H}}{\partial^{BCI} v_{x,P}} \frac{\partial^{BCI} v_{x,P}}{\partial \boldsymbol{x}} \tag{155}$$

Derivatives of atan2:

$$\frac{\partial \mathcal{H}}{\partial^{BCI} v_{y,P}} = \frac{^{BCI} v_{x,P}}{^{BCI} v_{x,P}^2 +^{BCI} v_{y,P}^2} \tag{156}$$

$$\frac{\partial \mathcal{H}}{\partial^{BCI} v_{x,P}} = -\frac{^{BCI} v_{y,P}}{^{BCI} v_{x,P}^2 +^{BCI} v_{y,P}^2} \tag{157}$$

Next, get derivatives of the velocity vector in the P frame because we need two components of it.

$$\frac{\partial^{BCI} \boldsymbol{v}_P}{\partial \boldsymbol{x}} = \frac{\partial \boldsymbol{R}^{BCF \to P}}{\partial \boldsymbol{x}} \left[^{BCI} \boldsymbol{v}_{BCF}\right] + \boldsymbol{R}^{BCF \to P} \frac{\partial^{BCI} \boldsymbol{v}_{BCF}}{\partial \boldsymbol{x}} \tag{158}$$

$$= \frac{\partial \boldsymbol{R}^{BCF \to P}}{\partial \boldsymbol{x}} \boldsymbol{R}^{BCI \to BCF} \left[^{BCI} \boldsymbol{v}_{BCI}\right] + \boldsymbol{R}^{BCF \to P} \frac{\partial \boldsymbol{R}^{BCI \to BCF}}{\partial \boldsymbol{x}} \left[^{BCI} \boldsymbol{v}_{BCI}\right] + \boldsymbol{R}^{BCF \to P} \boldsymbol{R}^{BCI \to BCF} \tag{159}$$

$\frac{\partial \boldsymbol{R}^{BCF \to P}}{\partial \boldsymbol{x}}$ is given in Section 8.3.1. $\frac{\partial^{BCI} \boldsymbol{v}_{BCF}}{\partial \boldsymbol{x}}$ and $\frac{\partial^{BCI} \boldsymbol{v}_{BCI}}{\partial \boldsymbol{x}}$ are given in Section 7.1.2. $\frac{\partial \boldsymbol{R}^{BCI \to BCF}}{\partial \boldsymbol{x}}$ is given in Section 1.3.1. Thus, all the intermediate derivatives are already known.

9 Flight Path Angle

9.1 Velocity Relative to BCF Frame

The flight path angle γ is defined as

$$\gamma = \text{asin}\left(\frac{^{BCF} v_{z,T}}{^{BCF} v}\right) \tag{160}$$

$$= \text{atan2}\left[^{BCF} v_{z,T}, \left(^{BCF} v_{x,T}^2 +^{BCF} v_{y,T}^2\right)^{1/2}\right], \tag{161}$$

where

$$^{BCF} \boldsymbol{v}_T = \boldsymbol{R}^{BCF \to T} \left[^{BCF} \boldsymbol{v}_{BCF}\right] \tag{162}$$

9.1.1 Derivatives

The first level of the chain rule is performed using the derivative of atan2. (See Section 10.1.)

$$\frac{\partial \gamma}{\partial \boldsymbol{x}} = \frac{\partial \gamma}{\partial {}^{BCF}v_{z,T}} \frac{\partial {}^{BCF}v_{z,T}}{\partial \boldsymbol{x}} + \frac{\partial \gamma}{\partial \left({}^{BCF}v_{x,T}^2 + {}^{BCF}\ v_{y,T}^2\right)^{1/2}} \frac{\partial \left({}^{BCF}v_{x,T}^2 + {}^{BCF}\ v_{y,T}^2\right)^{1/2}}{\partial \boldsymbol{x}} \tag{163}$$

$$\frac{\partial \gamma}{\partial {}^{BCF}v_{z,T}} = \frac{\left({}^{BCF}v_{x,T}^2 + {}^{BCF}\ v_{y,T}^2\right)^{1/2}}{{}^{BCF}v^2} \tag{164}$$

$$\frac{\partial \gamma}{\partial \left({}^{BCF}v_{x,T}^2 + {}^{BCF}\ v_{y,T}^2\right)^{1/2}} = -\frac{{}^{BCF}v_{z,T}}{{}^{BCF}v^2} \tag{165}$$

$\frac{\partial {}^{BCF}\boldsymbol{v}_T}{\partial \boldsymbol{x}}$ is given in Eq. (114). Here, we just need to combine the components.

$$\frac{\partial {}^{BCF}v_{z,T}}{\partial \boldsymbol{x}} = \frac{\partial {}^{BCF}v_{z,T}}{\partial {}^{BCF}\boldsymbol{v}_T} \frac{\partial {}^{BCF}\boldsymbol{v}_T}{\partial \boldsymbol{x}} \tag{166}$$

$$\frac{\partial \left({}^{BCF}v_{x,T}^2 + {}^{BCF}\ v_{y,T}^2\right)^{1/2}}{\partial \boldsymbol{x}} = \frac{\partial \left({}^{BCF}v_{x,T}^2 + {}^{BCF}\ v_{y,T}^2\right)^{1/2}}{\partial {}^{BCF}\boldsymbol{v}_T} \frac{\partial {}^{BCF}\boldsymbol{v}_T}{\partial \boldsymbol{x}} \tag{167}$$

$$\frac{\partial {}^{BCF}v_{z,T}}{\partial {}^{BCF}\boldsymbol{v}_T} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \tag{168}$$

$$\frac{\partial \left({}^{BCF}v_{x,T}^2 + {}^{BCF}\ v_{y,T}^2\right)^{1/2}}{\partial {}^{BCF}\boldsymbol{v}_T} = \frac{1}{\left({}^{BCF}v_{x,T}^2 + {}^{BCF}\ v_{y,T}^2\right)^{1/2}} \begin{bmatrix} {}^{BCF}v_{x,T} & {}^{BCF}v_{y,T} & 0 \end{bmatrix} \tag{169}$$

9.2 Velocity Relative to BCI Frame

The flight path angle γ is defined as

$$\gamma = \operatorname{asin}\left(\frac{{}^{BCI}v_{z,T}}{{}^{BCI}v}\right) \tag{170}$$

$$= \operatorname{atan2}\left[{}^{BCI}v_{z,T}, \left({}^{BCI}v_{x,T}^2 + {}^{BCI}\ v_{y,T}^2\right)^{1/2}\right], \tag{171}$$

where

$${}^{BCI}\boldsymbol{v}_T = \boldsymbol{R}^{BCI\to T}\left[{}^{BCI}\boldsymbol{v}_{BCI}\right] \tag{172}$$

$$\boldsymbol{R}^{BCI\to T} = \boldsymbol{R}^{BCF\to T}\boldsymbol{R}^{BCI\to BCF}\left[{}^{BCI}\boldsymbol{v}_{BCI}\right] \tag{173}$$

9.2.1 Derivatives

The first level of the chain rule is performed using the derivative of atan2. (See Section 10.1.)

$$\frac{\partial \gamma}{\partial \boldsymbol{x}} = \frac{\partial \gamma}{\partial {}^{BCI}v_{z,T}} \frac{\partial {}^{BCI}v_{z,T}}{\partial \boldsymbol{x}} + \frac{\partial \gamma}{\partial \left({}^{BCI}v_{x,T}^2 + {}^{BCI}v_{y,T}^2 \right)^{1/2}} \frac{\partial \left({}^{BCI}v_{x,T}^2 + {}^{BCI}v_{y,T}^2 \right)^{1/2}}{\partial \boldsymbol{x}} \tag{174}$$

$$\frac{\partial \gamma}{\partial {}^{BCI}v_{z,T}} = \frac{\left({}^{BCI}v_{x,T}^2 + {}^{BCI}v_{y,T}^2 \right)^{1/2}}{{}^{BCI}v^2} \tag{175}$$

$$\frac{\partial \gamma}{\partial \left({}^{BCI}v_{x,T}^2 + {}^{BCI}v_{y,T}^2 \right)^{1/2}} = -\frac{{}^{BCI}v_{z,T}}{{}^{BCI}v^2} \tag{176}$$

For derivatives of the BCI velocity, we use the chain rule and the derivative of the BCF velocity, which we already have (Eq. (114)).

$$\frac{\partial {}^{BCI}\boldsymbol{v}_T}{\partial \boldsymbol{x}} = \frac{\partial}{\partial \boldsymbol{x}} \left\{ \boldsymbol{R}^{BCF \to T} \boldsymbol{R}^{BCI \to BCF} \left[{}^{BCI}\boldsymbol{v}_{BCI} \right] \right\} \tag{177}$$

$$= \left\{ \frac{\partial \boldsymbol{R}^{BCF \to T}}{\partial \boldsymbol{x}} \boldsymbol{R}^{BCI \to BCF} + \boldsymbol{R}^{BCF \to T} \frac{\partial \boldsymbol{R}^{BCI \to BCF}}{\partial \boldsymbol{x}} \right\} \left[{}^{BCI}\boldsymbol{v}_{BCI} \right] + \boldsymbol{R}^{BCF \to T} \boldsymbol{R}^{BCI \to BCF} \frac{\partial \left[{}^{BCI}\boldsymbol{v}_{BCI} \right]}{\partial \boldsymbol{x}} \tag{178}$$

The term $\frac{\partial \boldsymbol{R}^{BCF \to T}}{\partial \boldsymbol{x}}$ is given by Eq. (116). The term $\frac{\partial \left[{}^{BCI}\boldsymbol{v}_{BCI} \right]}{\partial \boldsymbol{x}}$ is obtained from Section 7.1.2. The term $\frac{\partial \boldsymbol{R}^{BCI \to BCF}}{\partial t}$ is obtained from Eq. (9). (All other elements of $\frac{\partial \boldsymbol{R}^{BCI \to BCF}}{\partial \boldsymbol{x}}$ are zero.)

10 Utility Math

10.1 atan2

Let $\alpha = \operatorname{atan2}(\alpha_y, \alpha_x)$ be the atan2 function. Then

$$\frac{\partial \alpha}{\partial \alpha_x} = -\frac{\alpha_y}{\alpha_x^2 + \alpha_y^2} \tag{179}$$

$$\frac{\partial \alpha}{\partial \alpha_y} = \frac{\alpha_x}{\alpha_x^2 + \alpha_y^2} \tag{180}$$

10.2 Magnitude of Vector

$$\frac{\partial x}{\partial \boldsymbol{x}} = \frac{\boldsymbol{x}^T}{x} \tag{181}$$

10.3 Skew-symmetric Cross Matrix

$$\{\boldsymbol{\omega}\}^{\times} = \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix} \tag{182}$$

10.4 Angle Between Two Vectors

The angle α between any two vectors $\boldsymbol{x}_{3\times1}$ and $\boldsymbol{y}_{3\times1}$ expressed in the same reference frame may be calculated as:

$$\alpha = \text{atan2}\left[||\boldsymbol{x} \times \boldsymbol{y}||, \boldsymbol{x}^T \boldsymbol{y}\right] \tag{183}$$

Note that $\alpha \in [0, \pi]$. (I.e., the angle is not "directional" in the sense that α is always the shortest angle between the two vectors.)

10.5 Unit Vector

A unit vector is defined by

$$\hat{\boldsymbol{x}} = \frac{\boldsymbol{x}}{x}, \tag{184}$$

where $x = \left(\boldsymbol{x}^T \boldsymbol{x}\right)^{1/2}$. The derivative of a unit vector with respect to the vector itself is

$$\frac{\partial \hat{\boldsymbol{x}}}{\partial \boldsymbol{x}} = \frac{1}{x}\left(\boldsymbol{I} - \frac{1}{x^2}\boldsymbol{x}\boldsymbol{x}^T\right), \tag{185}$$

where $\boldsymbol{I}$ is an appropriately sized identity matrix.

10.6 Three-dimensional Tensors

10.6.1 Derivative of Matrix with Respect to Vector

The derivative of a matrix $\boldsymbol{M}_{m\times n}$ w.r.t. a vector $\boldsymbol{v}_{p\times1}$ is defined for the purposes of this work as a three-dimensional tensor $\boldsymbol{T}$:

$$\boldsymbol{T}(i,j,k) = \frac{\partial \boldsymbol{M}(i,j)}{\partial \boldsymbol{v}(k)}, \quad i \in [1,m], \quad j \in [1,n], \quad k \in [1,p] \tag{186}$$

10.6.2 Tensor Multiplication with Vector

The product of tensor $\boldsymbol{T}_{n\times n\times n}$ and vector $\boldsymbol{v}_{n\times 1}$ is an $n \times n$ matrix:

$$[\boldsymbol{T} \bullet_2 \boldsymbol{v}]\,(i,j) \triangleq \sum_{p=1}^{n} \boldsymbol{T}(i,p,j)\boldsymbol{v}(p) \tag{187}$$